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Multi-species interacting particle systems: duality, integrability and scaling limits

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Abstract

Interacting particle systems (IPS) are paradigmatic models for non-equilibrium statistical mechanics. Indeed, they allow to keep the computations manageable without losing the fundamental features of the systems they aim to reproduce. Usually, two central questions are investigated: the non-equilibrium stationary probability distribution, that is reached in the long-time horizon, and the hydrodynamic limit, which allows us to derive macroscopic laws that the IPS satisfies, once rescaled to the continuum.

In the literature many results and techniques have been developed regarding *single species IPS*, which are models where indistinguishable particles move on a discrete geometry occupying the available vacancies. At the microscopic level, duality and integrability allow to obtain closed expressions for the non-equilibrium stationary distribution, while, the scaling limits theory has made rigorous the passage from micro to macro dynamics.

At present, there is a growing interest in systems with multiple conservation laws, therefore, from the point of view of statistical mechanics, research focuses on *multi-species interacting particles systems.* In these IPS, there is the presence of multiple different species (or colours) of particles which perform the dynamic on a discrete geometry. The difference with respect to the single species set up is that, now, in addition to the occupation of the available vacancies, the different species of particles interact with each other, for instance by exchanging their positions and mutating species. This dynamics, is also reflected in the scaling limits, giving rise to systems of partial differential equations (PDE's) describing multi-component transport models. In this realm, new phenomena may arise, such as multi-component uphill diffusion, which is a situation where the flux (of mass, particle, energy, charges...) has the same sign as the difference between boundary densities, causing the current to go uphill and violating the Fick's law of diffusion.

In this thesis we focus on boundary driven multi-species IPS where each vertex of the discrete geometry can host at most a finite number of particles, which can be chosen among the different species. For this reason, these IPS are said to have compact state space. First we analyse the so-called *multi-species stirring process*, which consists of a generalization of the symmetric exclusion process (SEP), when many colours of particles are considered. After describing the model without boundary interaction, we derive its scaling limits, both the hydrodynamic limit and the equilibrium fluctuations from it. Then, we put the system out of equilibrium by adding boundary reservoirs. We describe the non-equilibrium generator via a Lie algebra and we use this property to find an absorbing duality result. This absorbing duality, allows us to characterized the non-equilibrium steady state by means of absorption probabilities, which are the probabilities that dual particles are absorbed at *extra-sites.* These extra-sites replace, in the dual process, the original boundary reservoirs. While this duality result is available regardless of the geometry and maximal occupancy of the process, the complete characterization of the non-equilibrium steady state is accomplished for the integrable version of multi-species stirring process via the combination of duality and matrix product ansatz (MPA). This allows us to write a closed

formula for the multi-point correlations. Finally, as a still partially open problem, we construct, for this integrable chain, the so-called *mapping of non-equilibrium onto equilibrium* by the use of *quantum inverse scattering method.*

After having analyzed the stirring process we add a further transition mechanism: the reaction. Starting from a system of linear PDE's showing uphill diffusion for one of the species (*partial uphill*), we construct a multi-species reaction-diffusion process with the feature of having evolution equations for the average occupation variable (thought as a proxy for the true density) given by the discretization on a finite lattice of the previously introduced linear system of PDE's. Then, the hydrodynamic limit is obtained. We observe that at this last level the uphill diffusion is lost. Furthermore, we prove the absorbing duality for this boundary driven reaction diffusion model and we derive the equilibrium density fluctuation from its hydrodynamic limit.

Finally, we report two perspective for future research. The first concerns the study of multi-species processes with *non-compact* state space, i.e. having the possibility of hosting an unbounded number of particles at each site. In this context we introduce the multi-species independent random walker where, exploiting the lack of interaction we manage to completely characterize the non-equilibrium steady state via duality. We then introduce the multi-species harmonic process and the multi-species simple inclusion process. For both we define the generator with boundary driving and we find the reversible (equilibrium) measure. This is the starting point for the proof of absorbing duality, an goal of future research.
As a second perspective we report a result regarding a single species asymmetric energy transport model with boundary driving. The reservoirs have been designed to maintain the absorbing duality property. Furthermore, using this duality it is possible to characterize the exponential current in the non-equilibrium steady state. We believe that this single species result can be a starting point for the generalization of this bulk-boundary driven model to the multi-species set-up. Moreover, the scaling limits of this bulk-boundary driven process could also be used to describe the macroscopic equations of multi-component transport models with drift.

Riassunto

I sistemi di particelle interagenti (IPS dall'inglese) sono modelli paradigmatici della meccanica statistica di non equilibrio. Infatti, consentono, al contempo, di mantenere i calcoli abbastanza semplici e di non perdere le caratteristiche fondamentali dei sistemi fisici che si prefiggono di riprodurre. Solitamente, due importanti questioni sono investigate: la distribuzione di probabilità stazionaria, che viene raggiunta dal sistema nel lungo periodo e il limite idrodinamico, che consente di ottenere le leggi macroscopiche soddisfatte dal sistema di particelle una volta riscalato al continuo.

In letteratura, sono presenti molti risultati e sono state sviluppate diverse tecniche a riguardo degli IPS a singola specie, che sono modelli dove delle particelle indistinguibili si muovo su una data geometria discreta occupando gli spazi disponibili. A livello microscopico, la dualità e la integrabilità consentono di ottenere espressioni chiuse per la distribuzione stazionaria, mentre la teoria dei limiti di scala ha reso rigoroso il passaggio dal micro al macro.

Al giorno d'oggi, c'è un crescente interesse per i sistemi in cui ci sono più leggi di conservazione, pertanto, dal punto di vista della meccanica statistica, la ricerca si focalizza su IPS di tipo multi-specie. Qui, si ha la presenza di diverse specie di particelle (dette anche colori) che, oltre alla occupazione gli spazi vuoti disponibili, si ha anche un interazione di tipo specie-specie, per esempio scambiando le reciproche posizioni o attraverso una mutazione della specie stessa. Questa dinamica si riflette anche sui limiti di scala, dando luogo a sistemi di equazioni alle derivate parziali (dall'inglese PDE's) che descrivono modelli di trasporto multi-componente. In questo contesto, nuovi fenomeni hanno lugo, come la diffusione uphill multi-componente, ovvero la situazione in cui il flusso (di massa, di particelle, di energia, di carica...) ha lo stesso segno della differenza di densità ai bordi, causando una corrente uphill e violando la legge di Fick.

In questa tesi ci concentriamo su IPS multi-species con boundary driving dove ogni vertice della geometria discreta può ospitare al massimo un numero finito di particelle. Per tale ragione, si dice che questi modelli hanno spazio degli stati compatto. Per prima cosa analizziamo il così detto *multi-species stirring process*, che consiste in una generalizzazione del processo di esclusione simmetrica (SEP dall'inglese) nel caso in cui particelle di diversi colori siano considerate. Dopo avere descritto il modello senza boundary driving, deriviamo i limiti di scala, sia il limite idrodinamico che le fluttuazioni di densità all'equilibrio. Portiamo poi il sistema lontano dall'equilibrio attraverso l'introduzione di boundary reservoirs (bagni termodinamici). Descriviamo il generatore di questo modello attraverso un algebra di Lie e usiamo questa descrizione per dimostrare un risultato di dualità assorbente. Questa dualità assorbente ci consente di caratterizzare lo stato stazionario di non-equilibrio attraverso le probabilità di assorbimento, ovvero le probabilità che le particelle duali siano assorbite negli extra-sites. Questi extra-site rimpiazzano, nel processo duale, i boundary reservoirs originali. Sebbene questo risultato di dualità sia valido a prescindere dalla geometria scelta e dalla occupazione massima del processo, la completa caratterizzazione dello stato stazionario viene ottenuta nel caso di multi-species stirring process

integrabile, attraverso la combinazione della dualità e del matrix product ansatz (MPA). Questo ci consente di scrivere formule esatte per le correlazioni multi-punto. Infine, come problema ancora parzialmente aperto, costruiamo, per questa catena integrabile, il mappping del processo di non equilibrio sul processo di equilibrio utilizzando il quantum inverse scattering method.

Dopo avere analizzato il multi-species stirring process aggiungiamo un ulteriore meccanismo di transizione: la reazione. Partendo da un sistema di PDE's lineari che mostra la diffusione uphill per una delle specie (*uphill parziale*), costruiamo un IPS multi-specie con la caratteristica che, le equazioni di evoluzione dell'occupazione media (pensata come un "approssimazione" della vera densità) evolvano come la versione discretizzata del sistema di PDE's precedentemente introdotto. Si deriva poi il limite idrodinamico, osservando che in quest'ultimo caso la diffusione uphill non ha luogo. Inoltre, deriviamo per questo modello boundary driven multi-specie di reazione diffusione, la dualità assorbente e le fluttuazioni dal limite idrodinamico.

Riportiamo infine due prospettive per futuri lavori di ricerca. La prima riguarda lo studio dei processi multi-specie in cui lo spazio degli stati non è compatto, ovvero situazioni in cui ogni sito può ospitare un numero illimitato di particelle. In tale contesto introduciamo il random walk indipendente multi-specie e, grazie alla mancanza di interazione, riusciamo a caratterizzare completamente la distribuzione stazionaria di non equilibrio, soltanto con l'uso della dualità. Introduciamo poi il processo armonico multi-specie e il processo di inclusione multi-specie. Per entrambi, definiamo il generatore con boundary driving e ricaviamo la misura reversibile come punto di partenza per la dimostrzione della dualità assorbente, che sarà oggetto di futuri sviluppi.
Come seconda prospettiva, riportiamo un risultato riguardante un processo a singola specie asimmetrico di trasporto dell'energia in cui un boundary driving è stato aggiunto. In particolare, i reservoirs sono stati pensati per mantenere la dualità assorbente. Usando questa dualità è possibile caratterizzare le correnti esponenziali nello stato stazionario di non equilibrio. Riteniamo che questo modello bulk-boundary driven a singola specie possa essere un buon punto di partenza per generalizzare al caso multi-specie. Inoltre, il limite idrodinamico di tale modello potrebbe condurre a equazioni di trasporto multi-componente con drift.

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Publications

This thesis is based on the following papers:

- F.C. , Cristian Giardinà, and Cecilia Vernia. *Uphill in Reaction-Diffusion Multi-species Interacting Particles Systems.* J. Stat. Phys. 190, 132 (2023). https://doi.org/10.1007/s10955-023-03141-3. [1]
- F.C., Cristian Giardinà, Frank Redig. *Density fluctuations for the multi-species stirring process.* arXiv: 2307.05111 (2023). Submitted to Journal of Theoretical Probability. Outcome: required minor reviews. [2]
- F.C., Rouven Frassek, Cristian Giardinà. *Duality for the multi-species stirring process with open boundaries.* arXiv: 2312.15532 (2023). Submitted to Journal of Physics A: Mathematical and Theoretical. [3]
- Gioia Carinci, F.C., Chiara Franceschini. *Duality for a boundary driven asymmetric model of energy transport.* (2024) J. Phys. A: Math. Theor. https://doi.org/10.1088/1751-8121/ad2399. arXiv: 2309.17349. [4]

Contents

Chapter 1

Research problem and outline of the thesis

1.1 Informal introduction

Statistical mechanics is the branch of physics that describes the macroscopic behaviour of a system starting from the dynamics of its microscopic components. For instance, one might be interested in deriving the PDE's describing the time and space evolution of the density of a fluid, starting from the interaction between the molecules that compose it. Since at a microscopic level the systems are composed by a huge number of particles (the number of constituent particles of a mole of any chemical element is of the order 10^{23}) it is impossible to describe their motion via the Newton equations. Therefore, in statistical mechanics one considers a probabilistic description of a system via a probability measure. Here a distinction is made between *equilibrium statistical mechanics* and *non-equilibrium statistical mechanics*. In the first context there is a universal description of the law of the particles given by the *Boltzmann-Gibbs distribution*, in the latter no such a general result is available. In this out-of-equilibrium situation, interacting particle systems (IPS) are paradigmatic models. Indeed, with respect to the Hamiltonian systems, they allow to keep the computations manageable, without loosing the fundamental features of the physical situations they aim to represent.

Interacting particle systems were originally introduced by Spitzer in 1970 [5]. They consists of particles that move and interact randomly in a discrete physical space (for example a graph). Moreover, the whole configuration of the process must be Markov, meaning that its evolution depends only on the present and not on the past history. In particular, to model non-equilibrium phenomena, two basic mechanism have been introduced:

- *Boundary driving*: here thermodynamic baths are put in contact with the system, they inject and remove particles from the external environment.
- *Bulk driving*: here an external field that drags the particles in a certain direction. This can also be viewed as an asymmetry of the jump rates of the IPS.

One may consider other mechanics, such as *active particles*, where particles have an internal state that can also randomly change in time [6]. These non-equilibrium models are characterized by the property of having non-zero currents. In this set-up, among others, two central questions are usually investigated: the characterization of the non-equilibrium steady state (i.e. the stationary measure describing the long-time behaviour of the system) and the hydrodynamic limit (that gives the macroscopic evolution equation corresponding to the given microscopic dynamics). To these aim, duality is a fundamental property of IPS that is crucially important. This technique consists in the study of a process via a second one by the use of a duality function, that connects the expectations of the two processes. This allows (hopefully) to simplify the computations. For instance, the m-point correlation functions can be characterized by the study of m dual particles, instead of considering the whole system. In this way, the derivation of the hydrodynamic limit can often be reduced by duality to the application of the invariance principle for a single dual particle, that behaves as an independent random walker. Furthermore, integrability techniques (coming from quantum physics) have been applied to IPS, allowing an explicit writing of the correlation function by the knowledge of the transition probability of the dual dynamics via Bethe ansatz techniques.

In this thesis the focus is on the study of interacting particle systems where many different species of particles perform the dynamics. We consider both equilibrium and non-equilibrium situations, the latter obtained by a boundary driving. In the following, we state in detail the research problems that motivated the work reported in the chapters of this thesis.

1.2 Research problem

This thesis has the following general aims:

1. *Duality for multi-species interacting particle systems (IPS).* We aim to extend and develop duality theory for interacting particle systems where many species (or colours) of particles are present. In particular, we aim to prove rigorously the existence of a duality relation between boundary driven process and absorbing processes by using the Lie algebraic approach.

2. *Exact formulas for integrable multi-species IPS.* Combining absorbing duality and integrability (matrix product ansatz and quantum inverse scattering method) we aim to write exact formulas for the non-equilibrium steady state of integrable multi-species interacting particle systems.

3. *Scaling limits for multi-species IPS.* We aim to derive the scaling limits for the multi-species interacting particle systems: both hydrodynamic limit and equilibrium density fluctuations are investigated.

4. *Multi-species IPS for uphill diffusion.* Starting form a multi-species reaction diffusion PDE's model that shows violation of Fick's law by steady state uphill diffusion, we aim to construct microscopic interacting particle systems that reproduce on the discrete lattice these PDE's models.

We now explain in detail the above purposes.

1.2.1 Duality for multi-species IPS

Duality is a powerful tool to investigate interacting particle systems. It consists in the study of a first IPS (or more generally of a Markov process) via a second model, that is called *dual process*. Duality relation is an equation that relates the expectation with respect the laws of the two processes (original and dual process) via an observable called *duality function*.

Definition 1 *(Duality between Markov processes) Let $(\eta(t))_{t\geqslant 0}$ and $(\xi(t))_{t\geqslant 0}$ be two continuous time Markov processes defined on the state spaces Ω and $\widetilde{\Omega}$ respectively. They are dual if there exists a function $D : \Omega \times \widetilde{\Omega} \to \mathbb{R}$ such that*

$$\mathbb{E}^{\eta}\left[D(\eta(t), \xi)\right] = \mathbb{E}^{\xi}\left[D(\eta, \xi(t)\right] \qquad \forall \eta \in \Omega,\ \forall \xi \in \widetilde{\Omega} \tag{1.2.1}$$

where the expectations $\mathbb{E}^{\eta}$ and $\mathbb{E}^{\xi}$ are taken with respect to the distributions of $(\eta(t))_{t\geqslant 0}$ and $(\xi(t))_{t\geqslant 0}$ with initial configurations η and ξ respectively.

This relation allows to simplify the computation of some interesting quantities (such as non-equilibrium steady states and scaling limits). In the past, duality was introduced in the *single species* models set-up, i.e. processes where only one type of particles performs the dynamic on a graph (see for instance [7, 8, 9, 10, 11, 12]). In this thesis we aim to rigorously derive duality relations for boundary driven multi-species IPS, i.e. models where many species (or colours) of particles are present at the same time and interact among each others. The first process that we study is the so called *multi-species stirring process*, which is a natural generalization of the symmetric exclusion process with maximal occupancy ν, called SEP(ν) (for the SEP(ν) see [8, 12] and see [13, 14] for some versions of the multi-species stirring process). We consider this multi-species stirring process on the geometry of a connected and undirected graph $G = (V, \mathcal{E})$ where V is the set of vertices and $\mathcal{E}$ is the set of edges. Each vertex can host maximally $\nu \in \mathbb{N}$ particles, that can be chosen among $N-1$ species. We denote by the index N the *vacancies* or *holes* and with indices from 1 to $N-1$ the different species of particles. We describe the occupation at site $x \in V$ with an N-dimensional vector $n^x = (n_1^x, \ldots, n_N^x)$ in which the value of the a-th component n_a^x denotes the number of particles of species $a \in \{1, \ldots, N-1\}$, while the component n_N^x counts the number of holes at site x. Moreover, we denote the process by $(\boldsymbol{n}(t))_{t\geqslant 0}$ where n_A^x, with $A \in \{1, \ldots, N\}$ has the meaning explained above. The state space of the process reads

$$\Omega := \bigotimes_{x\in V} \Omega_x \tag{1.2.2}$$

where

$$\Omega_x := \left\{ n^x = (n_1^x, \ldots, n_N^x) \in \mathbb{N}_0^N \ :\ \sum_{A=1}^{N} n_A^x = \nu \right\} . \tag{1.2.3}$$

The dynamics has two parts:

- on each edge of the graph, any two types of particles are swapped at rate 1; moreover a particle of any type and a hole are also swapped at rate 1;
- on each vertex x of the graph, a particle of type $a \in \{1, \ldots, N-1\}$ is injected at rate $\alpha_a^x n_N^x > 0$; at rate $\alpha_N^x n_a^x$, a particle of type $a \in \{1, \ldots, N-1\}$ is removed; additionally, a particle of type $a \in \{1, \ldots, N-1\}$ is removed from site x and replaced by a particle of type $b \in \{1, \ldots, N-1\}$ with rate $\alpha_b^x n_a^x$.

The infinitesimal generator of the process reads

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \mathcal{L}_x \tag{1.2.4}$$

where $\omega_{x,y} \geqslant 0$ are the so-called *conductances* and $\Gamma_x \geqslant 0$, called *local inhomogeneities*, that are the couplings to reservoirs. The generator $\mathcal{L}_{x,y}$ is called the *edge generator*, while $\mathcal{L}_x$ is called the *site generator*. These linear operators act on functions $f : \Omega \to \mathbb{R}$ as follows

$$\mathcal{L}_{x,y} f(\boldsymbol{n}) = \sum_{A,B=1}^{N} n_A^x n_B^y \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^y - \boldsymbol{\delta}_B^y) - f(\boldsymbol{n}) \right] \ , \tag{1.2.5}$$

$$\mathcal{L}_x f(\boldsymbol{n}) = \sum_{A,B=1}^{N} \alpha_A^x n_B^x \left[f(\boldsymbol{n} + \boldsymbol{\delta}_A^x - \boldsymbol{\delta}_B^x) - f(\boldsymbol{n}) \right] \ , \tag{1.2.6}$$

where

$$(\boldsymbol{\delta}_A^x)_B^y = \begin{cases} 1 & \text{if} \quad y = x, \ B = A \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{1.2.7}$$

We observe that the site generator (1.2.6) is not just injection and removal of particles as in the case of SEP (see [12]), but it includes also transitions where two different species of particles are exchanged. As we will see, this turns out to be the correct choice of the boundary dynamics to have duality.
Starting from the literature of the symmetric single species processes (see for instance [8, 11, 15]) the first target that we aim to is the description of this multi-species stirring process by using a suitable representation of the $gl(N)$ Lie algebra. Secondly, we would like to understand if, using this algebraic structure, one can prove a duality relation for the edge generator by exploiting symmetries. Finally, in order to characterize the non-equilibrium steady state of the boundary driven process, we aim to prove the existence of an absorbing dual process, with the property of voiding the graph in the long time horizon.
Once the above results are proven, duality can be extended to *reaction-diffusion process*. The specific choice of this further dynamics is explained in Section 1.2.4, motivated by the uphill diffusion.

Main results. In this thesis, we have proved that the boundary driven multi-species stirring process admits a dual process with absorbing boundaries. This dual process is defined on an *enlarged graph*, i.e. a copy of the original graph G where an extra site $u(x)$ is attached to each vertex x. We denote it by $\widetilde{G} = (\widetilde{V}, \widetilde{\mathcal{E}})$ where

$$\widetilde{V} := V \cup \{u(x) \ : \ x \in V\} \qquad \widetilde{\mathcal{E}} := \mathcal{E} \cup \{(x, u(x)) \ : \ x \in V\} \ . \tag{1.2.8}$$

These extra-sites have unbounded maximal occupancy and once a particle reaches them it stays there forever (absorbing property). The configuration space of the dual process is the enlarged state space

$$\widetilde{\Omega} = \bigotimes_{x \in V} \widetilde{\Omega}_x \ = \bigotimes_{x \in V} (\Omega_x \times \mathbb{N}_0^{N-1}) \ . \tag{1.2.9}$$

We denote the dual configuration by $\boldsymbol{\xi} \in \widetilde{\Omega}$ as

$$\boldsymbol{\xi} = \bigotimes_{x \in V} \Big((\xi_1^x, \ldots, \xi_N^x) \otimes (\xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)}) \Big) \tag{1.2.10}$$

where ξ_A^x denotes the number of dual particles or of holes at site x, while $\xi_a^{u(x)}$ denotes the number of particles at the extra site $u(x)$ attached to x. The generator of the dual process reads

$$\widetilde{\mathcal{L}} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \widetilde{\mathcal{L}}_x \tag{1.2.11}$$

where $\mathcal{L}_{x,y}$ is the same of (1.2.5) and, for any function $f : \widetilde{\Omega} \to \mathbb{R}$

$$\widetilde{\mathcal{L}}_x f(\boldsymbol{\xi}) = |\alpha^x| \sum_{a=1}^{N-1} \xi_a^x \left(f(\boldsymbol{\xi} - \boldsymbol{\delta}_a^x + \boldsymbol{\delta}_N^x + \boldsymbol{\delta}_a^{u(x)}) - f(\boldsymbol{\xi}) \right) . \tag{1.2.12}$$

Then we have the following result.

Theorem 1 *The multi-species stirring process* $(\boldsymbol{n}(t))_{t\geqslant 0}$ *defined on the state space* Ω *with generator* $\mathcal{L}$ *defined in* (1.2.4) *is dual to the process* $(\boldsymbol{\xi}(t))_{t\geqslant 0}$ *defined on the enlarged state space* $\widetilde{\Omega}$ *with generator defined in* (1.2.11)*. The duality function is given by*

$$D(\boldsymbol{n}, \boldsymbol{\xi}) = \prod_{x\in V} \left(\frac{(\nu - \sum_{a=1}^{N-1} \xi_a^x)!}{\nu!} \prod_{a=1}^{N-1} \frac{n_a^x!}{(n_a^x - \xi_a^x)!} \left(\frac{\alpha_a^x}{\sum_{A=1}^{N} \alpha_A^x} \right)^{\xi_a^{u(x)}} \right) \tag{1.2.13}$$

for all $\boldsymbol{n} \in \Omega$ *and* $\forall \boldsymbol{\xi} \in \widetilde{\Omega}$.

We observe that the site generator of the dual process (1.2.12) is purely absorbing. Indeed, the only transition it allows is the absorption at extra-site $u(x)$ of any species "a" of particle present at site x. This means that, in the long time limit all dual particles are absorbed by the extra-site, voiding the graph. This will be the key property used to characterize the non-equilibrium steady state of the boundary driven multi-species stirring process. We further observe that the generator (1.2.12) can be represented as a triangular matrix. This will be crucial in the algebraic derivation of exact formulas for the non-equilibrium correlations.

1.2.2 Exact formulas for integrable multi-species IPS

The quantum inverse scattering method is a technique originally developed to diagonalize the closed or open XXX integrable Heisenberg chain [16, 17]. The dynamics of this Heisenberg chain is ruled by a Hamiltonian matrix that we denote by H. To diagonalize H, a *transfer matrix* $T(u)$, with $u \in \mathbb{C}$ called *spectral parameter*, is constructed. This $T(u)$ is a "generating function" for the conserved quantities of the system (and generates the symmetries of the Hamiltonian). The transfer matrix $T(u)$ can be explicitly diagonalized (by the *algebraic Bethe ansatz* [16, 17, 18]) and, since it shares the eigenvectors with the Hamiltonian, one obtains the aimed result. This technique can be applied to IPS (see for instance [19, 20, 21, 22, 23]) by mapping the generator of the interacting particle system to a XXX integrable chain, and allowing to write explicit expression for the eigenvectors.

To find the non-equilibrium steady state of some interacting particle systems, the matrix product ansatz (MPA) has been developed [24]. This MPA, that is connected with the integrable structure of the process by the so called *Zamolodchikov algebra* (see [25]), establishes that the non-equilibrium steady state is given by an abstract product of matrices, acting on the state vectors of a supplementary space. Moreover, these matrices form an algebra (DEHP) with proper commutation relations. In principle this allows to find formulas for non-equilibrium correlations; however, since a priori a representation of this algebra is not known, the computations are usually involved. In the multi-species set-up it has been defined in [13] for the *integrable boundary driven multi-species stirring process*, that is the process with generator (1.2.4) specified to the case where maximal occupancy $\nu = 1$ and when the geometry reduces to a chain of length L with two boundary reservoirs at the extreme sites. In this set-up we define the Hamiltonian (transposed of the Markov generator), that reads

$$H = H_{\text{left}} + H_{\text{bulk}} + H_{\text{right}} \tag{1.2.14}$$

where

$$H_{\text{bulk}} = \sum_{x=1}^{L-1} \mathcal{H}_{x,x+1} \,. \tag{1.2.15}$$

Here $\mathcal{H}_{x,x+1}$ denotes the two-site Hamiltonian

$$\mathcal{H} = P - I \tag{1.2.16}$$

where we used the permutation matrix

$$P = \sum_{A,B=1}^{N} e_{AB} \otimes e_{BA} \,, \tag{1.2.17}$$

acting non-trivially on the nearest neighbour sites $x, x+1$. Here, $(e_{AB})_{CD} = \delta_{AC}\delta_{BD}$ are the elementary $N \times N$ matrices, given by the first fundamental representation of $gl(N)$ and we denote by I the identity matrix. The boundary terms of the Hamiltonian (1.2.14) are given by

$$H_{\text{left}} = \begin{pmatrix} \alpha_1 - 1 & \alpha_1 & \alpha_1 & \dots & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 - 1 & \alpha_2 & \dots & \dots & \alpha_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ \alpha_{N-1} & \alpha_{N-1} & \dots & \dots & \alpha_{N-1} - 1 & \alpha_{N-1} \\ \alpha_N & \alpha_N & \dots & \dots & \alpha_N & \alpha_N - 1 \end{pmatrix} \tag{1.2.18}$$

and

$$H_{\text{right}} = \begin{pmatrix} \beta_1 - 1 & \beta_1 & \beta_1 & \dots & \dots & \beta_1 \\ \beta_2 & \beta_2 - 1 & \beta_2 & \dots & \dots & \beta_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ \beta_{N-1} & \beta_{N-1} & \dots & \dots & \beta_{N-1} - 1 & \beta_{N-1} \\ \beta_N & \beta_N & \dots & \dots & \beta_N & \beta_N - 1 \end{pmatrix} . \tag{1.2.19}$$

Integrability and the existence of a non-equilibrium steady state in matrix product form was proved in [13]. This means that the non-equilibrium steady state $|\Psi\rangle$ reads

$$|\Psi\rangle = \frac{1}{Z_L} \langle\langle W| \underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}}_{L \text{ times}} |V\rangle\rangle \tag{1.2.20}$$

with the normalization

$$Z_L = \langle\langle W|(X_1 + \ldots + X_N)^L|V\rangle\rangle \,. \tag{1.2.21}$$

Here the operators (matrices) $(X_A)_{A\in\{1,\ldots,N\}}$ fulfil the commutators

$$[X_A, X_B] = (\alpha_A - \beta_A)X_B - (\alpha_B - \beta_B)X_A \tag{1.2.22}$$

for all $A, B \in \{1, \ldots, N\}$. Their actions on the boundary vectors are given, for all $A \in \{1, \ldots, N\}$, by

$$\langle\langle W| \left(\alpha_A(X_1 + \ldots + X_N) - X_A\right) = (\alpha_A - \beta_A)\langle\langle W| \tag{1.2.23}$$

$$\left(\beta_A(X_1 + \ldots + X_N) - X_A\right) |V\rangle\rangle = -(\alpha_A - \beta_A)|V\rangle\rangle \,. \tag{1.2.24}$$

We denote by $\langle\langle W|$ and $|V\rangle\rangle$ bra and ket vectors of an abstract supplementary space. The algebraic relations (1.2.22), (1.2.23) and (1.2.24) are the multi-species version of the DEHP algebra introduced in [24].
Starting from this result, in this thesis, we aim to combine the MPA defined above for the integrable multi-species stirring process with the absorbing duality, in order to find closed formulas for the non-equilibrium correlations. The key idea is to use the duality transformation to triangularize the boundaries of the Hamiltonian (1.2.14) and to combine it with a further linear transformation that simplifies the above commutation relations for these operators $(X_A)_{A\in\{1,\ldots,N\}}$.

Main results. Combining the absorbing duality result (specified for the integrable chain) and the MPA (1.2.20) we show that the non-equilibrium steady state correlations of the integrable boundary driven multi-species stirring process are written in a closed formula in terms of the absorption probabilities of the dual process. We report this result in the following Theorem where we denote by $\boldsymbol{Y} = (Y_1, \ldots, Y_L)$ the random vector distributed as the non-equilibrium steady state of the integrable boundary driven multi-species stirring process.

Theorem 2 *Let $m \in \{1, \ldots, L\}$. Consider m sites $1 \leqslant x_1 < x_2 < \ldots < x_m \leqslant L$ and m colours denoted by $a_k \in \{1, \ldots, N-1\}$ with $k = 1, 2, \ldots, m$, chosen among the $N-1$ available species. Then the m-point correlations with respect to the non-equilibrium steady state measure are given by*

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=a_k\}}\right] = \sum_{t_1,\ldots,t_m=0}^{1} \left(\prod_{k=1}^{m} \alpha_{a_k}^{t_k} \beta_{a_k}^{1-t_k}\right) \mathcal{P}_{x_1,\ldots,x_m}(t_1, \ldots, t_m) \tag{1.2.25}$$

where

$$\mathcal{P}_{x_1,\ldots,x_m}(t_1, \ldots, t_m) = \sum_{c_1=t_1}^{1} \cdots \sum_{c_m=t_m}^{1} f(c_1, \ldots, c_m) \prod_{j=1}^{m} (-1)^{c_j - t_j} g_j(x_j, c_j, \ldots, c_m) \tag{1.2.26}$$

with

$$f(c_1, \ldots, c_m) = \frac{(L+1-\sum_{a=1}^{m} c_a)!}{(L+1)!} \tag{1.2.27}$$

and

$$g_j(x_j, c_j, \ldots, c_m) = \left(L + 2 - x_j - \sum_{k=j}^{m} c_k\right)^{c_j}. \tag{1.2.28}$$

As we will clarify, $\mathcal{P}_{x_1,\ldots,x_m}(t_1, \ldots, t_m)$ can be interpreted as the absorption probability of m dual particles starting at sites $x_1, \ldots, x_m$, with $t_k = 1$ meaning that the particle starting from x_k has been absorbed at extra site 0 (left-most extra site of the dual chain). Using these exact correlations, one can write an explicit formula for the non-equilibrium steady probability distribution.
Finally, as an open problem and a perspective for future works, we study the mapping of the boundary driven Hamiltonian for the integrable chain H onto an equilibrium Hamiltonian, that is obtained from H by setting equal the boundary densities. This is carried out via a non-local similarity transformation, usually denoted by W and it is known in literature [26, 27, 28] as *mapping of non-equilibrium onto equilibrium.* The idea is to write an explicit expression of the similarity transformation W via the conserved charges of the system. These conserved charges are the matrix-coefficient of the power expansion with respect to the spectral parameter of the transfer matrix $T(u)$. While the result can be obtained for SSEP with some different techniques, for the boundary driven multi-species stirring process the topic is still not fully understood. As a perspective, in this thesis we report some partial results that will be carried on in future works.

1.2.3 Scaling limits for multi-species IPS

Starting from the microscopic dynamics of the system, the *scaling limit* techniques aim to scale down the size of the system and accelerate time in order to find macroscopic laws that rule the time and space evolution of the system [29, 30, 31]. In literature, it is known that the hydrodynamic limit of the SEP(ν) is the heat equation with diffusivity ν and its equilibrium fluctuation field re-scales to a proper Ornstein-Uhlenbeck process [29, 30, 32]. At present, there is a growing interest in systems with multiple conserved quantities, their hydrodynamic limit, and their fluctuations [33, 34] and, as well as in “multi-layer” models [35, 36, 37]. In this thesis we aim to understand which are the hydrodynamic equations for the multi-species stirring process and how the equilibrium fluctuations from the hydrodynamic limit behave. More precisely, the process considered for this purpose is on the geometry of an infinite one-dimensional lattice $\mathbb{Z}$ with nearest neighbour interaction. The generator is given by

$$\mathcal{L} = \sum_{x\in\mathbb{Z}} \mathcal{L}_{x,x+1} \tag{1.2.29}$$

where $\mathcal{L}_{x,x+1}$ is the edge generator (1.2.5) with $y = x + 1$. We denote the process by $(\boldsymbol{n}(t))_{t\geqslant 0}$, where for all $x \in \mathbb{Z}$ we use n_a^x to indicate the number of particles of type $a \in \{1, \dots, N-1\}$ at site x and n_N^x to indicate the number of holes at site x. The object used for the study of the hydrodynamic limit is the so called *density field* of the species $a \in \{1, \dots, N-1\}$. For any $\phi \in C_c^\infty(\mathbb{R})$ this field is defined as

$$\begin{aligned} X_a^{K,t}(\cdot) &: C_c^\infty(\mathbb{R}) \to \mathbb{R} \\ &\phi \to X_a^{K,t}(\phi) = \frac{1}{K}\sum_{x\in\mathbb{Z}} \phi\Big(\frac{x}{K}\Big) n_a^x(tK^2) \end{aligned} \tag{1.2.30}$$

where $K \in \mathbb{N}$ is the scaling parameter that eventually goes to infinity. We scaled time by K^2 and we scaled space by $1/K$ obtaining the so called *diffusive scaling*.
Similarly, for the derivation of the equilibrium fluctuations we introduce the *density fluctuation field* for a species $a \in \{1, \dots, N-1\}$. This is a random distribution, i.e., a random element of $(C_c^\infty(\mathbb{R}))^*$ defined via:

$$\begin{aligned} Y_a^{K,t}(\cdot) &: C_c^\infty(\mathbb{R}) \to \mathbb{R} \\ &\phi \to Y_a^{K,t}(\phi) = \frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}} \phi\Big(\frac{x}{K}\Big)\left(n_a^x(tK^2) - \mathbb{E}_{\mu_{\text{rev}}}\left[n_a^x\right]\right) \end{aligned} \tag{1.2.31}$$

where $\mathbb{E}_{\mu_{\text{rev}}}[n_a^x]$ is the expectation with respect to the equilibrium reversible measure of the multi-species stirring process and K is the scaling parameter that eventually goes to infinity. Also for the fluctuations, the diffusive scaling of space and time is chosen.
Moreover, the question about hydrodynamic limit and density fluctuations can be extended to multi-species reaction diffusion processes. Again, the specific choice of the reaction dynamics is explained in Section 1.2.4, and it is motivated by the study of uphill diffusion.

Main results. Using techniques based on the Dynkin martingale (see [29, 30, 31]) we show that the hydrodynamic equation for the multi-species stirring process on the infinite line with generator (1.2.29) is a system of $N-1$ uncoupled heat equations. More precisely, assuming that the initial macroscopic profile $\widehat{\rho}^{(a)}(u)$ is compatible with the initial measure (for details see Chapter 5) we state the following hydrodynamic result.

Theorem 3 *Let $\widehat{\rho}^{(a)}$ be an initial macroscopic profile of species $a \in \{1, \ldots, N-1\}$ and let be $(\mu_K)_{K \in \mathbb{N}}$ a sequence of compatible initial measures. P_K denotes the law of the process $\left(X_1^{K,t}(\phi), \ldots, X_{N-1}^{K,t}(\phi)\right)$ induced by $(\mu_K)_{K \in \mathbb{N}}$. Then, $\forall T > 0$, $\delta > 0$, $\forall a \in \{1, \ldots, N-1\}$ and $\forall \phi \in C_c^\infty(\mathbb{R})$*

$$\lim_{K \to \infty} P_K \left(\sup_{t \in [0,T]} \left| X_a^{K,t}(\phi) - \int_{\mathbb{R}} \phi(u) \rho^{(a)}(u,t) du \right| > \delta \right) = 0 \tag{1.2.32}$$

where $\rho^{(a)}(u,t)$ is a strong solution of the the PDE Cauchy problem

$$\begin{cases} \partial_t \rho^{(a)}(u,t) = \nu \Delta \rho^{(a)}(u,t) & u \in \mathbb{R}, \quad t \in [0,T] \\ \rho^{(a)}(u,0) = \widehat{\rho}^{(a)}(u) \end{cases} . \tag{1.2.33}$$

For what concerns the density fluctuations from hydrodynamic limit we have the following result.

Theorem 4 *The stationary density fluctuation field $(Y^{K,t})_{t \geqslant 0} = (Y_1^{K,t}, \ldots, Y_{N-1}^{K,t})_{t \geqslant 0}$ of the $N-1$ species of particles converges, as $K \to \infty$, to the solution of a $(N-1)$-dimensional SPDE of Ornstein-Uhlenbeck type given by*

$$dY^t = \nu(\boldsymbol{A} Y^t dt + \sqrt{2\Sigma} \nabla dW^t) \tag{1.2.34}$$

where

$$(W^t)_{t \in [0,T]} = \left((W_1^t, \ldots, W_{N-1}^t)\right)_{t \in [0,T]} \tag{1.2.35}$$

is an $(N-1)$-dimensional vector of independent space-time white noises. The matrices are the following

$$\boldsymbol{A} = \begin{pmatrix} \Delta & 0 & \ldots & 0 \\ 0 & \Delta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Delta \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} p_1(1-p_1) & -p_1 p_2 & \ldots & -p_1 p_{N-1} \\ -p_1 p_2 & p_2(1-p_2) & \ldots & -p_2 p_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{N-1} p_1 & -p_{N-1} p_2 & \ldots & p_{N-1}(1-p_{N-1}) \end{pmatrix} \tag{1.2.36}$$

and Σ is semi-positive definite

We can show that the stationary-reversible distribution is a product of multinomials and the matrix Σ is the covariance matrix of a multinomial distribution with parameters ν and $(p_1, \ldots, p_N)$ with $p_N = 1 - p_1 - \ldots - p_{N-1}$. The matrix Σ is however non-diagonal, showing that on the level of fluctuations interaction between the different species becomes visible.

1.2.4 Multi-species IPS for uphill diffusion

In the chemistry literature it has been observed the *multi-component uphill diffusion* [38, 39, 40], that is, a situation where the diffusive current of a certain chemical species has the same sign of the gradient of its concentration, making the flux goes "uphill".
For diffusive models with a single species, transport of mass on a finite volume (here assumed to be the unit d-dimensional cube) is often described by the continuity equation

$$\frac{\partial}{\partial t} \rho = -\nabla \cdot J \tag{1.2.37}$$

and the Fick's law

$$J = -\sigma \nabla \rho \tag{1.2.38}$$

Here $\rho : [0,1]^d \times \mathbb{R}_+ \to [0,1]$ is the density of mass, $J : [0,1]^d \times \mathbb{R}_+ \to \mathbb{R}$ is the current, and $\sigma > 0$ is the diffusivity coefficient (that we assume constant throughout this thesis). For multi-component systems with N species, considering the vectors $\boldsymbol{\rho} = (\rho^{(1)}, \ldots, \rho^{(N)})$ and $\boldsymbol{J} = (J^{(1)}, \ldots, J^{(N)})$, where $\rho^{(a)}(x,t)$ and $J^{(a)}(x,t)$ denote the density and the current of the a^{th} species, the generalization of (1.2.37) and (1.2.38) is

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\nabla \cdot \boldsymbol{J} \tag{1.2.39}$$

and

$$\boldsymbol{J} = -\boldsymbol{\Sigma} \cdot \nabla \boldsymbol{\rho}, \tag{1.2.40}$$

where $\boldsymbol{\Sigma}$ (sometimes called *diffusivity matrix*) is now the $N \times N$ matrix of diffusion and 'cross-diffusion' coefficients. When $\boldsymbol{\Sigma}$ is non-diagonal, then uphill diffusion is possible [38, 39, 40]. We distinguish between the case of *'partial' uphill diffusion* , which is obtained when the current of a given species has the same sign of the boundary density gradient of that species, and *'global' uphill diffusion* , which arises when the total mass flows from a region of lower total density to a region of higher total density. For the sake of brevity we will refer to these two scenarios as partial and global uphill, respectively. In literature [35], a double diffusivity model for global uphill diffusion has been derived, starting from a multi-layer IPS. This model has diagonal diffusivity matrix and a linear reaction term. As a possible development, one can consider a simple prototype of macroscopic multi-component system with non-diagonal diffusivity matrix, ruled by the following PDE's:

$$\begin{aligned} \partial_t \rho^{(1)} &= \sigma_{11} \partial_x^2 \rho^{(1)} + \sigma_{12} \partial_x^2 \rho^{(2)} + \Upsilon \left(\rho^{(2)} - \rho^{(1)}\right) \\ \partial_t \rho^{(2)} &= \sigma_{21} \partial_x^2 \rho^{(1)} + \sigma_{22} \partial_x^2 \rho^{(2)} + \Upsilon \left(\rho^{(1)} - \rho^{(2)}\right) \end{aligned} \tag{1.2.41}$$

where the diffusivity matrix is

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}. \tag{1.2.42}$$

We assume that this model is defined on a one dimensional domain $[0,1]$ and that is endowed with Dirichlet boundary conditions given by $\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}} \geqslant 0$ at $x = 0$ and $\rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}} \geqslant 0$ at $x = 1$. The stationary diffusive currents are given by

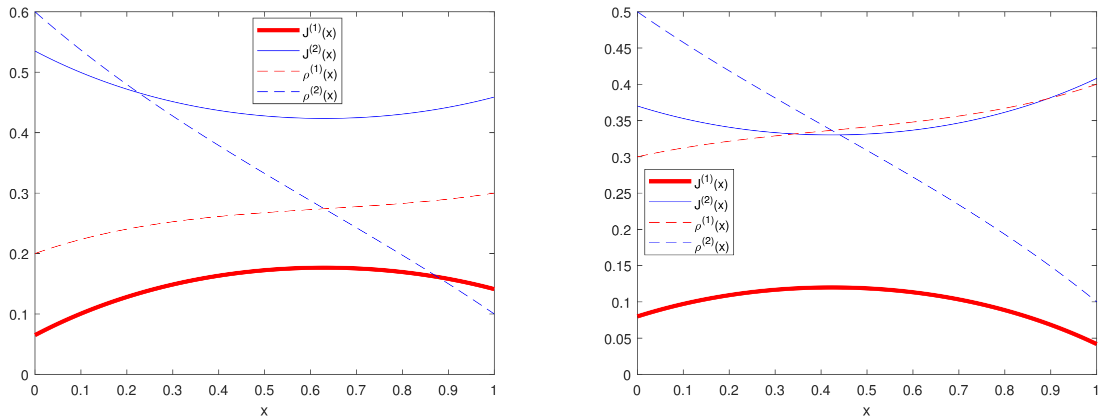
$$\begin{aligned} J^{(1)}(x) &= -\sigma_{11} \partial_x \rho^{(1)}(x) - \sigma_{12} \partial_x \rho^{(2)}(x) \\ J^{(2)}(x) &= -\sigma_{21} \partial_x \rho^{(1)}(x) - \sigma_{22} \partial_x \rho^{(2)}(x) \end{aligned} \tag{1.2.43}$$

We define in detail the two uphill diffusion scenarios:

- *global uphill*: this happens when the boundary values of the total density $\rho_{\text{left}} = \rho^{(1)}_{\text{left}} + \rho^{(2)}_{\text{left}}$ and $\rho_{\text{right}} = \rho^{(1)}_{\text{right}} + \rho^{(2)}_{\text{right}}$ and the total current $J(x) = J^{(1)}(x) + J^{(2)}(x)$ are such that either $\rho_{\text{left}} < \rho_{\text{right}}$ and $J(x) > 0$ $\forall x \in [0,1]$, or $\rho_{\text{left}} > \rho_{\text{right}}$ and $J(x) < 0$ $\forall x \in [0,1]$.

- *partial uphill for the a^{th} species*: for boundary values $\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}$, the system has stationary partial uphill diffusion for the species $a \in \{1,2\}$ if $\rho^{(a)}_{\text{left}} < \rho^{(a)}_{\text{right}}$ and $J^{(a)}(x) > 0$ $\forall x \in [0,1]$, or if $\rho^{(a)}_{\text{left}} > \rho^{(a)}_{\text{right}}$ and $J^{(a)}(x) < 0$ $\forall x \in [0,1]$.

By writing the explicit solution of the non-equilibrium steady state of the PDE's (1.2.41) it is possible to find a choice of boundary values in which partial uphill diffusion takes place. For instance we report in Figure 1.1a and in Figure 1.1b two examples of partial uphill situations.



(a) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (2, 6, 3, 1) \times 10^{-1}$.

(b) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (3, 5, 4, 1) \times 10^{-1}$.

Figure 1.1: Density profile (dashed lines) and currents (continuous line). The red color is for species 1 and the blue color for species 2. The boundary values are in figure 1.1a $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (2, 6, 3, 1) \times 10^{-1}$ and are $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (3, 5, 4, 1) \times 10^{-1}$ in figure 1.1b. In both cases, the diffusivity matrix and the reaction term are $\sigma_{11} = \sigma_{22} = \Upsilon = 1$ and $\sigma_{12} = \sigma_{21} = 1/2$.

We further observe that, from the results in [41, 42], to keep the non-negativity of the densities $\rho^{(1)}(x,t), \rho^{(2)}(x,t)$ for all $x \in [0,1]$ and for all $t \geqslant 0$, it is not possible to have a linear model that evolves as

$$\begin{aligned} \partial_t \rho^{(1)} &= \sigma_{11} \partial_x^2 \rho^{(1)} + \sigma_{12} \partial_x^2 \rho^{(2)} \\ \partial_t \rho^{(2)} &= \sigma_{21} \partial_x^2 \rho^{(1)} + \sigma_{22} \partial_x^2 \rho^{(2)} \end{aligned} \tag{1.2.44}$$

with non-diagonal diffusivity matrix and endowed with Dirichlet boundary conditions, otherwise one of the density becomes negative at some points (loosing physical meaning). This motivates the presence of the linear reaction term with intensity Υ in equations (1.2.41).
In this thesis we aim to find a boundary driven multi-species IPS that gives partial uphill diffusion. We start by searching for an IPS whose average occupation variable of particles evolves as the discretized version on finite segment of (1.2.41). This average occupation variable (sometimes called *average density*) plays the role of a proxy for the true density. The following step is to re-scale the system, in order to obtain the hydrodynamic limit. Moreover, because of the maximum principle for Markov processes, it is interesting to understand if and how the crossing diffusivity Laplacian (second derivative in space) can be found, both for the discretized equations on a finite lattice and for the hydrodynamic limit.

Main results. We start from a general boundary driven multi-species process, inspired by the works [43] and [44], defined on a chain of length L and with configurations denoted by $\boldsymbol{n} = (n_A^x)_{A \in \{1,2,3\}, x \in \{1,\ldots,L\}}$. Again, we consider 1, 2 the labels for the species of particles and 3 the label for the holes. Here the the state space is

$$\Omega_L = \left\{ (n_1, n_2, n_3) \in \{0,1\}^3 \ : \ n_1 + n_2 + n_3 = 1 \right\}^L . \tag{1.2.45}$$

We introduce the average occupation variable of each species of particles

$$\rho_x^{(a)} := \mathbb{E}_\mu \left[n_a^x \right] \qquad \forall a \in \{1,2\}, \quad \forall x \in \{1,\dots,L\} \tag{1.2.46}$$

where the average is taken with respect to the law of the process initialized with a distribution μ. Then, we impose conditions on the rates in order to guarantee that the evolution equations for this average occupation variable of particles are given by a closed (without higher correlations terms) discretized version on a finite lattice of (1.2.41) that reads

$$\begin{aligned} \frac{d}{dt}\rho_1^{(1)} =& \sigma_{11}\left(\rho_{\text{left}}^{(1)} - 2\rho_1^{(1)} + \rho_2^{(1)}\right) + \sigma_{12}\left(\rho_{\text{left}}^{(2)} - 2\rho_1^{(2)} + \rho_2^{(1)}\right) + \Upsilon\left(\rho_2^{(2)} - \rho_1^{(1)}\right) \\ \frac{d}{dt}\rho_1^{(2)} =& \sigma_{21}\left(\rho_{\text{left}}^{(1)} - 2\rho_1^{(1)} + \rho_2^{(1)}\right) + \sigma_{22}\left(\rho_{\text{left}}^{(2)} - 2\rho_1^{(2)} + \rho_2^{(2)}\right) + \Upsilon\left(\rho_1^{(1)} - \rho_1^{(2)}\right) \end{aligned} \tag{1.2.47}$$

$$\begin{aligned} \frac{d}{dt}\rho_x^{(1)} &= \sigma_{11}\Delta_L\rho_x^{(1)} + \sigma_{12}\Delta_L\rho_x^{(2)} + \Upsilon\left(\rho_x^{(2)} - \rho_x^{(1)}\right) \\ \frac{d}{dt}\rho_x^{(2)} &= \sigma_{21}\Delta_L\rho_x^{(1)} + \sigma_{22}\Delta_L\rho_x^{(2)} + \Upsilon\left(\rho_x^{(1)} - \rho_x^{(2)}\right) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \forall x = 2,\dots,L-1 \end{aligned} \tag{1.2.48}$$

$$\begin{aligned} \frac{d}{dt}\rho_L^{(1)} =& \sigma_{11}\left(\rho_{L-1}^{(1)} - 2\rho_L^{(1)} + \rho_{\text{right}}^{(1)}\right) + \sigma_{12}\left(\rho_{L-1}^{(2)} - 2\rho_L^{(2)} + \rho_{\text{right}}^{(2)}\right) + \Upsilon\left(\rho_L^{(2)} - \rho_L^{(1)}\right) \\ \frac{d}{dt}\rho_L^{(2)} =& \sigma_{21}\left(\rho_{L-1}^{(1)} - 2\rho_L^{(1)} + \rho_{\text{right}}^{(1)}\right) + \sigma_{22}\left(\rho_{L-1}^{(2)} - 2\rho_L^{(2)} + \rho_{\text{right}}^{(2)}\right) + \Upsilon\left(\rho_L^{(1)} - \rho_L^{(2)}\right) \end{aligned} \tag{1.2.49}$$

where $\Delta_L\rho_x^{(a)} = \rho_{x-1}^{(a)} - 2\rho_x^{(a)} + \rho_{x+1}^{(a)}$ is the discrete Laplace operator. We obtain the following result.

Theorem 5 *Let Σ be a 2×2 positive definite diffusion matrix and $\Upsilon > 0$ be a reaction coefficient. Let $\rho_{\text{left}}^{(1)}$ and $\rho_{\text{left}}^{(2)}$ (respectively, $\rho_{\text{right}}^{(1)}$ and $\rho_{\text{right}}^{(2)}$) be the densities of the species 1 and 2 at the left (respectively, right) boundary. Then, for any choice of $h, m \geqslant 0$ there exists boundary-driven interacting particle systems on the chain $\{1,\dots,L\}$ such that their evolution equations of the average occupation variable are (1.2.47), (1.2.48), (1.2.49) if and only if the diffusion matrix coefficients $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ and the reaction coefficient Υ are non-negative and fulfill the conditions*

$$\sigma_{11} + \sigma_{21} = \sigma_{12} + \sigma_{22} \qquad \sigma_{12} \leqslant \frac{\Upsilon - m}{2} \qquad \sigma_{21} \leqslant \frac{\Upsilon - h}{2}\,. \tag{1.2.50}$$

As an example, we report a symmetric representative of this family, obtained when $\sigma_{11} = \sigma_{22}$, $\sigma_{12} = \sigma_{21}$ and $h = m$. This process is defined on the state space (1.2.45) and it has the following generator $\mathcal{L}$:

$$\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}_{\text{right}}. \tag{1.2.51}$$

In the bulk we have

$$\mathcal{L}_{x,x+1} = \sigma_{11}\mathcal{L}_{x,x+1}^{S} + \sigma_{12}\mathcal{L}_{x,x+1}^{SM} + (\Upsilon - 2\sigma_{12} - m)\mathcal{L}_{x,x+1}^{LM} + m\mathcal{L}_{x,x+1}^{RM} \tag{1.2.52}$$

where $\mathcal{L}^S_{x,x+1}$ is the generator (1.2.4), specified when $\nu = 1$ and $N = 3$, acting on the bond $(x, x+1)$. Considering a functions $f : \Omega_{\mathbb{Z}} \to \mathbb{R}$, we have

$$\begin{aligned}
\mathcal{L}^{SM}_{x,x+1} f(\boldsymbol{n}) &= \sum_{A,B=1}^{3} n^x_A n^{x+1}_B \left[f(\boldsymbol{n} - \boldsymbol{\delta}^x_A + \boldsymbol{\delta}^x_{\overline{B}} - \boldsymbol{\delta}^{x+1}_B + \boldsymbol{\delta}^{x+1}_{\overline{A}}) - f(\boldsymbol{n}) \right] \\
\mathcal{L}^{LM}_{x,x+1} f(\boldsymbol{n}) &= \sum_{A=1}^{3} n^x_A \left[f(\boldsymbol{n} - \boldsymbol{\delta}^x_A + \boldsymbol{\delta}^x_{\overline{A}}) - f(\boldsymbol{n}) \right] \\
\mathcal{L}^{RM}_{x,x+1} f(\boldsymbol{n}) &= \sum_{B=1}^{3} n^{x+1}_B \left[f(\boldsymbol{n} - \boldsymbol{\delta}^{x+1}_B + \boldsymbol{\delta}^{x+1}_{\overline{B}}) - f(\boldsymbol{n}) \right] .
\end{aligned} \tag{1.2.53}$$

Here we have introduced the *mutation map* $A \mapsto \bar{A}$ defined by:

$$\begin{aligned}
1 &\to 2 \\
2 &\to 1 \\
3 &\to 3 \,.
\end{aligned} \tag{1.2.54}$$

The boundary generators $\mathcal{L}_{\text{left}}, \mathcal{L}_{\text{right}}$ are given by the ones of the boundary driven multi-species stirring process defined in (1.2.6), acting on the sites 1 and L respectively and with a proper choice of parameters α's (see Chapter 10 for details).
After having understood that the discretized version of (1.2.41) can be obtained by the generator (1.2.52), the natural follow-up question is to study the hydrodynamic limit of this process. Surprisingly, we discover that the hydrodynamic equations (for simplicity considered on the infinite line $\mathbb{Z}$) have diagonal diffusivity matrix, i.e.

$$\begin{cases}
\partial_t \rho^{(1)} = \sigma_{11} \partial_x^2 \rho^{(1)} + \widetilde{\Upsilon} \left(\rho^{(2)} - \rho^{(1)} \right) \\
\partial_t \rho^{(2)} = \sigma_{11} \partial_x^2 \rho^{(2)} + \widetilde{\Upsilon} \left(\rho^{(1)} - \rho^{(2)} \right) \\
\rho^{(a)}(0, x) = \widehat{\rho}^{(a)}(x) \qquad \forall x \in [0,1], \ \forall a \in \{1,2\}
\end{cases} \tag{1.2.55}$$

This limit is proved by diffusive scaling the systems with scaling parameter $K \in \mathbb{N}$ (space by $1/K$ and time by K^2). To avoid violations of the maximum principle, we have to further scale down the reaction constants obtaining $\Upsilon = \frac{\widetilde{\Upsilon}}{K^2}$, $m = \frac{\widetilde{m}}{K^2}$ and the crossing diffusivity constant obtaining $\sigma_{12} = \frac{\widetilde{\sigma_{12}}}{K^2}$. This makes the cross diffusion terms vanish in the hydrodynamic limit.
Therefore, on one hand, partial uphill diffusion can be present only at the level of discretized equations on a finite lattice, but it is lost in the hydrodynamic limit. On the other hand, by the first condition in (1.2.50) given by $\sigma_{11} + \sigma_{21} = \sigma_{12} + \sigma_{22}$, global uphill is never possible.
Having in mind the dynamic of the interacting particle system described by the generator (1.2.52), we can generalize it to arbitrary maximal occupancy ν and number of species N, proving that it also satisfy a $gl(N+1)$ symmetry. This last discovery, allows to extend the proof of absorbing duality, done for the boundary driven multi-species stirring process, to this reaction diffusion process. We also notice that the duality function turns out to be the same of the boundary driven multi-species stirring process (see (1.2.13)). In our knowledge, this is one of the first examples of absorbing duality for a boundary driven reaction diffusion process.

1.3 Outline of the thesis

In what follows we describe with more details the content of the thesis.

Part I: Review of single species IPS.

1. Chapter 2: *General concepts in interacting particle systems.* Here we aim to recall the notation and some fundamental notions about interacting particle systems. We start by briefly defining a Markov process and characterizing it via the transition semigroup and the infinitesimal generator. After having defined these concepts in an abstract setting, we specialize them to the case of interacting particle systems, i.e. when the state space is countable or finite. Then, we introduce the evolution of a probability distribution for a Markov process, with more attention to the invariant and reversible distributions, that will be investigated in the following chapters for multi-species processes. Finally, we recall the link between martingales and Markov processes.

2. Chapter 3: *Single species interacting particle systems: duality, integrability and scaling limits.* This chapter was written with the aim of introducing the reader to some results present in the literature for the single species set-up. This is the starting point to extend these results to the multi-species situation. After having introduced some classical single species models (independent random walker (IRW), symmetric exclusion process with maximal occupancy ν (SEP(ν)), symmetric inclusion process with attractive parameter 2k (SIP(2k)) and Brownian energy process with parameter 2k (BEP(2k))), we provide a concise review of duality. Taking as an example the boundary driven SEP(ν), we show the main ideas to obtain absorbing duality and to use it to characterize the non-equilibrium steady state. Then, considering the integrable version of the simple symmetric exclusion process (SSEP or SEP(1)), we briefly review the integrability techniques, i.e. quantum inverse scattering method and the matrix product ansatz. Finally, we report the main ideas behind the proof of the hydrodynamic limit for the SSEP.

Part II: Multi-species stirring process.

1. Chapter 4: *The model.* In this chapter we introduce the multi-species stirring process. We first define the process with closed boundaries, i.e. on a torus, on an infinite line and on a general graph $G = (V, \mathcal{E})$. Then, we add the boundary driving, by connecting each vertex of the graph with an external boundary reservoir characterized by an average density of particles. When all boundary densities are equal, we derive its equilibrium reversible measure.
   We introduce the Lie algebra $gl(N)$ and, by properly choosing a finite dimensional representation, we use it to construct the Hamiltonian operator H (we recall that H is the transposed of the generator) of the boundary driven multi-species stirring process. Moreover, we show that this matrix H is proportional to the co-product of second Casimir of this algebra, opening the possibility of finding symmetries of H.

2. Chapter 5: *Scaling limits for the equilibrium process.* This chapter is devoted to the derivation and the proofs of the scaling limits for the multi-species stirring process on an infinite line. First we show the hydrodynamic limit, that turns out to be a system of $N-1$ decoupled heat equations, one for each species of particles. Second we prove the equilibrium density fluctuations from hydrodynamic limit. In this case, the result is a system of $N-1$ stationary Ornstein-Uhlenbeck processes with the coupling in the white noise term. The techniques used here are of probabilistic nature and are based on the Dynkin martingale method. However, duality (that will be proved in the next chapter) is

used in the context of fluctuation to derive the equations for the covariances of the limiting process via a dual random walker.

3. Chapter 6: *Duality and non-equilibrium steady state.* First we prove the theorem that states the duality between the boundary driven model and a dual process, which presents the same edge dynamics, but has absorbing boundaries. This dual process, is defined on an enlarged graph, with an extra-site $u(x)$ attached to each site x. This extra-site has unbounded maximal occupancy and is absorbing, in the sense that once a dual particle reaches it, it stays there forever. In this context, duality can be seen as a matrix transformation that connects the original Hamiltonian to the dual Hamiltonian. Due to the absorbing property, the boundary matrices of the dual Hamiltonian are triangular. As an application of duality, we characterize the moments with respect to the non-equilibrium steady state of the model. More precisely, the non-equilibrium steady state correlations are written in terms of absorption probabilities of dual particles. As by-product, we define the thermalized non-equilibrium multi-species stirring process and we prove absorbing duality for it.

4. Chapter 7: *Exact formulas for the integrable model.* First we specialize the boundary driven multi-species stirring process and the duality result to the geometry of a one dimensional chain with two reservoirs connected at the end sites, further assuming that the maximal occupancy is $\nu = 1$ (sometimes called hard-core exclusion). In such a situation the process is integrable and the non-equilibrium steady state can be written in matrix-product form (see [13]). The main result of this chapter is the derivation of exact formulas for the non-equilibrium steady state correlations and, as a consequence, of the non-equilibrium steady distribution. To do this, we first consider a similarity transformation S_1 (closely related to duality) that brings the original Hamiltonian to a second non-stochastic Hamiltonian $H^{'}$ (again, closely related to the dual process) with triangular boundaries by leaving the bulk untouched. Then we introduce a second similarity transformation S_2 that "rotates" the boundaries obtaining a left triangular and a right diagonal matrices and leaving the bulk untouched. This second transformation brings $H^{'}$ to an non-stochastic Hamiltonian denoted by $H^{''}$. All in all, this sequence of two similarity transformations, brings the original Hamiltonian H to a simpler one $H^{''}$ that has left-triangular and right-diagonal boundaries. This allows to simplify the commutation relations (1.2.22), (1.2.23) and (1.2.24) of the MPA and to obtain an explicit expression for the ground state of $H^{''}$, denoted by $|\Psi^{''}\rangle$. Inverting the two similarity transformations, one obtains the expression of the non-equilibrium steady state correlations of H in terms of the absorption probabilities of the dual process.

5. Chapter 8: *Perspectives: mapping of non-equilibrium onto equilibrium.* In this chapter we aim to introduce the problem of mapping the non-equilibrium Hamiltonian H of the integrable boundary driven multi-species stirring process onto the Hamiltonian of the equilibrium process that is obtained from H by setting equal the boundary densities. This aim is not yet fully reached, we report it as a perspective for future works. In this chapter, we first recall from [13] the quantum inverse scattering method for the boundary driven multi-species stirring process. After having derived some useful exchange relations between the entries of the double row monodromy matrix, we consider the power expansion in the spectral parameter of the transfer matrix $T(u)$, by computing the conserved charges of the system. Using these charges, we develop a technique to write the analytic expression for the matrix that maps the non-equilibrium onto equilibrium for the simplest case $N = 2$.

How to extend this technique to general N is still an open problem for which we have only partial results and that we aim to investigate in the future.

Part III: Reaction diffusion multi-species models.

1. Chapter 9: *Reaction diffusion models for multi-species uphill diffusion.* In this chapter we first report a brief overview about diffusion, by recalling the fundamental macroscopic laws (Fick's diffusion) and we underline the role of statistical mechanics in this field. Then, we define the uphill diffusion for multi-component systems. Finally, we construct a linear reaction diffusion PDE's model (the one of equation (1.2.41)) that, under appropriate boundary conditions, shows *partial uphill diffusion* in its macroscopic non-equilibrium steady state. This will be the model that we will try to reproduce with the microscopic dynamics in the following chapter of this part.

2. Chapter 10: *A reaction-diffusion interacting particle system with uphill.* Inspired by [43, 44], we start by describing a general boundary driven multi-species interacting particle system with hard-core exclusion. Then, we impose some linear algebraic conditions in order to guarantee that the evolution of the average occupation variable of particles of each species is given by a discretized version on a finite lattice of the PDE's (1.2.41) and that these evolution equations are closed (no higher correlations, see (1.2.47), (1.2.48), (1.2.49)). After having proved the existence of a family of models with the required features, we select among this family the simplest process with symmetric jumps and we show its hydrodynamic limit by martingale techniques. We discover that, although at discrete level the average occupation variable of particles reproduces the desired PDE's model, the hydrodynamic equations become coupled only in the reaction term (i.e. the terms with σ_{12}, σ_{21} vanish), loosing the possibility of steady state partial uphill diffusion.

3. Chapter 11: *Generalization of the model and duality.* The symmetric boundary driven interacting particle system introduced in Chapter 10, is limited to 2 species and to maximal occupancy of $\nu = 1$. In this chapter, we first generalized this model to arbitrary number N of species and to partial exclusion rule ($\nu > 1$). Then, by exploiting a $gl(N+1)$ symmetry, we show that this more general model admits absorbing duality with the same duality function of the multi-species stirring process. To our knowledge this is one of the first examples of reaction diffusion models where absorbing duality is proved. Finally, we derive the hydrodynamic limit and equilibrium density fluctuations for the case with diagonal diffusivity matrix ($\sigma_{12} = \sigma_{21} = 0$).

Part IV: Future perspective and outlook.

1. Chapter 12: *Duality and integrability for multi-species non-compact processes.* This chapter is based on a work in progress with Cristian Giardinà and Rouven Frassek. In the previous parts of this thesis we studied processes with compact state space, i.e. IPS where each site can maximally host a finite number of particles. In this chapter, we begin to investigate the non-compact multi-species IPS, i.e. processes where each site can host an unbounded number of particles. This set-up is richer, since it possibly includes also multi-species interacting diffusions. Firstly, inspired by [14], we introduce the boundary driven multi-species independent random walk. After having described it with the Heisenberg Lie algebra, we prove absorbing duality and we use it to characterize the non-equilibrium steady state. Indeed, due to the lack of interaction, duality is enough to write explicitly

the non-equilibrium measure.
After the non-interacting model, we propose a boundary driven version of a multi-species harmonic process, having in mind the goal of generalizing the single species process defined in [26]. We show that the bulk rates of this multi-species harmonic, can be obtained as the limit of the transition rates of the asymmetric process introduced in [45]. Then, we underline the fact that the choice of the boundary generators of this multi-species harmonic was done to guarantee that, in the case of equilibrium when all boundary parameters are equal for each species, there exists a reversible measure that is the same of the closed process. This is the correct choice for the derivation of the absorbing duality in the Lie algebraic framework.
Finally, we introduce the boundary driven multi-species SIP(2k) and we derive its equilibrium reversible measure, discovering that it is the multi-variate version of the reversible measure of the single species SIP(2k), i.e. a product of negative multinomials.

2. Chapter 13: *Duality for boundary driven asymmetric multi-species models* Usually, in literature there are two ways of putting a particle system out-of-equilibrium: the boundary driving, extensively discussed in this thesis, or the bulk driving, in which an asymmetry in the jump rates is introduced. A current research goal [9] is to define processes where the two features are combined, i.e. asymmetric processes with boundary reservoirs interaction. In this direction, we present a result in the single species situation: a boundary driven version for the asymmetric Brownian energy process (ABEP), where the interaction with the external environment has been carefully chosen to guarantee the existence of classical absorbing duality. This ABEP process is obtain from the BEP(2k) via a deterministic map and, to conserve the classical duality, it turns out that the boundary reservoir become non-local. Nevertheless, we show how this duality allow to compute long-range correlations for the exponential current of the ABEP.

Part I

Review of single species IPS

Chapter 2

General concepts in interacting particles systems

2.1 Markov processes: probabilistic set-up

In this section we provide the main concepts concerning the probabilistic set-up of Markov processes. We first introduce the generator and the semigroup characterization of a Markov process. Then we define the invariant and the reversible distribution, that are important tools to study the long term behavior of the process. Finally, we recall a martingale characterization that will be useful in the study of scaling limits.

2.1.1 Markov semigroup and generator

In the following we consider a Polish space Ω and we call *path* or *sample space* $D_\Omega[0,\infty)$ the set of càdlàg trajectories from $[0,\infty)$ to Ω, namely

$$D_\Omega([0,\infty)) := \{\omega \,:\, [0,\infty) \to \Omega \,|\, t \to \omega_t \text{ càdlàg}\} \tag{2.1.1}$$

This space is equipped with the Skorokhod topology (see [46, 30]). We denote by $\mathcal{F}$ the natural sigma-algebra of set on the path space. We define the a Markov process $(\eta(t))_{t\geqslant 0}$ via its law on $D_\Omega[0,\infty)$.

Definition 2 *(Markov process) A Markov process with values on Ω is a collection of measures $\{\mathbb{P}^\xi \,:\, \xi \in \Omega\}$ on the path space $D_\Omega([0,\infty))$ such that satisfies the following properties*

1. $\mathbb{P}^\xi(\eta(0) = \xi) = 1$ *for all* $\xi \in \Omega$
2. *for all* $\xi \in \Omega$ *the map* $\xi \to \mathbb{P}^\xi(A)$ *from* Ω *to* $[0,1]$ *is measurable for all* $A \in \mathcal{F}$
3. $t, s \geqslant 0$ *we have the Markov property*

$$\mathbb{P}^\xi(\eta(t+s) \in A \,|\, \mathcal{F}_s) = \mathbb{P}^{\eta(s)}(\eta(t) \in A) \quad a.s.\text{-}\mathbb{P}^\xi \tag{2.1.2}$$

here $\mathcal{F}_t$ is the natural filtration of the process.

The evaluation map from $D_\Omega([0,\infty))$ to Ω associates to each time $t \geqslant 0$ the configuration at time t of the Markov process as $t \to \eta(t)$. Therefore it is implicitly defined a process $(\eta(t))_{t\geqslant 0}$, starting from $\xi \in \Omega$ with càdlàg trajectories in Ω whose law is $\mathbb{P}^\xi(\cdot)$.

Remark 1 *In the following we will denote by* $\mathbb{P}_\mu$ *the law of the Markov process initialized by a distribution* μ*, i.e.*

$$\mathbb{P}_\mu := \int_\Omega \mathbb{P}^\xi \mu(d\xi) \tag{2.1.3}$$

For a given Markov process, we define the *transition Kernel*, i.e.

$$p_t(\xi, dy) = \mathbb{P}^\xi(\eta(t) \in dy) \tag{2.1.4}$$

that satisfies the *Chapman-Kolmogorov* equation

$$p_{t+s}(\xi, A) = \int_\Omega p_s(\xi, dz) p_t(z, A) \tag{2.1.5}$$

Usually, the transition kernel is weakly characterized using the space of bounded an measurable functions from Ω to $\mathbb{R}$ denoted by $C_b(\Omega)$ via the definition of the *transition or Markov semigroup.* For all $t \geqslant 0$ we define

$$\begin{aligned} S_t :& C_b(\Omega) \to C_b(\Omega) \\ f \mapsto & (S_t f)(\xi) = \int_\Omega f(x) p_t(\xi, dx) = \mathbb{E}^\xi \left[f(\eta(t)) \right] \end{aligned} \tag{2.1.6}$$

Here we have denoted by $\mathbb{E}^\xi$ the expectation with respect to $\mathbb{P}^\xi$. The family of $(S_t)_{t\geqslant 0}$ is a semigroup of operators, since $S_{t+s} = S_t S_s$ by the Chapman-Kolmogorov equation. In this thesis we will always consider *Feller semigroup*, i.e. a semigroup defined on $C_b(\Omega)$ such that the range of S_t is again $C_b(\Omega)$. For the sake of simplicity we will always call it Markov semigroup, understanding that we also require the Feller property. Under these assumption, the following properties hold.

Theorem 6 *(Properties of Markov semigroup) Let* $(\eta(t))_{t\geqslant 0}$ *a Markov process on* Ω*. Then the associated semigroup* $(S_t)_{t\geqslant 0}$ *has the following properties*

1. *Identity at times* $t = 0$
$$S_0 f = f \qquad \forall f \in C_b(\Omega) \tag{2.1.7}$$

2. *Right strong continuity*
$$\lim_{t\to 0^+} S_t f = f \qquad \forall f \in C_b(\Omega) \tag{2.1.8}$$

3. *Semigroup property:* $\forall s, t \geqslant 0$
$$S_{t+s} f = S_t S_s f \qquad \forall f \in C_b(\Omega) \tag{2.1.9}$$

4. *Positivity,* $\forall t \geqslant 0$
$$S_t f \geqslant 0 \qquad \forall f \in C_b(\Omega) \quad \text{such that} \quad f \geqslant 0 \tag{2.1.10}$$

5. *Conservation of probability,*
$$S_t 1 = 1 \qquad \forall t \geqslant 0 \tag{2.1.11}$$

For the proof see [7] and [47].
We have seen that, given a Markov process, it is possible to construct a Markov semigroup. The following theorem states the opposite implication. This will implies a one-to-one correspondence between Markov process and Markov semigroups.

Theorem 7 *Let $(S_t)_{t\geqslant 0}$ a Markov semigroup defined on $C_b(\Omega)$, then there exists a unique Markov process $(\eta(t))_{t\geqslant 0}$ defined on Ω such that for all $f \in C_b(\Omega)$, for all $\xi \in \Omega$ and for all $t \geqslant 0$*

$$(S_t f)(\eta) = \mathbb{E}^{\xi}\left[f(\eta(t))\right] \tag{2.1.12}$$

where the above expectation is taken with respect to the law of $\eta(t)$ starting from ξ.

For the proof see [7, 47].

Given a Markov semigroup $(S_t)_{t\geqslant 0}$ we introduce the *Markov infinitesimal generator* as the strong limit (in the norm $\|\cdot\|_\infty$)

$$\mathcal{L}f = \lim_{t\to 0}\frac{S_t f - f}{t} \tag{2.1.13}$$

The generator is defined on a domain

$$\mathcal{D}_{\mathcal{L}} := \left\{ f \in C_b(\Omega),\ \ \exists g \in C_b(\Omega)\ :\ \lim_{t\to 0}\|\frac{S_t f - f}{t} - g\|_\infty = 0 \right\} \tag{2.1.14}$$

It is possible to characterize Markov process using the generator. This is stated in the Hille-Yosida Theorem.

Theorem 8 *(Hille-Yosida) There is a one-to-one correspondence between Markov semigroup on $C_b(\Omega)$ and Markov generators on $C_b(\Omega)$ given by:*

1. *for all $f \in \mathcal{D}_{\mathcal{L}}$ we have*

$$\mathcal{L}f = \lim_{t\to 0}\frac{S_t f - f}{t} \tag{2.1.15}$$

2. *for all $f \in C_b(\Omega)$ and $t \geqslant 0$ we have*

$$S_t f = \lim_{n\to\infty}\left(I - \frac{t}{n}\mathcal{L}\right)^n f \tag{2.1.16}$$

where I is the identity operator.

Moreover, we have that for $f \in \mathcal{D}_{\mathcal{L}}$, $S_t f$ is differentiable with respect to time and it satisfies

$$\frac{d}{dt}S_t f = \mathcal{L}S_t f \quad \textit{(Kolmogorov backward equation)} \tag{2.1.17}$$

$$\frac{d}{dt}S_t f = S_t\mathcal{L}f \quad \textit{(Kolmogorov forward equation)} \tag{2.1.18}$$

For the proof see [7] and [47].

Informally, we write

$$S_t = e^{t\mathcal{L}} \tag{2.1.19}$$

As a consequence of the properties of the Markov semigroup stated in Theorem 6, the Markov generator must satisfy the following properties.

Proposition 1 *Consider $\mathcal{L}$ a Markov generator, then it satisfies*

1. *Conservation of probability*

$$\mathcal{L}1 = 0 \tag{2.1.20}$$

2. *Maximum principle: let $f(\bar{\eta}) = \max_{\eta\in\Omega} f(\eta)$, then*

$$\mathcal{L}f(\bar{\eta}) \leqslant 0 \tag{2.1.21}$$

All in all, we have seen that there is a one-to-one correspondence between Markov process and Markov generators. Therefore, in the following we will always study a Markov process via its generator.

Semigroup and generators for particles systems

When the state space Ω is countable or finite, the Markov process becomes a jump process and, for all $\eta, \xi \in \Omega$, the generator reads

$$\mathcal{L}f(\eta) = \sum_{\xi \in \Omega} c(\eta, \xi) \left(f(\xi) - f(\eta) \right) \tag{2.1.22}$$

where $c(\eta, \xi)$ is called the *transition rate*. Informally, it can be interpreted as the probability of having a jump from a configuration η to a configuration ξ in an infinitesimal time interval dt. When Ω is finite or countable, the generator can be represented as a matrix

$$\mathcal{L} = \left(c(\eta, \xi) \right)_{\xi, \eta \in \Omega} \tag{2.1.23}$$

Moreover, in this setting, any functions f can be represented as a vector with components $f(\eta)$ for all $\eta \in \Omega$. In order to be a Markov generator, the two conditions stated in Proposition 1 now read

1. *the rates must be non-negative*

$$c(\eta, \xi) \geqslant 0 \qquad \eta, \xi \in \Omega \quad \text{such that} \quad \eta \neq \xi \tag{2.1.24}$$

2. $\mathcal{L}$ *must be a stochastic matrix*, i.e.

$$\sum_{\xi \in \Omega} c(\eta, \xi) = 0 \qquad \forall \eta \in \Omega \tag{2.1.25}$$

this conditions can be translated in

$$c(\eta, \eta) = - \sum_{\xi \in \Omega \,:\, \xi \neq \eta} c(\eta, \xi) \qquad \forall \eta \in \Omega \tag{2.1.26}$$

meaning that the diagonal element of the matrix are non-positive and equal to minus the sum of all other rates of the row.

The problem of the well definiteness of the generator of an interacting particles system reduces to ask that the process is *non-explosive*. We do not treat this topic in this thesis and we always assume that the processes that we study have well defined generator (usually because we consider finite state spaces). For details one could refer, for instance, to [7, 47].
In the case of particle system the transition semigroup is a matrix S_t with elements $S_t(\xi, \xi^{'})$ that are interpreted as the probabilities of having the process in a configuration $\xi^{'}$ at time t given that at $t = 0$ it was in ξ, namely

$$S_t(\xi, \xi^{'}) = \mathbb{P}(\eta(t) = \xi^{'} \mid \eta(0) = \xi) \tag{2.1.27}$$

If the matrix S_t does not depend on time we say that the process is *time homogeneous*. In the course of this thesis we will always assume time homogeneity of the processes.
In case of a countable or finite state space, the Kolmogorov backward equation stated in (2.1.17) becomes the evolution equation for the expectation of the function $f : \Omega \to \mathbb{R}$ and it reads

$$\frac{d}{dt} \mathbb{E}_\eta \left[f(\eta(t)) \right] = \left(\mathcal{L} \mathbb{E}_\eta \left[f(\cdot) \right] \right) \eta(t) \tag{2.1.28}$$

Similarly, by using the Kolmogorov forward equation (2.1.18) one obtains

$$\frac{d}{dt} \mathbb{E}_\eta \left[f(\eta(t)) \right] = \mathbb{E}_\eta \left[\mathcal{L} f(\eta(t)) \right] \tag{2.1.29}$$

2.1.2 Stationary and reversible measures

In this section we recall some basic facts about the time evolution of the probability distribution of a Markov process. We first write general statements and then we specialize to the situations of finite state space. In this section we always consider the case the process to have values in a state space Ω that we assume to be Polish and compact. We introduce $\mathfrak{P}(\Omega)$ the set of probability measures on Ω. This space is equipped with the weak topology denoted by d and defined as

$$d(\mu,\nu) = \sup_{f\in C_b(\Omega)} |\int_\Omega f d\mu - \int_\Omega f d\nu| \qquad \forall \mu,\nu \in \mathfrak{P}(\Omega) \tag{2.1.30}$$

We say that a sequence of measure converges $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ if $d(\mu_n,\mu) \to 0$ as $n \to \infty$.

Definition 3 *(Evolution of a measure) Let $\mu \in \mathfrak{P}(\Omega)$, then the time evolution from time 0 to time t of μ, is the unique element $\mu_t = \mu S_t$ such that*

$$\int_\Omega f d\mu_t = \int_\Omega f d(\mu S_t) = \int_\Omega S_t f d\mu \qquad \forall f \in C_b(\Omega) \tag{2.1.31}$$

The uniqueness of this element is guaranteed by the Riesz-Representation theorem.

Definition 4 *(Invariant measures) A measure $\mu \in \mathfrak{P}(\Omega)$ is said invariant or stationary if $\mu_t = \mu$ for all $t \geqslant 0$, i.e.*

$$\int_\Omega f d\mu_t = \int_\Omega f d\mu \qquad \forall f \in C_b(\Omega) \tag{2.1.32}$$

Definition 5 *(Reversible measure) A measure $\mu \in \mathfrak{P}(\Omega)$ is said reversible if for all $t \geqslant 0$ satisfies*

$$\int_\Omega f S_t g d\mu = \int_\Omega g S_t f d\mu \qquad \forall f \in C_b(\Omega) \tag{2.1.33}$$

By taking $g = 1$ it is trivial to show that every reversible measure is also invariant.

Proposition 2 *(Characterization of invariant measures) Consider a measure $\mu \in \mathfrak{P}(\Omega)$, then*

- *μ is invariant if and only if*

$$\int_\Omega \mathcal{L} f d\mu = 0 \qquad \forall f \in \mathcal{D}_\mathcal{L} \tag{2.1.34}$$

- *μ is reversible if and only if*

$$\int_\Omega g\mathcal{L} f d\mu = \int_\Omega f \mathcal{L} g d\mu \qquad \forall f,g \in \mathcal{D}_\mathcal{L} \tag{2.1.35}$$

  i.e. if the generator is self-adjoint with respect to μ.

The proof is standard and based on Hille-Yosida theorem. See [7] and [47].
We denote by $\mathcal{I}$ the set of invariant measure of a process. This set has some properties that are listed in the following proposition.

Proposition 3 *Let $\mathcal{I}$ the set of invariant measures, then it satisfy the following properties:*

1. *$\mathcal{I}$ is convex and weakly compact with respect to the weak topology d*

2. *$\mathcal{I} \neq \varnothing$*

3. *if there exists a $\mu \in \mathfrak{P}(\Omega)$ such that $\nu = \lim_{t\to\infty}\mu S_t$, then $\nu \in \mathcal{I}$*

For the proof see [7] and [47].
To describe the long time behaviour, an other useful class of measures is needed: the ergodic measures. To define them, we introduce the invariant set: $\mathcal{A} \subset \Omega$ is called invariant for the process $(\eta(t))_{t\geqslant 0}$ if, for all $t \geqslant 0$, we have

$$\eta(0) \in \mathcal{A} \quad \text{if and only if} \quad \eta(t) \in \mathcal{A} \tag{2.1.36}$$

Definition 6 *(Ergodic measures) A measure $\mu \in \mathcal{I}$ is ergodic if*

$$\mu(A) = \int_A d\mu \in \{0,1\} \qquad \forall \mathcal{A} \quad \textit{invariant} \tag{2.1.37}$$

Let $\mu \in \mathfrak{P}$ be ergodic, then we have the following convergence property: for all $\nu \in \mathfrak{P}(\Omega)$

$$\mu = \lim_{t\to\infty} \nu_t \tag{2.1.38}$$

This means that, regardless to the initial measure, in the long time horizon, the time evolved measure converges (weakly) to the ergodic one. To have ergodicity of an interacting particle system we need to require that the chain is irreducible and positive recurrent. There is an important relation between ergodic measures and the extreme points of $\mathcal{I}$.

Proposition 4 *Let $\mathcal{I} \subset \mathfrak{P}(\Omega)$ the set of invariant measure, we denote by $\mathcal{I}_e$ the set of the extreme points. Then, we have the following statements:*

- *$\mu \in \mathcal{I}$ is ergodic if and only if $\mu \in \mathcal{I}_e$*
- *Any $\mu \in \mathcal{I}$ can be written as a convex combination of ergodic measures*

For the proof see [7] and [47].
We say that a function $g : \Omega \to \mathbb{R}$ is a conserved quantity if, for all $t \geqslant 0$, $g(\eta(0)) = g(\eta(t))$. We have the following characterization of the ergodic measures via conserved quantities.

Proposition 5 *$\mu \in \mathcal{I}$ is ergodic if and only if every conserved quantity is constant μ-almost surely.*

For the proof see [7] and [47].

Evolution of measures in the context of particle systems: the master equation

Here we recall the main concepts about evolution of measures, invariance and reversibility when the state space of the Markov process is finite or countable. In this situation the generator of the process is the matrix written in (2.1.22) with elements $c(\eta,\xi)$, that are the transition rates. Denoting the process with $(\eta(t))_{t\geqslant 0}$ and its configurations by η, we consider a measure $\mu \in \mathfrak{P}(\Omega)$ and we introduce a ket-vector $\langle\mu|$ where each component $\mu(\xi)$ is the probability mass of the event $\eta = \xi$, i.e.

$$\mu(\xi) = \mathbb{P}(\eta = \xi) \qquad \forall \xi \in \Omega \tag{2.1.39}$$

The time evolution of this vector is denoted by $\langle\mu(t)|$ and each of its component $\mu_t(\xi)$ represents the probability of having a configuration ξ at time t, given that at initial time the process was distributed by μ, i.e.

$$\mu_t(\xi) = \mathbb{P}_\mu(\eta(t) = \xi) \qquad \forall \xi \in \Omega \tag{2.1.40}$$

This $\langle\mu(t)|$ satisfies an evolution equation that is derived as follows. By using the definition of transition semigroup given in (2.1.27), we have that

$$\begin{aligned}\mu_t(\xi) &= \sum_{\zeta\in\Omega} \mathbb{P}_\mu(\eta(t) = \xi|\eta(0) = \zeta)\mathbb{P}_\mu(\eta(0) = \zeta) \\ &= \sum_{\zeta\in\Omega} \mathbb{P}_\mu(\eta(t) = \xi|\eta(0) = \zeta)\mu(\zeta) = \sum_{\zeta\in\Omega} S_t(\zeta,\xi)\mu(\xi)\end{aligned} \tag{2.1.41}$$

where $S_t(\xi,\zeta)$ is the transition semigroup of the process. Therefore, we obtain

$$\langle\mu_t| = \langle\mu|S_t \tag{2.1.42}$$

Deriving both sides and using the Kolmogorov forward equation defined in (2.1.18), we obtain the *master equation*

$$\frac{d}{dt}\langle\mu_t| = \langle\mu_t|\mathcal{L} \tag{2.1.43}$$

or, introducing the Hamiltonian operator as $H = \mathcal{L}^T$, this master equation reads

$$\frac{d}{dt}|\mu_t\rangle = H|\mu_t\rangle \tag{2.1.44}$$

This equation describe how the distribution of the process evolves in time. It plays an important role in non-equilibrium statistical mechanics. Notice that (2.1.43) can be retrieved from (2.1.29) by choosing $f(\eta(t)) = \mathbb{1}_{\{\eta(t)=\xi\}}$.

Stationary and reversible measures for a particle system

As described in the previous paragraph, we can associate a vector $\langle\mu|$ to the distribution of an interacting particles system. Therefore, the stationarity condition written in (2.1.34) now reads

$$\langle\mu|\mathcal{L} = 0 \tag{2.1.45}$$

or, using the Hamiltonian

$$H|\mu\rangle = 0 \tag{2.1.46}$$

This invariant measure $|\mu\rangle$ is often called steady state. Finding this steady state turns out to be a spectral problem: "*find the eigenvector of the Hamiltonian with vanishing eigenvalue*", provided that it exist and it is unique.

In the discrete state space setting, a measure $\mu \in \mathfrak{P}(\Omega)$ is reversible if and only if it satisfies the *detailed balance condition*, that reads

$$\mu(\eta)\, c(\eta,\xi) = \mu(\xi)\, c(\xi,\eta) \qquad \forall \eta,\xi \in \Omega \tag{2.1.47}$$

2.1.3 Martingales and Markov processes

In this section we introduce two martingales arising from the generator of a Markov process. These martingales are useful in the proofs of the scaling limit.

Lemma 1 *Consider a Markov process* $(\eta(t))_{t\geqslant 0}$ *on the state space* Ω *and with generator* $\mathcal{L}$. *Consider an arbitrary function* $F \in C_c^{\infty}(\Omega)$. *Then, the following processes*

$$M_F^t := F(\eta(t)) - F(\eta(0)) - \int_0^t (\mathcal{L}F)(\eta(s))ds \tag{2.1.48}$$

is a (Dynkin) martingale with respect to the natural filtration $(\mathcal{F}_t)_{t\geqslant 0}$ *and its quadratic variation is*

$$[M_F^{\cdot}, M_F^{\cdot}]_t = \int_0^t \Gamma_F^s(\eta)ds \tag{2.1.49}$$

where

$$\Gamma_F^s(\eta) := (\mathcal{L}F^2)(\eta(s)) - 2F(\eta(s))(\mathcal{L})F(\eta(s)) \tag{2.1.50}$$

is the Carré-du-Champ operator.

Proof: see [30] and [31].

Remark 2 *In case of interacting particle systems, the Carré-du-Champ operator can be written as*

$$\Gamma_F^{\cdot}(\eta) = \sum_{\xi\in\Omega} c(\eta,\xi)\left(F(\xi) - F(\eta)\right)^2 \tag{2.1.51}$$

This follows from the definition of generator (2.1.22).

By the Doob decomposition theorem(see [46]), we have that the process

$$\mathcal{N}_F^t := \left(M_F^t\right)^2 - \int_0^t \Gamma_F^s(\eta)ds \tag{2.1.52}$$

is also a martingale with respect to the natural filtration. The martingale constructed in this section can also be defined by using a time-dependent function $F : [0,\infty) \times \Omega \to \mathbb{R}$ sufficiently smooth in both variables (for details see Appendix 1 of [30]).

Chapter 3

Single species interacting particle systems: duality, integrability and scaling limits

3.1 Single species models

With single species models we refer to Markov processes (both interacting particle systems and interacting diffusions) in which only one type of particles (or energy) is present. This means that on a given discrete geometry each site can be occupied by a certain number of indistinguishable particles (energy). The transitions consists in jumps at Poissonian times, where particles move from a site to another changing the configuration of the process. A different situation is the multi-species case, where we consider many species (also called colors) of particles allowing, besides the movement of each colour, some species-species interaction, both of position exchange or of mutation type.
In this section we introduce some classical interacting particle systems and interacting diffusions models. For each model we briefly describe the dynamics and the infinitesimal generator, deriving also the equilibrium reversible distribution. In all these models we assume that they perform their dynamics on a connected and undirected graph $G = (V, \mathcal{E})$, where V is a set of vertices and $\mathcal{E}$ is the set of edges. The sate space is always product over sites and it reads

$$\Omega = \bigotimes_{x \in V} \Omega_x \tag{3.1.1}$$

where the space Ω_x depends on the process. We distinguish two situations:

- *Processes without boundary driving*: in this case the graph G does not interact with the external environment and the generator of the process is given by

  $$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} \, . \tag{3.1.2}$$

  Here $\mathcal{L}_{x,y}$ acts only on the edge $(x, y) \in \mathcal{E}$ and then it is called *edge generator* and it determines the dynamics of the process. The quantities $\omega_{x,y}$ are called conductances and they "weigh" the connectivity of the graph. In this thesis we assume that they have fixed values, however generalizations are possible, for instance one could make them random (stochastic particle systems in random environment, see, for instance [48, 49]). In this equilibrium set-up we talk about a *process with closed boundaries.*

- *Boundary-driven processes*: here we put each vertex in contact with an external reservoir, that tries to impose a fixed *boundary density* of particles (or of energy). When all these boundary densities are the same, the process is said to be *at equilibrium*, instead, when at least two boundary densities are different the process is driven *out-of-equilibrium* (losing reversibility). The generator has a further term, working on a single site x

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \mathcal{L}_x \,. \tag{3.1.3}$$

  Here $\omega_{x,y}$ and $\mathcal{L}_{x,y}$ are the quantities defined in (3.1.2), while $\mathcal{L}_x$ is called *site generators*. Here, $\mathcal{L}_x$ rules the interaction of the process with the external environment. The quantities Γ_x are called local inhomogeneities, and they "weigh" the interaction between the reservoirs and the process on the graph. Again, we consider these Γ_x to be fixed, but generalization such as choosing them as random variables are possible. In this set-up we talk about a *process with open boundaries* or a *boundary-driven process*.

In the following, by specifying $\mathcal{L}_{x,y}$, $\mathcal{L}_x$ and Ω_x we uniquely define the model and its interaction with boundaries.

Parametric generators for particles systems. In the following we introduce three models: independent random walk (IRW), symmetric exclusion process (SEP(ν)) and symmetric inclusion process (SIP(2k)). We write a parametric form for the generator that allows, by correctly choosing the parameters, to obtain the specific model. For arbitrary $(x, y) \in \mathcal{E}$, $x, y \in V$ and for arbitrary local function $f : \Omega \to \mathbb{R}$, the edge generator reads

$$(\mathcal{L}_{x,y}f)(\eta) = \eta_x(a^y + \theta\eta_y)\left(f(\eta - \delta_x + \delta_y) - f(\eta)\right) + \eta_y(a^x + \theta\eta_x)\left(f(\eta - \delta_y + \delta_x) - f(\eta)\right) \,. \tag{3.1.4}$$

The site generator reads

$$\mathcal{L}_x f(\eta) = \alpha_1^x(a^x + \theta\eta_x)\left(f(\eta + \delta_x) - f(\eta)\right) + \alpha_2^x\eta_x\left(f(\eta - \delta_x) - f(\eta)\right) \tag{3.1.5}$$

where

$$(\delta_x)_y = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \text{otherwise} \end{cases} \tag{3.1.6}$$

Therefore, with the notation $\eta - \delta_x + \delta_y$ we mean that, starting from the configuration η, we remove a particle at site x and we add a particles at site y. The parameters $\theta \in \{-1, 0, 1\}$ defines the type of interaction, while $(a^x)_{x\in V}$ are called *attraction intensities*. In the following, to simplify the analysis, we assume that $a^x = a$ for all $x \in V$ with a still depending on the model. The quantities α_1^x, α_2^x are the boundary parameters and they are linked with the rates at which the reservoirs inject and remove particles, respectively.
Using these boundary parameters we will define, for each model, the boundary densities that are interpreted as the average number of particles present in the reservoirs. These densities depend on the models and are defined case by case. When all these densities are equal, we obtain an equilibrium state also for a model with open boundaries.

Interacting diffusions. For the interacting diffusion, we will only consider the Brownian energy process (BEP), therefore we will directly write its generator. In this situation, the boundary parameters are called "temperatures" and make this models also suitable in the microscopic derivation of the Fourier law. Many other interacting diffusion models are available, for instance the Kipnis-Marchioro-Presutti model (KMP) model [50] or the Brownian Momentum Process (BMP) [11]. However, the study of them goes beyond the purpose of this thesis.

3.1.1 Independent random walk (IRW)

The independent random walk is one of the simplest examples of interacting particle system where, in particular, the interaction is suppressed. The particles perform a continuous time random walk on the graph and the reservoirs inject and remove particles (birth and death processes). The state space is given by the product of $\Omega_x = \{\eta_x \,:\, \eta_x \in \mathbb{N}_0\}$, i.e.

$$\Omega = \mathbb{N}_0^{|V|} \tag{3.1.7}$$

where we denote by $|V|$ the number of vertices of G. The edge generator is obtained from (3.1.4) by setting $a^x = 1$ for all $x \in V$ an by setting $\theta = 0$, therefore it reads

$$\mathcal{L}_{x,y} f(\eta) = \eta_x \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right) + \eta_y \left(f(\eta + \delta_x - \delta_y) - f(\eta)\right) . \tag{3.1.8}$$

The site generator is obtained from (3.1.5) by fixing $a^x = 1$ for all $x \in V$, $\theta = 0$ and $\alpha_1^x, \alpha_2^x \in [0, \infty)$. It reads

$$\mathcal{L}_x f(\eta) = \alpha_1^x \left(f(\eta + \delta_x) - f(\eta)\right) + \alpha_2^x \eta_x \left(f(\eta - \delta_x) - f(\eta)\right) . \tag{3.1.9}$$

We introduce the boundary density at site x as

$$\rho^x := \frac{\alpha_1^x}{\alpha_2^x} \qquad \forall x \in V \tag{3.1.10}$$

We now characterize the equilibrium reversible measure when all the boundary densities are the same. The proof follows by imposing the detail balance condition (2.1.47).

Lemma 2 *(Reversible measure for the open equilibrium process) The IRW process* $(\eta(t))_{t \geqslant 0}$ *with open boundaries and with generator*

$$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \mathcal{L}_x \,, \tag{3.1.11}$$

where the edge generator $\mathcal{L}_{x,y}$ *is defined in* (3.1.8) *and where the site generator* $\mathcal{L}_x$ *is defined in* (3.1.9)*, admits a reversible measure if and only if* $\frac{\alpha_1^x}{\alpha_2^y} = \frac{\alpha_1}{\alpha_2}$ *for all* $x \in V$*. This measure is product over sites with marginals distributed as Poisson(*$\frac{\alpha_1}{\alpha_2}$*), i.e.*

$$\mu_{rev} = \bigotimes_{x \in V} \mu_{rev}^x \qquad \textit{where} \qquad \mu_{rev}^x \sim \textit{Poisson}(\frac{\alpha_1}{\alpha_2}) \tag{3.1.12}$$

Due to the non-interacting dynamics, it possible to derive explicitly the non-equilibrium steady state measure using duality. This non-equilibrium measure is given by a product over sites of Poisson distribution with non-homogeneous parameters. We refer to [12] for its derivation.

3.1.2 Symmetric exclusion process (SEP(ν))

The Symmetric Exclusion Process consists in a continuous time random walk on the graph G, where each site can host at most $\nu \in \mathbb{N}$ particles. The quantity ν is called *maximal occupancy.* The state space is given by

$$\Omega = \bigotimes_{x \in V} \Omega_x \qquad \text{where} \qquad \Omega_x = \{\eta_x \,:\, \eta_x \in \{0, 1, \ldots, \nu\}\} . \tag{3.1.13}$$

This means that at each site x at most ν (indistinguishable) particles can be hosted, setting in the so called *exclusion constraint*. With a different perspective, one can interpret this constrain by saying that, at each site the total number of particles and the total number of holes (or vacancies) must always be equal to ν. The SEP(ν) is one of the first examples where interaction is not null, meaning that the transition rate both depends on the occupancy of the departure and the arrival site. More precisely, the jump rate is proportional to the number of particles in the departure site and to the number of holes in the arrival one. This means that particles are encouraged to stay in different sites. For this reason we say that SEP(ν) is a *fermionic process*. The edge generator is given by (3.1.4) by fixing $a^x = \nu \in \mathbb{N}$ for all $x \in V$ and $\theta = -1$. Then, it reads

$$\mathcal{L}_{x,y} f(\eta) = \eta_x(\nu - \eta_y) \left(f(\eta - \delta_x + \delta_y) - f(\eta) \right) + \eta_y(\nu - \eta_x) \left(f(\eta - \delta_y + \delta_x) - f(\eta) \right) . \quad (3.1.14)$$

The site generator is found from (3.1.5) by setting $a^x = \nu \in \mathbb{N}$ for all $x \in V$, $\theta = -1$ and $\alpha_1^x, \alpha_2^x \in [0, \infty)$. Therefore, it reads

$$\mathcal{L}_x f(\eta) = \alpha_1^x(\nu - \eta_x) \left(f(\eta + \delta_x) - f(\eta) \right) + \alpha_2^x \eta_x \left(f(\eta - \delta_x) - f(\eta) \right) . \quad (3.1.15)$$

If $\nu = 1$, the process is said to have an *hard-core exclusion* and it is called Simple Symmetric Exclusion Process (SSEP). We introduce the boundary density at site $x \in V$ as

$$\rho^x = \nu \frac{\alpha_1^x}{\alpha_2^x + \alpha_2^x} . \quad (3.1.16)$$

We now characterize the equilibrium reversible measure when all boundary densities are the same. This result can be proved by imposing the detailed balance condition (2.1.47).

Lemma 3 *(Reversible measure for the open equilibrium process) The SEP(ν) process* $(\eta(t))_{t \geqslant 0}$ *with open boundaries and with generator*

$$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \mathcal{L}_x , \quad (3.1.17)$$

where the edge generator $\mathcal{L}_{x,y}$ *is defined in* (3.1.14) *and where the site generator* $\mathcal{L}_x$ *is defined in* (3.1.15)*, admits a reversible measure if and only if* $\frac{\alpha_1^x}{\alpha_2^x} = \frac{\alpha_1}{\alpha_2}$ *for all* $x \in V$*. This measure is product over sites with marginals distributed as Binomial(*$\nu, \frac{\alpha_1}{\alpha_2 + \alpha_1}$*), i.e.*

$$\mu_{rev} = \bigotimes_{x \in V} \mu_{rev}^x \qquad \textit{where} \qquad \mu_{rev}^x \sim \textit{Binomial}(\nu, \frac{\alpha_1}{\alpha_2 + \alpha_1}) . \quad (3.1.18)$$

3.1.3 Symmetric inclusion process (SIP(2k))

The Symmetric Inclusion Process consists in a continuous time random walk on the graph G, where each site can host an unbounded number of particles. The state space is given by

$$\Omega = \bigotimes_{x \in V} \Omega_x \qquad \text{where} \qquad \Omega_x = \{\eta_x \ : \ \eta_x \in \mathbb{N}_0\} . \quad (3.1.19)$$

The transition rate depends on the occupancy of the departure and the arrival site. More precisely, the jump rate is proportional to the number of particles in the departure site and to the number of particles in the arrival site plus the attractive parameter $2k$. This is an inclusion jump that leads to an attractive interaction. For this reason we say that the SIP is a *bosonic process*. The edge generator is given by (3.1.4) by fixing $a^x = 2k \in \mathbb{R}_+$ for all $x \in V$ and $\theta = +1$. It reads

$$\mathcal{L}_{x,y} f(\eta) = \eta_x(2k + \eta_y) \left(f(\eta - \delta_x + \delta_y) - f(\eta)\right) + \eta_y(2k + \eta_x) \left(f(\eta - \delta_y + \delta_x) - f(\eta)\right) \ . \tag{3.1.20}$$

The edge generator is found from (3.1.5) by setting $a^x = 2k \in \mathbb{R}$ for all $x \in V$, $\theta = +1$ and $\alpha_1^x, \alpha_2^x \in [0, \infty)$. It reads

$$\mathcal{L}_x f(\eta) = \alpha_1^x(2k + \eta_x) \left(f(\eta + \delta_x) - f(\eta)\right) + \alpha_2^x \eta_x \left(f(\eta - \delta_x) - f(\eta)\right) \ . \tag{3.1.21}$$

Assuming that $\alpha_2^x > \alpha_1^x$, we introduce the boundary density at site $x \in V$ as

$$\rho^x = (2k) \frac{\alpha_1^x}{\alpha_2^x - \alpha_1^x} \ . \tag{3.1.22}$$

We now characterize the equilibrium reversible measure when all the boundary densities are equal. The proof consists in imposing the detailed balance condition (2.1.47).

Lemma 4 *(Reversible measure for the open equilibrium process) The SIP(2k) process* $(\eta(t))_{t \geqslant 0}$ *with open boundaries and with generator*

$$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \mathcal{L}_x \ , \tag{3.1.23}$$

where the edge generator $\mathcal{L}_{x,y}$ *is defined in* (3.1.20) *and where the site generator* $\mathcal{L}_x$ *is defined in* (3.1.21)*, admits a reversible measure if and only if* $\frac{\alpha_1^x}{\alpha_2^x} = \frac{\alpha_1}{\alpha_2}$ *for all* $x \in V$*. This measure is product over sites with marginals distributed as Negative-Binomial(*$2k, \frac{\alpha_1}{\alpha_2}$*), i.e.*

$$\mu_{rev} = \bigotimes_{x \in V} \mu_{rev}^x \qquad \textit{where} \qquad \mu_{rev}^x \sim \textit{Negative-Binomial}(2k, \frac{\alpha_1}{\alpha_2}) \ . \tag{3.1.24}$$

3.1.4 Brownian energy process (BEP(2k))

The Brownian Energy Process with parameter $2k$ (BEP(2k)) was introduced in [11] to model heat exchange. Energy is diffusively exchanged between sites. As we will clarify, this model is also related to the SIP(2k) and to the Kipnis-Marchioro-Presutti (KMP) process (see [50]). Indeed, it can be obtained as the many-particles limit of SIP(2k), while the KMP can be obtained by thermalizing BEP(1), i.e. when $k = 1/2$ (see [12]). The process is denoted with $(z(t))_{t \geqslant 0}$, where at each site $x \in V$, $z_x(t)$ denotes the amount of energy at site x and at time t. The state space reads

$$\Omega = \bigotimes_{x \in V} \Omega_x \qquad \text{where} \qquad \Omega_x = \{z_x \ : \ z_x \in \mathbb{R}_+\} \ . \tag{3.1.25}$$

The BEP process has a parameter $2k > 0$, that rules the energy exchange. The edge generator reads

$$\mathcal{L}_{x,y} = z_x z_y \left(\partial_{z_x} - \partial_{z_y}\right)^2 - 2k(z_x - z_y) \left(\partial_{z_x} - \partial_{z_y}\right) \tag{3.1.26}$$

assuming that it acts on smooth function with compact support $f : \Omega \to \mathbb{R}$. The site generator is given by

$$\mathcal{L}_x = T_x \left(2k \partial_{z_x} + z_x \partial_{z_x}^2\right) - z_x \partial_{z_x} \ . \tag{3.1.27}$$

Here T_x is the boundary parameter and it physically represent the temperature of the reservoir. In this case we call boundary density at site $x \in V$ the temperature itself, i.e.

$$\rho^x = T_x \,. \tag{3.1.28}$$

We now prove the characterize the equilibrium reversible measure when all the boundary densities are equal. This result can be proved by showing that the generator is self-adjoint in the $L^2(\mathbb{R}^+, \mu_{rev})$.

Lemma 5 *(Reversible measure for the open equilibrium process) The BEP(2k) process* $(z(t))_{t\geqslant 0}$ *with open boundaries and with generator*

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \mathcal{L}_x \,, \tag{3.1.29}$$

where the edge generator is defined in (3.1.26) *and where the site generator* (3.1.27) *admits a reversible measure if and only if* $T_x = T$ *for all* $x \in V$*. This measure is product over sites with marginals distributed as Gamma(2k,T), i.e.*

$$\mu_{rev} = \bigotimes_{x\in V} \mu_{rev}^x \qquad \textit{where} \qquad \mu_{rev}^x \sim \textit{Gamma}(2k, T) \,. \tag{3.1.30}$$

3.2 Duality between Markov processes

Here we introduce the concept of stochastic duality between two Markov processes. Duality is a powerful technique that connects the average of two stochastic processes. It was originally introduced in [5] to characterize the stationary distribution of SSEP and IRW. Then, it was systematically defined for more models (exclusion, voter) in [7] to investigate ergodic properties. The techniques used to find duality are, in many cases, connected with some Lie algebraic symmetries of the process. In this direction, there are the seminal works in the context of the exclusion process [8, 10, 9] and [11, 12, 51] in the context of inclusion processes and interacting diffusions. For an excellent review we refer to [15]. In the past years, many application of duality have been found:

- *Out-of-equilibrium systems*: out-of-equilibrium stochastic systems are prototypes for non-equilibrium statistical physics models. In these models, currents can be created by putting the system in contact with external reservoirs or by inserting an asymmetry in the jump rates (drift). Using duality, one can show that the computation of n-point non-equilibrium correlation in a system of size L reduces in the study of the behavior of $n \ll L$ dual particles. One of the first examples is [52], where duality for the boundary driven SSEP has been applied to show the existence of long-range correlation and to compute two points correlations. Other examples of the use of duality to characterize the non-equilibrium stationary measure are [12, 53, 54, 9]. More recently some works in this direction are [26, 55, 56] and for the multi-species case [3]. Duality has also been used to study heat exchange models relating these continuous processes to discrete particle systems, see for instance [50, 57, 58].

- *Hydrodynamic limit and density fluctuations*: duality can be used to derive scaling limit properties of the particle systems. Here, the idea consists in re-scaling the dynamics of a single dual particles and applying the invariance principle for the random walk to find, in the limit, a Brownian motion. Also in this context the literature is wide. Some examples for the hydrodynamic limit can be found in [15, 29, 48, 59], while for the fluctuation a recent work is [60].

- *Population genetics*: duality is widely used also in the context of population genetics. The idea is to relate the time forward and time backward process and extract information on the genealogy going back in time. Some examples of works in this context are [61, 62, 63].

We introduce the abstract concept of duality between Markov processes with more interest for its definition in case of Markov chains.

Definition 7 *(Duality between Markov processes) Let $(\eta(t))_{t\geqslant 0}$ and $(\xi(t))_{t\geqslant 0}$ be two continuous time Markov processes defined on the state spaces Ω and $\widetilde{\Omega}$ respectively. They are dual if there exists a function $D : \Omega \times \widetilde{\Omega} \to \mathbb{R}$ such that*

$$\mathbb{E}^{\eta}\left[D(\eta(t), \xi)\right] = \mathbb{E}^{\xi}\left[D(\eta, \xi(t)\right] \qquad \forall \eta \in \Omega,\ \forall \xi \in \widetilde{\Omega} \tag{3.2.1}$$

where the expectations $\mathbb{E}^{\eta}$ and $\mathbb{E}^{\xi}$ are taken with respect to the distributions of $(\eta(t))_{t\geqslant 0}$ and $(\xi(t))_{t\geqslant 0}$ with initial configurations η and ξ respectively. In case $(\xi(t))_{t\geqslant 0}$ is a copy of $(\eta(t))_{t\geqslant 0}$ the process is said to be self-dual.

The definition of duality can also be stated at the level of the generators of the processes.

Definition 8 *(Duality between Markov generators) Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two Markov generators for two Markov processes defined on Ω and $\widetilde{\Omega}$ respectively. Let $\mathcal{D}(\mathcal{L})$ and $\mathcal{D}(\widetilde{\mathcal{L}})$ be the domains of the two generators respectively. Let $D : \Omega \times \widetilde{\Omega} \to \mathbb{R}$ be a function with the property that $D(\cdot, \xi) \in \mathcal{D}(\mathcal{L})$ and $D(\eta, \cdot) \in \mathcal{D}(\widetilde{\mathcal{L}})$ $\forall \eta \in \Omega$, $\forall \xi \in \widetilde{\Omega}$. Then, we say that the two generators are dual with duality function $D(\eta, \xi)$ if*

$$\left(\mathcal{L}D(\cdot, \xi)\right)(\eta) = (\widetilde{\mathcal{L}}D(\eta, \cdot))(\xi)\,, \tag{3.2.2}$$

$\forall \eta \in \Omega$, $\forall \xi \in \widetilde{\Omega}$. *Moreover, if $\mathcal{L} = \widetilde{\mathcal{L}}$ the generator is said to be self-dual.*

These two notions of duality are linked.

Proposition 6 *Duality between Markov generators implies duality between processes. If $\forall \eta \in \Omega$ we have $S_t D(\eta, \cdot) \in \mathcal{D}(\widetilde{\mathcal{L}})$, and $\forall \xi \in \widetilde{\Omega}$ we have $\widetilde{S}_t D(\cdot, \xi) \in \mathcal{D}(\mathcal{L})$ then, duality between processes implies duality between generators.*

Proof: see, for instance [15, 63].

□

In this thesis we will always assume that these two definitions of duality given in Definitions 7 and 8 are equivalent, meaning that we can study duality between Markov processes just by having duality between generators.
In the case of countable state space the generator $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ can be represented as matrices in some basis of their state spaces, as

$$\mathcal{L} = \left(c(\eta, \xi)\right)_{\eta \in \Omega,\, \xi \in \widetilde{\Omega}}\,, \qquad \widetilde{\mathcal{L}} = \left(\widetilde{c}(\eta, \xi)\right)_{\eta \in \Omega,\, \xi \in \widetilde{\Omega}}\,. \tag{3.2.3}$$

We introduce the *duality matrix* as

$$D := \left(D(\eta, \xi)\right)_{\eta \in \Omega,\, \xi \in \widetilde{\Omega}}\,. \tag{3.2.4}$$

The elements of duality matrix are the images of the duality function for fixed η and ξ. Therefore the duality relation (3.2.2) reads

$$\sum_{\eta' \in \Omega} c(\eta,\eta')D(\eta',\xi) = \sum_{\xi' \in \tilde{\Omega}} \tilde{c}(\xi,\xi')D(\eta,\xi') = \sum_{\xi' \in \tilde{\Omega}} D(\eta,\xi')c^T(\xi',\xi) \tag{3.2.5}$$

that can be written as

$$\mathcal{L}D = D\tilde{\mathcal{L}}^T \, . \tag{3.2.6}$$

In many contexts it is useful to consider the matrix obtained by transposing the generator. It is called Hamiltonian matrix and it is linked with the generator by

$$H = \mathcal{L}^T \, . \tag{3.2.7}$$

The duality relation (3.2.2) can be reformulated by using Hamiltonian and duality matrices as

$$H^T D = D\tilde{H} \, . \tag{3.2.8}$$

3.2.1 Absorbing duality and non-equilibrium steady state for the SEP(ν)

The idea of constructing duality relies on the possibility of writing the generator of the process as proper combination of products of some “building blocks ”, that are given by the basis generators of a representation of a Lie algebras. In physics literature this has been first carried out for the simple exclusion process (that has an $su(2)$ Lie algebraic structure) in [8]. Then, in [11, 12], this approach has been extended to other processes like SIP(2k) and BEP(2k) with Lie algebra structure $su(1,1)$. Recently, many duality results and application relying on the Lie algebraic structure of the process have been collected in [15].
For the sake of simplicity and for clarity, in this section, we concentrate on the SEP(ν). The techniques, applied to this representative, can be adapted to other processes. We first recall the connection between self-duality and the symmetries of an interacting particle system, then we exploit it to construct self-duality for the SEP(ν) on a closed graph.

Definition 9 *A matrix S over the field $\mathbb{R}$ of dimension $|\Omega| \times |\Omega|$ is a symmetry for $\mathcal{L}$ of a Markov process with finite state space Ω if*

$$[S,\mathcal{L}] = S\mathcal{L} - \mathcal{L}S = 0 \tag{3.2.9}$$

Remark 3 *Observe that if S is a symmetry for the generator, then S^T is a symmetry for the Hamiltonian, indeed*

$$\left[S^T, H\right] = S^T H - HS^T = \left(H^T S - SH^T\right)^T = ([\mathcal{L},S])^T = 0 \tag{3.2.10}$$

Theorem 9 *(Self-duality and symmetries) Consider a Markov process $(\eta(t))_{t\geqslant 0}$ defined on a finite state space Ω, with generator $\mathcal{L}$, then*

1. *If $\mu : \Omega \to (0,\infty)$ is a reversible measure, then*

$$D^{ch}(\eta,\xi) = \frac{\delta_{\eta,\xi}}{\mu(\eta)} \tag{3.2.11}$$

is a self-duality matrix called "cheap duality".

2. *If S is a symmetry for $\mathcal{L}$, then $D = SD^{ch}$ and $D' = D^{ch}S^T$ are self-duality matrices.*

The proof of this result form the definition of reversibility via the detailed balance condition (2.1.47) and we refer to [15, 11] for details. We now apply this result to construct the self-duality for the SEP(ν) on a closed graph, as introduced in Section 3.1.2. To find symmetries of the generator of this process, we rely on its description via a representation of the $su(2)$ Lie algebra.

Lie algebraic description of the SEP(ν)

We introduce the vector

$$|\eta\rangle \qquad \text{with} \qquad \eta \in \{0, 1, \ldots, \nu\} \tag{3.2.12}$$

and we associate to each vertex $x \in V$ the vector $|\eta_x\rangle$, that is a copy of $|\eta\rangle$ at site x. By taking the tensor product over the graph, the configuration η of the SEP(ν) introduced in Section 3.1.2 is now described by a the vector

$$|\boldsymbol{\eta}\rangle = \bigotimes_{x \in V} |\eta_x\rangle \tag{3.2.13}$$

Now the state space reads

$$\Omega = \bigotimes_{x \in V} \Omega_x \tag{3.2.14}$$

where

$$\Omega_x = \{|\eta_x\rangle \; : \; \eta_x \in \{0, 1, \ldots, \nu\}\} \tag{3.2.15}$$

We consider the $su(2)$ Lie algebra and we pick the representation with spin label $\nu/2$, that acts on the space Ω_x (see [64, 65]). We observe that this representation has dimension $\nu + 1$, that is indeed the dimension of the state space Ω_x. In this representation, the $su(2)$ algebra has the following generators, specified by their action on the basis vectors

$$J_+|\eta\rangle = (\nu - \eta)|\eta + 1\rangle \qquad J_-|\eta\rangle = \eta|\eta - 1\rangle \qquad J_0|\eta\rangle = (\eta - \frac{\nu}{2})|\eta\rangle \,. \tag{3.2.16}$$

They satisfy the commutation relations

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \tag{3.2.17}$$

Therefore, by using these basis elements of $su(2)$ we write the transpose of the Markov generator on the edge (x, y) (i.e. the Hamiltonian matrix) as

$$\mathcal{L}^T_{x,y} = H_{x,y} = J_+ \otimes J_- + J_- \otimes J_+ + 2J_0 \otimes J_0 - \frac{\nu^2}{2} \tag{3.2.18}$$

acting on the tensor product space $\Omega_x \otimes \Omega_y$. Recalling the definition of the second Casimir element of $su(2)$ (see [64, 65])

$$C = J_+J_- + J_-J_+ + 2J_0J_0 \,, \tag{3.2.19}$$

and considering its coproduct

$$\Delta(C) = 2J_+ \otimes J_- + 2J_- \otimes J_+ + 4J_0 \otimes J_0 + C \otimes \mathbb{1} + \mathbb{1} \otimes C \tag{3.2.20}$$

we write the edge Hamiltonian as

$$H_{x,y} = \frac{1}{2}\Delta(C) - \frac{1}{2}C \otimes \mathbb{1} - \frac{1}{2}\mathbb{1} \otimes C - \frac{\nu^2}{2} \tag{3.2.21}$$

Since the C is a central element (i.e. it commute with all the the basis elements of the Lie algebra), this last expression will be useful in finding symmetries of $H_{x,y}$ and thus, symmetries of the edge generator (3.1.14) of the SEP(ν).

Self-duality for the SEP(ν) with closed boundaries

From Section 3.1.2, we recall that for the closed-boundary SEP(ν) there exists a family of reversible distribution, spanned by a parameter $\lambda \in [0,1]$ defined as

$$\mu = \bigotimes_{x \in V} \mu_x \qquad \text{where} \qquad \mu_x \sim \text{Binomial}(\nu, \lambda) \quad \forall x \in \{1, \dots, L\} \tag{3.2.22}$$

Since we do not consider the boundary driving, the parameter λ is arbitrary and we choose it as $\lambda = \frac{1}{2}$. Then, up to a constant that does not depend on η_x, the cheap-duality matrix defined in (3.2.11) reads

$$D^{ch} = \prod_{x \in V} d_x^{ch} \tag{3.2.23}$$

where

$$d_x^{ch} = \sum_{\eta_x \in \Omega_x} \binom{\nu}{\eta_x}^{-1} |\eta_x\rangle\langle\eta_x| \tag{3.2.24}$$

or, equivalently element by element we have

$$d_x^{ch}(\eta_x, \xi_x) = \frac{\delta_{\eta_x, \xi_x}}{\binom{\nu}{\eta_x}} \tag{3.2.25}$$

By direct computations one can check that

$$(J^-)^T d_x^{ch} = d_x^{ch} J^+, \qquad (J^0)^T d_x^{ch} = d_x^{ch} J^0 \tag{3.2.26}$$

The Hamiltonian density $H_{x,y}$ of the SEP(ν) (3.2.21) is written in function of the second Casimir of the $su(2)$ Lie algebra. As a consequence, every basis element of the Lie algebra commutes with $H_{x,y}$. We chose as a symmetry

$$S_x = \exp\left(J_+^x\right) = \sum_{k=0}^{\infty} \frac{(J_+^x)^k}{k!} \tag{3.2.27}$$

where we denote by J_+^x a copy of J_+ acting on the site x. We introduce the matrix

$$d_x = d_x^{ch} S_x \tag{3.2.28}$$

that acts non-trivially only on site x and we define the *self-duality matrix*

$$D = \prod_{x \in V} d_x = \prod_{x \in V} d_x^{ch} S_x \tag{3.2.29}$$

acting non trivially on the whole graph. By applying Theorem 9 we have that the matrix D is indeed a self-duality matrix for the SEP(ν). Their elements are given by

$$D(\eta, \xi) = \prod_{x \in V} \frac{(\nu - \xi_x)!}{(\nu)!} \frac{\eta_x!}{(\eta_x - \xi_x)!} \tag{3.2.30}$$

This can be proved by direct computations.

Absorbing duality for the boundary driven SEP(ν)

Starting from the self-duality for the closed graph, we now prove the duality result for the SEP(ν) with boundary driving. Given that the original process is defined on the graph $G = (V, \mathcal{E})$, the dual process $(\xi(t))_{t\geqslant 0}$ is defined on an "enlarged graph" denoted by $\widetilde{G} = \left(\widetilde{V}, \widetilde{\mathcal{E}}\right)$, where

$$\widetilde{V} := V \cup \{u(x) \,:\, x \in V\} \qquad \widetilde{\mathcal{E}} := \mathcal{E} \cup \{(x, u(x)) \,:\, x \in V\} \tag{3.2.31}$$

In this dual graph an extra-site $u(x)$ has been attached to each site of the original graph. The dual state space reads

$$\widetilde{\Omega} = \bigotimes_{x\in V} \widetilde{\Omega}_x = \bigotimes_{x\in V} (\Omega_x \times \mathbb{N}_0) \tag{3.2.32}$$

These extra-site can host an infinite number of dual particles. The configuration $\boldsymbol{\xi}$ of the dual process reads

$$\boldsymbol{\xi} = \bigotimes_{x\in V} (\xi_x \otimes \xi_{u(x)}) \tag{3.2.33}$$

where ξ_x denotes the occupation of the dual process at site x and where $\xi_{u(x)}$ denotes the occupation of the dual process at the extra-site $u(x)$. We state the absorbing duality result in a theorem.

Theorem 10 *(Duality for the boundary driven SEP(ν)) The boundary driven SEP(ν) $(\eta(t))_{t\geqslant 0}$, defined on the state space Ω defined in* (3.1.13)*, with generator*

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \mathcal{L}_x \, , \tag{3.2.34}$$

where $\mathcal{L}_{x,y}$ is given in (3.1.14) *and where $\mathcal{L}_x$ is defined in* (3.1.15)*, is dual with respect to the process $(\xi(t))_{t\geqslant 0}$, defined on the state space $\widetilde{\Omega}$, with dual generator*

$$\widetilde{\mathcal{L}} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \widetilde{\mathcal{L}}_x \, , \tag{3.2.35}$$

where $\mathcal{L}_{x,y}$ is the one defined in (3.1.4) *and*

$$\widetilde{\mathcal{L}}_x f(\xi) = (\alpha_2^x + \alpha_1^x)\xi_x \left(f(\xi - \delta_x + \delta_{u(x)}) - f(\xi)\right) \tag{3.2.36}$$

The duality function is given by

$$D(\eta, \xi) = \prod_{x\in V} \left(\frac{(\nu - \xi_x)!}{(\nu)!} \frac{\eta_x!}{(\eta_x - \xi_x)!}\right) (\rho_x)^{\xi_{u(x)}} \tag{3.2.37}$$

with $\rho_x = \frac{\alpha_1^x}{\alpha_2^x + \alpha_1^x}$.

We observe that the site dual generator $\widetilde{\mathcal{L}}_x$ is purely absorbing since it only removes dual particles from the site x and puts them at the extra-site $u(x)$. Once a dual particle reaches $u(x)$ it stays there forever. For this reason we say that this is an *absorbing dual process.*

Proof of Theorem 10: we have already proven duality for the edge generators. To show duality for the site generator we still rely on the Lie algebra, proving duality relations between the site Hamiltonians (see (3.2.8)). Using the representation with spin j of the $su(2)$ Lie algebra we have

$$\mathcal{L}_x^T = H_x = \alpha_1^x \left(J_+ + J_0 - j\right) + \alpha_2^x \left(J_-^x - J_0^x - j\right) \tag{3.2.38}$$

This Hamiltonian H_x acts on the ket-vectors associated to the configuration of the original process $|\boldsymbol{\eta}\rangle = \bigotimes_{x\in V} |\eta_x\rangle$, where $\eta_x \in \{0, \ldots, \nu\}$. The dual site Hamiltonian reads

$$\widetilde{\mathcal{L}}_x^T = \widetilde{H}_x = (\alpha_1^x + \alpha_2^x)\left(\mathbf{a}_{u(x)}^+ J_-^x - J_0^x - j\right) \tag{3.2.39}$$

where $\mathbf{a}_{u(x)}^+$ is a bosonic creation operator that creates particles on the extra-site $u(x)$, i.e. for all $n \in \mathbb{N}$

$$\mathbf{a}^+|n\rangle = |n+1\rangle, \qquad \langle n|\mathbf{a}^+ = \langle n-1| \tag{3.2.40}$$

This dual Hamiltonian $\widetilde{H}_x$ acts on the ket-vectors associated to the dual configuration, given by

$$|\boldsymbol{\xi}\rangle = \bigotimes_{x\in V} \left(|\xi_x\rangle \otimes |\xi_{u(x)}\rangle\right) \tag{3.2.41}$$

where ξ_x denotes the occupation of the dual process at site x and where $\xi_{u(x)}$ denotes the occupation of the dual process at the extra-site $u(x)$. The duality matrix now reads

$$D = \prod_{x\in V} (d_x) \otimes \mathcal{D}_{u(x)} \tag{3.2.42}$$

where

$$d_x = d_x^{ch} \exp\left(J_+^x\right), \tag{3.2.43}$$

with d_x^{ch} is the the cheap duality matrix defined in (3.2.24), acts non-trivially at site x and as the identity on the rest of the chain. We introduce

$$\mathcal{D}_{u(x)} = \sum_{\xi_{u(x)}=0}^{\infty} (\rho_x)^{\xi_{u(x)}} \langle \xi_{u(x)}| \tag{3.2.44}$$

Here, with a slight abuse of notation we denote by $\langle \xi_{u(x)}|$ the bra-vector where $\xi_{u(x)}$ is the occupation variable of the extra-site. When projected on a ket-vector $\mathcal{D}_{u(x)}$ one obtains the duality function (3.2.37). We aim to show that

$$H_x^T D = D\widetilde{H}_x \tag{3.2.45}$$

Recalling the Hadamard formula for all $X \in su(2)$

$$\exp\left(-J_+\right)X\exp\left(J_+\right) = X - [J_+, X] + \frac{1}{2}[J_+, [J_+, X]] - \frac{1}{3!}[J_+ [J_+, [J_+, X]]] + \ldots \tag{3.2.46}$$

we obtain that

$$\exp\left(-J_+\right)J_- \exp\left(J_+\right) = J_- - 2J_0 - J_+, \qquad \exp\left(-J_+\right)J_0 \exp\left(J_+\right) = J_0 + J_+ \tag{3.2.47}$$

Using (3.2.26) have that

$$H_x^T d_x^{ch} \exp\left(J_x^+\right) = d_x^{ch} \exp\left(J_x^+\right) \exp\left(-J_x^+\right) \left(\alpha_1^x (J_-^x + J_0^x - \frac{\nu}{2}) + \alpha_2^x (J_+^x - J_0^x - \frac{\nu}{2})\right) \exp\left(J_x^+\right) \tag{3.2.48}$$

Using equations (3.2.47)we have

$$\begin{aligned} &\exp\left(-J_x^+\right) \left(\alpha_1^x (J_-^x + J_0^x - \frac{\nu}{2}) + \alpha_2^x (J_+^x - J_0^x - \frac{\nu}{2})\right) \exp\left(J_x^+\right) \\ =&(\alpha_1^x + \alpha_2^x) \left(\rho_x J_-^x - J_0^x - \frac{\nu}{2}\right) \end{aligned} \tag{3.2.49}$$

We observe that the matrix $\left(\rho_x J_-^x - J_0^x - \frac{\nu}{2}\right)$ is not stochastic. However, the following relation holds

$$\begin{aligned} \sum_{\xi_{u(x)}=0}^{\infty} (\rho_x)^{\xi_{u(x)}} \langle \xi_{u(x)} | \rho_x &= \sum_{\xi_{u(x)}=0}^{\infty} (\rho_x)^{\xi_{u(x)}+1} \langle \xi_{u(x)} + 1 | \mathbf{a}_{u(x)}^+ \\ &= \sum_{\xi_{u(x)}=0}^{\infty} \rho_x^{\xi_{u(x)}} \langle \xi_{u(x)} | \mathbf{a}_{u(x)}^+ \end{aligned} \tag{3.2.50}$$

where we used (3.2.40) and the fact that $\langle 0 | \mathbf{a}^+ = 0$. Therefore, using (3.2.49), multiplying with the tensor product both sides of (3.2.48) by

$$\sum_{\xi_{u(x)}=0}^{\infty} (\rho_x)^{\xi_{u(x)}} \langle \xi_{u(x)} | \tag{3.2.51}$$

and using (3.2.50), we obtain

$$\begin{aligned} &\alpha_1^x \left\{ \left(J^+ + J^0 - \frac{\nu}{2}\right) + \alpha_2^x \left(J_-^x - J_0^x - \frac{\nu}{2}\right) \right\}^T \left(d_x^{ch} \exp\left(J_+^x\right)\right) \otimes \mathcal{D}_{u(x)} \\ =& \left(\left(d_x^{ch} \exp\left(J_+^x\right)\right) \otimes \mathcal{D}_{u(x)}\right) (\alpha_2^x + \alpha_1^x) \left(\mathbf{a}_{u(x)}^+ J_-^x - J_0^x - \frac{\nu}{2}\right) \end{aligned} \tag{3.2.52}$$

and (3.2.45) follows.

□

Characterization of the non-equilibrium steady state by duality

Consider a boundary driven Markov process $(\eta(t))_{t \geqslant 0}$. We say that a measure μ_{NESS} is the non-equilibrium stationary distribution (NESS) if, for all bounded functions $f : \Omega \to \mathbb{R}$ and for every starting measure μ it holds that

$$\lim_{t \to \infty} \mathbb{E}_\mu \left[f(\eta(t))\right] = \mathbb{E}_{\mu_{\text{NESS}}} \left[f(\eta)\right] \tag{3.2.53}$$

For the processes studied in this thesis we assume that

- V is finite and that the set of extra-sites of the dual process is disjoint from V
- the conductances and the local inhomogeneities are bounded, i.e. $\omega_{x,y} < \infty$ and $\Gamma_x < \infty$

Under these two conditions one can prove that the non-equilibrium steady state exists and is unique. Moreover, all initial measure converges (when time goes to ∞) to the non-equilibrium steady state μ_{NESS}. For details see Theorem X.13 of [15]. In particular, this is true for the chain with two reservoirs. Since the dual process has absorbing extra-sites, in the long time horizon, all dual particles are absorbed in one of the extra-sites $u(x)$. Therefore, we have that, by using duality

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\eta,\xi)\right] = \lim_{t\to\infty} \mathbb{E}^{\eta}\left[D(\eta(t),\xi)\right] = \lim_{t\to\infty} \mathbb{E}^{\xi}\left[D(\eta,\xi(t))\right] = \mathbb{E}^{\xi}\left[\prod_{x\in V} \rho_x^{\xi_{u(x)}(\infty)}\right] \tag{3.2.54}$$

where we denoted by $\xi_{\mu(x)}(\infty)$ the occupation of extra-site $u(x)$ when $t \to \infty$ and ρ_x is the particle density (or the temperature) of the reservoir connected to the site x. In the formula above, we have denoted by $\mathbb{E}^{\eta}[\cdot]$ and $\mathbb{E}^{\xi}[\cdot]$ the expectations with respect to the law of the original and dual process initialized with a configuration η and ξ respectively. We stress that, since the dual process is absorbing, the last term in (3.2.54) is found by the knowledge of the probabilities that dual particles are absorbed at extra-sites 0 and $L+1$. These probabilities are called *absorption probabilities* and they fully characterize the NESS. This is an important simplification that duality implies when studying non-equilibrium processes.
To clarify the ideas, we report the result for the boundary driven SEP(ν) on a simpler geometry, where the graph G reduces to a chain (segment) with only two reservoirs attached at sites 1 and L. In this situation, we have the following result that allows to characterize the average of the duality function with respect to the non-equilibrium stationary distribution. We denote by ρ_{left} and ρ_{right} the densities (see (3.1.16)) of the reservoirs attached to the sites 1 and L respectively.

Theorem 11 *(Characterization of NESS via duality) Let $(\eta(t))_{t\geqslant 0}$ be the SEP(ν) and let $(\xi(t))_{t\geqslant 0}$ be its dual absorbing process introduced in Theorem 10. Then, for a given configuration ξ, we have that*

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\eta,\xi)\right] = \sum_{k=0}^{|\xi|} \left(\frac{\rho_{\text{left}}}{\nu}\right)^k \left(\frac{\rho_{\text{right}}}{\nu}\right)^{|\xi|-k} \mathcal{P}_{\xi}(k) \tag{3.2.55}$$

where

$$\mathcal{P}_{\xi}(k) = \mathbb{P}\left(\xi_0(\infty) = k,\ \xi_{L+1}(\infty) = |\xi| - k | \xi(0) = \xi\right) \tag{3.2.56}$$

are called absorption probabilities, with k denoting the number of particles absorbed in 0.

Proof of Theorem 11: Using the ergodicity of the process on a finite chain and the duality definition (3.2.1) we have that

$$\begin{aligned}\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\eta,\xi)\right] &= \lim_{t\to\infty} \mathbb{E}^{\eta}\left[D(\eta(t),\xi)\right] \\ &= \lim_{t\to\infty} \mathbb{E}^{\xi}\left[D(\eta,\xi(t))\right] \\ &= \sum_{k=0}^{|\xi|} \left(\frac{\rho_{\text{left}}}{\nu}\right)^k \left(\frac{\rho_{\text{right}}}{\nu}\right)^{|\xi|-k} \mathcal{P}_{\xi}(k)\end{aligned} \tag{3.2.57}$$

□

From equation (3.2.55) we see that absorbing duality brings a simplification in the characterization of the non-equilibrium steady state. Indeed, the determination of all the moments with respect to this measure reduces in the computation of absorption probabilities. However, usually this is not an easy task, since the absorption probabilities satisfy complicated difference equations. In some contexts, integrability leads to explicit expression of the absorption probabilities (see for instance [26, 20, 3]).

3.2.2 Duality as a change of representation

The existence of a duality relations between two Markov processes goes beyond the construction of a symmetry for the Hamiltonian, as it has been reported for the SEP(ν) in the previous sections of this chapter. Indeed, the duality relation between two Markov process can be seen as a consequence of a change of representation of the underlying Lie algebra. More precisely, the duality function can be interpreted as an intertwining map between two representations.
Given a Lie algebra $\mathfrak{g}$ with algebra generators $(x_i)_{i\in\{1,\dots,n\}}$ that satisfy the commutators

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k \tag{3.2.58}$$

where c_{ijk} is the *structure constant*, we call *conjugate Lie algebra* the algebra $\bar{\mathfrak{g}}$ with generators $(y_i)_{i\in\{1,\dots,n\}}$ that have $-c_{ijk}$ as a structure constant, i.e. they satisfy

$$[y_i, y_j] = -\sum_{k=1}^{n} c_{ijk} y_k \tag{3.2.59}$$

We then have the following proposition that connects duality between Markov processes and a change of representation of the Lie algebras, allowing to interpret the duality function (matrix) is an intertwining between two representations. We state it in the finite dimensional setting, however it can be extended to Hilbert spaces (see [15, 63, 66]).

Proposition 7 *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be two generators of Markov processes with configuration spaces Ω and $\widetilde{\Omega}$ respectively. Let $(X_i)_{i\in\{1,\dots,n\}}$ be a collection of linear operators acting on Ω and that form a representation of a Lie algebra $\mathfrak{g}$. Let $(Y_i)_{i\in\{1,\dots,n\}}$ be a collection of linear operators acting on $\widetilde{\Omega}$ and that form a representation of a Lie algebra $\bar{\mathfrak{g}}$. Assume that*

- *$\mathcal{L}$ can be written as a linear finite product of operators X_i.*
- *$\widetilde{\mathcal{L}}$ can be written as a linear finite product of operators Y_i with the reverse order with respect to the product of the X_i.*
- *There exists an intertwining map D such that*

$$X_i D = D Y_i^T \qquad \forall i \in \{1, \dots, n\}\,. \tag{3.2.60}$$

Then,

$$\mathcal{L} D = D \widetilde{\mathcal{L}}^T \tag{3.2.61}$$

For the proof see [11, 15, 63, 66].
Proposition 7 allows to construct the duality relation between the two Markov generators $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ starting from the knowledge of a duality relations between the generators of a representation of $\mathfrak{g}$ and a representation of $\bar{\mathfrak{g}}$, i.e. the existence of an intertwining between X_i and Y_i^T, for all $i \in \{1, \dots, n\}$. Furthermore, in case the duality function (matrix) is invertible, the duality relation leads to an equivalence between two representations.
One can show that the construction of self-duality via symmetries is nothing but a particular choice of the change of representation described in Proposition 7 (see [15]).
This interpretation of duality, allows to extend the results to more complicated processes. For instance we can prove duality between a interacting diffusions and particle systems.

Duality between BEP(2k) and SIP(2k)

In this section we present an example of construction of duality by using a change or representation of a Lie algebra: the duality between BEP(2k) and SIP(2k). We start by recalling the definition of the BEP(2k) process on a connected graph $G = (V, \mathcal{E})$. The state space of the process is $\Omega = \bigotimes_{x\in V} \Omega_x$ where $\Omega_x = \{z_x \in \mathbb{R}^+\}$. The variable z_x is interpreted as the energy at site x. The generator reads

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y}\mathcal{L}_{x,y} \tag{3.2.62}$$

where

$$\mathcal{L}_{x,y} = z_x z_y \left(\partial_{z_x} - \partial_{z_y}\right)^2 - 2k(z_x - z_y)\left(\partial_{z_x} - \partial_{z_y}\right) . \tag{3.2.63}$$

We introduce the Lie algebra $su(1,1)$ and we pick a representation acting on polynomials of variable z, that we denote by $f(z)$. This representation has the following algebra generators

$$\mathcal{K}_+ f(z) = zf(z) \qquad \mathcal{K}_- f(z) = z\partial_z^2 f(z) + 2k\partial_z f(z) \qquad \mathcal{K}_0 f(z) = z\partial_z f(z) + kf(z) . \tag{3.2.64}$$

By direct computations one can show that they satisfy the commutators of $su(1,1)$, i.e.

$$[\mathcal{K}_+, \mathcal{K}_-] = -2\mathcal{K}_0 \qquad [\mathcal{K}_0, \mathcal{K}_\pm] = \pm\mathcal{K}_\pm \tag{3.2.65}$$

Using this representation, we have that

$$\mathcal{L}_{x,y} = \mathcal{K}_+ \otimes \mathcal{K}_- + \mathcal{K}_- \otimes \mathcal{K}_+ - 2\mathcal{K}_0 \otimes \mathcal{K}_0 + 2k^2 . \tag{3.2.66}$$

We consider now the representation of the conjugate algebra $su(1,1)$, whose basis elements acts on function with discrete variable $f : \mathbb{N}_0 \to \mathbb{R}$ (assuming $f(-1) = 0$) as

$$K_+ f(n) = (2k+n)f(n+1) \qquad K_- f(n) = nf(n-1) \qquad K_0 f(n) = (n+k)f(n) . \tag{3.2.67}$$

They satisfy the commutators

$$[K_+, K_-] = 2K_0 \qquad [K_\pm, K_0] \pm K_\pm . \tag{3.2.68}$$

Considering the duality function

$$d(z,n) = z^n \frac{\Gamma(2k)}{\Gamma(2k+n)} , \tag{3.2.69}$$

it is straightforward to verify

$$(\mathcal{K}_a d(\cdot, n))(z) = (K_a d(z, \cdot)(n) \qquad \forall a \in \{+,-,0\} . \tag{3.2.70}$$

Therefore, by applying Proposition 7, we have that

$$(\mathcal{L}D(\cdot,\xi))(z) = \left(\widetilde{\mathcal{L}}D(z,\cdot)\right)(\xi) \tag{3.2.71}$$

where the dual generator $\widetilde{\mathcal{L}}$ coincide with the SIP(2k) generator on a closed graph (defined in Section 3.1), i.e.

$$\widetilde{\mathcal{L}} = \sum_{(x,y)\in\mathcal{E}} \widetilde{\mathcal{L}}_{x,y} . \tag{3.2.72}$$

Here, the operator $\widetilde{\mathcal{L}}_{x,y}$ is given by

$$\widetilde{\mathcal{L}}_{x,y} = K_+ \otimes K_- + K_- \otimes K_+ - 2K_0 \otimes K_0 + 2k^2 \, . \tag{3.2.73}$$

where its action on a function is

$$\widetilde{\mathcal{L}}_{x,y} f(\xi) = \xi_x(2k + \xi_y)\left(f(\xi - \delta_y + \delta_x) - f(\xi)\right) + \xi_y(2k + \xi_x)\left(f(\xi - \delta_x + \delta_y) - f(\xi)\right) \, . \tag{3.2.74}$$

The duality function is given by

$$D(z, \xi) = \prod_{x \in V} z_x^{\xi_x} \frac{\Gamma(2k)}{\Gamma(2k + \xi_x)} \, . \tag{3.2.75}$$

Above, we denoted the SIP(2k) by $(\xi(t))_{t \geqslant 0}$.
To complete the picture we report the duality between the boundary driven BEP(2k) and the SIP(2k) with absorbing boundaries. This allows to characterize the non-equilibrium steady distribution of the BEP(2k) via the absorption probabilities of the SIP(2k). Here we consider the BEP(2k) on the configuration space Ω previously defined and the SIP(2k) with the enlarged configuration space $\widetilde{\Omega} = \bigotimes_{x \in V} \widetilde{\Omega}_x$ where $\widetilde{\Omega}_x = \Omega_x^{'} \times N_0$ and where $\Omega_x^{'} := \{\xi_x \, : \, \xi_x \in \mathbb{N}_0\}$. This means that at each vertex an absorbing extra-site has been added.

Theorem 12 *The boundary driven BEP(2k) denoted by* $(z(t))_{t \geqslant 0}$*, on the state space* Ω*, with generator*

$$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \mathcal{L}_x \tag{3.2.76}$$

where the edge generator $\mathcal{L}_{x,y}$ *is defined in* (3.1.26) *and the site generator* $\mathcal{L}_x$ *is defined in* (3.1.27)*, is dual to the SIP(2k) denoted by* $(\xi(t))_{t \geqslant 0}$ *on the state space* $\widetilde{\Omega}$ *and with dual generator given by*

$$\widetilde{\mathcal{L}} = \sum_{(x,y) \in \mathcal{E}} \widetilde{\mathcal{L}}_{x,y} + \sum_{x \in V} \Gamma_x \widetilde{\mathcal{L}}_x \tag{3.2.77}$$

where $\widetilde{\mathcal{L}}_{x,y}$ *is defined the edge generator of the SIP(2k)* (3.1.20) *and where*

$$\mathcal{L}_x f(\xi) = \xi_x \left(f(\xi - \delta_x + \delta_{u(x)}) - f(\xi)\right) \, . \tag{3.2.78}$$

The duality function given by

$$D(z, \xi) = \prod_{x \in V} \left(z_x^{\xi_x} \frac{\Gamma(2k)}{\Gamma(2k + \xi_x)} \right) (T_x)^{\xi_{u(x)}} \, . \tag{3.2.79}$$

The proof consists in a direct computation and can be found, for instance, in [12, 15].

3.3 Integrable interacting particle systems

In this section we briefly introduce the concept of integrability of an interacting particle system. Our approach consists in finding symmetries for the Hamiltonian of the integrable process (for instance the boundary driven SSEP on a finite segment) by constructing a generating function called *transfer matrix*. Usually, this is denoted by $T(u)$ with $u \in \mathbb{C}$ named *spectral parameter*. The existence of this transfer matrix relies on the fact that the Hamiltonian of an integrable process can be written in terms of some building blocks, the R-matrices and the K-matrices, that

satisfy consistency relations named *Yang Baxter Equations* (YBE). Once the $T(u)$ is constructed, it can be explicitly diagonalized and, since it commutes with the Hamiltonian H of the process, all the eigenvectors and the spectrum of H can be explicitly determined. The technique explained above is called *quantum inverse scattering method* (QISM). For a good review we refer to [16, 21]. The second technique that we report is the *matrix product ansatz* (MPA). Originally introduced in [24] for the ASEP, it states that the non-equilibrium steady state of the Hamiltonian H of the integrable ASEP can be written as proper product of matrices acting on the states of an abstract supplementary space. As it has been observed in [25], QISM and MPA are deeply connected.
In this section we first report a short and non-exhaustive overview of the QISM and of the MPA applied to one of the first examples of integrable interacting particle system: the boundary driven SSEP. For excellent reviews we refer to [19, 21, 67].

3.3.1 The integrable SSEP

We consider the boundary driven SSEP on the geometry of a line segment of length L with two boundaries attached to the end sites 1 and L respectively. In this situation the process is said to be *integrable*. The state space is given by $\Omega = \bigotimes_{x\in V} \Omega_x$ where

$$\Omega_x = \{|\eta_x\rangle \; : \; \eta_x \in \{1,2\}\} \tag{3.3.1}$$

Here we associate $\eta_x = 1$ when the site x is occupied by a particle, while we associate $\eta_x = 2$ to indicate that there is a hole. The space Ω_x is isomorphic to $\mathbb{C}^2$ since

$$|1\rangle = |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |2\rangle = |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.3.2}$$

This means that the state space of the SSEP is the same of the XXX-Heisenberg spin chain (see for instance [16, 68]). The Hamiltonian matrix, for the SSEP is

$$H = H_{\text{left}} + \sum_{x=1}^{L-1} H_{x,x+1} + H_{\text{right}} \tag{3.3.3}$$

where the bulk Hamiltonian densities are

$$H_{x,x+1} = \sum_{a,b=1}^{2} \left(e_{ab}^{[x]} e_{ba}^{[x+1]} - e_{bb}^{[x]} e_{aa}^{[x+1]} \right) = P_{x,x+1} - 1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.3.4}$$

Here $P_{x,x+1}$ is the permutation operator acting on the bond $(x, x+1)$ given by

$$P = \sum_{a,b=1}^{2} e_{ab}^{[x]} e_{ba}^{[x+1]} \tag{3.3.5}$$

and $e_{ab}^{[x]}$ is the elementary matrix $(e_{ab})_{cd} = \delta_{ac}\delta_{bd}$ acting non trivially only at site x. The boundary Hamiltonians are given by

$$H_{\text{left}} = \alpha_2 \left(e_{21}^{[1]} - e_{11}^{[1]} \right) + \alpha_1 \left(e_{12}^{[1]} - e_{22}^{[1]} \right) = \begin{pmatrix} -\alpha_2 & \alpha_1 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \tag{3.3.6}$$

$$H_{\text{right}} = \beta_2 \left(e_{21}^{[L]} - e_{11}^{[L]} \right) + \beta_1 \left(e_{12}^{[L]} - e_{22}^{[L]} \right) = \begin{pmatrix} -\beta_2 & \beta_1 \\ \beta_2 & -\beta_1 \end{pmatrix} \tag{3.3.7}$$

We observe that the above matrices are stochastic, since the out-of-diagonal elements are non-negative and the sum of the columns is vanishing.

3.3.2 Quantum inverse scattering method

We start by introducing the building blocks of the transfer matrix: the R and the K-matrices. Here we denote by $\mathbb{V} = V_1 \otimes \cdots \otimes V_L = (\mathbb{C}^2)^{\otimes L}$ the so-called *quantum space*. This is nothing but the configuration space of the integrable SSEP introduced in (3.3.1). Moreover, we denote by $V_0 = \mathbb{C}^2$ the *auxiliary space*. This is not physical, but it has to be introduced for the computations. From here on-wards, all the spaces denoted by V_i with $i \in \{1, \ldots, L\}$ are always considered to be $V_i = \mathbb{C}^2$. In the following, when a tensor product is considered the identity matrix is denoted by $\mathbb{1}$, otherwise it is simply indicated by 1.

Definition 10 (R-matrix) *A matrix $R(u)$ acting on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is called R-Matrix if it satisfies the Yang-Baxter Equation (YBE)*

$$R_{1,2}(u-v)R_{1,3}(u-w)R_{2,3}(v-w) = R_{2,3}(v-w)R_{1,3}(u-w)R_{1,2}(u-v) \tag{3.3.8}$$

that is defined on the product space on $V_1 \otimes V_2 \otimes V_3$ and where

$$R_{1,2}(u-v) = R(u-v) \otimes \mathbb{1} \tag{3.3.9}$$

with $R(u-v)$ acting on $V_1 \otimes V_2$. $R_{1,3}(u,w)$ and $R_{2,3}(v,w)$ are defined similarly. The variables $u, v, w \in \mathbb{C}$ are called spectral parameters.

By direct computations one can check that

$$R(u) = \frac{u+P}{u+1}\,. \tag{3.3.10}$$

solves the YBE (3.3.8). Here P is the permutation operator (3.3.5) acting on $V_1 \otimes V_2$.

Definition 11 *($\hat{K}$-matrices) A matrix $\hat{K}(u)$, defined on the vector space $\mathbb{C}^2$ is called a $\hat{K}$-matrix if it solves the reflection equation (RE) (also called boundary-YBE)*

$$R_{1,2}(u-v)\hat{K}_1(u)R_{1,2}(u+v)\hat{K}_2(v) = \hat{K}_2(v)R_{1,2}(u+v)\hat{K}_1(u)R_{1,2}(u-v) \tag{3.3.11}$$

defined on $V_1 \otimes V_2$ and where $R(u)$ is a solution of the YBE (3.3.8)*. Here we denoted by $\hat{K}_1(u) = \hat{K}(u) \otimes \mathbb{1}$ on the tensor space $V_1 \otimes V_2$.*

Definition 12 *(K-matrices) A matrix $K(u)$, defined on the vector space V is called a K-matrix if it solves the reversed reflection equation (RRE) (also called reversed-boundary-YBE)*

$$R_{1,2}(v-u)K_1(u)R_{1,2}(-u-v-2)K_2(v) = K_2(v)R_{1,2}(-u-v-2)K_1(u)R_{1,2}(v-u) \tag{3.3.12}$$

defined on $V_1 \otimes V_2$ and where $R(u)$ is a solution of the YBE (3.3.8)*. Here we denoted by $K_1(u) = K(u) \otimes \mathbb{1}$ on the tensor space $V_1 \otimes V_2$.*

Starting from the boundaries of the integrable SSEP (3.3.7) and (3.3.6) one can show by direct computations (see [21] for details) that the following $\hat{K}, K$-matrices satisfy (3.3.11) and (3.3.12) respectively:

$$\hat{K}(u) = 2H_{\text{right}}u + (\beta_1+\beta_2)u + 1 \tag{3.3.13}$$

$$K(u) = 2H_{\text{left}}(u+1) + (\alpha_1+\alpha_2)(u+1) + 1 \tag{3.3.14}$$

explicitly we have that

$$\hat{K}(u) = \begin{pmatrix} 1 + u(\beta_1 - \beta_2) & 2u\beta_1 \\ 2u\beta_2 & 1 + u(\beta_2 - \beta_1) \end{pmatrix} \tag{3.3.15}$$

$$K(u) = \begin{pmatrix} 1 + (u+1)(\alpha_1 - \alpha_2) & 2(u+1)\alpha_1 \\ 2(u+1)\alpha_2 & 1 + (u+1)(\alpha_2 - \alpha_1) \end{pmatrix} \tag{3.3.16}$$

Starting from the R-matrix and from the $\hat{K}, K$-matrices one can recover the terms in the Hamiltonian of the integrable SSEP by

$$H_{\text{left}} = \frac{1}{2}\frac{d}{du}K(u)|_{u=0} - \frac{1}{2}(\alpha_1 + \alpha_2) \tag{3.3.17}$$

$$H_{\text{right}} = \frac{1}{2}\frac{d}{du}\hat{K}(u)|_{u=0} - \frac{1}{2}(\beta_1 + \beta_2) \tag{3.3.18}$$

$$H_{x,x+1} = \frac{d}{du}R_{x,x+1}(u)|_{u=0} \tag{3.3.19}$$

To define the transfer matrix we need to introduce the double row monodromy matrix.

Definition 13 *(Double-row monodromy) Let $V_0 = \mathbb{C}^2$ the auxiliary space and let $R(u)$ the $R-$matrices defined in* (3.3.10)*, then we call double-row monodromy the matrix acting on the product space $V_0 \otimes \mathbb{V}$*

$$U_0(u) = M_0(u)\hat{K}_0(u)\hat{M}_0(u) \tag{3.3.20}$$

where $M_0(u)$

$$M_0(u) = R_{0,1}(u)R_{0,2}(u)\cdots R_{0,L}(u)\,, \tag{3.3.21}$$

$$\hat{M}_0(u) = R_{0,L}(u)R_{0,L-1}(u)\cdots R_{0,1}(u) \tag{3.3.22}$$

and where $\hat{K}_0(u)$ is defined in (3.3.15) *acting non-trivially on V_0. Here, $R(u)$ is the matrix defined in* (3.3.10)*.*

The double-row monodromy matrix can be written in the canonical base of the auxiliary space V_0 as

$$U_0(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{3.3.23}$$

where $A(u), B(u), C(u)$ and $D(u)$ are matrices acting on the quantum space $\mathbb{V}$.
This double row monodromy matrix satisfies a consistency relation, called boundary RTT, that directly follows from YBE (3.3.8) and RE (3.3.11). This boundary RTT reads

$$R_{0,0'}(u-v)U_0(u)R_{0,0'}(u+v)U_{0'}(v) = U_{0'}(v)R_{0,0'}(u+v)U_0(u)R_{0,0'}(u-v) \tag{3.3.24}$$

where the above equation has to be understood on the space $V_0 \otimes V_{0'} \otimes \mathbb{V}$, with the auxiliary spaces $V_0, V_{0'} = \mathbb{C}^2$. This relation allows to determine the exchange relations between the entries of the double row monodromy $A(u), B(u), C(u)$ and $D(u)$ that are crucial in many contexts, like the mapping of non-equilibrium onto equilibrium (see [28]) and the algebraic Bethe ansatz (see [17, 69]).
We now introduce the central object of the QISM: the transfer matrix.

Definition 14 *(Transfer Matrix) The transfer matrix for the integrable SSEP is the matrix acting on the quantum space* $\mathbb{V}$ *given by*

$$T(u) = tr_0\left(K_0(u)U_0(u)\right) \tag{3.3.25}$$

where $U_0(x)$ *is the double-row monodromy matrix defined in* (3.3.20)*,* $K(u)$ *is the matrix* (3.3.16) *and where the trace is taken in the auxiliary space* V_0*.*

This transfer matrix has two fundamental features:

1. It commutes with itself at different values of the spectral parameter:

$$[T(u), T(v)] = T(u)T(v) - T(v)T(u) = 0 \qquad \forall u, v \in \mathbb{C} \tag{3.3.26}$$

2. The Hamiltonian H (3.3.3) of the integrable SSEP is, up to a matrix proportional to the identity, the derivative of $T(u)$ evaluated at $u = 0$, namely

$$H = C_1 \frac{d}{du} T(u)|_{u=0} + C_2 \tag{3.3.27}$$

   with $C_1, C_2 \in \mathbb{R}$ proper constants.

Equations (3.3.26) and (3.3.27) imply that the eigenvectors of the transfer matrix $T(u)$ are independent of the spectral parameter u and that they also diagonalize H (3.3.3) of the integrable SSEP. This is the key aspect of the integrability of interacting particle systems: *all the states of the boundary driven Hamiltonian* H (3.3.3) *can be determined by diagonalizing the transfer matrix.* The eigenvalues of H can also be recovered from the ones of $T(u)$ by using (3.3.27). In this thesis, we do not use the explicit expression of the eigenvectors of $T(u)$ however, they can be computed by the *algebraic Bethe ansatz*, see for instance [17, 69, 18].

3.3.3 Matrix product ansatz

We call state and we denote by $|\mu_t\rangle$, with initial state $|\mu\rangle$, a vector where we list the probabilities of all possible configurations of the integrable SSEP with Hamiltonian (3.3.3) at a certain time t, i.e.

$$|\mu_t\rangle = \sum_{\xi \in \Omega} \mathbb{P}_\mu(\eta(t) = \xi)|\xi\rangle \tag{3.3.28}$$

where $\mathbb{P}_\mu(\eta(t) = \xi)$ is the probability of having the configuration ξ at time t, starting from an initial distribution μ.

where Ω is the state space of the SSEP defined in (3.3.1). As already pointed out by master equation 2.1.43, this state evolves in time as

$$\frac{d}{dt}|\mu_t\rangle = H|\mu_t\rangle \tag{3.3.29}$$

where H is the Hamiltonian matrix of SSEP defined in (3.3.3). Of particular interest for statistical mechanics is the non-equilibrium steady state, i.e. the state $|\Psi\rangle$ such that $H|\Psi\rangle = 0$. This means to look for the eigenvector with vanishing eigenvalue of H. One way of obtaining this state is by quantum inverse scattering method. In the past, other techniques have been developed to find $|\Psi\rangle$ without using the whole structure of QISM, for instance the matrix product ansatz (MPA), that will be exposed in this section. In the case of SSEP, the MPA can also be derived from integrability, by using the Zamolodchikov algebras [25]. The matrix product ansatz was introduced in [24] for the asymmetric boundary driven simple exclusion process (ASEP). This ansatz states that the non-equilibrium steady state of the ASEP can be formally written in terms of a product of matrices acting on a supplementary space. The probability of each configuration of the process is expressed as a product of matrices paired with two vectors (bra and ket) belonging to the same supplementary space. This technique revealed to be successful and many applications have been analysed, for example, among others, the study of large deviations for ASEP (see [70, 71]). A review of the MPA techniques is found in [67], while for the specific situation of the SSEP, one can refer to [21]. In this section we report the statement and the proof of the MPA in the case of SSEP.

Proposition 8 *(MPA statement) The non-equilibrium steady state* $|\Psi\rangle$ *of the SSEP with Hamiltonian* H*, defined in* (3.3.3)*, is given by*

$$|\Psi\rangle = \frac{1}{Z_L}\langle\langle W| \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{L \text{ times}} |V\rangle\rangle \tag{3.3.30}$$

where E, D *are matrices, in general infinite dimensional, acting on a supplementary space and satisfying*

$$[D, E] = D + E \tag{3.3.31}$$

$$\langle\langle W|(\alpha_2 E - \alpha_1 D + 1) = 0 \tag{3.3.32}$$

$$(\beta_2 E - \beta_1 D - 1)|V\rangle\rangle = 0 \tag{3.3.33}$$

Here the normalization Z_L *is given by*

$$Z_L = \langle\langle W|(E + D)^L|V\rangle\rangle \tag{3.3.34}$$

Proof of Proposition 8: we aim to show that

$$H|\Psi\rangle = 0 \tag{3.3.35}$$

We have that

$$\begin{aligned}
Z_L H|\Psi\rangle =& H\langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
=& \langle\langle W| \left[H_{\text{left}} \begin{pmatrix} E \\ D \end{pmatrix} \right] \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\
&+ \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \left[H_{\text{right}} \begin{pmatrix} E \\ D \end{pmatrix} \right] |V\rangle\rangle \\
&+ \sum_{x=1}^{L-1} \langle\langle W| \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{x-1 \text{ times}} \left[H_{x,x+1} \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right] \otimes \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \begin{pmatrix} E \\ D \end{pmatrix}}_{L-x-1 \text{ times}} |V\rangle\rangle
\end{aligned} \tag{3.3.36}$$

Using (3.3.3) and (3.3.31) we have that

$$H_{x,x+1} \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = (P_{x,x+1} - 1) \begin{pmatrix} EE \\ ED \\ DE \\ DD \end{pmatrix} = \begin{pmatrix} 0 \\ DE - ED \\ ED - DE \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ D+E \\ -D-E \\ 0 \end{pmatrix} = \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \tag{3.3.37}$$

Using (3.3.6) and (3.3.32) we have

$$\langle\langle W | H_{\text{left}} \begin{pmatrix} E \\ D \end{pmatrix} = \langle\langle W | \begin{pmatrix} -\alpha_2 E + \alpha_1 D \\ \alpha_2 E - \alpha_2 D \end{pmatrix} = \langle\langle W | \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{3.3.38}$$

Using (3.3.7) and (3.3.33) we have

$$H_{\text{right}} \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle = \begin{pmatrix} -\beta_2 E + \beta_1 D \\ \beta_2 E - \beta_1 D \end{pmatrix} |V\rangle\rangle = \begin{pmatrix} -1 \\ 1 \end{pmatrix} |V\rangle\rangle \tag{3.3.39}$$

Then we obtain

$$\begin{aligned} Z_L H|\Psi\rangle =& \langle\langle W| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle\rangle \\ & - \langle\langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & + \sum_{x=1}^{L-1} \langle\langle W| \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{x-1 \text{ times}} \left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \right] \\ & \otimes \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{L-x-1 \text{ times}} |V\rangle\rangle \\ &= \sum_{x=1}^{L-1} \langle\langle W| \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{x-1 \text{ times}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{L-x-1 \text{ times}} |V\rangle\rangle \\ & - \sum_{x=1}^{L-1} \langle\langle W| \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{x-1 \text{ times}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \underbrace{\begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix}}_{L-x-1 \text{ times}} |V\rangle\rangle \end{aligned} \tag{3.3.40}$$

where in the last equality we have shifted the summation.

□

Using this form (3.3.30) of the non-equilibrium steady state, one can compute its correlations. Consider a coordinate $x \in \{1, \dots, L\}$ of the chain, then the average occupation of a particle at site x with respect to the non-equilibrium steady measure is given by

$$\langle \rho_x \rangle = \frac{1}{Z_L} \langle\langle W | (D+E)^{x-1} D (D+E)^{L-x} | V \rangle\rangle \tag{3.3.41}$$

With a similar argument we have that, given $1 \leqslant x < y \leqslant L$ the two-point correlation in the non-equilibrium steady state reads

$$\langle \rho_x \rho_y \rangle = \frac{1}{Z_L} \langle\langle W | (D+E)^{x-1} D (D+E)^{y-x-1} D (D+E)^{L-y} | V \rangle\rangle \tag{3.3.42}$$

Explicit expression for one and two-point correlations can be found by using (3.3.31), (3.3.32) and (3.3.33). We refer to [24, 67, 72] for details. Moreover, explicit representations of the matrices D, E and of the vectors $|V\rangle\rangle$ and $\langle\langle W|$ can be found on an infinite dimensional supplementary Fock space. For details see again [24, 67, 72].
Duality and the MPA for the SSEP can be combined to obtain the explicit form for the non-equilibrium steady state. This idea has been developed in [3] and it will be reported in Chapter 7 of this thesis for the multi-species stirring process with boundary driving. By specializing this result to the single species case, the non-equilibrium steady state correlations for the integrable SSEP can be retrieved (see Section 7.6).

3.4 Scaling limits

The scaling limits are mathematical method that allow to describe the time evolution of a macroscopic quantity (for instance the thermodynamic property of a fluid) starting from an underlying microscopic interacting particle systems. This approach allows to derive the so called hydrodynamic equation, starting from an empirical measure named (*empirical*) *density field*. The idea is to scale down the size of the system (then the distance between sites) and accelerate the time to obtain the convergence of the empirical density to a measure, absolutely continuous with respect to Lebesgue, whose density satisfies a PDE. Once the hydrodynamic PDE is found, further questions are the fluctuation and the large deviations of this empirical measure from the hydrodynamic limit. Here, for the sake of simplicity and to convey the main ideas, we briefly expose the hydrodynamic limit for SSEP on an infinite line. This is the starting point for more complex situations.

3.4.1 Hydrodynamic limit for the SSEP

In this section we introduce the ideas concerning the hydrodynamic limit for SSEP. This is the first example of a scaling limit for an interacting particle system. We report the main points of the proof, without going into details. For extended explanations see [29, 30, 31]. As already pointed out, the hydrodynamic limit describes the behavior of a particle system when space and time are properly rescaled. For the sake of simplicity we consider the SSEP $(\eta(t))_{t\geqslant 0}$ on the discrete one dimensional lattice $\mathbb{Z}$, with state space $\Omega = \bigotimes_{x\in\mathbb{Z}} \Omega_x$ where $\Omega_x = \{\eta_x \in \{0,1\}$. The interaction is assumed to be nearest neighbour type and the generator reads

$$\mathcal{L} = \sum_{x\in\mathbb{Z}} \mathcal{L}_{x,x+1} \tag{3.4.1}$$

where, considering a function $f : \Omega \to \mathbb{R}$

$$\mathcal{L}_{x,x+1} f(\eta) = \eta_x(1-\eta_{x+1}) \left(f(\eta - \delta_x + \delta_{x+1}) - f(\eta)\right) + \eta_{x+1}(1-\eta_x) \left(f(\eta - \delta_{x+1} + \delta_x) - f(\eta)\right) \tag{3.4.2}$$

We introduce a scaling parameter $K \in \mathbb{N}$, that will eventually go to infinity, we accelerate time by N^2 and we scale down the space by $1/N$ (diffusive scaling). We introduce the *density field* that is defined, for any $\phi \in C_c^\infty(\mathbb{R})$, as

$$\begin{aligned} X^{K,t}(\cdot) : C_c^\infty(\mathbb{R}) &\to \mathbb{R} \\ \phi \to X^{K,t}(\phi) &= \frac{1}{K} \sum_{x\in\mathbb{Z}} \phi(\frac{x}{K}) \eta^x(tK^2) \end{aligned} \tag{3.4.3}$$

To state the hydrodynamic limit, we need an assumption on the behavior of the density field at the initial time. This assumption is written in the following definition.

Definition 15 *Let $\widehat{\rho} : \mathbb{R} \to [0, \nu]$ be a continuous function called the initial macroscopic profile. A sequence $(\mu_K)_{N \in \mathbb{N}}$ of measures on Ω, is a sequence of compatible initial conditions if $\forall \delta > 0$:*

$$\lim_{N\to\infty} \mu_K \left(\left| X^{N,0}(\phi) - \int_{\mathbb{R}} \phi(u)\widehat{\rho}(u)du \right| > \delta \right) = 0 \tag{3.4.4}$$

with arbitrary $\phi \in C_c^\infty(\mathbb{R})$.

We state the following result

Theorem 13 *Let $\widehat{\rho}$ be an initial macroscopic profile and let $(\mu_K)_{N \in \mathbb{N}}$ be a sequence of compatible initial measures. Denotes with P_K the law of the process $\left(X^{K,t}(\phi)\right)_{t \geqslant 0}$ induced by $(\mu_K)_{N \in \mathbb{N}}$. Then, $\forall T > 0$, $\delta > 0$ and $\forall \phi \in C_c^\infty(\mathbb{R})$, we have*

$$\lim_{N\to\infty} P_K \left(\sup_{t \in [0,T]} \left| X^{K,t}(\phi) - \int_{\mathbb{R}} \phi(u)\rho(u,t)du \right| > \delta \right) = 0 \tag{3.4.5}$$

where $\rho(x, t)$ is a strong solution of the the PDE Cauchy problem

$$\begin{cases} \partial_t \rho(x,t) = \Delta\rho(x,t) & x \in \mathbb{R}, \quad t \in [0,T] \\ \rho(x,0) = \widehat{\rho}(x) \end{cases} \tag{3.4.6}$$

Proof of Theorem 13: the proof consists in the following steps:

- *Convergence of the Dynkin martingale.* Using the generator of the SSEP 3.4.1 we introduce the Dynkin martingale $M_\phi^{K,t}$ (defined in (2.1.48)) by choosing the function $F = X^{K,t}(\phi)$, motivating the K dependence in the martingale. We show that

$$\lim_{K\to\infty} \mathbb{E}_{P_K} \left[\sup_{t \in [0,T]} |M_\phi^{K,t}|^2 \right] = 0 \tag{3.4.7}$$

- *Tightness.* Using (3.4.7) we prove that the sequence of measure P_K is tight in the Skorokhod space $D_\Omega[0, \infty)$. As a consequence, we have the weak convergence $P_K \xrightarrow{K\to\infty} P^*$, where P^* is the unique limiting measure (Prokhorov's theorem, see [46]).

- *Solution of the limiting equation.* We show that, the limiting measure P^* is concentrated on continuous path (that is a subspace of the Skorokhod space $D_\Omega[0, \infty)$), that it is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and that its density $\rho(t, x)$ is the unique solution of the weak form of the Cauchy problem (3.4.6). The higher regularity of the solution can be proved by standard PDE's arguments.

We refer to literature for the details.

□

Similar techniques can be used to prove the limiting equations (that are now infinite dimensional SPDE's) of the fluctuation field [30, 15, 32]. Moreover, one can use the hydrodynamic limit of a weakly asymmetric version of the SSEP to characterized the large deviation functional for the SSEP [73].
In this section we have presented a technique based on the so called Dynkin martingale (2.1.49). However, also duality can be used to prove the hydrodynamic limit and the density fluctuations limit. Here the proof is based on the invariance principle for a single dual particle and on the existence of successful coupling for a pair of dual particles. For an excellent review of this technique we remind the reader to [15].

Part II

Multi-species stirring process

Chapter 4

The model

4.1 Motivations

Boundary-driven particle systems are paradigmatic models for non-equilibrium statistical mechanics. In a boundary-driven system, the model is put in contact with reservoirs, and a current is generated through the system. In the long time limit, a non-equilibrium steady state sets in, with a stationary value of the current.
The model we consider is the *multi-species stirring process with open boundaries*, i.e. the multi-species analogue of the exclusion process as introduced in [14], with additional boundary reservoirs as proposed in [13]. In our set-up, particles can be of $N-1$ types (or species) and they can move on a generic connected and undirected graph $G = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$. The maximal number of particles allowed at each vertex (also called *maximal occupancy*) is denoted by $\nu \in \mathbb{N}$. Two particles, or a particle and a hole, do interact when they sit on two sites connected by a graph edge and this interaction consists in swapping them. Furthermore, reservoirs are attached at each site of the graph, creating, removing and swapping particles. As we will see, the choice of these boundary reservoirs is motivated by the fact that they allow to determine an absorbing dual process, regardless of the graph G on which the process is defined and of the maximal occupancy ν at each site.
We start in Section 4.2 by defining the generator of the multi-species process: first in the equilibrium situation (torus, infinite line, closed graph) and then in the boundary driven set-up. Moreover, we derive the equilibrium reversible measure, when all the boundary densities are the same. In Section 4.3 we provide a Lie algebraic description of the boundary driven case, in terms of symmetric representations of $gl(N)$. This will be a useful tool in the proof of duality and for integrability property of the model.

4.2 The stirring process

The process studied in this chapter is the *multi-species stirring process*. We first define it without boundary driving, i.e. when the process takes place on a torus (periodic boundary conditions), on the infinite line $\mathbb{Z}$ or on a closed graph. After having described these closed boundary situations, we put the system in contact with external reservoir in order to set up an out-of-equilibrium version. This last case is richer, since the number of particles is not conserved any more and non-equilibrium currents sets in (making vanish the time reversibility).

4.2.1 The multi-species stirring process

We consider three different situations without boundary interaction. In all of these set-up we assume that each site can host a maximal number of particles called $\nu \in \mathbb{N}$. The particles have a *type* (sometimes called *species* or *colour*) which can takes values $\{1, 2, \ldots, N-1\}$. More precisely, at each site there are ν available places (called *holes* or *vacancies*) that can be either occupied by particles or not. As a consequence, each site can be totally empty (when no particles are present and thus ν holes are available), totally occupied (when ν particles, of any type, are present and thus no holes are available) or partially occupied (when some of the holes are occupied and some are available). All these configurations obey the *exclusion constraint* that tells that, at each site and at any time, the sum of the number of particles of any type plus the number of holes is ν. The interaction consists in the swapping of any two types of particles or in the swapping of any type of particle and a hole, both at rate 1.
At each site x of the considered geometry, we describe the occupation with an $N-$dimensional vector $n^x = (n_1^x, \ldots, n_N^x)$ in which the value of the a-th component n_a^x denotes the number of particles of species $a \in \{1, \ldots, N-1\}$, while the component n_N^x counts the number of holes at site x. In the following we use indices denoted by lowercase letters (for instance $a, b, c, d \in \{1, \ldots, N-1\}$) when only particles are taken into account. Moreover, we introduce indices denoted by uppercase letters (for instance $A, B, C, D \in \{1, \ldots, N\}$) to also incorporate the holes, that correspond to the index N. The link between the two indices is given by

$$a \in \{1, \ldots, N-1\} \qquad \text{and} \qquad A = (a, N)\,. \tag{4.2.1}$$

At site x of the chosen geometry, we introduce the local configuration space as the set of $N-$dimensional vectors with integer components and whose sum is always equal to ν. That is

$$\Omega_x := \left\{ n^x = (n_1^x, \ldots, n_N^x) \in \mathbb{N}_0^N \ : \ \sum_{A=1}^{N} n_A^x = \nu \right\}\,. \tag{4.2.2}$$

The process on a torus.

Let $\mathbb{T}_L := \mathbb{Z}/L\mathbb{Z}$ be a torus with L sites. The configuration space of the process on the torus $\mathbb{T}_L$ is

$$\Omega := \bigotimes_{x=1}^{L} \Omega_x \tag{4.2.3}$$

where Ω_x is the set defined in (4.2.2). We denote a particle configuration on the whole torus by $\boldsymbol{n} \in \Omega$, where $\boldsymbol{n} = (n_A^x)_{x \in \mathbb{T}_L,\, A \in \{1,\ldots,N\}}$. The infinitesimal generator of the process reads

$$\mathcal{L} = \sum_{x=1}^{L} \mathcal{L}_{x,x+1} \tag{4.2.4}$$

with periodic boundary conditions

$$\mathcal{L}_{L,L+1} = \mathcal{L}_{L,1} \tag{4.2.5}$$

For any given functions $f : \Omega \to \mathbb{R}$ we have the following action

$$\mathcal{L}_{x,x+1} f(\boldsymbol{n}) = \sum_{A,B=1}^{N} n_A^x n_B^{x+1} \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^{x+1} - \boldsymbol{\delta}_B^{x+1}) - f(\boldsymbol{n}) \right]\,, \tag{4.2.6}$$

where

$$(\boldsymbol{\delta}_A^x)_B^y = \begin{cases} 1 & \text{if} \quad y = x,\ B = A\ , \\ 0 & \text{otherwise}\ . \end{cases} \tag{4.2.7}$$

Thus the dynamics consists in an exchange of particles or in an exchange of a particle and a hole between nearest neighbour vertices at Poissonian times. This means that on the bond $(x, x+1)$ a particle or a hole indicated by A at site x is exchanged with a particle or a hole indicated by B at site $x+1$ at rate $n_A^x n_B^{x+1}$.

The process on the infinite line. We now consider a second situation without boundary driving, i.e. when the multi-species stirring process is defined on the infinite line $\mathbb{Z}$. For simplicity, we keep the dimension equal to 1, however the definition can be extended to the case of the regular infinite lattice $\mathbb{Z}^d$ with $d \in \mathbb{N}$. The configuration space of the process is now given by

$$\Omega = \bigotimes_{x \in \mathbb{Z}} \Omega_x \tag{4.2.8}$$

where Ω_x is the set defined in (4.2.2). Here we denote the the configuration variable by $\boldsymbol{n} = (n_A^x)_{x\in\mathbb{Z},\, A\in\{1,\dots,N\}}$ and we still assume that the interaction is of nearest neighbour type. Therefore, on each bond $(x, x+1)$ two different species of particles or a particle and a hole are exchanged with rate 1. The generator reads

$$\mathcal{L} = \sum_{x\in\mathbb{Z}} \mathcal{L}_{x,x+1} \tag{4.2.9}$$

where $\mathcal{L}_{x,x+1}$ is the generator defined in (4.2.6). This will be the set-up in which we will study the scaling limits of this process.

The process on a general graph. A more general situation is the when the multi-species stirring process is defined on a graph. We consider a connected graph $G = (V, \mathcal{E})$ with vertex set V and edge set $\mathcal{E}$. We denote the configuration variable by $\boldsymbol{n} = (n_A^x)_{x\in V,\, A\in\{1,\dots,N\}}$, where n_A^x denotes the number of particles or the number of holes at site x. The configuration space now reads

$$\Omega := \bigotimes_{x\in V} \Omega_x \tag{4.2.10}$$

where, Ω_x is the set defined in (4.2.2). The dynamics consists in the exchange of any two types of particles or of a particle and a hole between two sites connected by an edge, i.e. a site x and a site y such that $(x, y) \in \mathcal{E}$. Therefore, the generator reads

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} \tag{4.2.11}$$

where $\omega_{x,y} \geqslant 0$ are so-called *conductances* that weigh the connectivity of the graph. The linear operator $\mathcal{L}_{x,y}$ is called the *edge generator* and it acts on a functions $f : \Omega \to \mathbb{R}$ as follows

$$\mathcal{L}_{x,y} f(\boldsymbol{n}) = \sum_{A,B=1}^{N} n_A^x n_B^y \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^y - \boldsymbol{\delta}_B^y) - f(\boldsymbol{n}) \right]\ , \tag{4.2.12}$$

where $\boldsymbol{\delta}_A^x$ is defined in (4.2.7).

4.2.2 The boundary driven multi-species stirring process (non-equilibrium)

In this situation we put the multi-species stirring process in contact with the external environment, by inserting the boundary dynamics via particles reservoirs. To avoid repetition we only consider the more general case of a connected graph $G = (V, \mathcal{E})$ and we put each vertex in contact with an external reservoir characterized an average density of particles. The result is an out-of-equilibrium system, i.e. a situation where particles are injected, removed and exchanged with the external environment. This generates non-equilibrium current, losing the reversibility of the process. We recall that we denote the configuration variable on a graph by $\boldsymbol{n} = (n_A^x)_{x\in V,\, A\in\{1,\dots,N\}}$, where n_A^x denotes the number of particles or the number of holes at site x. The dynamics has now two contributions:

- on each edge of the graph, any two types of particles are swapped at rate 1; moreover a particle of any type and a hole are also swapped at rate 1;
- on each vertex x of the graph, a particle of type $a \in \{1, \dots, N-1\}$ is injected at rate $\alpha_a^x n_N^x > 0$; at rate $\alpha_N^x n_a^x$, a particle of type $a \in \{1, \dots, N-1\}$ is removed; additionally, a particle of type $a \in \{1, \dots, N-1\}$ is removed from site x and replaced by a particle of type $b \in \{1, \dots, N-1\}$ with rate $\alpha_b^x n_a^x$.

The swap dynamics taking place on the edges is of Kawasaki-type with N conservation laws (the total number of particles of each type and the total number of holes). The site-dynamics is instead of Glauber-type. In the long-time limit, a so-called non-equilibrium steady state sets in. In the case $N = 2$, we retrieve the boundary-driven version of the symmetric exclusion process [8, 12].

The process generator with boundary driving

We now give the mathematical description of the boundary driven multi-species stirring process. The configuration space of the process on the graph G is the same of the closed graph (4.2.10), i.e.

$$\Omega := \bigotimes_{x\in V} \Omega_x \tag{4.2.13}$$

where, we recall that

$$\Omega_x := \left\{ n^x = (n_1^x, \dots, n_N^x) \in \mathbb{N}_0^N \;:\; \sum_{A=1}^{N} n_A^x = \nu \right\} . \tag{4.2.14}$$

The infinitesimal generator of the process reads

$$\mathcal{L} = \sum_{(x,y)\in\mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x\in V} \Gamma_x \mathcal{L}_x \tag{4.2.15}$$

where $\omega_{x,y} \geqslant 0$ are conductances and we introduce $\Gamma_x \geqslant 0$, that are the couplings to reservoirs, sometimes called *local inhomogeneities*. The generator $\mathcal{L}_{x,y}$ is the edge generator (that was already introduced in (4.2.12), but we report it again for the sake of self-consistency), while we define $\mathcal{L}_x$, that is called the *site generator*. These linear operators act on functions $f : \Omega \to \mathbb{R}$ as follows

$$\mathcal{L}_{x,y} f(\boldsymbol{n}) = \sum_{A,B=1}^{N} n_A^x n_B^y \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^y - \boldsymbol{\delta}_B^y) - f(\boldsymbol{n}) \right] , \tag{4.2.16}$$

$$\mathcal{L}_x f(\boldsymbol{n}) = \sum_{A,B=1}^{N} \alpha_A^x n_B^x \left[f(\boldsymbol{n} + \boldsymbol{\delta}_A^x - \boldsymbol{\delta}_B^x) - f(\boldsymbol{n}) \right] \ , \tag{4.2.17}$$

where $\boldsymbol{\delta}_A^x$ is defined in (4.2.7). Besides the edge dynamics, already described in Section 4.2.1, each vertex exchanges particles or holes with the external environment (reservoirs). Namely, on each site $x \in V$ a particle or a hole indicated by B is replaced with a particle or a hole indicated A at rate $\Gamma_x \alpha_A^x n_B^x$. The transitions where the a particle is replaced by a hole/a hole is replaced by a particle is interpreted as a removal/injection of particles.

4.2.3 Reversible measures in the equilibrium set-up

For a particular choice of the reservoir parameters one has an N-parameter family of reversible measures. More precisely when the boundary parameters are the same on each site, i.e.

$$\alpha_A^x = \alpha_A \qquad \forall x \in V \ , \tag{4.2.18}$$

then the process described by the generator (4.2.15) is reversible with respect to the homogeneous product measure

$$\mu_{\text{rev}} = \bigotimes_{x \in V} \mu_{\text{rev}}^x \tag{4.2.19}$$

with marginals μ_{rev}^x given by the multinomial distribution

$$\mu_{\text{rev}}^x \sim \text{Multinomial}\,(\nu, \rho_1, \dots, \rho_N) \ . \tag{4.2.20}$$

Here

$$\rho_A = \frac{\alpha_A}{|\alpha|} \ ,$$

is the density of species a and we used the notation $|\alpha| = \sum_{A=1}^N \alpha_A$. Explicitly,

$$\mu_{\text{rev}}^x(n^x) = \frac{\nu!}{\prod_{A=1}^N n_A^x!} \prod_{A=1}^N \rho_A^{n_A^x} \ . \tag{4.2.21}$$

This can be proved by checking that detailed balance is satisfied. If condition (4.2.18) is not met, then in general reversibility is lost: indeed, in this situation, each reservoir at site x has its own set of densities vector $\rho^x = (\rho_1^x, \dots, \rho_N^x)$ with components

$$\rho_A^x = \frac{\alpha_A^x}{|\alpha^x|} \tag{4.2.22}$$

and $|\alpha^x| = \sum_{A=1}^N \alpha_A^x$. As a consequence, particles are injected and removed in the graph with different rates. When, for any $A \in \{1, \dots, N\}$, there are at least two reservoirs at two different sites x, y such that $\rho_A^x \neq \rho_A^y$, a non-zero current sets in, breaking the reversibility. Informally, one can say that the reservoirs try to impose their different densities at the boundaries of the the graph, putting the system out-of-equilibrium.

Remark 4 *One can check that conditions* (4.2.18) *are not the most general that implies reversibility. Indeed, by choosing the reservoir parameters as*

$$\alpha_A^x \left(|\alpha^y| - \alpha_A^y \right) = \alpha_A^y \left(|\alpha^x| - \alpha_A^x \right) \qquad \forall x, y \in V, \quad \forall A \in \{1, \dots, N\} \tag{4.2.23}$$

where

$$|\alpha^x| = \sum_{A=1}^{N} \alpha_A^x \,. \tag{4.2.24}$$

reversibility holds. In the case where the graph is a chain with nearest neighbors interactions and two reservoirs are attached to the two boundary sites then condition (4.2.23) *is the multi-species counterpart of* $\alpha\delta = \beta\gamma$ *of the boundary driven partially excluded process introduced in [8] (see Section 3.2 of [12]).*

4.3 Lie algebraic description of the process

In this thesis we will often use the fact that the Markov generator of the multi-species stirring process can be described in terms a Lie algebra $gl(N)$. In this section we provide details about this.

Consider the Lie algebra $gl(N)$ with generators denoted by E_{AB} with $A, B \in \{1, \ldots, N\}$ and commutation relations

$$[\mathrm{E}_{AB}, \mathrm{E}_{CD}] = \mathrm{E}_{AD}\delta_{CB} - \mathrm{E}_{CB}\delta_{AD} \qquad \forall A, B \in \{1, \ldots, N\} \,. \tag{4.3.1}$$

The finite-dimensional representations are labelled by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$ of ν with $\lambda_A \geqslant \lambda_{A+1}$, $\lambda_A \in \mathbb{N}$ and $\sum_{A=1}^{N} \lambda_A = \nu \in \mathbb{N}$. We are interested in the *symmetric* finite-dimensional representations with

$$\lambda = (\nu, 0, \ldots, 0) \,. \tag{4.3.2}$$

The dimension M_ν of this symmetric representations is given by the combination of N objects in ν positions with repetition, namely

$$M_\nu = \frac{(N+\nu-1)!}{\nu!(N-1)!} \,. \tag{4.3.3}$$

The generators of the symmetric representations will be denoted by E_{AB}. A basis of the vector space $\mathbb{C}^{M_\nu}$ are the column vectors denoted by

$$|n\rangle = |n_1, \ldots, n_N\rangle, \quad \text{with} \quad n_A \in \mathbb{N}_0 \quad \text{such that} \quad \sum_{A=1}^{N} n_A = \nu \,. \tag{4.3.4}$$

The basis vectors satisfy the orthogonality relation

$$\langle m|n\rangle = \langle m_1, \ldots, m_N|n_1, \ldots, n_N\rangle = \prod_{A=1}^{N} \delta_{m_A, n_A} \,, \tag{4.3.5}$$

where $\langle m_1, \ldots, m_N|$ is the row vector obtained by transposing $|m\rangle = |m_1, \ldots, m_N\rangle$ and δ_{m_A,n_A} is the Kronecker delta.

The explicit action of the algebra generators on the basis vectors $|n\rangle$ is the following:

$$\begin{cases} E_{AB}|n_1, \ldots, n_A, \ldots, n_B, \ldots, n_N\rangle = n_B|n_1, \ldots, n_A+1, \ldots, n_B-1, \ldots, n_N\rangle & A \neq B \\ E_{AA}|n_1, \ldots, n_A, \ldots, n_N\rangle = n_A|n_1, \ldots, n_A, \ldots, n_N\rangle \,. \end{cases} \tag{4.3.6}$$

The matrices defined in this way satisfy the commutation relations (4.3.1) and yield highest Dynkin weight (4.3.2).

As mentioned above, the process with generator (4.2.15) can be described in terms of $gl(N)$ Lie algebra generators. The state space (4.2.13) is given by the $|V|$-fold tensor product of the vector space with basis elements $|n^x\rangle$ at a given site. Namely, a vector $|\boldsymbol{n}\rangle \in \Omega$ can be written as

$$|\boldsymbol{n}\rangle = \left(\bigotimes_{x \in V} |n_1^x, \ldots, n_N^x\rangle \right) \tag{4.3.7}$$

with $\sum_{A=1}^{N} n_A^x = \nu$ for any $x \in V$. For a fixed $x \in V$ we write $|n^x\rangle = |n_1^x, \ldots, n_N^x\rangle$. The following orthogonality relation is a consequence of the single site relation (4.3.5)

$$\langle \boldsymbol{n} | \boldsymbol{m} \rangle = \prod_{x \in V} \prod_{A=1}^{N} \delta_{n_A^x, m_A^x} \,. \tag{4.3.8}$$

We introduce the Hamiltonian operator

$$H = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{H}_{x,y} + \sum_{x \in V} \Gamma_x H_x \tag{4.3.9}$$

where the edge Hamiltonian $\mathcal{H}_{x,y}$ that describes the interaction between two connected sites is

$$\mathcal{H}_{x,y} = \sum_{A,B=1}^{N} \left(E_{AB}^x E_{BA}^y - E_{BB}^x E_{AA}^y \right) \tag{4.3.10}$$

and where the site Hamiltonian H_x is

$$H_x = \sum_{A,B=1}^{N} \alpha_A^x \left(E_{AB}^x - E_{BB}^x \right) \tag{4.3.11}$$

Here E_{AB}^x denotes the generator E_{AB} in (4.3.6) acting non-trivially on site x (and as the identity on the other sites). The Hamiltonian in (4.3.9) is stochastic and is linked to the Markov generator by

$$H = \mathcal{L}^T \tag{4.3.12}$$

where T denotes transposition, see e.g. [74]. The action of the generator on functions f can then be expressed as

$$\mathcal{L} f(\boldsymbol{n}) = \langle f | H | \boldsymbol{n} \rangle \tag{4.3.13}$$

where

$$\langle f | = \sum_{\boldsymbol{m} \in \Omega} f(\boldsymbol{m}) \langle \boldsymbol{m} | \,. \tag{4.3.14}$$

We can write the edge Hamiltonian (4.3.10) as a function of the coproduct of the quadratic Casimir of $gl(N)$

$$C = \sum_{A,B=1}^{N} E_{AB} E_{BA} \,, \tag{4.3.15}$$

that acts diagonally as $C|n\rangle = \nu(\nu+N)|n\rangle$ on any state $|n\rangle$ and belogs to the center of $gl(N)$ (i.e. it commutes with all the algebra elements). More precisely, considering the standard coproduct

$$\Delta : gl(N) \to gl(N) \otimes gl(N) \tag{4.3.16}$$

with

$$E_{AB} \mapsto E_{AB} \otimes \mathbb{1} + \mathbb{1} \otimes E_{AB} \ , \tag{4.3.17}$$

we have

$$\Delta(C) = \sum_{A,B=1}^{N} \Delta(E_{AB})\Delta(E_{BA}) = 2 \sum_{A,B=1}^{N} E_{AB} \otimes E_{BA} + C \otimes \mathbb{1} + \mathbb{1} \otimes C \, . \tag{4.3.18}$$

Then, one can check that

$$\mathcal{H}_{x,y} = \frac{1}{2}\Delta_{x,y}(C) - \nu(2\nu + N) \tag{4.3.19}$$

where $\Delta_{x,y}(C)$ denotes $\Delta(C)$ acting on the sites of edge $(x, y) \in \mathcal{E}$ and acting trivially on the other sites of the graph.

Chapter 5

Scaling limits

5.1 Motivations

The symmetric exclusion process is a famous and well-studied particle system, where the hydrodynamic limit is the heat equation (see Chapter 3) and where the stationary fluctuations around the hydrodynamic limit are given by an infinite dimensional Ornstein-Uhlenbeck process [30], [29], [32] , [5]. The large deviations from the hydrodynamic limit are also well-studied [73], and because of integrability, in the simplest one-dimensional setting with reservoirs the non-equilibrium steady state can be written explicitly (see Chapter 3), and as a consequence, the large deviation around the stationary non-equilibrium density profile can be computed [75], [76]. Such explicit solvability of a model is very rare and in the case of the symmetric exclusion process a consequence of the fact that the Markov generator corresponds to an integrable spin chain (for the $d = 1$ nearest neighbor setting) and that the model is self-dual (for the general symmetric model on any graph). At present, there is a growing interest in models with multiple conserved quantities, their hydrodynamic limit, and their fluctuations (often referred to as "non-linear fluctuating hydrodynamics") [33, 34], as well as in "multi-layer" models [35, 36, 37], and in processes where effects such as uphill diffusion can be observed [1, 77].
In this chapter we study the scaling limits of the multi-species stirring process on the infinite line $\mathbb{Z}$, i.e. when boundaries are not present. In Section 5.2, first we report the hydrodynamic limit of the process by using techniques based on the martingale approach (see [30]). As we will see, at the level of the hydrodynamic equations, the species diffuses separately, giving rise to a set of decoupled heat equations. In Section 5.3, we consider the stationary density fluctuation field $(Y^{K,t})_{t\geqslant 0}$ of the $N-1$ species of particles and we state that, in the diffusive re-scaling of space and time, this field converges as $K \to \infty$ to the solution of a $(N-1)$-dimensional SPDE of Ornstein-Uhlenbeck type given by

$$dY^t = \nu(\boldsymbol{A}Y^t dt + \sqrt{2\Sigma}\nabla dW^t)\,. \tag{5.1.1}$$

The operator-valued matrix $\boldsymbol{A}$ is simply given ΔI with I the identity matrix and $\Delta = \partial_{xx}$, and corresponds to the hydrodynamic limit, which is a system of uncoupled heat equations. The matrix Σ is however non-diagonal, showing that on the level of fluctuations interaction between the different species becomes visible. The stationary distribution is a product of multinomials and the matrix Σ is the covariance matrix of a multinomial distribution. Equation (5.1.1) is the natural generalization of the Ornstein-Uhlenbeck process which describes the density fluctuations of the symmetric exclusion process, where the coefficient in front of the conservative noise is the square-root of the variance of Bernoulli distribution. The proof of this result is reported in detail in Section 5.4.

5.2 Hydrodynamic limit

Here we prove the hydrodynamic behaviour of the multi-species stirring process on the regular lattice $\mathbb{Z}$ with nearest neighbour interaction. This set-up has been presented in Section 4.2.1. We recall that the process is denoted by $(\boldsymbol{n}(t))_{t\geqslant 0}$ where n_A^x, with $A \in \{1,\ldots,N\}$ indicates the number of particles or the number of holes at site x. The configuration space is given by (4.2.13), specialized to the infinite chain, i.e.

$$\Omega := \left\{ n = (n_1,\ldots,n_N) \ : \ n_A \in \{0,1,\ldots,\nu\} \quad \text{satisfying} \quad \sum_{A=1}^{N} n_A = \nu \right\}^{\mathbb{Z}}, \tag{5.2.1}$$

while the generator is given by

$$\mathcal{L} = \sum_{x\in\mathbb{Z}} \mathcal{L}_{x,x+1} \tag{5.2.2}$$

where $\mathcal{L}_{x,x+1}$ is the edge generator of the multi-species stirring process (4.2.6), i.e.

$$\mathcal{L}_{x,x+1} f(\boldsymbol{n}) = \sum_{A,B=1}^{N} n_A^x n_B^{x+1} \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^{x+1} - \boldsymbol{\delta}_B^{x+1}) - f(\boldsymbol{n}) \right] . \tag{5.2.3}$$

We introduce the *density field* of the species $a \in \{1,\ldots,N-1\}$. For any $\phi \in C_c^\infty(\mathbb{R})$ this field is defined as

$$\begin{aligned} X_a^{K,t}(\cdot) &: C_c^\infty(\mathbb{R}) \to \mathbb{R} \\ &\phi \to X_a^{K,t}(\phi) = \frac{1}{K}\sum_{x\in\mathbb{Z}} \phi(\frac{x}{K}) n_a^x(tK^2) \end{aligned} \tag{5.2.4}$$

where $K \in \mathbb{N}$ is the scaling parameter. To state the hydrodynamic limit, we need an assumption on the behaviour of the density field at the initial time. This assumption is written in Definition 16.

Definition 16 *Let $\widehat{\rho}^{(a)} : \mathbb{R} \to [0,\nu]$, with $a \in \{1,\ldots,N-1\}$, be a continuous function called the initial macroscopic profile of species a. A sequence $(\mu_K)_{K\in\mathbb{N}}$ of measures on Ω, is a sequence of compatible initial conditions if $\forall a \in \{1,\ldots,N-1\}$, $\forall \delta > 0$:*

$$\lim_{K\to\infty} \mu_K \left(\left| X_a^{K,0}(\phi) - \int_{\mathbb{R}} \phi(u)\widehat{\rho}^{(a)}(u)du \right| > \delta \right) = 0 \tag{5.2.5}$$

with arbitrary $\phi \in C_c^\infty(\mathbb{R})$.

We state the following result

Theorem 14 *Let $\widehat{\rho}^{(a)}$ be an initial macroscopic profile of species $a \in \{1,\ldots,N-1\}$ and let be $(\mu_K)_{K\in\mathbb{N}}$ a sequence of compatible initial measures. P_K denotes the law of the process $\left(X_1^{K,t}(\phi),\ldots,X_{N-1}^{K,t}(\phi)\right)$ induced by $(\mu_K)_{K\in\mathbb{N}}$. Then, $\forall T > 0$, $\delta > 0$, $\forall a \in \{1,\ldots,N-1\}$ and $\forall \phi \in C_c^\infty(\mathbb{R})$*

$$\lim_{K\to\infty} P_K \left(\sup_{t\in[0,T]} \left| X_a^{K,t}(\phi) - \int_{\mathbb{R}} \phi(u)\rho^{(a)}(u,t)du \right| > \delta \right) = 0 \tag{5.2.6}$$

where $\rho^{(a)}(u,t)$ is a strong solution of the the PDE Cauchy problem

$$\begin{cases}\partial_t \rho^{(a)}(u,t) = \nu\Delta\rho^{(a)}(u,t) & u\in\mathbb{R}, \quad t\in[0,T] \\ \rho^{(a)}(u,0) = \widehat{\rho}^{(a)}(u) \end{cases} . \tag{5.2.7}$$

Proof of Theorem 14: the proof is based on the martingale techniques proposed in [30, 29, 31]. The aim is to show that the sequence of measure $(\mathbb{P}_K)_{K\in\mathbb{N}}$ is tight and the limit point has a density that is the solution of the PDE (5.2.7). We start by considering the Dynkin martingale defined in (2.1.48) associated to the process $(\boldsymbol{n}(t))_{t\geqslant 0}$ defined, for any $\phi\in C_c^\infty(\mathbb{R})$ and $\forall a\in\{1,\dots,N-1\}$, as

$$m_{a,\phi}^{K,t} := X_a^{K,t}(\phi) - X_a^{K,0}(\phi) - \int_0^t K^2\mathcal{L}X_a^{K,s/K^2}(\phi)\,ds. \tag{5.2.8}$$

The action of the generator (5.2.2) on the density field (5.2.4) is

$$\begin{aligned}
\mathcal{L}X_a^{K,\cdot}(\phi) &= \frac{1}{K}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N} n_A^x n_B^{x+1}\left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n_a^y - \delta_A^x + \delta_B^x + \delta_A^{x+1} - \delta_B^{x+1}) - n_a^y\right)\right] \\
&= \frac{1}{K}\sum_{x\in\mathbb{Z}}\left\{n_a^x(\nu - n_a^{x+1})\left[\phi\left(\frac{x+1}{K}\right) - \phi\left(\frac{x}{K}\right)\right]\right. \\
&\qquad\left. + n_a^{x+1}(\nu - n_a^x)\left[\phi\left(\frac{x}{K}\right) - \phi\left(\frac{x+1}{K}\right)\right]\right\} \\
&= \frac{\nu}{K}\sum_{x\in\mathbb{Z}} n_a^x\left[\phi\left(\frac{x-1}{K}\right) + \phi\left(\frac{x+1}{K}\right) - 2\phi\left(\frac{x}{K}\right)\right]
\end{aligned}$$

by the Taylor's series with Lagrange remainder computed in (5.4.22) we obtain

$$K^2\mathcal{L}X_a^{K,\cdot}(\phi) = \frac{\nu}{K}\sum_{x\in\mathbb{Z}}(n_a^x - \nu p_a)\Delta\phi(\frac{x}{K}) + R_0(\phi,a)$$

where

$$R_0(\phi,a) = \frac{\nu}{K}\sum_{x\in\mathbb{Z}} n_a^x\left[\frac{1}{6}\frac{1}{K}\left[\phi^{(3)}(\frac{x+\theta^+}{K}) - \phi^{(3)}(\frac{x-\theta^-}{K})\right]\right]. \tag{5.2.9}$$

with $\theta^+,\theta^-\in(0,1)$ and where $\phi^{(3)}$ denotes the third derivative of ϕ. Observing that $\phi\in C_c^\infty(\mathbb{R})$ and $n_a^x\leqslant\nu$, then $R_0(\phi,a)$ is infinitesimal when $K\to\infty$. Therefore

$$K^2\mathcal{L}X_a^{K,\cdot}(\phi) = \frac{\nu}{K}\sum_{x\in\mathbb{Z}} n_a^x\Delta\phi(\frac{x}{K}) + o\left(\frac{1}{K}\right). \tag{5.2.10}$$

Replacing (5.2.10) in (5.2.8) we obtain

$$m_{a,\phi}^{K,t}(X) + o\left(\frac{1}{K}\right) = X_a^{K,t}(\phi) - X_a^{K,0}(\phi) - \nu\int_0^t X_a^{K,s/K^2}(\Delta\phi)\,ds \tag{5.2.11}$$

where on the right-hand-side we recognize the discrete counterpart of the weak formulation of the heat equation with constant diffusivity ν for the species a. We shall prove that

$$\lim_{K\to\infty} P_K\left(\sup_{t\in[0,T]}\left|X_a^{K,t}(\phi) - X_a^{K,0}(\phi) - \nu\int_0^t X_a^{K,s/K^2}(\Delta\phi)\,ds\right| > \delta\right) = 0. \tag{5.2.12}$$

We find an upper bound by Chebyshev's and Doob's inequalities

$$\begin{aligned}
&P_K\left(\sup_{t\in[0,T]}\left|X_a^{K,t}(\phi)-X_a^{K,0}(\phi)-\nu\int_0^t X_a^{K,s/K^2}(\Delta\phi)\,ds\right|>\delta\right)\\
&\leqslant\frac{1}{\delta^2}\mathbb{E}_{\mu_K}\left[\sup_{[0,T]}\left|m_{a,\phi}^{K,t}\right|^2\right]\leqslant\frac{4}{\delta^2}\mathbb{E}_{\mu_K}\left[\left|m_{a,\phi}^{K,T}\right|^2\right].
\end{aligned}\tag{5.2.13}$$

Moreover, by Doob's decomposition

$$\mathbb{E}_{\mu_K}\left[\left|m_{a,\phi}^{K,T}\right|^2\right]=\mathbb{E}_{\mu_K}\left[\int_0^T K^2\Gamma_{a,a}^{\phi,s/K^2}ds\right]\tag{5.2.14}$$

where $\Gamma_{a,a}^{\phi,s}$ denotes the operator the Carré-du-Champ operator, defined in (2.1.50) on the density field (5.2.4). Here, for the sake of notation, we do not write the time dependence. We then obtain

$$\begin{aligned}
\Gamma_{a,a}^{\phi}&=\frac{1}{K^2}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N}n_A^x n_B^{x+1}\left[\sum_{y\in\mathbb{Z}}\phi(\frac{y}{K})\left((n_a^y-\delta_A^x+\delta_B^x+\delta_A^{x+1}-\delta_B^{x+1})-n_a^y\right)\right]^2\\
&=\frac{1}{K^2}\sum_{x\in\mathbb{Z}}n_a^x\sum_{B=1\,:\,B\neq a}^{N}n_B^{x+1}\left[\phi(\frac{x+1}{K})-\phi(\frac{x}{K})\right]^2\\
&+\frac{1}{K^2}\sum_{x\in\mathbb{Z}}\sum_{A=1\,:\,A\neq a}^{N}n_A^x n_a^{x+1}\left[-\phi(\frac{x+1}{K})+\phi(\frac{x}{K})\right]^2\\
&=\frac{1}{K^2}\sum_{x\in\mathbb{Z}}\left(n_x^a\sum_{B=1\,:\,B\neq a}^{N}n_B^{x+1}+n_a^{x+1}\sum_{A=1\,:\,A\neq a}^{N}n_A^x\right)\left[\phi(\frac{x+1}{K})-\phi(\frac{x}{K})\right]^2.
\end{aligned}$$

By Taylor's series with Lagrange remainder we obtain

$$K^2\Gamma_{a,a}^{\phi}=\frac{1}{K^2}\sum_{x\in\mathbb{Z}}\left(n_x^a\sum_{B=1\,:\,B\neq a}^{N}n_B^{x+1}+n_a^{x+1}\sum_{A=1\,:\,A\neq a}^{N}n_A^x\right)\nabla(\phi)^2(\frac{x}{K})+o\left(\frac{1}{K^2}\right).\tag{5.2.15}$$

Using (5.2.14), (5.2.15), the boundness $|n_a^x|\leqslant N\nu$ $\forall x\in\mathbb{Z}$ and $\forall a\in\{1,\ldots,N-1\}$ and the fact that $\nabla\phi$ is smooth and has compact support we obtain

$$\begin{aligned}
\mathbb{E}_{\mu_K}\left[\left|m_{a,\phi}^{N,T}\right|^2\right]&\leqslant K\frac{C}{K^2}\sup_{x\in\mathbb{Z},\,t\in[0,T]}\mathbb{E}_{\mu_K}\left[\left(n_a^x\sum_{B=1\,:\,B\neq a}^{N}n_B^{x+1}+n_a^{x+1}\sum_{A=1\,:\,A\neq a}^{N}n_A^x\right)\right]+o\left(\frac{1}{K^2}\right)\\
&\leqslant\frac{C}{K}+o\left(\frac{1}{K^2}\right).
\end{aligned}\tag{5.2.16}$$

Taking the limit and using (5.2.13) and (5.2.16)

$$\lim_{K\to\infty}P_K\left(\sup_{t\in[0,T]}\left|X_a^{K,t}(\phi)-X_a^{K,0}(\phi)-\nu\int_0^t X_a^{K,s}(\Delta\phi)\,ds\right|>\delta\right)\leqslant\lim_{K\to\infty}\frac{C}{K}=0.\tag{5.2.17}$$

With the above convergence and by standard computations we can prove that the sequence of measure $(P_K)_{K\in\mathbb{N}}$ defined in Theorem 14 is tight and that all limit points do coincide with $\rho^{(a)}(t,u)du$ with $\rho^{(a)}(t,u)$ is the unique solution of

$$\begin{cases}\partial_t\rho^{(a)}(t,u)=\nu\Delta\rho^{(a)}(t,u)\\ \rho^{(a)}(0,u)=\widehat{\rho}^{(a)}(u)\end{cases},\tag{5.2.18}$$

provided that $\widehat{\rho}^{(a)}(u)$ is compatible with the initial sequence of measures $(\mu_K)_{K\in\mathbb{N}}$ in the sense of Definition 16. Finally, existence and uniqueness of a strong solution of the above system of equations is standard.

□

5.3 Equilibrium fluctuations

We consider the setting where the process $(\boldsymbol{n}(t))_{t\geqslant 0}$ starts from equilibrium, i.e. a reversible measure where we have fixed the probabilities $p = (p_1, \ldots, p_N)$ once for all. Then the *density fluctuation field* for a species $a \in \{1, \ldots, N-1\}$ is a random distribution, i.e., a random element of the dual space of $C_c^\infty(\mathbb{R})$ (usually denoted by $(C_c^\infty(\mathbb{R}))^*$) defined via:

$$\begin{aligned} Y_a^{K,t}(\cdot) &: C_c^\infty(\mathbb{R}) \to \mathbb{R} \\ \phi &\to Y_a^{K,t}(\phi) = \frac{1}{\sqrt{K}} \sum_{x\in\mathbb{Z}} \phi(\frac{x}{K})(n_a^x(tK^2) - \nu p_a) \end{aligned} \tag{5.3.1}$$

where $\nu p_a = \mathbb{E}_{\mu_{\text{rev}}}[n_a^x]$ with μ_{rev} the reversible measure of the multi-species stirring process define in (4.2.19). We call Q_K the law of the random vector process

$$(Y^{K,t})_{t\geqslant 0} = \left(Y_1^{K,t}, \ldots, Y_{N-1}^{K,t}\right)_{t\geqslant 0}$$

and by $\mathbb{E}$ the expectation with respect to this law. Note that, because $(\boldsymbol{n}(t))_{t\geqslant 0}$ is initialized from the reversible measure (4.2.19), the process keeps the product measure structure for every time $t \geqslant 0$. We denote by

$$(C_c^\infty(\mathbb{R}))^*_{(N-1)} = \underbrace{(C_c^\infty(\mathbb{R}))^* \times \ldots \times (C_c^\infty(\mathbb{R}))^*}_{(N-1)\,\text{times}} \tag{5.3.2}$$

the dual space of $(C_c^\infty(\mathbb{R}))^{N-1}$. Our main result is the following theorem.

Theorem 15 *There exists a unique random element*

$$\left(Y_1^t, \ldots, Y_{N-1}^t\right)_{t\in[0,T]} \in C\left([0,T]; (C_c^\infty(\mathbb{R}))^*_{(N-1)}\right)$$

with law Q such that

$$Q_K \to Q \qquad \textit{weakly for } K \to \infty. \tag{5.3.3}$$

Moreover, for every $a \in \{1, \ldots, N-1\}$, $(Y_a^t)_{t\geqslant 0}$ is a generalized stationary Ornstein-Uhlenbeck process solving the following martingale problem:

$$M_{a,\phi}^t := Y_a^t(\phi) - Y_a^0(\phi) - \nu \int_0^t Y_a^s(\Delta\phi)\, ds \tag{5.3.4}$$

is a martingale $\forall \phi \in C_c^\infty(\mathbb{R})$ with respect to the natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ of $(Y_1^t, \ldots, Y_{N-1}^t)_{t\in[0,T]}$ with quadratic covariation

$$[M_{a,\phi}, M_{b,\phi}]_t = -2t\nu^2 p_a p_b \int_{\mathbb{R}} (\nabla\phi(u))^2\, du \tag{5.3.5}$$

and quadratic variation

$$[M_{a,\phi}]_t = 2t\nu^2 p_a(1-p_a) \int_{\mathbb{R}} (\nabla\phi(u))^2\, du \tag{5.3.6}$$

Remark 5 *The above martingale problem can be restated by requiring that* (5.3.4) *and*

$$\mathcal{N}^t_{a,b,\phi} = M^t_{a,\phi}M^t_{b,\phi} + 2t\nu^2 p_a p_b \int_{\mathbb{R}} \nabla(\phi(u))^2 du\,, \tag{5.3.7}$$

$$\mathcal{N}^t_{a,a,\phi} = \left(M^t_{a,\phi}\right)^2 - 2t\nu^2 p_a(1-p_a) \int_{\mathbb{R}} \nabla(\phi(u))^2 du \tag{5.3.8}$$

are martingales with respect to the natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Theorem 15 suggests that the limiting process

$$(Y^t)_{t\in[0,T]} = \left(Y^t_1, \ldots, Y^t_{N-1}\right)_{t\in[0,T]} \tag{5.3.9}$$

can be formally written as the solution of the distribution-valued SPDE

$$dY^t = \nu(\boldsymbol{A}Y^t dt + \sqrt{2\Sigma}\nabla dW^t) \tag{5.3.10}$$

where

$$(W^t)_{t\in[0,T]} = \left((W^t_1, \ldots, W^t_{N-1})\right)_{t\in[0,T]} \tag{5.3.11}$$

is an $(N-1)$-dimensional vector of independent space-time white noises. The matrices are the following

$$\boldsymbol{A} = \begin{pmatrix} \Delta & 0 & \ldots & 0 \\ 0 & \Delta & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Delta \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \ldots & -p_1p_{N-1} \\ -p_1p_2 & p_2(1-p_2) & \ldots & -p_2p_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ -p_{N-1}p_1 & -p_{N-1}p_2 & \ldots & p_{N-1}(1-p_{N-1}) \end{pmatrix} \tag{5.3.12}$$

and Σ is semi-positive definite. The covariances of (5.3.9) $\forall t \in [0,T]$ are given by:

(i) when $a \neq b$

$$\operatorname{Cov}\left(Y^t_a(\phi), Y^0_b(\psi)\right) = -\nu p_a p_b \langle S_t\phi, \psi\rangle_{L^2(dx)}\,, \tag{5.3.13}$$

(ii) when $a = b$

$$\operatorname{Cov}\left(Y^t_a(\phi), Y^0_a(\psi)\right) = \nu p_a(1-p_a) \langle S_t\phi, \psi\rangle_{L^2(dx)} \tag{5.3.14}$$

where $(S_t)_{t\geqslant 0}$ is the transition semigroup of the Brownian motion $(B_\nu(t))_{t\geqslant 0}$ with variance νt.

5.4 Proof of Theorem 15

The proof of Theorem 15 consists in the following steps: firstly we show that the sequence of measures $(Q_K)_{K\in\mathbb{N}}$ is tight and converges to a unique limit point Q; secondly we show that at the initial time $t = 0$ the process is Gaussian and has covariances given by

$$\operatorname{Cov}(Y^0_a(\phi), Y^0_b(\psi)) = -\nu p_a p_b \langle \phi, \psi\rangle_{L^2(dx)}, \qquad \operatorname{Cov}(Y^0_a(\phi), Y^0_a(\psi)) = \nu p_a(1-p_a)\langle \phi, \psi\rangle_{L^2(dx)}. \tag{5.4.1}$$

Finally, we prove that Q solves the martingale problem for any $t \in [0,T]$. As shown in Section 4, Chapter 11 of [30], these steps are equivalent to saying that Q is the unique solution of the martingale problem and, furthermore they allow to find the transition probabilities of the Markov process $(Y_t)_{t\in[0,T]}$. We observe that the Gaussianity of the limiting process at initial time $t = 0$ is a consequence of the central limit theorem and of the fact that, for every $x \in \mathbb{Z}$, $n^x = (n^x_1, \ldots, n^x_N)$ is distributed with the reversible Multinomial measure (4.2.19).

Preliminarily, we need some convergence properties of the Dynkin martingale associated with the density fluctuation field. Thus, we split the proof of Theorem 15 as follows:

1. Convergence of Dynkin's martingale, Section 5.4.1.
2. Tightness of $(Q_K)_{K\in\mathbb{N}}$, using the Aldous' criterion [78], Section 5.4.2.
3. Space-time covariances. This will be done using duality, Section 5.4.3.
4. Uniqueness of the limiting distribution Q and solution of the martingale problem, Section 5.4.4.

5.4.1 Convergence of martingales

The Dynkin martingale

In analogy to what has been done in Section 2.1.3, we apply this scheme to to the multi-species stirring process on an infinite line with time scaled by K^2. We denote it by $(\boldsymbol{n}(tK^2))_{t\geqslant 0}$ and its generator is given in (5.2.2). By choosing, for any $\phi \in C_c^\infty(\mathbb{R})$ and for any $a \in \{1,\ldots,N-1\}$, the function $f(\boldsymbol{n}(t)) = Y_a^{K,t}(\phi)$, we define the following Dynkin martingale

$$M_{a,\phi}^{K,t} := Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \int_0^t K^2 \mathcal{L} Y_a^{K,s/K^2}(\phi) ds \tag{5.4.2}$$

where $Y_a^{K,t}(\phi)$ denotes the equilibrium fluctuation field for the species a defined in (5.3.1). Observe that the last term above martingale is defined as

$$\int_0^{tK^2} \mathcal{L} Y_a^{K,s}(\phi) ds. \tag{5.4.3}$$

Performing a change of integration variable we obtain (5.4.2). For any $a,b \in \{1,\ldots,N-1\}$, the quadratic covariation is

$$\left[M_{a,\phi}^K, M_{b,\phi}^K\right]_t = \int_0^t K^2 \Gamma_{a,b}^{\phi,s/K^2} ds \tag{5.4.4}$$

where, for a generic $s \geqslant 0$

$$\Gamma_{a,b}^{\phi,s} := \mathcal{L}(Y_a^{K,s}(\phi) Y_b^{K,s}(\phi)) - Y_a^{K,s}(\phi)\mathcal{L}(Y_b^{K,s}(\phi)) - Y_b^{K,s}(\phi)\mathcal{L}(Y_a^{K,s}(\phi)). \tag{5.4.5}$$

Using (2.1.51), this can be written as

$$\Gamma_{a,b}^{\phi,s} = \sum_{x\in\mathbb{Z}} \sum_{A,B=1}^{N} n_A^x n_B^{x+1} \left[\widetilde{Y_{a,k,l}^{K,s}}(\phi) - Y_a^{K,s}(\phi)\right]\left[\widetilde{Y_{b,k,l}^{K,s}}(\phi) - Y_b^{K,s}(\phi)\right] \tag{5.4.6}$$

where $\widetilde{Y_{a,k,l}^{K,s}}(\phi)$ is a short-cut for the equilibrium fluctuation field computed in the configuration $\boldsymbol{n}(K^2 s) - \delta_A^x + \delta_B^x + \delta_A^{x+1} - \delta_B^{x+1}$.
We further introduce the following family of Doob's martingales

$$\mathcal{N}_{a,b,\phi}^{K,t} = M_{a,\phi}^{K,t} M_{b,\phi}^{K,t} - \int_0^t K^2 \Gamma_{a,b}^{\phi,s/K^2} ds \qquad \forall a,b \in \{1,\ldots,N-1\}, \tag{5.4.7}$$

which will be useful in the analysis.

Remark 6 *Often, in the following to alleviate notation we do not write explicitly the time dependence, i.e.*

$$\begin{aligned}\Gamma^{\phi}_{a,b} = \frac{1}{K}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N} n^x_A n^{x+1}_B \left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n^y_a - \delta^x_A + \delta^x_B + \delta^{x+1}_A - \delta^{x+1}_B) - n^y_a\right)\right] \\ \times\left[\sum_{z\in\mathbb{Z}}\phi\left(\frac{z}{K}\right)\left((n^z_b - \delta^x_A + \delta^x_B + \delta^{x+1}_A - \delta^{x+1}_B) - n^z_b\right)\right].\end{aligned} \tag{5.4.8}$$

Remark 7 *In principle we should consider $\Gamma^{\phi,\psi,s}_{a,b}$, underlining the fact that the test function could depend on the species too. However, $\Gamma^{\phi,\psi,s}_{a,b}$ is bilinear and symmetric with respect the test function therefore, by polarization identity, it is enough to evaluate $\Gamma^{\phi,\phi,s}_{a,b}$. We will denote it by $\Gamma^{\phi,s}_{a,b}$ for the sake of notation simplicity. Bilinearity is clear. We prove the symmetry. To alleviate the notation we do not write the here the explicitly the time dependence:*

$$\begin{aligned}\Gamma^{\phi,\psi}_{a,b} &= \frac{1}{K}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N} n^x_A n^{x+1}_B \left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n^y_a - \delta^x_A + \delta^x_B + \delta^{x+1}_A - \delta^{x+1}_B) - n^y_a\right)\right] \\ &\times\left[\sum_{z\in\mathbb{Z}}\psi\left(\frac{z}{K}\right)\left((n^z_b - \delta^x_A + \delta^x_B + \delta^{x+1}_A - \delta^{x+1}_B) - n^z_b\right)\right] \\ &= \frac{1}{K}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N} n^x_A n^{x+1}_B \\ &\times\left[\phi\left(\frac{x}{K}\right)(n^x_a - \delta^x_A + \delta^x_B - n^x_a) + \phi\left(\frac{x+1}{K}\right)(n^{x+1}_a + \delta^{x+1}_A - \delta^{x+1}_B - n^{x+1}_a)\right] \\ &\times\left[\psi\left(\frac{x}{K}\right)(n^x_b - \delta^x_A + \delta^x_B - n^x_b) + \psi\left(\frac{x+1}{K}\right)(n^{x+1}_b + \delta^{x+1}_A - \delta^{x+1}_B - n^{x+1}_b)\right] \\ &= \frac{1}{K}\sum_{x\in\mathbb{Z}}\left\{n^x_a n^{x+1}_b\left[\phi\left(\frac{x}{K}\right)(-1) + \phi\left(\frac{x+1}{K}\right)(+1)\right]\left[\psi\left(\frac{x}{K}\right)(+1) + \psi\left(\frac{x+1}{K}\right)(-1)\right]\right. \\ &+ \left. n^x_b n^{x+1}_a\left[\phi\left(\frac{x}{K}\right)(+1) + \phi\left(\frac{x+1}{K}\right)(-1)\right]\left[\psi\left(\frac{x}{K}\right)(-1) + \psi\left(\frac{x+1}{K}\right)(+1)\right]\right\} \\ &= -\frac{1}{K}\sum_{x\in\mathbb{Z}}\left(n^x_a n^{x+1}_b + n^x_b n^{x+1}_a\right)\left[\phi\left(\frac{x+1}{K}\right) - \phi\left(\frac{x}{K}\right)\right]\left[\psi\left(\frac{x+1}{K}\right) - \psi\left(\frac{x}{K}\right)\right].\end{aligned}$$

This expression is clearly symmetric in ϕ and ψ.

Remark 8 *In the following, we will denote by $C, (C_i)_{i\in\mathbb{N}}, \widehat{C}$ finite and positive constants.*

Convergence of Dynkin's martingale

Here we state and prove some convergence properties of the family of martingales $\left(M^{K,t}_{a,\phi}\right)_{a\in\{1,\dots,N-1\}}$ and $\left(\mathcal{N}^{K,t}_{a,b,\phi}\right)_{a,b\in\{1,\dots,N-1\}}$ when $K\to\infty$. We formulate this in Proposition 9. This result will be useful in the proof of tightness and uniqueness of the limit point of the sequence of measures $(Q_K)_{K\in\mathbb{N}}$.

Proposition 9 *For all $\phi\in C^\infty_c(\mathbb{R})$ and $\forall t\in[0,T]$ we have the following convergences:*

1. $\forall a \in \{1, \ldots, N-1\}$

$$\lim_{K\to\infty} \mathbb{E}\left[\left(M^{K,t}_{a,\phi} - Y^{K,t}_a(\phi) + Y^{K,0}_a(\phi) + \nu \int_0^t Y^{K,s/K^2}_a(\Delta\phi)ds\right)^2\right] = 0\,, \tag{5.4.9}$$

2. $\forall a, b \in \{1, \ldots, N-1\}$

$$\begin{aligned}\lim_{K\to\infty} \mathbb{E}\Bigg[\Bigg(\mathcal{N}^{K,t}_{a,b,\phi} - &\left(Y^{K,t}_a(\phi) - Y^{K,0}_a(\phi) - \nu \int_0^t Y^{K,s/K^2}_a(\Delta\phi)ds\right) \\ \times &\left(Y^{K,t}_b(\phi) - Y^{K,0}_b(\phi) - \nu \int_0^t Y^{K,s/K^2}_b(\Delta\phi)ds\right) + 2t\nu^2 p_a p_b \int_{\mathbb{R}} \nabla(\phi(u))^2 du\Bigg)^2\Bigg] = 0\end{aligned} \tag{5.4.10}$$

when $a \neq b$ and

$$\begin{aligned}\lim_{K\to\infty} \mathbb{E}\Bigg[\Bigg(\mathcal{N}^{K,t}_{a,a,\phi} - &\left(Y^{K,t}_a(\phi) - Y^{K,0}_a(\phi) - \nu \int_0^t Y^{K,s/K^2}_a(\Delta\phi)ds\right)^2 \\ - &\ 2t\nu^2 p_a(1-p_a) \int_{\mathbb{R}} (\nabla\phi(u))^2 dx\Bigg)^2\Bigg] = 0\end{aligned} \tag{5.4.11}$$

when $a = b$.

To prove Proposition 9 we need two intermediate results that we state in Lemma 6 and in Lemma 7.

Lemma 6 *For all $\phi \in C^\infty_c(\mathbb{R})$, for all $a, b \in \{1, \ldots, N-1\}$ we have*

$$\lim_{K\to\infty} \mathbb{E}\left[\left(K^2\Gamma^\phi_{a,b} + 2\nu^2 p_a p_b \int_{\mathbb{R}} (\nabla\phi(u))^2 du\right)^2\right] = 0 \qquad \textit{for} \quad a \neq b \tag{5.4.12}$$

$$\lim_{K\to\infty} \mathbb{E}\left[\left(K^2\Gamma^\phi_{a,a} - 2\nu^2 p_a(1-p_a) \int_{\mathbb{R}} (\nabla\phi(u))^2 du\right)^2\right] = 0 \qquad \textit{for} \quad a = b\,. \tag{5.4.13}$$

Proof: We will only prove (5.4.12), since the proof of (5.4.13) is similar. $L^2(\mu_{\text{rev}})$ convergence (5.4.12) is equivalent to showing the following $L^1(\mu_{\text{rev}})$ convergence

$$\lim_{K\to\infty} K^2\mathbb{E}\left[\Gamma^\phi_{a,b}\right] = -2\nu^2 p_a p_b \int_{\mathbb{R}} (\nabla\phi(u))^2 du \tag{5.4.14}$$

and a vanishing variance

$$\lim_{K\to\infty} \text{Var}(K^2\Gamma^\phi_{a,b}) = 0. \tag{5.4.15}$$

We start by proving (5.4.14). Using (5.4.6) we write

$$\begin{aligned}\Gamma^\phi_{a,b} = \frac{1}{K}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N} n^x_A n^{x+1}_B &\left[\sum_{y\in\mathbb{Z}} \phi\left(\frac{y}{K}\right)((n^y_a - \delta^x_A + \delta^x_B + \delta^x_A - \delta^x_B) - n^y_a)\right] \\ \times &\left[\sum_{z\in\mathbb{Z}} \phi\left(\frac{z}{K}\right)((n^z_b - \delta^x_A + \delta^x_B + \delta^x_A - \delta^x_B) - n^z_b)\right]\end{aligned}$$

$$= -\frac{1}{K}\sum_{x\in\mathbb{Z}}\left(n_a^x n_b^{x+1} + n_b^x n_a^{x+1}\right)\left(\phi\left(\frac{x+1}{K}\right) - \phi\left(\frac{x}{K}\right)\right)^2 .$$

By the Taylor's formula with the Lagrange remainder we have

$$\left(\phi\left(\frac{x+1}{K}\right) - \phi\left(\frac{x}{K}\right)\right)^2 = \frac{1}{K^2}\nabla\phi\left(\frac{x}{K}\right)^2 + \frac{1}{K^4}\frac{1}{4}\left(\Delta\phi\left(\frac{x+\theta^+}{K}\right)\right)^2$$
$$+ \frac{1}{K^3}\frac{1}{2}\left(\nabla\phi\left(\frac{x}{K}\right)\Delta\phi\left(\frac{x+\theta^+}{K}\right) + \nabla\phi\left(\frac{x}{K}\right)\Delta\phi\left(\frac{x+\theta^+}{K}\right)\right) \tag{5.4.16}$$

where $\theta^+ \in (0,1)$. We thus obtain

$$K^2\Gamma^{\phi}_{a,b} = -\frac{1}{K}\sum_{x\in\mathbb{Z}}\left(n_a^x n_b^{x+1} + n_b^x n_a^{x+1}\right)\nabla\phi\left(\frac{x}{K}\right)^2 + o\left(\frac{1}{K}\right). \tag{5.4.17}$$

Therefore

$$\lim_{K\to\infty} K^2\mathbb{E}\left[\Gamma^{\phi}_{a,b}\right] = \lim_{K\to\infty}\left[-\frac{1}{K}\sum_{x\in\mathbb{Z}}\mathbb{E}\left[n_a^x n_b^{x+1} + n_a^{x+1} n_b^x\right]\nabla\phi\left(\frac{x}{K}\right)^2\right] = -2\nu^2 p_a p_b\int_{\mathbb{R}}(\nabla\phi(u))^2 du \tag{5.4.18}$$

and (5.4.14) is proved. To prove (5.4.15) we need the second moment. We have

$$\begin{aligned}
&\mathbb{E}\left[\left(K^2\Gamma^{\phi}_{a,b}\right)^2\right]\\
=&\frac{1}{K^2}\sum_{x,y\in\mathbb{Z}}\nabla\phi\left(\frac{x}{K}\right)^2\nabla\phi\left(\frac{y}{K}\right)^2\mathbb{E}\left[(n_a^x n_b^{x+1} + n_b^x n_a^{x+1})(n_a^y n_b^{y+1} + n_b^y n_a^{y+1})\right] + o\left(\frac{1}{K^2}\right)\\
=&4\nu^4 p_a^2 p_b^2\frac{1}{K^2}\sum_{x,y\in\mathbb{Z}}\nabla\phi\left(\frac{x}{K}\right)^2\nabla\phi\left(\frac{y}{K}\right)^2 + o\left(\frac{1}{K^2}\right).
\end{aligned}$$

By taking the limit

$$\lim_{K\to\infty}\mathbb{E}\left[\left(K^2\Gamma^{\phi}_{a,b}\right)^2\right] = 4\nu^4 p_a^2 p_b^2\left(\int_{\mathbb{R}}(\nabla\phi(u))^2\,du\right)^2 .$$

Therefore, using (5.4.18), we have

$$\lim_{K\to\infty}\mathrm{Var}\left(K^2\Gamma^{\phi}_{a,b}\right) = \lim_{K\to\infty}\mathbb{E}\left[\left(K^2\Gamma^{\phi}_{a,b}\right)^2\right] - \lim_{K\to\infty}\left(\mathbb{E}\left[K^2\Gamma^{\phi}_{a,b}\right]\right)^2 = 0\,. \tag{5.4.19}$$

□

Lemma 7 *For all $\phi \in C_c^\infty(\mathbb{R})$, for all $a,b \in \{1,\ldots,N-1\}$ and for all $t \in [0,T]$ we have*

$$\begin{aligned}
\lim_{K\to\infty}\mathbb{E}\Bigg[\Bigg\{&M^{K,t}_{a,\phi}M^{K,t}_{b,\phi} - \left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)\\
&\times\left(Y_b^{K,t}(\phi) - Y_b^{K,0}(\phi) - \nu\int_0^t Y_b^{K,s/K^2}(\Delta\phi)ds\right)\Bigg\}^2\Bigg] = 0 \qquad \textit{for} \quad a \neq b\,,
\end{aligned} \tag{5.4.20}$$

$$\lim_{K\to\infty}\mathbb{E}\left[\left\{(M_{a,\phi}^{K,t})^2-\left(Y_a^{K,t}(\phi)-Y_a^{K,0}(\phi)-\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2\right\}^2\right]=0 \qquad \textit{for} \quad a=b\,. \tag{5.4.21}$$

Proof: We prove only the convergence (5.4.21) since (5.4.20) can be proved similarly. By Cauchy-Schwartz inequality

$$\begin{aligned}
&\mathbb{E}\left[\left((M_{a,\phi}^{K,t})^2-\left(Y_a^{K,t}(\phi)-Y_a^{K,0}(\phi)-\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2\right)^2\right]\\
&\leqslant\left(\underbrace{\mathbb{E}\left[\left((M_{a,\phi}^{K,t})-\left(Y_a^{K,t}(\phi)-Y_a^{K,0}(\phi)-\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)\right)^4\right]}_{A_K}\right.\\
&\left.\times\underbrace{\mathbb{E}\left[\left((M_{a,\phi}^{K,t})+\left(Y_a^{K,t}(\phi)-Y_a^{K,0}(\phi)-\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)\right)^4\right]}_{B_K}\right)^{1/2}.
\end{aligned}$$

We will prove that the term denoted by A_K goes to zero when $K\to\infty$ while the term B_K remains finite.

Proof that $\lim_{K\to\infty}A_K=0$**:** we first compute the action of the generator on the fluctuation field:

$$\begin{aligned}
\mathcal{L}Y_a^{K,\cdot}(\phi)&=\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N,\cdot}n_A^x n_B^{x+1}\left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n_a^y-\delta_A^x+\delta_B^x+\delta_A^{x+1}-\delta_B^{x+1}-\nu p_a)-n_a^y+\nu p_a\right)\right]\\
&=\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}\sum_{A,B=1}^{N}n_A^x n_B^{x+1}\left[\phi\left(\frac{x+1}{K}\right)\left((n_a^{x+1}+\delta_A^{x+1}-\delta_B^{x+1})-n_a^{x+1}\right)\right.\\
&\left.+\phi\left(\frac{x}{K}\right)\left((n_a^x-\delta_A^x+\delta_B^x)-n_a^x\right)\right]\\
&=\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}\left\{n_a^x\sum_{B=1\,:\,B\neq a}^{N}n_B^{x+1}\left[\phi\left(\frac{x+1}{K}\right)-\phi\left(\frac{x}{K}\right)\right]\right.\\
&\left.+n_a^{x+1}\sum_{A=1\,:\,A\neq a}^{N}n_A^x\left[\phi\left(\frac{x}{K}\right)-\phi\left(\frac{x+1}{K}\right)\right]\right\}\\
&=\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}\left\{n_a^x(\nu-n_a^{x+1})\left[\phi\left(\frac{x+1}{K}\right)-\phi\left(\frac{x}{K}\right)\right]\right.\\
&\left.+n_a^{x+1}(\nu-n_a^x)\left[\phi\left(\frac{x}{K}\right)-\phi\left(\frac{x+1}{K}\right)\right]\right\}\\
&=\frac{\nu}{\sqrt{K}}\sum_{x\in\mathbb{Z}}n_a^x\left[\phi\left(\frac{x-1}{K}\right)+\phi\left(\frac{x+1}{K}\right)-2\phi\left(\frac{x}{K}\right)\right].
\end{aligned}$$

Using Taylor's series with Lagrange remainder implies

$$\phi(\frac{x+1}{K})+\phi(\frac{x-1}{K})-2\phi(\frac{x}{K}) = \frac{1}{K^2}\Delta\phi(\frac{x}{K})+\frac{1}{6}\frac{1}{K^3}\left[\phi^{(3)}(\frac{x+\theta^+}{K})-\phi^{(3)}(\frac{x-\theta^-}{K})\right] \quad (5.4.22)$$

where $\theta^+, \theta^- \in (0,1)$ and where we denoted by $\phi^{(3)}$ the third derivative of $\phi(x)$. Observing further that

$$\sum_{x\in\mathbb{Z}} \nu p_a \left[\phi\left(\frac{x-1}{K}\right)+\phi\left(\frac{x+1}{K}\right)-2\phi\left(\frac{x}{K}\right)\right]=0\,, \quad (5.4.23)$$

we obtain

$$K^2\mathcal{L}Y_a^{K,\cdot}(\phi)=\frac{\nu}{\sqrt{K}}\sum_{x\in\mathbb{Z}}(n_a^x-\nu p_a)\Delta\phi(\frac{x}{K})+R_1(\phi,a,\cdot) \quad (5.4.24)$$

where

$$R_1(\phi,a,\cdot)=\frac{\nu}{K^{3/2}}\sum_{x\in\mathbb{Z}} n_a^x\left[\frac{1}{6}\left[\phi^{(3)}(\frac{x+\theta^+}{K})-\phi^{(3)}(\frac{x-\theta^-}{K})\right]\right]. \quad (5.4.25)$$

Therefore, we find an upper bound for A_K

$$\begin{aligned}\mathbb{E}\left[\left(M_{a,\phi}^{K,t}-\left(Y_a^{K,t}(\phi)-Y_a^{K,0}(\phi)-\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)\right)^4\right] &= \mathbb{E}\left[\left(\int_0^t R_1(\phi,a,s)ds\right)^4\right] \\ &\leqslant C\int_0^T \mathbb{E}\left[R_1(\phi,a,s)^4\right]ds\end{aligned} \quad (5.4.26)$$

where in the last inequality we used Fubini's Theorem and Holder's inequality with coefficients 4 and 4/3. The set $\cup_{k=0}^2 \text{supp}\left(\frac{d^k}{dx^k}\phi\right)$ is compact in $\mathbb{R}$. We call

$$\mathcal{A}:=N\left(\cup_{k=0}^2\text{supp}\left(\frac{d^k}{dx^k}\phi\right)\right)\cap\mathbb{Z}. \quad (5.4.27)$$

Then, we bound from above the expectation in the integral as follows

$$\mathbb{E}\left[R_1(\phi,a,\cdot)^4\right]\leqslant\frac{1}{K^6}\sum_{x_1,x_2,x_3,x_4\in\mathcal{A}}\mathbb{E}\left[\prod_{i=1}^4(n_a^{x_i}-\nu p_a)\right]\|\Delta\phi\|_\infty.$$

The the only terms that survive in the average are:

$$(n_a^{x_i}-\nu p_a)^2(n_a^{x_j}-\nu p_a)^2 \qquad (n_a^{x_i}-\nu p_a)^4 \qquad \forall i,j\in\{1,2,3,4\}\;:\;i\neq j.$$

The moment generating function of a Multinomial$(\nu, p_1, \dots p_N)$ vector $(X_1,\dots,X_N)$ is

$$M(t)=\mathbb{E}\left[\prod_{l=1}^N e^{X_l t_l}\right]=\left(\sum_{l=1}^N p_l e^{t_l}\right)^\nu.$$

We can compute explicitly

$$\begin{aligned}\mathbb{E}\left[(n_a^x-\nu p_a)^4\right]&=f(p_a,4)\,,\\ \mathbb{E}[(n_a^x-\nu p_a)^2(n_a^y-\nu p_a)^2]&=g(p_a,4)\end{aligned}$$

where $f(p_a,4), g(p_a,4)$ are polynomials of fourth order in p_a and bounded from above by a proper finite and positive constant. The measure of the set $\mathcal{A}$ is bounded by $|\mathcal{A}| \leqslant CK$. By consequence

$$\sum_{x_1,x_2,x_3,x_4\in\mathcal{A}} \mathbb{E}\left[\prod_{i=1}^{4}(n_a^{x_i} - \nu p_a)\right] = \sum_{x\in\mathcal{A}} f(p_a,4) + \sum_{x,y\in\mathcal{A}} g(p_a,4) \leqslant K^2 C\,.$$

Therefore

$$\mathbb{E}\left[R_1(\phi,a,\cdot)^4\right] \leqslant \frac{K^2}{K^6} C\|\Delta\phi\|_\infty.$$

and by taking the limit

$$\lim_{K\to\infty} \mathbb{E}\left[R_1(\phi,a,\cdot)^4\right] = 0\,. \tag{5.4.28}$$

Recalling (5.4.26) this implies that $\lim_{K\to\infty} A_K = 0$.

Proof that $\lim_{K\to\infty} B_K < \infty$**:** for any real numbers $q,p \in \mathbb{R}$,

$$(q+p)^4 \leqslant 8(q^4+p^4). \tag{5.4.29}$$

Applying this inequality

$$\begin{aligned}&\mathbb{E}\left[\left(M_{a,\phi}^{K,t} + Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right]\\ &\leqslant 8\left(\mathbb{E}\left[\left(M_{a,\phi}^{K,t}\right)^4\right] + \mathbb{E}\left[\left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right]\right).\end{aligned}$$

Applying again inequality (5.4.29) we have

$$\begin{aligned}\mathbb{E}\left[\left(M_{a,\phi}^{K,t}\right)^4\right] &\leqslant C\left(\mathbb{E}\left[Y_a^{K,t}(\phi)^4\right] + \mathbb{E}\left[Y_a^{K,0}(\phi)^4\right]\right.\\ &\quad\left. + \mathbb{E}\left[\left(\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right] + \mathbb{E}\left[\left(\int_0^t R_1(\phi,a,s)ds\right)^4\right]\right)\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right] &\leqslant \hat{C}\left(\mathbb{E}\left[Y_a^{K,t}(\phi)^4\right] + \mathbb{E}\left[Y_a^{0,N}(\phi)^4\right]\right.\\ &\quad\left. + \mathbb{E}\left[\left(\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right]\right).\end{aligned}$$

Arguing similarly to before we find

$$\begin{aligned}\mathbb{E}\left[Y_a^{K,\cdot}(\phi)^4\right] &= \frac{1}{K^2}\sum_{x_1,x_2,x_3,x_4\in\mathcal{A}} \mathbb{E}\left[\prod_{i=1}^{4}(n_a^{x_i} - \nu p_a)\right]\prod_{i=1}^{4}\phi\left(\frac{x_i}{K}\right)\\ &\leqslant \frac{C}{K^2}\|\phi\|_\infty\left(\sum_{x\in\mathcal{A}} f(p_a,4) + \sum_{x,y\in\mathcal{A}} g(p_a,4)\right) < \infty\end{aligned} \tag{5.4.30}$$

then, by taking the limit

$$\lim_{K\to\infty} \mathbb{E}\left[Y_a^{K,\cdot}(\phi)^4\right] \leqslant C_1. \tag{5.4.31}$$

Obviously, the same bound holds for $\mathbb{E}\left[Y_a^{K,0}(\phi)^4\right]$. We can argue similarly and find the following upper bound for the integral term

$$\mathbb{E}\left[\left(\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right] \leqslant C\int_0^T \mathbb{E}\left[Y_a^{K,s/K^2}(\nu\Delta\phi)^4\right]ds < \infty$$

then, in the limit

$$\lim_{K\to\infty} \mathbb{E}\left[\left(\nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^4\right] = C_2. \tag{5.4.32}$$

By putting together (5.4.28), (5.4.31) and (5.4.32) we obtain that B_K remains finite as $K \to \infty$.

□

Proof of Proposition 9: To prove (5.4.9) we have that, by the expressions (5.4.24), (5.4.25),

$$\begin{aligned} &\lim_{K\to\infty} \mathbb{E}\left[\left(M_{a,\phi}^{K,t} - Y_a^{K,t}(\phi) + Y_a^{K,0} + \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2\right] \\ \leqslant &C\lim_{K\to\infty}\int_0^t \mathbb{E}\left[R_1(\phi,a,s)^2\right]ds \leqslant \lim_{K\to\infty}\frac{C_1}{K} = 0. \end{aligned} \tag{5.4.33}$$

To prove (5.4.10) we only consider the case $a = b$, since the case $a \neq b$ is proved similarly. By the triangle inequality

$$\begin{aligned} &\mathbb{E}\Bigg[\Bigg(\mathcal{N}_{a,a,\phi}^{K,t} - \left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2 \\ &\qquad -2t\nu^2 p_a(1-p_a)\int_{\mathbb{R}}(\nabla\phi(u))^2du\Bigg)^2\Bigg] \\ \leqslant &\mathbb{E}\left[\left\{(M_{a,\phi}^{K,t})^2 - \left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2\right\}^2\right] \\ +&\mathbb{E}\left[\left(K^2\int_0^t \Gamma_{a,a}^{\phi;s}ds - 2t\nu^2 p_a(1-p_a)\int_{\mathbb{R}}(\nabla\phi(u))^2du\right)^2\right]. \end{aligned}$$

In the limit we apply Lemma 6 and Lemma 7 and we obtain

$$\begin{aligned} \lim_{K\to\infty}\mathbb{E}\Bigg[\Bigg(&\mathcal{N}_{a,a,\phi}^{K,t} - \left(Y_a^{K,t}(\phi) - Y_a^{K,0}(\phi) - \nu\int_0^t Y_a^{K,s/K^2}(\Delta\phi)ds\right)^2 \\ &+ 2t\nu^2 p_a(1-p_a)\int_{\mathbb{R}}(\nabla\phi(u))^2du\Bigg)^2\Bigg] = 0\,. \end{aligned} \tag{5.4.34}$$

□

5.4.2 Tightness

In this section we prove tightness for the sequence of probability measures $(Q_K)_{K\in\mathbb{N}}$ on the Skorokhod space (see [79] for details) of càdlàg trajectories $D\left([0,T],((C_c^\infty(\mathbb{R}))^*_{N-1}\right)$. A necessary and sufficient condition for tightness is given by the following Theorem proved by Aldous [78].

Theorem 16 (Aldous' criterion) *Consider a Polish space $\mathcal{E}$, endowed with a metric $d_{\mathcal{E}}(\cdot,\cdot)$ where we denote by μ_t the functions from $[0,T]$ to $\mathcal{E}$. A sequence of probability measures $(P_K)_{K\in\mathbb{N}}$ on the Skorokhod space $D([0,T],\mathcal{E})$ is tight if and only if*

1. *$\forall t\in[0,T]$ and $\forall\epsilon>0$ $\exists\mathcal{K}(t,\epsilon)\subset\mathcal{E}$ compact such that*

$$\sup_{K\in\mathbb{N}} P_K\left(\mu_t\notin\mathcal{K}(t,\epsilon)\right)\leqslant\epsilon\,. \tag{5.4.35}$$

2. *$\forall\epsilon>0$*

$$\lim_{\delta\to0}\limsup_{K\to\infty}\sup_{\tau\in\mathcal{T}_T,\,\theta\leqslant\delta} P_K\left(d_{\mathcal{E}}(\mu_\tau,\mu_{\tau+\theta})>\epsilon\right)=0 \tag{5.4.36}$$

where $\mathcal{T}_T$ is a family of stopping times bounded by T.

In Proposition 10 we will apply Theorem 16 to prove tightness of the sequence of measure $(Q_K)_{K\in\mathbb{N}}$. The computation can be done on the Skorokhod space $D([0,T],\mathbb{R}^{N-1})$. Indeed, $C_c^\infty(\mathbb{R})$ is a nuclear space (see [80] for details), then it suffice to prove tightness of the distribution of $Q_K\circ\boldsymbol{\phi}$ with $\boldsymbol{\phi}=(\phi,\ldots,\phi)$ vector of size $(N-1)$ and for arbitrary $\phi\in C_c^\infty(\mathbb{R})$.

Proposition 10 *The sequence of measure $(Q_K)_{K\in\mathbb{N}}$ on the space $D\left([0,T],(C_c^\infty(\mathbb{R}))^*_{(N-1)}\right)$ is tight since the following statements are true for any $\boldsymbol{\phi}\in(C_c^\infty(\mathbb{R}))^{N-1}$:*

1. *$\forall t\in[0,T]$ and $\epsilon>0$ there exists a compact set $\mathcal{K}(t,\epsilon)\in\mathbb{R}^{N-1}$ such that*

$$\sup_{K\in\mathbb{N}} Q_K\left(Y^{K,t}(\boldsymbol{\phi})\notin\mathcal{K}(t,\epsilon)\right)\leqslant\epsilon\,. \tag{5.4.37}$$

2. *$\forall\epsilon>0$*

$$\lim_{\delta\to0}\limsup_{K\to\infty}\sup_{\tau\in\mathcal{T}_T,\,\theta\leqslant\delta} Q_K\left(\|Y^{K,\tau}(\boldsymbol{\phi})-Y^{K,\tau+\theta}(\boldsymbol{\phi})\|_S>\epsilon\right)=0 \tag{5.4.38}$$

where $\|Y^{K,t}(\boldsymbol{\phi})\|_S=\max_{a\in\{1,\ldots,N-1\}}\left\{|Y_a^{K,t}(\phi)|\right\}$ and $\mathcal{T}_T$ is a family of stopping times bounded by T.

Proof. We show that the (5.4.37) and (5.4.38) are satisfied.

Proof of (5.4.37)**:** we fix arbitrary $t\in[0,T]$ and $\epsilon>0$. We apply the central limit theorem for the $(N-1)$-dimensional random vector $Y^{K,t}(\boldsymbol{\phi})$ taking values on $\mathbb{R}^{N-1}$, observing that the process $(\boldsymbol{n}(t))_{t\geqslant0}$ has a product invariant distribution given by (4.2.19). To do this we need the expectation and the covariances under Q_K of the equilibrium fluctuation field. We fix arbitrary $a,b\in\{1,\ldots,N-1\}$. We have

$$\mathbb{E}\left(Y_a^{K,t}(\phi)\right)=\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}\mathbb{E}\left[n_a^x(tK^2)-\nu p_a\right]\phi\left(\frac{x}{K}\right)=0$$

and

$$\mathrm{Var}\left(Y_a^{K,t}(\phi)\right) = \frac{1}{K}\sum_{x\in\mathbb{Z}} \phi^2\left(\frac{x}{K}\right)\mathbb{E}\left[(n_a^x(tK^2))^2\right]$$
$$\mathrm{Cov}(Y_a^{K,t}(\phi), Y_b^{K,t}(\phi)) = \frac{1}{K}\sum_{x\in\mathbb{Z}} \phi^2\left(\frac{x}{K}\right)\mathrm{Cov}\left(n_a^x(tK^2), n_b^x(tK^2)\right).$$

Taking the limit we obtain

$$\lim_{K\to\infty} \mathbb{E}\left(Y_a^{K,t}(\phi)\right) = 0, \quad \lim_{K\to\infty} \mathrm{Var}\left(Y_a^{K,t}(\phi)\right) = \nu p_a(1-p_a)\int_{\mathbb{R}} (\phi(u))^2 du$$
$$\lim_{K\to\infty} \mathrm{Cov}(Y_a^{K,t}(\phi), Y_b^{K,t}(\phi)) = -\nu p_a p_b \int_{\mathbb{R}} (\phi(u))^2 du$$

Therefore, the random vector $Y^{K,t}$ converges in distribution to a centered Gaussian random vector with covariance matrix $\boldsymbol{K}$ with elements

$$\boldsymbol{K}_{a,b} = -\nu p_a p_b \int_{\mathbb{R}} (\phi(u))^2 du, \qquad \boldsymbol{K}_{a,a} = \nu p_a(1-p_a)\int_{\mathbb{R}} (\phi(u))^2 du. \tag{5.4.39}$$

Thus for arbitrary $\epsilon > 0$ and $\forall t \in [0,T]$ we can choose $\mathcal{K}(t,\epsilon) \subset \mathbb{R}^{N-1}$ compact, such that

$$\sup_{K\in\mathbb{N}} Q_K\left(Y^{K,t}(\boldsymbol{\phi}) \notin \mathcal{K}(t,\epsilon)\right) \leqslant \epsilon.$$

Proof of (5.4.38)**:** without loss of generality and for the sake of notation, here we will work with a single species $a \in \{1,\dots,N-1\}$. For arbitrary stopping time $\tau \in \mathcal{T}$, We write the process

$$Y_a^{K,\tau}(\phi) = M_{a,\phi}^{K,\tau} + Y_a^{K,0}(\phi) + \int_0^\tau K^2 \mathcal{L} Y_a^{K,s/K^2}(\phi) ds.$$

By Chebyshev and triangular inequality

$$Q_K\left(|Y_a^{K,\tau}(\phi) - Y_a^{K,\tau+\theta}(\phi)| \geqslant \epsilon\right) \leqslant \frac{1}{\epsilon^2}\mathbb{E}\left[\left(Y_a^{K,\tau}(\phi) - Y_a^{K,\tau+\theta}(\phi)\right)^2\right]$$
$$\leqslant \frac{2}{\epsilon^2}\left(\underbrace{\mathbb{E}\left[\left(M_{a,\phi}^{K,\tau} - M_{a,\phi}^{K,\tau+\theta}\right)^2\right]}_{A_K} + \underbrace{\mathbb{E}\left[\left(\int_\tau^{\tau+\theta} K^2 \mathcal{L} Y_a^{K,s/K^2}(\phi) ds\right)^2\right]}_{B_K}\right)$$

We first prove that A_K goes to zero when $K \to \infty$. By the martingale property we have

$$\mathbb{E}\left[\left(M_{a,\phi}^{K,\tau} - M_{a,\phi}^{K,\tau+\theta}\right)^2\right] = \mathbb{E}\left[\left(M_{a,\phi}^{K,\tau+\theta}\right)^2 - \left(M_{a,\phi}^{K,\tau}\right)^2\right].$$

By Doob's decomposition theorem (see [46])

$$\mathbb{E}\left[\left(M_{a,\phi}^{K,t}\right)^2\right] = \mathbb{E}\left[\int_0^t K^2 \Gamma_{a,a}^{\phi,s/K^2} ds\right].$$

We write the following chain of inequalities by using Fubini theorem, Cauchy-Schwartz inequality, optional stopping theorem for martingales (see [46]) and the fact that, by Lemma 6, the sequence $K^2\Gamma_{a,a}^{\phi,s/K^2}$ is uniformly bounded in K in $L^2(\mu_{\text{rev}})$

$$\begin{aligned}\sup_{K\in\mathbb{N}}\mathbb{E}\left[\left(M_{a,\phi}^{K,\tau+\theta}\right)^2-\left(M_{a,\phi}^{K,\tau}\right)^2\right]&=\sup_{K\in\mathbb{N}}\mathbb{E}\left[\int_\tau^{\tau+\theta}K^2\Gamma_{a,a}^{\phi,s/K^2}ds\right]\\ \leqslant&\sqrt{\theta}\left(\int_0^T\sup_{K\in\mathbb{N}}\mathbb{E}\left[\left(K^2\Gamma_{a,a}^{\phi,s/K^2}\right)^2ds\right]\right)^{1/2}\leqslant\sqrt{\theta}C.\end{aligned}$$

By taking the limits and by the above upperbound we have

$$\lim_{\delta\to0}\limsup_{K\to\infty}\sup_{\tau\in\mathcal{T}_T,\,\theta\leqslant\delta}A_K\leqslant\lim_{\delta\to0}\limsup_{K\to\infty}\sup_{\tau\in\mathcal{T}_T,\,\theta\leqslant\delta}\sqrt{\theta}C=0$$

then, A_K goes to zero as $K\to\infty$.
Secondly, we prove that B_K vanishes when $K\to\infty$. By Fubini theorem and Cauchy-Schwarz inequality

$$\mathbb{E}\left[\left(\int_\tau^{\tau+\theta}K^2\mathcal{L}Y_a^{K,s/K^2}(\phi)ds\right)^2\right]\leqslant\sqrt{\theta}\left(\int_0^{T+\theta}\mathbb{E}\left[\left(K^2\mathcal{L}Y_a^{K,s/K^2}(\phi)\right)^2\right]ds\right)^{1/2}.$$

The integrand can be bounded from above as follows

$$\begin{aligned}\mathbb{E}\left[\left(K^2\mathcal{L}Y_a^{K,s}(\phi)\right)^2\right]=&\mathbb{E}\left[\left(\frac{1}{\sqrt{K}}\sum_{x\in\mathbb{Z}}(n_a^x-\nu p_a)\Delta_K\phi\left(\frac{x}{K}\right)\right)^2\right]\\ \leqslant&\frac{C}{N^3}\|\Delta\phi\|_\infty\sum_{x\in\mathcal{A}}\mathbb{E}\left[(n_a^x-\nu p_a)^2\right]\end{aligned}$$

where Δ_K denotes the discrete Laplacian with spacing $1/K$ and $\mathcal{A}$ is the set defined in (5.4.27). Therefore, arguing as in the proof of Lemma 7 and by taking the limits we have

$$\lim_{\delta\to0}\limsup_{K\to\infty}\sup_{\tau\in\mathcal{T}_T,\,\theta\leqslant\delta}\mathbb{E}\left[\left(\int_\tau^{\tau+\theta}K^2\mathcal{L}Y_a^{K,s/K^2}(\phi)ds\right)^2\right]\leqslant\lim_{\delta\to0}\sqrt{\delta}C_1=0.$$

Thus B_K vanishes as $K\to\infty$. This concludes the proof of tightness of the sequence $(Q_K)_{K\in\mathbb{N}}$.

□

5.4.3 The covariances of the limiting process

In this section we compute the covariance of the limiting process, using duality. As a corollary this gives these covariances at the initial time $t=0$, needed for the proof of Theorem 15. To use duality in this context, we adapt the results of Theorem 17 that will be rigorously proved in Chapter 6. On the infinite lattice $\mathbb{Z}$ (and thus without boundary driving) the multi-species stirring process is self-dual with duality function

$$D(n,\xi)=\prod_{x\in\mathbb{Z}}\left(\frac{(\nu-\sum_{a=1}^{N-1}\xi_a^x)!}{\nu!}\prod_{a=1}^{N-1}\frac{n_a^x!}{(n_a^x-\xi_a^x)!}\right).\tag{5.4.40}$$

where we denote by $(\boldsymbol{\xi}_t)_{t\geqslant 0}$ the (self-)dual process. The following proposition shows that the covariances (5.3.13) and (5.3.14) of the limiting process can be computed via the single-particle self-duality. Notice that because the limiting process is Gaussian, these covariances uniquely determine the process.

Proposition 11 *The covariances of the limiting process $(Y_1^t, \dots, Y_{N-1}^t)$ are:*

$$Cov\left(Y_a^t(\phi), Y_b^0(\psi)\right) = -\nu p_a p_b \langle S_t\phi, \psi\rangle_{L^2(dx)} \qquad a \neq b, \tag{5.4.41}$$

$$Cov\left(Y_a^t(\phi), Y_a^0(\psi)\right) = \nu p_a(1-p_a) \langle S_t\phi, \psi\rangle_{L^2(dx)} \qquad a = b. \tag{5.4.42}$$

where $(S_t)_{t\geqslant 0}$ is the transition semigroup of the Brownian motion with variance νt.

Proof: By the self-duality, the dual process initialized with one particle behaves as an independent random walker (IRW) jumping at rate ν on $\mathbb{Z}$. Thus the following computation holds for $a \neq b$:

$$\begin{aligned}
\mathbb{E}\left[Y_a^{K,t}(\phi), Y_b^{K,0}(\psi)\right] &= \frac{1}{K}\sum_{x,y\in\mathbb{Z}} \phi\left(\frac{x}{K}\right)\psi\left(\frac{y}{K}\right)\mathbb{E}\left[(n_a^x(tK^2)-\nu p_a)(n_b^y-\nu p_b)\right] \\
&= \frac{1}{K}\sum_{x,y\in\mathbb{Z}} \int_\Omega \mu_{\mathrm{rev}}(d\boldsymbol{n})\mathbb{E}^{\boldsymbol{n}}\left[(n_a^x(tK^2)-\nu p_a)\right](n_b^y-\nu p_b)\phi\left(\frac{x}{K}\right)\psi\left(\frac{y}{K}\right) \\
&= \frac{1}{K}\sum_{x,y\in\mathbb{Z}} \int_\Omega \mu_{\mathrm{rev}}(dn)(n_b^y-\nu p_b) \\
&\times \sum_{z\in\mathbb{Z}} p_{tK^2}^{IRW}(x,z)(n_a^z(tK^2)-\nu p_a)\phi\left(\frac{x}{K}\right)\psi\left(\frac{y}{K}\right) \\
&= \frac{1}{K}\sum_{x,y,z\in\mathbb{Z}} \mathrm{Cov}(n_a^z, n_b^y) p_{tK^2}^{IRW}(x,z)\phi\left(\frac{x}{K}\right)\psi\left(\frac{y}{K}\right) \\
&= -\nu p_a p_b \frac{1}{K}\sum_{x,y\in\mathbb{Z}} p_{tK^2}^{IRW}(x,y)\phi\left(\frac{x}{K}\right)\psi\left(\frac{y}{K}\right)
\end{aligned}$$

where we denoted by $\mathbb{E}^{\boldsymbol{n}}$ the average with respect to the reversible measure starting from a configuration $\boldsymbol{n}$ and we denoted $p_t^{IRW}(\cdot,\cdot)$ the transition kernel of the IRW jumping at rate ν. By taking the limit on both sides and by the invariance principle we have

$$\lim_{K\to\infty}\mathbb{E}\left[Y_a^{K,t}(\phi), Y_b^{K,0}(\psi)\right] = -\nu p_a p_b \langle S_t\phi, \psi\rangle_{L^2(dx)}. \tag{5.4.43}$$

For the case $a = b$ the proof is similar.

□

We find the covariances of the process at the initial time $t = 0$ by using the property of the transition semigroup of the Brownian motion (see Theorem 6). This is reported in the following corollary.

Corollary 1 *The covariance of the limiting process $(Y_1^0, \dots, Y_{N-1}^0)$ at time $t = 0$ are:*

$$Cov\left(Y_a^0(\phi), Y_b^0(\psi)\right) = -\nu p_a p_b \langle \phi, \psi\rangle_{L^2(dx)} \qquad a \neq b \tag{5.4.44}$$

$$Cov\left(Y_a^0(\phi), Y_a^0(\psi)\right) = \nu p_a(1-p_a) \langle \phi, \psi\rangle_{L^2(dx)} \qquad a = b. \tag{5.4.45}$$

Proof: the proof is straightforward from the properties of the semigroup $(S_t)_{t\geqslant 0}$ stated in Theorem 6 and by Proposition 11.

□

5.4.4 Uniqueness and continuity of the limit point

As shown in Section 5.4.2, the sequence of probability measures $(Q_K)_{K\in\mathbb{N}}$ giving the law of $(Y^{K,t})_{t\in[0,T]}$ is tight, then the Prokhorov's theorem [46] guarantees that every sub-sequence $(Q_{K_i})_{i\in\mathbb{N}}$ is convergent to a unique limit point that we denote by Q. It remains to prove that, $\forall a \in \{1,\ldots,N-1\}$, the limiting process $(Y_a^t)_{t\geqslant 0}$ has continuous trajectory (Q-almost surely) and that Q solves the martingale problem introduced in Theorem 15. The Q-a.s. continuity will be proved in Proposition 12, while the solution of the martingale problem will be proved in Proposition 13.

Proposition 12 *For every $T > 0$, $\phi \in C_c^\infty$ and $a \in \{1,\ldots,N-1\}$ the map $[0,T] \ni t \mapsto Y_a^t(\phi)$ is $Q-a.s.$ continuous.*

Proof. We prove that the set of discontinuity points of $Y_a^t(\phi)$ is negligible under Q. We introduce the usual modulus of continuity for any fixed $\delta > 0$:

$$\omega_\delta(Y_a(\phi)) := \sup_{|t-s|<\delta} |Y_a^t(\phi) - Y_a^s(\phi)| \tag{5.4.46}$$

and the modified uniform modulus of continuity

$$\omega_\delta^{'}(Y_a(\phi)) := \inf_{\{t_\ell\}_{0\leqslant\ell\leqslant r}} \max_{1\leqslant\ell\leqslant r} \sup_{t_{\ell-1}\leqslant s<t\leqslant t_\ell} |Y_a^t(\phi) - Y_a^s(\phi)| \tag{5.4.47}$$

where the first infimum is taken over all partitions $\{t_i, 0 \leqslant i \leqslant r\}$ of the interval $[0,T]$ such that

$$0 = t_0 < t_1 < \ldots < t_r = T \quad \text{with} \quad t_\ell - t_{\ell-1} \geqslant \delta \quad \text{for all } \ell = 1,\ldots,r.$$

They are related (see [78] for details) by the inequality

$$\omega_\delta(Y_a(\phi)) \leqslant 2\omega_\delta^{'}(Y_a(\phi)) + \sup_t |Y_a^t(\phi) - Y_a^{t-}(\phi)|. \tag{5.4.48}$$

Moreover, (see again [78]) it holds that for arbitrary $\epsilon > 0$

$$\lim_{\delta\to 0} \limsup_{i\to\infty} Q_{K_i}\left(w_\delta^{'}\left(Y_a^{K_i}(\phi)\right) \geqslant \epsilon\right) = 0.$$

Furthermore we have the upper bound

$$\sup_t |Y_a^{K_i,t}(\phi) - Y_a^{K_i,t-}(\phi)| \leqslant \frac{2\nu||\phi||_\infty}{\sqrt{K_i}}.$$

As a consequence of tightness we have that, for arbitrary $\epsilon > 0$

$$\lim_{\delta\to 0} Q\left(\omega_\delta(Y_a(\phi)) \geqslant \epsilon\right) = \lim_{\delta\to 0} \limsup_{i\to\infty} Q_{K_i}\left(\omega_\delta(Y_a^{K_i}(\phi)) \geqslant \epsilon\right) \tag{5.4.49}$$

therefore, by (5.4.48) we may write

$$\begin{aligned}\lim_{\delta\to 0} Q\left(\omega_\delta(Y_a(\phi)) \geqslant \epsilon\right) &\leqslant \lim_{\delta\to 0} \limsup_{i\to\infty} Q_{K_i}\left(\omega_\delta^{'}(Y_a^{K_i}(\phi)) \geqslant \frac{\epsilon}{2}\right) \\ &\quad + \lim_{\delta\to 0} \limsup_{i\to\infty} Q_{K_i}\left(\sup_t |Y_a^{K_i,t}(\phi) - Y_a^{K_i,t-}(\phi)| \geqslant \epsilon\right) \\ &= 0.\end{aligned} \tag{5.4.50}$$

Thus the almost sure continuity is proved.

□

Proposition 13 *For all $\phi \in C_c^\infty(\mathbb{R})$ and for all $a, b \in \{1, \dots, N-1\}$ the processes $(M^t_{a,\phi})_{t\in[0,T]}$ defined in (5.3.4) and $(\mathcal{N}^t_{a,b,\phi})_{t\in[0,T]}$, $(\mathcal{N}^t_{a,a,\phi})_{t\in[0,T]}$ defined in (5.3.7), (5.3.8) are martingales with respect to the natural filtration $\mathcal{F}_t := \sigma\left\{(Y_1^s, \dots, Y_{N-1}^s) \,:\, 0 \leqslant s \leqslant t \leqslant T\right\}$.*

Proof. The strategy of the proof is inspired by the proof of Proposition 2.3, Chapter 11 of [30] dealing with the mono-species zero-range process. The fundamental tools are the Portemanteau theorem and Proposition 9. We further remark that the trajectories of the process $(Y_a^{K,t})_{t\in[0,T]}$ are elements of the space $D([0,T], C_c^\infty(\mathbb{R}))^*$ that is not metric, then we cannot directly apply Portmanteau theorem. To overcome this issue, we adapt the strategy used in Section 5 of [81]. The complete proof is reported for the martingale $(M^t_{a,\phi})_{t\in[0,T]}$ while, concerning the martingales $(\mathcal{N}^t_{a,b,\phi})_{t\in[0,T]}$ and $(\mathcal{N}^t_{a,a,\phi})_{t\in[0,T]}$, we just give some estimates that allow to follow a similar strategy. Moreover, only the case $a \neq b$ is considered, since the case $a = b$ is similar.

Proof for $(M^t_{a,\phi})_{t\in[0,T]}$: The process $(M^t_{a,\phi})_{t\in[0,T]}$ defined in (5.3.4) is $\mathcal{F}_t-$measurable, therefore we only need to show that, for arbitrary $0 \leqslant s \leqslant t \leqslant T$

$$\mathbb{E}_Q\left[M^t_{a,\phi}|\mathcal{F}_s\right] = M^s_{a,\phi} \tag{5.4.51}$$

The property (5.4.51) is equivalent to showing that

$$\mathbb{E}_Q\left[M^t_{a,\phi}\mathcal{I}(Y)\right] = \mathbb{E}_Q\left[M^s_{a,\phi}\mathcal{I}(Y)\right]. \tag{5.4.52}$$

where the function $\mathcal{I}(Y)$ is defined as follows. We fix $m \in \mathbb{N}$ and we introduce the vectors $\boldsymbol{s} = (s_1, \dots, s_m)$ with $0 \leqslant s_1 \leqslant s_2 \leqslant \dots, \leqslant s_m \leqslant s$ and $\mathbf{H} = (H_1, \dots, H_m)$ with $H_1, \dots, H_m \in (C_c^\infty)^{N-1}$. For arbitrary $\Psi \in C_b(\mathbb{R}^m)$, we introduce the function from $\left(D\left([0,T], (C_c^\infty(\mathbb{R}))^*\right)\right)^m$ to $\mathbb{R}$

$$\mathcal{I}\left(Y^{K,\cdot}, \boldsymbol{H}, \boldsymbol{s}\right) := \Psi\left(Y^{K,s_1}(H_1), \dots, Y^{K,s_m}(H_m)\right). \tag{5.4.53}$$

For the sake of notation, we will denote this function with $\mathcal{I}(Y^K)$. Since $(M^{K,t}_{a,\phi})_{t\in[0,T]}$ defined in (5.4.2) is a martingale it holds that

$$\lim_{i\to\infty} \mathbb{E}_{Q_{K_i}}\left[M^{K_i,t}_{a,\phi}\mathcal{I}(Y^{K_i})\right] = \lim_{i\to\infty} \mathbb{E}_{Q_{K_i}}\left[M^{K_i,s}_{a,\phi}\mathcal{I}(Y^{K_i})\right] \tag{5.4.54}$$

therefore, to conclude (5.4.52) it is enough to show that

$$\lim_{i\to\infty} \mathbb{E}_{Q_{K_i}}\left[M^{K_i,t}_{a,\phi}\mathcal{I}(Y^{K_i})\right] = \mathbb{E}_Q\left[M^t_{a,\phi}\mathcal{I}(Y)\right]. \tag{5.4.55}$$

For arbitrary $\phi \in C_c^\infty(\mathbb{R})$ we introduce

$$\begin{aligned} \mathcal{M}_\phi \,&: D\left([0,T], (C_c^\infty(\mathbb{R})^*\right) \to D([0,T], \mathbb{R}) \\ &Y_a^\cdot \to \mathcal{M}_\phi(Y_a^\cdot) \;=\; Y_a^\cdot(\phi) - Y_a^\cdot(\phi) - \int_0^\cdot Y_a^q(\Delta\phi)dq. \end{aligned} \tag{5.4.56}$$

Observe that, for every $t \in [0,T]$

$$\mathcal{M}_\phi(Y_a^t) = M^t_{a,\phi}. \tag{5.4.57}$$

therefore, we need to show that

$$\lim_{i\to\infty} \mathbb{E}_{Q_{K_i}}\left[M^{K_i,t}_{a,\phi}\mathcal{I}(Y^{K_i})\right] = \mathbb{E}_Q\left[\mathcal{M}_\phi\left(Y_a^t\right)\mathcal{I}(Y)\right] \tag{5.4.58}$$

We prove this in two steps:

i)

$$\lim_{i\to\infty}\mathbb{E}_{Q_{K_i}}\left[M^{K_i,t}_{a,\phi}\,\mathcal{I}(Y^{K_i})\right]=\lim_{i\to\infty}\mathbb{E}_{Q_{K_i}}\left[\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})\right] \tag{5.4.59}$$

ii)

$$\lim_{i\to\infty}\mathbb{E}_{Q_{K_i}}\left[\mathcal{M}_\phi\left(Y^{K_i,t}_a\right)\mathcal{I}(Y^{K_i})\right]=\mathbb{E}_Q\left[\mathcal{M}_\phi\left(Y^t_a\right)\mathcal{I}(Y)\right]. \tag{5.4.60}$$

By Cauchy-Schwartz inequality, by the smoothness of Ψ and by Proposition 9 we obtain

$$\begin{aligned}&\lim_{i\to\infty}\mathbb{E}_{Q_{K_i}}\left[\left|M^{K_i,t}_{a,\phi}-\mathcal{M}_\phi(Y^{K_i,t}_a)\right|\mathcal{I}(Y^{K_i})\right]\\ &\leqslant\|\Psi\|_\infty\lim_{i\to\infty}\sqrt{\mathbb{E}_{Q_{K_i}}\left[\left(M^{K_i,t}_{a,\phi}-Y^{K_i,t}_a(\phi)+Y^{K_i,0}_a(\phi)+\nu\int_0^t Y^{K_i,q/K_i^2}_a(\Delta\phi)dq\right)^2\right]}=0.\end{aligned} \tag{5.4.61}$$

This implies (5.4.59), thus the first step is proved. Furthermore, we have the following upper-bound

$$\sup_{i\in\mathbb{N}}\mathbb{E}_{Q_{K_i}}\left[\left(\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})\right)^2\right]\leqslant\|\Psi\|^2_\infty\sup_{k\in\mathbb{N}}\mathbb{E}_{Q_{K_i}}\left[\left(\mathcal{M}_\phi(Y^{K_i,t}_a)\right)^2\right]<\infty \tag{5.4.62}$$

which implies that the family of martingales $\left(\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})\right)_{i\in\mathbb{N}}$ is uniformly integrable with respect to the law Q_{K_i}. Then, to prove (5.4.60), it is enough to show that $\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})$ converges in distribution to $\mathcal{M}_\phi(Y^t_a)\mathcal{I}(Y)$. To this aim, we define, for arbitrary test functions $\phi, H_1,\dots,H_m$,

$$\begin{aligned}P^a_1:D\left([0,T],(C^\infty_c(\mathbb{R}))^*\right)&\to D\left([0,T],\mathbb{R}\right)^{m+2}\\ Y^{K_i,\cdot}&\to P^a_1(Y^{K_i,\cdot})=\left(Y^{K_i,\cdot}_a(\phi),Y^{K_i,\cdot}_a(\Delta\phi),Y^{K_i,\cdot}(H_1),\dots,Y^{K_i,\cdot}(H_m)\right)\end{aligned} \tag{5.4.63}$$

and

$$\begin{aligned}&P_2:D\left([0,T],\mathbb{R}\right)^{m+2}\to\mathbb{R}\\ &P^a_1(Y^{K_i,\cdot})\to P_2(P^a_1(Y^{K_i,\cdot}))=\left(\mathcal{M}_\phi(Y^{K_i,t}_a)\right)\ \Psi(Y^{K_i,s_1}(H_1),\dots,Y^{K_i,s_m}(H_m))\end{aligned} \tag{5.4.64}$$

in such a way that

$$\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})=P_2\circ P^a_1(Y^{K_i,t}_a). \tag{5.4.65}$$

Using Theorem 1.7 in [79], each component of P_1 is continuous and therefore

$$P^a_1(Y^{K_i,t})\to P^a_1(Y^t)\qquad\text{as}\qquad i\to\infty$$

on the Skorokhod space $D([0,T],\mathbb{R})^{m+2}$. Since by Proposition 12 the limiting point $(Y^t_a)_{t\in[0,T]}$ is a.s. continuous, the convergence holds also uniformly in time. Using the continuity of Ψ we thus obtain

$$P_2\circ P^a_1(Y^{K_i,t})\to P_2\circ P^a_1(Y^t)\qquad\text{as}\qquad i\to\infty$$

uniformly in time. As a consequence, the set of discontinuity points of P_2 under Q_{K_i} is a negligible set. By Portmanteau theorem, this implies that $\mathcal{M}_\phi(Y^{K_i,t}_a)\mathcal{I}(Y^{K_i})$ converges in distribution to $\mathcal{M}_\phi(Y^t_a)\mathcal{I}(Y)$. Therefore (5.4.60) is proved.

Proof for $(\mathcal{N}^t_{a,b,\phi})_{t\in[0,T]}$ and $(\mathcal{N}^t_{a,a,\phi})_{t\in[0,T]}$: we have the following estimate using Proposition 9

$$\begin{aligned}
&\lim_{i\to\infty}\mathbb{E}\Bigg[\Bigg(\mathcal{N}^{K_i,t}_{a,b,\phi}-\Bigg(Y_a^{K_i,tK_i^2}(\phi)-Y_a^{K_i,0}(\phi)-\nu\int_0^t Y_a^{K_i,s/K_i^2}(\Delta\phi)ds\Bigg)\\
&\qquad\Bigg(Y_b^{K_i,t}(\phi)-Y_b^{K_i,0}(\phi)-\nu\int_0^t Y_b^{K_i,s/K_i^2}(\Delta\phi)ds\Bigg)+2t\nu^2p_ap_b\int_{\mathbb{R}}\nabla(\phi(u))^2du\Bigg)\mathcal{I}(Y^{K_i})\Bigg]\\
&\leqslant\|\Psi\|_\infty\lim_{i\to\infty}\Bigg(\mathbb{E}\Bigg[\Bigg(\mathcal{N}^{K_i,t}_{a,b,\phi}-\Bigg(Y_a^{K_i,tK_i^2}(\phi)-Y_a^{K_i,0}(\phi)-\nu\int_0^t Y_a^{K_i,s/K_i^2}(\Delta\phi)ds\Bigg)\\
&\quad\Bigg(Y_b^{K_i,t}(\phi)-Y_b^{K_i,0}(\phi)-\nu\int_0^t Y_b^{K_i,s/K_i^2}(\Delta\phi)ds\Bigg)+2t\nu^2p_ap_b\int_{\mathbb{R}}\nabla(\phi(u))^2du\Bigg)^2\Bigg]\Bigg)^{1/2}=0
\end{aligned}\tag{5.4.66}$$

that implies the counterpart of (5.4.59). Moreover, we have the following upper bound

$$\begin{aligned}
&\sup_{i\in\mathbb{N}}\mathbb{E}_{Q_{K_i}}\Bigg[\Bigg(M^{K_i,t}_{a,\phi}M^{K_i,t}_{b,\phi}+2t\nu^2p_ap_b\int_{\mathbb{R}}\nabla(\phi(u))^2du\Bigg)^2\Bigg]\\
\leqslant&C\sup_{i\in\mathbb{N}}\Bigg\{\mathbb{E}_{Q_{K_i}}\Bigg[\Bigg(Y_a^{K_i,t}(\phi)-Y_a^{K_i,0}(\phi)-\nu\int_0^t Y_a^{K_i,q/K_i^2}(\Delta\phi)dq\Bigg)^4\Bigg]\\
&\qquad\mathbb{E}_{Q_{K_i}}\Bigg[\Bigg(Y_b^{K_i,t}(\phi)-Y_b^{K_i,0}(\phi)-\nu\int_0^t Y_b^{K_i,q/K_i^2}(\Delta\phi)dq\Bigg)^4\Bigg]\Bigg\}<\infty
\end{aligned}\tag{5.4.67}$$

where in the last inequality we used Proposition 9. This is the counterpart of (5.4.62) and allows to show uniform integrability. The rest of the proof is similar.

□

Perspectives

To have a more complete picture of the scaling limits of the multi-species stirring process one possible future development is to study the boundary driven model, obtained by adding a parameter that rules the intensity of the reservoirs. This allows to obtain Dirichlet, Neumann and Robin boundary condition in the limit, similarly to what has been done for the SEP(ν) in [82]. This idea can be extended to fluctuations (see [60] for the SEP(ν)). An other possible generalization is the introduction of random environment (see [48] for single species).
Starting from the work [73], we aim, to determine the scaling limit for the weakly asymmetric multi-species stirring process by the so called *super-exponential estimate*. This allows also to determine the large deviation functional for the process on a torus. A challenge in this direction is the correct choice of the external fields that make the process weakly asymmetric.

Chapter 6

Duality and non-equilibrium steady state

6.1 Motivations

Our strategy to solve for the non-equilibrium steady state is of a probabilistic nature and relies on the use of *duality* [8]. Starting from the pioneering work of Kipnis, Marchiorio and Presutti [50], this approach has been substantially developed in recent years to study stochastic processes in the boundary-driven set-up [11, 12]. The boundary-driven process is mapped to a dual process with absorbing extra sites and the problem of describing the stationary state of the original system is simplified to the problem of computing the absorption probabilities of the dual particles. The main goal of this chapter is to state and prove a duality result for the boundary driven multi-species stirring process and exploit it to characterize the non-equilibrium steady state. More precisely, we prove duality between the boundary driven multi-species stirring process introduced in Chapter 4 and a dual process that has the same dynamics in the bulk, but has absorbing boundaries. The technique used relies on the research of a symmetry of the edge Hamiltonian of the process and on the application of the *Hadamard formula* on the boundaries. First, we state and prove (see Section 6.2) the duality theorem for the boundary driven multi-species stirring process. We consider the case of the connected and undirected graph $G = (V, \mathcal{E})$ where each site is put in contact with an external reservoir. Indeed, this underline the robustness of duality with respect to the choice of the geometry and allows to recover the other non-equilibrium situation (like the chain with two reservoirs attached at the extremal sites) by specializing the process on the desired geometry. Second, in Section 6.3, we exploit duality to formally write formulas for the moments of non-equilibrium steady state via absorption probabilities. Finally, in Section 6.4, as a by-product, we define the thermalized boundary driven multi-species stirring process and we show that it is in a duality relation with and absorbing dual process.

6.2 Absorbing duality

In this section we formulate duality for the multi-species stirring process $(\boldsymbol{n}(t))_{t\geqslant 0}$ with open boundaries, defined by the generator (4.2.15) on the geometry of a connected and undirected graph $G = (V, \mathcal{E})$. For the sake of self-consistency we recall that the state space is given by

$$\Omega := \bigotimes_{x\in V} \Omega_x \tag{6.2.1}$$

where,

$$\Omega_x := \left\{ n^x = (n_1^x, \ldots, n_N^x) \in \mathbb{N}_0^N \ : \ \sum_{A=1}^N n_A^x = \nu \right\} . \tag{6.2.2}$$

We denote a particle configuration on the graph as $\boldsymbol{n} \in \Omega$, where $\boldsymbol{n} = (n_A^x)_{x \in V, A \in \{1,\ldots,N\}}$. We also recall that the infinitesimal generator of the process reads

$$\mathcal{L} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \mathcal{L}_x \tag{6.2.3}$$

where, considering a function $f : \Omega \to \mathbb{R}$ we have

$$\mathcal{L}_{x,y} f(\boldsymbol{n}) = \sum_{A,B=1}^N n_A^x n_B^y \left[f(\boldsymbol{n} - \boldsymbol{\delta}_A^x + \boldsymbol{\delta}_B^x + \boldsymbol{\delta}_A^y - \boldsymbol{\delta}_B^y) - f(\boldsymbol{n}) \right] , \tag{6.2.4}$$

$$\mathcal{L}_x f(\boldsymbol{n}) = \sum_{A,B=1}^N \alpha_A^x n_B^x \left[f(\boldsymbol{n} + \boldsymbol{\delta}_A^x - \boldsymbol{\delta}_B^x) - f(\boldsymbol{n}) \right] \tag{6.2.5}$$

where $\boldsymbol{\delta}_A^x$ is defined in (4.2.7).
The dual process $(\boldsymbol{\xi}(t))_{t \geqslant 0}$ is defined on the *enlarged graph* $\widetilde{G} = (\widetilde{V}, \widetilde{\mathcal{E}})$ where

$$\widetilde{V} := V \cup \{u(x) \ : \ x \in V\} \qquad \widetilde{\mathcal{E}} := \mathcal{E} \cup \{(x, u(x)) \ : \ x \in V\} . \tag{6.2.6}$$

This means that to each site $x \in V$ we associate an "extra-site" via a bijection $u : V \to V$. We denote as $u(x)$ the extra-site associated to $x \in V$. The configuration space of the dual process is the enlarged state space

$$\widetilde{\Omega} = \bigotimes_{x \in V} \widetilde{\Omega}_x \ = \bigotimes_{x \in V} (\Omega_x \times \mathbb{N}_0^{N-1}) . \tag{6.2.7}$$

Note that on the extra-site we allow an unbounded number of particles. Thus dual particles will accumulate in these extra-sites in the course of time. We write the configurations $\boldsymbol{\xi} \in \widetilde{\Omega}$ as

$$\boldsymbol{\xi} = \bigotimes_{x \in V} \left((\xi_1^x, \ldots, \xi_N^x) \otimes (\xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)}) \right) \tag{6.2.8}$$

where the component ξ_a^x is interpreted as the number of dual particles of type $a \in \{1, \ldots, N-1\}$ at site x, while ξ_N^x is interpreted as the number of holes at site x of the dual process. The component $\xi_a^{u(x)}$ gives the number of dual particles of type $a \in \{1, \ldots, N-1\}$ at the extra-site $u(x)$ connected to $x \in V$. We observe that the configuration variable at the extra site $u(x)$ does not have to satisfy any exclusion constraint, i.e. an unbounded number of particles can be hosted. Therefore, an infinite number of holes is available at each extra-site $u(x)$. As a consequence, the hole occupation variable $\xi_N^{u(x)}$ is not considered. We state the following duality result.

Theorem 17 (Absorbing duality) *The multi-species stirring process $(\boldsymbol{n}(t))_{t \geqslant 0}$ defined on the state space Ω with generator $\mathcal{L}$ defined in* (4.2.15) *is dual to the process $(\boldsymbol{\xi}(t))_{t \geqslant 0}$ defined on the enlarged state space $\widetilde{\Omega}$ with generator*

$$\widetilde{\mathcal{L}} = \sum_{(x,y) \in \mathcal{E}} \omega_{x,y} \mathcal{L}_{x,y} + \sum_{x \in V} \Gamma_x \widetilde{\mathcal{L}}_x \tag{6.2.9}$$

where $\mathcal{L}_{x,y}$ is defined in (4.2.16) *and, for any function $f : \widetilde{\Omega} \to \mathbb{R}$*

$$\widetilde{\mathcal{L}}_x f(\boldsymbol{\xi}) = |\alpha^x| \sum_{a=1}^{N-1} \xi_a^x \left(f(\boldsymbol{\xi} - \boldsymbol{\delta}_a^x + \boldsymbol{\delta}_N^x + \boldsymbol{\delta}_a^{u(x)}) - f(\boldsymbol{\xi}) \right) . \tag{6.2.10}$$

The duality function is given by

$$D(\boldsymbol{n}, \boldsymbol{\xi}) = \prod_{x \in V} \left(\frac{(\nu - \sum_{a=1}^{N-1} \xi_a^x)!}{\nu!} \prod_{a=1}^{N-1} \frac{n_a^x!}{(n_a^x - \xi_a^x)!} (\rho_a^x)^{\xi_a^{u(x)}} \right) \tag{6.2.11}$$

where we recall the definition of the reservoir densities (cf. (4.2.22)*)*

$$\rho_a^x = \frac{\alpha_a^x}{|\alpha^x|} . \tag{6.2.12}$$

The dual dynamics is described as follows. On one hand, the edge part $\mathcal{L}_{x,y}$ of the dual Markov generator gives rise to the multi-species stirring dynamics on the graph. On the other hand, the site part $\widetilde{\mathcal{L}}_x$ of the dual generator replaces a particle of any type $a \in \{1, \dots, N-1\}$ at site x with a particle of type N and creates a particle of the same type a at the extra-site $u(x)$. This last transition is performed with rate $|\alpha^x|\xi_a^x$. This means that eventually the dual process voids the graph, putting all the dual particles of species $\{1, \dots, N-1\}$ in the extra-sites. In other words the extra-sites play the role of absorbing boundaries.

Remark 9 *In the reversible situation, i.e. when $\forall x \in V$ we have $\rho_a^x = \rho_a$, the expectation of the duality function $D(\boldsymbol{n}, \boldsymbol{\xi})$ with $\boldsymbol{n}$ distributed as $\mu_{rev} = \bigotimes_{x \in V}$ Multinomial$(\nu, \rho_1, \dots, \rho_N)$ is*

$$\mathbb{E}_{\mu_{rev}} [D(\boldsymbol{n}, \boldsymbol{\xi})] = \prod_{a=1}^{N-1} (\rho_a)^{\sum_{x \in V} \xi_a^x + \sum_{x \in V} \xi_a^{u(x)}} \qquad \forall \boldsymbol{n} \in \Omega, \quad \forall \boldsymbol{\xi} \in \widetilde{\Omega} . \tag{6.2.13}$$

6.2.1 Proof of Theorem 17

To prove duality between the process $(\boldsymbol{n}(t))_{t \geqslant 0}$ and the process $(\boldsymbol{\xi}(t))_{t \geqslant 0}$ we show that (3.2.8) is fulfilled. To show this, we will use the Hamiltonians and their Lie algebraic description. Indeed, in this formalism the proof reduces to finding symmetries of the generator (for the bulk duality) and group like transformations (for the "boundary duality"). To a configuration $\boldsymbol{\xi}$ of the configuration space (6.2.7) of a dual process we associate the vector

$$|\boldsymbol{\xi}\rangle = \bigotimes_{x \in V} \left(|\xi_1^x, \dots, \xi_N^x\rangle \otimes |\xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)}\rangle \right) . \tag{6.2.14}$$

Here $|\xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)}\rangle$ is the vector associated to the configuration at the extra site $u(x)$. For $q_1, \dots, q_{N-1} \in \mathbb{N}_0$, we assume that it satisfies an orthogonality relation

$$\langle q_1^{u(x)}, \dots, q_{N-1}^{u(x)} | \xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)}\rangle = \prod_{a=1}^{N-1} \delta_{\xi_a^{u(x)}, q_a} . \tag{6.2.15}$$

Remark 10 *In the following, we denote the configuration vectors on the extra-site $u(x)$ by $|\xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)}\rangle$. This allows to stress that these vectors belong to the extra-space N_0^{N-1} "attached" to site x. Moreover, this notation allows to directly connect the ket-vector $|\xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)}\rangle$ with the vector $(\xi_1^{u(x)}, \dots, \xi_{N-1}^{u(x)})$ defined in* (6.2.8)*, in which we recall that the components $\xi_a^{u(x)}$ denote the number of dual particles of type a at site $u(x)$.*

The Hamiltonian of the dual process reads

$$\widetilde{H} = \sum_{x,y\in\mathcal{E}} \omega_{x,y}\mathcal{H}_{x,y} + \sum_{x\in V}\Gamma_x \widetilde{H}_x \tag{6.2.16}$$

where $\mathcal{H}_{x,y}$ is the one defined in (4.3.10), while

$$\widetilde{H}_x = |\alpha^x| \sum_{a=1}^{N-1} \left((\mathbf{a}^+)_a^{u(x)}\, E^x_{Na} - E^x_{aa} \right) . \tag{6.2.17}$$

Here we introduced the pair of bosonic operators $\mathbf{a}$, $\mathbf{a}^+$ satisfying $[\mathbf{a}, \mathbf{a}^+] = 1$ and acting as

$$\mathbf{a}^+|q\rangle = |q+1\rangle \qquad \mathbf{a}|q\rangle = q|q-1\rangle \tag{6.2.18}$$

and

$$\langle q|\mathbf{a}^+ = \langle q-1| \qquad \langle q|\mathbf{a} = (q+1)\langle q+1| \tag{6.2.19}$$

on a generic vector $\langle q|$ with $q \in \mathbb{N}_0$, so that in (6.2.17) $(\mathbf{a}^+)_a^{u(x)}$ denotes $\mathbf{a}^+$ acting on the extra-site $u(x)$ and on the species $a \in \{1, \ldots, N-1\}$.
We will show below that the Hamiltonians (4.3.9) and (6.2.16) are dual in the sense of (3.2.8). From an algebraic point of view, the duality matrix D (6.2.11) is described as

$$D = \prod_{x\in V} d_x \otimes \mathcal{D}_{u(x)} . \tag{6.2.20}$$

Here

$$d_x = R_x \exp\left(E^x\right) \tag{6.2.21}$$

with the diagonal part

$$R_x = \sum_{n^x\in\Omega_x} \frac{\prod_{A=1}^N n_A^x!}{\nu!} |n_1^x, \ldots, n_N^x\rangle\langle n_1^x, \ldots, n_N^x| \tag{6.2.22}$$

and

$$E^x = \sum_{a=1}^{N-1} E^x_{aN} . \tag{6.2.23}$$

Furthermore

$$\mathcal{D}_{u(x)} = \sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \prod_{a=1}^{N-1} (\rho_a^x)^{\xi_a^{u(x)}} \langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)}| . \tag{6.2.24}$$

We observe that the matrix R_x is diagonal. Its elements are related to the inverse of the weights of the reversible measure (4.2.19). In particular, to obtain these elements, we have considered the weights of (4.2.19) when all the parameters $\rho_a = \frac{1}{N}$. Then, the constant $\left(\frac{1}{N}\right)^\nu$ has been neglected, since it does not change the duality relation. This R_x is the multi-species version of the *cheap duality matrix*, that was introduced in the case of SEP(ν) in (3.2.11).
Since (6.2.20) is product over sites we show that

$$\mathcal{H}^T_{x,y} D = D\mathcal{H}_{x,y} \qquad \forall (x,y) \in \mathcal{E} \tag{6.2.25}$$

and

$$H^T_x D = D\widetilde{H}_x \qquad \forall x \in V. \tag{6.2.26}$$

We perform the proof of duality in three steps: first we will show that matrix (6.2.20) has elements (6.2.11); second we will prove the bulk duality (6.2.25); finally we will show the boundary duality (6.2.26). As a preliminary result we note that

$$(E^x_{BA})^T = R_x E^x_{AB} R_x^{-1} \qquad \forall x \in V\ , \tag{6.2.27}$$

that follows immediately from the definition of E_{AB}.

Elements of the duality matrix. Consider the matrix D defined in (6.2.20). We aim to show that its matrix elements coincide with duality function (6.2.11), i.e.

$$\langle \boldsymbol{n}|D|\boldsymbol{\xi}\rangle = D(\boldsymbol{n},\boldsymbol{\xi}) \qquad \forall \boldsymbol{n} \in \Omega, \quad \boldsymbol{\xi} \in \widetilde{\Omega}\,. \tag{6.2.28}$$

Fix an arbitrary site $x \in V$, then we have that

$$\begin{aligned}&\langle n^x_1,\dots,n^x_N|\left(d_x \otimes \mathcal{D}_{u(x)}\right)\left(|\xi^x_1,\dots,\xi^x_N\rangle \otimes |\xi^{u(x)}_1,\dots,\xi^{u(x)}_{N-1}\rangle\right)\\ =&\langle n^x_1,\dots,n^x_N|(\exp(E^x_{N1}+\ldots+E^x_{N\,N-1}))^T R_x \otimes \sum_{q^{u(x)}_1,\dots,q^{u(x)}_{N-1}=0}^{\infty} \prod_{a=1}^{N-1} (\rho^x_a)^{q^{u(x)}_a} \langle q^{u(x)}_1,\dots,q^{u(x)}_N|\\ &|\xi^x_1,\dots,\xi^x_N\rangle \otimes |\xi^{u(x)}_1,\dots,\xi^{u(x)}_{N-1}\rangle\end{aligned} \tag{6.2.29}$$

where we used (6.2.27).
On one hand, on the extra-site $u(x)$ we have

$$\sum_{q^{u(x)}_1,\dots,q^{u(x)}_{N-1}=0}^{\infty} \left(\prod_{a=1}^{N-1} (\rho^x_a)^{q^{u(x)}_a}\right) \langle q^{u(x)}_1,\dots,q^{u(x)}_{N-1}|\xi^{u(x)}_1,\dots,\xi^{u(x)}_{N-1}\rangle = \prod_{a=1}^{N-1} (\rho^x_a)^{\xi^{u(x)}_a} \tag{6.2.30}$$

where we used the orthogonality relation (6.2.15). On the other hand, on the site x, we have

$$\begin{aligned}&\langle n^x_1,\dots,n^x_N|(\exp(E^x_{N1}+\ldots+E^x_{N\,N-1}))^T R_x|\xi^x_1,\dots,\xi^x_N\rangle\\ &= \langle n^x_1,\dots,n^x_N|\left(\sum_{k_1=0}^{\infty}\frac{\left\{(E^x_{N1})^T\right\}^{k_1}}{k_1!}\cdots\sum_{k_{N-1}=0}^{\infty}\frac{\left\{(E^x_{N\,N-1})^T\right\}^{k_{N-1}}}{k_{N-1}!}\right.\\ &\qquad\left.\times\sum_{s\in\Omega_x}\frac{s^x_1!\cdots s^x_N!}{\nu!}|s^x_1,\dots,s^x_N\rangle\langle s^x_1,\dots,s^x_N|\right)|\xi^x_1,\dots,\xi^x_N\rangle\\ &= \sum_{k_1=0}^{n^x_1}\cdots\sum_{k_{N-1}=0}^{n^x_{N-1}}\langle n^x_1-k_1,\dots,n^x_{N-1}-k_{N-1},\dots,n^x_N+k_1+\ldots+k_{N-1}|\\ &\qquad\times\frac{n^x_1!\cdots n^x_{N-1}!}{(n^x_1-k_1)!\cdots(n^x_{N-1}-k_{N-1})!}\frac{1}{k_1!\cdots k_{N-1}!}\frac{\xi^x_1!\cdots\xi^x_N!}{\nu!}|\xi^x_1,\dots,\xi^x_N\rangle\\ &= \frac{(\nu-\sum_{a=1}^{N-1}\xi^x_a)!}{\nu!}\prod_{a=1}^{N-1}\frac{n^x_a!}{(n^x_a-\xi^x_a)!}\end{aligned} \tag{6.2.31}$$

where we used the definitions of the action of E_{AB} and the orthogonality relations (4.3.5), together with the fact that $\xi^x_N = \nu - \sum_{a=1}^{N-1}\xi^x_a$. Finally, by taking the product over $x \in V$ (6.2.28) is proved.

□

Proof of bulk duality (6.2.25)**.** To show this relation we need two 'ingredients'. First the existence of a similarity transformation between the Hamiltonian $\mathcal{H}_{x,y}$ and its transposed. As we will show, this similarity transformation is $R_x R_y$. Second, the possibility of finding a symmetry for the edge Hamiltonian. Exploiting (4.3.19), we can take any symmetry of the Casimir and apply the co-product. As we will see, it will be convenient to choose $\sum_{a=1}^{N-1} E_{aN}$.

We first look for the similarity between $\mathcal{H}_{x,y}$ and its transposed. Using (6.2.27) we obtain

$$\mathcal{H}_{x,y}^T = (R_x R_y)\, \mathcal{H}_{x,y}\, (R_x R_y)^{-1} \tag{6.2.32}$$

therefore we have found the similarity transformation between $\mathcal{H}_{x,y}$ and its transposed.
We now look for a symmetry of $\mathcal{H}_{x,y}$, i.e. a matrix $S_{x,y}$ of the same dimension such that it satisfies

$$\mathcal{H}_{x,y} S_{x,y} = S_{x,y} \mathcal{H}_{x,y}\ . \tag{6.2.33}$$

Using (4.3.19), we observe that $\mathcal{H}_{x,y}$ is proportional to the coproduct of the second Casimir, up to a diagonal term. Therefore, it is enough to look for a symmetry of $\Delta(C)$. Using the bilinearity of the coproduct operator and the fact that C belongs to the centre of the Lie algebra, it is easy to show that for any linear combination of generators $E \in gl(N)$ the following holds:

$$\text{if} \quad [C, E] = 0 \qquad \text{then} \qquad [\Delta(C), \Delta(E))] = 0\ . \tag{6.2.34}$$

Let

$$E = \sum_{a=1}^{N-1} E_{aN} \tag{6.2.35}$$

and, for a fixed $(x, y) \in \mathcal{E}$, we define

$$S_{x,y} = \exp\left(\Delta_{x,y}(E)\right) = \exp\left(E^x\right) \exp\left(E^y\right) \tag{6.2.36}$$

where $\Delta_{x,y}(E)$ denotes the coproduct acting on sites x and y. As a consequence, this operator $S_{x,y}$ satisfies

$$[S_{x,y}, \mathcal{H}_{x,y}] = 0 \tag{6.2.37}$$

i.e. it is a symmetry of $\mathcal{H}_{x,y}$. Exploiting these considerations we may write

$$\begin{aligned}
\mathcal{H}_{x,y}^T D &= (R_x R_y)\mathcal{H}_{x,y}(R_x R_y)^{-1} \left(d_x \otimes \mathcal{D}_{u(x)}\right) \left(d_y \otimes \mathcal{D}_{u(y)}\right) \prod_{z \in V\,:\, z \neq x,y} \left(d_z \otimes \mathcal{D}_{u(z)}\right) \\
&= \left(R_x \exp\left(E^x\right) \otimes \mathcal{D}_{u(x)}\right) \left(R_y \exp\left(E^y\right) \otimes \mathcal{D}_{u(y)}\right) \mathcal{H}_{x,y} \prod_{z \in V\,:\, z \neq x,y} \left(d_z \otimes \mathcal{D}_{u(z)}\right) \\
&= D \mathcal{H}_{x,y}
\end{aligned} \tag{6.2.38}$$

where we used (6.2.32) and (6.2.37) in the second equality. Thus, (6.2.25) is proved.

□

Proof of boundary duality (6.2.26)**.** To prove (6.2.26) we transform via the Hadamard formula the transposed of the site Hamiltonian (6.2.17) and then to introduce properly a creation operator acting on an extra-site $u(x)$. Considering $\mathcal{A}, \mathcal{B} \in gl(N)$, the Hadamard formula reads

$$\exp(-\mathcal{B})\mathcal{A}\exp(\mathcal{B}) = \mathcal{A} - [\mathcal{B}, \mathcal{A}] + \frac{1}{2!}[\mathcal{B}, [\mathcal{B}, \mathcal{A}]] - \frac{1}{3!}[\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] + \dots \tag{6.2.39}$$

In the following we evaluate this formula for $\mathcal{B} = \sum_{a=1}^{N-1} E_{aN}$ defined in (6.2.23) and $\mathcal{A} = E_{CD}$ with $C, D \in \{1, \dots, N\}$. We find

$$[\mathcal{B}, \mathcal{A}] = \sum_{a=1}^{N-1} [E_{aN}, E_{CD}] = \delta_{CN} \sum_{a=1}^{N-1} E_{aD} - (1 - \delta_{DN}) E_{CN} \tag{6.2.40}$$

and

$$\begin{aligned}
[\mathcal{B}\,[\mathcal{B}, \mathcal{A}]] &= \left[\sum_{a=1}^{N-1} E_{aN}, \delta_{CN} \sum_{b=1}^{N-1} E_{bD} - (1 - \delta_{DN}) E_{CN} \right] \\
&= \sum_{a=1}^{N-1} \left(\delta_{CN} \sum_{b=1}^{N-1} [E_{aN}, E_{bD}] - (1 - \delta_{DN}) \, [E_{aN}, E_{CN}] \right) \\
&= \sum_{a=1}^{N-1} \left(-\delta_{CN} \delta_{aD} \sum_{b=1}^{N-1} E_{bN} - (1 - \delta_{DN}) \delta_{CN} E_{aN} \right) \\
&= -\left(\delta_{CN} (1 - \delta_{DN}) + (1 - \delta_{DN}) \delta_{CN} \right) \sum_{b=1}^{N-1} E_{bN} \\
&= -\, 2 \delta_{CN} (1 - \delta_{DN}) \sum_{b=1}^{N-1} E_{bN} \; .
\end{aligned} \tag{6.2.41}$$

From the third commutator on we always obtain zero, i.e. $[\mathcal{B}, [\mathcal{B}, [\mathcal{B}, \mathcal{A}]]] = 0$. All in all we have that

$$\exp\left(-\sum_{a=1}^{N-1} E_{aN} \right) E_{CD} \exp\left(\sum_{a=1}^{N-1} E_{aN} \right) \tag{6.2.42}$$

$$= E_{CD} - \delta_{CN} \sum_{a=1}^{N-1} E_{aD} + (1 - \delta_{DN}) E_{CN} - \delta_{CN} (1 - \delta_{DN}) \sum_{b=1}^{N-1} E_{bN} \; . \tag{6.2.43}$$

Using (6.2.27) we write the transpose of site Hamiltonian (4.3.11)

$$H_x^T = \sum_{A,B=1}^{N} \alpha_A^x \left(E_{AB}^x - E_{BB}^x \right)^T = R_x \sum_{A,B=1}^{N} \alpha_A^x \left(E_{BA}^x - E_{BB}^x \right) R_x^{-1} \; . \tag{6.2.44}$$

We multiply both sides by $R_x \exp(E^x)$

$$H_x^T R_x \exp(E^x) = R_x \sum_{A,B=1}^{N} \alpha_A^x \left(E_{BA}^x - E_{BB}^x \right) \exp(E^x) \; . \tag{6.2.45}$$

By using (6.2.42) and the fact that $\sum_{a,b=1}^{N} E_{AB}$ is central for the algebra we have

$$\exp(-E^x) \sum_{A=1}^{N} \sum_{B=1}^{N} \alpha_A^x \left(E_{BA}^x - E_{BB}^x \right) \exp(E^x) = |\alpha^x| \sum_{a=1}^{N-1} \left(\rho_a^x E_{Na}^x - E_{aa}^x \right) \; . \tag{6.2.46}$$

Thus, we rewrite (6.2.45) as

$$H_x^T R_x \exp(E^x) = R_x \exp(E^x) |\alpha^x| \sum_{a=1}^{N-1} \left(\rho_a^x E_{Na}^x - E_{aa}^x \right) \; . \tag{6.2.47}$$

Taking the tensor product of both sides of (6.2.47) we obtain

$$
\begin{aligned}
&H_x^T d_x \otimes \left(\sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{a=1}^{N-1} (\rho_a^x)^{\xi_a^{u(x)}} \right) \langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \right) \\
=&d_x \, |\alpha^x| \sum_{a=1}^{N-1} (\rho_a^x E_{Na}^x - E_{aa}^x) \otimes \left(\sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{a=1}^{N-1} (\rho_a^x)^{\xi_a^{u(x)}} \right) \langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \right)
\end{aligned}
\tag{6.2.48}
$$

With a slight abuse of notation and in the spirit of Remark 10, we here denote the occupation vector at extra-site $u(x)$ by $\langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)} |$.
Recalling the action of the bosonic creation operator acting at site $u(x)$ and on the species $a \in \{1, \ldots, N-1\}$ we have that

$$
\langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)} + 1, \ldots, \xi_{N-1}^{u(x)} | (\mathbf{a}^+)_a^{u(x)} = \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \, . \tag{6.2.49}
$$

Using the above equation we rewrite the right hand side term of (6.2.48) with Lie generator E_{Na}^x as

$$
\begin{aligned}
&\sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{b=1}^{N-1} (\rho_b^x)^{\xi_b^{u(x)}} \right) \rho_a^x \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \\
=& \sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{b=1\,:\, b\neq a}^{N-1} (\rho_b^x)^{\xi_b^{u(x)}} \right) (\rho_a^x)^{\xi_a^{u(x)}+1} \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)} + 1, \ldots, \xi_{N-1}^{u(x)} | (\mathbf{a}^+)_a^{u(x)} \\
=& \sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{b=1}^{N-1} (\rho_b^x)^{\xi_b^{u(x)}} \right) \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | (\mathbf{a}^+)_a^{u(x)}
\end{aligned}
\tag{6.2.50}
$$

where, in the up to last equality, we performed a change of summation variable and we used the fact that

$$
\begin{aligned}
&\sum_{\xi_a^{u(x)}=0}^{\infty} (\rho_a^x)^{\xi_a^{u(x)}} \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | (\mathbf{a}^+)_a^{u(x)} \\
=& \sum_{\xi_a^{u(x)}=1}^{\infty} (\rho_a^x)^{\xi_a^{u(x)}} \langle \xi_1^{u(x)}, \ldots, \xi_a^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | (\mathbf{a}^+)_a^{u(x)} \, .
\end{aligned}
\tag{6.2.51}
$$

Therefore, inserting (6.2.50) in (6.2.48) we obtain

$$
\begin{aligned}
(H_x^T d_x) \otimes \mathcal{D}_{u(x)} &= (H_x^T d_x) \otimes \sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{a=1}^{N-1} (\rho_a^x)^{\xi_a^{u(x)}} \right) \langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \\
&= \left\{ d_x \otimes \sum_{\xi_1^{u(x)},\ldots,\xi_{N-1}^{u(x)}=0}^{\infty} \left(\prod_{a=1}^{N-1} (\rho_a^x)^{\xi_a^{u(x)}} \right) \langle \xi_1^{u(x)}, \ldots, \xi_{N-1}^{u(x)} | \right\} \\
&\times |\alpha^x| \sum_{a=1}^{N-1} \left((\mathbf{a}^+)_a^{u(x)} E_{Na}^x - E_{aa}^x \right)
\end{aligned}
$$

$$= \left(d_x \otimes \mathcal{D}_{u(x)}\right) \widetilde{H}_x \ . \tag{6.2.52}$$

Since the duality matrix (6.2.20) is product over sites, the above equality implies (6.2.26).

□

6.3 Non-equilibrium steady state

In this section we aim to write the moments with respect to the non-equilibrium steady for the boundary driven multi-species stirring process by using the absorbing duality stated in Theorem 17. In particular, we show how to write expression for the m-point non-equilibrium correlations, introducing a technique that can be adapted for higher moments. For the sake of simplicity, we consider the case where the geometry is reduced to a chain of length L, with two boundary reservoirs connected at the end sites of the chain, namely 1 and L. The interaction is assumed to be nearest neighbours. However, the results can be extended to the process on the general graph introduced in Chapter 4. We recall that we denote the original processes by the variables $(\boldsymbol{n}(t))_{t\geqslant 0}$, while the dual processes is denoted by the variables $(\boldsymbol{\xi}(t))_{t\geqslant 0}$. We specialize the state spaces to the chain by writing, for the original process

$$\Omega_L = \bigotimes_{x=1}^{L} \Omega_x \ . \tag{6.3.1}$$

and for the dual process

$$\widetilde{\Omega}_L = \mathbb{N}_0^{N-1} \otimes \Omega_L \otimes \mathbb{N}_0^{N-1} \ . \tag{6.3.2}$$

Here Ω_x is the space define in (4.2.2). In this set-up, the process $(\boldsymbol{n}(t))_{t\geqslant 0}$ has generator

$$\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}_{\text{right}} \ . \tag{6.3.3}$$

where $\mathcal{L}_{x,x+1}$ is the edge generator defined in (4.2.16) acting on the bond $(x, x+1)$ and $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$ are the generators (4.2.17) acting on sites 1 and L with parameters $(\alpha_A)_{A\in\{1,\dots,N\}}$ and $(\beta_A)_{A\in\{1,\dots,N\}}$ respectively. Adapting the result of Theorem 17, the dual process $(\boldsymbol{\xi}(t))_{t\geqslant 0}$, is defined on the dual state space $\widetilde{\Omega}_L$, and has generator

$$\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \widetilde{\mathcal{L}}_{\text{right}} \tag{6.3.4}$$

where $\mathcal{L}_{x,x+1}$ is given by (4.2.15), acting on the bond $(x, x+1)$ and where the boundary generators $\widetilde{\mathcal{L}}_{\text{left}}$ and $\widetilde{\mathcal{L}}_{\text{right}}$ are given in (6.2.10): they absorb particles from sites 1 and L and put them at extra-sites 0 and $L+1$ respectively. The duality function is (6.2.11) specified on the chain, i.e.

$$D(\boldsymbol{n}, \boldsymbol{\xi}) = \prod_{a=1}^{N-1} \left(\rho_a^{\text{left}}\right)^{\xi_a^0} \prod_{x=1}^{L} \frac{(\nu - \sum_{a=1}^{N-1} \xi_a^x)!}{\nu!} \prod_{a=1}^{N-1} \frac{n_a^x!}{(n_a^x - \xi_a^x)!} \prod_{a=1}^{N-1} \left(\rho_a^{\text{right}}\right)^{\xi_a^{L+1}} \ , \tag{6.3.5}$$

where we denoted by

$$\rho_a^{\text{left}} = \frac{\alpha_a}{|\alpha|}, \qquad \rho_a^{\text{right}} = \frac{\beta_a}{|\beta|} \ . \tag{6.3.6}$$

As already pointed out, the dual process voids the chain by piling up particles of species $\{1, \ldots, N-1\}$ in the extra sites 0 and $L+1$ and replacing these particles by the holes N. This property allows to characterize the steady state non-equilibrium distribution (that is unique because of the ergodicity of the chain) by the probability that dual particles are absorbed in the extra-sites, called *absorption probabilities.* We call μ_{NESS} the non-equilibrium stationary distribution of the the process $(\boldsymbol{n}(t))_{t\geqslant 0}$ and we denote by $\mathbb{E}_{\mu_{\text{NESS}}}[\cdot]$ the expectation with respect to this measure. Using duality and the ergodicity of the chain, we write the average with respect to μ_{NESS} of the duality function.

Proposition 14 *Let $(\boldsymbol{n}(t))_{t\geqslant 0}$ be the boundary driven multi-species stirring process on a chain of length L defined on Ω_L, with generator $\mathcal{L}$ defined in* (6.3.3)*. Let $(\boldsymbol{\xi}(t))_{t\geqslant 0}$ be its dual absorbing process defined on $\widetilde{\Omega}_L$, with generator $\widetilde{\mathcal{L}}$ defined in* (6.3.4)*. Then, for a given configuration $\boldsymbol{\xi} \in \widetilde{\Omega}_L$, we have that*

$$\mathbb{E}_{\mu_{NESS}}\left[D(\boldsymbol{n}, \boldsymbol{\xi})\right] = \sum_{t_1=0}^{|\xi_1|} \cdots \sum_{t_{N-1}=0}^{|\xi_{N-1}|} \prod_{a=1}^{N-1} \left(\rho_a^{left}\right)^{t_a} \left(\rho_a^{right}\right)^{|\xi_a|-t_a} \mathcal{P}_{\boldsymbol{\xi}}(t_1, \ldots, t_{N-1}) \tag{6.3.7}$$

where

$$\mathcal{P}_{\boldsymbol{\xi}}(t_1, \ldots, t_{N-1}) = \mathbb{P}\left(\boldsymbol{\xi}(\infty) = \sum_{a=1}^{N-1} \left(t_a \delta_a^0 + (|\xi_a| - t_a)\delta_a^{L+1}\right) \Big| \boldsymbol{\xi}(0) = \boldsymbol{\xi}\right) \tag{6.3.8}$$

are the absorption probabilities i.e. the probabilities that, starting from an initial configuration $\boldsymbol{\xi}$, t_a dual particles of species a are absorbed at 0*. We have denoted $|\xi_a| = \sum_{x=1}^{L} \xi_a^x$.*

Proof of Proposition 14: by using duality, ergodicity and the absorbing property of the dual process we have that

$$\begin{aligned} \mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{n}, \boldsymbol{\xi})\right] &= \lim_{t\to\infty} \mathbb{E}^{\boldsymbol{n}}\left[D(\boldsymbol{n}(t), \boldsymbol{\xi})\right] = \lim_{t\to\infty} \mathbb{E}^{\boldsymbol{\xi}}\left[D(\boldsymbol{n}, \boldsymbol{\xi}(t))\right] \\ &= \sum_{t_1=0}^{|\xi_1|} \cdots \sum_{t_N=0}^{|\xi_N|} \prod_{a=1}^{N-1} \left(\rho_a^{\text{left}}\right)^{t_a} \left(\rho_a^{\text{right}}\right)^{|\xi_a|-t_a} \mathcal{P}_{\boldsymbol{\xi}}(t_1, \ldots, t_N) \end{aligned} \tag{6.3.9}$$

In (6.3.9), we denoted by $\mathbb{E}^{\boldsymbol{n}}$ and by $\mathbb{E}^{\boldsymbol{\xi}}$ the expectations with respect to the laws of the original and dual processes initialized by the arbitrary configurations $\boldsymbol{n}$ and by the dual configuration $\boldsymbol{\xi}$ respectively.

□

Non-equilibrium correlations

Equation (6.3.7) allows to write, in function of the absorption probabilities, expression for the moments with respect to the non-equilibrium steady state. In the following, we show how it can be used to compute the m-point non-equilibrium steady state correlations. For every $m \in \{1, \ldots, L\}$ we consider the coordinates $1 \leqslant x_1 < \ldots < x_m \leqslant L$ and we select m colours $a_1, \ldots, a_m \in \{1, \ldots, N-1\}$ among the $N-1$ available ones. We introduce the dual configuration $\hat{\boldsymbol{\xi}} = \sum_{k=1}^{m} \delta_{a_k}^{x_k}$, meaning that

$$\hat{\xi}_A^x = \begin{cases} 1 & \text{if} \quad x = x_k \text{ and } A = a_k \\ 0 & \text{otherwise} \end{cases} \tag{6.3.10}$$

or, in words, we put a dual particle of species a_k at site x_k, leaving empty all the other sites. The duality function is (6.3.5), therefore, we have that

$$D(\boldsymbol{n}, \hat{\boldsymbol{\xi}}) = \frac{\prod_{k=1}^{m} n_{a_k}^{x_k}}{C(\nu, m)} \,. \tag{6.3.11}$$

where $C(\nu, m)$ is a constant that depends on m and on the maximal occupancy per site ν. Therefore, up to this constant, $\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{n}, \hat{\boldsymbol{\xi}})\right]$ gives the m-point correlations in the non-equilibrium steady state. Using formula (6.3.7) we have that

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[\prod_{k=1}^{m} n_{a_k}^{x_k}\right] = C(\nu, m) \sum_{t_1=0}^{|\hat{\xi}_1|} \cdots \sum_{t_{N-1}=0}^{|\hat{\xi}_{N-1}|} \prod_{a=1}^{N-1} \left(\rho_a^{\text{left}}\right)^{t_a} \left(\rho_a^{\text{right}}\right)^{|\hat{\xi}_a| - t_a} \mathcal{P}_{\hat{\boldsymbol{\xi}}}(t_1, \ldots, t_{N-1}) \,. \tag{6.3.12}$$

Therefore, the m-point non-equilibrium steady state correlations can be computed once the absorption probabilities are known.

Remark 11 *Considering arbitrary coordinates $(x_k)_{k \in \{1,\ldots,m\}}$ (not necessary different) and considering proper dual configurations, the technique explained above can be used to compute, in function of the absorption probabilities, the m-mixed factorial moments under the non-equilibrium stationary distribution.*

Computing these absorption probabilities is usually an hard task, since they satisfy complicated difference equations. For example, we report here the solution for the simplest case of one-point correlation that is easily solvable, since the single dual particle behaves as an independent random walker.
We take $m = 1$ and we fix a site $x \in \{1, \ldots, L\}$ and a species $a \in \{1, \ldots, N-1\}$. We consider the dual configuration $\hat{\boldsymbol{\xi}} = \delta_a^x$, then

$$\nu D(\boldsymbol{n}, \delta_a^x) = n_a^x \,. \tag{6.3.13}$$

The average occupation reads

$$\nu \mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{n}, \delta_a^x)\right] = \sum_{t_a=0}^{1} \left(\rho_a^{\text{left}}\right)^{t_s} \left(\rho_s^{\text{right}}\right)^{1-t_s} \mathbb{P}_{\delta_a^x}\left(\xi_\infty = t_a \delta_0 + (1 - t_a)\delta_{L+1}\right) \,. \tag{6.3.14}$$

To explicitly find the result, we aim to determine $\mathcal{P}(1, \delta_a^x) = \mathbb{P}_{\delta_a^x}\left(\xi_\infty = t_a \delta_a^0 + (1 - t_a)\delta_a^{L+1}\right)$. This is the probability that a random walker (started at x) on the chain $\{1, \ldots, L\}$ with absorbing boundaries is absorbed at extra-site 0. It fulfils the following discrete Laplace equation

$$\begin{cases} \Delta_L \mathcal{P}(1, \delta_a^x) = 0 \\ \mathcal{P}(1, \delta_a^1) = \frac{|\alpha|}{1+|\alpha|} + \mathcal{P}(1, \delta_a^2)\frac{1}{1+|\alpha|} \\ \mathcal{P}(1, \delta_a^L) = \mathcal{P}(1, \delta_a^{L-1})\frac{1}{1+|\beta|} \end{cases} \tag{6.3.15}$$

where Δ_L is the discrete Laplace operator on the chain of length L. The equations (6.3.15) can be solved and using $\mathcal{P}(1, \delta_a^x) = 1 - \mathcal{P}(0, \delta_a^x)$ we obtain

$$\nu \mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{n}, \delta_a^x)\right] = \nu \frac{\alpha_a(L|\beta| - |\beta|x + 1) + \beta_a(|\alpha|x + 1 - |\alpha|)}{|\alpha||\beta|L - |\alpha||\beta| + |\alpha| + |\beta|} \,. \tag{6.3.16}$$

Remark 12 *: In case $|\alpha| = |\beta| = \nu = 1$ (when the chain is integrable) we have*

$$\mathbb{E}_{\mu_{NESS}}\left[D(\boldsymbol{n}, \delta_a^x)\right] = \frac{\alpha_a(L-x+1)+\beta_a x}{L+1} \,. \tag{6.3.17}$$

This is the same result obtained in [13] for the one-point correlations for the integrable version of the boundary driven multi-species stirring process. For details see Chapter 7.

We end this section by observing that, in the case of the chain with $\nu = 1$, the integrability of the model allows to write closed formulas for all the absorption probabilities. This will be explained in detail in Chapter 7.

6.4 Duality for the thermalized multi-species stirring process

In this section we aim to extend to the multi-species stirring process the concept of *instantaneous thermalization limit* (see [12, 50]). For the sake of simplicity we consider the set-up of a boundary driven chain of length L with two reservoirs at the end sites 1 and L respectively, as defined in Section 6.3. The thermalized process gives rise, for any pair nearest neighbour sites of the chain, to a redistribution of the total number of particles. The thermalized model that we present here is obtained from the boundary driven model with generator (6.3) as follows: for each bond $(x, x+1)$ the total number of particles $\epsilon = n^x + n^{x+1}$ is redistributed according to the reversible measure, conditioned on the conservation of ϵ. At the boundary, instead, the configuration n^x, with $x \in \{1, L\}$, is replaced with $\boldsymbol{r} \in \Omega_x$. This transition takes place with a rate that depends on the boundary parameters and on $\boldsymbol{r}$ itself.

The thermalization measure

We introduce the reversible measure (4.2.19) conditioned on the conservation of the particles on the bond $(x, x+1)$

$$\begin{aligned}\mu_{TH}(r|\epsilon) :&= \mathbb{P}\left(n^x = r | n^x + n^{x+1} = \epsilon\right)\\ &= \mathbb{P}\left((n_1^x, \ldots, n_N^x) = (r_1, \ldots, r_N) \mid (n_1^x + n_1^{x+1}, \ldots, n_N^x + n_N^{x+1}) = (\epsilon_1, \ldots, \epsilon_N)\right) \,.\end{aligned} \tag{6.4.1}$$

Here $r = (r_1, \ldots, r_N)$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ are such that $\forall A \in \{1, \ldots, N\}$ we have $r_A \in \{0, \ldots, \nu\}$ and $\epsilon_A \in \{0, \ldots, 2\nu\}$, under the constraints

$$\epsilon_N = 2\nu - \sum_{a=1}^{N-1} \epsilon_a, \qquad r_N = \nu - \sum_{a=1}^{N-1} r_a \,. \tag{6.4.2}$$

Using the conditional probability and the fact that the reversible measure (4.2.19) is the product over the sites of multinomial distributions we have

$$\mu_{TH}(r|\epsilon) = \frac{\mathbb{P}\left(\{n^x = r\} \cap \{n^x + n^{x+1} = \epsilon\}\right)}{\mathbb{P}\left(n^x + n^{x+1} = \epsilon\right)} = \frac{\mathbb{P}\left(n^x = r\right)\mathbb{P}\left(n^{x+1} = \epsilon - r\right)}{\mathbb{P}\left(n^x + n^{x+1} = \epsilon\right)} \,. \tag{6.4.3}$$

Therefore the thermalization measure reads

$$\mu_{TH}(r|\epsilon) = \frac{\binom{\nu}{r_1, \ldots, r_N}\binom{\nu}{(\epsilon_1 - r_1), \ldots, (\epsilon_N - r_N)}}{\binom{2\nu}{\epsilon_1 \ldots, \epsilon_N}} \mathbb{1}_{\{r_1 + \ldots + r_N = \nu\}} \tag{6.4.4}$$

where we recall that the multinomial coefficient reads

$$\binom{\nu}{n_1,\ldots,n_N} = \frac{\nu!}{n_1!\cdots n_N!}\ . \tag{6.4.5}$$

The thermalized multi-species stirring process is defined on the configuration space Ω_L introduced in (6.3.1) and its generator reads

$$\mathcal{L}^{TH} = \mathcal{L}^{TH}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}^{TH}_{x,x+1} + \mathcal{L}^{TH}_{\text{right}} \tag{6.4.6}$$

where

$$\begin{aligned}\mathcal{L}^{TH}_{x,x+1} f(\boldsymbol{n}) = & \sum_{r_1=0}^{n_1^x+n_1^{x+1}} \cdots \sum_{r_N=0}^{n_N^x+n_N^{x+1}} \mathbb{1}_{\{r_1+\ldots+r_N=\nu\}} \\ & \times \left\{ f(n^1,\ldots,n^{x-1},r,n^x+n^{x+1}-r,n^{x+2},\ldots,n^L) - f(\boldsymbol{n})\right\} \mu_{TH}(\boldsymbol{r}|\boldsymbol{\epsilon})\end{aligned} \tag{6.4.7}$$

while

$$\begin{aligned}\mathcal{L}^{TH}_{\text{left}} f(\boldsymbol{n}) = & \sum_{r_1=0}^{\nu} \cdots \sum_{r_N=0}^{\nu} \mathbb{1}_{\{r_1+\ldots+r_N=\nu\}} \left\{ f(r,n^2,\ldots,n^L) - f(\boldsymbol{n})\right\} \\ & \times \binom{\nu}{r_1,\ldots,r_N} \prod_{a=1}^{N-1} \left(\frac{\alpha_a}{\alpha_N}\right)^{r_a} \left(\frac{\alpha_N}{|\alpha|}\right)^{\nu}\end{aligned} \tag{6.4.8}$$

and $\mathcal{L}^{TH}_{\text{right}}$ acting similarly on the site L and with parameters β's. In the thermalized bulk generator particles on a bond x and $x+1$ are redistribute according to the measure $\mu_{TH}(r|\epsilon)$, while in the thermalized boundary generators the configuration at site 1 (L) is replaced by r, according to a rate depending on r itself and on the reservoirs parameters.

The dual thermalized process

Here we state the duality result for the thermalized multi-species stirring process with boundary driving.

Proposition 15 (Duality for the thermalized multi-species stirring process) *The thermalized multi-species stirring process $(\boldsymbol{n}(t))_{t\geqslant 0}$ on the state space Ω_L, with generator $\mathcal{L}^{TH}$ defined in* (6.4.6) *is dual to the process $(\boldsymbol{\xi}(t))_{t\geqslant 0}$ on the enlarged configuration space $\widetilde{\Omega}_L$, with dual generator*

$$\widetilde{\mathcal{L}}^{TH} = \widetilde{\mathcal{L}}^{TH}_{left} + \sum_{x=1}^{L-1} \mathcal{L}^{TH}_{x,x+1} + \widetilde{\mathcal{L}}^{TH}_{right} \tag{6.4.9}$$

where $\mathcal{L}^{TH}_{x,x+1}$ is (6.4.7) *and $\widetilde{\mathcal{L}}^{TH}_{left}$ and $\widetilde{\mathcal{L}}^{TH}_{right}$ are absorbing with rate* 1*:*

$$\begin{aligned}\widetilde{\mathcal{L}}^{TH}_{left} f(\boldsymbol{\xi}) &= \left\{ f(\xi^0 + \sum_{a=1}^{N-1} \boldsymbol{\delta}^0_a \xi^1_a,\ \xi^1 - \sum_{a=1}^{N-1} \boldsymbol{\delta}^1_a \xi^1_a + \boldsymbol{\delta}^1_N \sum_{a=1}^{N-1} \xi^1_a,\ \xi^2,\ldots,\xi^L) - f(\boldsymbol{\xi})\right\} \\ \widetilde{\mathcal{L}}^{TH}_{right} f(\boldsymbol{\xi}) &= \left\{ f(\xi^0,\ldots,\xi^{L-1},\ \xi^L - \sum_{a=1}^{N-1} \boldsymbol{\delta}^L_a \xi^L_a + \boldsymbol{\delta}^L_N \sum_{a=1}^{N-1} \xi^L_a,\ \xi^{L+1} + \sum_{a=1}^{N-1} \boldsymbol{\delta}^{L+1}_a \xi^L_a) - f(\boldsymbol{\xi})\right\}\ .\end{aligned} \tag{6.4.10}$$

The duality function is the same of the multi-species stirring process, i.e.

$$D(\boldsymbol{n},\boldsymbol{\xi}) = \prod_{a=1}^{N-1}\left(\rho_a^{left}\right)^{\xi_a^0}\prod_{x=1}^{L}\frac{(\nu-\sum_{a=1}^{N-1}\xi_a^x)!}{\nu!}\prod_{a=1}^{N-1}\frac{n_a^x!}{(n_a^x-\xi_a^x)!}\prod_{a=1}^{N-1}\left(\rho_a^{right}\right)^{\xi_a^{L+1}} \quad , \tag{6.4.11}$$

cf. (6.2.11).

This dual process is absorbing since the dual boundary generator $\mathcal{L}_{\text{left}}^{TH}$ ($\mathcal{L}_{\text{right}}^{TH}$) remove all dual particles at site 1 (L) and put these particles at the extra-site 0 ($L+1$) with rate 1.

Proof of Proposition 15: the proof of the duality consists in showing

$$\left(\mathcal{L}_{x,x+1}^{TH}D(\cdot,\boldsymbol{\xi})\right)(\boldsymbol{n}) = \left(\widetilde{\mathcal{L}}_{x,x+1}^{TH}D(\boldsymbol{n},\cdot)\right)(\boldsymbol{\xi}) \qquad x\in\{1,\dots,L-1\} \tag{6.4.12}$$

$$\left(\mathcal{L}_{\text{left}}^{TH}D(\cdot,\boldsymbol{\xi})\right)(\boldsymbol{n}) = \left(\widetilde{\mathcal{L}}_{\text{left}}^{TH}D(\boldsymbol{n},\cdot)\right)(\boldsymbol{\xi}) \; . \tag{6.4.13}$$

Duality for the right boundary is similar.
Equation (6.4.12) follows from the proof of the edge duality for the multi-species stirring process. To prove (6.4.13) we directly apply the generators on the duality function. For the sake of notation, we introduce

$$d(n^x,\xi^x) := \frac{(\nu-\sum_{a=1}^{N-1}\xi_a^x)!}{\nu!}\prod_{a=1}^{N-1}\frac{n_a^x!}{(n_a^x-\xi_a^x)!} \; . \tag{6.4.14}$$

First, we act with $\mathcal{L}_{\text{left}}^{TH}$ on $D(\boldsymbol{n},\boldsymbol{\xi})$. By writing explicitly $\rho_a^{\text{left}} = \frac{\alpha_a}{|\alpha|}$, this action gives

$$\begin{aligned}\left(\mathcal{L}_{\text{left}}^{TH}D(\cdot,\boldsymbol{\xi})\right)(\boldsymbol{n}) &= \sum_{r_1=0}^{\nu}\cdots\sum_{r_N=0}^{\nu}\mathbb{1}_{\{r_1+\ldots+r_N=\nu\}}\left\{\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{|\alpha|}\right)^{\xi_a^0}\right)\frac{\left(\nu-\sum_{c=1}^{N-1}\xi_c^1\right)!}{\nu!}\right.\\ &\times\left.\left(\prod_{b=1}^{N-1}\frac{r_b!}{(r_b-\xi_b^1)!}\right)\left(\prod_{x=2}^{L}d(n^x,\xi^x)\right)\left(\prod_{d=1}^{N-1}(\rho_d^R)^{\xi_d^{L+1}}\right)-D(\boldsymbol{n},\boldsymbol{\xi})\right\}\\ &\times\frac{\nu!}{r_1!\cdots r_N!}\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{\alpha_N}\right)^{r_a}\left(\frac{\alpha_N}{|\alpha|}\right)^{\nu}\right) \; .\end{aligned} \tag{6.4.15}$$

We consider the first addend in the curly bracket of the (6.4.15). We add and remove ξ_a^1 at the exponent of $\left(\frac{\alpha_a}{|\alpha|}\right)$ and then we multiplying this first addend by $\left(\frac{\alpha_N}{|\alpha|}\right)^{-(r_1^1+\ldots+r_{N-1})+(r_1^1+\ldots+r_{N-1})}$ to get

$$\begin{aligned}&\sum_{r_1=0}^{\nu}\cdots\sum_{r_N=0}^{\nu}\mathbb{1}_{\{r_1+\ldots+r_N=\nu\}}\\ &\times\left\{\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{|\alpha|}\right)^{\xi_a^0-\xi_a^1+\xi_a^1}\right)\frac{\left(\nu-\sum_{c=1}^{N-1}\xi_c^1\right)!}{\nu!}\left(\prod_{b=1}^{N-1}\frac{r_b!}{(r_b-\xi_b^1)!}\right)\left(\prod_{x=2}^{L}d(n^x,\xi^x)\right)\left(\prod_{d=1}^{N-1}(\rho_d^R)^{\xi_d^{L+1}}\right)\right\}\\ &\times\frac{\nu!}{r_1!\cdots r_N!}\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{\alpha_N}\right)^{r_a}\right)\left(\frac{\alpha_N}{|\alpha|}\right)^{-(r_1^1+\ldots+r_{N-1})+(r_1^1+\ldots+r_{N-1})}\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{|\alpha|}\right)^{\xi_a^0+\xi_a^1}\right)\sum_{r_1=0}^{\nu}\cdots\sum_{r_N=0}^{\nu}\mathbb{1}_{\{r_1+\ldots+r_N=\nu\}}\frac{\left(\nu-\xi_1^1-\ldots-\xi_{N-1}^1\right)!}{(r_1-\xi_1^1)!\cdots(r_{N-1}-\xi_{N-1}^1)!(\nu-r_1-\ldots-r_{N-1})!} \\
&\times\left(\prod_{b=1}^{N-1}\left(\frac{\alpha_b}{|\alpha|}\right)^{r_b-\xi_b^1}\right)\left(\frac{\alpha_N}{|\alpha|}\right)^{\nu-r_1-\ldots-r_{N-1}}\left(\prod_{x=2}^{N}d_x(n^x,\xi^x)\right)\left(\prod_{d=1}^{N-1}(\rho_d^R)^{\xi_d^{L+1}}\right) \\
&= \left(\prod_{a=1}^{N-1}\left(\rho_a^{\text{left}}\right)^{\xi_a^0+\xi_a^1}\right)\left(\prod_{x=2}^{L}d(n^x,\xi^x)\right)\left(\prod_{d=1}^{N-1}(\rho_d^{\text{right}})^{\xi_d^{L+1}}\right) .
\end{aligned}
\tag{6.4.16}
$$

Where we used

$$
\begin{aligned}
&\frac{\left(\nu-\sum_{c=1}^{N-1}\xi_c^1\right)!}{\nu!}\left(\prod_{b=1}^{N-1}\frac{r_b!}{(r_b-\xi_b^1)!}\right)\frac{\nu!}{r_1!\cdots r_N!} \\
=&\frac{\left(\nu-\xi_1^1-\ldots-\xi_{N-1}^1\right)!}{(r_1-\xi_1^1)!\cdots(r_{N-1}-\xi_{N-1}^1)!(\nu-r_1-\ldots-r_{N-1})!}
\end{aligned}
\tag{6.4.17}
$$

and

$$
\begin{aligned}
&\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{|\alpha|}\right)^{\xi_a^0-\xi_a^1+\xi_a^1}\right)\left(\prod_{b=1}^{N-1}\left(\frac{\alpha_b}{\alpha_N}\right)^{r_b}\right)\left(\prod_{c=1}^{N-1}\left(\frac{\alpha_N}{|\alpha|}\right)^{\nu-r_c+r_c}\right) \\
=&\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{|\alpha|}\right)^{\xi_a^0+\xi_a^1}\right)\left(\prod_{b=1}^{N-1}\left(\frac{\alpha_b}{|\alpha|}\right)^{r_b-\xi_b^1}\right)\left(\frac{\alpha_N}{|\alpha|}\right)^{\nu-r_1-\ldots-r_{N-1}}
\end{aligned}
$$

and the multinomial theorem in the last equality.
We now consider the second addend in the curly brackets of (6.4.15). We multiplying it by $\prod_{a=1}^{N-1}\left(\frac{\alpha_N}{|\alpha|}\right)^{-r_a+r_a}$ to get

$$
\begin{aligned}
&D(\boldsymbol{n},\boldsymbol{\xi})\sum_{r_1=0}^{\nu}\cdots\sum_{r_N=0}^{\nu}\mathbb{1}_{\{r_1+\ldots+r_N=\nu\}}\frac{\nu!}{r_1!\cdots r_N!} \\
&\times\left(\prod_{a=1}^{N-1}\left(\frac{\alpha_a}{\alpha_N}\right)^{r_a}\right)\left(\frac{\alpha_N}{|\alpha|}\right)^{\nu}\left(\frac{\alpha_N}{|\alpha|}\right)^{-(r_1+\ldots+r_{N-1})+(r_1+\ldots+r_{N-1})} \\
=&D(\boldsymbol{n},\boldsymbol{\xi})
\end{aligned}
\tag{6.4.18}
$$

where we used the multinomial theorem. Therefore, replacing (6.4.16) and (6.4.18) in (6.4.15) we have that

$$
\left(\mathcal{L}_{\text{left}}^{TH}D(\cdot,\boldsymbol{\xi})\right)(\boldsymbol{n}) = \left\{\prod_{a=1}^{N-1}\left(\frac{\alpha_k}{|\alpha|}\right)^{\xi_a^0+\xi_a^1}\prod_{x=2}^{N}d_x(n^x,\xi^x)\prod_{d=1}^{N-1}(\rho_a^R)^{\xi_d^{L+1}}-D(\boldsymbol{n},\boldsymbol{\xi})\right\}
\tag{6.4.19}
$$

$$
= \left(\widetilde{\mathcal{L}}_{\text{left}}^{TH}D(\boldsymbol{n},\cdot)\right)(\boldsymbol{\xi}) .
\tag{6.4.20}
$$

□

Chapter 7

Exact formulas for the integrable case

7.1 Motivations

As we have already pointed out, out-of-equilibrium particles systems can be used to model non-equilibrium statistical mechanics problems. Under this perspective, it is interesting to find explicit expressions for the non-equilibrium steady state. Usually, this is not an easy task because the correlations are long-ranged [52].

We recall that for the single species set-up, the solution of the exclusion process with open boundaries [24, 83] has been of crucial importance in our understanding of the structural properties of non-equilibrium state, such as long-range correlations [52], non-local density large deviation functions [84, 75] and current large deviations [85, 86]. One may argue that the results available for the exclusion process are rooted in the algebraic symmetries of the corresponding integrable spin $s = 1/2$ Heisenberg chain, see e.g. [19, 21] for two excellent reviews. This integrability property can be combined with duality and leads to closed-form expressions for the correlations and the non-equilibrium steady state, as shown in [20, 28].

A special case of the boundary driven multi-species stirring process is obtained when the graph G is a finite chain, with two reservoirs attached to the first and the last site respectively, and the maximal number of particles allowed at each site is fixed to be $\nu = 1$. In this case the multi-species stirring process with open boundaries becomes integrable and, see [13], the non-equilibrium steady state has been characterized via MPA. In this chapter we aim to extend this result by finding a closed formula for the multi-point non-equilibrium steady state correlations. To achieve this goal, we combine the matrix product ansatz (derived in [13]) and the $gl(N)$ invariance of the process in the bulk cf. (6.2.37), that leads to absorbing duality of Chapter 6. This allows further to determine the absorption probabilities and the probability mass-function of the non-equilibrium steady state in closed forms.

We start with Section 7.2 by introducing the process on a chain and by specializing the duality results of Chapter 6. We then recall the MPA (see Section 7.3) and, in Section 7.4 and in Section 7.5, we state and prove the aimed exact formula for the multi-point non-equilibrium steady state correlations. Furthermore, we use it to derive the probability mass function of the non-equilibrium steady state. Finally, in Section 7.6, we specialize the results to SSEP, matching with the literature.

7.2 Integrable process on a line segment

In this section we specialize the multi-species stirring process to the geometry of the one-dimensional chain with sites $\{1, \ldots, L\}$ where two reservoirs are attached to the boundary sites 1 and L. The reservoirs, exchanging particles with the external environment, put the chain out of equilibrium. The Hamiltonian that we consider here is obtained from (4.3.9) assuming that the conductances are

$$\omega_{x,y} = \begin{cases} 1 & \text{if} \quad |x-y| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{7.2.1}$$

and the coupling to reservoirs are

$$\Gamma_x = \begin{cases} 1 & \text{if} \quad x \in \{1, L\} \\ 0 & \text{otherwise} \end{cases} \quad . \tag{7.2.2}$$

The case $\nu = 1$, i.e. one particle at most for each site, is integrable and has been considered previously in [13]. For $\nu = 1$ we denote the $gl(N)$ generators as $(e_{AB})_{A,B \in \{1,\ldots,N\}}$ obeying $(e_{AB})_{CD} = \delta_{AC}\delta_{BD}$, i.e. they are the basis generators of the first fundamental representation of $gl(N)$ (with highest weight $(1, 0, \ldots, 0)$). The Hamiltonian can then be written as

$$H = H_{\text{left}} + H_{\text{bulk}} + H_{\text{right}} \tag{7.2.3}$$

where

$$H_{\text{bulk}} = \sum_{x=1}^{L-1} \mathcal{H}_{x,x+1} \ . \tag{7.2.4}$$

Here $\mathcal{H}_{x,x+1}$ denotes the two-site Hamiltonian

$$\mathcal{H} = P - 1 \tag{7.2.5}$$

with the permutation matrix

$$P = \sum_{A,B=1}^{N} e_{AB} \otimes e_{BA} \, , \tag{7.2.6}$$

acting non-trivially on the vertices of the edge $x, x+1$. In this context it will be useful to introduce the following notation for the occupation variables of the process. Each configuration is denoted by

$$|\boldsymbol{\tau}\rangle = |\tau_1, \ldots, \tau_L\rangle \tag{7.2.7}$$

with $\tau_x \in \{1, \ldots, N\}$, $\forall x \in \{1, \ldots, L\}$. Since the maximal occupancy at each site is $\nu = 1$, the configuration $\boldsymbol{n}$ introduced in Section 4.2.2 and $\boldsymbol{\tau}$ are related by

$$n_A^x = \delta_{\tau_x, A} \ . \tag{7.2.8}$$

The state space of the process is now

$$\Omega' = \{|\tau_1, \ldots, \tau_L\rangle \ : \ \tau_x \in \{1, \ldots, N\}\} \ . \tag{7.2.9}$$

The action of the Hamiltonian density $\mathcal{H}$ on the tensor product of the configuration of two sites follows immediately from (7.2.5) and reads

$$\mathcal{H}|\tau\rangle \otimes |\tau'\rangle = |\tau'\rangle \otimes |\tau\rangle - |\tau\rangle \otimes |\tau'\rangle \ . \tag{7.2.10}$$

The boundary terms of the Hamiltonian (7.2.3) are given by

$$H_{\text{left}} = \begin{pmatrix} \alpha_1 - 1 & \alpha_1 & \alpha_1 & \dots & \dots & \alpha_1 \\ \alpha_2 & \alpha_2 - 1 & \alpha_2 & \dots & \dots & \alpha_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ \alpha_{N-1} & \alpha_{N-1} & \dots & \dots & \alpha_{N-1} - 1 & \alpha_{N-1} \\ \alpha_N & \alpha_N & \dots & \dots & \alpha_N & \alpha_N - 1 \end{pmatrix} \tag{7.2.11}$$

and

$$H_{\text{right}} = \begin{pmatrix} \beta_1 - 1 & \beta_1 & \beta_1 & \dots & \dots & \beta_1 \\ \beta_2 & \beta_2 - 1 & \beta_2 & \dots & \dots & \beta_2 \\ \vdots & \vdots & & \ddots & & \vdots \\ \beta_{N-1} & \beta_{N-1} & \dots & \dots & \beta_{N-1} - 1 & \beta_{N-1} \\ \beta_N & \beta_N & \dots & \dots & \beta_N & \beta_N - 1 \end{pmatrix} \tag{7.2.12}$$

where, without loss of generality, we assume that the parameters satisfy

$$\sum_{A=1}^{N} \alpha_A = 1, \qquad \sum_{A=1}^{N} \beta_A = 1 \ . \tag{7.2.13}$$

These boundary Hamiltonian matrices can also be written in terms of the first fundamental representation of $gl(N)$ as

$$H_{\text{left}} = \sum_{A,B=1}^{N} \alpha_A \left(e^1_{AB} - e^1_{BB} \right), \qquad H_{\text{right}} = \sum_{A,B=1}^{N} \beta_A \left(e^L_{AB} - e^L_{BB} \right) \ . \tag{7.2.14}$$

Under these assumption the Hamiltonian 7.2.3 is integrable. This has been shown in [13] within the Quantum Inverse Scattering Method.

7.2.1 Duality for the integrable process on a line segment

Here we consider the chain defined at the beginning of Section 7.2 with the hard-core constrain (at most one particle per site), i.e. $\nu = 1$.
The duality result stated in Section 6.2 is adapted to the present situation as follows. The extra-sites are denoted by 0 and $L+1$, and they are connected with sites 1 and L of the chain respectively. The matrix R_x defined in (6.2.22) reduces to the identity, therefore the relation between the generators of the algebra and its transposed reduces to

$$e^T_{AB} = e_{BA} \ . \tag{7.2.15}$$

The duality matrix reads

$$D = \mathcal{D}_0 \otimes \prod_{x=1}^{L} \exp\left(\sum_{a=1}^{N-1} e^x_{aN} \right) \otimes \mathcal{D}_{L+1} \tag{7.2.16}$$

where $\mathcal{D}_0$ and $\mathcal{D}_{L+1}$ are given in (6.2.24). The elements of this duality matrix are given by

$$D(\boldsymbol{\tau}, \boldsymbol{\xi}) = \left(\prod_{a=1}^{N-1} \alpha_a^{\xi_a^0} \right) \left(\prod_{x=1}^{L} \prod_{a=1}^{N-1} \mathbb{1}_{\{\delta_{\tau_x,a} \geqslant \xi_a^x\}} \right) \left(\prod_{a=1}^{N-1} \beta_a^{\xi_a^{L+1}} \right) \ . \tag{7.2.17}$$

The dual Hamiltonian is

$$\widetilde{H} = \widetilde{H}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{H}_{x,x+1} + \widetilde{H}_{\text{right}} \tag{7.2.18}$$

where $\mathcal{H}$ is defined in (7.2.5) and

$$\widetilde{H}_{\text{left}} = \sum_{a=1}^{N-1} \left((\mathbf{a}^+)_a^0 \, e_{Na}^1 - e_{aa}^1\right) \qquad \widetilde{H}_{\text{right}} = \sum_{a=1}^{N-1} \left((\mathbf{a}^+)_a^{L+1} \, e_{Na}^L - e_{aa}^L\right) \, . \tag{7.2.19}$$

The configuration of the dual process is still indicated by the vector $|\boldsymbol{\xi}\rangle$ defined in (6.2.14). We recall that, in this dual configuration, at each site of the bulk at most one particles is allowed, while the maximal occupancy is unbounded at the extra-sites 0 and $L+1$.
In the long-time limit the dual process voids the chain, i.e. all particles of types $\{1, \ldots, N-1\}$ are eventually absorbed at the extra-sites 0 and $L+1$ and replaced by types N. We also notice that, up the the bosonic creation operators $(\mathbf{a}^+)_a$, the dual boundary Hamiltonians (7.2.19) are triangular. Using duality, one can compute the m-point correlations between non-empty particles in terms of the absorption probabilities of m dual particles. Therefore, to determine the non-equilibrium steady state correlations of the integrable multi-species stirring process, it is enough to compute these absorption probabilities. In the following we denote by μ the non-equilibrium steady state distribution and we call $\boldsymbol{Y} = (Y_1, \ldots, Y_L)$ the random vector with law μ. Furthermore we write $\mathbb{E}[\cdot]$ for the expectation with respect to μ. As a consequence of Proposition 14 of Section 6.3 we state the following result.

Corollary 2 (Correlations via duality) *Let $m \in \{1, \ldots, L\}$ and consider m sites $1 \leqslant x_1 < x_2 < \ldots < x_m \leqslant L$ and m colours denoted by $a_k \in \{1, \ldots, N-1\}$ with $k = 1, 2, \ldots, m$, chosen among the $N-1$ available species. Then, the m-point correlations in the non-equilibrium steady state read*

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k} = a_k\}}\right] = \sum_{t_1=0}^{1} \cdots \sum_{t_m=0}^{1} \left(\prod_{k=1}^{m} \alpha_{a_k}^{t_k} \beta_{a_k}^{1-t_k}\right) \mathcal{P}_{x_1,\ldots,x_m}(t_1, \ldots, t_m) \tag{7.2.20}$$

where the absorption probabilities reads

$$\mathcal{P}_{x_1,\ldots,x_m}(t_1, \ldots, t_m) := \mathbb{P}\left(\boldsymbol{\xi}(\infty) = \sum_{k=1}^{m} \left(t_k \delta_{a_k}^0 + (1-t_k)\delta_{a_k}^{L+1}\right) \Big| \boldsymbol{\xi}(0) = \boldsymbol{\xi}\right) \, . \tag{7.2.21}$$

Here the initial dual configuration is $\boldsymbol{\xi} = \sum_{k=1}^{m} \delta_{a_k}^{x_k}$, meaning that

$$\begin{cases} \xi_A^x = 1 & \textit{if } x = x_k \textit{ and } A = a_k \\ \xi_A^x = 0 & \textit{otherwise} \end{cases} \, . \tag{7.2.22}$$

In (7.2.21) the variable t_k is 1 (resp. 0) when the dual particle of species a_k initially positioned at site x_k is absorbed at at the extra site 0 (resp. $L+1$).

Proof of Corollary 2: For any $\boldsymbol{\tau} \in \Omega'$ the duality matrix defined in (7.2.17) evaluated on the dual configuration $\boldsymbol{\xi} \in \widetilde{\Omega}$ given in (7.2.22) read

$$D(\boldsymbol{\tau}, \boldsymbol{\xi}) = \prod_{k=1}^{m} \mathbb{1}_{\{\tau_{x_k} = a_k\}} \, . \tag{7.2.23}$$

Therefore, by ergodicity and duality we have

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^{m}\mathbb{1}_{\{Y_{x_k}=a_k\}}\right] &= \lim_{t\to\infty}\mathbb{E}_{\boldsymbol{\tau}}\left[D(\boldsymbol{\tau}(t),\boldsymbol{\xi})\right] = \lim_{t\to\infty}\mathbb{E}_{\boldsymbol{\xi}}\left[D(\boldsymbol{\tau},\boldsymbol{\xi}(t))\right] \\ &= \sum_{t_1=0}^{1}\cdots\sum_{t_m=0}^{1}\left(\prod_{k=1}^{m}\alpha_{a_k}^{t_k}\beta_{a_k}^{1-t_k}\right)\mathcal{P}_{x_1,\ldots,x_m}(t_1,\ldots,t_m)\ .\end{aligned} \tag{7.2.24}$$

□

By using the correlations in the non-equilibrium steady state (7.4.4), it is possible to determine the non-equilibrium steady probability distribution.

Corollary 3 (Non-equilibrium steady state) *For any configuration $\boldsymbol{\tau}\in\Omega'$ the probability mass function of the non-equilibrium steady states is fully determined by the correlations of equation* (7.4.4) *through the relation*

$$\mu(\boldsymbol{\tau}) = \sum_{m=0}^{L}\sum_{1\leqslant x_1<x_2<\ldots<x_m\leqslant L}\left(\prod_{x\notin\{x_1,\ldots,x_m\}}\delta_{\tau_x,N}\right)\sum_{b_1,\ldots,b_m=1}^{N-1}\left(\prod_{k=1}^{m}(\delta_{\tau_{x_k},b_k}-\delta_{\tau_{x_k},N})\right)\mathbb{E}\left[\prod_{k=1}^{m}\mathbb{1}_{\{Y_{x_k}=b_k\}}\right] \tag{7.2.25}$$

where $\delta_{\tau_{x_k},b_k}-\delta_{\tau_{x_k},N} = (-1)^{\delta_{\tau_{x_k},N}}$.

Example for an $L=2$ chain. To clarify equation (7.2.25) we report the example of a chain with $L=2$, where we compute the probability mass functions of three configurations. Without loss of generality we write only the formulae involving particles of species 1. We have:

- probability of two occupied site:

$$\mu(1,1) = \mathbb{E}\left[\mathbb{1}_{\{Y_1=1\}}\mathbb{1}_{\{Y_2=1\}}\right]\ . \tag{7.2.26}$$

  Here we have only one term corresponds to $m=2$ in formula (7.2.25).

- probability of the first site occupied and the second empty:

$$\begin{aligned}\mu(1,N) =& \mathbb{E}\left[\mathbb{1}_{\{Y_1=1\}}\mathbb{1}_{\{Y_2=N\}}\right] = \mathbb{E}\left[\mathbb{1}_{\{Y_1=1\}}\left(1-\sum_{a=1}^{N-1}\mathbb{1}_{\{Y_2=a\}}\right)\right] \\ =& \mathbb{E}\left[\mathbb{1}_{\{Y_1=1\}}\right]-\sum_{a=1}^{N-1}\mathbb{E}\left[\mathbb{1}_{\{Y_1=1\}}\mathbb{1}_{\{Y_2=a\}}\right]\ .\end{aligned} \tag{7.2.27}$$

  In the last equality we have two terms: the first corresponds to $m=1$ (with $x_1=1$) and the second corresponds to $m=2$ in formula (7.2.25).

- probability of both sites empty:

$$\begin{aligned}\mu(N,N) =& \mathbb{E}\left[\mathbb{1}_{\{Y_1=N\}}\mathbb{1}_{\{Y_2=N\}}\right] = \mathbb{E}\left[\left(1-\sum_{a_1=1}^{N-1}\mathbb{1}_{\{Y_1=a_1\}}\right)\left(1-\sum_{a_2=1}^{N-1}\mathbb{1}_{\{Y_2=a_2\}}\right)\right] \\ =& 1-\sum_{a_1=1}^{N-1}\mathbb{E}\left[\mathbb{1}_{\{Y_1=a_1\}}\right]-\sum_{a_2=1}^{N-1}\mathbb{E}\left[\mathbb{1}_{\{Y_2=a_2\}}\right]+\sum_{a_1,a_2=1}^{N-1}\mathbb{E}\left[\mathbb{1}_{\{Y_1=a_1\}}\mathbb{1}_{\{Y_2=a_2\}}\right]\ .\end{aligned} \tag{7.2.28}$$

In the last equality we have four terms: the first term corresponds to $m = 0$, the second and the third term correspond to $m = 1$ (with $x_1 = 1$ and $x_1 = 2$ respectively) and the fourth term corresponds to $m = 2$ in formula (7.2.25).

Proof of Corollary 3: Given $\boldsymbol{\tau} \in \Omega'$, we have

$$\begin{aligned}\mu(\boldsymbol{\tau}) =& \mathbb{E}\left[\prod_{x=1}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right] = \mathbb{E}\left[\left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)\left(\prod_{x=1\,:\,\tau_x=N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)\right] \\ =& \mathbb{E}\left[\left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)\left(\prod_{x=1\,:\,\tau_x=N}^{L}\left(1-\sum_{a=1}^{N-1}\mathbb{1}_{\{Y_x=a\}}\right)\right)\right]\end{aligned} \tag{7.2.29}$$

where in the last equality we exploited the fact that

$$\mathbb{1}_{\{Y_x=N\}} = 1 - \sum_{a=1}^{N-1} \mathbb{1}_{\{Y_x=a\}} \,. \tag{7.2.30}$$

The configuration $\boldsymbol{\tau}$ has $\ell = \sum_{x=1}^{L}(1-\delta_{\tau_x,N})$ occupied sites and then the remaining $L-\ell$ sites are empty. Therefore, we may write

$$\begin{aligned}&\left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)\left(\prod_{x=1\,:\,\tau_x=N}^{L}\left(1-\sum_{a=1}^{N-1}\mathbb{1}_{\{Y_x=a\}}\right)\right) \\ =& \prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}} + \left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)(-1)\left\{\sum_{x_1=1}^{L}\delta_{\tau_{x_1},N}\sum_{a_1=1}^{N-1}\mathbb{1}_{\{Y_{x_1}=a_1\}}\right\}+ \\ &+\left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)(-1)^2\left\{\sum_{1\leqslant x_1<x_2\leqslant L}\delta_{\tau_{x_1},N}\delta_{\tau_{x_2},N}\sum_{a_1,a_2=1}^{N-1}\mathbb{1}_{\{Y_{x_1}=a_1\}}\mathbb{1}_{\{Y_{x_2}=a_2\}}\right\}+\ldots+ \\ &+\left(\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}}\right)(-1)^{L-\ell}\sum_{1\leqslant x_1<\ldots<x_{L-\ell}\leqslant L}\left(\prod_{k=1}^{L-\ell}\delta_{\tau_{x_k},N}\right)\sum_{a_1,\ldots,a_{L-\ell}=1}^{N-1}\prod_{k=1}^{L-\ell}\mathbb{1}_{\{Y_{x_k}=a_k\}}\,.\end{aligned} \tag{7.2.31}$$

In the right-hand-side of the equation above the first addend is the product of the indicator of ℓ sites, since none of the empty sites of the configuration $\boldsymbol{\tau}$ has been considered. Similarly, the second addend is the product of the indicator of $\ell+1$ occupied sites. Indeed, in addition to the ℓ occupied sites of the configuration $\boldsymbol{\tau}$, one empty site is in turn chosen and its hole is filled with all possible species particles. The idea is repeated in the next addends with $2, 3, \ldots L-\ell$ empty sites that are, in turn, filled with all possible species of particles. We notice that the exponent of the factors (-1) is given by the number of holes that have been filled with all possible species of particles. We introduce an index $m \in \{\ell, \ell+1, \ldots, L\}$ that counts the number of correlated site and we define the coordinates $1 \leqslant x_1 < x_2 < \ldots < x_m \leqslant L$. Then, we associate to each of these m one of the addend in (7.2.31). In particular, for $m = \ell$ we associate the first addend and we rewrite it as

$$\begin{aligned}\prod_{x=1\,:\,\tau_x\neq N}^{L} \mathbb{1}_{\{Y_x=\tau_x\}} =& \sum_{1\leqslant x_1<\ldots<x_\ell\leqslant L}\left(\prod_{x\notin\{x_1,\ldots,x_\ell\}}\delta_{\tau_x,N}\right) \\ &\times\sum_{b_1,\ldots,b_\ell=1}^{N-1}\left(\prod_{k=1}^{\ell}(\delta_{\tau_{x_k},b_k}-\delta_{\tau_{x_k},N})\right)\left(\prod_{k=1}^{\ell}\mathbb{1}_{\{Y_{x_k}=b_k\}}\right)\end{aligned} \tag{7.2.32}$$

Similarly, the second addend in (7.2.31) is associated to $m = \ell + 1$ and it is rewritten as

$$
\begin{aligned}
&\left(\prod_{x=1\,:\,\tau_x \neq N}^{L} \mathbb{1}_{\{Y_x = \tau_x\}} \right) (-1) \left\{ \sum_{x_1=1}^{L} \delta_{\tau_{x_1},N} \sum_{a_1=1}^{N-1} \mathbb{1}_{\{Y_{x_1}=a_1\}} \right\} \\
&= \sum_{1 \leqslant x_1 < \ldots < x_{\ell+1} \leqslant L} \left(\prod_{x \notin \{x_1,\ldots,x_{\ell+1}\}} \delta_{\tau_x,N} \right) \sum_{b_1,\ldots,b_{\ell+1}=1}^{N-1} \left(\prod_{k=1}^{\ell+1} (\delta_{\tau_{x_k},b_k} - \delta_{\tau_{x_k},N}) \right) \left(\prod_{k=1}^{\ell+1} \mathbb{1}_{\{Y_{x_k}=b_k\}} \right)
\end{aligned}
\tag{7.2.33}
$$

The idea goes on for $m = \ell + 2, \ldots L$ and we obtain that

$$
\begin{aligned}
&\left(\prod_{x=1\,:\,\tau_x \neq N}^{L} \mathbb{1}_{\{Y_x = \tau_x\}} \right) \left(\prod_{x=1\,:\,\tau_x = N}^{L} \left(1 - \sum_{a=1}^{N-1} \mathbb{1}_{\{Y_x=a\}} \right) \right) \\
&= \sum_{m=\ell}^{L} \sum_{1 \leqslant x_1 < x_2 < \ldots < x_m \leqslant L} \left(\prod_{x \notin \{x_1,\ldots,x_m\}} \delta_{\tau_x,N} \right) \sum_{b_1,\ldots,b_m=1}^{N-1} \left(\prod_{k=1}^{m} (\delta_{\tau_{x_k},b_k} - \delta_{\tau_{x_k},N}) \right) \left(\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=b_k\}} \right)
\end{aligned}
\tag{7.2.34}
$$

The above sum can be extended until $m = 0$ since all the terms $m = 0, \ldots, \ell - 1$ are vanishing. Finally, by taking the expectation with respect to the steady state distribution we obtain (7.2.25).

□

7.3 Matrix product ansatz

The matrix product ansatz for the multi-species stirring process has been formulated in [13], and here we briefly recall the main steps. Denoting by $|\Psi(t)\rangle$ the column vector that encodes the probability distribution of the chain with Hamiltonian H (7.2.3) at time $t \geqslant 0$, its evolution equation is given by the master equation (see (2.1.43)) that reads

$$
\frac{d|\Psi(t)\rangle}{dt} = H|\Psi(t)\rangle \,. \tag{7.3.1}
$$

This Markov chain is irreducible and positive recurrent, therefore there exists a unique stationary measure, that will be reached when time goes to infinity, regardless of the initial configuration. We denote by $|\Psi\rangle$ the column vector that gives the stationary distribution (non-equilibrium steady state). This vector is the right eigenvector with vanishing eigenvalue of H, i.e. it solves

$$
H|\Psi\rangle = 0 \,. \tag{7.3.2}
$$

The MPA states the following

$$
|\Psi\rangle = \frac{1}{Z_L} \langle\langle W| \underbrace{\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}}_{L \text{ times}} |V\rangle\rangle \tag{7.3.3}
$$

with the normalization

$$
Z_L = \langle\langle W| (X_1 + \ldots + X_N)^L |V\rangle\rangle \,. \tag{7.3.4}
$$

Here the operators X_A fulfil, for all $A, B \in \{1, \ldots, N\}$ the commutators

$$[X_A, X_B] = (\alpha_A - \beta_A)X_B - (\alpha_B - \beta_B)X_A \tag{7.3.5}$$

and their action on the boundary vectors are given, for all $A \in \{1, \ldots, N\}$, by

$$\langle\langle W|\,(\alpha_A(X_1 + \ldots + X_N) - X_A) = (\alpha_A - \beta_A)\langle\langle W| \tag{7.3.6}$$

$$(\beta_A(X_1 + \ldots + X_N) - X_A)\,|V\rangle\rangle = -(\alpha_A - \beta_A)|V\rangle\rangle \tag{7.3.7}$$

Without loss of generality, we further assume that $\langle\langle W|V\rangle\rangle = 1$.
To show that (7.3.3) is the steady state we report the proof done in [13], where an argument similar to the proof of Proposition (8) was used. We introduce

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix}, \qquad \overline{\boldsymbol{X}} = \begin{pmatrix} (\alpha_1 - \beta_1) \\ \vdots \\ (\alpha_N - \beta_N) \end{pmatrix} \tag{7.3.8}$$

By using the commutation relations of X_A's (7.3.5), the action on the boundary vectors (7.3.6) and (7.3.7) and the definition of the Hamiltonian H written in (7.2.3) we have that

$$\mathcal{H}(\boldsymbol{X} \otimes \boldsymbol{X}) = \boldsymbol{X} \otimes \overline{\boldsymbol{X}} - \overline{\boldsymbol{X}} \otimes \boldsymbol{X} \tag{7.3.9}$$

and

$$\langle\langle W|H_{\text{left}}\boldsymbol{X} = \langle\langle W|\overline{\boldsymbol{X}}, \qquad H_{\text{right}}\boldsymbol{X}|V\rangle\rangle = -\overline{\boldsymbol{X}}|V\rangle\rangle \tag{7.3.10}$$

Using the above relations we obtain

$$\begin{aligned} Z_L H|\Psi\rangle = {}& \langle\langle \overline{X} \otimes \Big(\underbrace{\boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X}}_{L-1\,\text{times}}\Big)|V\rangle\rangle - \langle\langle W|\,\Big(\underbrace{\boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X}}_{L-1\text{ times}}\Big) \otimes \overline{\boldsymbol{X}}|V\rangle\rangle \\ & + \sum_{x=1}^{L-1} \langle\langle W|\,\Big(\underbrace{\boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X}}_{x-1\text{ times}}\Big) \otimes \big(\boldsymbol{X} \otimes \overline{\boldsymbol{X}} - \overline{\boldsymbol{X}} \otimes \boldsymbol{X}\big) \otimes \Big(\underbrace{\boldsymbol{X} \otimes \cdots \otimes \boldsymbol{X}}_{L-x-1\text{ times}}\Big)|V\rangle\rangle = 0 \end{aligned} \tag{7.3.11}$$

When the system is at equilibrium, i.e. $\alpha_A = \beta_A$ for all $A \in \{1, \ldots, N\}$, the operators X_A becomes proportional to the identity, with proportionality constant α_A. As a consequence they commute and $|\Psi\rangle$ is trivially the steady state.

Remark 13 *As already mention, when $N = 2$ the multi-species stirring process reduces to the SSEP. In this case, one can retrieve the matrices D and E of the MPA for the SSEP introduced in Proposition 8 by setting*

$$D = \frac{X_2}{\alpha_2 - \beta_2} \qquad \textit{and} \qquad E = -\frac{X_1}{\alpha_1 - \beta_1} \tag{7.3.12}$$

provided that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$

Remark 14 *Only $N - 1$ of the N equations* (7.3.6) *are independent. This can be seen by summing them over the index $A \in \{1, \ldots, N\}$, i.e.*

$$\sum_{A=1}^{N} \langle\langle W|\,(\alpha_A(X_1 + \ldots + X_N) - X_A) = 0, \tag{7.3.13}$$

and using (7.2.13)*. Similarly for the right boundary* (7.3.7) *there are only $N - 1$ independent equations.*

The MPA gives an abstract form of the non-equilibrium steady state in terms of an algebra of operators acting on a supplementary space. However, the computation of correlations are in general involved.

7.4 Exact non-equilibrium steady state correlations

Strategy. In the next Section 7.4.1 we will show that the DEHP algebra in (7.3.5)-(7.3.7) can be simplified substantially using the $gl(N)$ invariance of the bulk. In fact, such simplification can always be achieved if there exists an absorbing dual process as established in Chapter 6

The idea is summarized in the following scheme. We define a sequence of local similarity transformations such that

$$H \overset{S_1}{\longleftrightarrow} H^{'} \overset{S_2}{\longleftrightarrow} H^{''} \tag{7.4.1}$$

where $H^{'} = S_1 H S_1^{-1}$ has both boundary terms in a triangular form, and $H^{''} = S_2 H' S_2^{-1}$ has the left boundary in a triangular form and the right boundary is diagonal, we further refer the reader to [87, 69, 21, 20] where this idea was explored for the monospecies case. Using these transformations the commutators (7.3.5) and the action on the boundary vectors (7.3.6), (7.3.7) simplify significantly. More precisely for the components of $\tilde{X} = S_2 S_1 X$ we obtain the bulk relations

$$\left[\widetilde{X}_a, \widetilde{X}_N\right] = (\alpha_a - \beta_a)\widetilde{X}_N \ . \tag{7.4.2}$$

for $a = 1, \dots, N-1$. At the boundaries

$$\langle\langle W| \left((\alpha_a - \beta_a)\widetilde{X}_N - \widetilde{X}_a\right) = (\alpha_a - \beta_a)\langle\langle W| \ , \qquad \widetilde{X}_a|V\rangle\rangle = (\alpha_a - \beta_a)|V\rangle\rangle \, . \tag{7.4.3}$$

where again we have $a = 1, \dots, N-1$. As a consequence, the ground state $|\Psi^{''}\rangle$ of $H^{''}$ can be written exactly in closed-form. Finally, reversing the transformations in (7.4.1) we retrieve $|\Psi\rangle$, i.e. the vector whose components are the probabilities of a certain configuration of the process in the non-equilibrium steady state. The first transformation S_1 is closely related to the duality matrix (6.2.20) and we will see that $H^{'}$ is $\widetilde{H}^T$, up to the extra-site term described by the bosonic creation operator in (6.2.17). The Hamiltonian $H^{'}$ is not stochastic, however it turns out that the components of its eigenvector with zero eigenvalue $|\Psi^{'}\rangle$ are the correlations in the non-equilibrium steady state.

7.4.1 Correlations in the non-equilibrium steady state

In this section we write a formula for the stationary non-equilibrium steady state correlations between m- points of the chain.

Theorem 18 (Correlations in the non-equilibrium steady state) *Let $m \in \{1, \dots, L\}$. Consider m sites $1 \leqslant x_1 < x_2 < \dots < x_m \leqslant L$ and m colours denoted by $a_k \in \{1, \dots, N-1\}$ with $k = 1, 2, \dots, m$, chosen among the $N-1$ available species. Then the m-point correlations with respect to the non-equilibrium steady state measure are given by*

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k} = a_k\}}\right] = \sum_{t_1, \dots, t_m = 0}^{1} \left(\prod_{k=1}^{m} \alpha_{a_k}^{t_k} \beta_{a_k}^{1-t_k}\right) \mathcal{P}_{x_1, \dots, x_m}(t_1, \dots, t_m) \tag{7.4.4}$$

where

$$\mathcal{P}_{x_1, \dots, x_m}(t_1, \dots, t_m) = \sum_{c_1 = t_1}^{1} \cdots \sum_{c_m = t_m}^{1} f(c_1, \dots, c_m) \prod_{j=1}^{m} (-1)^{c_j - t_j} g_j(x_j, c_j, \dots, c_m) \tag{7.4.5}$$

with

$$f(c_1,\dots,c_m) = \frac{(L+1-\sum_{a=1}^{m} c_a)!}{(L+1)!} \tag{7.4.6}$$

and

$$g_j(x_j,c_j,\dots,c_m) = \left(L+2-x_j-\sum_{k=j}^{m} c_k\right)^{c_j} \ . \tag{7.4.7}$$

Examples

We give examples of correlations for $m = 1,2,3$ applying formula (7.4.4).

One-point correlations We consider the average with respect to μ of the occupation variable of the species $a_1 \in \{1,\dots,N-1\}$ at coordinate $x_1 \in \{1,\dots,L\}$. Using (7.4.5) we obtain the absorption probabilities

$$\mathcal{P}_{x_1}(0) = 1 - \frac{(L+1-x_1)}{(L+1)} \qquad \mathcal{P}_{x_1}(1) = \frac{(L+1-x_1)}{(L+1)} \tag{7.4.8}$$

where $\mathcal{P}_{x_1}(1)$ is the probability that a single random walk started at x_1 is absorbed. Then, using (7.4.4), we have

$$\langle \rho_{a_1}^{x_1} \rangle = \mathbb{E}\left[\mathbb{1}_{\{Y_{x_1}=a_1\}}\right] = \frac{(L+1-x_1)}{(L+1)}\alpha_{a_1} + \frac{x_1}{(L+1)}\beta_{a_1} \ , \tag{7.4.9}$$

where, for the sake of notation, we have introduced $\langle \rho_{a_1}^{x_1} \rangle$.

Two-point correlations We consider the average with respect to μ of the occupation variable of the species $a_1, a_2 \in \{1,\dots,N-1\}$ at coordinates $x_1, x_2 \in \{1,\dots,L\}$ such that $x_1 < x_2$. Using (7.4.5) we obtain the absorption probabilities

$$\begin{array}{ll} \mathcal{P}_{x_1,x_2}(0,0) = \dfrac{x_1(x_2-1)}{L(L+1)} & \mathcal{P}_{x_1,x_2}(1,0) = \dfrac{x_2(L+1-x_1)}{L(L+1)} \\ \mathcal{P}_{x_1,x_2}(0,1) = \dfrac{x_1(L+1-x_2)}{L(L+1)} & \mathcal{P}_{x_1,x_2}(1,1) = \dfrac{(L-x_1)(L+1-x_2)}{L(L+1)} \end{array} . \tag{7.4.10}$$

Therefore, we compute the second cumulant, i.e. the two-point connected correlation

$$\mathbb{E}\left[\left(\mathbb{1}_{\{Y_{x_1}=a_1\}} - \langle \rho_{a_1}^{x_1} \rangle\right)\left(\mathbb{1}_{\{Y_{x_2}=a_2\}} - \langle \rho_{a_2}^{x_2} \rangle\right)\right] = -\frac{x_1(L-x_2+1)}{L(L+1)^2}\left(\alpha_{a_1}-\beta_{a_1}\right)\left(\alpha_{a_2}-\beta_{a_2}\right) \ . \tag{7.4.11}$$

Three-point correlations We consider the average with respect to μ of the occupation variable of the species $a_1, a_2, a_3 \in \{1,\dots,N-1\}$ at coordinates $x_1, x_2, x_3 \in \{1,\dots,L\}$ such that $x_1 < x_2 < x_3$. Using (7.4.5) we compute the absorption probabilities

$$\mathcal{P}_{x_1,x_2,x_3}(0,0,0) = \frac{x_1(x_2-1)(x_3-2)}{L(L^2-1)}$$
$$\mathcal{P}_{x_1,x_2,x_3}(0,1,1) = \frac{x_1(L-x_2)(L-x_3+1)}{L\,(L^2-1)}$$

$$
\begin{aligned}
\mathcal{P}_{x_1,x_2,x_3}(0,1,0) &= \frac{x_1(L(1-x_3)+x_2(x_3-2)+1)}{L(L^2-1)} \\
\mathcal{P}_{x_1,x_2,x_3}(0,0,1) &= \frac{x_1(x_2-1)(L-x_3+1)}{L\,(L^2-1)} \\
\mathcal{P}_{x_1,x_2,x_3}(1,1,0) &= \frac{(L-1)x_2(x_3-1)-x_1(x_2-1)(x_3-2)}{L\,(L^2-1)} \\
\mathcal{P}_{x_1,x_2,x_3}(1,0,1) &= \frac{(L-x_3+1)(x_2(L-x_1-1)+x_1)}{L\,(L^2-1)} \\
\mathcal{P}_{x_1,x_2,x_3}(1,0,0) &= \frac{x_2(x_3-1)(L-1)-x_1(x_2-1)(x_3-2)}{L\,(L^2-1)} \\
\mathcal{P}_{x_1,x_2,x_3}(1,1,1) &= \frac{(L-x_1-1)(L-x_2)(L-x_3+1)}{L(L^2-1)}
\end{aligned} \tag{7.4.12}
$$

Therefore, we compute the third cumulant, i.e. the three-point connected correlation as

$$
\begin{aligned}
&\mathbb{E}\left[\left(\mathbb{1}_{\{Y_{x_1}=a_1\}} - \langle \rho_{a_1}^{x_1}\rangle\right)\left(\mathbb{1}_{\{Y_{x_2}=a_2\}} - \langle \rho_{a_2}^{x_2}\rangle\right)\left(\mathbb{1}_{\{Y_{x_3}=a_3\}} - \langle \rho_{a_3}^{x_3}\rangle\right)\right] \\
&= -\,\frac{2x_1(L+1-2x_2)(L+1-x_3)}{(L+1)^3(L-1)L}(\alpha_{a_1}-\beta_{a_1})(\alpha_{a_2}-\beta_{a_2})(\alpha_{a_3}-\beta_{a_3})\,.
\end{aligned} \tag{7.4.13}
$$

Remark 15 *The first and second cumulants, computed in* (7.4.9) *and* (7.4.11) *respectively, match with those found in Section 4.3 of [13].*

7.5 Proof of Theorem 18

The proof of formula (7.4.4) is split into the following steps:

- In Section 7.5.1, we introduce two similarity transformations S_1 and S_2. The first turns the Hamiltonian H into $H^{'} = S_1 H S_1^{-1}$ with boundary terms in a triangular form; the second turns the Hamiltonian H' into $H^{''} = S_2 H' S_2^{-1}$ having the left boundary in a triangular form and the right boundary is diagonal. Associated to $H^{'}$ and $H^{''}$ there are two ground states denoted by $|\Psi^{'}\rangle$ and $|\Psi^{''}\rangle$ respectively.

- In Section 7.5.2 we apply the MPA to $H^{''}$. Here the commutation relations defining the matrix algebra are simpler. The explicit expression for $|\Psi^{''}\rangle$ is determined (see (7.5.24)).

- In Section 7.5.3, we invert the similarity transformation S_2 to recover the ground state $|\Psi^{'}\rangle$ from the explicit expression of $|\Psi^{''}\rangle$ (see (7.5.48)).

- In Section 7.5.4 we prove that the correlations are in turn the components of the vector $|\Psi^{'}\rangle$. By exploiting a binomial formula we rewrite the correlations in terms of polynomials in the left and right boundary densities with coefficients given by the absorption probabilities, as claimed in Corollary 2. Moreover, we show how the components of $|\Psi^{'}\rangle$ can also be interpreted as marginals of the non-equilibrium steady state distribution, in order to include also $\Psi^{''}(N,\ldots,N)$. Finally, for completeness, we show that Corollary 3 is in turn corresponding to the transformation $|\Psi\rangle = S_1^{-1}|\Psi'\rangle$ in Section 7.5.5.

7.5.1 The similarity transformations

Consider the matrix

$$\mathcal{S}_1 := \exp\left(\sum_{a=1}^{N-1} e_{Na}\right) , \tag{7.5.1}$$

and

$$\mathcal{S}_1^x := \exp\left(\sum_{a=1}^{N-1} e_{Na}^x\right) , \tag{7.5.2}$$

to denote $\mathcal{S}_1$ (see (7.5.1)) when acting at site x. Observe that $\mathcal{S}_1^x$ is the transposed of the bulk part of the duality matrix d_x at site $x \in \{1, \ldots, L\}$ (see (6.2.21)). By taking the product over the chain we define

$$S_1 = \prod_{x=1}^{L} \mathcal{S}_1^x \tag{7.5.3}$$

that is related with the bulk duality matrix by

$$S_1 = \prod_{x=1}^{L} d_x^T . \tag{7.5.4}$$

As the transformation S_1 is invertible, we can introduce

$$H' = H'_{\text{left}} + H_{\text{bulk}} + H'_{\text{right}} \tag{7.5.5}$$

that is related to H by

$$H' = S_1 H S_1^{-1} . \tag{7.5.6}$$

The bulk part of the Hamiltonian is left unchanged (because it is written using the coproduct of the second Casimir, see equation (4.3.19)) while the boundaries are given by

$$H'_{\text{left}} = \sum_{a=1}^{N-1} \left(\alpha_a e_{aN}^1 - e_{aa}^1\right) \qquad H'_{\text{right}} = \sum_{a=1}^{N-1} \left(\beta_a e_{aN}^L - e_{aa}^L\right) . \tag{7.5.7}$$

Indeed, using (6.2.46), (7.2.13) and (7.2.15), we obtain for the left boundary

$$\begin{aligned} H'_{\text{left}} &= S_1 H_{\text{left}} S_1^{-1} = \exp\left(\sum_{c=1}^{N-1} e_{Nc}^1\right) \sum_{A,B=1}^{N} \alpha_a \left(e_{AB}^1 - e_{BB}^1\right) \exp\left(-\sum_{c=1}^{N-1} e_{Nc}^1\right) \\ &= \left\{\exp\left(-\sum_{c=1}^{N-1} e_{cN}^1\right) \sum_{A,B=1}^{N} \alpha_A \left(e_{BA}^1 - e_{BB}^1\right) \exp\left(\sum_{c=1}^{N-1} e_{cN}^1\right)\right\}^T \\ &= \sum_{a=1}^{N-1} \left(\alpha_a e_{Na}^1 - e_{aa}^1\right)^T \\ &= \sum_{a=1}^{N-1} \left(\alpha_a e_{aN}^1 - e_{aa}^1\right) . \end{aligned} \tag{7.5.8}$$

Similarly, for the right boundary we have

$$H'_{\text{right}} = S_1 H_{\text{right}} S_1^{-1} = \sum_{a=1}^{N-1} \left(\beta_a e_{aN}^L - e_{aa}^L\right) . \tag{7.5.9}$$

Remark 16 *The boundary Hamiltonians $H^{'}_{left}$ and $H^{'}_{right}$ in* (7.5.7) *resemble the transposed of $\widetilde{H}_{left}$ and $\widetilde{H}_{right}$, i.e. to the transposed of the boundary part of the dual Hamiltonian defined in* (6.2.17)*. They can be identified when replacing the extra-site bosonic creation operators* $(\mathbf{a}^+)^0_a$ *(resp.* $(\mathbf{a}^+)^{L+1}_a$*) with the corresponding reservoir parameters α_a (resp. β_a).*

We have the following correspondence between eigenvectors with zero eigenvalues

$$|\Psi\rangle = S_1^{-1}|\Psi^{'}\rangle \tag{7.5.10}$$

where $|\Psi^{'}\rangle$ satisfies $H^{'}|\Psi^{'}\rangle = 0$.

We introduce

$$\mathcal{S}_2 := \exp\left(-\sum_{a=1}^{N-1}\beta_a e_{aN}\right) . \tag{7.5.11}$$

and

$$\mathcal{S}_2^x := \exp\left(-\sum_{a=1}^{N-1}\beta_a e^x_{aN}\right) , \tag{7.5.12}$$

which denotes $\mathcal{S}_2$ (see (7.5.11)) acting at site x. By taking the product over the chain we obtain

$$S_2 = \prod_{x=1}^{L}\mathcal{S}_2^x . \tag{7.5.13}$$

We define

$$H^{''} = H^{''}_{\text{left}} + H_{\text{bulk}} + H^{''}_{\text{right}} , \tag{7.5.14}$$

that is related to H' by

$$H^{''} = S_2 H^{'} S_2^{-1} . \tag{7.5.15}$$

The boundary Hamiltonians of $H^{''}$ read

$$H^{''}_{\text{left}} = e^1_{NN} - \mathbb{1} + \sum_{a=1}^{N-1}(\alpha_a - \beta_a)e^1_{aN} \qquad H^{''}_{\text{right}} = e^L_{NN} - \mathbb{1} . \tag{7.5.16}$$

The left one is lower triangular and depends on the differences between the boundary parameters, while the right one is diagonal. Equations (7.5.16) are proved using the Hadamard formula (6.2.39) with $\mathcal{B} = \sum_{c=1}^{N-1}\beta_c e_{cN}$, $\mathcal{A}_N = e_{aN}$, and $\mathcal{A}_a = e_{aa}$.

Therefore, using (6.2.42) we obtain for the left boundary

$$H^{''}_{\text{left}} = S_2 H^{'}_{\text{left}} S_2^{-1} = \sum_{a=1}^{N-1}\left(\alpha_a e^1_{aN} - e^1_{aa} - \beta_a e^1_{aN}\right) = e^1_{NN} - \mathbb{1} + \sum_{a=1}^{N-1}(\alpha_a - \beta_a)e^1_{aN} . \tag{7.5.17}$$

Similarly, using again (6.2.42) we have for the right boundary

$$H^{''}_{\text{right}} = S_2 H^{'}_{\text{right}} S_2^{-1} = \sum_{a=1}^{N-1}\left(\beta_a e^L_{aN} - e^L_{aa} - \beta_a e^L_{aN}\right) = e^L_{NN} - \mathbb{1} . \tag{7.5.18}$$

The relation between eigenvectors with vanishing eigenvalue is

$$|\Psi^{'}\rangle = S_2^{-1}|\Psi^{''}\rangle \tag{7.5.19}$$

and $|\Psi''\rangle$ solves

$$H''|\Psi''\rangle = 0 \,. \tag{7.5.20}$$

It is convenient to introduce also the transformation

$$S = S_2 S_1 = \prod_{x=1}^{L} \exp\left(-\sum_{a=1}^{N-1} \beta_a e^x_{aN}\right) \exp\left(\sum_{a=1}^{N-1} e^x_{aN}\right) . \tag{7.5.21}$$

This matrix S connects H with H'' by

$$H = S^{-1} H'' S \tag{7.5.22}$$

and, the relation between eigenvectors with vanishing eigenvalue is

$$|\Psi\rangle = S^{-1}|\Psi''\rangle \,. \tag{7.5.23}$$

7.5.2 Closed formula for $|\Psi''\rangle$

The ground state of the Hamiltonian H'' is given by

$$|\Psi''\rangle = \sum_{\boldsymbol{\tau}\in\Omega'} \frac{1}{(1+L)!} \prod_{x=1}^{L} (\alpha_{\tau_x} - \beta_{\tau_x})^{(1-\delta_{\tau_x,N})} \left(1 + \sum_{j=x}^{L} \delta_{\tau_j,N}\right) |\boldsymbol{\tau}\rangle \tag{7.5.24}$$

where

$$\delta_{\tau_x,N} := \begin{cases} 1 & \text{if} \quad \tau_x = N \\ 0 & \text{otherwise} \end{cases} \,. \tag{7.5.25}$$

Proof of formula (7.5.24) We consider the vector with elements given by the operators of the MPA $(X_1, \ldots, X_N)$ and we act on it with the similarity transformation $S = S_2 S_1$ obtaining new operators $(\widetilde{X}_1, \ldots, \widetilde{X}_N)$ that will satisfy simpler commutation relations. We define the transformed matrix product operators via $\tilde{X} = SX$ such that

$$\widetilde{X}_a = X_a - \beta_a(X_a + \ldots + X_N) \,, \qquad \widetilde{X}_N = X_1 + \ldots + X_N \tag{7.5.26}$$

where $a \in \{1, \ldots, N-1\}$. We can also reverse the transformation by S^{-1} and get:

$$X_a = \widetilde{X}_a + \beta_a \widetilde{X}_N \,, \qquad X_N = \beta_N \widetilde{X}_N - (\widetilde{X}_1 + \ldots + \widetilde{X}_{N-1}) \,. \tag{7.5.27}$$

Summing over $b \in \{1, \ldots, N\}$ in commutation relations (7.3.5) we have

$$\left[X_a, \widetilde{X}_N\right] = (\alpha_a - \beta_a)\widetilde{X}_N \tag{7.5.28}$$

with $\widetilde{X}_N = X_1 + \ldots + X_N$. Therefore, using (7.5.27) we obtain the commutation relations for $\widetilde{X}_a$

$$\left[\widetilde{X}_a, \widetilde{X}_N\right] = (\alpha_a - \beta_a)\widetilde{X}_N \,, \tag{7.5.29}$$

cf. (7.4.2). Moreover using (7.5.26) and (7.5.27), the action of $\widetilde{X}_a$ on the boundary vectors are given by

$$\langle\langle W| \left((\alpha_a - \beta_a)\widetilde{X}_N - \widetilde{X}_a\right) = (\alpha_a - \beta_a)\langle\langle W| \,, \tag{7.5.30}$$

$$\widetilde{X}_a|V\rangle\rangle = (\alpha_a - \beta_a)|V\rangle\rangle\,, \tag{7.5.31}$$

cf. (7.4.3). Using the transformed operators $(\widetilde{X}_a)_{a\in\{1,\dots,N\}}$, the vector $|\Psi''\rangle$ of the matrix product ansatz is written as

$$|\Psi''\rangle = \frac{1}{Z_L}\sum_{\boldsymbol{\tau}\in\Omega'}\langle\langle W|\widetilde{X}_{\tau_1}\cdots\widetilde{X}_{\tau_L}|V\rangle\rangle\,|\boldsymbol{\tau}\rangle \tag{7.5.32}$$

where the normalization is given by

$$Z_L = \langle\langle W|(X_1+\ldots+X_N)^L|V\rangle\rangle = \langle\langle W|\widetilde{X}_N^L|V\rangle\rangle\,. \tag{7.5.33}$$

To determine the eigenvector (7.5.32) in closed form, we compute the coefficients and the normalization Z_L. Using (7.5.29) we have

$$\widetilde{X}_a\widetilde{X}_N = \widetilde{X}_N\left((\alpha_a-\beta_a)+\widetilde{X}_a\right)\,. \tag{7.5.34}$$

Fix $\ell, n\in\mathbb{N}$. By applying the above formula (7.5.34) we have

$$\widetilde{X}_a^n\widetilde{X}_N^\ell = \widetilde{X}_N^\ell\left(\widetilde{X}_a+\ell(\alpha_a-\beta_a)\right)^n\,. \tag{7.5.35}$$

Using (7.5.35) we obtain

$$\prod_{x=1}^{L}{}'\,\widetilde{X}_{\tau_x} = \widetilde{X}_N^{\sum_{x=1}^L\delta_{\tau_x,N}}\prod_{x=1}^{L}{}'\left(\widetilde{X}_{\tau_x}+(\alpha_{\tau_x}-\beta_{\tau_x})\sum_{j=x}^{L}\delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}}\,, \tag{7.5.36}$$

where for convenience we introduce the ordered product

$$\prod_{x=1}^{L}{}'\,\widetilde{X}_{\tau_x} = \widetilde{X}_{\tau_1}\cdots\widetilde{X}_{\tau_L}\,. \tag{7.5.37}$$

Multiplying by the boundary vectors and by using (7.5.31) we have

$$\langle\langle W|\widetilde{X}_{\tau_1}\cdots\widetilde{X}_{\tau_L}|V\rangle\rangle = \langle\langle W|\widetilde{X}_N^{\sum_{x=1}^L\delta_{\tau_x,N}}|V\rangle\rangle\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-\delta_{\tau_x,N}}\left(1+\sum_{j=x}^{L}\delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}}\,. \tag{7.5.38}$$

We now compute $\langle\langle W|\widetilde{X}_N^{\sum_{i=1}^L\delta_{\tau_x,N}}|V\rangle\rangle$. For all $n\in\mathbb{N}$ we have that

$$\begin{aligned}
\langle\langle W|\widetilde{X}_N^n|V\rangle\rangle &= \langle\langle W|\widetilde{X}_N\widetilde{X}_N^{n-1}|V\rangle\rangle = \langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle + \langle\langle W|\frac{1}{(\alpha_a-\beta_a)}\widetilde{X}_a\widetilde{X}_N^{n-1}|V\rangle\rangle \\
&= \langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle + \frac{1}{(\alpha_a-\beta_a)}\langle\langle W|\widetilde{X}_N^{n-1}\left(\widetilde{X}_a+(\alpha_a-\beta_a)(n-1)\right)|V\rangle\rangle \\
&= \langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle + (n+1-1)\,\langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle \\
&= (2+n-1)\,\langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle \\
&= \frac{(1+n)!}{n!}\langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle
\end{aligned} \tag{7.5.39}$$

where in the second equality we used (7.5.35). This leads to the recursion relation

$$\langle\langle W|\widetilde{X}_N^n|V\rangle\rangle = \frac{(1+n)!}{n!}\langle\langle W|\widetilde{X}_N^{n-1}|V\rangle\rangle \tag{7.5.40}$$

with $\langle\langle W|\widetilde{X}_N^0|V\rangle\rangle = \langle\langle W|V\rangle\rangle = 1$. Therefore, we obtain

$$\langle\langle W|\widetilde{X}_N^n|V\rangle\rangle = \frac{(1+n)!}{n!}\frac{n!}{(n-1)!}\cdots\frac{3!}{2!}\frac{2!}{1!}\langle\langle W|\widetilde{X}_N^0|V\rangle\rangle = (1+n)!\ . \tag{7.5.41}$$

By using this result we have that

$$\langle\langle W|\widetilde{X}_{\tau_1}\cdots\widetilde{X}_{\tau_L}|V\rangle\rangle = (1+\textstyle\sum_{x=1}^N \delta_{\tau_x,N})!\displaystyle\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-\delta_{\tau_x,N}}\left(1+\sum_{j=x}^{L}\delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}}\ . \tag{7.5.42}$$

The normalization constant is then given by

$$Z_L = \langle\langle W|(X_1+\ldots+X_N)^L|V\rangle\rangle = \langle\langle W|\widetilde{X}_N^L|V\rangle\rangle = (L+1)!\ . \tag{7.5.43}$$

Therefore, we write

$$|\Psi''\rangle = \sum_{\boldsymbol{\tau}\in\Omega'}\frac{(1+\sum_{x=1}^L \delta_{\tau_x,N})!}{(L+1)!}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-\delta_{\tau_x,N}}\left(1+\sum_{j=x}^{L}\delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}}|\boldsymbol{\tau}\rangle\ . \tag{7.5.44}$$

We observe that, for all x, we have

$$\left(1+\textstyle\sum_{j=x}^L \delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}} = \frac{(1+\sum_{j=x+1}^L \delta_{\tau_j,N})!}{(\sum_{j=x}^L \delta_{\tau_j,N})!}\ . \tag{7.5.45}$$

It follows that

$$\begin{aligned}
&\frac{(1+\sum_{x=1}^L \delta_{\tau_x,N})!}{(L+1)!}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-\delta_{\tau_x,N}}\left(1+\sum_{j=x}^{L}\delta_{\tau_j,N}\right)^{1-\delta_{\tau_x,N}}\\
=&\frac{(1+\sum_{x=1}^L \delta_{\tau_x,N})!}{(L+1)!}\prod_{x=1}^{L}\frac{(1+\sum_{j=x+1}^L \delta_{\tau_j,N})!}{(\sum_{j=x}^L \delta_{\tau_j,N})!}\\
=&\frac{1}{(L+1)!}\prod_{x=1}^{L}\frac{(1+\sum_{j=x}^L \delta_{\tau_j,x})!}{(\sum_{j=x}^L \delta_{\tau_j,x})!}\\
=&\frac{1}{(L+1)!}\prod_{x=1}^{L}\left(1+\sum_{j=x}^{L}\delta_{\tau_j,N}\right)\ .
\end{aligned} \tag{7.5.46}$$

Therefore, we obtain (7.5.24).

□

7.5.3 Closed formula for $|\Psi'\rangle$

By knowing the ground state of the Hamiltonian H'' we use (7.5.19) to retrieve the ground state of the Hamiltonian H'. The result is the following

$$|\Psi'\rangle = \sum_{\boldsymbol{\tau}\in\Omega'}\Psi'(\boldsymbol{\tau})|\boldsymbol{\tau}\rangle \tag{7.5.47}$$

where

$$\Psi'(\boldsymbol{\tau}) = \frac{1}{(L+1)!}\sum_{c_1=\delta_{\tau_1,N}}^{1}\cdots\sum_{c_L=\delta_{\tau_L,N}}^{1}\prod_{x=1}^{L}\left(1+\sum_{j=x}^{L}c_j\right)(\alpha_{\tau_x}-\beta_{\tau_x})^{1-c_x}\beta_{\tau_x}^{c_x-\delta_{\tau_x,N}}\ . \tag{7.5.48}$$

Proof of formula (7.5.47) We have that $|\Psi^{'}\rangle = S_2^{-1}|\Psi^{''}\rangle$, thus we show how the transformation S_2^{-1} acts on the vector $|\Psi^{''}\rangle$. Using the exponential series we have that

$$\mathcal{S}_2^{-1} = \mathbb{1} + \sum_{a=1}^{N-1} \beta_a e_{aN} \ , \tag{7.5.49}$$

which action on the occupation variable $|\sigma\rangle$ of a site is

$$\mathcal{S}_2^{-1}|\sigma\rangle = |\sigma\rangle + \delta_{N,\sigma} \sum_{a=1}^{N-1} \beta_a |a\rangle \ . \tag{7.5.50}$$

Therefore, taking the tensor product over the chain we obtain

$$S_2^{-1}|\boldsymbol{\sigma}\rangle = \left(|\sigma_1\rangle + \delta_{N,\sigma_1} \sum_{a_1=1}^{N-1} \beta_{a_1}|a_1\rangle \right) \otimes \ldots \otimes \left(|\sigma_L\rangle + \delta_{N,\sigma_L} \sum_{a_L=1}^{N-1} \beta_{a_L}|a_L\rangle \right) \ . \tag{7.5.51}$$

By projecting over a vector $\langle \boldsymbol{\tau}|$ we have

$$\langle \boldsymbol{\tau}|S_2^{-1}|\boldsymbol{\sigma}\rangle = \prod_{x=1}^{L} \left[\delta_{\tau_x,\sigma_x} + \delta_{N,\sigma_x}\beta_{\tau_x}(1-\delta_{\tau_x,N}) \right] \ . \tag{7.5.52}$$

From this it follows that

$$\begin{aligned}\Psi^{'}(\boldsymbol{\tau}) =& \langle \boldsymbol{\tau}|\Psi^{'}\rangle = \langle \boldsymbol{\tau}|S_2^{-1}|\Psi^{''}\rangle = \sum_{\boldsymbol{\sigma}\in\Omega'} \langle \boldsymbol{\tau}|S_2^{-1}|\boldsymbol{\sigma}\rangle\langle\boldsymbol{\sigma}|\Psi^{''}\rangle \\ =& \sum_{\boldsymbol{\sigma}\in\Omega'} \frac{1}{(L+1)!} \prod_{x=1}^{L} (\alpha_{\sigma_x} - \beta_{\sigma_x})^{1-\delta_{\sigma_x,N}} \left(1 + \sum_{j=x}^{L} \delta_{\sigma_j,N} \right) \left[\delta_{\tau_x,\sigma_x} + \beta_{\tau_x}\delta_{N,\sigma_x}(1-\delta_{\tau_x,N}) \right] \ .\end{aligned} \tag{7.5.53}$$

where we used the expression (7.5.24) for $\Psi^{''}(\boldsymbol{\sigma})$. We observe that, for any fixed $x \in \{1,\ldots,L\}$ and for any fixed $\boldsymbol{\tau},\boldsymbol{\sigma} \in \Omega^{'}$, we have

$$\begin{aligned}&\left[\delta_{\tau_x,\sigma_x} + \delta_{\sigma_x,N}(1-\delta_{\tau_x,N})\beta_{\tau_x}\right] (\alpha_{\sigma_x} - \beta_{\sigma_x})^{1-\delta_{\sigma_x,N}} \\ =& \left[\delta_{\tau_x,\sigma_x} + \delta_{\sigma_x,N}(1-\delta_{\tau_x,N})\beta_{\tau_x}\right] (\alpha_{\tau_x} - \beta_{\tau_x})^{1-\delta_{\sigma_x,N}} \ .\end{aligned} \tag{7.5.54}$$

Indeed, on one hand if $\sigma_x \neq N$ only the term δ_{τ_x,σ_x} does survive in the square brackets of the above equation, that is non-vanishing only if $\sigma_x = \tau_x$. This implies that both the left-hand-side and the right-hand-side are either $(\alpha_{\tau_x} - \beta_{\tau_x})$ or 0. On the other hand, if $\sigma_x = N$ we have $(\alpha_{\sigma_x} - \beta_{\sigma_x})^{1-\delta_{\sigma_x,N}} = (\alpha_{\tau_x} - \beta_{\tau_x})^{1-\delta_{\sigma_x,N}} = 1$, thus both sides of the equality are either β_{τ_x} or 0. Moreover, we have that

$$\begin{aligned}&\delta_{\sigma_x,\tau_x} + \delta_{\sigma_x,N}(1-\delta_{\tau_x,N})\beta_{\tau_x} \\ =&\beta_{\tau_x}^{\delta_{\sigma_x,N}(1-\delta_{\tau_x,N})} \left(\delta_{\tau_x,\sigma_x} + \delta_{\sigma_x,N}(1-\delta_{\tau_x,N})\right) \\ =&\beta_{\tau_x}^{\delta_{\sigma_x,N}(1-\delta_{\tau_x,N})} \left(\delta_{\tau_x,N}\delta_{\sigma_x,N} + (1-\delta_{\tau_x,N})(1-\delta_{\sigma_x,N}) + (1-\delta_{\tau_x,N})\delta_{\sigma_x,N}\right) \\ =&\beta_{\tau_x}^{\delta_{\sigma_x,N}(1-\delta_{\tau_x,N})} \left(\delta_{\sigma_x,N} + (1-\delta_{\sigma_x,N})(1-\delta_{\tau_x,N})\right)\end{aligned} \tag{7.5.55}$$

Indeed, considering the first and last equalities we have that: if $\tau_x = \sigma_x$, both sides are 1; if $\tau_x \neq N$ and $\sigma_x = N$ both sides are β_{τ_x}; while, in all other cases, both sides are 0.

As a consequence we can write (7.5.53) as

$$\begin{aligned}\Psi'(\boldsymbol{\tau}) =& \frac{1}{(L+1)!}\sum_{\boldsymbol{\sigma}\in\Omega'}^{N}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-\delta_{\sigma_x,N}}\left(1+\sum_{j=x}^{L}\delta_{\sigma_j,N}\right)\\ &\times\left[\beta_{\tau_x}^{\delta_{\sigma_x,N}(1-\delta_{\tau_x,N})}\left(\delta_{\sigma_x,N}+(1-\delta_{\sigma_x,N})(1-\delta_{\tau_x,N})\right)\right]\ . \end{aligned} \tag{7.5.56}$$

We observe that the argument of the summation above does not distinguish the colours of the σ's but only whether, at each site x, σ_x is occupied or empty. Therefore, we can replace the summation over $\sigma_1,\ldots,\sigma_L$ with a summation over the indices $c_1,\ldots,c_L\in\{0,1\}$. Thus, we obtain

$$\Psi'(\boldsymbol{\tau}) =\frac{1}{(L+1)!}\sum_{c_1=0}^{1}\cdots\sum_{c_L=0}^{1}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-c_x}\left(1+\sum_{j=x}^{L}c_j\right)\left[\beta_{\tau_x}^{c_x}\left(c_x+(1-c_x)(1-\delta_{\tau_x,N})\right)\right]\ . \tag{7.5.57}$$

Moreover, because of the term $(1-\delta_{\tau_x,N})(1-c_x)$ we can make the indices c_x vary in the set $\{\delta_{\tau_x,N},1\}$. Indeed, when $c_x=0$, this term is non-vanishing only if $\tau_x\neq N$. Therefore, we obtain

$$\Psi'(\boldsymbol{\tau}) = \frac{1}{(L+1)!}\sum_{c_1=\delta_{\tau_1,N}}^{1}\cdots\sum_{c_L=\delta_{\tau_L,N}}^{1}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-c_x}\left(1+\sum_{j=x}^{L}c_j\right)\beta_{\tau_x}^{c_x(1-\delta_{\tau_x,N})}\ . \tag{7.5.58}$$

Observing now that $c_x(1-\delta_{\tau_x,N})$ can be replaced by $c_x-\delta_{\tau_x,N}$, we finally have

$$\Psi'(\boldsymbol{\tau}) = \frac{1}{(L+1)!}\sum_{c_1=\delta_{\tau_1,N}}^{1}\cdots\sum_{c_L=\delta_{\tau_L,N}}^{1}\prod_{x=1}^{L}(\alpha_{\tau_x}-\beta_{\tau_x})^{1-c_x}\left(1+\sum_{j=x}^{L}c_j\right)\beta_{\tau_x}^{c_x-\delta_{\tau_x,N}}\ . \tag{7.5.59}$$

□

Remark 17 *By changing the summation indices from $c_x\in\{\delta_{\tau_x,N},1\}$ to $c_x'\in\{0,1-\delta_{\tau_x,N}\}$ and by the fact that occupation of each site is bounded by 1 we rewrite (7.5.48) as*

$$\begin{aligned}\Psi'(\boldsymbol{\tau}) =& \sum_{c_1'=0}^{1-\delta_{\tau_1,N}}\cdots\sum_{c_L'=0}^{1-\delta_{\tau_L,N}}\frac{(L+1-\sum_{x=1}^{L}c_x')!}{(L+1)!}\\ &\times\prod_{x=1}^{L}\left((\alpha_{\tau_x}-\beta_{\tau_x})\left(2+L-x-\sum_{j=x}^{L}c_j'\right)\right)^{c_x'}\beta_{\tau_x}^{(1-c_x'-\delta_{\tau_x,N})}\end{aligned} \tag{7.5.60}$$

where we used the fact that

$$\left(1+\sum_{j=x}^{L}c_j\right)=\left(2+L-x-\sum_{j=x}^{L}c_j'\right)\ . \tag{7.5.61}$$

Equation (7.5.60) will be useful in the following.

7.5.4 Closed formula for correlations

We consider $m \in \{1, \ldots, L\}$ sites with coordinates $x_1, \ldots, x_m \in \{1, \ldots, L\}$ such that $x_k < x_{k+1}$ for all $k \in \{1, \ldots, m-1\}$. We fix $a_k \in \{1, \ldots, N-1\}$ and we would like to compute

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=a_k\}}\right] = \sum_{\boldsymbol{\tau}\in\Omega'} \left(\prod_{k=1}^{m} \mathbb{1}_{\{\tau_{x_k}=a_k\}}\right) \langle \boldsymbol{\tau}|\Psi\rangle \,. \tag{7.5.62}$$

We fix the dual configuration $\boldsymbol{\xi} = \sum_{k=1}^{m} \delta_{a_k}^{x_k}$, corresponding to a vector $|\boldsymbol{\xi}\rangle$, and we have

$$\prod_{k=1}^{m} \mathbb{1}_{\{\tau_{x_k}=a_k\}} = D(\boldsymbol{\tau}, \boldsymbol{\xi}) \,. \tag{7.5.63}$$

Thus it follows that

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=a_k\}}\right] = \sum_{\boldsymbol{\tau}\in\Omega'} D(\boldsymbol{\tau}, \boldsymbol{\xi})\langle \boldsymbol{\tau}|\Psi\rangle = \sum_{\boldsymbol{\tau}\in\Omega'} \langle \boldsymbol{\tau}|D|\boldsymbol{\xi}\rangle\langle \boldsymbol{\tau}|\Psi\rangle \,. \tag{7.5.64}$$

We denote by $|\hat{\boldsymbol{\xi}}\rangle = |\xi^1, \ldots, \xi^L\rangle$, i.e. the vector constructed from $|\boldsymbol{\xi}\rangle$ by removing the components at sites 0 and $L+1$. Recalling that in the case $\nu = 1$ the matrix R_x defined in (6.2.22) reduces to the identity, we have that the matrix d_x, defined in (6.2.21), becomes

$$d_x = \exp\big(\sum_{a=1}^{N-1} e_{aN}^{x}\big) \,. \tag{7.5.65}$$

Then, considering equation (7.5.2), the following holds

$$S_1^T = \prod_{x=1}^{L} d_x \,. \tag{7.5.66}$$

As a consequence we have that

$$\langle \boldsymbol{\tau}|S_1^T|\hat{\boldsymbol{\xi}}\rangle = \langle \boldsymbol{\tau}|D|\boldsymbol{\xi}\rangle \,. \tag{7.5.67}$$

Therefore, it follows that, using the resolution of the identity

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=a_k\}}\right] &= \sum_{\boldsymbol{\tau}\in\Omega'} \langle \boldsymbol{\tau}|D|\boldsymbol{\xi}\rangle\langle \boldsymbol{\tau}|\Psi\rangle = \sum_{\boldsymbol{\tau}\in\Omega'} \langle \boldsymbol{\tau}|S_1^T|\hat{\boldsymbol{\xi}}\rangle\langle \boldsymbol{\tau}|\Psi\rangle \\ &= \sum_{\boldsymbol{\tau}\in\Omega'} \langle \hat{\boldsymbol{\xi}}|S_1|\boldsymbol{\tau}\rangle\langle \boldsymbol{\tau}|\Psi\rangle = \langle \hat{\boldsymbol{\xi}}|S_1|\Psi\rangle = \langle \hat{\boldsymbol{\xi}}|\Psi'\rangle \,.\end{aligned} \tag{7.5.68}$$

Using $\Psi'(\boldsymbol{\tau})$ in (7.5.60) we have

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=a_k\}}\right] &= \sum_{c_1=0}^{1} \cdots \sum_{c_m=0}^{1} \frac{(L+1-\sum_{a=1}^{m} c_a)!}{(L+1)!} \prod_{k=1}^{m} (\alpha_{a_k} - \beta_{a_k})^{c_k} \beta_{a_k}^{1-c_k} \left(2 + L - x_k - \sum_{j=k}^{m} c_j\right)^{c_k} \\ &= \sum_{c_1,\ldots,c_m=0}^{1} f(c_1, \ldots, c_m) \prod_{k=1}^{m} (\alpha_{a_k} - \beta_{a_k})^{c_k} \beta_{a_k}^{1-c_k} g_k(x_k, c_k, \ldots, c_m) \,,\end{aligned} \tag{7.5.69}$$

where in the last equality we have used the definition of (7.4.6) and (7.4.7). As a last step, using the binomial theorem, we rewrite the previous expression as a polynomial in the boundary densities:

$$\begin{aligned}
&\mathbb{E}\left[\prod_{k=1}^{m}\mathbb{1}_{\{Y_{x_k}=a_k\}}\right] \\
&= \sum_{c_1,\dots,c_m=0}^{1} f(c_1,\dots,c_m)\prod_{k=1}^{m}\left(\sum_{t_k=0}^{c_k}(-1)^{c_k-t_k}\alpha_{a_k}^{t_k}\beta_{a_k}^{1-t_k}g_k(x_k,c_k,\dots,c_m)\right) \\
&= \sum_{c_1,\dots,c_m=0}^{1} f(c_1,\dots,c_m)\sum_{t_1=0}^{c_1}\dots\sum_{t_m=0}^{c_m}\left(\prod_{k=1}^{m}\alpha_{a_k}^{t_k}\beta_{a_k}^{1-t_k}\right)\left(\prod_{k=1}^{m}(-1)^{c_k-t_k}g_k(x_k,c_k,\dots,c_m)\right) \\
&= \sum_{t_1,\dots,t_m=0}^{1}\left(\prod_{k=1}^{m}\alpha_{a_k}^{t_k}\beta_{a_k}^{1-t_k}\right)\sum_{c_1=t_1}^{1}\dots\sum_{c_m=t_m}^{1} f(c_1,\dots,c_m)\left(\prod_{k=1}^{m}(-1)^{c_k-t_k}g_k(x_k,c_k,\dots,c_m)\right) .
\end{aligned} \tag{7.5.70}$$

From third to fourth line we have changed the summation order. Therefore, comparing the above formula and (7.2.21), we can read off the absorption probabilities given in (7.4.5).

□

Interpretation of Ψ' as marginals of the steady state distribution From equation (7.5.68), we observe that, for any configuration $\boldsymbol{\tau}$ that has at least a particle, we interpret the coefficient $\Psi'(\tau)$ of the vector $|\Psi'\rangle$ as a correlation. However, this interpretation does not cover the case when $\boldsymbol{\tau} = (N,\dots,N)$, i.e. when no particles are present in the configuration. We show that, for any configuration $\boldsymbol{\tau} \in \Omega'$ we have

$$\Psi'(\boldsymbol{\tau}) = \mathbb{E}\left[\prod_{x=1}^{L}\left(\mathbb{1}_{\{Y_x=\tau_x\}}\right)^{(1-\delta_{\tau_x,N})}\right] . \tag{7.5.71}$$

The above formula (7.5.71) allows to interpret these coefficients as marginals of the non-equilibrium steady distribution μ. As a by-product, we obtain that $\Psi'(N,\dots,N) = 1$, i.e. it is the normalization of the measure μ. In particular, these marginal are the non-equilibrium steady state correlations as soon as the configuration $\boldsymbol{\tau}$ does contain at least a particle.
We prove (7.5.71): for any $\boldsymbol{\tau} \in \Omega'$ we have that

$$\Psi'(\boldsymbol{\tau}) = \langle\boldsymbol{\tau}|\Psi'\rangle = \sum_{\boldsymbol{\sigma}\in\Omega'}\langle\boldsymbol{\tau}|S_1|\boldsymbol{\sigma}\rangle\langle\boldsymbol{\sigma}|\Psi\rangle = \sum_{\boldsymbol{\sigma}\in\Omega'}\langle\boldsymbol{\sigma}|S_1^T|\boldsymbol{\tau}\rangle\langle\boldsymbol{\sigma}|\Psi\rangle . \tag{7.5.72}$$

We introduce the dual configuration constructed by attaching two extra-sites (without any particle) to the vector $\boldsymbol{\tau}$, transforming its dimension from L to $L+2$. We denote it by $\boldsymbol{\xi}(\boldsymbol{\tau}) = (0,\tau_1,\dots,\tau_L,0)$, where we wrote 0 in the two extra-sites to indicate that no dual particles are present there. More explicitly, this dual configuration is given by

$$\begin{cases}\xi_A^x(\boldsymbol{\tau}) = \delta_{\tau_x,A} & \forall A \in \{1,\dots N\} \quad \forall x \in \{1,\dots,L\} \\ \xi_1^0(\boldsymbol{\tau}) = \dots = \xi_{N-1}^0(\boldsymbol{\tau}) = 0 \\ \xi_1^{L+1}(\boldsymbol{\tau}) = \dots = \xi_{N-1}^{L+1}(\boldsymbol{\tau}) = 0\end{cases} . \tag{7.5.73}$$

Now we consider the elements $\langle \boldsymbol{\sigma}|D|\xi(\boldsymbol{\tau})\rangle$ of the duality matrix D and, since no dual particles are present in the extra-sites 0 and $L+1$, we have

$$\Psi^{'}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\sigma}\in\Omega'} \langle \boldsymbol{\sigma}|D|\boldsymbol{\xi}(\boldsymbol{\tau})\rangle\langle \boldsymbol{\sigma}|\Psi\rangle . \tag{7.5.74}$$

We observe that for any $x \in \{1, \ldots, L\}$

$$\prod_{a=1}^{N-1} \mathbb{1}_{\{\delta_{\sigma_x,a} \geqslant \xi_a^x(\boldsymbol{\tau})\}} = \left(\mathbb{1}_{\{\sigma_x=\tau_x\}}\right)^{(1-\delta_{\tau_x,N})} . \tag{7.5.75}$$

Indeed, we have that

$$\prod_{a=1}^{N-1} \mathbb{1}_{\{\delta_{\sigma_x,a} \geqslant \xi_a^x(\boldsymbol{\tau})\}} = \begin{cases} \mathbb{1}_{\{\sigma_x=\tau_x\}} & \text{if} \quad \tau_x \neq N \\ 1 & \text{if} \quad \tau_x = N \end{cases} . \tag{7.5.76}$$

Therefore using the duality function (7.2.17) and equation 7.5.74, we obtain

$$\Psi^{'}(\boldsymbol{\tau}) = \mathbb{E}\left[\prod_{x=1}^{L} \left(\mathbb{1}_{\{Y_x=\tau_x\}}\right)^{(1-\delta_{\tau_x,N})}\right] . \tag{7.5.77}$$

This last equation completes and generalizes (7.5.68) by showing that $\Psi^{'}(N, \ldots, N) = 1$.

Equivalently, we can interpret the components of $\Psi'(\boldsymbol{\tau})$ as follows. Let ℓ be the number of sites occupied by a particle in the configuration $\boldsymbol{\tau}$, i.e. $\ell = \sum_{x=1}^{L}(1-\delta_{\tau_x,N})$. We denote by $x_1, \ldots, x_\ell$, such that $x_k < x_{k+1}$ for all $k \in \{1, \ldots, \ell-1\}$, the coordinates where a particle of any species is present. Finally, we call

$$\Omega_\ell^{'}(\boldsymbol{\tau}) := \left\{\boldsymbol{\sigma} \in \Omega^{'} \;:\; \sigma_{x_1} = \tau_{x_1}, \ldots, \sigma_{x_\ell} = \tau_{x_\ell}\right\} . \tag{7.5.78}$$

Observe that, if $\ell = 0$ then $\Omega_\ell^{'}(\boldsymbol{\tau})$ does coincide with the whole state space $\Omega^{'}$, while, if $\ell = L$ then $\Omega_\ell^{'}(\boldsymbol{\tau})$ reduces to the configuration $\boldsymbol{\tau}$. Denoting the marginal on ℓ sites of the steady state distribution by

$$\mu_{x_1,\ldots,x_\ell}(\boldsymbol{\tau}) = \sum_{\boldsymbol{\sigma}\in\Omega_\ell^{'}(\boldsymbol{\tau})} \mu(\boldsymbol{\sigma}) \tag{7.5.79}$$

we have that

$$\Psi^{'}(\boldsymbol{\tau}) = \mathbb{E}\left[\prod_{x=1}^{L} \left(\mathbb{1}_{\{Y_x=\tau_x\}}\right)^{(1-\delta_{\tau_x,N})}\right] = \mathbb{E}\left[\prod_{k=1}^{\ell} \mathbb{1}_{\{Y_{x_k}=\tau_{x_k}\}}\right] = \sum_{\boldsymbol{\sigma}\in\Omega_\ell^{'}(\boldsymbol{\tau})} \mu(\boldsymbol{\sigma}) = \mu_{x_1,\ldots,x_\ell}(\boldsymbol{\tau}) . \tag{7.5.80}$$

It follows that $\Psi^{'}(\boldsymbol{\tau})$ is the marginal on ℓ sites $x_1, \ldots, x_\ell$ of the non-equilibrium steady state distribution. In particular, when the configuration $\boldsymbol{\tau}$ has ℓ particles, $\Psi^{'}(\boldsymbol{\tau})$ does coincide with the ℓ-point correlation function.

Remark 18 *Using the definition $|\Psi^{'}\rangle = S_1|\Psi\rangle$ and that*

$$\langle \tau|S_1 = \begin{cases} \sum_{\sigma=1}^{N}\langle \sigma| & \textit{if} \quad \tau = N \\ \langle \tau| & \quad\textit{else} \end{cases} , \tag{7.5.81}$$

we immediately verify

$$\mathbb{E}\left[\prod_{k=1}^{m}\mathbb{1}_{\{Y_{x_k}=a_k\}}\right] = \sum_{\boldsymbol{\tau}\in\Omega'}\prod_{k=1}^{m}\delta_{\tau_{x_k},a_k}\langle\boldsymbol{\tau}|\Psi\rangle = \langle N,\ldots,N,\underset{\uparrow\, x_1}{a_1},N,\ldots,N,\underset{\uparrow\, x_m}{a_m},N,\ldots,N|\Psi'\rangle\,. \tag{7.5.82}$$

Here the notation means that the entries $a_k \neq N$ *in* (7.5.82) *are at position* x_k *for* $k \in \{1,\ldots,m\}$.

7.5.5 Steady state

By the knowledge of the closed formula for correlations (7.4.4), we find the probability mass-function in the non-equilibrium steady state by using Corollary 3. Indeed we show here that $|\Psi\rangle = S_1^{-1}|\Psi'\rangle$ is nothing but Corollary 3. On a single site we have that

$$\mathcal{S}_1^{-1} = \mathbb{1} + (-1)\sum_{a=1}^{N-1} e_{Na} \tag{7.5.83}$$

thus,

$$\mathcal{S}_1^{-1}|\sigma\rangle = |\sigma\rangle + (-1)(1-\delta_{\sigma,N})|N\rangle \tag{7.5.84}$$

By taking the tensor product over the chain we obtain

$$S_1^{-1}|\boldsymbol{\sigma}\rangle = (|\sigma_1\rangle + (-1)(1-\delta_{\sigma_1,N})|N\rangle)\otimes\ldots\otimes(|\sigma_L\rangle + (-1)(1-\delta_{\sigma_L,N})|N\rangle) \tag{7.5.85}$$

therefore, projecting on $\langle\boldsymbol{\tau}|$ we have

$$\langle\boldsymbol{\tau}|S_1^{-1}|\boldsymbol{\sigma}\rangle = \prod_{x=1}^{L}\left(\delta_{\tau_x,\sigma_x} + (-1)\delta_{\tau_x,N}\sum_{a=1}^{N-1}\delta_{\sigma_x,a}\right)\,. \tag{7.5.86}$$

By inserting the resolution of identity $\sum_{\boldsymbol{\sigma}\in\Omega'}|\boldsymbol{\sigma}\rangle\langle\boldsymbol{\sigma}|$ we have that

$$\Psi(\boldsymbol{\tau}) = \langle\boldsymbol{\tau}|\Psi\rangle = \langle\boldsymbol{\tau}|S_1^{-1}|\Psi'\rangle = \sum_{\boldsymbol{\sigma}\in\Omega'}\langle\boldsymbol{\tau}|S_1^{-1}|\boldsymbol{\sigma}\rangle\langle\boldsymbol{\sigma}|\Psi'\rangle\,. \tag{7.5.87}$$

Using (7.5.86) we obtain

$$\begin{aligned}\Psi(\boldsymbol{\tau}) = \sum_{m=0}^{L}\sum_{1\leqslant x_1<x_2<\ldots<x_m\leqslant L}&\left(\prod_{x\notin\{x_1,\ldots,x_m\}}\delta_{\tau_x,N}\right)\sum_{b_1,\ldots,b_m=1}^{N-1}\left(\prod_{k=1}^{m}(\delta_{\tau_{x_k},b_k}-\delta_{\tau_{x_k},N})\right)\\ &\times\Psi'(N,\ldots,N,\underset{\uparrow\, x_1}{b_1},N,\ldots,N,\underset{\uparrow\, x_m}{b_m},N,\ldots,N)\end{aligned} \tag{7.5.88}$$

In the second equality we have rewritten the product by introducing the summation over m, the coordinates $x_1,\ldots,x_m$ and the occupation variables $b_1,\ldots,b_m$. Here, the notation

$$\Psi'(N,\ldots,N,\underset{\uparrow\, x_1}{b_1},N,\ldots,N,\underset{\uparrow\, x_m}{b_m},N,\ldots,N) \tag{7.5.89}$$

indicates that in positions $x_1,\ldots,x_m$ there are particles of species $b_1,\ldots,b_m$, while there are holes N in all other positions.

Remark 19 (Equilibrium case) *Using equation* (7.2.25) *with correlation computed in* (7.4.4) *in the equilibrium case ($\beta_a = \alpha_a$), we obtain*

$$\Psi(\boldsymbol{\tau}) = \prod_{a=1}^{N} \alpha_a^{\sum_{x=1}^{L} \delta_{\tau_x,a}} \tag{7.5.90}$$

As it has to be, this is the probability of the configuration $\boldsymbol{\tau}$ under the reversible measure (4.2.19)*, with site marginals distributed as Multinomial* $(1, \alpha_1, \ldots, \alpha_N)$*.*

7.6 The single species case (SSEP)

The computations made for the multi-species stirring process can be specialized to the single species case $N = 2$, i.e. when only one type of particles and the holes are present. In this situation we retrieve the SSEP, that was studied using the same approach as presented here in [20]. We report the Hamiltonians and the similarity transformations of SEEP:

$$H = H_{\text{left}} + \sum_{x=1}^{L-1} H_{x,x+1} + H_{\text{right}} \tag{7.6.1}$$

where the bulk Hamiltonian densities are

$$H_{x,x+1} = \sum_{a,b=1}^{2} \left(e_{ab} \otimes e_{ba} - e_{bb} \otimes e_{aa} \right) \ . \tag{7.6.2}$$

Here e_{ab} are the basis elements of the fundamental representation of $gl(2)$. The boundary Hamiltonians are

$$H_{\text{left}} = \begin{pmatrix} \alpha_1 - 1 & \alpha_1 \\ \alpha_2 & \alpha_2 - 1 \end{pmatrix}, \qquad H_{\text{right}} = \begin{pmatrix} \beta_1 - 1 & \beta_1 \\ \beta_2 & \beta_2 - 1 \end{pmatrix} \ . \tag{7.6.3}$$

where we assume that $(\alpha_1 + \alpha_2) = (\beta_1 + \beta_2) = 1$. In this case, the similarity transformations read

$$\mathcal{S}_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \mathcal{S}_2 = \begin{pmatrix} 1 & -\beta_1 \\ 0 & 1 \end{pmatrix}, \qquad \mathcal{S} = \mathcal{S}_2 \mathcal{S}_1 = \begin{pmatrix} 1 - \beta_1 & -\beta_1 \\ 1 & 1 \end{pmatrix} \ . \tag{7.6.4}$$

Therefore, we have that $H^{'}$ has the following boundaries

$$H^{'}_{\text{left}} = \begin{pmatrix} -1 & \alpha_1 \\ 0 & 0 \end{pmatrix}, \qquad H^{'}_{\text{right}} = \begin{pmatrix} -1 & \beta_1 \\ 0 & 0 \end{pmatrix} \ , \tag{7.6.5}$$

while the boundary Hamiltonians of $H^{''}$ are

$$H^{''}_{\text{left}} = \begin{pmatrix} -1 & \alpha_1 - \beta_1 \\ 0 & 0 \end{pmatrix}, \qquad H^{''}_{\text{right}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \ . \tag{7.6.6}$$

As a check of our results we report the steady state correlations and the vectors $|\Psi^{''}\rangle$, $|\Psi^{'}\rangle$ and $|\Psi\rangle$ for the SSEP, i.e. the situation where the process has only one species of particles and holes, with maximal occupancy $\nu = 1$ and compare the results to [20].

Correlations of SSEP. We retrieve the closed formula for correlations in the non-equilibrium steady state found in [20, (4.26)]:

$$\mathbb{E}\left[\prod_{k=1}^{m} \mathbb{1}_{\{Y_{x_k}=1\}}\right] = \sum_{q=0}^{m}(\alpha-\beta)^q\beta^{m-q} \sum_{1\leqslant \ell_1<\ldots<\ell_q\leqslant m}\prod_{k=1}^{q}\frac{(L+1-x_{\ell_k}-q+k)}{(2+L-k)} \ . \tag{7.6.7}$$

Proof of formula (7.6.7)**:** Starting from (7.4.4) we exploit the fact that for $N = 2$ each site can be either occupied by a particle or empty. We introduce $q = \sum_{k=1}^{m} c_{x_k}$. There are many values of c's that give the same q, therefore we define q variables denoted by $1 \leqslant \ell_1 < \ell_2 < \ldots < \ell_q \leqslant m$. To each of these variables we associate coordinates $x_{\ell_1}, \ldots, x_{\ell_q} \in \{x_1, \ldots, x_m\}$ that are such that $c_{x_{\ell_k}} = 1$ for all $k \in \{1, \ldots, q\}$ and $c_x = 0$ for $x \notin \{x_{\ell_1}, \ldots, x_{\ell_q}\}$. Therefore, for any fixed $c_{x_1}, \ldots, c_{x_m}$, we find q (by the summation $\sum_{k=1}^{m} c_{x_k} = q$) and $\ell_1, \ldots, \ell_q$ from which we fix $x_{\ell_1}, \ldots, x_{\ell_q}$ as explained. Then, we obtain

$$\prod_{k=1}^{m}\left(2+L-x_k-\sum_{j=k}^{m}c_j\right)^{c_k} = \prod_{k=1}^{q}(1+L-x_{\ell_k}-q+k) \ . \tag{7.6.8}$$

Indeed, in the product on the left hand side only q terms at coordinates $x_{\ell_1}, \ldots x_{\ell_q}$ do survive. Moreover, any fixed x_{ℓ_k} has $q-k$ occupied sites at its right. Furthermore, using the properties of the factorials, we have that

$$\frac{(L+1-\sum_{k=1}^{m} c_k)!}{(L+2)!} = \prod_{k=1}^{q}\frac{1}{(2+L-k)} \ . \tag{7.6.9}$$

Varying all possible c's in (7.4.4) is equivalent of varying all possible q and all possible $\ell_1, \ldots, \ell_q$, therefore we obtain

$$\sum_{c_1\ldots,c_m=0}^{m}\frac{(L+1-\sum_{k=1}^{m} c_k)!}{(L+1)!}\prod_{k=1}^{m}\left(2+L-x_k-\sum_{j=k}^{m}c_j\right) = \sum_{q=0}^{m}\sum_{1\leqslant\ell_1<\ldots<\ell_q\leqslant L}\prod_{k=1}^{q}\frac{(1+L-x_{\ell_k}-q+k)}{(2+L-k)} \tag{7.6.10}$$

and (7.6.7) follows.

□

The vector $|\Psi''\rangle$ for SSEP. The ground state of H'' is

$$|\Psi''\rangle = \sum_{q=0}^{L}\sum_{1\leqslant\ell_1<\ldots<\ell_q\leqslant L}(\alpha_1-\beta_1)^q\prod_{k=1}^{q}\frac{(1+L-\ell_k-q+k)}{(2+L-k)}|\boldsymbol{\ell},q\rangle \tag{7.6.11}$$

where $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_q)$.

Proof of formula (7.6.11)**:** We specialize equation (7.5.44) to the $N = 2$ case and introduce $q = \sum_{x=1}^{L}\delta_{\tau_x,1}$. For each $q \in \{1, \ldots, L\}$ we call $1 \leqslant \ell_1 < \ldots, \ell_q \leqslant L$ the coordinates where particles of type 1 are present. Thus, arguing as in the proof of (7.6.7) and using (7.6.9) we have that

$$\frac{\left(1+\sum_{x=1}^{L}\delta_{\tau_x,2}\right)!}{(1+L)!}\prod_{x=1}^{L}\left[(\alpha_{\tau_x}-\beta_{\tau_x})\left(1+\sum_{j=x}^{L}\delta_{\tau_j,2}\right)\right]^{1-\delta_{\tau_x,2}}$$

$$=\frac{\left(1+L-\sum_{x=1}^{L}(1-\delta_{\tau_x,2})\right)!}{(1+L)!}\prod_{x=1}^{L}\left[(\alpha_{\tau_x}-\beta_{\tau_x})\left(2+L-x-\sum_{j=x}^{L}(1-\delta_{\tau_j,2})\right)\right]^{1-\delta_{\tau_x,2}}$$

$$=(\alpha_1-\beta_1)^q\prod_{k=1}^{q}\frac{(1+L-\ell_k-q+k)}{(2+L-k)}\ . \tag{7.6.12}$$

By varying q and $\ell_1,\ldots,\ell_q$, we have a one-to-one mapping with $\boldsymbol{\tau}$, therefore we obtain (7.6.11).

□

The vector $|\Psi'\rangle$ for SSEP. The ground state of H' reads

$$|\Psi'\rangle=\sum_{\boldsymbol{\tau}\in\Omega'}\Psi'(\boldsymbol{\tau})|\boldsymbol{\tau}\rangle \tag{7.6.13}$$

where

$$\Psi'(\boldsymbol{\tau})=\sum_{q=0}^{|\boldsymbol{\tau}|}(\alpha-\beta)^q\beta^{L-q}\sum_{1\leqslant\ell_1<\ldots<\ell_q\leqslant|\boldsymbol{\tau}|}\prod_{k=1}^{q}\frac{(L+1-x_{\ell_k}-q+k)}{(2+L-k)}\ . \tag{7.6.14}$$

and where $|\boldsymbol{\tau}|=\sum_{x=1}^{L}\delta_{\tau_x,1}$.

Proof of formula (7.6.14)**:** For any configuration $\boldsymbol{\tau}\in\Omega$ we denote by $|\boldsymbol{\tau}|=\sum_{x=1}^{L}\delta_{\tau_x,2}$ the number of occupied sites. Then, considering equation (7.5.60) we have the result by adapting the proof of (7.6.7) in the following way. We select $q\in\{0,\ldots,|\boldsymbol{\tau}|\}$ particles among the occupied ones. We call $\ell_1,\ldots\ell_q\in\{1,\ldots,|\tau|\}$ such that $\ell_1<\ell_2<\ldots<\ell_q$. These $(\ell_k)_{k\in 1,\ldots,q}$ are labels for the q particles previously selected. For each ℓ_k we introduce x_{ℓ_k}, i.e. the coordinate where the particle with label ℓ_k is placed. Thus, we have that

$$\prod_{x=1}^{L}\left(L+2-x-\sum_{j=x}^{L}c_j\right)^{c_x}=\prod_{k=1}^{q}\left(L+2-x_{\ell_k}-\sum_{j=x_{\ell_k}}^{L}c_j\right)=\prod_{k=1}^{q}(1+L-x_{\ell_k}-q+k)\ . \tag{7.6.15}$$

It follows that

$$\sum_{c_1=0}^{1}\cdots\sum_{c_L=0}^{1}\mathbb{1}_{\{c_1+\cdots+c_L=q\}}\prod_{x=1}^{L}\left(L+2-x-\sum_{j=x}^{L}c_j\right)^{c_x}=\sum_{1\leqslant\ell_1<\ldots<\ell_q\leqslant|\boldsymbol{\tau}|}\prod_{k=1}^{q}(1+L-x_{\ell_k}-q+k)\ . \tag{7.6.16}$$

Therefore, using (7.6.9) one finds $|\Psi'\rangle$ for the SSEP.

□

The vector $|\Psi\rangle$ for SSEP. In the single species case $N=2$ we have that

$$|\Psi\rangle=\sum_{\boldsymbol{\tau}\in\Omega'}\Psi(\boldsymbol{\tau})|\boldsymbol{\tau}\rangle \tag{7.6.17}$$

where

$$\Psi(\boldsymbol{\tau})=\sum_{b=0}^{L}(-1)^b\sum_{1\leqslant q_1<\ldots<q_b\leqslant L}\left(\prod_{k=1}^{q}\delta_{\tau_{q_k},2}\right)\sum_{s=0}^{|\boldsymbol{\tau}|+b}(\alpha-\beta)^s\beta^{L-s}\sum_{1\leqslant\ell_1<\ldots<\ell_s\leqslant|\boldsymbol{\tau}|}\prod_{k=1}^{s}\frac{(L+1-x_{\ell_k}-s+k)}{(2+L-k)}\ . \tag{7.6.18}$$

Proof of (7.6.17)**:** the proof follows from (7.5.88) and from the coefficients $\Psi^{'}(\boldsymbol{\tau})$ computed in (7.6.14).

$\square$

Chapter 8

Perspectives: mapping of non-equilibrium onto equilibrium

8.1 Motivations

In Section 6.2.1 of Chapter 7, we have constructed two similarity transformations that map the Hamiltonian H of the integrable boundary driven multi-species stirring process to another matrix $H^{''}$ that has a triangular left boundary and a diagonal right boundary (that we recall is not stochastic). For the SSEP (that is the case with $N = 2$), it has been shown in [20, 28] that it is possible to construct a further non-local similarity transformation with the property of mapping the Hamiltonian $H^{''}$ onto a matrix H_{eq}, which is obtained from H by setting equal boundary densities at left and right reservoirs. This result is achieved by mapping the non-equilibrium Hamiltonian $H^{''}$ onto a Hamiltonian H_0 that has both boundaries diagonal. Indeed, once the mapping from H and H_0 is obtained, it is not hard to map H_0 onto H_{eq} (see Section 4 of [28]). This technique, used for the SSEP, is based on the quantum inverse scattering method and, more precisely, on the possibility of writing the matrix W in terms of the *conserved charges of the process*. In this chapter we aim to investigate about the possibility of constructing such a matrix W, to map non-equilibrium onto equilibrium, in the case of the boundary driven multi-species stirring process. Diagrammatically the goal of this chapter can be summarized in finding a matrix W such that

$$H \overset{S_1}{\longleftrightarrow} H^{'} \overset{S_2}{\longleftrightarrow} H^{''} \overset{W}{\longleftrightarrow} H_0 \tag{8.1.1}$$

where the matrices S_1, S_2 are the local similarity transformations introduced in(7.5.3) and (7.5.13), H, $H^{'}$ and $H^{''}$ are the matrices defined in (7.2.3), (7.5.5) and (7.5.14) respectively. Namely, we aim to find a matrix W such that

$$H^{''} W = W H_0 \tag{8.1.2}$$

If this is possible, the original Hamiltonian H can be mapped by a non-local similarity transformation to the diagonal boundary Hamiltonian H_0, i.e.

$$H_0 = W^{-1}\, S_2\, S_1\, H\, S_1^{-1}\, S_2^{-1}\, W \tag{8.1.3}$$

We further observe that the matrices $H^{''}$ and H_0 are isospectral. This will be important in the following when we will write a general formula for the matrix W.

In this chapter, we first introduce, in Section8.2, the quantum inverse scattering method (QISM) for the boundary driven multi-species stirring process. Starting from the reference [13], we write

the fundamental ingredients of the QISM: the R-matrices and the K-matrices. Using these ingredients, we then construct the double row monodromy, the transfer matrix and we derive the conserved charges of the system, together with their exchange relations.
Inspired by [20, 28], we write in Section 8.3 the matrix W as a function of the conserved charges, showing that the analytic expression of W found in literature for the case $N = 2$ can be retrieved. Finally, we try to extend this formula to the case of the boundary driven multi-species stirring process. At this point, the problem is still open and we are able to write only partial results. Indeed, the exchange relations between charges are more complicated and they involve a further creation operator. Thus, we still do not find an analytic formula (in function of the conserved charges) for W, with the features of being independent from the spectrum and of the eigenvectors of H_0. We leave the finalization of this goal to future works.

8.2 Quantum inverse scattering method

In the following, when a tensor product is considered the identity matrix is denoted by $\mathbb{1}$, otherwise it is simply indicated by 1.
The process that we consider is the integrable boundary driven multi-species stirring process, with configuration variables

$$|\boldsymbol{\tau}\rangle = |\tau_1, \ldots, \tau_L\rangle \tag{8.2.1}$$

where $\tau_x \in \{1, \ldots, N\}$ denotes the occupation at site x. The state space and the Hamiltonian of this integrable process have been introduced in Chapter 7. However, for the sake of completeness we recall them: the state space reads

$$\Omega' = \{|\tau_1, \ldots, \tau_L\rangle \,:\, \tau_x \in \{1, \ldots, N\}\} \ . \tag{8.2.2}$$

and the Hamiltonian is given by

$$H = H_{\text{left}} + H_{\text{bulk}} + H_{\text{right}} \tag{8.2.3}$$

where

$$H_{\text{bulk}} = \sum_{x=1}^{L-1} \mathcal{H}_{x,x+1} \ . \tag{8.2.4}$$

Here $\mathcal{H}_{x,x+1}$ denotes the two-site Hamiltonian

$$\mathcal{H} = P - 1 \tag{8.2.5}$$

with the permutation matrix

$$P = \sum_{A,B=1}^{N} e_{AB} \otimes e_{BA} \, . \tag{8.2.6}$$

The boundary terms of this Hamiltonian are given by

$$H_{\text{left}} = \sum_{A,B=1}^{N} \alpha_A \left(e^1_{AB} - e^1_{BB}\right), \qquad H_{\text{right}} = \sum_{A,B=1}^{N} \beta_A \left(e^L_{AB} - e^L_{BB}\right) \ . \tag{8.2.7}$$

where we assume that the boundary parameters satisfy

$$\sum_{A=1}^{N} \alpha_A = 1, \qquad \sum_{A=1}^{N} \beta_A = 1 \ . \tag{8.2.8}$$

We recall that, in Chapter 7, we have defined two similarity transformations S_1 and S_2 (see (7.5.5) and (7.5.14)) that bring the original Hamiltonian H to $H^{'}$ and $H^{''}$. We have that

$$H^{'} = S_1 H S_1^{-1} \tag{8.2.9}$$

where $H^{'}$ has the same bulk $H_{\text{bulk}} = H^{'}_{\text{bulk}}$ but has triangular boundaries

$$H^{'}_{\text{left}} = \sum_{a=1}^{N-1} \left(\alpha_a e^1_{aN} - e^1_{aa}\right) \qquad H^{'}_{\text{right}} = \sum_{a=1}^{N-1} \left(\beta_a e^L_{aN} - e^L_{aa}\right) \ . \tag{8.2.10}$$

where with e^x_{AB} we denote the elementary matrix e_{AB} acting non-trivially at site x and as the identity at all the other sites. Further transforming $H^{'}$ with S_2 we have that

$$H^{''} S_2 H^{'} S_2^{-1} \tag{8.2.11}$$

where $H^{''}$ has the same bulk $H_{\text{bulk}} = H^{''}_{\text{bulk}}$ but has a triangular left boundary and a diagonal right boundary, i.e.

$$H^{''}_{\text{left}} = e^1_{NN} - 1 + \sum_{a=1}^{N-1} (\alpha_a - \beta_a) e^1_{aN} \qquad H^{''}_{\text{right}} = e^L_{NN} - 1 \ . \tag{8.2.12}$$

Since we have used similarity transformations, the integrability results stated in [13] for H can be adapted to $H^{'}$ and $H^{''}$. Then, in the following section we briefly report this integrability results for the Hamiltonian $H^{''}$.

8.2.1 Fundamental ingredients

R and K matrices

The set-up of the QISM is similar to the one of the SSEP, then the notion introduced in that case can be adapted to this multi-species situation. For details we remind the reader to Chapter 3.
In the following, we will define the QISM for the matrix $H^{''}$. The computations for the original Hamiltonian H defined in (8.2.3) are formally the same. To show integrability of the Hamiltonian $H^{''}$ defined in (8.2.11), we define some matrices that ensure that $H^{''}$ is part of a commuting family of operators generated by the *transfer matrices*. We introduce the *quantum space* $\mathbb{V} = V_1 \otimes \ldots \otimes V_L$ and the *auxiliary space* V_0. In the case of the multi-species stirring process we chose $V_j = \mathbb{C}^N$, $\forall j = 0, 1, \ldots, L$. We define the following operators:

- *R-matrices*

$$R(u) = \frac{(P + u)}{(u + 1)} \tag{8.2.13}$$

  where P is the permutation operator defined as

$$P = \sum_{A,B=1}^{N} e_{AB} \otimes e_{BA} \, , \tag{8.2.14}$$

  with elementary matrices $(e_{AB})_{CD} = \delta_{AC}\delta_{BD}$ for $A, B \in \{1, \ldots, N\}$.

- $\hat{K}$*-matrix*

$$\widehat{K}(u) = (1+u)\left(1 + \frac{2u}{u+1} H''_{right}\right) = \begin{pmatrix} 1-u & 0 & \dots & 0 \\ 0 & 1-u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1+u \end{pmatrix} \tag{8.2.15}$$

- *K-matrix*

$$K(u) = \frac{\left((2u+N)H''_{left} + (u+N)\right)}{(u+1)^2(2u+N)} = \begin{pmatrix} -u & 0 & 0 & \dots & 0 & (N+2u)(\alpha_1 - \beta_1) \\ 0 & -u & 0 & \dots & 0 & (N+2u)(\alpha_2 - \beta_2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -u & (N+2u)(\alpha_{N-1} - \beta_{N-1}) \\ 0 & 0 & 0 & \dots & 0 & N+u \end{pmatrix} \tag{8.2.16}$$

Here, the variable $u \in \mathbb{C}$ is called spectral parameter. The matrices $R_{i,j}(u)$, $K_i(u)$ and $\widehat{K}_i(u)$ acting on the tensor product space $V_i \otimes V_j$, satisfy:

- *Yang-Baxter-Equation*

$$R_{i,j}(u-v)R_{i,k}(u-z)R_{j,k}(v-z) = R_{j,k}(v-z)R_{i,k}(u-z)R_{i,j}(u-v)\,. \tag{8.2.17}$$

- *Boundary-Yang-Baxter-Equation*

$$R_{i,j}(u-v)\widehat{K}_i(u)R_{i,j}(u+v)\widehat{K}_j(v) = \widehat{K}_j(v)R_{i,j}(u+v)\widehat{K}_i(u)R_{i,j}(u-v)\,. \tag{8.2.18}$$

- *Dual Boundary-Yang-Baxter-Equation*

$$K_j(v)R_{i,j}(-u-v-N)K_i(u)R_{i,j}(v-u) = R_{i,j}(v-u)K_i(u)R_{i,j}(-v-u-N)K_j(v)\,. \tag{8.2.19}$$

We have the following relation between $R(u)$ and $\hat{K}(u), K(u)$ matrices and the Hamiltonian H'':

- relation between $R(u)$ and $\mathcal{H}$

$$\mathcal{H} = \frac{d}{du}R(u)\Big|_{u=0}\,, \tag{8.2.20}$$

- relation between $\hat{K}(u)$ and H_{right}

$$H''_{\text{right}} = \frac{1}{2}\frac{d}{du}\hat{K}(u)\Big|_{u=0}\,, \tag{8.2.21}$$

- relation between $K(u)$ and H_{left}

$$H''_{\text{left}} = -\frac{1}{2}\frac{d}{du}K(u)\Big|_{u=0} - \frac{2N+1}{2N}\,, \tag{8.2.22}$$

Double row monodromy and transfer matrix

Here, we construct the fundamental object of the QISM: the transfer matrix. As already pointed out, this matrix is the generating function of a family of commuting operators that contains the Hamiltonian H'', defined in (8.2.3), itself. This allows to find all the eigenvectors of H'' by diagonalizing the transfer matrix (for instance by algebraic Bethe ansatz [17]). To construct the transfer matrix we introduce the *double row monodromy* acting on the space $V_0 \otimes \mathbb{V}$

$$U_0(u) = M_0(u)\widehat{K}_0(u)\widehat{M}_0(u) \tag{8.2.23}$$

where

$$M_0(u) = R_{0,1}(u)\dots R_{0,L}(u), \qquad \widehat{M}_0(u) = R_{0,L}(u)\dots R_{0,1}(u)\,. \tag{8.2.24}$$

The double row monodromy $U_0(x)$ satisfies the following *boundary RTT* relation

$$R_{0,0'}(u-v)U_0(u)R_{0,0'}(u+v)U_{0'}(u) = U_{0'}(v)R_{0,0'}(u+v)U_0(u)R_{0,0'}(u-v)\,, \tag{8.2.25}$$

on the space $V_0 \otimes V_{0'} \otimes \mathbb{V}$, where $V_0, V_{0'} = \mathbb{C}^N$ are auxiliary spaces. Moreover, we can write $U_0(u)$ in the canonical basis of V_0 as

$$U_0(u) = \begin{pmatrix} D_{1,1}(u) & D_{1,2}(u) & \dots & D_{1,N-1}(u) & C_1(u) \\ D_{2,1}(u) & D_{2,2}(u) & \dots & D_{2,N-1}(u) & C_2(u) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{N-1,1}(u) & D_{N-1,2}(u) & \dots & D_{N-1,N-1}(u) & C_{N-1}(u) \\ B_1(u) & B_2(u) & \dots & B_{N-1}(u) & D_{N,N}(u) \end{pmatrix} \tag{8.2.26}$$

where each element the above matrix acts on the quantum space $\mathbb{V}$. Usually, the entries $C_a(u)$ and $B_a(u)$ have exchanged position in the matrix (8.2.26), however in this chapter we adopt this convention since we chose the reference state of the chain to be the tensor product of the tensor product of $N-$dimensional vectors $(0,\dots,0,1)^{\otimes L}$, instead of $(1,0,\dots,0)^{\otimes L}$. This allows to interpret $B_a(u)$ as a rising operator.
We introduce the *transfer matrix*

$$T(u) = tr_0\left(K_0(u)U_0(u)\right) \tag{8.2.27}$$

that satisfies the commutation property

$$[T(u), T(v)] = 0\,. \tag{8.2.28}$$

The connection between H'' (defined in (8.2.11)) and $T(u)$ is given by

$$H'' = \frac{1}{2N}\frac{d}{du}T(u)\Big|_{u=0} - \frac{2N+1}{2N}\,. \tag{8.2.29}$$

Therefore, we have that

$$[H'', T(u)] = 0 \qquad \forall u \in \mathbb{C} \tag{8.2.30}$$

Equations (8.2.28) and (8.2.30) imply that the eigenvectors of $T(u)$ are independent of the spectral parameter and they also diagonalize H''. Using the entries of the double row monodromy of equation (8.2.26), the transfer matrix $T(u)$ can be written as

$$T(u) = (N+u)D_{N,N}(u) - \sum_{i=1}^{N-1} xD_{ii}(u) + \sum_{i=1}^{N-1}(2u+N)(\alpha_i - \beta_i)B_i(u) \tag{8.2.31}$$

8.2.2 The exchange relations

Exploiting the simplicity of $\hat{K}(u)$ and $K(u)$ in equation (8.2.15) and (8.2.16) we can use the boundary RTT relation (8.2.25) to derive explicit exchange relations for the matrix element of $U_0(u)$ defined in equation (8.2.26). These relations turn out to be crucial in the formulation of the mapping of non-equilibrium onto equilibrium.

Proposition 16 *(Fundamental commutation relations) For every $u, v \in \mathbb{C}$ such that $v \neq u$ and $v \neq -u-1$, the entries of the double row monodromy matrix $U_0(u)$ defined in* (8.2.26) *satisfy the fundamental commutation relations (FCR) given by:*

1. *for all $a \in \{1, \dots, N-1\}$*

$$\begin{aligned} D_{N,N}(u)B_a(v) &= f(u,v)B_a(v)D_{N,N}(u) + f_1(u,v)B_a(u)D_{N,N}(v) \\ &\quad - f_2(u,v)\left(B_a(u)D_{a,a}(v) + \sum_{c=1\,:\,c\neq a}^{N-1} B_c(u)D_{c,a}(v)\right) \end{aligned} \tag{8.2.32}$$

2. *for all $a \in \{1, \dots, N-1\}$*

$$\begin{aligned} D_{a,a}(u)B_a(v) &= h(u,v)B_a(v)D_{a,a}(u) + h_1(u,v)B_a(u)D_{a,a}(v) \\ &\quad + h_2(u,v)B_a(u)D_{N,N}(v) + h_3(u,v)B_a(v)D_{N,N}(u) \\ &\quad + \frac{1}{2}h_3(u,v)\sum_{c=1\,:\,c\neq a}^{N-1} B_c(u)D_{c,a}(v) + h_4(u,v)\sum_{c=1\,:\,c\neq a}^{N-1} B_c(v)D_{c,a}(u) \end{aligned} \tag{8.2.33}$$

3. *for all $a, b \in \{1, \dots, N-1\}$ such that $a \neq b$*

$$\begin{aligned} D_{b,b}(u)B_a(v) &= B_a(v)D_{b,b}(u) + \frac{1}{2}h_3(u,v)\left(B_a(v)D_{N,N}(u) - B_a(u)D_{N,N}(v)\right) \\ &\quad + h_1(x,y)\left(B_b(u)D_{b,a}(v) - B_b(v)D_{b,a}(u)\right) \\ &\quad + \frac{1}{2}h_3(u,v)\sum_{c=1\,:\,c\neq b}^{N-1}\left(B_c(u)D_{c,a}(v) - B_c(v)D_{c,a}(u)\right) \end{aligned} \tag{8.2.34}$$

4. *for every $a, b \in \{1, \dots, N-1\}$ such that $a \neq b$*

$$\begin{aligned} D_{a,b}(u)B_a(v) =&(v-u)h_1(u,v)B_a(v)D_{a,b}(u) \\ &+\frac{1}{2}h_3(u,v)\left(B_b(v)D_{N,N}(u) - (1+u-v)B_b(u)D_{N,N}(v)\right) \\ &+\frac{1}{(u-v)}\left(B_b(v)D_{a,a}(u) - B_b(u)D_{a,a}(v)\right) \\ &+f_2(u,v)\sum_{c=1\,:\,c\neq a}^{N-1} B_c(v)D_{c,b}(u) \end{aligned} \tag{8.2.35}$$

5. *for every $a, b \in \{1, \dots, N-1\}$ such that $a \neq b$*

$$D_{a,b}(u)B_b(v) = \frac{(v-u)}{(1+v-u)}B_b(v)D_{a,b}(u) + \frac{1}{(1+v-u)}D_{a,b}(v)B_b(u) \tag{8.2.36}$$

6. for all $a, b, c \in \{1, \ldots, N-1\}$ such that $a \neq b \neq c$

$$D_{a,b}(u)B_c(v) = B_c(v)D_{a,b}(u) + \frac{1}{(u-v)}\left(B_b(v)D_{a,c}(u) - B_b(u)D_{a,c}(v)\right) , \tag{8.2.37}$$

Here we have introduce the following rational functions:

$$\begin{aligned} f(u,v) &= \frac{(u+v)(u-v-1)}{(u-v)(1+u+v)}, & f_1(u,v) &= \frac{(u+v)}{(u-v)(1+u+v)}, \\ f_2(u,v) &= \frac{1}{(1+u+v)}, & h(u,v) &= \frac{(1+u-v)(u+v+2)}{(u-v)(1+u+v)}, \\ h_1(u,v) &= -\frac{(u+v+2)}{(u-v)(1+u+v)}, & h_2(u,v) &= \frac{(2+u-v)}{(u-v)(1+u+v)}, \\ h_3(u,v) &= -\frac{2}{(u-v)(1+u+v)}, & h_4(u,v) &= \frac{(1+u-v)}{(u-v)(1+u+v)} . \end{aligned} \tag{8.2.38}$$

Proof of Proposition 16: we perform the proof for the simplest non-trivial case $N = 3$. The computations can then be generalized for arbitrary N. In this case, relation (8.2.25) is an equality between 9×9 matrices, therefore we have 81 equations relating the elements of $U_0(u)$. For instance, we consider the element $(9, 3)$ of this matrix-equality and we read-off the following equation:

$$\begin{aligned} (1+u-v)(u+v)B_1(u)D_{3,3}(v) =& (u-v)B_1(v)D_{1,1}(u) + (u-v)B_2(v)D_{2,1}(u) \\ &+ (u-v)(1+u+v)D_{3,3}(v)B_1(u) + (u+v)B_1(v)D_{3,3}(u) . \end{aligned} \tag{8.2.39}$$

Renaming $u \to v$ and $v \to u$, and multiplying both sides bv $\frac{1}{(v-u)(1+u+v)}$ we obtain

$$\begin{aligned} D_{3,3}(u)B_1(v) =& \frac{(u-v-1)(u+v)}{(u-v)(1+u+v)}B_1(v)D_{3,3}(u) + \frac{(u+v)}{(u-v)(1+u+v)}B_1(u)D_{3,3}(v) \\ &- \frac{(u-v)}{(u-v)(1+u+v)}B_1(u)D_{1,1}(v) - \frac{(u-v)}{(u-v)(1+u+v)}B_1(u)D_{2,1}(v) . \end{aligned} \tag{8.2.40}$$

Introducing the functions (8.2.38), equation (8.2.32) with $N = 3$ follows. By analogous (long) computations all the other FCR can be determined.

□

8.2.3 The conserved charges

In this section we construct the conserved charges, that are defined as the coefficients of the polynomial expansion in the spectral parameter of the transfer matrix $T(u)$ (8.2.31).
By construction, we observe that the entries of the double row monodromy matrix (8.2.23) are polynomials of maximal degree $2L + 1$ in the spectral parameter. For our purposes it is enough to consider the following entries for all $a, b \in \{1, \ldots, N-1\}$:

$$D_{N,N}(u) = \sum_{k=0}^{2L+1} D^k_{N,N}u^k \quad D_{a,a}(u) = \sum_{k=0}^{2L+1} D^k_{a,a}u^k \quad B_a(u) = \sum_{k=0}^{2L-1} B^k_a u^k \quad D_{a,b}(u) = \sum_{k=0}^{2L} D^k_{a,b}u^k . \tag{8.2.41}$$

As a consequence of the above polynomial expansions, we have that the transfer matrix $T(u)$ (8.2.31) can be written as polynomial of degree $2L+2$ in the spectral parameter as

$$T(u) = \sum_{k=0}^{2L+2} Q^k u^k \tag{8.2.42}$$

where, for all $k \in \{0,\dots,2L+2\}$, we call the matrices Q^k the *conserved charges* of the system. We now write explicit expressions, in terms of the basis elements e_{AB} of the first fundamental representation of $gl(N)$, for the first three charges Q^{2L+2}, Q^{2L+1} and Q^{2L}. Indeed, as it will be clear in the following, they are enough for our purposes. Replacing the polynomial expansions (8.2.41) in equation (8.2.31) we have that

$$Q^{2L+2} = D^{2L+1}_{N,N} - \sum_{a=1}^{N-1} D^{2L+1}_{a,a}\,, \tag{8.2.43}$$

$$Q^{2L+1} = N D^{2L+1}_{N,N} + D^{2L}_{N,N} - \sum_{a=1}^{N-1} D^{2L}_{a,a}\,, \tag{8.2.44}$$

$$Q^{2L} \quad = N D^{2L}_{N,N} + D^{2L-1}_{N,N} - \sum_{a=1}^{N-1} D^{2L-1}_{a,a} + \sum_{a=1}^{N-1} 2(\alpha_a - \beta_a) B^{2L-1}_{a}\,. \tag{8.2.45}$$

By the definition of the double row monodromy (8.2.23) explicit equations for the charges will be obtained. In what follows, we denote by e^{tot}_{AB}

$$e^{tot}_{AB} = \sum_{x=1}^{L} e^{x}_{AB} \tag{8.2.46}$$

where, we recall, that

$$e^{x}_{AB} = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes e_{AB} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \tag{8.2.47}$$

with $\mathbb{1}$ the identity matrix of size N and e_{AB} at position x. Then we have:

- for $D_{N,N}(x)$:

$$D^{2L+1}_{N,N} = 1\,, \tag{8.2.48}$$

$$D^{2L}_{N,N} = 1 + 2e^{tot}_{NN}\,. \tag{8.2.49}$$

$$D^{2L-1}_{N,N} = 2e^{tot}_{NN}\left(e^{tot}_{NN} + 1\right) - L\,. \tag{8.2.50}$$

- For $D_{a,b}(x)$ with $a \in \{1,\dots,N-1\}$

$$D^{2L+1}_{a,a} = -1\,, \tag{8.2.51}$$

$$D^{2L}_{a,a} = 1 - 2e^{tot}_{aa}\,, \tag{8.2.52}$$

$$D^{2L-1}_{a,a} = 2Ne^{tot}_{aa} + 2e^{tot}_{NN} - L - 2\sum_{b=1}^{N-1} e^{tot}_{ab} e^{tot}_{ba}\,. \tag{8.2.53}$$

- For $B_a(x)$ with $a \in \{1, \dots, N-1\}$

$$B_a^{2L-1} = 2e_{aN}^{tot} e_{NN}^{tot} - 2 \sum_{x,y=1\,:\,x<y}^{L} \sum_{b=1}^{N} e_{bN}^{x} e_{ab}^{y} \,. \tag{8.2.54}$$

- For $D_{a,b}(x)$ with $a, b \in \{1, \dots, N-1\}$ such that $a \neq b$

$$D_{a,b}^{2L} = -2e_{a,b}^{tot} \,. \tag{8.2.55}$$

Remark 20 *When we restrict to the case of $N = 2$, we recover equations (2.24)-(2.30) of [20].*

Substituting the above explicit expressions in (8.2.43),(8.2.44) and in (8.2.45), we obtain the equations for the first three charges in terms of elementary matrices:

$$Q^{2L+2} = N \,, \tag{8.2.56}$$

$$Q^{2L+1} = N + 2L - 1 \,, \tag{8.2.57}$$

$$\begin{aligned} Q^{2L} \; &= (N - NL - 2L) + 2(N+2)e_{NN}^{tot} + 2e_{NN}^{tot} e_{NN}^{tot} + 2 \sum_{a,b=1}^{N-1} e_{ba}^{tot} e_{ab}^{tot} \\ &+ \sum_{a=1}^{N-1} 2(\alpha_a - \beta_a) \left(2e_{aN}^{tot} e_{NN}^{tot} - 2 \sum_{x,y=1\,:\,x<y}^{L} \sum_{b=1}^{N} e_{bN}^{x} e_{ab}^{y} \right) \,. \end{aligned} \tag{8.2.58}$$

The charges Q^{2L+2} and Q^{2L+1} are proportional to the identity matrix, while Q^{2L} contains non diagonal matrices and the boundary parameters $\alpha_a, \beta_a \; \forall a = 1, \dots, N-1$. Q^{2L} is the first non-trivial charge and then it is the leading (non-trivial) coefficient of the polynomial expansion of $T(u)$. By construction it follows that

$$\left[Q^{2L}, H'' \right] = Q^{2L} H'' - H'' Q^{2L} = 0 \tag{8.2.59}$$

while the commutation between the Hamiltonian H'' and Q^{2L+2}, Q^{2L+1} are trivial. We introduce the matrix

$$Q_0^{2L} = (N - NL - 2L) + 2(N+2)e_{NN}^{tot} + 2e_{NN}^{tot} e_{NN}^{tot} + 2 \sum_{a,b=1}^{N-1} e_{ba}^{tot} e_{ab}^{tot} \tag{8.2.60}$$

and the matrix

$$Q_+^{2L} = \sum_{a=1}^{N-1} 2\Delta_a \left(2e_{aN}^{tot} e_{NN}^{tot} - 2 \sum_{x,y=1\,:\,x<y}^{L} \sum_{b=1}^{N} e_{bN}^{x} e_{ab}^{y} \right) \tag{8.2.61}$$

where, for the sake of notation, we have denoted $\Delta_a = (\alpha_a - \beta_a)$. Therefore we write

$$Q^{2L} = Q_0^{2L} + Q_+^{2L} \,. \tag{8.2.62}$$

Observe that Q^{2L} and Q_0^{2L} are isospectral. This property will be crucial in the following, since it allow to rewrite the map W with these charges Q_0^{2L} and Q_0^{2L} in place of H'' and H_0. This is the reason why the integrability of the process allows to write W in function of the charges of the system.

Eigenvalues of Q_0^{2L}. Inspired by Chapter 10 of [65], in this paragraph we write an explicit expression of the eigenvalues of Q_0^{2L}, defined in (8.2.60), using the representation theory for $gl(N)$. We observe that the matrix Q_0^{2L} is not diagonal, since the last term

$$\sum_{a,b=1}^{N-1} e_{ab}^{tot} e_{ba}^{tot} \tag{8.2.63}$$

is not. This last term can be identified with the second Casimir of the $gl(N-1)$ subalgebra, denoted $\hat{C}_2^{N-1}$, i.e.

$$\hat{C}_2^{N-1} = \sum_{a,b=1}^{N-1} e_{ab}^{tot} e_{ba}^{tot} \,. \tag{8.2.64}$$

We indicate its eigenvalues by C_2^{N-1}. Since the configuration space of the process is given by the tensor product over the chain of length L of the first fundamental representation of $gl(N)$, we can decompose it in a direct sum of irreducible representation of $gl(N)$ (branching rule, see Chapter 5 [65]). As a consequence, the states of the process can be decomposed into the direct sum of basis vectors that belong to irreducible representations of $gl(N)$. On each of these basis vectors $\hat{C}_2^{N-1}$ acts diagonally. Their eigenvalues C_2^{N-1} are given by

$$C_2^{N-1} = \sum_{a=1}^{N-1} \ell_a(\ell_a - N + 2) + \frac{1}{6}(N-1)(N-2)(N-3) \tag{8.2.65}$$

where we have introduced the *shifted weights*

$$\ell_a = \mu_a + (N-1) - a \tag{8.2.66}$$

with μ_a is the a-th component of a weight μ of the irreducible representations that decompose the tensor product. To each weight, an eigenvalue of the form (8.2.65) is assigned, and its multiplicity is given by the dimension of the irreducible representation assigned to the weight.

Example for $N = 3$ and $L = 2$. In this case of length 2 chain we have the tensor product of two first fundamental representations of $gl(3)$ with Dynkin label $(1, 0, 0)$. Therefore, using the branching rule we obtain the following decomposition into irreducible representations

$$\underbrace{(1,0,0)}_{3} \otimes \underbrace{(1,0,0)}_{3} = \underbrace{(2,0,0)}_{6} \oplus \underbrace{(1,1,0)}_{3} \tag{8.2.67}$$

where we have denoted the dimension of the representations in under-brace. We aim to find the eigenvalues of the Casimir $\hat{C}_2^2$ of the $gl(2)$ sub-algebra. By direct computations one can show that this Casimir has the following eigenvalues

$$6 \quad \text{with multiplicity } 3, \qquad 2 \quad \text{with multiplicity } 5, \qquad 0 \quad \text{with multiplicity } 1 \tag{8.2.68}$$

We now show this by using formula (8.2.65). We first consider the highest weight $\mu^1 = (2, 0, 0)$ on the right-hand-side of (8.2.67). The weights of this representation are $\mu^1 = (2, 0, 0)$ (the highest weight) and $\mu^{1,2} = (1, 0, 0)$, $\mu^{1,3} = (0, 0, 0)$. We have that:

- $\mu^1 = (2, 0, 0)$ is associated with a symmetric representation of dimension 3, then, using (8.2.65) and (8.2.66), we compute the the eigenvalue with multiplicity 3:

$$\ell_1 = \mu_1^1 + (N-1) + 1 = 2 + 2 - 1 = 3, \qquad \ell_2 = \mu_2^1 + (N-1) + 2 = 0 + 2 - 2 = 0 \,. \tag{8.2.69}$$

It follows

$$C_2^2 = \sum_{a=1}^{2} \ell_a(\ell_a - 1) = 6 \,. \tag{8.2.70}$$

- $\mu_{1,2} = (1,0,0)$ is associated with a symmetric representation of dimension 2, then, using (8.2.65) and (8.2.66), we compute the the eigenvalue with multiplicity 2:

$$\ell_1 = 1 + 2 - 1 = 2, \qquad \ell_2 = 0 + 2 - 2 = 0 \,. \tag{8.2.71}$$

It follows

$$C_2^2 = \sum_{a=1}^{2} \ell_a(\ell_a - 1) = 2 \,. \tag{8.2.72}$$

- $\mu_{1,2} = (0,0,0)$ is associated with a symmetric representation of dimension 1, then, using (8.2.65) and (8.2.66), we compute the the eigenvalue with multiplicity 1

$$\ell_1 = 0 + 2 - 1 = 1, \qquad \ell_2 = 0 + 2 - 2 = 0 \,. \tag{8.2.73}$$

If follows

$$C_2^2 = \sum_{a=1}^{2} \ell_a(\ell_a - 1) = 0 \,. \tag{8.2.74}$$

Secondly, we consider the highest weight $\mu^2 = (1,1,0)$ on the right hand side of (8.2.67). The weights of this representation are $\mu^2 = (1,1,0)$ (the highest weight) and $\mu^{2,2} = (1,0,0)$. By repeating similar computations, we obtain an eigenvalue $C_2^2 = 2$ with multiplicity 1 and an eigenvalue $C_2^2 = 2$ with multiplicity 2.
All, in all we have obtained the eigenvalues 6, 2 and 0 with multiplicities 3, 5, and 1 respectively, matching with the direct computations of the spectrum of $\hat{C}_2^2$ of equation 8.2.68.

This argument allows to write an explicit expression for the eigenvalues of Q_0^{2L}. We recall that the first fundamental representation of $gl(N)$ has dimension N, i.e. the associated vector space is N-dimensional. Therefore, considering the whole chain of length L, we can assign by a one-to-one map a label $M = 1, 2, \ldots, N^L$ to each configuration vector $|\boldsymbol{\tau}\rangle$. We denote this map by $M \to |\boldsymbol{\tau}(M)\rangle = |\tau_1(M), \ldots, \tau_L(M)\rangle$, where $\tau_x(M)$ indicates the occupation at site x assigned by the label M. Then, we denote the eigenvectors that simultaneously diagonalize Q_0^{2L} and e_{AA}^{tot} by $|\Psi_M^0\rangle$ and the eigenvalues of Q_0^{2L} by Λ_M, i.e.

$$e_{AA}^{tot}|\Psi_M^0\rangle = \left(\sum_{x=1}^{L} \delta_{A,\tau_x(M)}\right)|\Psi_M^0\rangle, \qquad Q_0^{2L}|\Psi_M^0\rangle = \Lambda_M|\Psi_M^0\rangle \,. \tag{8.2.75}$$

Moreover, we denote by Λ_M and $|\Psi_M\rangle$ the eigenvalues and the eigenvectors of Q^{2L}, that is

$$Q^{2L}|\Psi_M\rangle = \Lambda_M|\Psi_M\rangle, \qquad \forall M \in \{1, \ldots, N^L\} \,. \tag{8.2.76}$$

Computing the the eigenvalues C_2^{N-1} of the second Casimir of the subalgebra $gl(N-1)$ by the decomposition into irreducible representations (as explained above) we have that, for all $M \in \{1, \ldots, N^L\}$

$$\Lambda_M = (N - NL - 2L) + 2(N+2)\left(\sum_{x=1}^{L} \delta_{N,\tau_x(M)}\right) + \left(\sum_{x=1}^{L} \delta_{N,\tau_x(M)}\right) + C_2^{N-1} \,. \tag{8.2.77}$$

Exchange relations for the conserved charges

As corollary of Proposition 16 we find the exchange relations between the conserved charges of the system.

Corollary 4 *(Exchange relations for charges) The following exchange relations between the conserved charges* (8.2.42) *and the coefficients of the polynomial expansion of the entries of* $U_0(u)$ *given in* (8.2.41) *do hold:*

1. *For every* $a \in \{1, \ldots, N-1\}$

$$Q_0^{2L} B_a^{2L-1} = B_a^{2L-1}\left(Q_0^{2L} - 2(D_{a,a}^{2L} + D_{N,N}^{2L})\right) - 2\sum_{b=1\,:\,b\neq a}^{N-1} B_b^{2L-1} D_{b,a}^{2L}\,. \tag{8.2.78}$$

2. *For all* $a \in \{1, \ldots, N-1\}$

$$D_{N,N}^{2L} B_a^{2L-1} = B_a^{2L-1}\left(D_{N,N}^{2L} + D_{a,a}^{2L+1} - D_{N,N}^{2L+1}\right)\,. \tag{8.2.79}$$

3. *For all* $a \in \{1, \ldots, N-1\}$

$$D_{a,a}^{2L} B_a^{2L-1} = B_a^{2L-1}\left(D_{a,a}^{2L} + D_{a,a}^{2L+1} - D_{N,N}^{2L+1}\right)\,. \tag{8.2.80}$$

4. *For all* $a, b \in \{1, \ldots, N-1\}$

$$D_{a,b}^{2L} B_a^{2L-1} = B_a^{2L-1} D_{a,b}^{2L} + B_b^{2L-1}\left(D_{a,a}^{2L+1} - D_{N,N}^{2L+1}\right)\,. \tag{8.2.81}$$

5. *For* $a, b, c \in \{1, \ldots, N-1\}$ *such that* $a \neq b \neq c \neq a$,

$$\begin{aligned} \left[D_{b,b}^{2L}, B_a^{2L-1}\right] &= 0, \qquad \left[B_a^{2L-1}, B_b^{2L-1}\right] = 0, \\ \left[B_a^{2L-1}, D_{b,a}^{2L}\right] &= 0, \qquad \left[B_c^{2L-1}, D_{b,a}^{2L}\right] = 0\,. \end{aligned} \tag{8.2.82}$$

Proof of Corollary 4: the proof follows from Proposition 16 by equalizing the powers of the polynomial expansion (8.2.41) and (8.2.42).

□

8.3 Mapping of non-equilibrium onto equilibrium

We now look for a similarity transformation W that maps the non-equilibrium Hamiltonian $H^{''}$ (introduced in (7.5.14)) onto a Hamiltonian H_0 that has diagonal boundaries. We introduce

$$H^0 = H^0_{left} + H_{bulk} + H^0_{right} \tag{8.3.1}$$

where

$$H^0_{left} = e^1_{NN} - 1 \qquad H^0_{right} = e^L_{NN} - 1\,. \tag{8.3.2}$$

Therefore we have that

$$H^{''} = H^0 + H^B \tag{8.3.3}$$

where

$$H^B = \sum_{a=1}^{N-1} (\alpha_a - \beta_a) e^1_{aN} = \sum_{a=1}^{N-1} \Delta_a e^1_{aN}\,, \tag{8.3.4}$$

and where $\Delta_a = (\alpha_a - \beta_a)$. We look for a matrix W such that

$$H^{''} W = W H_0\,. \tag{8.3.5}$$

Since $H^{''}$ and H_0 are isospectral, this matrix W reads (see [28] for the $N = 2$ case)

$$W = \lim_{\epsilon \to 0} \epsilon \sum_{M=1}^{N^L} \left((\mathcal{E}_M + \epsilon) - H^{''} \right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \tag{8.3.6}$$

where we have we have denoted by $\mathcal{E}_M$ and $|\Lambda^0_M\rangle$ the eigenvalue and the eigenvector of H_0 associated with the label $M \in \{1, \dots, N^L\}$, i.e.

$$H_0 |\Psi^0_M\rangle = \mathcal{E}_M |\Psi^0_M\rangle\,. \tag{8.3.7}$$

Furthermore, we denote by $\mathcal{E}_M$ and by $|\Psi_M\rangle$ the eigenvalue and the eigenvector of $H^{''}$ associated with m, i.e.

$$H^{''} |\Psi_M\rangle = \mathcal{E}_M |\Psi_M\rangle\,. \tag{8.3.8}$$

We show that the matrix W defined in (8.3.6) satisfies (8.3.5) in Section 8.3.3.
Exploiting integrability, we observe that since, Q_0^{2L} (defined in (8.2.60)) commutes with H_0 and Q^{2L} (defined in (8.2.62)) commutes with $H^{''}$, the aimed change of basis may be obtained by looking for a matrix W such that

$$Q^{2L} W = W Q_0^{2L}\,. \tag{8.3.9}$$

Therefore, the W matrix defined in (8.3.6) is written as

$$\begin{aligned} W &= \lim_{\epsilon \to 0} \epsilon \sum_{M=1}^{N^L} \left((\Lambda_M + \epsilon) - Q^{2L} \right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \\ &= \lim_{\epsilon \to 0} \epsilon \sum_{M=1}^{N^L} \left((\Lambda_M + \epsilon) - Q_0^{2L} - Q_+^{2L} \right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \end{aligned} \tag{8.3.10}$$

where Q_+^{2L} is the matrix defined in (8.2.61).

Remark 21 *(Mapping beyond the integrable chain) We notice that since the boundary driven multi-species stirring process with Hamiltonian H defined in* (4.3.9) *admits absorbing duality, it is possible, in analogy to what we did here, to construct suitable similarity transformations S_1 and S_2 that bring the Hamiltonian H to a matrix $H^{''}$, which has left triangular boundary and right diagonal boundary. We introduce H_0, obtained by setting diagonal both boundaries of H. These Hamiltonians are related by*

$$H^{''} = H_0 + H_B \tag{8.3.11}$$

where H_B is triangular and rises the occupation of a state by removing holes and introducing particles of some type (in analogy to equation (8.3.4)*). In [28], it has been stated that for the case $N = 2$ there exists a similarity transformation W such that $H^{''} W = W H_0$. However, this argument states only an existence result, without giving a closed formula. In this general case this W could be evaluated by perturbation theory. In the integrable case instead, the existence of the conserved charges allows to write this W in a closed formula (summing up the perturbative series). We expect that also in this multi-species situation a similar existence result might be found. However, the possibility of having vanishing eigenvectors cannot be excluded a priori.*

8.3.1 Construction of the mapping for the integrable SSEP

Considering the Hamiltonian H defined in (7.2.3) when $N = 2$, we obtain the *integrable SSEP* (see Section 7.6 and the references [21, 20, 28]). Therefore, adapting the similarity transformations S_1 and S_2, one obtains the $H^{''}$ matrix for the SSEP. In this case an analytic form, not depending on the spectrum and of the eigenvectors of H_0, of the matrix W of equation (8.3.10) has been found in [20, 28]. Here, we propose a slightly different technique to obtain the same result. The advantage is that it seems more suitable to be extended to the multi-species stirring process situation. From Section 7.6, we recall that the Hamiltonian $H^{''}$ in the $N = 2$ case is given by

$$H^{''} = H^{''}_{\text{left}} + \sum_{x=1}^{L-1} H_{x,x+1} + H^{''}_{\text{right}} \tag{8.3.12}$$

where the bulk Hamiltonian densities are

$$H_{x,x+1} = \sum_{A,B=1}^{2} \left(e_{AB} \otimes e_{BA} - e_{BB} \otimes e_{AA} \right) . \tag{8.3.13}$$

and where boundary Hamiltonians of $H^{''}$ are

$$H^{''}_{\text{left}} = \begin{pmatrix} -1 & \alpha_1 - \beta_1 \\ 0 & 0 \end{pmatrix}, \qquad H^{''}_{\text{right}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{8.3.14}$$

Proposition 17 *(Mapping for SSEP) Consider $H^{''}$ and H_0 for the SSEP. Assume that $|\Lambda_M\rangle$, $|\Lambda^0_M\rangle$ and $\mathcal{E}_M$ satisfy*

$$H^{''}|\Psi_M\rangle = \mathcal{E}_M|\Psi_M\rangle, \qquad H_0|\Psi^0_M\rangle = \mathcal{E}_M|\Psi^0_M\rangle \tag{8.3.15}$$

where $M \in \{1, \ldots, 2^L\}$ denotes the total number of particles present in the chain. Then, we have that, for all $M \in \{1, \ldots, 2^L\}$, the following formula holds

$$W = \sum_{k=0}^{L} \frac{(\alpha_1 - \beta_1)^k}{k!} \left(\frac{B_1^{2L-1}}{2} \right)^k \frac{\Gamma(\sigma_0^{tot} + 2 - k)}{\Gamma(\sigma_0^{tot} + 2)} \tag{8.3.16}$$

where $\sigma_0^{tot} = \sum_{x=1}^{L} \sigma_0^x$, with σ_0^x is the opposite of the diagonal Pauli matrix, i.e.

$$\sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{8.3.17}$$

acting non-trivially at site x. Moreover, B_1^{2L-1} is given in (8.2.54) *when $N = 2$.*

Proof of Proposition 17 The double row monodromy reads

$$U_0 = \begin{pmatrix} D_{1,1}(x) & C_1(x) \\ B_1(x) & D_{2,2}(x) \end{pmatrix} \tag{8.3.18}$$

For the sake of notation we denote by $\Delta = (\alpha_1 - \beta_1)$. Using (8.2.62) we rewrite (8.3.10) as

$$W = \lim_{\epsilon \to 0} \epsilon \sum_{M=1}^{2^L} \left((\Lambda_M + \epsilon) - Q_0^{2L} - 2\Delta B_1^{2L-1} \right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \, . \tag{8.3.19}$$

Let $\boldsymbol{A}, \boldsymbol{B}$ two square matrices of the same size and let $\boldsymbol{A}$ be invertible, then we have the following series expansion

$$(\boldsymbol{A}-\boldsymbol{B})^{-1}=\sum_{n=0}^{\infty}(\boldsymbol{A}^{-1}\boldsymbol{B})^{n}\boldsymbol{A}^{-1}\,. \tag{8.3.20}$$

Using (8.3.20) we obtain

$$\begin{aligned} W &= \lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{2^L}\Big((\Lambda_M+\epsilon)-Q_0^{2L}-2\Delta B_1^{2L-1}\Big)^{-1}|\Psi_M^0\rangle\langle\Psi_M^0| \\ &= \lim_{\epsilon\to 0}\sum_{M=1}^{2^L}\epsilon\sum_{k=0}^{L}\Delta^k 2^k\left\{\left((\Lambda_M+\epsilon)-Q_0^{2L}\right)^{-1}B_1^{2L-1}\right\}^k\left((\Lambda_M+\epsilon)-Q_0^{2L}\right)^{-1}|\Psi_M^0\rangle\langle\Psi_M^0|\,. \end{aligned} \tag{8.3.21}$$

By specializing to the case $N=2$ the exchange relation between charges written in (8.2.78),(8.2.79) and (8.2.80), we have

$$Q_0^{2L}B_1^{2L-1}=B_1^{2L-1}\left(Q_0^{2L}-2\left(D_{2,2}^{2L}+D_{1,1}^{2L}\right)\right)\,, \tag{8.3.22}$$

$$D_{2,2}^{2L}B_1^{2L-1}=B_1^{2L-1}\left(D_{2,2}^{2L}-Q_0^{2L+2}\right) \tag{8.3.23}$$

and

$$D_{1,1}^{2L}B_1^{2L-1}=B_1^{2L-1}\left(D_{1,1}^{2L}-Q_0^{2L+2}\right)\,. \tag{8.3.24}$$

Therefore, using equations (8.2.56) we obtain

$$\begin{aligned} (Q_0^{2L}B_1^{2L-1})^k &= (B_1^{2L-1})^k\prod_{j=1}^{k}\left(Q_0^{2L}-2j(D_{2,2}^{2L}+D_{1,1}^{2L})+\mathbb{1}_{\{j>1\}}4j\sum_{i=1}^{j-1}(j-i)Q_0^{2L+2}\right) \\ &= (B_1^{2L-1})^k\prod_{j=1}^{k}\left(Q_0^{2L}-2j(D_{2,2}^{2L}+D_{1,1}^{2L})+4j(j-1)\right) \\ &= (B_1^{2L-1})^k\prod_{j=1}^{k}\left(Q_0^{2L}-4j(1+e_{22}^{tot}-e_{11}^{tot})+4j(j-1)\right)\,. \end{aligned} \tag{8.3.25}$$

Then, by using formula (8.3.20) and the above result we have that

$$\left\{(\Lambda_M+\epsilon-Q_0^{2L})^{-1}B_1^{2L-1}\right\}^k=(B_1^{2L-1})^k\prod_{j=1}^{k}\left((\Lambda_M+\epsilon)-Q_0^{2L}+4j(1+e_{22}^{tot}-e_{11}^{tot})-4j(j-1)\right)^{-1}\,. \tag{8.3.26}$$

Therefore by projecting on $|\Psi_M^0\rangle$ we have that

$$\begin{aligned} W|\Psi_M^0\rangle \lim_{\epsilon\to 0}\epsilon\sum_{k=0}^{L}2^k\Delta^k(B_1^{2L-1})^k\prod_{j=1}^{k}\left\{(\Lambda_M+\epsilon)-Q_0^{2L}+4j(e_{22}^{tot}-e_{11}^{tot})-4j^2+8j\right\}^{-1} \\ \times\left((\Lambda_M+\epsilon)-Q_0^{2L}\right)^{-1}|\Psi_M^0\rangle\,. \end{aligned} \tag{8.3.27}$$

Using again (8.3.20), re-summing the series and taking the limit $\epsilon \to 0$ we obtain

$$\begin{aligned} W|\Psi_M^0\rangle &= \lim_{\epsilon\to 0} \epsilon \sum_{k=0}^{L} 2^k \Delta^k (B_1^{2L-1})^k \prod_{j=1}^{k} \left\{(\Lambda_M+\epsilon) - \Lambda_M + 4j(e_{22}^{tot} - e_{11}^{tot}) - 4j^2 + 8j\right\}^{-1} \\ &\quad \times \left((\Lambda_M+\epsilon) - Q_0^{2L}\right)^{-1} |\Psi_M^0\rangle \\ &= \sum_{k=0}^{L} 2^k \Delta^k (B_1^{2L-1})^k \frac{1}{4^k}\frac{1}{k!} \prod_{j=1}^{k} \left\{e_{22}^{tot} - e_{11}^{tot} + 2 - j\right\}^{-1} |\Psi_M^0\rangle . \end{aligned} \tag{8.3.28}$$

In the last equality we have exploited that, from (8.3.20), it follows

$$\lim_{\epsilon\to 0} \epsilon \left((\Lambda_M+\epsilon) - Q_0^{2L}\right)^{-1} |\Psi_M^0\rangle = |\Psi_M^0\rangle . \tag{8.3.29}$$

Recalling that there is a one-to-one correspondence between $M \in \{1,\dots,2^L\}$ and a configuration $|\boldsymbol{\tau}(M)\rangle$ of the chain, we introduce $\boldsymbol{m}_1 = \sum_{x=1}^{L} \delta_{\tau_x(M),1}$ that counts the number of particles (here we only have type 1) present in the configuration $|\boldsymbol{\tau}(M)\rangle$. Then, we observe that $e_{11}^{tot}|\Psi_M^0\rangle = \boldsymbol{m}_1|\Psi_M^0\rangle$ and $e_{22}^{tot}|\Psi_M^0\rangle = (L-\boldsymbol{m}_1)|\Psi_M^0\rangle$, then we obtain

$$W|\Psi_M^0\rangle = \sum_{k=0}^{L} 2^k \Delta^k (B_1^{2L-1})^k \prod_{j=1}^{k} \left\{L - 2\boldsymbol{m}_1 + 2 - j\right\}^{-1} |\Psi_M^0\rangle . \tag{8.3.30}$$

By using the Euler-Γ function we have that

$$(L-2\boldsymbol{m}_1+2-j) = \frac{\Gamma(L-2\boldsymbol{m}_1+2-j+1)}{\Gamma(L-2\boldsymbol{m}_1+2-j)} \quad \Rightarrow \quad \prod_{j=1}^{k}(L-2\boldsymbol{m}_1+2-j)^{-1} = \frac{\Gamma(L-2\boldsymbol{m}_1+2-k)}{\Gamma(L-2\boldsymbol{m}_1+2)} . \tag{8.3.31}$$

Finally, introducing the matrix $\sigma_0^{tot} = e_{22}^{tot} - e_{11}^{tot}$ (that is minus the diagonal Pauli matrix), we obtain (8.3.16).

□

Remark 22 *We observe that considering the vacuum state, identified by $M = 1$ and defined,*

$$|\Psi_1^0\rangle = \underbrace{\begin{pmatrix}0\\1\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}0\\1\end{pmatrix}}_{L-times} \tag{8.3.32}$$

and by applying (8.3.16)*, we obtain $|\Psi_1\rangle = W|\Psi_1^0\rangle$, where $|\Psi_1\rangle$ is the non-equilibrium steady state of the SSEP. Indeed, it is a straightforward computation to show that $H_0|\Psi_1^0\rangle = 0$ and, form* (8.3.5) *it follows that $H''|\Psi_1\rangle = H''W|\Psi_1^0\rangle = WH_0|\Psi_1^0\rangle = 0$.*

8.3.2 Perspective: extension to the multi-species stirring process

In the general case of $N > 2$, inspired by the proof of Proposition 17, we can use the multi-variable version of the power expansion (8.3.20) and rewrite equation (8.3.10) as

$$\begin{aligned} W = \lim_{\epsilon\to 0} \epsilon \sum_{M=1}^{N^L} \sum_{k_1=0}^{L} \cdots \sum_{k_{N-1}=0}^{L} \mathbb{1}_{\{k_1+\cdots+k_{N-1}\leqslant L\}} 2^{k_1+\ldots+k_{N-1}} \Delta_1^{k_1} \cdots \Delta_{N-1}^{k_{N-1}} \\ SYM_{k_1,\dots,k_{N-1}} \left(\left\{\left(\Lambda_M+\epsilon-Q_{2L}^0\right)^{-1} B_1^{2L-1}\right\}, \dots, \left\{\left(\Lambda_M+\epsilon-Q_{2L}^0\right)^{-1} B_{N-1}^{2L-1}\right\}\right) \\ \times \left(\Lambda_M+\epsilon-Q_0^{2L}\right)^{-1} |\Psi_M^0\rangle\langle\Psi_M^0| \end{aligned} \tag{8.3.33}$$

where, we recall that Λ_M are the eigenvectors of Q_0^{2L} corresponding to M. Here SYM is the following transformation. We introduce, the indices i_j with $j = 1, \ldots, k_1 + \ldots + k_N$, such that

$$\begin{aligned} & i_1 = \ldots = i_{k_1} = 1 \\ & i_{k_1+1} = \ldots = i_{k_1+k_2} = 2 \\ & \vdots \\ & i_{k_1+\ldots+k_{N-2}+1} = \ldots = i_{k_1+\ldots+k_{N-1}} = N-1 \end{aligned} \quad . \tag{8.3.34}$$

We define

$$\begin{aligned} & SYM_{k_1,\ldots,k_{N-1}}\left(\left\{\left(\Lambda_M^Q + \epsilon - Q_{2L}^0\right)^{-1} B_1^{2L-1}\right\}, \ldots, \left\{\left(\Lambda_M + \epsilon - Q_{2L}^0\right)^{-1} B_{N-1}^{2L-1}\right\}\right) \\ = & \frac{1}{k_1! \cdots k_{N-1}!} \sum_{\sigma \in Sym} \left\{\left(\Lambda_M + \epsilon - Q_{2L}^0\right)^{-1} B_{\sigma(i_1)}^{2L-1}\right\} \cdots \left\{\left(\Lambda_M + \epsilon - Q_0^{2L}\right)^{-1} B_{\sigma(i_{k_1}+\ldots+i_{k_{N-1}})}^{2L-1}\right\} \end{aligned} \tag{8.3.35}$$

where Sym is the symmetric group of $N-1$ objects.
We observe that (8.3.33) still depends on the spectrum and on the eigenvectors of H_0 and it is not an analytic expression as (8.3.16). As a future perspective we aim to prove a multi-species version of Proposition 17. In this situation the exchange relation (8.2.78) has an extra creation operator

$$-2 \sum_{k=1\,:\,k \neq i}^{N-1} B_k^{2L-1} D_{k,i}^{2L} \,. \tag{8.3.36}$$

This extra creation operator, "mixes up" the different species of particles, then the computations seems to be more involved and the technique used for proof of Proposition 17 does not seem to be directly applicable. Moreover, one could aim to find the non-equilibrium steady state of H by acting with W defined in (8.3.33) on the ground state of H'', given by the tensor product of the N-dimensional vectors

$$|\Psi_1^0\rangle = \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}}_{L \text{ times}} , \tag{8.3.37}$$

and then invert the similarities S_2 and S_1. The hope is that the exchange relations between charges of Corollary 4 do simplify when the charges act on $|\Psi_1^0\rangle$, allowing to obtain the analytic expression of the vector $|\Psi''\rangle$ such that $H''|\Psi''\rangle = 0$.

8.3.3 Proof of equation (8.3.5)

Starting from the definition of the resolvent operator, we aim to show that (8.3.6) fulfils (8.3.5). Consider the *resolvent operator* (see [88])

$$\mathcal{R}(H'', z) := \left(z - H''\right)^{-1} \tag{8.3.38}$$

where z must be in the resolvent set, i.e. the subset of $\mathbb{C}$ such that the $\mathcal{R}(H'', z)$ is well defined. The above equation implies that

$$(z - H'')\mathcal{R}(H'', z) = I \,. \tag{8.3.39}$$

In order to be well defined z must not be an eigenvalue of $H^{''}$. Fix $\epsilon > 0$ and $M \in \{1, \dots, N^L\}$ and consider $z = \mathcal{E}_M + \epsilon$, where $\mathcal{E}_M$ is the eigenvalue with label M of the Hamiltonian $H^{''}$. We obtain

$$(\mathcal{E}_M + \epsilon - H^{''})\mathcal{R}(H^{''}, \mathcal{E}_M + \epsilon) = 1 \,. \tag{8.3.40}$$

We multiply both sides by $|\Psi^0_M\rangle\langle\Psi^0_M|$, with $|\Psi^0_M\rangle$ eigenvector with label M of the Hamiltonian H_0, and we have

$$(\mathcal{E}_M + \epsilon - H^{''})\mathcal{R}(H^{''}, \mathcal{E}_M + \epsilon)|\Psi^0_M\rangle\langle\Psi^0_M| = |\Psi^0_M\rangle\langle\Psi^0_M|, \tag{8.3.41}$$

by summing over all possible $M \in \{1, \dots, N^L\}$ and multiplying by ϵ both sides we have

$$\epsilon \sum_{M=1}^{N^L} (\mathcal{E}_M + \epsilon - H^{''})\mathcal{R}(H^{''}, \mathcal{E}_M + \epsilon)|\Psi^0_M\rangle\langle\Psi^0_M| = \epsilon \sum_{M=1}^{N^L} |\Psi^0_M\rangle\langle\Psi^0_M| \,. \tag{8.3.42}$$

We take the limit for $\epsilon \to 0$ on both sides

$$\lim_{\epsilon\to 0} \epsilon \sum_{M=1}^{N^L} (\mathcal{E}_M + \epsilon - H^{''})\mathcal{R}(H^{''}, \mathcal{E}_M + \epsilon)|\Psi^0_M\rangle\langle\Psi^0_M| = \lim_{\epsilon\to 0} \epsilon \sum_{M=1}^{N^L} |\Psi^0_M\rangle\langle\Psi^0_M| \,. \tag{8.3.43}$$

We first observe that

$$\lim_{\epsilon\to 0} \left(\mathcal{E}_M + \epsilon - H^{''}\right) = \left(\Lambda_M - H^{''}\right) < \infty \,. \tag{8.3.44}$$

Moreover, using the power expansion (8.3.20) we have that, for each $M \in \{1, \dots, N^L\}$,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon \left(\mathcal{E}_M + \epsilon - H^{''}\right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| = \lim_{\epsilon\to 0} \sum_{n=0}^{\infty} \left((\mathcal{E}_M + \epsilon)^{-1} H^{''}\right)^n (\mathcal{E}_M + \epsilon)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \\
&= \lim_{\epsilon\to 0} \epsilon \sum_{n=0}^{\infty} \left((\Lambda_M + \epsilon)^{-1} (\mathcal{E}_M + H_B)\right)^n (\mathcal{E}_M + \epsilon)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \\
&= \lim_{\epsilon\to 0} \epsilon \left(\mathcal{E}_M + \epsilon - \mathcal{E}_M - H_B\right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \\
&= \lim_{\epsilon\to 0} \epsilon \left(\epsilon - H_B\right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \,.
\end{aligned} \tag{8.3.45}$$

Using a second time (8.3.20) we have that

$$\begin{aligned}
&\lim_{\epsilon\to 0} \epsilon \left(\mathcal{E}_M + \epsilon - H^{''}\right)^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| = \lim_{\epsilon\to 0} \epsilon \sum_{n=0}^{\infty} \left(\epsilon^{-1} H_B\right)^n \epsilon^{-1} |\Psi^0_M\rangle\langle\Psi^0_M| \\
&= \lim_{\epsilon\to 0} \sum_{n=0}^{\infty} \frac{1}{\epsilon^n} (H_B)^n |\Psi^0_M\rangle\langle\Psi^0_M| = 0
\end{aligned} \tag{8.3.46}$$

where in the last equality we have exploited the fact that, since the state space $\Omega^{'}$ of integrable chain is finite, there exists $\overline{n} \in \mathbb{N}_0$ such that $\forall n > \overline{n}$ we have $(H_B)^n |\Psi^0_M\rangle = 0$.
Since both factor are finite, the limit of the product is the product of the limit and we obtain

$$\sum_{M=1}^{N^L} (\mathcal{E}_M - H^{''}) \lim_{\epsilon\to 0} \epsilon \mathcal{R}(H^{''}, \mathcal{E}_M + \epsilon)|\Psi^0_M\rangle\langle\Psi^0_M| = 0 \,. \tag{8.3.47}$$

Since the limit of the sum is the sum of the limit

$$\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{E}_M\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)|\Psi_M^0\rangle\langle\Psi_M^0| = H^{''}\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)|\Psi_M^0\rangle\langle\Psi_M^0|\,. \tag{8.3.48}$$

Recalling that $|\Psi_M^0\rangle$ are the eigenvectors with eigenvalue $\mathcal{E}_M$ of H_0 we have

$$\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)H_0|\Psi_M^0\rangle\langle\Psi_M^0| = H^{''}\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)|\Psi_M^0\rangle\langle\Psi_M^0|\,. \tag{8.3.49}$$

Since $H_0 = H_0^T$ we write

$$\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)|\Psi_M^0\rangle\langle\Psi_M^0|H_0 = H^{''}\lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\mathcal{R}(H^{''},\mathcal{E}_M+\epsilon)|\Psi_M^0\rangle\langle\Psi_M^0|\,. \tag{8.3.50}$$

By calling

$$W = \lim_{\epsilon\to 0}\epsilon\sum_{M=1}^{N^L}\left(\mathcal{E}_M+\epsilon-H^{''}\right)^{-1}|\Psi_M^0\rangle\langle\Psi_M^0| \tag{8.3.51}$$

we finally obtain

$$WH_0 = H^{''}W\,. \tag{8.3.52}$$

Part III

Reaction diffusion multi-species models

Chapter 9

Reaction diffusion models for multi-species uphill diffusion

9.1 Classical diffusion and uphill diffusion

Diffusion is a phenomenon consisting in the net movement of a quantity (particles, atoms, energy, molecules, people, animals) from region with different densities, and it is used in many scientific areas. We report here some examples. In physics diffusion is used to described the flux of thermal energy (Fourier law), of particles, of mass or electric charges in out-of-equilibrium systems in contact with thermodynamic baths. In chemistry, diffusion models the migration of molecules of chemical species due to the difference of concentration. Also in life science, like biology, diffusion is used to describe some phenomena, like active live matter or cellular automata. Finally, especially in the last decades, diffusion has been used to model situations arising from economics, sociology and data science (see for instance the modelling of traffic flows [89]).
Statistical mechanics allows to derive the macroscopic laws that rule diffusion from the interaction of microscopic quantities, like interacting particles. This have been made rigorous by the introduction of the hydrodynamic limit [29, 30, 31].
From a phenomenological point of view the diffusion equation (heat equation) arises from the combination of the continuity equation and the so called *Fick's law*. More precisely, for diffusive models with a single species, transport of mass on a finite volume (here assumed to be the unit d-dimensional cube) is described by the continuity equation

$$\frac{\partial}{\partial t}\rho = -\nabla \cdot J \tag{9.1.1}$$

and the Fick's law

$$J = -\sigma \nabla \rho. \tag{9.1.2}$$

Here $\rho : [0,1]^d \times \mathbb{R}_+ \to [0,1]$ is the density of mass, $J : [0,1]^d \times \mathbb{R}_+ \to \mathbb{R}$ is the current, and $\sigma > 0$ is the diffusivity coefficient (that we assume constant throughout this thesis). Equations (9.1.1) and (9.1.2) can be obtained as the hydrodynamical limit of diffusive interacting particle systems of "gradient type" [29], such as the simple symmetric exclusion process or the Kipnis-Marchioro-Presutti model [50]. Fick's law (9.1.2) tells us that the total flow is opposite to the density gradient.

For multi-component systems with N species, considering the vectors $\boldsymbol{\rho} = (\rho^{(1)}, \ldots, \rho^{(N)})$ and $\boldsymbol{J} = (J^{(1)}, \ldots, J^{(N)})$, where $\rho^{(i)}(x,t)$ and $J^{(i)}(x,t)$ denote the density and the current of the i$^{\text{th}}$ species, the generalization of (9.1.1) and (9.1.2) is

$$\frac{\partial}{\partial t}\boldsymbol{\rho} = -\nabla \cdot \boldsymbol{J} \tag{9.1.3}$$

and

$$\boldsymbol{J} = -\boldsymbol{\Sigma} \cdot \nabla\boldsymbol{\rho}. \tag{9.1.4}$$

where $\boldsymbol{\Sigma}$ is now the $N \times N$ matrix of diffusion (also called *diffusivity matrix*) and 'cross-diffusion' coefficients. Equation (9.1.4) describes the so called *diffusive currents*. Replacing the diffusive currents in (9.1.3) one obtains a multi-component linear diffusion equation. However, as soon as the diffusivity matrix is non-diagonal, the physical meaning of the density vector $\boldsymbol{\rho}$ is lost since one of its components loses the non-negativity. Indeed, referring to Theorem 2.1 of [41], we have that: *considering the system of partial differential equations*

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = \boldsymbol{\Sigma}\Delta\boldsymbol{\rho} \tag{9.1.5}$$

where $\boldsymbol{\Sigma}$ *is a constant and positive definite matrix. Then classical solution of the initial boundary value problem for* (9.1.5) *in* $[0,1] \times [0,T]$ *under either Dirichlet or Neumann boundary conditions preserve the non-negativity if an only if the matrix* $\boldsymbol{\Sigma}$ *is diagonal.*
This implies that purely diffusive multi-component systems must be non-linear. This statement has been also pointed out from a chemical perspective in [42]. This suggests that, if we want to keep the linearity and the non-negativity of the densities of a multi-component system a possibility is to add a reaction term. This motivates the choice of the model that we will study in the following.
Before introducing this model, we make a brief and non-exhaustive overview of the uphill diffusion. As we have already pointed out, the flux (current) of a quantity ρ is defined by the Fick's law as

$$J = -\sigma\nabla\rho \tag{9.1.6}$$

It is important to observe the presence of the minus sign in front of the gradient. This tells that the flux goes from region with higher density to regions with lower density. Uphill diffusion occurs when J points in the opposite direction, i.e. along the gradient.
The first experimental observation of uphill diffusion dates back to the to the 30's in the experiment of Hartley [90]. Another crucial observation of this phenomenon has been done in the Darken experiment, where the diffusion of carbon atoms was studied [91]. More recently, in the chemical literature many interesting works has been carried out [38, 39, 40].
In the world of interacting particle systems the research is wide. For instance uphill diffusion has been studied in the context of phase transitions concerning the *spinodal decomposition*, a situation in which a phase separation spontaneously occurs inside the unstable region of the phase diagram, without any nucleation mechanism, see for instance [92]. The uphill phenomenon has been reported in many single species system in the presence of a phase transition [77, 93, 94, 95, 96] for 1D particle systems with Kac potentials and [97] for 2D lattice gases related to the Ising model). Recently, uphill diffusion has been observed also in non-linear models (see for instance [98, 99]).
In this thesis we aim to concentrate our efforts in the study of uphill diffusion for multi-components systems. Recently, a double diffusivity model has been introduced in [35]. This microscopic model consists in a two layered particle system. Each layer has its own diffusivity constant and particles can change layer with a certain rate. When rescaled to the continuum,

by means of hydrodynamic limit, the system leads to a system of two reaction diffusion PDE's with different diffusivity constant. It has been shown that uphill diffusion for the total density (the sum of the two densities) is possible and associated with a boundary layer. Moreover, a precise region of uphill diffusion is characterized.
Other interesting works where uphill diffusion arises as a result of the competition between the gradients of each species are [100, 101].

9.1.1 Steady state uphill diffusion in a multi-component systems

In this work we restrict ourselves to the case of two species diffusing on the unit interval. In the case of a larger number of species one may expect that more complex regions of uphill diffusion can arise. Let us call $\rho^{(\alpha)}(x,t) : [0,1] \times [0,\infty) \to [0,1]$ the density of the species $\alpha \in \{0,1,2\}$. We impose the constraint $\rho^{(0)} + \rho^{(1)} + \rho^{(2)} = 1$, which will represent later the hard-core interaction of the associated interacting particle system. It is then enough to study the evolution of $\rho^{(1)}$ and $\rho^{(2)}$, which will be assumed to be smooth functions. We consider a Cauchy problem with Dirichlet boundary conditions, where each density is endowed with an initial datum $\rho^{(\alpha)}(x,0) = \rho_0^{(\alpha)}(x)$ and boundary conditions $\rho^{(\alpha)}(0,t) = \rho_L^{(\alpha)}$ and $\rho^{(\alpha)}(1,t) = \rho_R^{(\alpha)}$ for $\alpha = 1,2$. We are interested in the stationary properties. We consider

$$\begin{aligned} \partial_t \rho^{(1)} &= \sigma_{11}\partial_x^2\rho^{(1)} + \sigma_{12}\partial_x^2\rho^{(2)} + \Upsilon\left(\rho^{(2)} - \rho^{(1)}\right) \\ \partial_t \rho^{(2)} &= \sigma_{21}\partial_x^2\rho^{(1)} + \sigma_{22}\partial_x^2\rho^{(2)} + \Upsilon\left(\rho^{(1)} - \rho^{(2)}\right) \end{aligned} \tag{9.1.7}$$

where the matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \tag{9.1.8}$$

is assumed to have positive determinant and the sum of all its elements to be positive (the reason for this assumption will become clear in what follows, see below equation (9.1.13)). The stationary diffusive currents are given by

$$\begin{aligned} J^{(1)}(x) &= -\sigma_{11}\partial_x\rho^{(1)}(x) - \sigma_{12}\partial_x\rho^{(2)}(x) \\ J^{(2)}(x) &= -\sigma_{21}\partial_x\rho^{(1)}(x) - \sigma_{22}\partial_x\rho^{(2)}(x) \end{aligned} \tag{9.1.9}$$

We distinguish two cases:

- *global uphill*: this happens when the boundary values of the total boundary density $\rho_L = \rho_L^{(1)} + \rho_L^{(2)}$ and $\rho_R = \rho_R^{(1)} + \rho_R^{(2)}$ and the total current $J(x) = J^{(1)}(x) + J^{(2)}(x)$ are such that either $\rho_L < \rho_R$ and $J(x) > 0\ \forall x \in [0,1]$, or $\rho_L > \rho_R$ and $J(x) < 0\ \forall x \in [0,1]$.
- *partial uphill for the a^{th} species*: for boundary values $\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}} \geqslant 0$, the system has stationary partial uphill diffusion for the species $a \in \{1,2\}$ if $\rho^{(a)}_{\text{left}} < \rho^{(a)}_{\text{right}}$ and $J^{(a)}(x) > 0\ \forall x \in [0,1]$, or if $\rho^{(a)}_{\text{left}} > \rho^{(a)}_{\text{right}}$ and $J^{(a)}(x) < 0\ \forall x \in [0,1]$.

For a complete review about uphill diffusion for single species and multi-species systems we refer to [102].
Clearly, in the case where each density simply obeys a one dimensional heat equation

$$\begin{aligned} \partial_t \rho^{(1)}(x,t) &= \sigma_{11}\partial_x^2\rho^{(1)}(x,t) \\ \partial_t \rho^{(2)}(x,t) &= \sigma_{22}\partial_x^2\rho^{(2)}(x,t) \end{aligned} \tag{9.1.10}$$

no uphill diffusion (neither global nor partial) is possible. When $\sigma_{11} = \sigma_{22}$, equations (9.1.10) can be obtained as the hydrodynamic limit of the $N+1$, with $N=2$, species stirring process (see Theorem 14). Global uphill diffusion can be obtained by keeping the matrix Σ diagonal and adding a reaction term, i.e.

$$\begin{aligned}\partial_t \rho^{(1)} &= \sigma_{11}\partial_x^2 \rho^{(1)} + \Upsilon\left(\rho^{(2)} - \rho^{(1)}\right) \\ \partial_t \rho^{(2)} &= \sigma_{22}\partial_x^2 \rho^{(2)} + \Upsilon\left(\rho^{(1)} - \rho^{(2)}\right)\end{aligned} \tag{9.1.11}$$

In [35] the above equations have been obtained as the hydrodynamical limit of a switching interacting particle system, and the region with global uphill has been explicitly characterized.

To obtain partial uphill diffusion one needs to consider the more general case (9.1.7) with a *non-diagonal* matrix Σ. We give the stationary solution of (9.1.7) with Dirichlet boundary conditions, i.e. the solution of

$$\begin{aligned}&\sigma_{11}\frac{d^2}{dx^2}\rho^{(1)}(x) + \sigma_{12}\frac{d^2}{dx^2}\rho^{(2)}(x) + \Upsilon(\rho^{(2)}(x) - \rho^{(1)}(x)) = 0 \\ &\sigma_{21}\frac{d^2}{dx^2}\rho^{(1)}(x) + \sigma_{22}\frac{d^2}{dx^2}\rho^{(2)}(x) + \Upsilon(\rho^{(1)}(x) - \rho^{(2)}(x)) = 0 \\ &\rho^{(1)}(0) = \rho^{(1)}_{\text{left}} \quad \rho^{(2)}(0) = \rho^{(2)}_{\text{left}} \quad \rho^{(1)}(1) = \rho^{(1)}_{\text{right}} \quad \rho^{(2)}(1) = \rho^{(2)}_{\text{right}}\end{aligned} \tag{9.1.12}$$

Once we write explicitly $\rho^{(1)}(x), \rho^{(2)}(x)$, the existence of partial uphill can be obtained. Introducing the constants $\boldsymbol{A} = \Upsilon\frac{\sigma_{12}+\sigma_{22}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} > 0$ and $\boldsymbol{B} = -\Upsilon\frac{\sigma_{11}+\sigma_{21}}{\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21}} < 0$, the steady state density profiles reads

$$\begin{aligned}\rho^{(1)}(x) &= \boldsymbol{E} + \boldsymbol{F}x + \boldsymbol{C}\left(1 + \frac{\boldsymbol{A}-\boldsymbol{B}}{\boldsymbol{B}}\right)e^{-\sqrt{\boldsymbol{A}-\boldsymbol{B}}x} + \boldsymbol{D}\left(1 + \frac{\boldsymbol{A}-\boldsymbol{B}}{\boldsymbol{B}}\right)e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}x} \\ \rho^{(2)}(x) &= \boldsymbol{E} + \boldsymbol{F}x + \boldsymbol{C}e^{-\sqrt{\boldsymbol{A}-\boldsymbol{B}}x} + \boldsymbol{D}e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}x}\end{aligned} \tag{9.1.13}$$

where the constants $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}, \boldsymbol{F}$ are determined by the boundary conditions as follows:

$$\begin{aligned}\boldsymbol{E} &= \frac{\boldsymbol{A}\,\rho^{(2)}_{\text{left}} - \boldsymbol{B}\,\rho^{(1)}_{\text{left}}}{\boldsymbol{A}-\boldsymbol{B}} \\ \boldsymbol{C} &= \frac{\boldsymbol{B}\left(\rho^{(1)}_{\text{left}}\,e^{2\sqrt{\boldsymbol{A}-\boldsymbol{B}}} - \rho^{(2)}_{\text{left}}\,e^{2\sqrt{\boldsymbol{A}-\boldsymbol{B}}} - \rho^{(1)}_{\text{right}}\,e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}} + \rho^{(2)}_{\text{right}}\,e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}}\right)}{(\boldsymbol{A}-\boldsymbol{B})\left(e^{2\sqrt{\boldsymbol{A}-\boldsymbol{B}}} - 1\right)} \\ \boldsymbol{F} &= -\frac{\boldsymbol{A}\,\rho^{(2)}_{\text{left}} - \boldsymbol{A}\,\rho^{(2)}_{\text{right}} - \boldsymbol{B}\,\rho^{(1)}_{\text{left}} + \boldsymbol{B}\,\rho^{(1)}_{\text{right}}}{\boldsymbol{A}-\boldsymbol{B}} \\ \boldsymbol{D} &= \frac{\boldsymbol{B}\left(\rho^{(1)}_{\text{left}} - \rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{right}}\,e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}} + \rho^{(2)}_{\text{right}}\,e^{\sqrt{\boldsymbol{A}-\boldsymbol{B}}}\right)}{\boldsymbol{A}-\boldsymbol{B}-\boldsymbol{A}\,e^{2\sqrt{\boldsymbol{A}-\boldsymbol{B}}} + \boldsymbol{B}\,e^{2\sqrt{\boldsymbol{A}-\boldsymbol{B}}}}\end{aligned} \tag{9.1.14}$$

From equation (9.1.13) we see that the conditions $\sigma_{11}+\sigma_{12}+\sigma_{21}+\sigma_{22} > 0$ and $\sigma_{11}\sigma_{22}-\sigma_{12}\sigma_{21} > 0$ guarantees $\boldsymbol{A}-\boldsymbol{B} > 0$, i.e., non-oscillating solutions.

Non diagonal diffusivity matrix equation

We shall show that in this set up partial uphill diffusion is possible. To this aim, because of the great number of parameters we specialize (9.1.13) to a particular choice, namely

$$\sigma_{11} = \sigma_{22} = \Upsilon = 1 \qquad \sigma_{21} = \sigma_{12} = \frac{1}{2}\,. \tag{9.1.15}$$

The stationary profiles become, $\forall a = 1, 2$,

$$\begin{aligned} \rho^{(a)}(x) =& \frac{\rho^{(1)}_{\text{left}}}{2} + \frac{\rho^{(2)}_{\text{left}}}{2} - \frac{x\left(\rho^{(1)}_{\text{left}} + \rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}}\right)}{2} \\ &+(-1)^a \frac{\mathrm{e}^{2-2\,x}\left(\rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}} - \rho^{(1)}_{\text{left}}\,\mathrm{e}^2 + \rho^{(2)}_{\text{left}}\,\mathrm{e}^2\right)}{2\,(\mathrm{e}^4 - 1)} \\ &+(-1)^a \frac{\mathrm{e}^{2\,x}\left(\rho^{(1)}_{\text{left}} - \rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{right}}\,\mathrm{e}^2 + \rho^{(2)}_{\text{right}}\,\mathrm{e}^2\right)}{2\,(\mathrm{e}^4 - 1)} \end{aligned} \tag{9.1.16}$$

and the diffusive currents read, $\forall a = 1, 2$

$$\begin{aligned} J^{(a)}(x) =& \frac{3\,\rho^{(1)}_{\text{left}}}{4} + \frac{3\,\rho^{(2)}_{\text{left}}}{4} - \frac{3\,\rho^{(1)}_{\text{right}}}{4} - \frac{3\,\rho^{(2)}_{\text{right}}}{4} \\ &+ (-1)^a \frac{\mathrm{e}^{2-2\,x}\left(\rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}} - \rho^{(1)}_{\text{left}}\,\mathrm{e}^2 + \rho^{(2)}_{\text{left}}\,\mathrm{e}^2\right)}{2\,(\mathrm{e}^4 - 1)} \\ &-(-1)^a \frac{\mathrm{e}^{2\,x}\left(\rho^{(1)}_{\text{left}} - \rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{right}}\,\mathrm{e}^2 + \rho^{(2)}_{\text{right}}\,\mathrm{e}^2\right)}{2\,(\mathrm{e}^4 - 1)} \end{aligned} \tag{9.1.17}$$

The problem of having partial uphill for, say, the species 1 is then the following: by assuming that $\rho^{(1)}_{\text{left}} < \rho^{(1)}_{\text{right}}$

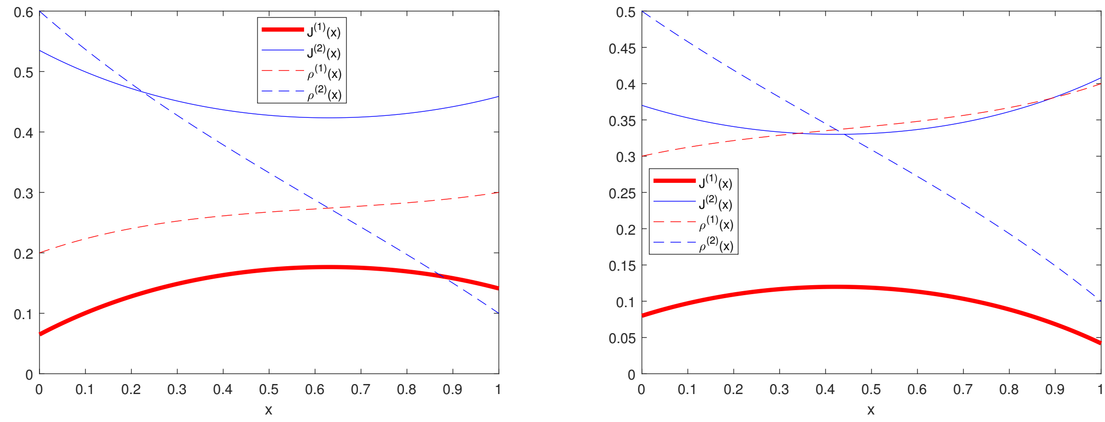
$$\text{find} \quad (\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) \quad \text{such that} \quad \min_{x\in[0,1]} J^{(1)}(x) > 0. \tag{9.1.18}$$

There are choices of boundary densities that allow for partial uphill diffusion of the species 1. Here we plot in Figure 9.1a, 9.1b, 9.2a and 9.2b the stationary densities and currents for a specific choices of the boundary values and of the diffusivity matrix and reaction term. From the picture one can clearly see partial uphill diffusion (in the absence of global uphill).

Diagonal diffusivity matrix equations

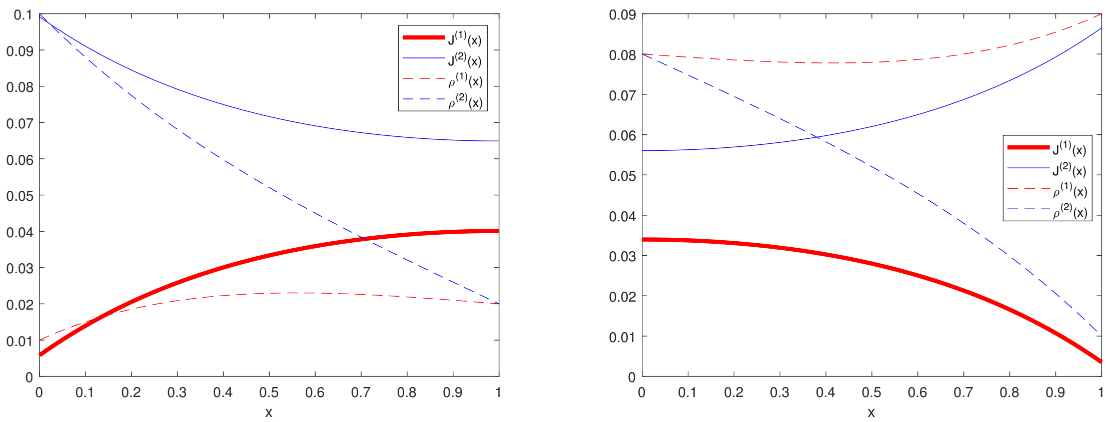
We specialize the stationary solution (9.1.13) to the case where $\sigma_{12} = \sigma_{21} = 0$ and $\Upsilon > 0$. Motivated by the hydrodynamic result of the paper [35], we consider the case $\sigma_{11} = \sigma_{22}$. Introducing the constant $k^2 = \frac{\Upsilon}{\sigma_{11}}$, the stationary profiles takes the form $\forall a = 1, 2$:

$$\begin{aligned} \rho^{(a)}(x) =& \frac{1}{2}\left(\rho^{(1)}_{\text{left}} + \rho^{(2)}_{\text{left}} + x(-\rho^{(1)}_{\text{left}} - \rho^{(2)}_{\text{left}} + \rho^{(1)}_{\text{right}} + \rho^{(2)}_{\text{right}})\right. \\ &\left. - (-1)^a \operatorname{csch}\left(\sqrt{2}k\right)\left((\rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{left}})\sinh\left(\sqrt{2}k(x-1)\right) + (\rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}})\sinh\left(\sqrt{2}kx\right)\right)\right) \end{aligned} \tag{9.1.19}$$



(a) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (2, 6, 3, 1) \times 10^{-1}$. (b) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (3, 5, 4, 1) \times 10^{-1}$.

Figure 9.1: Density profile (dashed lines) and currents (continuous line). The red color is for species 1 and the blue color for species 2. The boundary values are in figure 9.1a $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (2, 6, 3, 1) \times 10^{-1}$ and are $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (3, 5, 4, 1) \times 10^{-1}$ in figure 9.1b. In both cases, the diffusivity matrix and the reaction term are $\sigma_{11} = \sigma_{22} = \Upsilon = 1$ and $\sigma_{12} = \sigma_{21} = 1/2$.



(a) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (1, 10, 2, 2) \times 10^{-2}$. (b) $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (8, 8, 9, 1) \times 10^{-2}$.

Figure 9.2: Density profile (dashed lines) and currents (continuous line). The red color is for species 1 and the blue color for species 2. The boundary values are in figure 9.2a $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (1, 10, 2, 2) \times 10^{-2}$ and are $(\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) = (8, 8, 9, 1) \times 10^{-2}$ in figure 9.2b. In both cases, the diffusivity matrix and the reaction term are $\sigma_{11} = \sigma_{22} = \Upsilon = 1$ and $\sigma_{12} = \sigma_{21} = 1/2$.

The diffusive currents then read:

$$\begin{aligned} J^{(a)}(x) =& \frac{1}{2}\Big(\rho^{(1)}_{\text{left}} + \rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}} + \\ &+ (-1)^a \sqrt{2}k \operatorname{csch}\left(\sqrt{2}k\right)\left((\rho^{(2)}_{\text{left}} - \rho^{(1)}_{\text{left}})\cosh\left(\sqrt{2}k(x-1)\right) + (\rho^{(1)}_{\text{right}} - \rho^{(2)}_{\text{right}})\cosh\left(\sqrt{2}kx\right)\right)\Big) \end{aligned} \tag{9.1.20}$$

The problem of having local uphill for, say, the species 1 is then the following: by assuming that $\rho^{(1)}_{\text{left}} < \rho^{(1)}_{\text{right}}$

$$\text{find} \quad (\rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}) \quad \text{such that} \quad \min_{x\in[0,1]} J^{(1)}(x) > 0. \tag{9.1.21}$$

One can check that $\forall k > 0$ and $\forall \rho^{(1)}_{\text{left}}, \rho^{(2)}_{\text{left}}, \rho^{(1)}_{\text{right}}, \rho^{(2)}_{\text{right}}$ such that $\rho^{(1)}_{\text{left}} < \rho^{(1)}_{\text{right}}$, the minimum of $J^{(1)}(x)$ is always negative.

Chapter 10

A reaction-diffusion interacting particle system with uphill

10.1 Motivations

In Chapter 9, a system of two linear reaction diffusion PDE's whose non-equilibrium steady states shows partial uphill diffusion has been proposed. This chapter is motivated by the research of an interacting particle systems whose average occupation of particles evolves as the discretized version of the system of PDE's (9.1.7) on a finite size lattice. As a natural follow-up question, we then ask if in the hydrodynamic limit the empirical density field converges to the PDE's (9.1.7).

In the first part of this chapter, inspired by the pioneering work [43] (generalized to a multi-species process in [44] and [22]), we introduce boundary driven model on a graph with N species of particles and a hole (denoted here by 0) with "general" dynamics, that includes diffusion, exchange of particles, mutation and creation/annihilation transition mechanisms. As a particular case, also the multi-species stirring process (Section 10.2) can be recovered. First, we specialize this model to a boundary driven chain of length L and to the case where 2 species of particles (and a hole) are present. Then, we impose conditions on the transition rates in order to obtain closed evolution equation for the average particles occupation given by the discrete counter part of the PDE's model (9.1.7) (Section 10.3). We discover that there exists a two parameters family of models with the required features (Section 10.4). Finally, we prove the hydrodynamic limit for the simplest and symmetric representative of this family of models, discovering that the empirical density field converges to a system with diagonal diffusion matrix (Section 10.5), despite the form of the evolution equation of the average occupation variables.

10.2 Hard-core multi-species particles on a graph G

Notation: In what follows, we use Greek letters $(\alpha, \beta, \gamma, \delta, \ldots)$ to denote the species of the particles or the holes. By Latin letters $(x, y, z, \ldots)$ to denote the sites of the graph.

In this section we define our microscopic model on a generic graph $G = (V, \mathcal{E})$. Here, the set $V = \{1, 2, \ldots, L\}$ is a collection of L vertices. The set of edges $\mathcal{E}$ is such that the graph is connected, oriented (directed) and without self-edges. On this graph G we consider a system of interacting particles, each of which has its own type/species. We assume there are N species. Furthermore, on each vertex of the graph there is at most one particle (hard-core exclusion rule). Thus, the occupation variable at each vertex takes values in $\{0, 1, 2, \ldots, N\}$, with type 0 denoting the empty site.

The dynamical rule is due to a one-body interaction and a two-body interaction:

- on each site $x \in V$ the occupation of type γ changes to type α at rate $a_x W_\gamma^\alpha(x)$;

- on each edge $(x, y) \in E$ the occupations of type (γ, δ) changes to type (α, β) at rate $\omega_{x,y}\Gamma_{\gamma\delta}^{\alpha\beta}$.

Here the non-negative numbers $\{\omega_{x,y}\}_{(x,y)\in E}$ and $\{a_x\}_{x\in V}$ are, respectively, edge weights (conductances) and site weights (local inhomogeneities) of the graph[1]. For a visual representation of the process with two species see Figure 10.1.

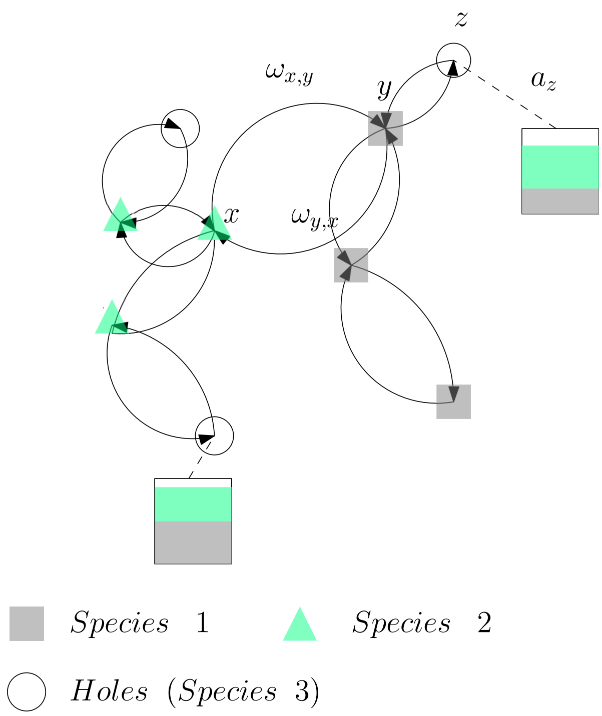


Figure 10.1: Hard-core two-species particles on a graph with 8 vertices and 2 reservoirs. Grey squares identify the species 1, green triangles the species 2, and white circles the empty state. The reservoirs are represented by rectangles, where the interior colours denote the density of species.

10.2.1 Process definition

On the graph $G = (V, E)$, we consider the Markov process $\{\eta(t); t \geqslant 0\}$ with state space $\Omega = \{0, 1, 2, \ldots, N\}^V$. A configuration of the process is denoted by $\eta = (\eta_x)_{x\in V}$, where each component can take the values $\eta_x \in \{0, 1, ..., N\}$ and where $\eta_x = \alpha$ means the presence of a particle or a hole denoted by the label α at the site x. We recall that $\eta_x = 0$ is interpreted as an empty site. The process is defined by the generator $\mathcal{L}$ working on functions $f : \Omega \to \mathbb{R}$ as

$$(\mathcal{L}f)(\eta) = (\mathcal{L}_{\text{edge}}f)(\eta) + (\mathcal{L}_{\text{site}}f)(\eta), \tag{10.2.1}$$

[1] To avoid confusion with the transition rates that we will introduce in the following, here we denote the local inhomogeneities by a_x

where

$$(\mathcal{L}_{\text{edge}}f)(\eta) = \sum_{(x,y)\in E} \omega_{x,y} \cdot (\mathcal{L}_{x,y}f)(\eta)$$

and

$$(\mathcal{L}_{\text{site}}f)(\eta) = \sum_{x\in V} a_x \cdot (\mathcal{L}_x f)(\eta)$$

We shall explain the two generators $\mathcal{L}_{\text{edge}}$ and $\mathcal{L}_{\text{site}}$ in the following subsections.

The edge generator

We introduce the $(N+1)^2 \times (N+1)^2$ matrix Γ whose elements are rates of transition for the particle jumps on each edge. More precisely, we denote by $\Gamma_{\gamma\delta}^{\alpha\beta}$ the rate to change the configuration η with $\eta_x = \gamma, \eta_y = \delta$ to the configuration η' with $\eta'_x = \alpha, \eta'_y = \beta$, while $\eta'_z = \eta_z$ for all $z \neq x, y$. Thus, the single-edge generator is given by

$$\begin{aligned} &\mathcal{L}_{x,y} f(\eta_1, \ldots, \gamma, \ldots, \delta, \ldots, \eta_L) \\ &= \sum_{\alpha,\beta=0}^{N} \Gamma_{\gamma\delta}^{\alpha\beta} \left[f(\eta_1, \ldots, \alpha, \ldots, \beta, \ldots, \eta_L) - f(\eta_1, \ldots, \gamma, \ldots, \delta, \ldots, \eta_L) \right] \end{aligned} \tag{10.2.2}$$

where

$$\begin{aligned} \Gamma_{\gamma\delta}^{\alpha\beta} &\geqslant 0 \qquad \text{if} \quad (\alpha,\beta) \neq (\gamma,\delta) \\ \sum_{(\gamma,\delta)\in\{0,1,2,\ldots,N\}^2 \,:\, (\gamma\delta)\neq(\alpha,\beta)} \Gamma_{\gamma\delta}^{\alpha\beta} &= -\Gamma_{\alpha\beta}^{\alpha\beta} \quad \forall (\alpha,\beta) \in \{0,1,2,\ldots,N\}^2 \,. \end{aligned}$$

The site generator

Having in mind that the site generator will describe a 'boundary' driving leading the system to a non-equilibrium steady state, we assume that on each site there is a process which injects and removes particles at a rate which is space-dependent. Thus, for each vertex $x \in V$, we introduce the $(N+1) \times (N+1)$ matrix $W(x)$ whose elements are rates of transitions on that vertex. More precisely, we denote by $W_\gamma^\alpha(x)$ the rate to change the configuration η with $\eta_x = \gamma$ into the configuration η' with $\eta'_x = \alpha$, while $\eta'_z = \eta_z$ for all $z \neq x$. The single-vertex generator is given by

$$\begin{aligned} &\mathcal{L}_x f(\eta_1, \ldots, \gamma, \ldots, \eta_L) \\ &= \sum_{\alpha=0}^{N} W_\gamma^\alpha(x) \left[f(\eta_1, \ldots, \alpha, \ldots., \eta_L) - f(\eta_1, \ldots, \gamma, \ldots, \eta_L) \right] \end{aligned} \tag{10.2.3}$$

where

$$\begin{aligned} W_\gamma^\alpha(x) &\geqslant 0 \quad \text{if} \quad \alpha \neq \gamma \\ \sum_{\gamma\in\{0,1,2,\ldots,N\}:\gamma\neq\alpha} W_\gamma^\alpha(x) &= -W_\alpha^\alpha(x) \quad \forall \alpha \in \{0,1,2,\ldots,N\} \,. \end{aligned}$$

10.2.2 Comparison to other processes

Here, we discuss the relation of the general dynamics described above to some multi-species processes considered in the past literature (we consider here the case of homogeneous conductances and inhomogeneities $\omega_{x,y} = a_x = 1$). We shall mostly limit the discussion to *symmetric* systems (for asymmetric models there is also a large literature, see for instance [103] and references therein). In most cases, previous analyses have been restricted to a regular lattice or a one-dimensional chain.

- *General multi-species models.* The edge dynamics of the reaction-diffusion particle system in Section 10.2.1 has been considered on a d-dimensional lattice in [43] for the case $N = 1$ species and in [44] for the case of an arbitrary number of species. In those papers, sufficient conditions on the rates $\Gamma^{\alpha\beta}_{\gamma\delta}$ to guarantee the existence of dual process have been identified.

- *Multi-species exclusion processes.* The edge dynamics of multi-species simple symmetric exclusion process (SSEP) on a d-dimensional lattice, with at most one-particle per site, has been considered in [104]. It corresponds to the model of Section 10.2.1 with $\Gamma^{\alpha 0}_{0\alpha} = \Gamma^{0\alpha}_{\alpha 0} \neq 0$ for all $\alpha = 0, 1, \ldots, N$, while all other off-diagonal elements of the matrix Γ vanish, as well as the elements of the matrices $W(x)$. For this model, the hierarchy of equations for the correlations does not close, and the hydrodynamic limit has been shown in [104] to be given by two coupled *non-linear* heat equations. An open boundary version of the model with simple symmetric exclusion dynamic in the bulk has been presented in [100]. It corresponds to the model of Section 2.1 with $\Gamma^{0b}_{b0} = \Gamma^{b0}_{0b} = D_b$ and with boundary rates $W^b_0(1) = \alpha_b, W^0_b = \gamma_b, W^b_0(N) = \beta_b, W^0_b(N) = \delta_b$ (here b labels the species). All the other off-diagonal elements Γ and $W(z)$ vanish.

- *Multi-species stirring process.* In the bulk of the multi-species stirring process with maximal occupancy $\nu = 1$ (see Chapter 4) every couple of particles or every couple of a hole and a particle is exchanged in position with the same rate, which can be taken equal to 1 without loss of generality. Thus, the bulk dynamics of the stirring process corresponds to the case $\Gamma^{\delta\gamma}_{\gamma\delta} = 1$ for all $\gamma, \delta = 0, 1, \ldots, N$, while all other off-diagonal elements of the matrix Γ vanish. The hydrodynamic limit of the stirring process on a lattice is given by N independent diffusions (see Chapter 5), i.e. the generalization of (9.1.10) to N types. The multi-species stirring process on a chain, with maximal occupancy $\nu = 1$ and with boundary driving has been studied in Chapter 7, with the choice $W^b_\gamma(1) = \alpha_b$ and $W^b_\gamma(N) = \beta_b$ with $b \in \{0, \ldots, N\}$.

- *Multi-layers switching process*: A different set-up for multi-species particle systems has been recently proposed in [35, 36, 37]. One considers N "piled" copies of the graph G, each with its own single-type dynamics. The possibility of changing type is described by a *switching rate* between layers. This set-up eliminates the constraint of one particle per site, in the sense that the projection of the dynamics on the columns of the piled graph allows the presence of several particle of different types on the same "base" site. In the case where each layer is a one-dimensional chain and two-layers are considered, the hydrodynamic limit has been shown to be given by the "weakly" coupled reaction diffusion equation (9.1.11). When boundary reservoirs are added, global uphill diffusion and boundary layers are possible [35].

10.3 Evolution equations for the average occupation

For the model introduced in Section 10.2.1, we define the average of the occupation variable of each species of particles $\zeta \in \{1, \ldots, N\}$ at time $t \geqslant 0$ and at the vertex $z \in V$

$$\mu_z^{(\zeta)}(t) = \mathbb{E}\left[\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}(\eta(t))\right]. \tag{10.3.1}$$

Similarly, we consider the time-dependent correlations (multiple occupancy variables) between species of particles $\zeta, \zeta' \in \{1, \ldots, N\}$ at points $z, z' \in V$

$$c_{z,z'}^{(\zeta,\zeta')}(t) = \mathbb{E}\left[\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}(\eta(t))\mathbb{1}_{\{\mathcal{I}_{z'}^{\zeta'}\}}(\eta(t))\right]. \tag{10.3.2}$$

Here $\mathcal{I}_z^\zeta = \{\eta \in \Omega \,:\, \eta_z = \zeta\}$ and $\mathbb{1}_\mathcal{I}$ denotes the indicator function of the set $\mathcal{I}$. The notation $\mathbb{E}\left[f(\eta(t))\right] = \int \nu_0(d\eta)\mathbb{E}_\eta\left[f(\eta(t))\right]$ denotes the expectation in the process $\{\eta(t)\}_{t\geqslant 0}$ started from the initial measure ν_0. The evolution equation of the density of the ζ-species can be obtained by acting with the generator. We have

$$\frac{d\mathbb{E}\left[\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}(\eta(t))\right]}{dt} = \mathbb{E}\left[\left(\mathcal{L}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta(t))\right]. \tag{10.3.3}$$

In the following section we evaluate the right hand side of this equation by considering first edge contributions and then site contributions.

10.3.1 Action of $\mathcal{L}_{x,y}$

If $z \notin \{x, y\}$ then obviously $\left(\mathcal{L}_{x,y}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = 0$. Otherwise, recalling that the graph G is directed and the notation of [44], we have the following: when we fix $z = x$ then

$$\left(\mathcal{L}_{z,y}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = A_1^\zeta + \sum_{\delta=1}^N F_{+1}^{\zeta\delta}\mathbb{1}_{\{\mathcal{I}_y^\delta\}}(\eta) + \sum_{\gamma=1}^N B_1^{\zeta\gamma}\mathbb{1}_{\{\mathcal{I}_z^\gamma\}}(\eta) + \sum_{\gamma,\delta=1}^N G_{+1}^{\zeta\gamma\delta}\mathbb{1}_{\{\mathcal{I}_y^\gamma\}}(\eta)\mathbb{1}_{\{\mathcal{I}_z^\delta\}}(\eta) \tag{10.3.4}$$

and when we fix $z = y$ then

$$\left(\mathcal{L}_{x,z}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = A_2^\zeta + \sum_{\gamma=1}^N F_{-1}^{\zeta\gamma}\mathbb{1}_{\{\mathcal{I}_x^\gamma\}}(\eta) + \sum_{\delta=1}^N C_2^{\zeta\delta}\mathbb{1}_{\{\mathcal{I}_z^\delta\}}(\eta) + \sum_{\gamma,\delta=1}^N G_{-1}^{\zeta\gamma\delta}\mathbb{1}_{\{\mathcal{I}_z^\gamma\}}(\eta)\mathbb{1}_{\{\mathcal{I}_x^\delta\}}(\eta) \tag{10.3.5}$$

where the constants are defined as follows:

1. *zero-order terms*:

$$A_1^\zeta = \sum_{\beta=0}^N \Gamma_{00}^{\zeta\beta} \qquad\qquad A_2^\zeta = \sum_{\beta=0}^N \Gamma_{00}^{\beta\zeta}$$

2. *first-order terms*:

$$B_1^{\zeta\gamma} = \begin{cases} \sum_{\beta=0}^N (\Gamma_{\gamma 0}^{\zeta\beta} - \Gamma_{00}^{\zeta\beta}) & \text{if } \zeta \neq \gamma \\ -\sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{\zeta 0}^{\zeta'\beta} + \Gamma_{00}^{\zeta\beta}\right) & \text{if } \zeta = \gamma \end{cases}$$

$$C_2^{\zeta\delta} = \begin{cases} \sum_{\beta=0}^N (\Gamma_{0\delta}^{\beta\zeta} - \Gamma_{00}^{\beta\zeta}) & \text{if } \zeta \neq \delta \\ -\sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{0\zeta}^{\beta\zeta'} + \Gamma_{00}^{\beta\zeta} \right) & \text{if } \zeta = \delta \end{cases}$$

$$F_{-1}^{\zeta\gamma} = B_2^{\zeta\gamma} = \sum_{\beta=0}^N (\Gamma_{\gamma 0}^{\beta\zeta} - \Gamma_{00}^{\beta\zeta})$$

$$F_{+1}^{\zeta\delta} = C_1^{\zeta\delta} = \sum_{\beta=0}^N (\Gamma_{0\delta}^{\zeta\beta} - \Gamma_{00}^{\zeta\beta})$$

3. *second-order terms:*

$$G_{+1}^{\zeta\gamma\delta} = D_1^{\zeta,\gamma,\delta}$$
$$= \begin{cases} \sum_{\beta=0}^N (\Gamma_{\gamma\delta}^{\zeta\beta} - \Gamma_{\gamma 0}^{\zeta\beta} - \Gamma_{0\delta}^{\zeta\beta} + \Gamma_{00}^{\zeta\beta}); & \text{if } \zeta \neq \gamma \\ -\sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{\zeta\delta}^{\zeta'\beta} + \Gamma_{0\delta}^{\zeta\beta} \right) + \sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{\zeta 0}^{\zeta'\beta} + \Gamma_{00}^{\zeta\beta} \right) & \text{if } \zeta = \gamma \end{cases}$$

$$G_{-1}^{\zeta\gamma\delta} = D_2^{\zeta,\gamma,\delta}$$
$$= \begin{cases} \sum_{\beta=0}^N (\Gamma_{\gamma\delta}^{\beta\zeta} - \Gamma_{\gamma 0}^{\beta\zeta} - \Gamma_{0\delta}^{\beta\zeta} + \Gamma_{00}^{\beta\zeta}) & \text{if } \zeta \neq \delta \\ -\sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{\gamma\zeta}^{\beta\zeta'} + \Gamma_{\gamma 0}^{\beta\zeta} \right) + \sum_{\beta=0}^N \left(\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N \Gamma_{0\zeta}^{\beta\zeta'} + \Gamma_{00}^{\beta\zeta} \right) & \text{if } \zeta = \delta \end{cases}$$

10.3.2 Action of $\mathcal{L}_x$

If $z \neq x$ then obviously $\left(\mathcal{L}_x \mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = 0$. Otherwise

$$\left(\mathcal{L}_z \mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = A^\zeta(z) + \sum_{\beta=1}^N F^{\zeta\beta}(z) \mathbb{1}_{\{\mathcal{I}_z^\beta\}}(\eta) \tag{10.3.6}$$

where now the constants are defined as:

1. *zero-order term*:

$$A^\zeta(z) = W_0^\zeta(z)$$

2. *first-order term*:

$$F^{\zeta\beta}(z) = \begin{cases} W_\beta^\zeta(z) - W_0^\zeta(z) & \text{if } \zeta \neq \beta \\ -\sum_{\zeta'=0\,:\,\zeta'\neq\zeta}^N W_\zeta^{\zeta'}(z) - W_0^\zeta(z) & \text{if } \zeta = \beta \end{cases}.$$

10.3.3 Action of $\mathcal{L}$

We now collect the results of the previous sections. We may write

$$\begin{aligned} \left(\mathcal{L}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) &= \sum_{x,y\,:\,(x,y)\in E} \omega_{x,y} \left(\mathcal{L}_{x,y}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) + \sum_x a_x \left(\mathcal{L}_x \mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) \\ &= \sum_{y\,:\,(z,y)\in E} \omega_{z,y} \left(\mathcal{L}_{z,y}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) + \sum_{x\,:\,(x,z)\in E} \omega_{x,z} \left(\mathcal{L}_{x,z}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) \end{aligned} \tag{10.3.7}$$

$$+ \quad a_z \left(\mathcal{L}_z \mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta).$$

Substituting (10.3.4), (10.3.5), (10.3.6) in the above expressions we obtain

$$\begin{aligned}
\left(\mathcal{L}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) \quad = \quad & \sum_{y\,:\,(z,y)\in E} \omega_{z,y} \left(A_1^\zeta + \sum_{\delta=1}^N F_{+1}^{\zeta\delta} \mathbb{1}_{\{\mathcal{I}_y^\delta\}}(\eta) + \sum_{\gamma=1}^N B_1^{\zeta\gamma} \mathbb{1}_{\{\mathcal{I}_z^\gamma\}}(\eta) \right. \\
& \left. + \sum_{\gamma,\delta=1}^N G_{+1}^{\zeta\gamma\delta} \mathbb{1}_{\{\mathcal{I}_y^\gamma\}}(\eta) \mathbb{1}_{\{\mathcal{I}_z^\delta\}}(\eta) \right) \\
+ \quad & \sum_{x\,:\,(x,z)\in E} \omega_{x,z} \left(A_2^\zeta + \sum_{\gamma=1}^N F_{-1}^{\zeta\gamma} \mathbb{1}_{\{\mathcal{I}_x^\gamma\}}(\eta) + \sum_{\delta=1}^N C_2^{\zeta\delta} \mathbb{1}_{\{\mathcal{I}_z^\delta\}}(\eta) \right. \\
& \left. + \sum_{\gamma,\delta=1}^N G_{-1}^{\zeta\gamma\delta} \mathbb{1}_{\{\mathcal{I}_z^\gamma\}}(\eta) \mathbb{1}_{\{\mathcal{I}_x^\delta\}}(\eta) \right) \\
+ \quad & a_z \left(A^\zeta(z) + \sum_{\beta=1}^N F^{\zeta\beta}(z) \mathbb{1}_{\{\mathcal{I}_z^\beta\}}(\eta) \right). \qquad (10.3.8)
\end{aligned}$$

10.3.4 Evolution equations

Using equation (10.3.8) for the right hand side of (10.3.3) we obtain the evolution equation for the average occupation. Recalling the notation in (10.3.1) and (10.3.2) (for the sake of space we do not write the explicit t-dependence) we arrive to

$$\begin{aligned}
\frac{d}{dt}\mu_z^{(\zeta)} \quad = \quad & \sum_{y\,:\,(z,y)\in E} \omega_{z,y} \left(A_1^\zeta + \sum_{\delta=1}^N F_{+1}^{\zeta\delta}\, \mu_y^{(\delta)} + \sum_{\gamma=1}^N B_1^{\zeta\gamma}\, \mu_z^{(\gamma)} + \sum_{\gamma,\delta=1}^N G_{+1}^{\zeta\gamma\delta}\, c_{y,z}^{(\gamma,\delta)} \right) \\
+ \quad & \sum_{x\,:\,(x,z)\in E} \omega_{x,z} \left(A_2^\zeta + \sum_{\gamma=1}^N F_{-1}^{\zeta\gamma}\, \mu_x^{(\gamma)} + \sum_{\delta=1}^N C_2^{\zeta\delta}\, \mu_z^{(\delta)} + \sum_{\gamma,\delta=1}^N G_{-1}^{\zeta\gamma\delta}\, c_{z,x}^{(\gamma,\delta)} \right) \\
+ \quad & a_z \left(A^\zeta(z) + \sum_{\beta=1}^N F^{\zeta\beta}(z)\, \mu_z^{(\beta)} \right). \qquad (10.3.9)
\end{aligned}$$

We notice that the equations for the time-dependent averages $\mu_z^{(\zeta)}(t)$ are not closed, as they involve the correlations $c_{z,z'}^{(\zeta,\zeta')}(t)$.

Remark 23 (The process on the lattice) *The generator* (10.2.1) *is an generalization of the lattice generator studied in [44] to a general graph with the addition of open boundaries. Indeed, take as a special graph the d-dimensional regular lattice $\mathbb{Z}^d$ and ignore the boundaries. Then, calling $e^{(k)}$ the unit vector in the k^{th} direction ($k = 1, \ldots, d$) and defining*

$$\begin{aligned}
E^\zeta &= A_1^\zeta + A_2^\zeta \\
F_0^{\zeta\beta} &= C_2^{\zeta\beta} + B_1^{\zeta\beta}
\end{aligned} \qquad (10.3.10)$$

equation (10.3.8) *becomes*

$$\left(\mathcal{L}\mathbb{1}_{\{\mathcal{I}_z^\zeta\}}\right)(\eta) = \sum_{k=1}^d \left\{ E^\zeta + \sum_{\beta=1}^N \sum_{j=-1}^{+1} F_j^{\zeta\beta} \mathbb{1}_{\{\mathcal{I}^\beta_{z+je^{(k)}}\}}(\eta) \right.$$

$$+\sum_{\beta,\beta'=1}^{N}\sum_{j=\pm1} G_j^{\zeta\beta\beta'}\mathbb{1}_{\{\mathcal{I}^{\beta}_{z+je^{(k)}}\}}(\eta)\mathbb{1}_{\{\mathcal{I}^{\beta'}_{z}\}}(\eta)\Bigg\} \tag{10.3.11}$$

which is equation (3.12) *in [44].*

10.4 Boundary-driven chains with linear reaction-diffusion

In this and the following sections we specialize to the case with only two species, labelled by 1 and 2 with the holes labelled by 0. Furthermore, we specialize to the one-dimensional geometry by considering a undirected linear chain.

More precisely, the graph has L vertices labelled by $\{1, 2, \ldots, L\}$ with a distinguish role of the sites $\{1, L\}$which model two reservoirs. The interaction is of nearest neighbor type, i.e.

$$\omega_{x,y} = \begin{cases} 1 & \text{if} \quad |x-y| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad\qquad a_x = \begin{cases} 1 & \text{if} \quad x \in \{1, L\} \\ 0 & \text{otherwise} \end{cases}$$

It is convenient to call the sites $\{2, \ldots, L-1\}$ as "bulk" and the two end sites $\{1, L\}$ as " boundary". The generator of the process thus reads as:

$$\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{z=1}^{L-1} \mathcal{L}_{z,z+1} + \mathcal{L}_{\text{right}} \tag{10.4.1}$$

We specialize the result of Eq. (10.3.9) to the boundary-driven chain. Introducing $\forall \zeta, \beta = 1, 2$:

$$\begin{aligned} F_0^{\zeta\beta} &= B_1^{\zeta\beta} + C_2^{\zeta\beta} & \qquad E^{\zeta} &= A_1^{\zeta} + A_2^{\zeta} \\ A_{\text{left}}^{\zeta} &= A^{\zeta}(1) & A_{\text{right}}^{\zeta} &= A^{\zeta}(N) \\ F_{\text{left}}^{\zeta\beta} &= F^{\zeta\beta}(1) & F_{\text{right}}^{\zeta\beta} &= F^{\zeta\beta}(N) \end{aligned}$$

the evolution equations for the densities of the two species at site $z \in \{1, 2, \ldots, L\}$ are given by:

$$\begin{aligned} \frac{d}{dt}\mu_1^{(\zeta)} &= A_{\text{left}}^{\zeta} + A_1^{\zeta} + \sum_{\beta=1}^{2}\left(\left(B_1^{\zeta\beta} + F_{\text{left}}^{\zeta\beta}\right)\mu_1^{(\beta)} + F_{+1}^{\zeta\beta}\mu_2^{(\beta)}\right) \\ &+ \sum_{\beta,\beta'=1}^{2} G_{+1}^{\zeta\beta\beta'} c_{1,2}^{(\beta,\beta')} \end{aligned} \tag{10.4.2}$$

$$\begin{aligned} \frac{d}{dt}\mu_z^{(\zeta)} &= E^{\zeta} + \sum_{\beta=1}^{2}\left(F_{-1}^{\zeta\beta}\mu_{z-1}^{(\beta)} + F_0^{\zeta\beta}\mu_z^{(\beta)} + F_{+1}^{\zeta\beta}\mu_{z+1}^{(\beta)}\right) \qquad \text{if } z \in \{2, \ldots, L-1\} \\ &+ \sum_{\beta,\beta'=1}^{2}\left(G_{-1}^{\zeta\beta\beta'} c_{z-1,z}^{(\beta,\beta')} + G_{+1}^{\zeta\beta\beta'} c_{z,z+1}^{(\beta,\beta')}\right\} \end{aligned} \tag{10.4.3}$$

$$\frac{d}{dt}\mu_L^{(\zeta)} = A^{\zeta}_{\text{right}} + A_2^{\zeta} + \sum_{\beta=1}^{2}\left(\left(C_2^{\zeta\beta} + F^{\zeta\beta}_{\text{right}}\right)\mu_L^{(\beta)} + F_{-1}^{\zeta\beta}\mu_{L-1}^{(\beta)}\right)$$
$$+ \sum_{\beta,\beta'=1}^{2} G_{-1}^{\zeta\beta\beta'} c_{L-1,L}^{(\beta,\beta')} \tag{10.4.4}$$

In the next section, we simplify the evolution equations for the average density by selecting a subclass of processes with closed equations and a linear structure.

10.4.1 Imposing the matching

One could go further and compute the hierarchy of equations for higher-order correlation function [44]. For general choices of the rate matrices Γ and W, the equations do not close. In the following, we shall focus on those choices of rates that satisfy the following two requirements:

1. *Closure of the correlation equations.* This amounts to requiring that the correlation terms in (10.4.2), (10.4.3), (10.4.4) vanish. It is shown in [44] that the vanishing of correlations actually implies closure of the multi-point correlation function at all orders.

2. *The average occupations follow the discretization of the reaction diffusion equation.* Considering the reaction diffusion system (9.1.7), we approximate the laplacians with the central difference operators. We call $\rho_z^{(\alpha)}$ the density of species or of a hole denoted by the index $\alpha \in \{0,1,2\}$ at vertex $z \in \{1,\ldots,L\}$ with the constraint $\rho_z^{(0)} + \rho_z^{(1)} + \rho_z^{(2)} = 1$. Furthermore we fix the densities at the left end (vertex 1) to the values of $\rho^{(1)}_{\text{left}}$, $\rho^{(2)}_{\text{left}}$ and similarly at the right end (vertex L) we impose $\rho^{(1)}_{\text{right}}$, $\rho^{(2)}_{\text{right}}$. Then the discretization of the two component reaction diffusion equations (9.1.7), reads as

$$\begin{aligned}\frac{d}{dt}\rho_1^{(1)} =& \sigma_{11}\left(\rho^{(1)}_{\text{left}} - 2\rho_1^{(1)} + \rho_2^{(1)}\right) + \sigma_{12}\left(\rho^{(2)}_{\text{left}} - 2\rho_1^{(2)} + \rho_2^{(2)}\right) + \Upsilon\left(\rho_1^{(2)} - \rho_1^{(1)}\right)\\ \frac{d}{dt}\rho_1^{(2)} =& \sigma_{21}\left(\rho^{(1)}_{\text{left}} - 2\rho_1^{(1)} + \rho_2^{(1)}\right) + \sigma_{22}\left(\rho^{(2)}_{\text{left}} - 2\rho_1^{(2)} + \rho_2^{(2)}\right) + \Upsilon\left(\rho_1^{(1)} - \rho_2^{(2)}\right)\end{aligned} \tag{10.4.5}$$

$$\begin{aligned}\frac{d}{dt}\rho_z^{(1)} &= \sigma_{11}\left(\rho^{(1)}_{z-1} - 2\rho_z^{(1)} + \rho_{z+1}^{(1)}\right) + \sigma_{12}\left(\rho^{(2)}_{z-1} - 2\rho_z^{(2)} + \rho_{z+1}^{(2)}\right) + \Upsilon\left(\rho_z^{(2)} - \rho_z^{(1)}\right)\\ \frac{d}{dt}\rho_z^{(2)} &= \sigma_{21}\left(\rho^{(1)}_{z-1} - 2\rho_z^{(1)} + \rho_{z+1}^{(1)}\right) + \sigma_{22}\left(\rho^{(2)}_{z-1} - 2\rho_z^{(2)} + \rho_{z+1}^{(2)}\right) + \Upsilon\left(\rho_z^{(1)} - \rho_z^{(2)}\right)\\ &\qquad \forall z = 2,\ldots,L-1\end{aligned} \tag{10.4.6}$$

$$\begin{aligned}\frac{d}{dt}\rho_L^{(1)} =& \sigma_{11}\left(\rho^{(1)}_{L-1} - 2\rho_L^{(1)} + \rho_{\text{right}}^{(1)}\right) + \sigma_{12}\left(\rho^{(2)}_{L-1} - 2\rho_L^{(2)} + \rho_{\text{right}}^{(2)}\right) + \Upsilon\left(\rho_L^{(2)} - \rho_L^{(1)}\right)\\ \frac{d}{dt}\rho_L^{(2)} =& \sigma_{21}\left(\rho^{(1)}_{L-1} - 2\rho_L^{(1)} + \rho_{\text{right}}^{(1)}\right) + \sigma_{22}\left(\rho^{(2)}_{L-1} - 2\rho_L^{(2)} + \rho_{\text{right}}^{(2)}\right) + \Upsilon\left(\rho_L^{(1)} - \rho_L^{(2)}\right)\end{aligned} \tag{10.4.7}$$

We impose that the evolution equations for the averaged occupations given in (10.4.2), (10.4.3), (10.4.4) do coincide with the discretized reaction-diffusion equations (10.4.5), (10.4.6), (10.4.7).

We observe that, this system of difference-differential equation rules the evolution of the average density of the particles of species ζ and therefore is a proxy for the PDE's defined in (9.1.7). Therefore, one can adapt the computations done in Section 9.1.1 to this discrete situation.
By imposing the closure condition 1. and the discrete linear reaction-diffusion condition 2. we get the set of equations described above.

Conditions from the bulk. We first consider equation (10.4.3) which we require to have the form of (10.4.6). We obtain the following conditions:

- *Closure conditions*: equation (10.4.6) has no second order terms, thus:

$$G_{+1}^{\alpha\beta\beta'} = 0 \qquad G_{-1}^{\alpha\beta\beta'} = 0 \quad \forall\alpha,\beta,\beta' = 1,2 \tag{10.4.8}$$

  The above requirement leads to 16 conditions on the transition rates $\Gamma_{\gamma\delta}^{\alpha\beta}$.

- *Laplacian conditions*: the one point correlation function should evolve as the coupled discrete Laplacian in (10.4.6) with linear reaction. This is accomplished by imposing:

$$\begin{aligned} &F_{-1}^{11} = F_{+1}^{11} = \sigma_{11} \quad F_{-1}^{12} = F_{+1}^{12} = \sigma_{12} \quad F_{-1}^{21} = F_{+1}^{21} = \sigma_{21} \quad F_{-1}^{22} = F_{+1}^{22} = \sigma_{22} \\ &F_0^{11} = -2\sigma_{11} - \Upsilon \quad F_0^{12} = -2\sigma_{12} + \Upsilon \quad F_0^{21} = -2\sigma_{21} + \Upsilon \quad F_0^{22} = -2\sigma_{22} - \Upsilon \end{aligned} \tag{10.4.9}$$

  The above requirement leads to 12 conditions on the transition rates $\Gamma_{\gamma\delta}^{\alpha\beta}$.

- *Zero-order terms*: equation (10.4.6) has no zero-order term, thus:

$$E^1 = 0 \qquad E^2 = 0 \tag{10.4.10}$$

  The above requirement leads to 2 conditions on the transition rates $\Gamma_{\gamma\delta}^{\alpha\beta}$.

Our task is to determine the 81 transition rates $\Gamma_{\gamma\delta}^{\alpha\beta}$ $\forall\alpha,\beta,\gamma,\delta = 0,1,2$ that define the bulk infinitesimal generator. By exploiting the stochasticity properties of the generator (sum of the elements on the rows must be zero), the problem reduces to finding 72 transition rates. By considering (10.4.8), (10.4.9), (10.4.10), only $16 + 12 + 2 = 30$ conditions are available. This means that the problem to solve is under-determined.

For the analysis that will follow, it is convenient to introduce an unknown vector $\mathbf{u} \in \mathbb{R}_+^{72}$ that contains the desired 72 transition rates, and an appropriate matrix $K \in \mathbb{R}^{30\times72}$ and vector $\mathbf{b} \in \mathbb{R}^{30}$. Then, it is possible to rewrite (10.4.8), (10.4.9), (10.4.10) as:

$$K\mathbf{u} = \mathbf{b}. \tag{10.4.11}$$

The matrix K is full rank, thus there exists a family of solutions with 42 free parameters. Furthermore we have to guarantee the non-negativity of the solution, as the transition rates are non-negative. For later use, recalling the definitions of F, G, E's, we observe that the conditions (10.4.8), (10.4.9), (10.4.10) actually only involve sums of three transition rates.

Conditions from the boundaries. We now want to find conditions to match (10.4.2) and (10.4.4) with (10.4.5) and (10.4.7), respectively. We consider the conditions on the left boundary; the right boundary is treated similarly. We get:

- *Closure conditions*: the vanishing of correlation in (10.4.2) is already guaranteed by (10.4.8).

- *Laplacian conditions*:

$$\begin{array}{llll} F_{\text{left}}^{11}+B_1^{11}=-2\sigma_{11}-\Upsilon & F_{\text{left}}^{12}+B_1^{12}=-2\sigma_{12}+\Upsilon & F_{+1}^{11}=\sigma_{11} & F_{+1}^{12}=\sigma_{12} \\ F_{\text{left}}^{22}+B_1^{22}=-2\sigma_{22}-\Upsilon & F_{\text{left}}^{21}+B_1^{21}=-2\sigma_{21}+\Upsilon & F_{+1}^{21}=\sigma_{21} & F_{+1}^{22}=\sigma_{22} \end{array}$$

  Since the equations that involve $F_{+1}^{\zeta,\delta}$ are already imposed in (10.4.9), inserting the definition of the $F_{\text{left}}^{\zeta,\delta}$, the above conditions reduce to

$$\begin{aligned} -W_0^1(1)-W_1^0(1)-W_1^2(1)+B_1^{11} &= -2\sigma_{11}-\Upsilon \\ B_1^{12}+W_2^1(1)-W_0^1(1) &= -2\sigma_{12}+\Upsilon \\ W_1^2(1)-W_0^2(1)+B_1^{21} &= -2\sigma_{21}+\Upsilon \\ -W_2^0(1)-W_0^2(1)-W_2^1(1)+B_1^{22} &= -2\sigma_{22}-\Upsilon \end{aligned} \tag{10.4.12}$$

- *Zero-order terms*:

$$A_{\text{left}}^1+A_1^1=\sigma_{11}\rho_{\text{left}}^{(1)}+\sigma_{12}\rho_{\text{left}}^{(2)} \qquad A_{\text{left}}^2+A_1^2=\sigma_{21}\rho_{\text{left}}^{(1)}+\sigma_{22}\rho_{\text{left}}^{(2)}$$

  As a consequence of (10.4.10), A_2^ζ are zero. Therefore, the above conditions reduce to

$$W_0^1(1)=\sigma_{11}\rho_{\text{left}}^{(1)}+\sigma_{12}\rho_{\text{left}}^{(2)} \qquad W_0^2(1)=\sigma_{21}\rho_{\text{left}}^{(1)}+\sigma_{22}\rho_{\text{left}}^{(2)} \tag{10.4.13}$$

All in all, combining (10.4.12) and (10.4.13) we see that the rates of the boundary generators are uniquely determined by the bulk rates. Indeed, for a choice of the bulk rates (which in turn appear in the $B_1^{\zeta,\delta}$), we have:

$$\begin{array}{ll} W_0^1(1)=\sigma_{11}\rho_{\text{left}}^{(1)}+\sigma_{12}\rho_{\text{left}}^{(2)} & W_0^2(1)=\sigma_{21}\rho_{\text{left}}^{(1)}+\sigma_{22}\rho_{\text{left}}^{(2)} \\ W_0^1(1)+W_1^0(1)+W_1^2(1)=2\sigma_{11}+\Upsilon+B_1^{11} & W_2^1(1)-W_0^1(1)=-2\sigma_{12}+\Upsilon-B_1^{12} \\ W_1^2(1)-W_0^2(1)=-2\sigma_{21}+\Upsilon-B_1^{21} & W_2^0(1)+W_0^2(1)+W_2^1(1)=2\sigma_{22}+\Upsilon+B_1^{22} \end{array} \tag{10.4.14}$$

On the right boundary, a similar argument yields:

$$\begin{array}{ll} W_0^1(N)=\sigma_{11}\rho_{\text{right}}^{(1)}+\sigma_{12}\rho_{\text{right}}^{(2)} & W_0^2(N)=\sigma_{21}\rho_{\text{right}}^{(1)}+\sigma_{22}\rho_{\text{right}}^{(2)} \\ W_0^1(N)+W_1^0(N)+W_1^2(N)=2\sigma_{11}+\Upsilon+C_2^{11} & W_2^1(N)-W_0^1(N)=-2\sigma_{12}+\Upsilon-C_2^{12} \\ W_1^2(N)-W_0^2(N)=-2\sigma_{21}+\Upsilon-C_2^{21} & W_2^0(N)+W_0^2(N)+W_2^1(N)=2\sigma_{22}+\Upsilon+C_2^{22} \end{array} \tag{10.4.15}$$

Let us notice that (10.4.14) and (10.4.15) are determined systems of algebraic equations in the unknowns $W_\cdot^\cdot(1), W_\cdot^\cdot(N)$.

10.4.2 Determination of the rates

Our first main result is contained in Theorem 19. It identifies a necessary and sufficient condition (in terms of two parameters $h, m \geqslant 0$) on the diffusivity matrix Σ and the reaction coefficient Υ such that the one-dimensional boundary driven chain with two-species has averaged densities satisfying the discrete linear reaction-diffusion equations (10.4.5), (10.4.6), (10.4.7). Furthermore, by setting $h = m$, it provides the example of a one-parameter family of *symmetric* models with such a property. To state the example it is convenient to introduce the *mutation map* $\alpha \mapsto \bar{\alpha}$ defined by:

$$\begin{aligned} 1 &\to 2 \\ 2 &\to 1 \\ 0 &\to 0\,. \end{aligned} \tag{10.4.16}$$

Theorem 19 *Let Σ be a 2×2 positive definite diffusion matrix and $\Upsilon > 0$ be a reaction coefficient. Let $\rho^{(1)}_{\text{left}}$ and $\rho^{(2)}_{\text{left}}$ (respectively, $\rho^{(1)}_{\text{right}}$ and $\rho^{(2)}_{\text{right}}$) be the densities of the species 1 and 2 at the left (respectively, right) boundary. Then, for any choice of $h, m \geqslant 0$ there exists boundary-driven interacting particle systems on the chain $\{1, \dots, L\}$ such that their evolution equations of the average occupation variable are (10.4.5), (10.4.6), (10.4.7) if and only if the diffusion matrix coefficients $\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}$ and the reaction coefficient Υ are non-negative and fulfill the conditions*

$$\sigma_{11} + \sigma_{21} = \sigma_{12} + \sigma_{22} \qquad \sigma_{12} \leqslant \frac{\Upsilon - m}{2} \qquad \sigma_{21} \leqslant \frac{\Upsilon - h}{2}\,. \tag{10.4.17}$$

Moreover, an explicit example of a symmetric generator (parameterized by $h = m \geqslant 0$) is given by

$$L = L_{left} + \sum_{z=1}^{L-1} L_{z,z+1} + L_{right} \tag{10.4.18}$$

with edge generator

$$\begin{aligned} L_{z,z+1} f(\eta) &= \sigma_{11}(f(\eta_1, \dots, \eta_{z+1}, \eta_z, \dots, \eta_N) - f(\eta)) \\ &+ \sigma_{12}(f(\eta_1, \dots, \bar{\eta}_{z+1}, \bar{\eta}_z, \dots, \eta_N) - f(\eta)) \\ &+ (\Upsilon - 2\sigma_{12} - m)(f(\eta_1, \dots, \bar{\eta}_z, \eta_{z+1}, \dots, \eta_N) - f(\eta)) \\ &+ m(f(\eta_1, \dots, \eta_z, \bar{\eta}_{z+1}, \dots, \eta_N) - f(\eta))\,. \end{aligned} \tag{10.4.19}$$

The site generator at the left boundary is given by

$$\begin{aligned} L_{left} f(\eta) &= (\sigma_{11}\rho^{(1)}_{\text{left}} + \sigma_{12}\rho^{(2)}_{\text{left}})\mathbb{1}_{\{\mathcal{I}^0_1\}}(\eta)\left[f(\eta_1 + \delta^1, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \\ &+ (\sigma_{12}\rho^{(1)}_{\text{left}} + \sigma_{11}\rho^{(2)}_{\text{left}})\mathbb{1}_{\{\mathcal{I}^0_1\}}(\eta)\left[f(\eta_1 + \delta^2, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \\ &+ (\sigma_{11} + \sigma_{12})\rho^{(0)}_{\text{left}}\mathbb{1}_{\{\mathcal{I}^1_1\}}(\eta)\left[f(\eta_1 - \delta^1, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \\ &+ (\sigma_{11} + \sigma_{12})\rho^{(0)}_{\text{left}}\mathbb{1}_{\{\mathcal{I}^2_1\}}(\eta)\left[f(\eta_1 - \delta^2, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \\ &+ (m + \sigma_{12}\rho^{(1)}_{\text{left}} + \sigma_{11}\rho^{(2)}_{\text{left}})\mathbb{1}_{\{\mathcal{I}^1_1\}}(\eta)\left[f(\eta_1 + \delta^2 - \delta^1, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \\ &+ (m + \sigma_{11}\rho^{(1)}_{\text{left}} + \sigma_{12}\rho^{(2)}_{\text{left}})\mathbb{1}_{\{\mathcal{I}^2_1\}}(\eta)\left[f(\eta_1 - \delta^2 + \delta^1, \dots, \eta_L) - f(\eta_1, \dots, \eta_L)\right] \end{aligned} \tag{10.4.20}$$

where $\rho^{(0)}_{\text{left}} := 1 - \rho^{(1)}_{\text{left}} - \rho^{(2)}_{\text{left}}$. Here $\pm\delta^\alpha$ denotes the addition/removal of species α. The site generator at the right boundary is defined similarly (now with parameters $\rho^{(1)}_R$ and $\rho^{(2)}_R$).

A two-parameter family of models

In Theorem 4.1 we have written explicitly a one-parameter family (with parameter m) of generators denoted by L. However, for fixed diffusivity matrix Σ and reaction coefficient Υ that satisfy condition (10.4.17) there exists a two-parameter ($h, m \geqslant 0$) family of generators with average density evolution equations given by (10.4.5),(10.4.6) and (10.4.7). In the following of this section we report the general form of these generators, depending on the two parameters $h, m \geqslant 0$. We observe that the rate matrix is symmetric only when $h = m$, $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = \sigma_{21}$.
The matrices representing the generators $\mathcal{L}_{z,z+1}$ are of dimension 9×9 while the matrices representing the generators $\mathcal{L}_{\text{left}}$, $\mathcal{L}_{\text{right}}$ are of dimension 3×3. The elements of these matrices are ordered as follows:

- for $\mathcal{L}_{z,z+1}$, the row and the column indexes are

$$00, 01, 02, 10, 11, 12, 20, 21, 22$$

  For example, the element on the 3rd row and 4th column gives the rate of transition $02 \to 10$

- for the site matrices $\mathcal{L}_{\text{left}}$ and $\mathcal{L}_{\text{right}}$, the rows and the columns a indexes are $0, 1, 2$.

$$\mathcal{L}_{\text{left}} =$$
$$\begin{pmatrix} -\sigma_{11}\rho^{(1)}_{\text{left}} - \sigma_{12}\rho^{(2)}_{\text{left}} - \sigma_{21}\rho^{(1)}_{\text{left}} - \sigma_{22}\rho^{(2)}_{\text{left}} & \sigma_{11}\rho^{(1)}_{\text{left}} + \sigma_{12}\rho^{(2)}_{\text{left}} & \sigma_{21}\rho^{(1)}_{\text{left}} + \sigma_{22}\rho^{(2)}_{\text{left}} \\ \sigma_{11} + \sigma_{21} - \sigma_{11}\rho^{(1)}_{\text{left}} - \sigma_{12}\rho^{(2)}_{\text{left}} - \sigma_{21}\rho^{(1)}_{\text{left}} - \sigma_{22}\rho^{(2)}_{\text{left}} & \sigma_{11}\rho^{(1)}_{\text{left}} - \sigma_{21} - h - \sigma_{11} + \sigma_{12}\rho^{(2)}_{\text{left}} & h + \sigma_{21}\rho^{(1)}_{\text{left}} + \sigma_{22}\rho^{(2)}_{\text{left}} \\ \sigma_{22} + \sigma_{12} - \sigma_{22}\rho^{(2)}_{\text{left}} - \sigma_{21}\rho^{(1)}_{\text{left}} - \sigma_{12}\rho^{(2)}_{\text{left}} - \sigma_{11}\rho^{(1)}_{\text{left}} & m + \sigma_{11}\rho^{(1)}_{\text{left}} + \sigma_{12}\rho^{(2)}_{\text{left}} & \sigma_{21}\rho^{(1)}_{\text{left}} - \sigma_{12} - m - \sigma_{22} + \sigma_{22}\rho^{(2)}_{\text{left}} \end{pmatrix} \tag{10.4.21}$$

$$\mathcal{L}_{z,z+1} =$$
$$\begin{pmatrix} \Gamma^{00}_{00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma^{01}_{01} & h & \sigma_{11} & 0 & 0 & \sigma_{21} & 0 & 0 \\ 0 & m & \Gamma^{02}_{02} & \sigma_{12} & 0 & 0 & \sigma_{22} & 0 & 0 \\ 0 & \sigma_{11} & \sigma_{21} & \Gamma^{10}_{10} & 0 & 0 & \Upsilon - 2\sigma_{21} - h & 0 & 0 \\ 0 & 0 & 0 & 0 & \Gamma^{11}_{11} & h & 0 & \Upsilon - 2\sigma_{21} - h & \sigma_{21} \\ 0 & 0 & 0 & 0 & m & \Gamma^{12}_{12} & 0 & \sigma_{11} & \Upsilon - \sigma_{12} - \sigma_{21} - h \\ 0 & \sigma_{12} & \sigma_{22} & \Upsilon - 2\sigma_{12} - m & 0 & 0 & \Gamma^{20}_{20} & 0 & 0 \\ 0 & 0 & 0 & 0 & \Upsilon - \sigma_{12} - \sigma_{21} - m & \sigma_{22} & 0 & \Gamma^{21}_{21} & h \\ 0 & 0 & 0 & 0 & \sigma_{12} & \Upsilon - 2\sigma_{12} - m & 0 & m & \Gamma^{22}_{22} \end{pmatrix} \tag{10.4.22}$$

Due to the stochasticity of the generator, the diagonal elements are the following

$$\begin{array}{lll} \Gamma^{00}_{00} = 0 & \Gamma^{01}_{01} - \sigma_{11} - \sigma_{21} - h & \Gamma^{02}_{02} = -\sigma_{22} - \sigma_{12} - m \\ \Gamma^{10}_{10} = -\Upsilon - \sigma_{11} + \sigma_{21} + h & \Gamma^{12}_{12} = -\sigma_{11} - \Upsilon + \sigma_{12} + \sigma_{21} - m + h & \Gamma^{11}_{11} = -\Upsilon + \sigma_{21} \\ \Gamma^{20}_{20} = -\Upsilon - \sigma_{22} + \sigma_{12} + m & \Gamma^{21}_{21} = -\Upsilon - \sigma_{22} + \sigma_{21} + \sigma_{12} + m - h & \Gamma^{22}_{22} = -\Upsilon + \sigma_{12} \end{array}$$

$$\mathcal{L}_{\text{right}} = \begin{pmatrix} -\sigma_{11}\rho^{(1)}_{\text{right}} - \sigma_{12}\rho^{(2)}_{\text{right}} - \sigma_{21}\rho^{(1)}_{\text{right}} - \sigma_{22}\rho^{(2)}_{\text{right}} & \sigma_{11}\rho^{(1)}_{\text{right}} + \sigma_{12}\rho^{(2)}_{\text{right}} & \sigma_{21}\rho^{(1)}_{\text{right}} + \sigma_{22}\rho^{(2)}_{\text{right}} \\ \sigma_{11} + \sigma_{21} - \sigma_{11}\rho^{(1)}_{\text{right}} - \sigma_{12}\rho^{(2)}_{\text{right}} - \sigma_{21}\rho^{(1)}_{\text{right}} - \sigma_{22}\rho^{(2)}_{\text{right}} & \sigma_{11}\rho^{(1)}_{\text{right}} - \sigma_{21} - h - \sigma_{11} + \sigma_{12}\rho^{(2)}_{\text{right}} & h + \sigma_{21}\rho^{(1)}_{\text{right}} + \sigma_{22}\rho^{(2)}_{\text{right}} \\ \sigma_{22} + \sigma_{12} - \sigma_{22}\rho^{(2)}_{\text{right}} - \sigma_{21}\rho^{(1)}_{\text{right}} - \sigma_{12}\rho^{(2)}_{\text{right}} - \sigma_{11}\rho^{(1)}_{\text{right}} & m + \sigma_{11}\rho^{(1)}_{\text{right}} + \sigma_{12}\rho^{(2)}_{\text{right}} & \sigma_{21}\rho^{(1)}_{\text{right}} - \sigma_{12} - m - \sigma_{22} + \sigma_{22}\rho^{(2)}_{\text{right}} \end{pmatrix} \tag{10.4.23}$$

Before discussing the proof of the theorem, a few comments are collected in the following remarks.

Remark 24 The theorem is in agreement with the previous literature results stating that in the absence of the reaction term, for the existence of the two dimensional coupled heat equations the cross diffusivities must vanish ([42], [41]). Here we find the corresponding statement at the level of the particle process. Indeed, by assuming $\Upsilon = 0$, then the condition (10.4.17) can be satisfied iff $\sigma_{12} = \sigma_{21} = h = m = 0$ and $\sigma_{11} = \sigma_{22}$.

Remark 25 The transitions allowed by the edge generator (10.4.19) are the following:

$$(\gamma, \delta) \quad \rightarrow \quad \begin{cases} (\delta, \gamma) & \text{stirring at rate } \sigma_{11} \\ (\overline{\delta}, \overline{\gamma}) & \text{stirring and mutation at rate } \sigma_{12} \\ (\overline{\gamma}, \delta) & \text{left mutation at rate } \Upsilon - 2\sigma_{12} - m \\ (\gamma, \overline{\delta}) & \text{right mutation at rate } m \end{cases} \tag{10.4.24}$$

Thus we see that the rate of stirring is associated to the diffusion coefficient σ_{11}, while the rate of stirring with mutation is related to the cross-diffusion coefficient σ_{12}. The rates of the left and right mutations are precisely tuned to guarantee that, for all $m \geqslant 0$, the evolution equations of the average occupation variables are (10.4.5), (10.4.6), (10.4.7). A visual representation of this process is showed in Figure 10.2. In particular, the choice $m = 0$ kills the right mutations, the choice $m = \Upsilon - 2\sigma_{12}$ kills the left mutations, while the choice $m = \frac{\Upsilon}{2} - \sigma_{12}$ gives the same rate to left and right mutations. Let us also observe that only when $m = 0$, the boundary generators satisfy the conditions $\forall z \in \{1, L\}$:

$$W^0_1(z) = W^0_2(z) \qquad W^1_0(z) = W^1_2(z) \qquad W^2_0(z) = W^2_1(z)\,. \tag{10.4.25}$$

Remark 26 We observe that the boundary generator defined in Theorem 19 has the same transitions of the boundary generator of the stirring process with maximal occupancy $\nu = 1$ (see the site generator (4.2.17)). The rates are determined by recalling that, here, the hole is denoted by 0, by setting $m = 0$ and by fixing the following boundary parameters

$$\alpha_0 = (\sigma_{11} + \sigma_{12})\rho^{(0)}_{\text{left}}, \qquad \alpha_1 = (\sigma_{11}\rho^{(1)}_{\text{left}} + \sigma_{12}\rho^{(2)}_{\text{left}}), \qquad \alpha_2 = (\sigma_{12}\rho^{(1)}_{\text{left}} + \sigma_{11}\rho^{(2)}_{\text{left}})$$

For the right boundary the result is obtained by replacing α's with β's and $\rho^{(\cdot)}_{\text{left}}$ by $\rho^{(\cdot)}_{\text{right}}$.

Remark 27 Considering the "color-blind" process, i.e. the process that does not distinguish between the particles of type 1 and those of type 2, we obtain a process with just occupied or empty sites. This is indeed the classical boundary-driven simple symmetric exclusion process [24], where in the bulk particles jump to the left or to the right at rate $\sigma := \sigma_{11} + \sigma_{12}$, provided there is space, and at the left boundary particles are created at rate $\sigma\rho_{\text{left}}$ and removed at rate $\sigma(1 - \rho_{\text{left}})$, where ρ_{left} is the particle density (and similarly at the right boundary with density ρ_{right}).

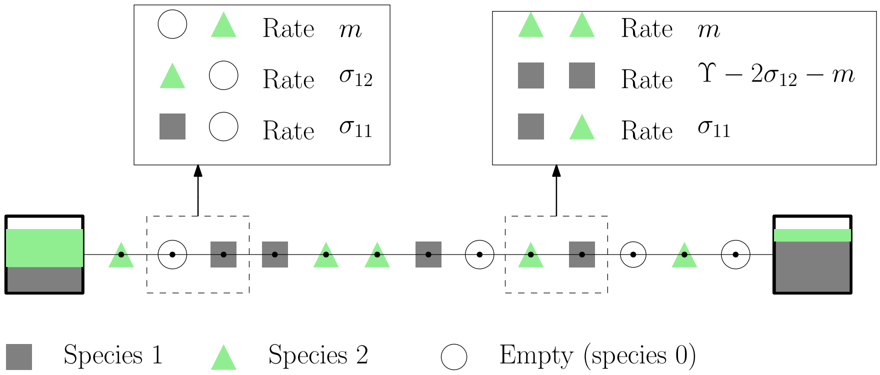


Figure 10.2: The boundary driven process with generator (10.4.19), (10.4.20) . Grey squares identify species 1, green triangles species 2, and white circles the empty state. The reservoirs are represented by rectangles, where the interior colours denote the particles or vacuum densities. In the boxes, we give two examples of allowed bulk transition with the corresponding rates.

10.4.3 Proof of Theorem 19

The proof is split in two parts: the bulk process and the boundary process. For each of them, we first give the strategy of the proof and then the details.

Bulk process

Strategy for the bulk process. To find the rates of the bulk process we need to solve (10.4.11), i.e. the system $K\mathbf{u} = \mathbf{b}$ where K is a matrix of size 30×72 and $\mathbf{b}$ is a vector described in the following. This system has a great under-determination order (72-30=42). To overcome this difficulty, we exploit the fact that, as already noticed in the text following (10.4.11), the required conditions (10.4.8), (10.4.9), (10.4.10) only involve sums of three rates. As a consequence, we may introduce a new system where the unknowns are the summed triples. This new system, which will be denoted by $\Xi\mathbf{y} = \mathbf{b}$ where Ξ is a matrix of size 30×36, has an under-determination order equal to 6, and thus can be solved explicitly under the non-negativity constraint on $\mathbf{y}$. It is precisely the request $\mathbf{y} \geqslant 0$ that further reduces the under-determination order to 2 (parametrized by the parameters $h, m \geqslant 0$) and produces the constraint (10.4.17).

Once the vector $\mathbf{y}$, whose components are sum of three rates, has been found, the next step is the identification of the transition rates themselves. This of course can be done in several ways. To produce an explicit example we have followed the two criteria below:

- The matrix associated to the generator has the greatest number of zeros.
- Choice of the following rates:

$$\Gamma_{12}^{21} = \sigma_{11} \qquad \Gamma_{21}^{12} = \sigma_{22} \qquad \Gamma_{11}^{22} = \sigma_{21} \qquad \Gamma_{22}^{11} = \sigma_{12}. \tag{10.4.26}$$

After simple but long computations, this choice leads to the generator (10.4.22). When we set $h = m$ and we choose a symmetric diffusivity matrix (which in turn guarantees a symmetric particle process) the generator (10.4.19) is obtained.

Proof for the bulk process

To solve (10.4.11) it is useful to rewrite the system by using the following variables, that are made by sums of three non diagonal rates:

$$y_1 = \sum_{\beta=0}^{2} \Gamma_{10}^{\beta 1} \quad y_2 = \sum_{\beta=0}^{2} \Gamma_{00}^{\beta 1} \quad y_3 = \sum_{\beta=0}^{2} \Gamma_{01}^{1\beta} \quad y_4 = \sum_{\beta=0}^{2} \Gamma_{00}^{1\beta} \quad y_5 = \sum_{\beta=0}^{2} \Gamma_{10}^{0\beta} \quad y_6 = \sum_{\beta=0}^{2} \Gamma_{10}^{2\beta}$$

$$y_7 = \sum_{\beta=0}^{2} \Gamma_{01}^{\beta 0} \quad y_8 = \sum_{\beta=0}^{2} \Gamma_{01}^{\beta 2} \quad y_9 = \sum_{\beta=0}^{2} \Gamma_{20}^{\beta 1} \quad y_{10} = \sum_{\beta=0}^{2} \Gamma_{02}^{1\beta} \quad y_{11} = \sum_{\beta=0}^{2} \Gamma_{02}^{\beta 1} \quad y_{12} = \sum_{\beta=0}^{2} \Gamma_{20}^{1\beta}$$

$$y_{13} = \sum_{\beta=0}^{2} \Gamma_{20}^{\beta 2} \quad y_{14} = \sum_{\beta=0}^{2} \Gamma_{00}^{\beta 2} \quad y_{15} = \sum_{\beta=0}^{2} \Gamma_{02}^{2\beta} \quad y_{16} = \sum_{\beta=0}^{2} \Gamma_{00}^{2\beta} \quad y_{17} = \sum_{\beta=0}^{2} \Gamma_{20}^{0\beta} \quad y_{18} = \sum_{\beta=0}^{2} \Gamma_{02}^{\beta 0}$$

$$y_{19} = \sum_{\beta=0}^{2} \Gamma_{10}^{\beta 2} \quad y_{20} = \sum_{\beta=0}^{2} \Gamma_{01}^{2\beta} \quad y_{21} = \sum_{\beta=0}^{2} \Gamma_{11}^{\beta 0} \quad y_{22} = \sum_{\beta=0}^{2} \Gamma_{21}^{\beta 0} \quad y_{23} = \sum_{\beta=0}^{2} \Gamma_{22}^{\beta 1} \quad y_{24} = \sum_{\beta=0}^{2} \Gamma_{11}^{0\beta}$$

$$y_{25} = \sum_{\beta=0}^{2} \Gamma_{12}^{0\beta} \quad y_{26} = \sum_{\beta=0}^{2} \Gamma_{12}^{\beta 1} \quad y_{27} = \sum_{\beta=0}^{2} \Gamma_{21}^{1\beta} \quad y_{28} = \sum_{\beta=0}^{2} \Gamma_{22}^{1\beta} \quad y_{29} = \sum_{\beta=0}^{2} \Gamma_{11}^{\beta 2} \quad y_{30} = \sum_{\beta=0}^{2} \Gamma_{12}^{\beta 0}$$

$$y_{31} = \sum_{\beta=0}^{2} \Gamma_{21}^{\beta 2} \quad y_{32} = \sum_{\beta=0}^{2} \Gamma_{22}^{\beta 0} \quad y_{33} = \sum_{\beta=0}^{2} \Gamma_{11}^{2\beta} \quad y_{34} = \sum_{\beta=0}^{2} \Gamma_{12}^{2\beta} \quad y_{35} = \sum_{\beta=0}^{2} \Gamma_{21}^{0\beta} \quad y_{36} = \sum_{\beta=0}^{2} \Gamma_{22}^{0\beta}$$

Let us introduce the following:

- *unknown vector*: $\mathbf{y} \in \mathbb{R}_{+}^{36}$

$$\mathbf{y} = (y_i)_{i=1,\dots 36}$$

- *known term*: $\mathbf{b} \in \mathbb{R}^{30}$ (that is exactly the one in (10.4.11))

$$\mathbf{b} = (\sigma_{11}, \sigma_{11}, -2\sigma_{11} - \Upsilon, \sigma_{12}, \sigma_{12}, -2\sigma_{12} + \Upsilon, \sigma_{22}, \sigma_{22}, -2\sigma_{22} - \Upsilon, \sigma_{21}, \sigma_{21}, -2\sigma_{21} + \Upsilon, \\ 0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)^T$$

- *coefficient matrix*: $\Xi \in \mathbb{R}^{30\times 36}$ (that is full rank)

By using the above vectors and matrix, the system (10.4.11) can be rewritten as

$$\Xi \mathbf{y} = \mathbf{b}. \tag{10.4.27}$$

The systems (10.4.11) and (10.4.27) are two ways of writing the conditions (10.4.8), (10.4.9), (10.4.10). By consequence, there exists an other full rank matrix, say $\Lambda \in \mathbb{R}^{36\times 72}$, that allows to retrieve a 36 parameter family of solutions of (10.4.11) once we know the one of (10.4.27) as follows

$$\Lambda \mathbf{u} = \mathbf{y}. \tag{10.4.28}$$

We first solve (10.4.27) and then we retrieve the specific solution (10.4.22) of (10.4.11), by solving (10.4.28) with some specific choices of the 36 parameters.

Solution of (10.4.27): the under-determination order is 6 and thus 6 components of the vector $\mathbf{y}$ are, actually, free parameters. Without any constraint (10.4.27) would have a 6 parameter family of solutions. However, the non-negativity of the solution (the y_i are sums of transition rates) will reduce the dependence on just two free parameters.

Indeed, by direct computations and by recalling that the variables $\{y_j\}_{j=1,\dots 36}$ must be non-negative we find the following 12 unknowns by using just 10 equations, namely:

$$y_1 - y_2 = \sigma_{11} \qquad y_3 - y_4 = \sigma_{11} \qquad y_9 - y_2 = \sigma_{12} \qquad y_{10} - y_4 = \sigma_{12} \qquad y_{13} - y_{14} = \sigma_{22}$$
$$y_{15} - y_{16} = \sigma_{22} \qquad y_{19} - y_{14} = \sigma_{21} \qquad y_{20} - y_{16} = \sigma_{21} \qquad y_2 + y_{14} = 0 \qquad y_4 + y_{16} = 0$$

that are solved if and only if

$$y_2 = y_4 = y_{14} = y_{16} = 0 \qquad y_1 = y_3 = \sigma_{11} \qquad y_{19} = y_{20} = \sigma_{21}$$
$$y_9 = y_{10} = \sigma_{12} \qquad y_{13} = y_{15} = \sigma_{22}.$$

By the non negativeness of the above y_j, it follows that

$$\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22} \geqslant 0. \tag{10.4.29}$$

Now, it remains to solve a system with 20 equations and 24 unknowns. By introducing as parameters $(y_7, y_8, y_{11}, y_{17}) := (g, h, m, s)$, this 20×24 system becomes a 20×20 parametric system. This last one has the following explicit parametric solution:

$$\begin{aligned}
&(y_5,\ y_6,\ y_{12},\ y_{18},\ y_{21},\ y_{22},\ y_{23},\ y_{24},\ y_{25},\ y_{26},\ y_{27},\ y_{28},\ y_{29},\ y_{30},\ y_{31},\ y_{32},\ y_{33},\ y_{34},\ y_{35},\ y_{36})\\
&= (2\sigma_{11} + 2\sigma_{21} - g,\ \Upsilon - 2\sigma_{21} - h,\ \ \Upsilon - 2\sigma_{12} - m,\ \ 2\sigma_{12} + 2\sigma_{22} - s,\ \ g - \sigma_{21} - \sigma_{11},\\
&g - \sigma_{22} - \sigma_{12},\ \ \sigma_{12} + m,\ \ \sigma_{11} + \sigma_{21} - g,\ \ 2\sigma_{11} - \sigma_{12} + 2\sigma_{21} - \sigma_{22} - g,\ \ \sigma_{11} + m,\\
&\sigma_{11} - 2\sigma_{12} + \sigma - m, \Upsilon - \sigma_{12} - m,\ \ \sigma_{21} + h,\\
&2\sigma_{12} - \sigma_{11} - \sigma_{21} + 2\sigma_{22} - s,\ \ \sigma_{22} + h,\ \ \sigma_{12} + \sigma_{22} - s,\\
&\Upsilon - \sigma_{21} - h,\ \ \sigma_{22} - 2\sigma_{21} + \Upsilon - h,\ \ s - \sigma_{21} - \sigma_{11},\ \ s - \sigma_{22} - \sigma_{12})\,.
\end{aligned} \tag{10.4.30}$$

Since all the y_i are sums of non negative transition rates, we impose that the components of (10.4.30) are non negative. This is true if and only if:

$$s = \sigma_{11} + \sigma_{21} \qquad g = \sigma_{11} + \sigma_{21} \tag{10.4.31}$$

and

$$\Upsilon, h, m \geqslant 0 \qquad \sigma_{12} \leqslant \frac{\Upsilon - m}{2} \qquad \sigma_{21} \leqslant \frac{\Upsilon - h}{2} \qquad \sigma_{11} + \sigma_{21} = \sigma_{12} + \sigma_{22}. \tag{10.4.32}$$

Since (10.4.31) fixes the value of two of the four parameters, the non negative solution only depends on h, m. Putting together (10.4.29) and (10.4.32) we obtain (10.4.17). Finally, this explicit non-negative solution of (10.4.27) is

$$\begin{aligned}
\mathbf{y} = (&\sigma_{11},\ 0,\ \sigma_{11}\ 0,\ \sigma_{11} + \sigma_{21},\ \Upsilon - 2\sigma_{21} - h,\ \sigma_{11} + \sigma_{21},\ h,\ \sigma_{12},\ \sigma_{12},\ m,\ \Upsilon - 2\sigma_{12} - m,\\
&\sigma_{11} - \sigma_{12} + \sigma_{21},\ 0,\ \sigma_{11} - \sigma_{12} + \sigma_{21},\ 0,\ \sigma_{11} + \sigma_{21},\ \sigma_{11} + \sigma_{21}\sigma_{21},\ \sigma_{21},\ 0,\ 0,\\
&\sigma_{12} + m,\ 0,\ 0,\ \sigma_{11} + m,\ \sigma_{11} - 2\sigma_{12} + \Upsilon - m,\ \Upsilon - \sigma_{12} - m,\ \sigma_{21} + h,\ 0,\\
&\sigma_{11} - \sigma_{12} + \sigma_{21} + h,\ 0,\ \Upsilon - \sigma_{21} - h,\ \sigma_{11} - \sigma_{12} - \sigma_{21} + \Upsilon - h,\ 0,\ 0)
\end{aligned} \tag{10.4.33}$$

Solution of (10.4.11): from (10.4.33) we know the explicit solution of (10.4.27). To find the solution of (10.4.11), we solve (10.4.28). This last system is full rank. It has 72 unknowns in 36 equations, thus the order of under-determination is 36. We must look for non-negative solution. To remove the under-determination, and produce examples (10.4.22) we impose the following conditions:

i The matrix associated to the generator has the greater number of zeros;

ii Fix the following rates:

$$\Gamma_{12}^{21} = \sigma_{11} \quad \Gamma_{21}^{12} = \sigma_{22} \quad \Gamma_{11}^{22} = \sigma_{21} \quad \Gamma_{22}^{11} = \sigma_{12}. \tag{10.4.34}$$

With the above two requests, the solution of (10.4.28) is unique (for fixed parameters h, m and for fixed diffusivity matrix and reaction constant) and the bulk generator takes the form (10.4.22). Indeed, by considering (10.4.33) we have:

- The row $\Gamma_{00}^{\alpha,\beta}$ has all the elements are zero;
- The row $\Gamma_{01}^{\alpha,\beta}$ is found by solving

$$\Gamma_{01}^{10} + \Gamma_{01}^{11} + \Gamma_{01}^{12} = \sigma_{11} \qquad \Gamma_{01}^{00} + \Gamma_{01}^{10} + \Gamma_{01}^{20} = \sigma_{11} + \sigma_{21}$$
$$\Gamma_{01}^{02} + \Gamma_{01}^{12} + \Gamma_{01}^{22} = h \qquad \Gamma_{01}^{20} + \Gamma_{01}^{21} + \Gamma_{01}^{22} = \sigma_{21}.$$

  By the conditions *i* and *ii* previously required, we obtain $\Gamma_{01}^{10} = \sigma_{11}$, $\Gamma_{01}^{20} = \sigma_{12}$, $\Gamma_{01}^{02} = h$ and all the other off-diagonal rates are equal to zero. By similar arguments, also the rows $\Gamma_{02}^{\alpha\beta}, \Gamma_{10}^{\alpha\beta}, \Gamma_{20}^{\alpha\beta}$ are determined.

- The row $\Gamma_{11}^{\alpha\beta}$ is found by solving:

$$\Gamma_{11}^{02} + \Gamma_{11}^{12} + \Gamma_{11}^{22} = \sigma_{21} + h \qquad \Gamma_{11}^{20} + \Gamma_{11}^{21} + \Gamma_{11}^{22} = \Upsilon - \sigma_{21} - h$$
$$\Gamma_{11}^{00} + \Gamma_{11}^{10} + \Gamma_{11}^{20} = 0 \qquad \Gamma_{11}^{00} + \Gamma_{11}^{01} + \Gamma_{11}^{02} = 0.$$

  By the conditions *i* and *ii* previously required we obtain $\Gamma_{11}^{22} = \sigma_{21}$, $\Gamma_{11}^{12} = h$, $\Gamma_{11}^{21} = \Upsilon - 2\sigma_{21} - h$ and all the other off-diagonal rates are equal to zero. By similar arguments, also the rows $\Gamma_{12}^{\alpha\beta}, \Gamma_{21}^{\alpha\beta}, \Gamma_{22}^{\alpha\beta}$ are determined.

We observe that, when $h = m = 0$ (10.4.22) do coincide with the non negative least square solution (see [105]) of (10.4.28). The generator (10.4.19) is recovered from (10.4.22) when $\sigma_{21} = \sigma_{12}$, $\sigma_{22} = \sigma_{11}$ and $h = m$ in (10.4.22).

Boundary processes

Strategy for the boundary process. To find the rates of the boundary process we need to solve (10.4.14) and (10.4.15). Having already determined the rates of the bulk process, by direct computation we find the boundary generators (10.4.21) and (10.4.23), which depend on $h, m \geqslant 0$. When we set $h = m$ and choose a symmetric diffusivity matrix, then the generator (10.4.20) is obtained.

Details of the proof for the boundary process

Once the bulk is known, the conditions for the boundaries form two determined systems of linear algebraic equations. We solve explicitly only the left boundary; the solution of the right one is similar.

Left boundary: recalling the definitions of B_1 and C_2, we have the following

$$B_1^{11} = -y_5 - y_6 - y_4 \qquad B_1^{12} = y_{12} - y_4 \qquad B_1^{21} = y_6 - y_{16} \qquad B_1^{22} = -y_{17} - y_{12} - y_{16}$$

$$C_2^{11} = -y_7 - h - y_2 \qquad C_2^{12} = m - y_2 \qquad C_2^{21} = h - y_{14} \qquad C_2^{22} = -y_{18} - m - y_{14};$$

by consequence system (10.4.14) is rewritten as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} W_0^1(1) \\ W_0^2(1) \\ W_1^0(1) \\ W_1^2(1) \\ W_2^0(1) \\ W_2^1(1) \end{pmatrix} = \begin{pmatrix} \sigma_{11}\rho_{\text{left}}^{(1)} + \sigma_{12}\rho_{\text{left}}^{(2)} \\ -\sigma_{11} - \sigma_{21} - h \\ m \\ \sigma_{21}\rho_{\text{left}}^{(1)} + \sigma_{22}\rho_{\text{left}}^{(2)} \\ h \\ -\sigma_{22} - \sigma_{12} - m \end{pmatrix}.$$

The coefficient matrix of the above system has full rank; thus there exists a unique solution. Recalling the definition of $W_\gamma^\alpha(1)$ we obtain (10.4.21). As a consequence of (10.4.17), and in particular $\sigma_{11} + \sigma_{21} = \sigma_{12} + \sigma_{22}$, this generator has non negative non-diagonal transition rates if

$$0 \leqslant \rho_{\text{left}}^{(1)} + \rho_{\text{left}}^{(2)} \leqslant 1. \tag{10.4.35}$$

(10.4.35) is always true since we assumed that since we assumed that the sum of the densities of the two species in the reservoir is at most one.

Right boundary: by similar arguments we solve (10.4.15) and we obtain the right boundary, i.e. (10.4.23). This matrix has non-negative off-diagonal rates if:

$$0 \leqslant \rho_{\text{right}}^{(1)} + \rho_{\text{right}}^{(2)} \leqslant 1. \tag{10.4.36}$$

(10.4.36) is always true since we assumed that the sum of the densities in the reservoir is at most one.

□

10.5 Hydrodynamic limit for the process of Theorem 19

We aim to derive the hydrodynamic equations for the family of processes defined in (10.4.19). In this section, we assume to work on the whole one-dimensional lattice $\mathbb{Z}$. To formulate the results, it is convenient to change notation. The state space of the Markov process defined by the edge generator (10.4.19) on the full line can be identified with the three-dimensional simplex

$$\Omega_{\mathbb{Z}} = \left\{(n_0, n_1, n_2) \in \{0,1\}^3 \ : \ n_0 + n_1 + n_2 = 1\right\}^{\mathbb{Z}} .$$

In this notation, the component n^z at site $z \in \mathbb{Z}$ of a configuration $\boldsymbol{n} \in \Omega_{\mathbb{Z}}$ is thus a triplet with two 0's and a 1, whose position is associated with a hole, or with a particle of type 1, or with a particle of type 2. For example, $(n_0^z, n_1^z, n_2^z) = (0,1,0)$ indicates that in the site $z \in \mathbb{Z}$ there is one particle of species 1. Then, recalling the notation in (10.4.16) for the mutation map, the process $\{\boldsymbol{n}(t), t \geqslant 0\}$ taking values in $\Omega_{\mathbb{Z}}$ is defined by the following generator L working of local functions $f : \Omega_{\mathbb{Z}} \to \mathbb{R}$:

$$L = \sum_{z \in \mathbb{Z}} L_{z,z+1} \tag{10.5.1}$$

with

$$L_{z,z+1} = \sigma_{11} L^{S}_{z,z+1} + \sigma_{12} L^{SM}_{z,z+1} + (\Upsilon - 2\sigma_{12} - m) L^{LM}_{z,z+1} + m L^{RM}_{z,z+1} \tag{10.5.2}$$

where

$$\begin{aligned}
L^{S}_{z,z+1} f(\boldsymbol{n}) &= \sum_{\alpha,\beta=0}^{2} n^z_\alpha n^{z+1}_\beta \left[f(\boldsymbol{n} - \boldsymbol{\delta}^z_\alpha + \boldsymbol{\delta}^z_\beta + \boldsymbol{\delta}^{z+1}_\alpha - \boldsymbol{\delta}^{z+1}_\beta) - f(\boldsymbol{n}) \right] \\
L^{SM}_{z,z+1} f(\boldsymbol{n}) &= \sum_{\alpha,\beta=0}^{2} n^z_\alpha n^{z+1}_\beta \left[f(\boldsymbol{n} - \boldsymbol{\delta}^z_\alpha + \boldsymbol{\delta}^z_{\overline{\beta}} - \boldsymbol{\delta}^{z+1}_\beta + \boldsymbol{\delta}^{z+1}_{\overline{\alpha}}) - f(\boldsymbol{n}) \right] \\
L^{LM}_{z,z+1} f(\boldsymbol{n}) &= \sum_{\alpha=0}^{2} n^z_\alpha \left[f(\boldsymbol{n} - \boldsymbol{\delta}^z_\alpha + \boldsymbol{\delta}^z_{\overline{\alpha}}) - f(\boldsymbol{n}) \right] \\
L^{RM}_{z,z+1} f(\boldsymbol{n}) &= \sum_{\beta=0}^{2} n^{z+1}_\beta \left[f(\boldsymbol{n} - \boldsymbol{\delta}^{z+1}_\beta + \boldsymbol{\delta}^{z+1}_{\overline{\beta}}) - f(\boldsymbol{n}) \right]
\end{aligned} \tag{10.5.3}$$

To formulate the hydrodynamic limit, we consider a scaling parameter $K \in \mathbb{N}$ and we introduce the empirical density fields

$$X^K_1(t) = \frac{1}{K} \sum_{z \in \mathbb{Z}} n^z_1(K^2 t) \delta_{\frac{z}{K}} \qquad X^K_2(t) = \frac{1}{K} \sum_{z \in \mathbb{Z}} n^z_2(K^2 t) \delta_{\frac{z}{K}} \tag{10.5.4}$$

The empirical density fields $\{X^K_1(t), t \geqslant 0\}$ and $\{X^K_2(t), t \geqslant 0\}$ are measure-valued processes constructed from the process $\{\boldsymbol{n}(t), t \geqslant 0\}$. We also need to specify a good set of initial distributions. In the following we denote by

$$\langle X^K_\alpha(t), g \rangle = \int_{\mathbb{R}} X^K_\alpha(t) g(u) du \,. \tag{10.5.5}$$

Definition 17 *Let $\widehat{\rho}^{(\alpha)} : \mathbb{R} \to [0,1]$, with $\alpha \in \{1,2\}$, be a continuous bounded real function called the initial macroscopic profile. A sequence $(\mu_K)_{K \in \mathbb{N}}$ of measures on $\Omega_{\mathbb{Z}}$, is a sequence of compatible initial conditions if $\forall \alpha \in \{1,2\}$, $\forall \delta > 0$:*

$$\lim_{K \to \infty} \mu_K \left(\left| \langle X^K_\alpha(0), g \rangle - \int_{\mathbb{R}} g(u) \widehat{\rho}^{(\alpha)}(u) du \right| > \delta \right) = 0 \tag{10.5.6}$$

where $g \in C^\infty_c(\mathbb{R})$.

We then have the following theorem for the hydrodynamic limit.

Theorem 20 (Hydrodynamic limit of the Markov process $\{n(t), t \geqslant 0\}$) . *Let $\widehat{\rho}^{(\alpha)}$ with $\alpha \in \{1,2\}$ be initial macroscopic profiles and $(\mu_K)_{K \in \mathbb{N}}$ be a sequence of compatible initial conditions. Let $\mathbb{P}_{\mu_K}$ be the law of the measure valued process $(X^K_1(t), X^K_2(t))$ defined in (10.5.4). Then $\forall T, \delta > 0$, $\forall \alpha \in \{1,2\}$ and for all $g \in C^\infty_c(\mathbb{R})$*

$$\lim_{K \to \infty} \mathbb{P}_{\mu_K} \left(\sup_{t \in [0,T]} \left| \langle X^K_\alpha(t), g \rangle - \int_{\mathbb{R}} g(u) \rho^{(\alpha)}(u,t) du \right| > \delta \right) = 0, \tag{10.5.7}$$

where $\rho^{(1)}, \rho^{(2)}$ are the strong solutions of

$$\begin{cases} \partial_t \rho^{(1)} = \sigma_{11} \partial^2_u \rho^{(1)} + \widetilde{\Upsilon} \left(\rho^{(2)} - \rho^{(1)} \right) \\ \partial_t \rho^{(2)} = \sigma_{11} \partial^2_u \rho^{(2)} + \widetilde{\Upsilon} \left(\rho^{(1)} - \rho^{(2)} \right) \\ \rho^{(\alpha)}(0,u) = \widehat{\rho}^{(\alpha)}(u) \qquad \forall u \in [0,1], \ \forall \alpha \in \{1,2\} \end{cases} \tag{10.5.8}$$

We observe that, since the PDE's (10.5.8) are coupled only in the reaction term, uphill diffusion cannot take place (see Section 9.1.1).

Proof of Theorem 10.5.7: The proof is standard and it is based on the Dynkin's martingale and its quadratic variation (see (2.1.49)). For the tightness and the uniqueness of the limiting point we refer to [29] and [31]. We provide here some details for the computations of the Dynkin's martingale and its quadratic variation via Carré-Du-Champ.

We introduce the following real and positive parameters:

$$\widetilde{\sigma}_{12} = K^2\sigma_{12}, \quad \widetilde{\Upsilon} = K^2\Upsilon \quad \widetilde{m} = K^2 m. \tag{10.5.9}$$

We consider the re-scaled generator

$$L^{(K)} = \sum_{z\in\mathbb{Z}} L^{(K)}_{z,z+1} \tag{10.5.10}$$

where

$$L^{(K)}_{z,z+1} = \sigma_{11}L^{S}_{z,z+1} + \widetilde{\sigma}_{12}\frac{1}{K^2}L^{SM}_{z,z+1} + \frac{1}{K^2}(\widetilde{\Upsilon} - 2\widetilde{\sigma}_{12} - \widetilde{m})L^{LM}_{z,z+1} + \widetilde{m}\frac{1}{K^2}L^{RM}_{z,z+1}. \tag{10.5.11}$$

By choosing $\forall z \in \mathbb{Z}$ and $\forall \alpha \in \{1,2\}$ the action of the rescaled generator on n^z_α is the following:

$$\begin{aligned}(L^{(K)}n^z_\alpha)(n) =& \sigma_{11}\left(n^{z+1}_\alpha - 2n^z_\alpha + n^{z-1}_\alpha\right) + \widetilde{\sigma}_{12}\frac{1}{K^2}\left(n^{z+1}_{\bar\alpha} - 2n^z_{\bar\alpha} + n^{z-1}_{\bar\alpha}\right)\\ &+\frac{1}{K^2}\left(\widetilde{\Upsilon} - 2\widetilde{\sigma}_{12}\right)(n^z_{\bar\alpha} - n^z_\alpha)\end{aligned}$$

By consequence considering a test function g

$$\begin{aligned}&\int_0^t ds\; K^2L^{(K)}\langle X^K_\alpha(s), g\rangle\\ =&\sigma_{11}\int_0^t ds\; K^2\;\frac{1}{K}\sum_{z\in\mathbb{Z}} n^z_\alpha(s)\left[g\left(\frac{z+1}{K}\right) - 2g\left(\frac{z}{K}\right) + g\left(\frac{z-1}{K}\right)\right]\\ &+\widetilde{\sigma}_{12}\int_0^t ds\; K^2\frac{1}{K^3}\sum_{z\in\mathbb{Z}}\left(n^z_{\bar\alpha}(s)\left[g\left(\frac{z+1}{K}\right) + g\left(\frac{z-1}{K}\right)\right] - 2n^z_\alpha(s)g(\frac{z}{K})\right)\\ &+\int_0^t ds\; K^2\;\frac{1}{K^3}(\widetilde{\Upsilon} - 2\widetilde{\sigma}_{12})\sum_{z\in\mathbb{Z}} g\left(\frac{z}{K}\right)[n^z_{\bar\alpha} - n^z_\alpha]\end{aligned}$$

By using the Taylor expansion we rewrite the above equality as

$$\begin{aligned}\int_0^t ds\; K^2L^{(K)}\langle X^K_\alpha(s), g\rangle =& \sigma_{11}\int_0^t \frac{1}{K}\sum_{z\in\mathbb{Z}} n^z_\alpha\Delta g\left(\frac{z}{K}\right) + \widetilde{\sigma}_{12}\int_0^t \frac{1}{K^3}\sum_{z\in\mathbb{Z}} n^z_{\bar\alpha}\Delta g\left(\frac{z}{K}\right)\\ &+\widetilde{\Upsilon}\int_0^t \frac{1}{K}\sum_{z\in\mathbb{Z}} g\left(\frac{z}{K}\right)[n^z_{\bar\alpha} - n^z_\alpha] + o(\frac{1}{K})\\ =& \sigma_{11}\int_0^t \frac{1}{K}\sum_{z\in\mathbb{Z}} n^z_\alpha\Delta g\left(\frac{z}{K}\right)\\ &+\widetilde{\Upsilon}\int_0^t \frac{1}{K}\sum_{z\in\mathbb{Z}} g\left(\frac{z}{K}\right)[n^z_{\bar\alpha} - n^z_\alpha] + o(\frac{1}{K}).\end{aligned}$$

Defining the Dynkin's martingale $\forall \alpha \in \{1,2\}$

$$M_g^t(X_\alpha^K) := \langle X_\alpha^K(t), g\rangle - \langle X_\alpha^K(0), g\rangle - \int_0^t K^2 L^{(K)} \langle X_\alpha^K(s), g\rangle ds, \tag{10.5.12}$$

by the previous computations, we have

$$\begin{aligned} M_g^t(X_\alpha^K) + o(\frac{1}{K}) =& \langle X_\alpha^K(t), g\rangle - \langle X_\alpha^K(0), g\rangle - \sigma_{11} \int_0^t \langle X_\alpha^K(s), \Delta g\rangle ds \\ & - \widetilde{\Upsilon} \int_0^t \langle X_{\bar{\alpha}}^K(s) - X_\alpha^K(s), g\rangle ds. \end{aligned}$$

The right-hand side is the discrete counterpart of the weak solution of (10.5.8).
To have tightness of the law of the measure-valued processes (10.5.4) we need to show that

$$\lim_{K\to\infty} \mathbb{E}_{\mu_K}\left[M_g^t(X_\alpha^K)^2\right] = 0. \tag{10.5.13}$$

We first observe that

$$\mathbb{E}_{\mu_K}\left[M_g^t(X_\alpha^K)^2\right] \leqslant \mathbb{E}_{\mu_K}\left[\sup_{t\in[0,T]} |M_g^t(X_\alpha^K)|^2\right] \leqslant 4\mathbb{E}_{\mu_K}\left[M_g^T(X_\alpha^K)^2\right] = 4\mathbb{E}_{\mu_K}\left[\int_0^T K^2 \Gamma_{\alpha,\alpha}^{g,s} ds\right],$$

where $\Gamma_{\alpha,\alpha}^{g,s}$ is the Carré-Du-Champ operator that can be written as

$$\Gamma_{\alpha,\alpha}^{g,s} = L^{(K)} \langle X_\alpha^K(t), g\rangle^2 - 2\langle X_\alpha^K(t), g\rangle L^{(K)} \langle X_\alpha^K(t), g\rangle. \tag{10.5.14}$$

By using the definition of the re-scaled generator (10.5.11) we obtain the following

$$\begin{aligned} K^2\Gamma_{\alpha,\alpha}^{g,s} =& \sigma_{11} \frac{1}{K^2} \sum_{z\in\mathbb{Z}} \left[n_\alpha^z(1-n_\alpha^{z+1}) + n_{\bar{\alpha}}^z(1-n_{\bar{\alpha}}^{z+1})\right] \left(\nabla g(\frac{z}{K})\right)^2 \\ &+ \widetilde{\sigma}_{12} \frac{1}{K^2} \sum_{z\in\mathbb{Z}} \Bigg\{ 2\left[n_\alpha^z n_{\bar{\alpha}}^{z+1} + n_{\bar{\alpha}}^z n_\alpha^{z+1}\right] g(\frac{z}{K}) g(\frac{z+1}{K}) \\ & \qquad + n_{\bar{\alpha}}^z \left[g(\frac{z+1}{K})^2 + g(\frac{z-1}{K})^2\right] + n_\alpha^z 2g(\frac{z}{K})^2 \Bigg\} \\ &+ \left(\widetilde{\Upsilon} - 2\widetilde{\sigma}_{12}\right) \frac{1}{K^2} \sum_{z\in\mathbb{Z}} (n_{\bar{\alpha}}^z + n_\alpha^z) g(\frac{z}{K})^2 + o(\frac{1}{K^2}). \end{aligned} \tag{10.5.15}$$

Let's introduce the set $\mathcal{S}_g$ as the smallest compact subset of $\mathbb{R}$ that contains the supports of a fixed g and of the first two derivatives. Then, $|\mathcal{S}_g| \leqslant C' K$, with a C' positive and finite constant. Moreover, by the hard-core constraint $n_\alpha^z \leqslant 1$, $\forall z \in \mathbb{Z}$ and $\forall \alpha \in \{1,2\}$. By consequence, exploiting the smoothness of g we derive the following bound

$$\mathbb{E}_{\mu_K}\left[\int_0^T K^2 \Gamma_{\alpha,\alpha}^{g,s/K^2} ds\right] \leqslant C\frac{1}{K}, \tag{10.5.16}$$

with $C < \infty$. This concludes the proof.

□

Remark 28 Let's define a "color-blind" density field

$$X^K(t) := \frac{1}{K}\sum_{z\in\mathbb{Z}} n^z(tK^2)\delta_{\frac{z}{K}} \tag{10.5.17}$$

where $n^z(t) := n^z_\alpha(t) + n^z_{\overline{\alpha}}(t)$. By re-scaling only the $L^{RM}_{z,z+1}$ and $L^{LM}_{z,z+1}$ terms of the generator, the same proof of Theorem 20 we would give, as limiting PDE, the heat equation

$$\begin{cases} \partial_t \rho(u,t) = (\sigma_{11} + \sigma_{12})\partial_u^2 \rho(u,t) \\ \rho(u,0) = \rho_0(u) \end{cases} \tag{10.5.18}$$

This is in agreement with the Remark 27.

Remark 29 We observe that in order to obtain the hydrodynamic limit of the process $\{\boldsymbol{n}(t); t \geqslant 0\}$ we had to scale the parameters as in (10.5.9). Indeed, the 'naive' scaling where the diffusivity parameters σ_{11} and σ_{12} are both kept constant (while the reaction parameters are scaled as $\Upsilon = \frac{1}{K^2}\tilde{\Upsilon}$ and $m = \frac{1}{K^2}\tilde{m}$) is not viable as it would make (10.5.15) infinite when $K \to \infty$. In other words, the problem with the 'naive' re-scaling is that the rate of left mutations

$$(\tilde{\Upsilon}\frac{1}{K^2} - 2\sigma_{12} - \tilde{m}\frac{1}{K^2}) \tag{10.5.19}$$

becomes negative (!) for sufficiently big K. One could still wonder if other scaling of the parameters would lead to equations (9.1.7) in the hydrodynamic limit. We argue that this is not possible, because the maximum principle (which is a necessary condition for the Markov property) would be violated. To show this, we rewrite the PDEs (9.1.7) in the form

$$\begin{cases} \partial_t \begin{pmatrix} \rho^{(1)} \\ \rho^{(2)} \end{pmatrix} = A \begin{pmatrix} \rho^{(1)} \\ \rho^{(2)} \end{pmatrix} \qquad \forall u \in [0,1] \\ \rho^{(1)}(0,u) = \hat{\rho}^{(1)}(u), \quad \rho^{(2)}(0,u) = \hat{\rho}^{(2)}(u) \end{cases} \tag{10.5.20}$$

where the operator A is defined as

$$A := \begin{pmatrix} \sigma_{11}\partial_u^2 & \sigma_{12}\partial_u^2 \\ \sigma_{12}\partial_u^2 & \sigma_{11}\partial_u^2 \end{pmatrix} + \Upsilon \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10.5.21}$$

Now, for a function $f = (f^{(1)}, f^{(2)})$ in the domain of A, let $u_* \in (0,1)$ be such that

$$f^{(1)}(u_*) := \max_{u\in(0,1)} f^{(1)}(u) \tag{10.5.22}$$

Then the first component of $(Af)(u_*)$ reads

$$\sigma_{11}\partial_u^2 f^{(1)}(u_*) + \sigma_{12}\partial_u^2 f^{(2)}(u_*) + \tilde{\Upsilon}\left(f^{(2)}(u_*) - f^{(1)}(u_*)\right). \tag{10.5.23}$$

Clearly (10.5.23) can be positive, since (10.5.22) guarantees that $\partial_{xx} f^{(1)}(x^*) \leqslant 0$, but the other terms of (10.5.23) can be positive and arbitrary large. As a consequence of the violation of the maximum principle it follows that A can not be the generator of a Markov process.

Remark 30 If we perform the hydrodynamic limit with an "Euler" re-scaling, i.e. we re-scale the time only by a factor $\frac{1}{K}$ and we define $\widehat{\sigma}_{12} = K\sigma_{12}$, $\widehat{\Upsilon} = K\Upsilon$ and $\widehat{m} = Km$ we obtain the following ODE's system

$$\begin{cases} \frac{d}{dt}\rho^{(1)}(t) = \widehat{\Upsilon}(\rho^{(2)} - \rho^{(1)}) \\ \frac{d}{dt}\rho^{(2)}(t) = \widehat{\Upsilon}(\rho^{(1)} - \rho^{(2)}) \\ \rho^{(1)}(0) = \rho_0^{(1)}, \quad \rho^{(2)}(0) = \rho_0^{(2)} \end{cases} \tag{10.5.24}$$

that is a purely reacting system. The ODE's are linear and the solution is given by

$$\begin{cases} \rho^{(1)}(t) = \frac{\rho_0^{(1)} + \rho_0^{(2)}}{2} + \frac{\rho_0^{(1)} - \rho_0^{(2)}}{2} e^{-2\widehat{\Upsilon}t} \\ \rho^{(2)}(t) = \frac{\rho_0^{(1)} + \rho_0^{(2)}}{2} - \frac{\rho_0^{(1)} - \rho_0^{(2)}}{2} e^{-2\widehat{\Upsilon}t} \end{cases} \tag{10.5.25}$$

Chapter 11

Generalization of the model and duality

11.1 Motivations

The interacting particle systems introduced in Theorem 19 of Chapter 10, is limited to the case of 2 species of particles and to hard-core exclusion. Therefore, a follow-up question is the extension of this model to the more general case where N species of particles are considered and where the hard-core constraint is lost (i.e. $\nu > 1$). Once this more general model is defined, we aim to investigate if it satisfies a $gl(N+1)$ symmetry which leads to an absorbing duality result. Then, we aim to extend the analysis done in Chapter 5 to the reaction diffusion with N species, $\nu > 1$ described above, in the particular case where the crossing diffusivity σ_{12} are vanishing, i.e. $\sigma_{12} = 0$. Indeed, as we have shown in Theorem 20, when the system is diffusively re-scaled, these crossing diffusivity disappear.
In this chapter we start by generalizing, to arbitrary number of species N and to arbitrary $\nu > 1$, the process defined in Theorem 19 (Section 11.2). Then, using an Lie algebraic argument, we prove absorbing duality for this more general process (Section 11.3). Finally, we prove hydrodynamic limit and equilibrium density fluctuations in the case the crossing diffusivity σ_{12} are vanishing (Section 11.4).

11.2 Generalization of the process of Theorem 19

In this section we aim to generalize the process defined in Theorem 19 to the case where $N > 2$ species of particles are present and where the hard-core constraint is lost, i.e. where the maximal occupancy is $\nu > 1$.
We aim to find a process whose average occupations $\rho_z^{(\zeta)}$ at site $z \in \{1, \ldots, L\}$ and for any species $\zeta \in \{1, \ldots, N\}$ evolves as a system of difference-differential equation given by

$$\frac{d}{dt}\rho_z^{(\zeta)} = \nu\sigma_{11}\Delta_L\rho_z^{(\zeta)} + \nu\sigma_{12}\sum_{\gamma=1,\gamma\neq\zeta}^{N}\Delta_L\rho_z^{(\gamma)} + \Upsilon\sum_{\gamma=1,\gamma\neq\zeta}^{N}\left(\rho_z^{(\gamma)} - \rho_z^{(\zeta)}\right) . \tag{11.2.1}$$

We assume that the boundary condition are given. We denote them by ρ_ζ^{left} and $\rho_{\text{right}}^{(\zeta)}$ for all $\zeta \in \{1, \ldots, N\}$ at left and right respectively, further assuming that $\sum_{\gamma=0}^{N}\rho_{\text{left}}^{(\gamma)} = \sum_{\gamma=0}^{N}\rho_{\text{right}}^{(\gamma)} = \nu$. In the following of this section, we first construct the process on a chain in order to obtain

(11.2.1) for the evolution of the average occupation variable of species ζ. Once we have found such a process, we construct the Lie algebraic description and we use it to prove absorbing duality.

The reaction-diffusion process on a chain

We assume that the geometry is a chain of length L where two reservoirs are connected to the end sites 1 and L. We denote the processes by the variables $(\boldsymbol{n}(t))_{t\geqslant 0}$, while we denote its configuration space by

$$\Omega_L = \bigotimes_{z=1}^{L} \Omega_z \,. \tag{11.2.2}$$

where

$$\Omega_z := \left\{ n^z = (n_1^z, \ldots, n_N^z) \in \mathbb{N}_0^{N+1} \;:\; \sum_{\gamma=0}^{N} n_\gamma^z = \nu \right\} . \tag{11.2.3}$$

The configuration of the on the whole segment is $\boldsymbol{n} = (n_\gamma^z)_{z\in\{1,\ldots,L\},\,\gamma\in\{0,\ldots,N\}}$ with n_γ^z indicates the number the number of particles of species $\gamma \in \{1,\ldots,N\}$ at site z and n_0^z indicates the number of holes at z. The reservoirs have parameters $(\alpha_\gamma)_{\gamma\in\{0,\ldots,N\}}$ and $(\beta_\gamma)_{\gamma\in\{0,\ldots,N\}}$ for the left and right boundaries respectively. The generator reads

$$\mathcal{L}^{rd} = \mathcal{L}^{rd}_{\text{right}} + \sum_{z=1}^{L-1} \mathcal{L}^{rd}_{z,z+1} + \mathcal{L}^{rd}_{\text{right}} \tag{11.2.4}$$

where

$$\mathcal{L}^{rd}_{z,z+1} = \nu\sigma_{11}\mathcal{L}^{s}_{z,z+1} + \nu\sigma_{12}\sum_{c=1}^{N-1}\mathcal{L}^{c}_{z,z+1} + (\Upsilon - 2\nu\sigma_{12})\mathcal{L}^{m}_{z,z+1} \tag{11.2.5}$$

where $\mathcal{L}^{s}_{z,z+1}$ is the edge-generator (4.2.16) of the multi-species stirring process (see (4.2.16)), specified to N species and a hole and acting on the bond $(z, z+1)$. For any function $f : \Omega \to \mathbb{R}$ and for any $c \in \{1,\ldots,N-1\}$ we introduce the *stirring-mutation* generator

$$\mathcal{L}^{c}_{z,z+1} f(\boldsymbol{n}) = \sum_{\gamma,\delta=0}^{N} n_\gamma^z n_\delta^{z+1} \Big(f(\boldsymbol{n} - \boldsymbol{\delta}_\gamma^z + \boldsymbol{\delta}^z_{h_c(\delta)} + \boldsymbol{\delta}^{z+1}_{h_c(\gamma)} - \boldsymbol{\delta}_\delta^{z+1}) - f(\boldsymbol{n}) \Big) \tag{11.2.6}$$

with the mapping

$$\begin{aligned} h_c(\cdot) : \{0,1,\ldots,N\} &\to \{0,1,\ldots,N\} \\ \gamma &\to h_c(\gamma) \end{aligned} \tag{11.2.7}$$

defined as

$$h_c(\gamma) = \begin{cases} \gamma + c & \text{if} \quad \gamma + c < N+1 \\ \gamma + c - N & \text{if} \quad \gamma + c \geqslant N+1 \\ 0 & \text{if} \quad \gamma = 0 \end{cases} . \tag{11.2.8}$$

Observe that $h_c(\cdot)$ is surjective and injective, and thus invertible. Moreover, one recovers the map $\bar{\cdot}$ defined in equation (10.4.16) when $N = 2$. We define the *pure-mutation* generator

$$\mathcal{L}^{m}_{z,z+1} f(\boldsymbol{n}) = \sum_{\gamma,\delta=1}^{N} n_\gamma^z \left(f(\boldsymbol{n} - \boldsymbol{\delta}_\gamma^z + \boldsymbol{\delta}_\delta^z) - f(\boldsymbol{n}) \right) \,. \tag{11.2.9}$$

The left boundary generator $\mathcal{L}^{rd}_{\text{left}}$ is given by (4.2.17) acting at site 1 and where the α's parameters are defined in function of the given boundary condition of the system of difference-differential (11.2.1)

$$\alpha_\zeta = \sigma_{11}\rho^{(\zeta)}_{\text{left}} + \sigma_{12}\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\rho^{(\gamma)}_{\text{left}} \qquad \text{if} \quad \zeta\in\{1,\dots,N\}$$
$$\alpha_0 = \nu\left(\sigma_{11}+(N-1)\sigma_{12}\right) - \sum_{\gamma=1}^{N}\alpha_\gamma\ . \tag{11.2.10}$$

As a consequence, from the above conditions we write the boundary values in terms of the reservoir parameters as

$$\rho^{(\zeta)}_{\text{left}} = \frac{\alpha_\zeta\sigma_{11} - \sigma_{12}\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\alpha_\zeta}{(\sigma_{11}-(N-1)\sigma_{12})} \qquad \rho^{(0)}_{\text{left}} = \frac{\alpha_0}{(\sigma_{11}+(N-1)\sigma_{12})} \tag{11.2.11}$$

for every $\zeta\in\{1,\dots,N\}$. Moreover, with this choice we have that $\sum_{\gamma=0}^{N}\alpha_\gamma = \nu(\sigma_{11}+(N-1)\sigma_{12})$ and $\sum_{\gamma=0}^{N}\rho^{(\gamma)}_{\text{left}} = \nu$. The boundary generator $\mathcal{L}^{rd}_{\text{right}}$ is defined analogously, but the α's are replaced by β's and the boundary values $\rho^{(\zeta)}_{\text{right}}$.

Action on the occupation variable For arbitrary site $z\in\{2,\dots,L-1\}$, the action on the occupation variable $f(\boldsymbol{n}) = n^z_\zeta$ of particle of type $\zeta\in\{1,\dots,N\}$ at site z reads

$$\mathcal{L}^{rd}n^z_\zeta = \nu\sigma_{11}\Delta_L n^z_\zeta + \nu\sigma_{12}\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\Delta_L n^z_\gamma + \Upsilon\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}(n^z_\gamma - n^z_\zeta) \tag{11.2.12}$$

where $\Delta_L n^z_\zeta = n^{z+1}_\zeta + n^{z-1}_\zeta - 2n^z_\zeta$. For $z=1$ and for all $\zeta\in\{1,\dots,N-1\}$ we have

$$\mathcal{L}^{rd}n^1_\zeta = \sigma_{11}\nu\left(\rho^{(\zeta)}_{\text{left}} - 2n^1_\zeta + n^2_\zeta\right) + \sigma_{12}\nu\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\left(\rho^{(\gamma)}_{\text{left}} - 2n^1_\gamma + n^2_\gamma\right) + \Upsilon\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\left(n^1_\gamma - n^1_\zeta\right)\ . \tag{11.2.13}$$

On the occupation variable n^L_ζ the action is similar. Indeed for the bulk we have that:

1. *Stirring generator*: using the generator $\mathcal{L}^s_{z,z+1}$ we have

$$\mathcal{L}^s_{z,z+1}n^z_\zeta = \sum_{\gamma,\delta=0}^{N} n^z_\gamma n^{z+1}_\delta\left((n^z_\zeta - \boldsymbol{\delta}^z_\gamma + \boldsymbol{\delta}^z_\delta + \boldsymbol{\delta}^{z+1}_\gamma - \boldsymbol{\delta}^{z+1}_\delta) - n^z_\zeta\right)\ . \tag{11.2.14}$$

In the brackets of the right-hand-side of the above equation we have: a contribution $-n^z_\zeta n^{z+1}_\delta$, when $\gamma=\zeta$; a contribution $n^z_\gamma n^{z+1}_\zeta$, when $\delta=\zeta$. Thus we obtain

$$\begin{aligned}\mathcal{L}^s_{z,z+1}n^z_\zeta &= \sum_{\gamma=1}^{N} n^z_\gamma n^{z+1}_\zeta - \sum_{\delta=1}^{N} n^z_\zeta n^{z+1}_\delta \\ &= \sum_{\gamma=1}^{N} n^z_\gamma n^{z+1}_\zeta + (\nu - \sum_{\gamma'=1}^{N} n^z_{\gamma'})n^{z+1}_\zeta - \sum_{\delta=1}^{N} n^z_\zeta n^{z+1}_\delta - n^z_\zeta(\nu - \sum_{\delta'=1}^{N} n^{z+1}_{\delta'}) \\ &= \nu\left(n^{z+1}_\zeta - n^z_\zeta\right)\end{aligned} \tag{11.2.15}$$

where we have used the fact that $n^z_0 = \nu - \sum_{\gamma'=1}^{N} n^z_{\gamma'}$.

2. *Stirring-mutation generator*: for every $c \in \{1, \dots, N-1\}$ we have

$$\mathcal{L}^c_{x,x+1} n^z_\zeta = \sum_{\gamma,\delta=0}^{N} n^z_\gamma n^{z+1}_\delta \left((n^z_\zeta - \boldsymbol{\delta}^z_\gamma + \boldsymbol{\delta}^z_{h_c(\delta)} + \boldsymbol{\delta}^{z+1}_{h_c(\gamma)} - \boldsymbol{\delta}^{z+1}_\delta) - n^z_\zeta \right) . \tag{11.2.16}$$

In the brackets of the right-hand-side of the above equation we have: a contribution $-n^z_\zeta n^{z+1}_\delta$, when $\gamma = \zeta$; a contribution $n^z_\gamma n^{z+1}_{\gamma'}$, where we denoted by $\gamma' \in \{1, \dots, N\}$ the species of particle such that $h_c(\gamma') = \zeta$. Let us observe that, by the definition of the map $h_c(\cdot)$ and for the fact that $1 \leqslant c \leqslant N-1$, we have that $\gamma' \neq \zeta$ and $\gamma' \neq 0$. Thus we obtain

$$\begin{aligned} \mathcal{L}^c_{z,z+1} n^z_\zeta &= \sum_{\gamma=0}^{N} n^z_\gamma n^{z+1}_{\gamma'} - \sum_{\delta=0}^{N} n^z_\zeta n^{z+1}_\delta \\ &= \sum_{\gamma=1}^{N} n^z_\gamma n^{z+1}_{\gamma'} + \left(\nu - \sum_{\dot\gamma=1}^{N} n^z_{\dot\gamma} \right) n^{z+1}_{\gamma'} - \sum_{\delta=1}^{N} n^z_\zeta n^{z+1}_\delta - n^z_\zeta \left(\nu - \sum_{\delta'=1}^{N} n^{z+1}_{\delta'} \right) \\ &= \nu (n^{z+1}_{\gamma'} - n^z_\zeta) \end{aligned} \tag{11.2.17}$$

where we used the fact that $n^z_0 = \nu - \sum_{\dot\gamma=1}^{N} n^z_{\dot\gamma}$.

3. *Pure-mutation generator*: here we have that

$$\mathcal{L}^m_{z,z+1} n^z_\zeta = \sum_{\gamma,\delta=1}^{N} n^z_\gamma \left((n^z_\zeta - \boldsymbol{\delta}^z_\gamma + \boldsymbol{\delta}^z_\delta) - n^z_\zeta \right) = \sum_{\gamma=1}^{N} (n^z_\gamma - n^z_\zeta) \tag{11.2.18}$$

Then, summing over all indices $c \in \{1, \dots, N-1\}$, we obtain (11.2.13).
For the action of the boundary generator we have that

$$\begin{aligned} \mathcal{L}^{rd}_{\text{left}} n^1_\zeta = &\sum_{\gamma,\delta=0}^{N} \alpha_\gamma n^1_\delta \left((n^1_\zeta - \boldsymbol{\delta}^1_\delta + \boldsymbol{\delta}^1_\gamma) - n^1_\zeta \right) = \alpha_\zeta \sum_{\delta=0}^{N} n^1_\delta - n^1_\zeta \sum_{\gamma=0}^{N} \alpha_\gamma \\ =&\alpha_\zeta \sum_{\gamma=1}^{N} n^1_\gamma + \alpha_\zeta \left(\nu - \sum_{\dot\gamma=1}^{N} n^1_{\dot\gamma} \right) - n^1_\zeta \sum_{\gamma=0}^{N} \alpha_\gamma = \alpha_\zeta \nu - n^1_\gamma \nu (\sigma_{11} + (N-1)\sigma_{12}) \\ =&\nu \sigma_{11} \rho^\zeta_{\text{left}} + \nu \sigma_{12} \left(\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N} \rho^{(\gamma)}_{\text{left}} \right) - n^1_\zeta \nu (\sigma_{11} + (N-1)\sigma_{12}) \\ =&\nu \sigma_{11} (\rho^{(\zeta)}_{\text{left}} - n^1_\zeta) + \nu \sigma_{12} \sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N} (\rho^{(\gamma)}_{\text{left}} - n^1_\zeta) \end{aligned} \tag{11.2.19}$$

where we used (11.2.10). Then equation (11.2.13) follows.

Therefore, defining the average occupation variable of species $\zeta \in \{1, \dots, N\}$ at site $z \in \{1, \dots, L\}$ as

$$\rho^{(\zeta)}_z := \mathbb{E}_\mu \left[n^\zeta_z \right] \tag{11.2.20}$$

(where the average is taken with respect to the law of the process initialized with the distribution μ) and considering the action of $\mathcal{L}^{rd}$ expressed by (11.2.12) and (11.2.13) we obtain the desired system of N difference-differential equations (11.2.1). For these reasons, this model is a generalization to N species and to maximal occupation ν of the one introduced in Theorem 19.

Remark 31 *We observe that, when we specify the generator* (11.2.4) *to the case of* $N = 2$ *species and to the case of maximal occupancy* $\nu = 1$ *we retrieve the generator* L (10.4.18) *introduced in Theorem 19.*

11.3 Duality for the reaction-diffusion process

11.3.1 Reversible measure for the reaction-diffusion process (equilibrium)

The process described by the generator $\mathcal{L}^{rd}$ introduced in (11.2.4) is reversible with respect to the homogeneous product measure

$$\Lambda_{\text{rev}} = \bigotimes_{z=1}^{L} \Lambda_{\text{rev}}^{z} \tag{11.3.1}$$

when

$$\alpha_\gamma = \beta_\gamma \quad \forall \gamma \in \{0, \dots, N\} \qquad \text{and} \qquad \alpha_\gamma = \alpha_\delta = \alpha \quad \forall \gamma, \delta \in \{1, \dots, N\} \, . \tag{11.3.2}$$

This measure has marginals Λ_{rev}^{z} given by

$$\Lambda_{\text{rev}}^{z} \sim \text{Multinomial}\,(\nu, p_0, p \dots, p) \tag{11.3.3}$$

where

$$p = \frac{\alpha}{\alpha_0 + (N)\alpha} \qquad \text{and} \qquad p_0 = \frac{\alpha_0}{\alpha_0 + (N)\alpha} \tag{11.3.4}$$

Namely,

$$\Lambda_{\text{rev}}^{z}(n^z) = \frac{\nu!}{\prod_{\gamma=0}^{N} n_\gamma^z!} p_0^{n_0^z} p^{\sum_{\delta=1}^{N} n_\delta^z} \tag{11.3.5}$$

This can be proved by imposing the detailed balance conditions for the bond $(z, z+1)$ and for the boundaries $\{1, L\}$.

11.3.2 Algebraic proof of duality for the reaction-diffusion process

In this section we formulate duality for the multi-species reaction diffusion process with generator (11.2.4). As a by-product we obtain absorbing duality for the process defined in Theorem 19. The approach used for the proof is similar to the one of Chapter 6. We denote the dual process by the variables $(\boldsymbol{\xi}(t))_{t \geqslant 0}$. In analogy to what has been done in Section 3.2.54 this dual process is defined on an enlarged chin with two extra-site 0 and $L + 1$ attached to 1 and L respectively. Then, the dual configuration spaces reads

$$\widetilde{\Omega}_L = \mathbb{N}_0^N \otimes \Omega_L \otimes \mathbb{N}_0^N \, . \tag{11.3.6}$$

where $\Omega_L = \bigotimes_{z=1}^{L} \Omega_z$ with Ω_z defined in (11.2.3).

Proposition 18 (Duality for the reaction-diffusion process) *The reaction-diffusion multi-species stirring process* $(\boldsymbol{n}(t))_{t \geqslant 0}$*, on the state space* Ω_L*, with generator* $\mathcal{L}^{rd}$ *defined in* (11.2.4) *is dual to the process* $(\boldsymbol{\xi}(t))_{t \geqslant 0}$ *on the state space* $\widetilde{\Omega}_L$ *with dual generator*

$$\widetilde{\mathcal{L}}_{left}^{rd} = \widetilde{\mathcal{L}}^{rd} + \sum_{z=1}^{L-1} \mathcal{L}_{z,z+1}^{rd} + \widetilde{\mathcal{L}}_{right}^{rd} \tag{11.3.7}$$

where $\mathcal{L}^{rd}_{x,x+1}$ *is defined in* (11.2.5) *and,* $\widetilde{\mathcal{L}}^{rd}_{left}$ *and* $\widetilde{\mathcal{L}}^{rd}_{right}$ *are the absorbing dual generators* $\widetilde{\mathcal{L}}_x$ *defined in* (6.2.10) *and acting at sites* 1 *and* L*, with the specific choice of the reservoirs parameters* (11.2.10)*. The duality function is the same of the stirring process, i.e.*

$$D(\boldsymbol{n},\boldsymbol{\xi}) = \left(\prod_{\gamma=1}^{N}\left(\frac{\alpha_\gamma}{|\alpha|}\right)^{\xi^0_\gamma}\right)\left(\prod_{z=1}^{L}\frac{(\nu-\sum_{\delta=1}^{N}\xi^z_\delta)!}{\nu!}\prod_{\gamma=1}^{N}\frac{n^z_\gamma!}{(n^z_\gamma-\xi^z_\gamma)!}\right)\left(\prod_{\gamma=1}^{N}\left(\frac{\beta_\gamma}{|\beta|}\right)^{\xi^{L+1}_\gamma}\right)\ , \tag{11.3.8}$$

cf. (6.2.11).

Proof of Proposition 18: it is enough to prove duality for the bulk generator, since the boundary generators are the same of the multi-species stirring process. In the spirit of Chapter 6, it is convenient to write the Hamiltonian of the reaction-diffusion process through the basis elements (4.3.6) of the $gl(N+1)$ Lie algebra. We have that

$$H^{rd} = H_{\text{left}} + \sum_{z=1}^{L-1}\left(\sigma_{11}\mathcal{H}^s_{z,z+1} + \sigma_{12}\sum_{c=1}^{N-1}\mathcal{H}^c_{z,z+1} + (\Upsilon-2\sigma_{12})\mathcal{H}^m_{z,z+1}\right) + H_{\text{right}}\ . \tag{11.3.9}$$

Here, $\mathcal{H}^s_{z,z+1}$ is (4.3.10) and $H_{\text{left}}, H_{\text{right}}$ is (4.3.11) (with the choice of the boundary parameters done in (11.2.10) and acting at site 1 and L respectively). Then, we have that

$$\mathcal{H}^c_{z,z+1} = \sum_{\gamma,\delta=0}^{N}\left(E_{h_c(\gamma)\delta}\otimes E_{h_c(\delta)\gamma} - E_{\delta\delta}\otimes E_{\gamma\gamma}\right) \tag{11.3.10}$$

and

$$\mathcal{H}^m_{z,z+1} = \sum_{\gamma,\delta=0}^{N}\left(E_{\delta\gamma}\otimes\mathbb{1} - E_{\gamma\gamma}\otimes\mathbb{1}\right)\ . \tag{11.3.11}$$

We prove that, for arbitrary $z\in\{1,\ldots,L\}$,

$$(\mathcal{H}^c_{z,z+1})^T D = D\mathcal{H}^c_{z,z+1} \qquad \forall c\in\{1,\ldots,N-1\} \tag{11.3.12}$$

$$(\mathcal{H}^m_{z,z+1})^T D = D\mathcal{H}^m_{z,z+1}\ . \tag{11.3.13}$$

where the duality matrix matrix D is the same of the multi-species stirring process (6.2.20), i.e.

$$D = \prod_{z=1}^{L} d_z\otimes\mathcal{D}_{u(z)}\ . \tag{11.3.14}$$

Here

$$d_z = R_z\exp\left(E^z\right) \tag{11.3.15}$$

with the diagonal part

$$R_z = \sum_{n^z\in\Omega_z}\frac{\prod_{\gamma=0}^{N}n^z_\gamma!}{\nu!}|n^z_0,\ldots,n^z_N\rangle\langle n^z_0,\ldots,n^z_N| \tag{11.3.16}$$

and

$$E^z = \sum_{\gamma=1}^{N}E^z_{\gamma 0}\,. \tag{11.3.17}$$

Furthermore

$$\mathcal{D}_{u(z)} = \sum_{\xi_1^{u(z)},\dots,\xi_N^{u(z)}=0}^{\infty} \prod_{\gamma=1}^{N} \left(\frac{\alpha_\gamma}{|\alpha|}\right)^{\xi_\gamma^{u(z)}} \langle \xi_1^{u(z)},\dots,\xi_N^{u(z)}| \ . \tag{11.3.18}$$

Since the matrix R_z is the same as in Chapter 6, we shall show that $\mathcal{H}^c_{z,z+1}$ and $\mathcal{H}^m_{z,z+1}$ commute with $\exp(E^z)\exp(E^{z+1})$, where E is defined in (11.3.17). Using the commutators (4.3.1), the bilinearity and associativity of the Kronecker product and the bilinearity of the brackets we obtain

$$\left[\left(\sum_{\gamma=1}^{N} E_{\gamma 0} \otimes \sum_{\delta=1}^{N} E_{\delta 0}\right), \sum_{\gamma',\delta'=0}^{N} \left(E_{h_c(\gamma')\delta'} \otimes E_{h_c(\delta')\gamma'} - E_{\delta'\delta'} \otimes E_{\gamma'\gamma'}\right)\right]$$
$$= \sum_{\gamma',\delta'=0}^{N} \left\{\left(E_{\gamma' 0} - E_{h_c(\delta')0}\right) \otimes \left(E_{\gamma' 0} - E_{h_c(\delta')0}\right)\right\} = 0 \ .$$

In the last equality we used the fact that map $h_c(\cdot)$ is surjective. Concerning the commutator of $\mathcal{H}^m_{x,x+1}$ the proof is similar and gives

$$\left[\sum_{\gamma=1}^{N} E_{\gamma 0} \otimes \sum_{\delta=1}^{N} E_{\delta 0}, \sum_{\gamma',\delta'=0}^{N} \left(E_{\delta'\gamma'} \otimes \mathbb{1} - E_{\gamma'\gamma'} \otimes \mathbb{1}\right)\right] = 0 \ . \tag{11.3.19}$$

□

Direct proof of self duality

We observe that self-duality for the bulk process defined in (10.5.1) can also be proved without relying on the $gl(N+1)$ algebraic structure. Indeed we have that the Markov process $\{\boldsymbol{n}(t), t \geqslant 0\}$ defined by the generator (10.5.1) is self-dual with the self duality function

$$D(\boldsymbol{n},\boldsymbol{\xi}) = \prod_{z\in\mathbb{Z}} \prod_{\gamma=1}^{2} \mathbb{1}_{\{n_\gamma^z \geqslant \xi_\gamma^z\}} \tag{11.3.20}$$

Here we have denoted by $(\boldsymbol{\xi}(t))_{t\geqslant 0}$ a copy of the process $\{\boldsymbol{n}(t), t \geqslant 0\}$ with generator (10.5.1). It is enough to prove that

$$\left(LD(\cdot,\boldsymbol{\xi})\right)(\boldsymbol{n}) = \left(LD(\boldsymbol{n},\cdot)\right)(\boldsymbol{\xi}) \qquad \forall (\boldsymbol{n},\boldsymbol{\xi}) \in \Omega_{\mathbb{Z}} \times \Omega_{\mathbb{Z}} \tag{11.3.21}$$

The generator (10.5.1) is a superposition of four generators. Remarkably, the duality relation can be verified for each of them. Indeed, one has:

$$(L^S_{z,z+1} D(\cdot,\boldsymbol{\xi}))(\boldsymbol{n})$$
$$= \left[\mathbb{1}_{\{n_1^{z+1} \geqslant \xi_1^z\}} \mathbb{1}_{\{n_2^{z+1} \geqslant \xi_2^z\}} \mathbb{1}_{\{n_1^{z} \geqslant \xi_1^{z+1}\}} \mathbb{1}_{\{n_2^{z} \geqslant \xi_2^{z+1}\}} - \mathbb{1}_{\{n_1^{z} \geqslant \xi_1^z\}} \mathbb{1}_{\{n_2^{z} \geqslant \xi_2^z\}} \mathbb{1}_{\{n_1^{z+1} \geqslant \xi_1^{z+1}\}} \mathbb{1}_{\{n_2^{z+1} \geqslant \xi_2^{z+1}\}}\right]$$
$$\times \prod_{x\notin\{z,z+1\}} \prod_{\gamma=1}^{2} \mathbb{1}_{\{n_\gamma^z \geqslant \xi_\gamma^z\}}$$
$$= \left[\mathbb{1}_{\{n_1^{z} \geqslant \xi_1^{z+1}\}} \mathbb{1}_{\{n_2^{z} \geqslant \xi_2^{z+1}\}} \mathbb{1}_{\{n_1^{z+1} \geqslant \xi_1^{z}\}} \mathbb{1}_{\{n_2^{z+1} \geqslant \xi_2^{z}\}} - \mathbb{1}_{\{n_1^{z} \geqslant \xi_1^z\}} \mathbb{1}_{\{n_2^{z} \geqslant \xi_2^z\}} \mathbb{1}_{\{n_1^{z+1} \geqslant \xi_1^{z+1}\}} \mathbb{1}_{\{n_2^{z+1} \geqslant \xi_2^{z+1}\}}\right]$$

$$\times \prod_{x\notin\{z,z+1\}} \prod_{\gamma=1}^{2} \mathbb{1}_{\{n_\gamma^z \geqslant \xi_\gamma^z\}}$$
$$= (L^S_{z,z+1} D(\boldsymbol{n},\cdot))(\boldsymbol{\xi}) \,.$$

Similarly, one has

$$(L^{SM}_{z,z+1} D(\cdot,\boldsymbol{\xi}))(\boldsymbol{n})$$
$$= \Big[\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_1^z\}}\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_2^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_1^z\geqslant\xi_2^{z+1}\}} - \mathbb{1}_{\{n_1^z\geqslant\xi_1^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_2^z\}}\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\ \xi_2^{z+1}\}}\Big]$$
$$\times \prod_{x\notin\{z,z+1\}} \prod_{\gamma=1}^{2} \mathbb{1}_{\{n_\gamma^z \geqslant \xi_\gamma^z\}}$$
$$= \Big[\mathbb{1}_{\{n_1^z\geqslant\xi_2^{z+1}\}}\mathbb{1}_{\{n_2^z\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_2^z\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_1^z\}} - \mathbb{1}_{\{n_1^z\geqslant\xi_1^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_2^z\}}\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_2^{z+1}\}}\Big]$$
$$\times \prod_{x\notin\{z,z+1\}} \prod_{\gamma=1}^{2} \mathbb{1}_{\{n_\gamma^z \geqslant \xi_\gamma^z\}}$$
$$= L^{SM}_{z,z+1}(D(\boldsymbol{n},\cdot)(\boldsymbol{\xi}) \,.$$

For the generator that mutates at site z we have

$$\begin{aligned}(L^{LM}_{z,z+1} D(\cdot,\boldsymbol{\xi}))(\boldsymbol{n}) &= \Big[\mathbb{1}_{\{n_2^z\geqslant\xi_1^z\}}\mathbb{1}_{\{n_1^z\geqslant\xi_2^z\}} - \mathbb{1}_{\{n_1^z\geqslant\xi_1^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_2^z\}}\Big] \prod_{x\neq z}\prod_{\gamma=1}^{2}\mathbb{1}_{\{n_\gamma^z\geqslant\xi_\gamma^z\}} \\ &= \Big[\mathbb{1}_{\{n_1^z\geqslant\xi_2^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_1^z\}} - \mathbb{1}_{\{n_1^z\geqslant\xi_1^z\}}\mathbb{1}_{\{n_2^z\geqslant\xi_2^z\}}\Big] \prod_{x\neq z}\prod_{\gamma=1}^{2}\mathbb{1}_{\{n_\gamma^z\geqslant\xi_\gamma^z\}} \\ &= (L^{LM}_{z,z+1} D(\boldsymbol{n},\cdot))(\boldsymbol{\xi}) \,,\end{aligned}$$

and analogously, for the generator that mutates at site $z+1$, we find

$$\begin{aligned}(L^{RM}_{z,z+1} D(\cdot,\boldsymbol{\xi}))(\boldsymbol{n}) &= \Big[\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_2^{z+1}\}} - \mathbb{1}_{\{n_1^{z+1}\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_2^{z+1}\}}\Big] \prod_{x\neq z+1}\prod_{\gamma=1}^{2}\mathbb{1}_{\{n_\gamma^z\geqslant\xi_\gamma^z\}} \\ &= \Big[\mathbb{1}_{\{n_1^{z+1}\geqslant\xi_2^{z+1}\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_1^{z+1}\}} - \mathbb{1}_{\{n_1^{z+1}\geqslant\xi_1^{z+1}\}}\mathbb{1}_{\{n_2^{z+1}\geqslant\xi_2^{z+1}\}}\Big] \prod_{x\neq z+1}\prod_{\gamma=1}^{2}\mathbb{1}_{\{n_\gamma^z\geqslant\xi_\gamma^z\}} \\ &= (L^{RM}_{z,z+1} D(\boldsymbol{n},\cdot)(\boldsymbol{\xi})\end{aligned}$$

□

Remark 32 It is interesting to notice that to ensure the existence of a self-dual process for the interacting particle system introduce in Theorem 19 on the geometry of an infinite line $\mathbb{Z}$, the closure condition (10.4.8) is not enough. Considering the most general reaction-diffusion process satisfying closed equations, described by the generator $\mathcal{L}_{z,z+1}$ in (10.4.22), we have to further assume that

$$\sigma_{22} = \sigma_{11} \qquad \sigma_{21} = \sigma_{12} \qquad h = m. \tag{11.3.22}$$

Indeed, the duality relation (11.3.21) is equivalent to the following relation between matrices

$$(d\otimes d)^{-1}\,\mathcal{L}_{z,z+1}\,(d\otimes d) = \widetilde{L}^T_{z,z+1} \qquad \forall z\in\mathbb{Z} \tag{11.3.23}$$

where ^{T} denotes transposition and where

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{11.3.24}$$

In order to interpret $\widetilde{L}_{z,z++1}$ as a generator of a stochastic particle system, we have to impose that the out of diagonal elements are non-negative and the sum of the elements of each row is equal to zero. It is possible to show that this is equivalent to requiring that (11.3.22) holds. Moreover, if (11.3.22) is fulfilled, both the matrices $\mathcal{L}_{z,z+1}$ and $\widetilde{L}_{z,z+1}$ do coincide with the matrix associated to the generator $L_{z,z+1}$ given in (10.5.2), i.e. self-duality.

11.4 Scaling limits for reaction-diffusion process on an infinite line

In this section we investigate the hydrodynamic limit and the density fluctuations for the process $(\boldsymbol{n}(t))_{t\geqslant 0}$ defined in (11.2.4) by assuming that $\sigma_{12} = 0$. Indeed, as we have seen in Section 10.5, the crossing diffusivity contribution are not seen in the scaling limit. For the sake of simplicity we assume that the boundary interaction is neglected and the geometry becomes an infinite line $\mathbb{Z}$. The state space is given by $\Omega = \bigotimes_{z\in\mathbb{Z}} \Omega_x$, with Ω_x defined in (4.2.2).The generator is the one given in (11.2.4) without the stirring-mutation contribution, i.e. it reads

$$\mathcal{L}^r = \sum_{z=1}^{L-1} \mathcal{L}^r_{z,z+1} \tag{11.4.1}$$

where

$$\mathcal{L}^r_{z,z+1} = \mathcal{L}^s_{z,z+1} + \mathcal{L}^m_{z,z+1} \tag{11.4.2}$$

with $\mathcal{L}^s_{z,z+1}$ is the generator of the stirring process defined in (4.2.16) and $\mathcal{L}^m_{z,z+1}$ is the pure mutation generator defined in (11.2.9). This process admits a family of reversible measures that are introduced in (11.3.1).

The hydrodynamic limit

We state the hydrodynamic result. For arbitrary $\phi \in C_c^\infty(\mathbb{R})$ we introduce the density field

$$\mathcal{X}_\zeta^{K,t}(\phi) := \frac{1}{K}\sum_{z\in\mathbb{Z}} n_\zeta^z(tK^2)\phi\left(\frac{z}{K}\right) \qquad \forall\zeta \in \{1,\ldots,N\}. \tag{11.4.3}$$

We make the assumption that there exists an initial macroscopic profile for assigned sequence of initial measures.

Definition 18 *Let $\widehat{\rho}^{(\zeta)} : \mathbb{R} \to [0,\nu]$, with $\zeta \in \{1,\ldots,N\}$, be a continuous function called the initial macroscopic profile of species a. A sequence $(\mu_K)_{N\in\mathbb{N}}$ of measures on $\Omega_\mathbb{Z}$, is a sequence of compatible initial conditions if $\forall\zeta \in \{1,\ldots,N\}$, $\forall\delta > 0$:*

$$\lim_{N\to\infty} \mu_K \left(\left| \mathcal{X}_\zeta^{K,0}(\phi) - \int_\mathbb{R} \phi(u)\widehat{\rho}^{(\zeta)}(u)du \right| > \delta \right) = 0 \tag{11.4.4}$$

with arbitrary $\phi \in C_c^\infty(\mathbb{R})$.

Then we state the hydrodynamic result.

Theorem 21 *Let* $\widehat{\rho}^{(\zeta)} : \mathbb{R} \to [0, \nu]$*, with* $\zeta \in \{1, \ldots, N\}$*, be an initial macroscopic profile and let* $(\mu_K)_{K \in \mathbb{N}}$ *be a sequence of compatible initial measures. Let* P_K *be the law of the process* $\left(\mathcal{X}_1^{K,t}(\phi), \ldots, \mathcal{X}_N^{K,t}(\phi)\right)$ *induced by* $(\mu_K)_{K \in \mathbb{N}}$*. Then,* $\forall T > 0$*,* $\delta > 0$*,* $\forall \zeta \in \{1, \ldots, N\}$ *and* $\forall \phi \in C_c^\infty(\mathbb{R})$

$$\lim_{K \to \infty} P_K \left(\sup_{t \in [0,T]} \left| \mathcal{X}_\zeta^{K,t}(\phi) - \int_{\mathbb{R}} \phi(u) \rho^{(\zeta)}(u,t) du \right| > \delta \right) = 0 \tag{11.4.5}$$

where $\rho^{(\zeta)}(u,t)$ *is a strong solution of the PDE*

$$\begin{cases} \partial_t \rho^{(\zeta)}(u,t) = \nu \Delta \rho^{(\zeta)}(u,t) + \widetilde{\Upsilon} \left(\sum_{\gamma=1 \,:\, \gamma \neq \zeta}^N \rho^{(\gamma)}(u,t) - \rho^{(\zeta)}(u,t) \right) & u \in \mathbb{R},\ t \in [0,T] \\ \rho^{(\zeta)}(u,0) = \widehat{\rho}^{(\zeta)}(u) \end{cases} \tag{11.4.6}$$

where $\widetilde{\Upsilon} \in (0, \infty)$*.*

Proof of Theorem 21: the generator of the process is given by (11.2.4), i.e. it is composed by the sum of $\mathcal{L}^s$ defined in (5.2.2) (and specified on a line with N species of particles) and $\mathcal{L}^m$ defined in (11.2.18). Therefore, here we only need to perform the computations for the second one. We diffusively scale the switching rate $\Upsilon = \frac{\widetilde{\Upsilon}}{K^2}$, then, the generator reads

$$\mathcal{L}^m f(\boldsymbol{n}) = \frac{\widetilde{\Upsilon}}{K^2} \sum_{z \in \mathbb{Z}} \sum_{\gamma, \delta = 1}^N n_\gamma^z \left[f(\boldsymbol{n} - \boldsymbol{\gamma}_\delta^z + \boldsymbol{\delta}_\delta^z) - f(\boldsymbol{n}) \right] \tag{11.4.7}$$

where $\Upsilon \in (0, +\infty)$
We compute the action of this generator on the density field (11.4.3)

$$\begin{aligned} \mathcal{L}^m \mathcal{X}_\zeta^{K,\cdot}(\phi) &= \frac{\widetilde{\Upsilon}}{K^3} \sum_{x \in \mathbb{Z}^d} \sum_{k,l=1}^N n_\gamma^z \left[\sum_{y \in \mathbb{Z}} \phi\left(\frac{y}{K}\right) \left((n_\zeta^y - \boldsymbol{\delta}_\gamma^z + \boldsymbol{\delta}_\delta^z) - n_\zeta^y \right) \right] \\ &= \frac{\Upsilon}{K^3} \sum_{z \in \mathbb{Z}} \left(\sum_{\gamma=1 \,:\, \gamma \neq \zeta}^N n_\gamma^z - n_\zeta^z \right) \phi\left(\frac{z}{K}\right) \\ &= \frac{\widetilde{\Upsilon}}{K^2} \left(\sum_{\gamma=1 \,:\, \gamma \neq \zeta}^N \mathcal{X}_\gamma^{K,\cdot}(\phi) - \mathcal{X}_\zeta^{K,\cdot}(\phi) \right). \end{aligned} \tag{11.4.8}$$

Then,

$$\int_0^t K^2 \mathcal{L}^m \mathcal{X}_\zeta^{K,s/K^2}(\phi) ds = \int_0^t \widetilde{\Upsilon} \left(\sum_{\gamma=1 \,:\, \gamma \neq a}^N \mathcal{X}_\gamma^{K,s/K^2}(\phi) - \mathcal{X}_\zeta^{K,s/K^2}(\phi) \right) ds. \tag{11.4.9}$$

Arguing as in the proof of the Theorem 14, we need to bound the quadratic variation. We explicitly compute

$$\begin{aligned}
&\mathcal{L}^m(\mathcal{X}_\zeta^{K,t}(\phi)\mathcal{X}_\zeta^{K,t}(\phi)) - 2\mathcal{X}_\zeta^{K,t}(\phi)\mathcal{L}^m(\mathcal{X}_\zeta^{K,t}(\phi))\\
=&\frac{\widetilde{\Upsilon}}{K^2}\sum_{x\in\mathbb{Z}}\sum_{\gamma,\delta=1}^{N} n_\gamma^z\left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n_\zeta^y-\boldsymbol{\delta}_\gamma^z+\boldsymbol{\delta}_\delta^z)-n_\zeta^y\right)\right]^2\\
=&\frac{\widetilde{\Upsilon}}{K^2}\sum_{z\in\mathbb{Z}}\sum_{\gamma=1}^{N} n_\gamma^z\phi^2\left(\frac{z}{K}\right)\\
\leqslant&\frac{C}{K^2}N
\end{aligned} \tag{11.4.10}$$

Arguing as in the proof of Theorem 14 we can show that

$$\begin{aligned}
\lim_{K\to\infty} P_K\Bigg(\sup_{t\in[0,T]}\Bigg|&\mathcal{X}_\zeta^{K,t}(\phi)-\mathcal{X}_\zeta^{K,0}(\phi)-\nu\int_0^t\mathcal{X}_\zeta^{K,s/K^2}(\Delta\phi)\,ds\\
&+\int_0^t\widetilde{\Upsilon}\left(\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\mathcal{X}_\gamma^{K,s/K^2}(\phi)-\mathcal{X}_\zeta^{K,s/K^2}(\phi)\right)ds\Bigg|>\delta\Bigg)=0.
\end{aligned} \tag{11.4.11}$$

The proof of tightness for the sequence of measure $(P_K)_{K\in\mathbb{N}}$ defined in Theorem (21) and the uniqueness of the limit point are standard and analogous to the ones of Theorem 14.

□

The density fluctuation

We consider the process $(\boldsymbol{n}_t)_{t\geqslant 0}$ initialized from the reversible measure Λ_{rev} defined in (11.3.1). The density fluctuation field for a species $\zeta\in\{1,\dots,N\}$ is an element of the space $(C_c^\infty(\mathbb{R}))^*$ defined, for any test function $\phi\in C_c^\infty(\mathbb{R})$, as

$$\mathcal{Y}_\zeta^{K,t}(\phi) := \frac{1}{\sqrt{K}}\sum_{z\in\mathbb{Z}}\phi\left(\frac{z}{K}\right)\left(n_\zeta^z(tK^2)-\nu p\right) \tag{11.4.12}$$

where $\nu p = \mathbb{E}_{\Lambda_{\text{rev}}}\left[n_\zeta^z\right]$. We call π_K the law of the random process $\left(\mathcal{Y}^{K,t}\right)_{t\geqslant 0} = \left(\left(\mathcal{Y}_1^{K,t},\dots,\mathcal{Y}_N^{K,t}\right)\right)_{t\geqslant 0}$ and $\mathbb{E}_{\pi_K}$ the expectation with respect to this law. The density fluctuation field (11.4.12) satisfies the convergence result stated in the following Theorem. We denote by

$$(C_c^\infty(\mathbb{R}))_N^* = \underbrace{(C_c^\infty(\mathbb{R}))^*\times\ldots\times(C_c^\infty(\mathbb{R}))^*}_{N\,\text{times}} \tag{11.4.13}$$

the dual space of $(C_c^\infty(\mathbb{R}))^N$.

Theorem 22 *There exists a unique* $\left(\mathcal{Y}^t\right)_{t\in[0,T]} = \left((\mathcal{Y}_1^t,\dots,\mathcal{Y}_N^t)\right)_{t\in[0,T]}$ *on the space* $C\left([0,T];(C_c^\infty(\mathbb{R}))_N^*\right)$ *with law* π *such that*

$$\pi_K\to\pi \qquad \textit{weakly for } K\to\infty. \tag{11.4.14}$$

Moreover, $\left(\mathcal{Y}^t\right)_{t\in[0,T]}$ *is a generalized stationary Ornstein-Uhlenbeck process solving, for every* $\zeta\in\{1,\dots,N\}$*, the following martingale problem:*

$$M_{\zeta,\phi}^t := \mathcal{Y}_\zeta^t(\phi)-\mathcal{Y}_\zeta^0(\phi)-\nu\int_0^t\mathcal{Y}_\zeta^s(\Delta\phi)ds-\Upsilon\int_0^t\left(\sum_{\gamma=1\,:\,\gamma\neq\zeta}^{N}\mathcal{Y}_\delta^s(\phi)-\mathcal{Y}_\zeta^s(\phi)\right)ds \tag{11.4.15}$$

is a martingale $\forall\phi \in C_c^\infty(\mathbb{R})$ *with respect to the natural filtration of* $(\mathcal{Y}_1^t, \dots, \mathcal{Y}_N^t)$ *with quadratic covariation*

$$\Big[M_{\zeta,\phi}, M_{\zeta',\phi}\Big]_t = -2t\nu^2p^2 \int_{\mathbb{R}} \nabla(\phi(u))^2 du - 2pt\nu\Upsilon \int_{\mathbb{R}} (\phi(u))^2 du \tag{11.4.16}$$

and quadratic variation

$$[M_{\zeta,\phi}]_t = 2t\nu^2 p(1-p) \int_{\mathbb{R}} \nabla(\phi(u))^2 du + Npt\nu\Upsilon \int_{\mathbb{R}} (\phi(u))^2 du. \tag{11.4.17}$$

Therefore, the limiting process of Theorem 22

$$(\mathcal{Y}^t)_{t\in[0,T]} = \big((\mathcal{Y}_1^t, \dots, \mathcal{Y}_N^t)\big)_{t\in[0,T]} \tag{11.4.18}$$

can be formally written as the solution of the distribution-valued SPDE

$$d\mathcal{Y}^t = \mathcal{A}\mathcal{Y}^t dt + \nu\sqrt{2\Sigma}\nabla dW^t + \sqrt{\nu\Upsilon}\sqrt{\mathcal{B}}d\mathcal{W} \tag{11.4.19}$$

where

$$(W^t)_{t\in[0,T]} = \big((W_1^t, \dots, W_N^t)\big)_{t\in[0,T]} \tag{11.4.20}$$

$$(\mathcal{W}^t)_{t\in[0,T]} = \big((\mathcal{W}_1^t, \dots, \mathcal{W}_N^t)\big)_{t\in[0,T]} \tag{11.4.21}$$

are two N-dimensional vectors of independent space-time white noises. The matrices read

$$\mathcal{A} = \begin{pmatrix} \nu\Delta - \Upsilon & \tilde{\Upsilon} & \dots & \tilde{\Upsilon} \\ \tilde{\Upsilon} & \nu\Delta - \tilde{\Upsilon} & \dots & \tilde{\Upsilon} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{\Upsilon} & \tilde{\Upsilon} & \dots & \nu\Delta - \tilde{\Upsilon} \end{pmatrix} \tag{11.4.22}$$

$$\Sigma = \begin{pmatrix} p(1-p) & -p^2 & \dots & -p^2 \\ -p^2 & p(1-p) & \dots & -p^2 \\ \vdots & \vdots & \ddots & \vdots \\ -p^2 & -p^2 & \dots & p(1-p) \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} Np & -2p & \dots & -2p \\ -2p & Np & \dots & -2p \\ \vdots & \vdots & \ddots & \vdots \\ -2p & -2p & \dots & Np \end{pmatrix}. \tag{11.4.23}$$

Proof of Theorem 22: the strategy is similar to the one used for Theorem 15. Therefore, we only report the computation of the quadratic covariation (via the Carré Du Champ operator denoted by $\Theta_{\gamma,\delta}^{\phi,t}$) of the Dynkin martingale associated to $(\boldsymbol{n}_t)_{t\geqslant 0}$

$$\begin{aligned} \Theta_{\gamma,\delta}^{\phi,t} &= (\mathcal{L}^s + \mathcal{L}^m)\left(\mathcal{Y}_\zeta^{K,t}(\phi)\mathcal{Y}_\delta^{K,t}(\phi)\right) \\ &\quad - \mathcal{Y}_\gamma^{K,t}(\phi)\left(\mathcal{L}^s + \mathcal{L}^m\right)\left(\mathcal{Y}_\delta^{K,t}(\phi)\right) - \mathcal{Y}_\delta^{K,t}(\phi)\left(\mathcal{L}^s + \mathcal{L}^m\right)\left(\mathcal{Y}_\gamma^{K,t}(\phi)\right) \\ &= \mathcal{L}^s(\mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{L}^s(\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\delta^{K,t}(\phi)\mathcal{L}^s(\mathcal{Y}_\gamma^{K,t}(\phi)) \\ &\quad + \mathcal{L}^m(\mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{L}^m(\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\delta^{K,t}(\phi)\mathcal{L}^m(\mathcal{Y}_\gamma^{K,t}(\phi)) \end{aligned} \tag{11.4.24}$$

introducing

$$\Gamma_{\gamma,\delta}^{\phi,t,\text{mutation}} := \mathcal{L}^m(\mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\gamma^{K,t}(\phi)\mathcal{L}^m(\mathcal{Y}_\delta^{K,t}(\phi)) - \mathcal{Y}_\delta^{K,t}(\phi)\mathcal{L}^m(\mathcal{Y}_\gamma^{K,t}(\phi)) \tag{11.4.25}$$

and recalling the definition of $\Gamma^{\phi,t}_{\gamma,\delta}$ written in (5.4.5) we have that the Carré Du Champ operator $\Theta^{\phi,t}_{\gamma,\delta}$ is the sum of the two Carré Du Champ associated to the generators $\mathcal{L}^s$ and $\mathcal{L}^m$ respectively, i.e.

$$\Theta^{\phi}_{\gamma,\delta} = \Gamma^{\phi,t}_{\gamma,\delta} + \Gamma^{\phi,t,\text{mutation}}_{\gamma,\delta}. \tag{11.4.26}$$

Therefore to perform the proof we only need to compute $\Gamma^{\phi,t,\text{mutation}}_{\gamma,\delta}$. We consider the case $\gamma \neq \delta$ (the case $\gamma = \delta$ is similar) and we compute explicitly

$$\begin{aligned} K^2\Gamma^{\phi,\text{mutation}}_{\gamma,\delta} &= \frac{\Upsilon}{K^3}\sum_{z\in\mathbb{Z}}\sum_{\gamma',\delta'=1}^{N} n^z_k \left[\sum_{y\in\mathbb{Z}}\phi\left(\frac{y}{K}\right)\left((n^y_\gamma - \boldsymbol{\delta}^z_{\delta'} + \boldsymbol{\delta}^z_{\gamma'}) - n^y_\gamma\right)\right] \\ &\times \left[\sum_{z\in\mathbb{Z}}\phi\left(\frac{z}{K}\right)\left((n^z_\delta - \boldsymbol{\delta}^z_{\delta'} + \boldsymbol{\delta}^z_{\gamma'}) - n^x_\delta\right)\right] \\ &= -\frac{\Upsilon}{K}\sum_{z\in\mathbb{Z}}\left(n^z_\zeta + n^z_\delta\right)\phi^2\left(\frac{z}{K}\right). \end{aligned}$$

As a consequence, the limit of the first and second moment are given by

$$\lim_{K\to\infty}\mathbb{E}_{\pi_K}\left[K^2\Gamma^{\phi,\text{mutation}}_{\gamma,\delta}\right] = -2p\nu\int_{\mathbb{R}}(\phi(u))^2du \tag{11.4.27}$$

and

$$\lim_{K\to\infty}\text{Var}_{\pi_K}\left(K^2\Gamma^{\phi,\text{mutation}}_{\gamma,\delta}\right) = 4p^2\nu^2\left(\int_{\mathbb{R}}(\phi(u))^2du\right)^2 \tag{11.4.28}$$

□

Part IV

Future perspective and outlook

Chapter 12

Duality and integrability for multi-species non-compact processes

12.1 Motivations

In this thesis we have studied in detail the multi-species stirring process. One feature of this model is the fact that each site can host maximally a fixed number of particles (called ν). The same feature is present also when, in Part III, the set-up has been extended to reaction diffusion processes. As a future perspective we aim to extend the analysis to more rich processes, where an unbounded number of particles can be hosted, then called *non-compact processes*. In the single species situations, some examples of such processes have been introduced in the past literature. For instance the symmetric inclusion process (SIP) (see Chapter 3) and the Harmonic process [26, 106, 107, 108] , that has been showed to be integrable. When at each vertex the considered quantity is continuous (for example energy), other processes have been introduced, for instance the Brownian energy process (BEP) [11], the Brownian momentum process (BMP) [11], the Kipnis-Marchioro-Presutti process (KMP) [50]. Recently, an integrable version of heat exchange model have been solved [57], on the basis of the results of [26]. Therefore, it is interesting to generalize to the multi-species situation the processes cited above, guessing that their richer structure could lead to new phenomena and more general results. What follows is part of a work in progress with Rouven Frassek and Cristian Giardinà with the aim of studying multi-species non-compact processes via duality and integrability techniques.
In Section 12.2 first study the simplest example of a boundary driven non-compact multi-species process: the independent walkers. Starting from the definition of the bulk process reported in [14], we add boundary interaction, allowing to put the system out-of-equilibrium. Then, using the N-species Heisenberg Lie algebra, we construct the Hamiltonian of the process and we prove absorbing duality. This absorbing duality maps the original multi-species IRW to a dual process with the same bulk dynamics but with absorbing boundaries, allowing to write closed formulas for the non-equilibrium steady state via explicit expression of the absorption probabilities. We notice that, due the lack of interaction, this task is directly carried out by duality alone.
In Section 12.3, we define a multi-species version of the symmetric harmonic process. We show that this process can be retrieved from the asymmetric process defined in [45], by taking a proper limit. Then, we show that the choice of the specific boundary driving is done with the goal of obtaining, at equilibrium, a reversible measure that is the same of the closed process. Usually, this is the correct choice of the boundary to prove absorbing duality via a Lie algebraic approach.

Finally, in Section 12.4, we introduce the *multi-species SIP(2k)*, as the multi-colours analogue of the SIP(2k) (see Chapter 3). After having introduced the dynamics, we prove that the reversible measure for the boundary driven chain at equilibrium is the same (up to a different choice of the parameters) of the multi-species harmonic process. Moreover, this reversible measure is also the multi-variate version of the one of the single species SIP(2k). Again, this choice of the boundary driving has been made with the aim of proving absorbing duality via a Lie algebraic approach.

12.2 The multi-species independent random walk

The multi-species independent random walk (see [14]) is one of the simplest multi-species interacting particle system since, in particular, interaction is null. The process consists in N random walkers (one for each species) performing their independent dynamics on a certain graph. Here, for the sake of simplicity, we consider the nearest neighbour chain of length L with two boundary reservoirs connected with site 1 and L.
We denote the configuration of the process by $\eta = \left(\eta^1, ..., \eta^L\right)$ where, for each $x = 1, ..., L$ we have the vector $\eta^x = (\eta_1^x, \eta_2^x, ..., \eta_N^x)$, where $\eta_a^x \in \mathbb{N}_0$ represents the number of particles of the species $a \in \{1, \ldots, N\}$. Each site can host an infinite number of particles and the state space reads

$$\Omega = (\mathbb{N}_0^N)^L \tag{12.2.1}$$

The dynamics is ruled by the generator

$$\mathcal{L} = \sum_{a=1}^{N} \mathcal{L}_{left}^a + \sum_{x=1}^{L-1} \sum_{a=1}^{N} \mathcal{L}_{x,x+1}^a + \sum_{a=1}^{N} \mathcal{L}_{right}^a \tag{12.2.2}$$

where, for all $a \in \{1, \ldots, N\}$, we have:

- in the bulk

$$\mathcal{L}_{x,x+1}^a f(\eta) = \eta_a^x \left[f(\eta - \delta_a^x + \delta_a^{x+1}) - f(\eta)\right] + \eta_a^{x+1} \left[f(\eta - \delta_a^{x+1} + \delta_a^x) - f(\eta)\right] \tag{12.2.3}$$

- on the left boundary generator reads

$$\mathcal{L}_{left}^a = \alpha_a \left[f(\eta + \delta_a^1) - f(\eta)\right] + \gamma_a \eta_a^1 \left[f(\eta - \delta_a^1) - f(\eta)\right] \tag{12.2.4}$$

- on the right boundary generator

$$\mathcal{L}_{left}^a = \delta_a \left[f(\eta + \delta_a^L) - f(\eta)\right] + \beta_a \eta_a^L \left[f(\eta - \delta_a^L) - f(\eta)\right] \tag{12.2.5}$$

We observe that the generator is written as a sum over sites and over species of linear operators, making the different species of particles independent. This is not the case in the multi-species stirring process where an interaction among different species of particles does exist.

Reversible measure (equilibrium)

We introduce the boundary density for the multi-species IRW as

$$\rho_a^{\text{left}} = \frac{\alpha_a}{\gamma_a}, \qquad \rho_a^{\text{right}} = \frac{\delta_a}{\beta_a} \qquad \forall a \in \{1, \ldots, N\} \tag{12.2.6}$$

The generator (12.2.2) admits a reversible measure when all boundary densities are the same, i.e. when

$$\rho_a^{\text{left}} = \rho_a^{\text{right}} = \lambda \qquad \forall a \in \{1, \ldots, N\}\,. \tag{12.2.7}$$

This reversible measure is the product over sites and over species of probability mass functions distributed as Poisson(λ) with λ defined in (12.2.7), that is

$$\mu_{\text{rev}} = \bigotimes_{x=1}^{L} \bigotimes_{a=1}^{N} \mu_{\text{rev}}^{x,a} \tag{12.2.8}$$

with

$$\mu_{\text{rev}}^{x,a}(\eta_a^x) = \frac{\lambda^{\eta_a^x}}{\eta_a^x!} e^{-\lambda} \tag{12.2.9}$$

12.2.1 The Lie algebraic description

Inspired by the ideas of Chapter 4, in order to prove duality for the multi-species IRW, we provide a Lie algebraic description of the Hamiltonian (the transposed of the generator).
We describe this Hamiltonian with the N-species Heisenberg algebra (see for instance [14]). It is defined by the basis elements $\mathbf{a}_a^+, \mathbf{a}_a$ with $a \in \{1, \ldots, N\}$. The commutation relations $\forall a, b \in \{1, \ldots, N\}$:

$$\left[\mathbf{a}_a^+, \mathbf{a}_b\right] = -\delta_{a,b}\mathbb{1} \tag{12.2.10}$$

where $\mathbb{1}$ is the identity operator. The configuration of the multi-species IRW process with generator (12.2.2) can be described by the state vector

$$|\boldsymbol{\eta}\rangle = \bigotimes_{x \in V} |\eta_x\rangle \tag{12.2.11}$$

where

$$|\eta_x\rangle = |\eta_1^x, \ldots, \eta_N^x\rangle \tag{12.2.12}$$

With this vector notation, the state space reads

$$\Omega := \{|n_1, \ldots, n_N\rangle \,:\, n_1, \ldots, n_N \in \mathbb{N}_0\} \tag{12.2.13}$$

The elements of Lie algebra act on the configuration of each site as

$$\begin{aligned} \mathbf{a}_a|\eta_1, \ldots, \eta_a, \ldots, \eta_N\rangle &= \eta_a|\eta_1, \ldots, \eta_a - 1, \ldots, \eta_N\rangle \\ \mathbf{a}_a^+|\eta_1, \ldots, \eta_k, \ldots, \eta_N\rangle &= |\eta_1, \ldots, \eta_k + 1, \ldots, \eta_N\rangle \end{aligned} \tag{12.2.14}$$

Using the basis elements of the N-species Heisenberg Lie algebra we define the Hamiltonian of the multi-species IRW as

$$H = \sum_{a=1}^{N} H_{\text{left}}^a + \sum_{x=1}^{L} \sum_{a=1}^{N} H_{x,x+1}^a + \sum_{a=1}^{N} H_{\text{right}}^a \tag{12.2.15}$$

where

$$H_{x,x+1}^a = (\mathbf{a}_a^\dagger \otimes \mathbf{a}_a - \mathbb{1} \otimes \mathbf{a}_a^\dagger \mathbf{a}_a) + (\mathbf{a}_a \otimes \mathbf{a}_a^\dagger - \mathbf{a}_a^\dagger \mathbf{a}_a \otimes \mathbb{1}) \tag{12.2.16}$$

and where

$$H_{\text{left}}^a = \alpha_a \left(\mathbf{a}_a^\dagger - \mathbb{1}\right) + \gamma_a \left(\mathbf{a}_a - \mathbf{a}_a^\dagger \mathbf{a}_a\right) \tag{12.2.17}$$

$$H_{\text{right}}^a = \delta_a \left(\mathbf{a}_a^\dagger - \mathbb{1}\right) + \beta_a \left(\mathbf{a}_a - \mathbf{a}_a^\dagger \mathbf{a}_a\right) \tag{12.2.18}$$

with H_{left} acting non-trivially only at site 1 and H_{right} acting non-trivially only at site L.

12.2.2 Duality

In this section we state and prove the absorbing duality for the boundary driven multi-species IRW. The technique used for the proof, is an adaptation of the one of Chapter 6 for the stirring process. The dual process is denoted by $(\boldsymbol{\xi}(t))_{t\geqslant 0}$, where

$$\boldsymbol{\xi} = (\xi_1^0, \ldots, \xi_N^0) \otimes \left(\bigotimes_{x=1}^{L} (\xi_1^x, \ldots, \xi_N^x) \right) \otimes (\xi_1^{L+1}, \ldots, \xi_N^{L+1}) \tag{12.2.19}$$

Here, two extra-site 0 and $L+1$ have been attached to 1 an L respectively. With ξ_a^x we denote the number of dual particles of species a at site x and with ξ_a^0 (ξ_a^{L+1}) we denote the number of dual particles at the extra-site 0 ($L+1$). The dual state space reads

$$\widetilde{\Omega} := \mathbb{N}_0^N \times \Omega \times \mathbb{N}_0^N = (\mathbb{N}_0^N)^{L+2} \tag{12.2.20}$$

We have the following proposition, establishing the duality result.

Proposition 19 *(Duality for multi-species IRW) The boundary driven multi-species IRW $(\boldsymbol{\eta}(t))_{t\geqslant 0}$ acting on the state space Ω defined in* (12.2.1) *with generator* (12.2.2) *is dual to the process $(\boldsymbol{\xi}(t))_{t\geqslant 0}$ acting on the dual state space $\widetilde{\Omega}$ defined in* (12.2.20) *and with dual generator*

$$\widetilde{\mathcal{L}} = \sum_{a=1}^{N} \widetilde{\mathcal{L}}_{left}^a + \sum_{x=1}^{L} \sum_{a=1}^{N} \mathcal{L}_{x,x+1}^a + \sum_{a=1}^{N} \widetilde{\mathcal{L}}_{right}^a \tag{12.2.21}$$

where $\mathcal{L}_{x,x+1}$ is the generator (12.2.3) *and where*

$$\widetilde{\mathcal{L}}_{\text{left}}^a = \gamma_a \xi_a^1 \left[f(\xi - \delta_a^1 + \delta_a^0) - f(\xi) \right] \tag{12.2.22}$$

$$\widetilde{\mathcal{L}}_{\text{right}}^a = \beta_a \xi_a^L \left[f(\xi - \delta_a^L + \delta_a^{L+1}) - f(\xi) \right] \tag{12.2.23}$$

The duality function is given by

$$D(\boldsymbol{\eta}, \boldsymbol{\xi}) = \left(\prod_{a=1}^{N} \left(\frac{\alpha_a}{\gamma_a} \right)^{\xi_a^0} \right) \left(\prod_{x=1}^{L} \prod_{a=1}^{N} \frac{\eta_a^x!}{(\eta_a^x - \xi_a^x)!} \right) \left(\prod_{a=1}^{N} \left(\frac{\delta_a}{\beta_a} \right)^{\xi_a^{L+1}} \right) \tag{12.2.24}$$

We observe that the dual boundary generators defined in (12.2.22) and in (12.2.23) are purely absorbing and, in the long time limit they void the dual chain.

Proof of Proposition 19: this proof is an adaptation of the proof of Theorem 17 done in Chapter 6 for the multi-species stirring process. Therefore, we only report here the main steps. In a vector notation, we indicate the dual configuration as

$$|\boldsymbol{\xi}\rangle = |\xi_1^0, \ldots, \xi_N^0\rangle \otimes \bigotimes_{x=1}^{L} |\xi_1^x, \ldots, \xi_N^x\rangle \otimes |\xi_1^{L+1}, \ldots, \xi_N^{L+1}\rangle \tag{12.2.25}$$

the dual Hamiltonian as

$$\widetilde{H} = \sum_{a=1}^{N} \widetilde{H}_{\text{left}}^a + \sum_{x=1}^{L} \sum_{a=1}^{N} H_{x,x+1}^a + \sum_{a=1}^{N} \widetilde{H}_{\text{right}}^a \tag{12.2.26}$$

where $H_{x,x+1}^a$ is the operator defined in (12.2.16) and where

$$\widetilde{H}_{\text{left}}^a = \left((\mathbf{a}_a^0)^\dagger \mathbf{a}_a^1 - (\mathbf{a}_a^1)^\dagger \mathbf{a}_a^1 \right) \tag{12.2.27}$$

$$\widetilde{H}^a_{\text{right}} = \left((\mathbf{a}_a^{L+1})^\dagger \mathbf{a}_a^L - (\mathbf{a}_a^L)^\dagger \mathbf{a}_a^1\right) \tag{12.2.28}$$

where we denoted by $\mathbf{a}_a^x$ and $(\mathbf{a}_a^x)^+$ the operators $\mathbf{a}_a$ and $(\mathbf{a}_a)^+$ acting on site x. The duality matrix reads

$$\begin{aligned} D = &\left(\sum_{\xi_a^0=0}^{\infty} \left(\frac{\alpha_a^x}{\gamma_a^x}\right)^{\xi_a^{L+1}} \langle \xi_1^0, \ldots, \xi_N^0|\right) \\ &\otimes \left(\prod_{x=1}^{L}\prod_{a=1}^{N} R_{x,k} \exp\left((\mathbf{a}_k^x)^\dagger\right)\right) \\ &\otimes \left(\sum_{\xi_a^{L+1}} \left(\frac{\delta_k^x}{\beta_a^x}\right)^{\xi_a^{L+1}} \langle \xi_1^{L+1}, \ldots, \xi_N^{L+1}|\right) \end{aligned} \tag{12.2.29}$$

We introduce the matrix for all $a \in \{1, \ldots, N\}$ and for all $x \in \{1, \ldots, L\}$

$$R_{a,x} = \eta_a^x! |\eta_1^x, \ldots, \eta_N^x\rangle\langle \eta_1^x, \ldots, \eta_N^x| \tag{12.2.30}$$

that satisfies the relation

$$(\mathbf{a}_a^\dagger)^T = R_{a,x}\mathbf{a}_a R_{a,x}^{-1} \tag{12.2.31}$$

By direct computations one can show that for all $a \in \{1, \ldots, N\}$

$$\Delta(\mathbf{a}_a^\dagger) := \mathbf{a}_a^\dagger \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{a}_a^\dagger \tag{12.2.32}$$

is a symmetry for the Hamiltonian $H^a_{x,x+1}$ defined in (12.2.16), i.e.

$$\left[H^a_{x,x+1}, \Delta_{x,x+1}(\mathbf{a}_a^+)\right] = 0 \tag{12.2.33}$$

where $\Delta_{x,x+1}(\mathbf{a}_a^+)$ is (12.2.32) on the bond $(x, x+1)$. Then, for all $x \in \{1, \ldots, L-1\}$ and for all $a \in \{1, \ldots, N\}$, it follows that

$$(H^a_{x,x+1})^T D = D H^a_{x,x+1} \tag{12.2.34}$$

Moreover, by using the Hadamard formula (6.2.39), adapted for the Heisenberg Lie algebra, we have that

$$e^{-a_k^\dagger} a_k^\dagger e^{a_k^\dagger} = a_k^\dagger \tag{12.2.35}$$

$$e^{-a_k^\dagger} a_k e^{a_k^\dagger} = a_k + \mathbb{1} \tag{12.2.36}$$

$$e^{-a_k^\dagger} a_k^\dagger a_k e^{a_k^\dagger} = a_k^\dagger a_k + a_k^\dagger \tag{12.2.37}$$

Then, it follows that

$$\exp\left(-\mathbf{a}_a^+\right)\left(\alpha_a\left(\mathbf{a}_a^\dagger - \mathbb{1}\right) + \gamma_a\left(\mathbf{a}_a - \mathbf{a}_a^\dagger\mathbf{a}_a\right)\right)\exp\left(\mathbf{a}_a^+\right) = \left(\gamma_a\mathbf{a}_a - \alpha_a\mathbf{a}_a^+\mathbf{a}_a\right) \tag{12.2.38}$$

Using the above equation, one can prove that for all $a \in \{1, \ldots, N\}$

$$(H^a_{\text{left}})^T D = D(\widetilde{H}^a_{\text{left}}) \tag{12.2.39}$$

□

12.2.3 Non-equilibrium steady distribution

Here we derive the non-equilibrium steady distribution of the boundary driven multi-species independent random walk. Because of the lack of interaction, this measure can be explicitly written just by using absorbing duality.

Proposition 20 *(Non-equilibrium measure for the multi-species IRW) The stationary measure* μ_{NESS} *of the multi-species boundary driven IRW with generator* $\mathcal{L}$ *defined in* (12.2.2) *is the product measure*

$$\mu_{\text{NESS}} = \prod_{x=1}^{L}\prod_{a=1}^{N} \mu_{\text{NESS}}^{x,a} \tag{12.2.40}$$

where

$$\mu_{\text{NESS}}^{a,x} = \textit{Poisson}(\lambda_{x,a})\,. \tag{12.2.41}$$

For all $a \in \{1,\ldots,N\}$ *and* $\forall x \in \{1,\ldots,L\}$*, the parameter of the inhomogeneous Poisson distribution reads*

$$\lambda_{x,a} := \frac{\rho_a^{\text{left}}\left(L+\frac{1}{\beta_a}-x\right)+\rho_a^{\text{right}}\left(x-1+\frac{1}{\gamma_a}\right)}{\left(L+\frac{1}{\beta_a}+\frac{1}{\gamma_a}-1\right)} \tag{12.2.42}$$

where the boundary densities ρ_a^{right} *and* ρ_a^{right} *are the ones defined in* (12.2.6)*.*

Proof of proposition 20: by the definition of factorial moments of the Poisson distribution, it is enough to show that

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[\prod_{x=1}^{L}\prod_{a=1}^{N}\frac{\eta_a^x!}{(\eta_a^x-\xi_a^x)!}\right] = \prod_{x=1}^{L}\prod_{a=1}^{L}\lambda_{a,x}^{\xi_a^x} \tag{12.2.43}$$

The right hand side of the equation above is the expectation with respect to the μ_{NESS} of the duality function (12.2.24), therefore using absorbing duality and arguing as in Section 6.3 of Chapter 6, we have that

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{\eta},\boldsymbol{\xi})\right] = \sum_{t_1=0}^{|\xi_1|}\cdots\sum_{t_N=0}^{|\xi_N|}\prod_{a=1}^{N}(\rho_a^{\text{left}})^{t_a}(\rho_a^{\text{right}})^{|\xi_a|-t_a}\mathcal{P}_{\boldsymbol{\xi}}(t_1,\ldots,t_N) \tag{12.2.44}$$

where we denoted by $|\xi_a| = \sum_{x=1}^{L}\xi_a^x$ the number of particle of type a in the whole chain and where we introduce the absorption probabilities

$$\mathcal{P}_{\boldsymbol{\xi}}(t_1,\ldots,t_N) = \mathbb{P}\left(\boldsymbol{\xi}(\infty) = \sum_{a=1}^{N}\left(t_a\delta_a^0+(|\xi_a|-t_a)\delta_a^{L+1}\right)\Big|\boldsymbol{\xi}(0)=\boldsymbol{\xi}\right) \tag{12.2.45}$$

namely the probability of having t_a particles of species a absorbed at site 0 and $|\xi_a|-t_a$ particles of species a absorbed at $L+1$ when $t\to\infty$. Using the independence of dual particles of each species, we introduce $m_a^x = 0,\ldots,\xi_a^x$ and we write

$$\begin{aligned}&\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{\eta},\boldsymbol{\xi})\right]\\ &=\prod_{x=1}^{L}\left(\sum_{m_1^x=0}^{\xi_1^x}\cdots\sum_{m_N^x=0}^{\xi_N^x}\prod_{a=1}^{N}(\rho_a^{\text{left}})^{m_a^x}(\rho_a^{\text{right}})^{\xi_a^x-m_a^x}\binom{\xi_a^x}{m_a^x}p_{x,a}(0)^{m_a^x}(1-p_{a,x}(0))^{\xi_a^x-m_a^x}\right)\end{aligned} \tag{12.2.46}$$

where $p_{x,a}(0)$ is the probability of a random walk of species a starting form site x of being absorbed in the extra-site 0. By standard computation we have that

$$p_{x,a}(0) = \frac{L + \frac{1}{\beta_a} - x}{L + \frac{1}{\beta_a} + \frac{1}{\gamma_a} - 1} \tag{12.2.47}$$

Using the binomial theorem in (12.2.46) we have that

$$\mathbb{E}_{\mu_{\text{NESS}}}\left[D(\boldsymbol{\eta}, \boldsymbol{\xi})\right] = \prod_{x=1}^{L} \prod_{a=1}^{N} \left(\rho_a^{\text{left}} p_{a,x}(0) + \rho_a^{\text{right}}(1 - p_{a,x}(0))\right)^{\xi_a^x} \tag{12.2.48}$$

By substituting (12.2.47), the result follows.

□

12.3 The multi-species harmonic process

In this section we introduce the multi-species harmonic process. For the sake of simplicity we define it in the first non-trivial case of $N = 2$ species and on the geometry of a chain of length L with two reservoirs attached to the site 1 and L respectively. We assume that the interaction is of nearest neighbour type. We denote the process by $(\boldsymbol{m}(t))_{t \geqslant 0}$ and a generic configuration by $\boldsymbol{m} = (m^1, \ldots, m^L)$ where, for each site $x \in \{1, \ldots, L\}$ we associate a vector $m^x = (m_1^x, m_2^x)$. Here m_1^x and m_2^x denote the number of particles of species 1 and 2, respectively, at site x. The configuration space reads

$$\Omega = \bigotimes_{x=1}^{L} \Omega_x \tag{12.3.1}$$

where

$$\Omega_x = \left\{(m_1^x, m_2^x) \in \mathbb{N}_0^2\right\} \tag{12.3.2}$$

We observe that each site can host an unbounded number of particles of any species. For all $s \in \mathbb{N}/2$, the generator is given by

$$\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}_{\text{right}} \tag{12.3.3}$$

where:

- *bulk generator*: we have that

$$\begin{aligned} \mathcal{L}_{x,x+1} f(\boldsymbol{m}) = \sum_{k_1=0}^{m_1^x} \sum_{k_2=0}^{m_2^x} & \mathbb{1}_{\{k_1+k_2>0\}} \varphi_s(m_1^x, m_2^x, k_1, k_2) \\ & \times \left[f(\boldsymbol{m} - k_1\delta_1^x - k_2\delta_2^x + k_1\delta_1^{x+1} + k_2\delta_2^{x+1}) - f(\boldsymbol{m})\right] \end{aligned} \tag{12.3.4}$$

  where we introduce the transition rate

$$\varphi_s(m_1^x, m_2^x, k_1, k_2) = \frac{\Gamma(k_1 + k_2)}{\Gamma(k_1 + 1)\Gamma(k_2 + 1)} \frac{\Gamma(m_1^x + m_2^x - k_1 - k_2 + 2s)}{\Gamma(m_1^x + m_2^x + 2s)} \prod_{a=1}^{2} \frac{\Gamma(m_a^x + 1)}{\Gamma(m_a^x - k_a + 1)} \tag{12.3.5}$$

In words, the dynamics on the bulk consists in removing k_1 particles of species 1 and k_2 particles of species 2 from site x and put these particles at site $x+1$. The rate of this transition depends on the number of particles present at site x before the transition happens m_1^x, m_2^x and on the number of particles k_1, k_2 moved from x to $x+1$ in the transition.

- *Left boundary generator:* we have that

$$\begin{aligned}\mathcal{L}_{\text{left}} f(m) = &\sum_{k_1=0}^{m_1^1} \sum_{k_2=0}^{m_2^1} \mathbb{1}_{\{k_1+k_2>0\}} \varphi_s(m_1^1, m_2^1, k_1, k_2)[f(m - k_1\delta_1^1 - k_2\delta_2^1) - f(m)] \\ &+ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \mathbb{1}_{\{k_1+k_2>0\}} \frac{\beta_1^{k_1}(1)}{k_1!} \frac{\beta_2^{k_2}(1)}{k_2!} \Gamma(k_1+k_2)[f(m + k_1\delta_1^1 + k_2\delta_2^1) - f(m)]\end{aligned} \tag{12.3.6}$$

In words, the left boundary generator allows two transitions. On one hand, it removes k_1 particles of species 1 and k_2 particles of species 2 from site 1 with rate $\varphi_s(m_1^x, m_2^x, k_1, k_2)$ of equation 12.3.5. On the other hand, it injects k_1 particles of species 1 and k_2 particles of species 2 from site 1. For this transition, the rate depends on the number of injected particles k_1, k_2 and also on the boundary parameters $\beta_1(1)$ and $\beta_2(1)$.

- *Right boundary*: here the generator $\mathcal{L}_{\text{right}}$ is similar to $\mathcal{L}_{\text{left}}$, but it acts on the site L and it has boundary parameters $\beta_1(L), \beta_2(L)$.

Comparison with the single species harmonic process

The process with generator (12.3.3) seems to be the natural generalization of the single species harmonic process defined in [26]. Indeed, when we take $N = 1$ we only have one species of particles (and then one species of k's and one species of m^x's) and we obtain, in the bulk,

$$\varphi_s(k, m) = \frac{1}{k} \frac{\Gamma(m+1)\Gamma(m-k+2s)}{\Gamma(m+2s)\Gamma(m-k+1)}, \tag{12.3.7}$$

on the left boundary,

$$\frac{\beta(1)^k}{k!} \Gamma(k) = \frac{\beta(1)}{k} \tag{12.3.8}$$

and similarly on the right boundary

$$\frac{\beta(L)^k}{k!} \Gamma(k) = \frac{\beta(L)}{k} \tag{12.3.9}$$

Equations (12.3.7), (12.3.8) and (12.3.9) give the rates of the single species harmonic process introduced in [26].

12.3.1 The bulk as the limit of an asymmetric process

The transition rates introduced in equation (12.3.5) can be obtained as a proper limit of the transitions rates of the asymmetric process defined in [45], where we have the following rates:

- *right jump*: Right jump: $\forall i = 1,\dots,n$, k_i particles among m_i jump to the right with the following rate:

$$\varphi_{\mathrm{R}}^{\nu,q}(m_1,m_2,k_1,k_2) = \frac{q^{\sum_{1\leqslant a<b\leqslant 2}(m_a-k_a)k_b}\,\nu^{k_1+k_2-1}\,(q,q)_{k_1+k_2-1}}{(\nu q^{m_1+m_2-k_1-k_2},q)_{k_1+k_2}}\prod_{a=1}^{2}\binom{m_a}{k_a}_q \tag{12.3.10}$$

- *left jump*:

$$\varphi_{\mathrm{L}}^{\nu,q}(m_1,m_2,k_1,k_2) = \frac{q^{\sum_{1\leqslant i<j\leqslant 2}(m_i-k_i)k_j}\,(q,q)_{k_1+k_2-1}}{(\nu q^{m_1+m_2-k_1-k_2},q)_{k_{1+k_2}}}\prod_{a=1}^{2}\binom{m_a}{k_a}_q \tag{12.3.11}$$

Here we have introduce the following notation:

1. *q-Pochammer*:

$$(z,q)_n = \prod_{j=1}^{n}\left(1-zq^{j-1}\right) \tag{12.3.12}$$

with the properties

$$(zq^{n-k},q)_k = \frac{(z,q)_n}{(z,q)_{n-k}} \tag{12.3.13}$$

$$\lim_{q\to 1}\frac{(q^{2s},q)_n}{(1-q)^n} = \frac{\Gamma(2s+n)}{\Gamma(2s)} \tag{12.3.14}$$

2. *q-binomial*:

$$\binom{m}{k}_q = \frac{(q,q)_n}{(q,q)_k(q,q)_{n-k}} \tag{12.3.15}$$

3. *q-number*:

$$[k]_q = \frac{1-q}{1-q^k} \qquad \text{with the property} \qquad \lim_{q\to 1}[k]_q = k \tag{12.3.16}$$

We now prove the connection between the symmetric and asymmetric process.

Lemma 8 *Let $\varphi_s(\cdot)$ be the transition rate* (12.3.5) *and let $\varphi_{\mathrm{L}}^{\nu,q}(\cdot)$ and $\varphi_{\mathrm{R}}^{\nu,q}(\cdot)$ be the transitions rates* (12.3.10) *and* (12.3.11) *respectively. Then, for all $m_1,m_2,k_1,k_2\in\mathbb{N}$ and for all $s\in\frac{\mathbb{N}}{2}$, we have*

$$\varphi_s(m_1,m_2,k_1,k_2) = \lim_{q\to 1}(1-q)\varphi_{\mathrm{R}}^{q^{2s},q}(m_1,m_2,k_1,k_2) = \lim_{q\to 1}(1-q)\varphi_{\mathrm{L}}^{q^{2s},q}(m_1,m_2,k_1,k_2) \tag{12.3.17}$$

Remark 33 *In the single species case it has been proved in [107] that the harmonic process with $s=1/2$ can be obtained as a proper limit of the asymmetric process defined in [109].*

Proof of Lemma 8: the proof is done by direct computations. We only show the limit for $\varphi_{\mathrm{R}}^{q^{2s},q}(\cdot)$, since the one for $\varphi_{\mathrm{L}}^{q^{2s},q}(\cdot)$ is analogous. By the definition of q-Pochhammer symbol we have that, for all $k_1,k_2,m_1,m_2,\nu\in\mathbb{N}$,

$$(\nu q^{m_1+m_2-k_1-k_2},q)_{k_1+k_2} = \frac{(\nu,q)_{m_1+m_2-k_1-k_2}}{(\nu,q)_{m_1+m_2}} \tag{12.3.18}$$

It follows that

$$\begin{aligned}
\varphi_{\mathrm{R}}^{\nu,q}(m_1,m_2,k_1,k_2) &= \frac{q^{(m_1-k_1)k_2}\nu^{k_1+k_2-1}(q,q)_{k_1+k_2-1}}{(\nu q^{m_1+m_2-k_1-k_2},q)_{k_1+k_2}}\binom{m_1}{k_1}_q\binom{m_2}{k_2}_q \\
&= q^{(m_1-k_1)k_2}\nu^{k_1+k_2-1}(q,q)_{k_1+k_2-1}\frac{(\nu,q)_{m_1+m_2-k_1-k_2}}{(\nu,q)_{m_1+m_2}} \\
&\times\frac{(q,q)_{m_1}}{(q,q)_{k_1}(q,q)_{m_1-k_1}}\frac{(q,q)_{m_2}}{(q,q)_{k_2}(q,q)_{m_2-k_2}} \\
&= q^{(m_1-k_1)k_2}\nu^{k_1+k_2-1}\frac{(q,q)_{k_1+k_2-1}}{(q,q)_{k_1}(q,q)_{k_2}} \\
&\times\frac{(\nu,q)_{m_1+m_2-k_1-k_2}}{(q,q)_{m_1-k_1}(q,q)_{m_2-k_2}}\frac{(q,q)_{m_1}(q,q)_{m_2}}{(\nu,q)_{m_1+m_2}}
\end{aligned} \tag{12.3.19}$$

We replace ν by q^{2s} and we have that

$$\begin{aligned}
\varphi_{\mathrm{R}}^{q^{2s},q}(m_1,m_2,k_1,k_2) =& q^{(m_1-k_1)k_2+2s(k_1+k_2-1)}\frac{(q,q)_{k_1+k_2-1}}{(q,q)_{k_1}(q,q)_{k_2}} \\
&\times\frac{(q^{2s},q)_{m_1+m_2-k_1-k_2}}{(q,q)_{m_1-k_1}(q,q)_{m_2-k_2}}\frac{(q,q)_{m_1}(q,q)_{m_2}}{(q^{2s},q)_{m_1+m_2}}
\end{aligned} \tag{12.3.20}$$

We multiply the right-hand-side by $\frac{(1-q)^{k_1+k_2-1}}{(1-q)^{k_1+k_2-1}}$, $\frac{(1-q)^{m_1+m_2-k_1-k_2}}{(1-q)^{m_1+m_2-k_1-k_2}}$ and $\frac{(1-q)^{m_1+m_2}}{(1-q)^{m_1+m_2}}$ to get

$$\begin{aligned}
\varphi_{\mathrm{R}}^{q^{2s},q}(m_1,m_2,k_1,k_2) =& q^{(m_1-k_1)k_2+2s(k_1+k_2-1)}\frac{(q,q)_{k_1+k_2-1}}{(1-q)^{k_1+k_2-1}}\frac{(1-q)^{k_1}}{(q,q)_{k_1}}\frac{(1-q)^{k_2}}{(q,q)_{k_2}} \\
&\times(1-q)^{-1}\frac{(q^{2s},q)_{m_1+m_2-k_1-k_2}}{(1-q)^{m_1+m_2-k_1-k_2}}\frac{(1-q)^{m_1-k_1}}{(q,q)_{m_1-k_1}}\frac{(1-q)^{m_2-k_2}}{(q,q)_{m_2-k_2}} \\
&\times\frac{(1-q)^{m_1+m_2}}{(q,q)_{m_1+m_2}}\frac{(q,q)_{m_1}}{(1-q)^{m_1}}\frac{(q,q)_{m_2}}{(1-q)^{m_2}}
\end{aligned} \tag{12.3.21}$$

By multiplying by $(1-q)$, by taking the limit $q \to 1$ and by exploiting (12.3.14) we obtain that

$$\begin{aligned}
&\lim_{q\to1}(1-q)\varphi_{\mathrm{R}}^{q^{2s},q}(m_1,m_2,k_1,k_2) \\
=&\lim_{q\to1} q^{(m_1-k_1)k_2+2s(k_1+k_2-1)}\frac{(q,q)_{k_1+k_2-1}}{(1-q)^{k_1+k_2-1}}\frac{(1-q)^{k_1}}{(q,q)_{k_1}}\frac{(1-q)^{k_2}}{(q,q)_{k_2}} \\
&\times\frac{(q^{2s},q)_{m_1+m_2-k_1-k_2}}{(1-q)^{m_1+m_2-k_1-k_2}}\frac{(1-q)^{m_1-k_1}}{(q,q)_{m_1-k_1}}\frac{(1-q)^{m_2-k_2}}{(q,q)_{m_2-k_2}}\frac{(1-q)^{m_1+m_2}}{(q,q)_{m_1+m_2}}\frac{(q,q)_{m_1}}{(1-q)^{m_1}}\frac{(q,q)_{m_2}}{(1-q)^{m_2}} \\
=&\frac{\Gamma(k_1+k_2)}{\Gamma(k_1+1)\Gamma(k_2+1)}\frac{\Gamma(m_1+m_2-k_1-k_2+2s)}{\Gamma(m_1+m_2+2s)}\prod_{i=1}^{2}\frac{\Gamma(m_i+1)}{\Gamma(m_i-k_i+1)} \\
=&\varphi_{2s}(m_1,m_2,k_1,k_2)
\end{aligned} \tag{12.3.22}$$

□

12.3.2 The choice of the boundaries

The choice of the rates of the boundary generators (12.3.6) allows to extend the reversible product measure of the bulk to the boundary driven process, provided that the parameters $\beta_a(1) = \beta_a(L)$ for all $a \in \{1, 2\}$. As already pointed out, this is the usual choice of the boundary to obtain an absorbing dual process for the generator (12.3.3).

Lemma 9 *The boundary driven multi-species harmonic process with generator* (12.3.3) *admits a reversible product measure given by*

$$\mu^{rev} = \bigotimes_{x\in V} \mu_x^{rev} \qquad \mu_x^{rev} \sim NegMult(2s, p_0, \beta_1, \beta_2) \ : \ p_0 + \beta_1 + \beta_2 = 1 \tag{12.3.23}$$

if an only if $\beta_a(1) = \beta_a(L) = \beta_a$ *for all* $a \in \{1, 2\}$.

Proof of Lemma 9: we first impose the detailed balance condition for the bulk generator for the transition

$$\boldsymbol{m} \to \boldsymbol{m} - k_1\delta_1^x - k_2\delta_2^x + k_1\delta_1^{x+1} + k_2\delta_2^{x+1} \tag{12.3.24}$$

Then, the detailed balance reads

$$\begin{aligned} &\mu_x(m_1^x, m_2^x)\mu_{x+1}(m_1^{x+1}, m_2^{x+1})\varphi(m_1^x, m_2^x, k_1, k_2) \\ =&\mu_x(m_1^x - k_1, m_2^x - k_2)\mu_{x+1}(m_1^{x+1} + k_1, m_2^{x+1} + k_2)\varphi(m_1^{x+1} + k_1, m_2^{x+1} + k_2, k_1, k_2) \end{aligned} \tag{12.3.25}$$

By replacing the rates (12.3.5) and the probability mass function of the reversible measure (12.3.23) we have that

$$\begin{aligned} &\Gamma(2s + m_1^x + m_2^x)\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^x}}{m_1^x!}\frac{\beta_2^{m_2^x}}{m_2^x!}\Gamma(2s + m_1^{x+1} + m_2^{x+1})\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^{x+1}}}{m_1^{x+1}!}\frac{\beta_2^{m_2^{x+1}}}{m_2^{x+1}!} \\ \times&\frac{\Gamma(k_1 + k_2)}{k_1!k_2!}\frac{\Gamma(m_1^x + m_2^x - k_1 - k_2 + 2s)}{\Gamma(m_1 + m_2 + 2s)}\frac{\Gamma(m_1^x + 1)}{\Gamma(m_1^x - k_1 + 1)}\frac{\Gamma(m_2^x + 1)}{\Gamma(m_2^x - k_1 + 1)} \\ =&\Gamma(2s + m_1^x + m_2^x - k_1 - k_2)\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^x - k_1}}{(m_1^x - k_1)!}\frac{\beta_2^{m_2^x - k_2}}{(m_2^x - k_2)!}\Gamma(2s + m_1^{x+1} + m_2^{x+1} + k_1 + k_2) \\ \times&\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^{x+1}+k_1}}{(m_1^{x+1} + k_1)!}\frac{\beta_2^{m_2^{x+1}+k_2}}{(m_2^{x+1} + k_2)!}\frac{\Gamma(k_1 + k_2)}{k_1!k_2!}\frac{\Gamma(2s + m_1^{x+1} + m_2^{x+1})}{\Gamma(m_1^{x+1} + m_2^{x+1} + k_1 + k_2 + 2s)} \\ \times&\frac{\Gamma(m_1^{x+1} + k_1 + 1)}{\Gamma(m_1^{x+1} + 1)}\frac{\Gamma(m_2^{x+1} + k_2 + 1)}{\Gamma(m_2^{x+1} + 1)} \end{aligned} \tag{12.3.26}$$

By using the property of the Euler gamma function $\Gamma(n) = (n + 1)!$ it becomes an identity. For the others boundary transition the computations are similar.
We now show the detail balance condition on the left boundary for the transition

$$\boldsymbol{m} \to \boldsymbol{m} + k_1\delta_1^x + k_2\delta_2^x\,, \tag{12.3.27}$$

obtaining

$$\mu_x(m_1^x, m_2^x)\frac{\beta_1^{k_1}}{k_1!}\frac{\beta_2^{k_2}}{k_2!}\Gamma(k_1 + k_2) = \mu_x(m_1^x + k_1, m_2^x + k_2)\varphi(m_1^x + k_1, m_2^x + k_2, k_1, k_2) \tag{12.3.28}$$

Substituting the rates of the left generator in equation (12.3.6), the bulk rate (12.3.5) and the probability mass function of the reversible measure (12.3.23) we have

$$\Gamma(m_1^x + m_2^x + 2s)\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^x}}{m_1^x!}\frac{\beta_2^{m_2^x}}{m_2^x!}\frac{\beta_1^{k_1}(1)}{k_1!}\frac{\beta_2^{k_2}(1)}{k_2!}\Gamma(k_1 + k_2)$$

$$
\begin{aligned}
=&\Gamma(m_1^x + m_2^x + k_1 + k_2 + 2s)\frac{p_0^{2s}}{\Gamma(2s)}\frac{\beta_1^{m_1^x+k_1}}{(m_1^x+k_1)!}\frac{\beta_2^{m_2^x+k_2}}{(m_1^x+k_2)!}\\
&\times \frac{\Gamma(k_1+k_2)}{k_1!k_2!}\frac{\Gamma(m_1^x+m_2^x+2s)}{\Gamma(m_1^x+m_2^x+k_1+k_2+2s)}\frac{\Gamma(m_1^x+k_1+1)}{\Gamma(m_1^x+1)}\frac{\Gamma(m_2^x+k_2+1)}{\Gamma(m_2^x+1)}
\end{aligned}
\tag{12.3.29}
$$

By using again the property $\Gamma(n) = (n+1)!$ the above equation reduces to

$$
\left(\frac{\beta_1(1)}{\beta_1}\right)^{k_1} = \left(\frac{\beta_2}{\beta_2(1)}\right)^{k_2} \tag{12.3.30}
$$

that is true if and only if $\beta_a(1) = \beta_a$ for all $a = 1, 2$.
For the other possible boundary transition and for the right boundary generator, the proof is similar.

□

12.4 The multi-species symmetric inclusion process

Starting from the single species SIP(2k) introduced in Chapter 3, we define its multi-colour version, the so called *multi-species SIP(2k)* with $k \in \mathbb{N}/2$. Again, for the sake of simplicity, we consider the geometry of a chain of length L with two reservoirs attached to the end sites 1 and L respectively. The interaction is set to be nearest neighbours. Moreover, to simplify the computations, we consider the first non-trivial case, i.e. when $N = 3$, with two species of particles and a hole, still denoted by the index $N = 3$. In this situation, we do not have any more the interpretation of the holes given in Chapter 4 for the stirring process, since here each site allows an unbounded number of particles. We will clarify this aspect in the following.
To each site x of chain we associate a configuration vector given by (n_1^x, n_2^x, n_3^x), where n_1^x, n_2^x denote the number of particles of type 1 and 2 present at this site, while n_3^x denotes the number of holes. On the whole chain, we assign the configuration $\boldsymbol{n} = (n_1^x, n_2^x, n_3^x)_{x\in\{1,\ldots,L\}}$. The state space is given by

$$
\Omega = \bigotimes_{x=1}^{L} \Omega_x \tag{12.4.1}
$$

where

$$
\Omega_x = \left\{(n_1^x, n_2^x, n_3^x) \in \mathbb{N}_0^3 \ : \ n_3 = 2k + n_1^x + n_2^x\right\} \tag{12.4.2}
$$

The generator is given by

$$
\mathcal{L} = \mathcal{L}_{\text{left}} + \sum_{x=1}^{L-1} \mathcal{L}_{x,x+1} + \mathcal{L}_{\text{right}} \tag{12.4.3}
$$

where:

- *the bulk generator*

$$
\mathcal{L}_{x,x+1} f(\boldsymbol{n}) = \sum_{A,B=1}^{3} n_A^x n_B^{x+1} \left(f(\boldsymbol{n} - \delta_A^x + \delta_B^x + \delta_A^{x+1} - \delta_B^{x+1}) - f(\boldsymbol{n})\right) \tag{12.4.4}
$$

- *the boundary generators*

$$
\mathcal{L}_{\text{left}} f(\boldsymbol{n}) = \sum_{A,B=1}^{3} \alpha_A n_B^1 \left(f(\boldsymbol{n} - \delta_B^1 + \delta_A^1) - f(\boldsymbol{n})\right) \tag{12.4.5}
$$

and

$$\mathcal{L}_{\text{left}} f(\boldsymbol{n}) = \sum_{A,B=1}^{3} \beta_A n_B^L \left(f(\boldsymbol{n} - \delta_B^L + \delta_A^L) - f(\boldsymbol{n}) \right) \tag{12.4.6}$$

We observe that the generator (12.4.3) is formally the same of the boundary driven multi-species stirring process defined in (4.2.15), specified to $N = 3$, on a line with nearest neighbour interaction and with two boundary reservoirs. The difference is in the definition of the holes variables n_3 that now have unbounded values. Again, these holes have an occupation variable depends on the occupancy of the species of particles, but the meaning is not any more the one of the multi-species stirring process (see Chapter 4). To make the differences more clear we compare the two situations in the case $N = 3$. Here, the holes occupation variables reads

multi-species stirring process : $n_3 = \nu - n_1 - n_2 \leqslant \nu$

multi-species SIP(2k) : $n_3 = 2k + n_1 + n_2$ unbounded

Moreover, we observe that the expression of the holes of the multi-species SIP(2k) can be obtained from the one of the multi-species stirring by replacing n_3 with $-n_3$ and with replacing ν by $-2k$.

Reversible measure

To prove write the reversible measure for this process we introduce the boundary densities for any species $a \in \{1, 2\}$:

$$\rho_a^{\text{left}} = \frac{\alpha_a}{\alpha_3}, \qquad \rho_a^{\text{right}} = \frac{\beta_a}{\beta_3} \tag{12.4.7}$$

Lemma 10 *The boundary driven multi-species SIP(2k) with generator* (12.4.3) *admits a reversible product measure given by*

$$\mu^{rev} = \bigotimes_{x \in V} \mu_x^{rev} \qquad \mu_x^{rev} \sim NegMult(2k, \frac{\alpha_1}{\alpha_3}, \frac{\alpha_2}{\alpha_3}, p_3) \;:\; \frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_3} + p_3 = 1 \tag{12.4.8}$$

if an only if $\rho_a^{\text{left}} = \rho_a^{\text{right}}$ *for all* $a \in 1, 2$

Proof of Lemma 10: we first impose the detailed balance condition on the bond $(x, x+1)$. We only consider the transition

$$\boldsymbol{n} \to \boldsymbol{n} + \delta_1^x - \delta_3^{x+1} \tag{12.4.9}$$

the other transition can be treated similarly. The detailed balance reads

$$\begin{aligned}
&\Gamma(2k + n_1^x + n_2^x) \frac{p_3^{2k}}{\Gamma(2k)} \frac{p_1^{n_1^x}}{n_1^x!} \frac{p_2^{n_2^x}}{n_2^x!} \Gamma(2k + n_1^{x+1} + n_2^{x+1}) \frac{p_3^{2k}}{\Gamma(2k)} \frac{p_1^{n_1^{x+1}}}{n_1^{x+1}!} \frac{p_2^{n_2^{x+1}}}{n_2^{x+1}!} (2k + n_1^x + n_2^x) n_1^{x+1} \\
=&\Gamma(2k + n_1^x + n_2^x + 1) \frac{p_3^{2k}}{\Gamma(2k)} \frac{p_1^{n_1^x+1}}{(n_1^x+1)!} \frac{p_2^{n_2^x}}{n_2^x!} \\
\times&\Gamma(2k + n_1^{x+1} + n_2^{x+1} - 1) \frac{p_3^{2k}}{\Gamma(2k)} \frac{p_1^{n_1^{x+1}-1}}{(n_1^{x+1}-1)!} \frac{p_2^{n_2^{x+1}}}{n_2^{x+1}!} (n_1^x + 1)(2k + n_1^{x+1} + n_2^{x+1} - 1)
\end{aligned} \tag{12.4.10}$$

that reduces to an identity by using the property $\Gamma(n+1) = n\Gamma(n)$.
We now consider the transition on the boundary

$$\boldsymbol{n} \to \boldsymbol{n} - \delta_1^1 + \delta_3^1 \tag{12.4.11}$$

For all the other transition the computations are similar. The detailed balance reads

$$\begin{aligned}
&\Gamma(2k+n_1^1+n_2^1)\frac{p_3^{2k}}{\Gamma(2k)}\frac{p_1^{n_1^1}}{n_1^1!}\frac{p_2^{n_2^1}}{n_2^1!}\alpha_3 n_1^1\\
=&\Gamma(2k+n_1^1+n_2^1-1)\frac{p_3^{2k}}{\Gamma(2k)}\frac{p_1^{n_1^1-1}}{(n_1^1-1)!}\frac{p_2^{n_2^1}}{n_2^1!}\alpha_1(2k+n_1^1+n_2^1-1)
\end{aligned} \tag{12.4.12}$$

It reduces to

$$\frac{\alpha_1}{\alpha_3}=p_1 \tag{12.4.13}$$

By taking the detailed balance for the same transition on the right boundary one obtains

$$\frac{\beta_1}{\beta_3}=p_1 \tag{12.4.14}$$

then the result follows.

□

We observe that the reversible measure found in Lemma 10 is the multivariate version of the reversible measure for the single species SIP(2k) derived in Lemma 4.

12.4.1 Perspectives

For what concerns the boundary driven multi-species harmonic process with generator (12.3.3) some future development are possible. Inspired by [26], one could try to describe multi-species harmonic by using a higher rank non-compact Lie algebras and exploit this description to derive absorbing duality. Starting from the duality function for the bulk of the multi-species stirring process (5.4.40), considering the (self-)duality functions for the bulk of the SEP(ν) written in (3.2.37) and the (self-)duality function for the harmonic process defined in [26], an "educated" guess of the duality function for the bulk of this multi-species harmonic process with generator (12.3.3) is

$$D(\boldsymbol{m},\boldsymbol{\xi})=\prod_{x=1}^{L}\left(\frac{\Gamma(2s)}{\Gamma(2s+\xi_1^x+\xi_2^x)}\prod_{a=1}^{2}\frac{\Gamma(m_a^x+1)}{\Gamma(m_a^x-\xi_a^x+1)}\right). \tag{12.4.15}$$

Here we denoted by $\boldsymbol{\xi}$ the configurations of the self-dual process of $(\boldsymbol{m}(t))_{t\geqslant 0}$.
Moreover, by mapping the integrable version of the process ($s=1/2$) to the higher rank non-compact spin chain, it should be possible to apply the quantum inverse scattering method to determine the non-equilibrium steady state and to derive a mapping of the non-equilibrium generator onto the equilibrium one (which has diagonal boundaries).
Once these tasks are achieved, the large deviations function could be investigated, as it has been done recently in [106]. Moreover, as mentioned in Section (12.1), starting from this multi-species harmonic process one could derive the multi-species analogues of other non-compact processes studied in the single species literature like BEP, BMP and KMP in the non-integrable case and the multi-species integrable heat conduction model.
Hopefully, these non-compact processes have a richer structure, both on mathematical and physical side and they could help (after a scaling limit is performed) in the understanding of multi-component diffusion equations.

Also, for what concerns the multi-species SIP(2k), we aim to prove absorbing duality, inferring it from the one of the multi-species harmonic process (once available). Moreover, this could open the possibility of extending this multi-species duality results to the continuum, like it has been done for the single species processes in [110]. Finally, condensation phenomena in the multi-species set-up could be studied by extending the results proved in [111].

Chapter 13

Duality for boundary driven asymmetric multi-species models

13.1 Motivations

Informal introduction. In many physical situations and in statistical mechanics models it is interesting to consider a drift term in the process, besides diffusion. Mathematically, this is usually achieved by inserting an asymmetry in the jump rate of the particles. Even in the closed boundary case, this asymmetry puts the system out-of-equilibrium, giving rise to the bulk driven systems. A more general situation is when, together with the bulk driving, also a boundary driving in inserted, obtaining a system with two non-equilibrium sources. Duality for bulk driven system is an active matter of research (see for instance [51, 112, 113]). When also the boundary driving is added, the usual definition of duality is still not completely understood and, recently in [114], the asymmetric simple exclusion process (ASEP) has been studied via the notion of *reverse duality*. It has also been proved that many integrability techniques can be adapted to ASEP (see for instance [24, 21, 115, 116, 117]). From the point of view of multi-species processes, it might be interesting to study, via duality and integrability techniques, processes with both bulk and boundary driving. Already for the single species processes, the problem of finding the usual duality for these bulk-boundary driven case is still an open question. In this chapter we put some light in this direction. Starting from the duality for the bulk driven asymmetric Brownian energy process (ABEP), we construct some boundaries with the feature of conserving classical duality. As we will clarify, this leads to reservoirs with non-local interaction. As a downside, this non-locality is harder to interpret in a physical framework. However, with the results of this chapter we aim to contribute in "laying the foundations" for future works in the multi-species bulk-boundary driven set-up.

The Asymmetric Brownian Energy Processes is an interacting diffusion system describing an asymmetric energy exchange between the sites of a lattice. Its symmetric version (BEP) was originally introduced in [11] where its symmetries and duality properties where unveiled. These are related to the intrinsic algebraic structure of the infinitesimal generator that can be written in terms of a continuous representation of the non-compact $\mathfrak{su}(1,1)$ Lie algebra.

In [11] (see also Chapter 3 for a short review) the BEP in the closed system (i.e. in absence of external reservoirs) was proven to be dual to the symmetric inclusion process (SIP)[1]. This is an

[1]For the sake of brevity, sometimes, in this chapter we do not write SIP(2K) or BEP(2k) but just SIP and

interacting particle system modelling particles moving on a lattice with an attractive interaction. The reason behind the above mentioned duality relation lies is in the $\mathfrak{su}(1,1)$ algebraic structure shared by the two processes. BEP and SIP are indeed two elements of a broader class of models all related to the $\mathfrak{su}(1,1)$ Lie algebra and including also other notable models. One of these is the Kipnis-Marchoro-Presutti model (KMP) [50] where the total energy is instantaneously redistributed among sites and that can be recovered as an instantaneous thermalization limit of the BEP. Another model inherently related to the BEP is the Wright-Fisher diffusion [62] that is the prototype model of mathematical population genetics. Duality between the Wright-fisher and the Moran model can be seen as a particular instance of duality between BEP and SIP (see e.g. [63]).

In [12] (see also Chapter 3 for a short review) the analysis was extended to the non-equilibrium situation in which the system is put in contact with two external heat reservoirs imposing two different temperatures $T_\ell \neq T_r$ at the endpoints of the bulk. The corresponding process is also called BEP with open boundaries and has been shown to be dual to the SIP with absorbing boundaries.

The asymmetric version of the model we study (ABEP) was first introduced in [51] in a closed boundary setting. This emerged as a scaling limit of the ASIP (an asymmetric version of the inclusion process) in a particular regime of weak asymmetry. In the same work, an alternative construction was proposed for the ABEP, that was shown to be obtainable from BEP, via a non-local transformation g depending on the asymmetry parameter. A duality relation between ABEP and SIP was then deduced in [51] as a consequence, independently, of the two above mentioned constructions. The duality function does not have a standard product structure (as is usually the case in the symmetric context) but a nested product structure related to the non-local map g. This property is a first instance of a duality relation between a genuinely non-equilibrium asymmetric system (in the sense that it has a non-zero average current) and a symmetric process. This link is made possible by the fact that the dependence on the asymmetry parameter is retained in the duality function, through the map g.

Here we extend the analysis to the system with open boundaries. In this context the problem becomes the definition of reservoirs itself. The aim is indeed to impose external temperatures $T_\ell \neq T_r$ in such a way as not to alter the condition of existence of a duality relation with the SIP with absorbing boundaries. Our strategy does not directly rely on algebraic considerations on the Markov generator but rather on the link between ABEP and BEP via the non-local map g. This transformation procedure allows us to construct reservoirs of the correct form. These turn out to act in a non-standard way. The left reservoir acts only on the left endpoint of the lattice, but its action takes into account the total energy of the system. The right reservoir, instead affects all the sites of the lattice. As a result of this construction we prove a duality relation with the SIP with absorbing boundary by means of two different duality functions. The first one is in a so-said *classical* form whereas the second one is in terms of *generalized Laguerre polynomials*.

As far as we know, duality in the presence of an asymmetry together with open boundary condition is still a quite challenging outcome as the classical techniques relying on algebraic considerations do not work. This is due to the fact that the quantum group symmetry needed to construct the duality relation is broken. Results are mainly available for the case of asymmetric simple exclusion process (ASEP). The first attempt is due to Okhubo in [54] where a dual operator has been obtained; however it could not be directly interpreted as a transition matrix for a stochastic process. We mention [118] where the author generalizes the self-duality of the

BEP, where the dependence on the parameter 2k is understood

asymmetric simple exclusion process with an open boundary condition at the left boundary and a closed right boundary. More recent results include [117] where a duality relation between an half-line open ASEP and a sub-Markov process where particles perform an asymmetric exclusion dynamics in the bulk and are killed at the boundary is proven. In [114, 119] it is shown a reverse duality relation for an open ASEP with open boundary and a shock ASEP with reflecting boundary.
The chapter is organized as follows: in Section 13.2 we introduce the model of interest, i.e. the ABEP with open boundaries. In Section 13.3 we show how the boundary driven ABEP can be obtained from its symmetric version(the BEP, see Section 3.1.4) via a non-local map transformation. At the end of this section we state some general results that allow to infer properties for a process that can be obtained from another process via a map transformation. In the subsequent two sections these general properties are then specialized to gather information for ABEP starting from known results for BEP: Section 13.4 is devoted to the study of the case $T_\ell = T_r$ in which the system is proven to be reversible, and the reversible measure is computed; in Section 13.5 instead we discuss duality relations. We end with Section 13.6 where we use the duality results to gather some information on the stationary measure in the general case. In particular we compute what we call the one-point and two-point stationary exponential correlations of the partial energies.

13.2 The model

As already introduce in Sections 3.1.4 of Chapter 2, the Brownian Energy Process BEP$(2k)$ is an interacting diffusion system of "continuous spins" placed on the sites of a lattice Λ_L, $2k$ is a positive parameter tuning the intensity of the interaction. We consider its asymmetric version ABEP$(\sigma, 2k)$, $\sigma > 0$ the asymmetry parameter, that can be defined when the lattice is a one-dimensional chain $\Lambda_L = \{1, \ldots, L\}$ and the interaction is nearest-neighbor. To each site of the lattice $i \in \Lambda_L$ is associated an energy $x_i \geqslant 0$. We denote by $x = (x_1, \ldots, x_L) \in \mathbb{R}_+^L$ the vector collecting all energies and we call $\Omega := \mathbb{R}_+^L$ the state space of the system. When the system is *closed*, or, in other words, in absence of external reservoirs, the dynamics conserves the total energy of the system $E(x) := \sum_{i \in \Lambda_L} x_i$.

In this chapter we consider the *open system*, i.e. we put the *bulk* lattice Λ_L in contact with two external reservoirs placed at artificial sites $\Lambda_L^{\text{res}} = \{0, L+1\}$. Each reservoir $j \in \Lambda_L^{\text{res}}$ can be interpreted as a thermal bath characterized by its own fixed temperature $T_j \geqslant 0$, that is attached to the bulk Λ_L only through the boundary sites 1 and L. The action of the reservoirs induces an energy exchange between the bulk lattice and the exterior, that destroys the total energy conservation. For simplicity we will also denote by $T_\ell := T_0$ the temperature of the left reservoir and by $T_r := T_{L+1}$ the temperature of the right reservoir.

In order to define the model, we need to introduce two crucial quantities the partial energies $E_i(x)$, $i \in \Lambda_L$, and the non-local map g.

Definition 19 *We define the map $g : \Omega \to \Omega$ via*

$$g(x) = (g_i(x))_{i \in \Lambda_L} \qquad \textit{with} \qquad g_i(x) := \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma} \tag{13.2.1}$$

where $E_i(x)$ denotes the energy of the system at the right of site $i \in \Lambda_L$, i.e.

$$E_i(x) = \sum_{l=i}^{L} x_l \quad \textit{for} \quad i = 1, \ldots, N \qquad \textit{with the convention} \quad E_{L+1}(x) = 0\,. \tag{13.2.2}$$

Notice that the total energy $E(x)$ coincides with the first component $E_1(x)$ of the vector of partial energies.

The stochastic evolution of the collection of energies of the system is governed by a Markov process $\{x(t), t \geqslant 0\}$ that we will define by giving its infinitesimal generator $\mathcal{L}^{\text{ABEP}}$. This acts on smooth functions $f : \Omega \to \mathbb{R}$ and is given by the sum of three terms, one of them governing the interaction between bulk sites and the other two modelling the action of left and right reservoirs. We define

$$\mathcal{L}^{\text{ABEP}} = \mathcal{L}^{\text{ABEP}}_{\text{left}} + \sum_{i=1}^{L-1} \mathcal{L}^{\text{ABEP}}_{i,i+1} + \mathcal{L}^{\text{ABEP}}_{\text{right}} . \tag{13.2.3}$$

where, for $i \in \{1, \ldots, N-1\}$, the action on smooth functions $f : \Omega \to \mathbb{R}$ is

$$\begin{aligned}[][\mathcal{L}^{\text{ABEP}}_{i,i+1} f](x) = & \frac{1}{\sigma^2}\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 f(x) \\ & + \frac{1}{\sigma}\left(\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right) - 2k\left(2 - e^{-\sigma x_i} - e^{\sigma x_{i+1}}\right)\right) \\ & \times \left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) f(x) \end{aligned} \tag{13.2.4}$$

whereas

$$\begin{aligned}[][\mathcal{L}^{\text{ABEP}}_{\text{left}} f](x) = & T_\ell \left(e^{\sigma E(x)}\left(2k - 1 + e^{\sigma x_1}\right)\frac{\partial}{\partial x_1} + \frac{e^{\sigma E(x)}}{\sigma}\left(e^{\sigma x_1} - 1\right)\frac{\partial^2}{\partial x_1^2}\right) f(x) \\ & - \frac{e^{\sigma x_1} - 1}{\sigma}\frac{\partial}{\partial x_1} f(x) \end{aligned} \tag{13.2.5}$$

and

$$[\mathcal{L}^{\text{ABEP}}_{\text{right}} f](x) = \left(2kT_r - \frac{1 - e^{-\sigma x_L}}{\sigma}\right)\sum_{l=1}^{L} e^{\sigma E_l(x)}\left(\partial_{x_l} - \partial_{x_{l-1}}\right) f(x) \tag{13.2.6}$$

$$+ T_r\left(1 - e^{-\sigma x_L}\right)\sum_{l=1}^{L} e^{2\sigma E_l(x)}\left(\partial_{x_l} - \partial_{x_{l-1}}\right) f(x) \tag{13.2.7}$$

$$+ T_r \frac{1 - e^{-\sigma x_L}}{\sigma}\sum_{l,j=1}^{L} e^{\sigma(E_l(x) + E_j(x))}\left(\partial_{x_l} - \partial_{x_{l-1}}\right)\left(\partial_{x_j} - \partial_{x_{j-1}}\right) .$$

The action of reservoirs is non-local in two different ways. The left reservoir acts only on the left boundary site 1, but its action takes in account the total energy $E(x)$ that is not an invariant of the dynamics. The right reservoir, instead, affects the whole chain by forcing a further interaction between bulk sites. This new interaction has a different nature with respect to the one induced by the bulk term of the generator. First of all it is non-local since all bulk sites interact with each other, moreover it is of *topological* and no longer of *metric* nature. Indeed the interaction between any couple of sites (l, j) depends on $E_l(x)$ and $E_j(x)$ i.e. on the total energy at the right of l, resp. of j. As a consequence, (13.2.6) does not have to be considered as a reservoir in the standard sense but rather as a non-local-topological term of the bulk interaction. This is parameterized by a drift parameter T_r.

In the field of interacting particle systems, the interest for models with topological interaction has emerged in the last few years both in the context of stochastic models of non-equilibrium

[120] and in the context of kinetic theory [121]. Here the interaction between two particles is called topological if it does not depend on their distance but on their ranking. In a particle configuration the ranking of a particle can be computed by counting the number of particles at its right (or at its left).
The main motivation for the study of models with topological interaction comes from population dynamics and in particular the study of the motion of crowds of animals (see e.g. [122]). Due to the non-locality of the interaction it is rare to find models with topological interaction with good algebraic structures and then showing duality properties. To our knowledge there is only one example of models of this type. This is the dynamic-ASEP, recently introduced in the literature [123, 113] for which duality results have been proven. This is a dynamic version of ASEP (asymmetric exclusion process) for which the interaction of a particle with the rest of the system depends on the number of particles at its right (or left), i.e. it has a topological nature. In this perspective the ABEP with reservoirs can be seen as a first example of system of interacting diffusions with metric plus topological interaction. As done in [124] for the dynamic-ASEP using a generalization of the microscopic Cole-Hopf transform, we believe it would be interesting to study the scaling limit of ABEP and to understand the macroscopic effect of these boundary reservoirs. In the rest of the chapter we will prove that it exhibits a duality property and derive explicitly some exponential moments.

13.3 From BEP to ABEP

The BEP$(2k)$ on Λ_L is the symmetric version of the ABEP$(\sigma, 2k)$ obtained in the limit as $\sigma \to 0$. As in the previous section we consider the system with nearest-neighbor interaction in contact with two boundary reservoirs kept at temperature T_ℓ and T_r. We denote by $\{z(t),\ t \geqslant 0\}$ the Brownian Energy process on the space state $\Omega = \mathbb{R}_+^L$ describing the evolution of the vectors $z := (z_1, \dots, z_L)$ of single-site energies. The infinitesimal generator, acting on smooth functions $f : \Omega \to \mathbb{R}$, is defined as follows

$$\mathcal{L}^{\mathrm{BEP}} = \mathcal{L}^{\mathrm{BEP}}_{\mathrm{left}} + \sum_{i=1}^{L-1} \mathcal{L}^{\mathrm{BEP}}_{i,i+1} + \mathcal{L}^{\mathrm{BEP}}_{\mathrm{right}} \tag{13.3.1}$$

where, for $i \in \{1, \dots, N-1\}$,

$$\mathcal{L}^{\mathrm{BEP}}_{i,i+1} f(z) = \Big[z_i z_{i+1} \left(\partial_{z_{i+1}} - \partial_{z_i}\right)^2 - 2k(z_i - z_{i+1}) \left(\partial_{z_{i+1}} - \partial_{z_i}\right) \Big] f(z) \tag{13.3.2}$$

whereas

$$\mathcal{L}^{\mathrm{BEP}}_{\mathrm{left}} f(z) = \left[T_\ell \left(2k \frac{\partial}{\partial_{z_1}} + z_1 \frac{\partial^2}{\partial^2_{z_1}} \right) - z_1 \frac{\partial}{\partial_{z_1}} \right] f(z) \tag{13.3.3}$$

and

$$\mathcal{L}^{\mathrm{BEP}}_{\mathrm{right}} f(z) = \left[T_r \left(2k \frac{\partial}{\partial_{z_L}} + z_L \frac{\partial^2}{\partial^2_{z_L}} \right) - z_L \frac{\partial}{\partial_{z_L}} \right] f(z)\ . \tag{13.3.4}$$

The latter terms give the action of left and right reservoirs that are attached, respectively, to site 1 and site L.

It can be easily checked that $\mathcal{L}^{\mathrm{BEP}}$ is recovered from $\mathcal{L}^{\mathrm{ABEP}}$ by suitably taking the limit as $\sigma \to 0$. On the other hand $\mathcal{L}^{\mathrm{ABEP}}$ can be constructed from $\mathcal{L}^{\mathrm{BEP}}$ by acting with the non-local map g introduced in Definition 19. This claim has been proven in [51] for the closed system. Below we show that such a construction can be extended to the reservoir terms of the generator.

Theorem 23 (From BEP to ABEP) *Let g be the map in Definition 19, then for all $f \in \mathcal{D}(\mathcal{L}^{\rm BEP})$ we have*

$$\mathcal{L}^{\rm ABEP}(f \circ g) = \left[\mathcal{L}^{\rm BEP} f\right] \circ g \ . \tag{13.3.5}$$

Proof of Theorem 23: the proof is divided in three steps. Firstly, using the function g introduced in Definition 19 and its inverse we find the relations between the derivative with respect to z_i and the derivatives with respect to x_i. Secondly, we use these relations to obtain the bulk generator of ABEP$(\sigma, 2k)$ starting from the one of the BEP$(2k)$. Finally, we obtain the the boundary generator of the ABEP$(\sigma, 2k)$ starting from the one of BEP$(2k)$. This last result is obtained for a system where each site of Λ_L is connected with a boundary reservoirs. By specializing to case when only two reservoirs are connected with sites 1 and L we obtain the generator (13.2.3).

Relations between derivatives. Recalling the definition of g:

$$\begin{aligned} g : \Omega \ &\to \ \Omega \\ x \ &\to \ g(x) = (g_i(x))_{i \in \Lambda_L}, \qquad \text{with} \qquad g_i(x) = \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma} \end{aligned}$$

with the convention $E_{L+1}(x) = 0$ and $E_1(x) = E(x)$.

Notice that the map g is not full range, i.e. $g(\Omega) \neq \Omega$, indeed

$$E(g(x)) = \frac{1}{\sigma}\left(1 - e^{-2\sigma E(x)}\right) \leqslant \frac{1}{\sigma} \tag{13.3.6}$$

so that in particular $g(\Omega) \subseteq \{x \in \Omega : E(x) \leqslant 1/\sigma\}$. Moreover g is a bijection from Ω to $g(\Omega)$. Indeed, denoting by $g^{\rm inv} : g(\Omega) \to \Omega$ the inverse transform of g. In other words, if $z = g(x) \in g(\Omega)$, then $x = g^{\rm inv}(z)$ with ith component being

$$g_i^{\rm inv}(z) = \frac{1}{\sigma} \ln\left\{\frac{1 - \sigma E_{i+1}(z)}{1 - \sigma E_i(z)}\right\} \tag{13.3.7}$$

Let $F := f \circ g$, or, equivalently, $f = F \circ g^{\rm inv}$ namely $F(x) = f(g(x))$ for $x \in \Omega$ and $f(z) = F(g^{\rm inv}(z))$ for $z \in g(\Omega)$, therefore, in order to prove (13.3.36), it is sufficient to show that, for all $x \in \Omega$,

$$\left[\mathcal{L}_{i,j}^{\rm BEP}(F \circ g^{\rm inv})\right](g(x)) = \left[\mathcal{L}_{i,j}^{\rm ABEP} F\right](x) \tag{13.3.8}$$

To this aim we compute the first and second derivatives of $f = F \circ g^{\rm inv}$. We have that for all $k \in \Lambda_L$

$$\frac{\partial f}{\partial z_k}(z) = \sum_{l \in \Lambda_L} \frac{\partial F}{\partial x_l}(g^{\rm inv}(z)) \cdot \frac{\partial g_l^{\rm inv}}{\partial z_k}(z) \tag{13.3.9}$$

and for all $k, m \in \Lambda_L$

$$\frac{\partial^2 f}{\partial^2 z_k z_m}(z) = \sum_{l,j \in \Lambda_L} \frac{\partial^2 F}{\partial^2 x_l x_j}(g^{\rm inv}(z)) \cdot \frac{\partial g_l^{\rm inv}}{\partial z_k}(z) \cdot \frac{\partial g_j^{\rm inv}}{\partial z_m}(z) + \sum_{l \in \Lambda_L} \frac{\partial F}{\partial x_l}(g^{\rm inv}(z)) \cdot \frac{\partial^2 g_l^{\rm inv}}{\partial^2 z_k z_m}(z) \ . \tag{13.3.10}$$

We now compute all the first and second derivatives of all the components of the inverse function g^{inv}, we obtain

$$\frac{\partial g_l^{\text{inv}}}{\partial z_k}(z) = \begin{cases} 0 & \text{if } k < l \\ \frac{1}{1-\sigma E_l(z)} & \text{if } k = l \\ \frac{\sigma z_l}{(1-\sigma E_l(z))(1-\sigma E_{l+1}(z))} & \text{if } k > l \end{cases} \tag{13.3.11}$$

and, for $m \leqslant k$ (it is symmetric in k and m),

$$\frac{\partial^2 g_l^{\text{inv}}}{\partial z_m \partial z_k}(z) = \begin{cases} 0 & \text{if } m < l \\ \frac{\sigma}{(1-\sigma E_l(z))^2} & \text{if } l = m \leqslant k \\ \frac{z_l \sigma^2 (2-\sigma z_l - 2\sigma E_{l+1}(z))}{[(1-\sigma E_l(z))(1-\sigma E_{l+1}(z))]^2} & \text{if } m > l \end{cases} \tag{13.3.12}$$

Using the telescopicity, these derivatives simplify, since we have that

$$E_l\left(g\left(x\right)\right) = \sum_{i=l}^{L} g_i(x) = \frac{1}{\sigma}\left(1 - e^{-\sigma E_l(x)}\right) \ . \tag{13.3.13}$$

Then, using (13.3.13) we obtain

$$\left.\frac{\partial g_l^{\text{inv}}}{\partial z_k}\right|_{z=g(x)} = \begin{cases} 0 & \text{if } k < l \\ e^{\sigma E_l(x)} & \text{if } k = l \\ e^{\sigma E_l(x)}\left(1 - e^{-\sigma x_l}\right) & \text{if } k > l \end{cases} \tag{13.3.14}$$

and

$$\left.\frac{\partial^2 g_l^{\text{inv}}}{\partial z_m \partial z_k}\right|_{z=g(x)} = \begin{cases} 0 & \text{if } m < l \\ \sigma e^{2\sigma E_l(x)} & \text{if } l = m \leqslant k \\ \sigma e^{2\sigma E_l(x)}\left(1 - e^{-2\sigma x_l}\right) & \text{if } m > l \ . \end{cases} \tag{13.3.15}$$

Bulk. We aim to show that for all $i \in \Lambda_L$, $x \in \Omega$,

$$\left[\mathcal{L}_{i,i+1}^{\text{BEP}} f\right](g(x)) = \left[\mathcal{L}_{i,i+1}^{\text{ABEP}} f \circ g\right](x) \tag{13.3.16}$$

where

$$\mathcal{L}_{i,i+1}^{\text{BEP}} f(z) = \left[z_i z_{i+1}\left(\partial_{z_{i+1}} - \partial_{z_i}\right)^2 - 2k(z_i - z_{i+1})\left(\partial_{z_{i+1}} - \partial_{z_i}\right)\right] f(z) \tag{13.3.17}$$

and where

$$\begin{aligned} [\mathcal{L}_{i,i+1}^{\text{ABEP}} f](x) = & \frac{1}{\sigma^2}\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right)^2 f(x) \\ & + \frac{1}{\sigma}\left(\left(1 - e^{-\sigma x_i}\right)\left(e^{\sigma x_{i+1}} - 1\right) - 2k\left(2 - e^{-\sigma x_i} - e^{\sigma x_{i+1}}\right)\right)\left(\frac{\partial}{\partial x_{i+1}} - \frac{\partial}{\partial x_i}\right) f(x) \end{aligned} \tag{13.3.18}$$

We consider term by term the bulk generator for the BEP(2k) given in (13.3.17) and we have that

$$z_i z_{i+1} = \frac{1}{\sigma^2}\left\{\left(e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}\right)\left(e^{-\sigma E_{i+2}(x)} - e^{-\sigma E_{i+1}(x)}\right)\right\}$$

$$= \frac{e^{-2\sigma E_{i+1}(x)}}{\sigma_2}(1 - e^{-\sigma x_i})(e^{\sigma \xi_{i+1}} - 1) \tag{13.3.19}$$

Moreover, we have that

$$\begin{aligned}(z_i - z_{i+1}) =& \frac{1}{\sigma}\left\{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)} - e^{-\sigma E_{i+2}(x)} + e^{-\sigma E_{i+1}(x)}\right\} \\ =& \frac{e^{-\sigma E_{i+1}(x)}}{\sigma}(2 - e^{-\sigma x_i} - e^{\sigma x_{i+1}})\end{aligned} \tag{13.3.20}$$

Considering the term with the second derivative of (13.3.2) we obtain

$$\begin{aligned}&\left(\frac{\partial^2 f}{\partial z_i^2}(z) + \frac{\partial^2 f}{\partial z_{i+1}^2}(z) - 2\frac{\partial^2 f}{\partial z_i \partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} \\ =& e^{2\sigma E_{i+1}(x)}\left\{\left(\frac{\partial^2 F}{\partial x_i^2}(x) + \frac{\partial^2 F}{\partial x_i^2}(x) - 2\frac{\partial^2 F}{\partial x_i \partial x_{i+1}}(x)\right) + \sigma\left(\frac{\partial F}{\partial x_{i+1}}(x) - \frac{\partial F}{\partial x_i}(x)\right)\right\}\end{aligned} \tag{13.3.21}$$

Indeed, we have that

$$\frac{\partial^2 f}{\partial z_i^2}(z) = \sum_{\ell,j=1}^{i} \frac{\partial^2 F}{\partial x_\ell \partial x_j}(g^{\text{inv}}(z))\frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_i}(z) + \sum_{\ell=1}^{i} \frac{\partial F}{\partial x_\ell}(g^{\text{inv}}(z))\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_i^2}(z)\,, \tag{13.3.22}$$

$$\frac{\partial^2 f}{\partial z_{i+1}^2}(z) = \sum_{\ell,j=1}^{i+1} \frac{\partial^2 F}{\partial x_\ell \partial x_j}(g^{\text{inv}}(z))\frac{\partial g_\ell^{\text{inv}}}{\partial z_{i+1}}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_{i+1}}(z) + \sum_{\ell=1}^{i+1} \frac{\partial F}{\partial x_\ell}(g^{\text{inv}}(z))\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_{i+1}^2}(z) \tag{13.3.23}$$

and

$$\frac{\partial^2 f}{\partial z_i \partial z_{i+1}}(z) = \sum_{\ell,j=1}^{i+1} \frac{\partial^2 F}{\partial x_\ell \partial x_j}(g^{\text{inv}}(z))\frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_{i+1}}(z) + \sum_{\ell=1}^{i+1} \frac{\partial F}{\partial x_\ell}(g^{\text{inv}}(z))\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_i \partial z_{i+1}}(z) \tag{13.3.24}$$

We observe that by using the equation for the first derivative of g^{inv} given in (13.3.14) we obtain

$$\begin{aligned}&\frac{\partial^2 F}{\partial x_i^2}(g^{\text{inv}}(z))\left(\frac{\partial g_i^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_i^{\text{inv}}}{\partial z_i}(z) + \frac{\partial g_i^{\text{inv}}}{\partial z_{i+1}}(z)\frac{\partial g_i^{\text{inv}}}{\partial z_{i+1}}(z) - 2\frac{\partial g_i^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_i^{\text{inv}}}{\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} \\ =& \frac{\partial^2 F}{\partial x_i^2}(x)\left(e^{2\sigma E_{i+1}(x)} + e^{2\sigma E_i(x)}(1 - e^{-\sigma x_i})^2 - 2e^{\sigma E_i(x)}e^{\sigma E_{i+1}(x)}\right) \\ =& \frac{\partial^2 F}{\partial x_i^2}(x)(-e^{2\sigma E_{i+1}(x)})\,,\end{aligned} \tag{13.3.25}$$

$$\begin{aligned}&\frac{\partial^2 F}{\partial x_i \partial x_{i+1}}(g^{\text{inv}}(z))\left(2\frac{\partial g_{i+1}^{\text{inv}}}{\partial z_{i+1}}(z)\frac{\partial g_i^{\text{inv}}}{\partial z_{i+1}}(z) - 2\frac{\partial g_i^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_{i+1}^{\text{inv}}}{\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} \\ &= \frac{\partial^2 F}{\partial x_i \partial x_{i+1}}(x)\left(2e^{\sigma E_{i+1}(x)}e^{\sigma E_i(x)}(1 - e^{-\sigma x_i}) - 2e^{\sigma E_i(x)}e^{\sigma E_{i+1}(x)}\right) \\ &= \frac{\partial^2 F}{\partial x_i \partial x_{i+1}}(x)e^{2\sigma E_{i+1}(x)}\end{aligned} \tag{13.3.26}$$

and

$$\frac{\partial F}{\partial x_{i+1}^2}(g^{\text{inv}}(z))\left(\frac{\partial g_{i+1}^{\text{inv}}}{\partial z_{i+1}}(z)\frac{\partial g_{i+1}^{\text{inv}}}{\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} = \frac{\partial F}{\partial x_{i+1}^2}(x)e^{2\sigma E_{i+1}(x)} \tag{13.3.27}$$

Moreover, using again (13.3.14), we have for all $\ell, j \in \{1,\ldots,i-1\}$ we have

$$\left(\frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_i}(z)+\frac{\partial g_\ell^{\text{inv}}}{\partial z_{i+1}}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_{i+1}}(z)-2\frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z)\frac{\partial g_j^{\text{inv}}}{\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} = 0 \tag{13.3.28}$$

Using the equation for the second derivatives of g^{inv} given in (13.3.15) we have

$$\begin{aligned}&\frac{\partial F}{\partial x_i}(g^{\text{inv}}(z))\left(\frac{\partial^2 g_i^{\text{inv}}}{\partial z_i^2}(z)+\frac{\partial^2 g_i^{\text{inv}}}{\partial z_{i+1}^2}(z)-2\frac{\partial^2 g_i^{\text{inv}}}{\partial z_i\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)}\\&=\frac{\partial F}{\partial x_i}(x)\left(\sigma e^{2\sigma E_i(x)}+\sigma e^{2\sigma E_i(x)}(1-e^{-\sigma x_i})-2\sigma e^{2\sigma E_i(x)}\right)\\&=\frac{\partial F}{\partial x_i}(x)(-\sigma e^{2\sigma E_{\ell+1}(x)}),\end{aligned} \tag{13.3.29}$$

$$\frac{\partial F}{\partial x_{i+1}}(g^{\text{inv}}(z))\frac{\partial^2 g_{i+1}^{\text{inv}}}{\partial z_{i+1}^2}(z)\Bigg|_{z=g(x)} = \frac{\partial F}{\partial x_{i+1}}(g^{\text{inv}}(z))\sigma e^{2\sigma E_{i+1}(x)} \tag{13.3.30}$$

and for all $\ell \in \{1,\ldots,i-1\}$ we have

$$\left(\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_i^2}(z)+\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_{i+1}^2}(z)-2\frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_i\partial z_{i+1}}(z)\right)\Bigg|_{z=g(x)} = 0 \tag{13.3.31}$$

Using (13.3.22), (13.3.23) and (13.3.24) with the results given in equations (13.3.25), (13.3.26), (13.3.27) and (13.3.28) we obtain the aimed equation for the second derivative part (13.3.21). We now consider the term with the first derivatives in (13.3.2) and we obtain

$$\left(\frac{\partial f}{\partial z_{i+1}}(z)-\frac{\partial f}{\partial z_i}(z)\right)\Bigg|_{z=g(x)} = e^{-\sigma E_{i+1}(x)}\left(\frac{\partial F}{\partial z_{i+1}}(x)-\frac{\partial F}{\partial z_i}(x)\right) \tag{13.3.32}$$

Here we have used the relations

$$\begin{aligned}\frac{\partial F}{\partial x_i}(g^{\text{inv}}(z))\left(\frac{\partial g_i^{\text{inv}}}{\partial z_{i+1}}(z)-\frac{\partial g_i^{\text{inv}}}{\partial z_i}(z)\right)\Bigg|_{z=g(x)} &=\frac{\partial F}{\partial x_i}(x)\left(e^{\sigma E_i(x)}(1-e^{-\sigma x_i})-e^{\sigma E_i(x)}\right)\\&=\frac{\partial F}{\partial x_i}(x)(-e^{\sigma E_{i+1}(x)}),\end{aligned} \tag{13.3.33}$$

$$\frac{\partial F}{\partial x_{i+1}}(g^{\text{inv}}(z))\frac{\partial g_{i+1}^{\text{inv}}}{\partial z_{i+1}}(z)\Bigg|_{z=g(x)} = \frac{\partial F}{\partial x_{i+1}}(g^{\text{inv}}(z))e^{\sigma E_{i+1}(x)} \tag{13.3.34}$$

and, for all $\ell \in \{1,\ldots,i-1\}$,

$$\left(\frac{\partial g_\ell^{\text{inv}}}{\partial z_{i+1}}(z)-\frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z)\right)\Bigg|_{z=g(x)} = 0 \tag{13.3.35}$$

that follow from (13.3.14).
Using (13.3.19), (13.3.20), (13.3.21) and (13.3.32) and (13.3.32) equation (13.3.18) is proved.

Boundary. We aim to show that, for all $(i,j) \in \Lambda_L \times \Lambda_L^{\text{res}}$, $x \in \Omega$,

$$\left[\mathcal{L}_{i,j}^{\text{BEP}} f\right](g(x)) = \left[\mathcal{L}_{i,j}^{\text{ABEP}} f \circ g\right](x) \tag{13.3.36}$$

where

$$\mathcal{L}_{i,j}^{\text{BEP}} f(z) = \left[T_j \left(2k \frac{\partial}{\partial_{z_i}} + z_i \frac{\partial^2}{\partial_{z_i}^2}\right) - z_i \frac{\partial}{\partial_{z_i}}\right] f(z) \tag{13.3.37}$$

and where

$$\begin{aligned}
[\mathcal{L}_{i,j}^{\text{ABEP}} f](x) = (2kT_j - g_i(x)) &\left[\sum_{l=1}^{i-1} e^{\sigma E_l(x)} \left(1 - e^{-\sigma x_l}\right) \partial_{x_l} + e^{\sigma E_i(x)} \partial_{x_i}\right] f(x) + \\
&T_j g_i(x) \left[\sum_{l,j=1}^{i-1} e^{\sigma E_l(x)} \left(1 - e^{-\sigma x_l}\right) e^{\sigma E_j(x)} \left(1 - e^{-\sigma x_j}\right) \partial_{x_l x_j}^2 + e^{2\sigma E_i(x)} \partial_{x_i}^2\right. \\
&2 \sum_{l=1}^{i-1} e^{\sigma E_l(x)} \left(1 - e^{-\sigma x_l}\right) e^{\sigma E_i(x)} \partial_{x_l x_i}^2 \\
&\left. + \sum_{l=1}^{i-1} \sigma e^{2\sigma E_l(x)} \left(1 - e^{-2\sigma x_l}\right) \partial_{x_l} + \sigma e^{2\sigma E_i(x)} \partial_{x_i}\right] f(x) \ .
\end{aligned} \tag{13.3.38}$$

Here we have that

$$\frac{\partial f}{\partial z_i}(z) = \sum_{\ell=1}^{i} \frac{\partial F}{\partial x_\ell}(g_\ell^{\text{inv}}(z)) \frac{\partial g_\ell^{\text{inv}}}{\partial z_i}(z) \tag{13.3.39}$$

$$\frac{\partial^2 f}{\partial z_i^2}(z) = \sum_{\ell,j=1}^{i} \frac{\partial^2 F}{\partial x_j \partial x_\ell}(g^{\text{inv}}(z)) \frac{g_\ell^{\text{inv}}}{\partial z_i}(z) \frac{g_j^{\text{inv}}}{\partial z_i}(z) + \sum_{\ell=1}^{i} \frac{\partial F}{\partial x_\ell}(g^{\text{inv}}(z)) \frac{\partial^2 g_\ell^{\text{inv}}}{\partial z_i^2}(z) \tag{13.3.40}$$

Using these equation and by substituting the expressions (13.3.14) and (13.3.15) in (13.3.37) we obtain (13.3.38) we obtain. Therefore (13.3.8) follows. Moreover, we recver the left and the right boundary generators by putting $\mathcal{L}_{0,1}^{\text{ABEP}} := \mathcal{L}_{\text{left}}^{\text{ABEP}}$ and $\mathcal{L}_{L,L+1}^{\text{ABEP}} := \mathcal{L}_{\text{right}}^{\text{ABEP}}$ respectively.

□

Remark 34 *(Algebraic construction of the bulk generator of the ABEP) In Section 3.2.2 we reported the algebraic construction of the bulk generator of the BEP(2k) via a representation, acting on smooth functions, of the $su(1,1)$ Lie algebra. Using (13.3.5) one can show that the generator of the ABEP($\sigma, 2k$) has the same Lie algebraic structure of BEP(2k), i.e. for all $i \in \Lambda_L$*

$$\mathcal{L}_{i,i+1}^{\text{ABEP}} = \widetilde{K}_i^+ \widetilde{K}_{i+1}^- + \widetilde{K}_i^- \widetilde{K}_{i+1}^+ - \widetilde{K}_i^0 \widetilde{K}_{i+1}^0 + 2k^2 \tag{13.3.41}$$

where, for all $a \in \{+,-,0\}$, the generators of the algebra $\widetilde{K}_i^a$ are connected with the one of $su(1,1)$ reported in (3.2.64) via the function g introduced in Definition (19) as

$$\widetilde{K}_i^a = C_g \circ K_i^a \circ C_{g^{-1}} \ . \tag{13.3.42}$$

Here, one can check that C_g is an algebra-homomorphism that, for all $f : \Omega \to \mathbb{R}$, acts as

$$\left(C_{g^{-1}} f\right)(x) = \left(f \circ g^{-1})\right)(x) \qquad \left(C_{g^{-1}} f\right)(x) = (f \circ g)(x) \tag{13.3.43}$$

More explicitly, we have that

$$(\widetilde{K}_i^a f \circ g)(x) = (K_i^a f)(g(x)) \tag{13.3.44}$$

For the details of this algebraic structure of the ABEP($\sigma, 2k$) we refer to [51].

13.3.1 Some general definitions and properties

The construction of ABEP$(\sigma, 2k)$ as a non-local transformation of BEP$(2k)$, allows to derive several fundamental properties of the asymmetric process, such as duality properties (the structure of the stationary measure (see Definition 4). These properties are obtained by starting from the analogous properties of the symmetric process and projecting them via the map g. Having this goal in mind, in this section we prove some general results relating two Markov processes that are connected via a map transformation.
In the next theorem we will see that if a stationary measure, a reversible measure or a duality function are known for one of the a processes, then the corresponding object can be computed for a process obtained from the original one via a transformation.

Theorem 24 *Let g be a map $g : \Omega \to \Omega$, with $\Omega \subseteq \mathbb{R}_+^L$ and let $\mathcal{L}$ and $\widehat{\mathcal{L}}$ be the infinitesimal generators of two Markov processes on the state spaces, respectively Ω and $\widehat{\Omega} := g(\Omega)$. Suppose that $\forall f \in \mathcal{D}(\mathcal{L})$ it holds that $f \circ g \in \mathcal{D}(\widehat{\mathcal{L}})$ and*

$$\widehat{\mathcal{L}}(f \circ g) = (\mathcal{L} f) \circ g \tag{13.3.45}$$

then we have the following properties.

i) Let μ be a measure on Ω absolutely continuous w.r.t. Lebesgue. Let $\mathcal{J}$ be the Jacobian matrix of the map g. If μ is a stationary (reversible) measure for $\mathcal{L}$ then

$$\hat{\mu} := (\mu \cdot \det \mathcal{J}) \circ g \tag{13.3.46}$$

is a stationary (reversible) measure for $\widehat{\mathcal{L}}$.

ii) Let L be the infinitesimal generators a Markov processes on the state space Ω^{dual}. If $\mathcal{L}$ is dual to L with duality function $D : \Omega \times \Omega^{\text{dual}} \to \mathbb{R}$, then $\widehat{\mathcal{L}}$ is dual to L with duality function $D : \widehat{\Omega} \times \Omega^{\text{dual}} \to \mathbb{R}$

$$\widehat{D}(\cdot, \xi) := D(\cdot, \xi) \circ g \ , \qquad\qquad \xi \in \Omega^{\text{dual}} \ . \tag{13.3.47}$$

Proof of Theorem 24:

i) Due to the absolute continuity of μ we can write, with a slight abuse of notation, that $\mu(dx) = \mu(x)\ dx$. The stationarity condition for μ with respect to $\mathcal{L}$ then reads

$$\int [\mathcal{L} f](z)\, \mu(z)\ dz = 0, \qquad \text{for all} \quad f \in \mathcal{D}(\mathcal{L}) \tag{13.3.48}$$

that, taking the change of variables $z = g(x)$, gives

$$\int [\mathcal{L} f](g(x))\ \cdot \mu(g(x)) \cdot \det \mathcal{J}(\mathrm{g(x)})\, \mathrm{dx} = 0, \qquad \text{for all} \quad \mathrm{f} \in \mathcal{D}(\mathcal{L}) \tag{13.3.49}$$

that, thanks to (13.3.45) and (13.3.46), it is equivalent to

$$\int [\widehat{\mathcal{L}}(f \circ g)](x)\ \cdot \hat{\mu}(x)\, dx = 0, \qquad \text{for all} \quad f \in \mathcal{D}(\mathcal{L}) \ . \tag{13.3.50}$$

Due to the fact that $D(\widehat{\mathcal{L}}) = \{F = f \circ g :\ f \in D(\mathcal{L})\}$, the last identity can be rewritten as

$$\int [\widehat{\mathcal{L}} F](x)\ \cdot \hat{\mu}(x)\, dx = 0, \qquad \text{for all} \quad F \in \mathcal{D}(\widehat{\mathcal{L}}) \tag{13.3.51}$$

that is the stationary condition of $\hat{\mu}$ with respect to $\widehat{\mathcal{L}}$. The statement regarding reversible measures can be proven in an analogous way.

ii) To prove the second statement we use the duality relation between $\mathcal{L}$ and L and take the composition of the duality function (as a function of the variable x) with the function g. For $x \in \Omega$ and $\xi \in \Omega^{\text{dual}}$, we have

$$\begin{aligned}\left[\widehat{\mathcal{L}}\widehat{D}(\cdot,\xi)\right](x) &= \left[\widehat{\mathcal{L}}(D(\cdot,\xi)\circ g)\right](x) && (13.3.52)\\ &= \left[\mathcal{L}D(\cdot,\xi)\right](g(x))\\ &= \left[LD(g(x),\cdot)\right](\xi)\\ &= \left[L\widehat{D}(x,\cdot)\right](\xi)\ . && (13.3.53)\end{aligned}$$

This concludes the proof of the second item.

□

In the next two sections we specialize the argument of the above theorem for our model of interest. In Section 13.4 we focus on the case in which the external reservoirs impose the same temperatures (i.e. when $T_\ell = T_r = T$). We prove that in this situation ABEP$(\sigma, 2k)$ is reversible and we find the reversible measure. In Section 13.5 we find two duality relations for ABEP$(\sigma, 2k)$.

13.4 Equal temperature reservoirs

In this section use item i) of Theorem 24 to withdraw some conclusions concerning the case in which the two reservoirs have the same temperature. The idea is to import this property from the reversibility of the corresponding symmetric process. From Section 3.1.4 of Chapter 2 we know indeed that, in absence of reservoirs, the BEP$(2k)$ is reversible. In particular it admits a one-parameter family of reversible probability measures μ_T, $T \geqslant 0$, that are products of Gamma distributions of shape parameters $2k$ and scale parameter T, i.e. $\mu_T^{\text{BEP}}(z)\ dz$ with

$$\mu_T^{\text{BEP}}(z) = \prod_{i=1}^{L} \frac{e^{-z_i/T} z_i^{(2k-1)}}{\Gamma(2k)T^{2k}} \tag{13.4.1}$$

When the process is in contact with two reservoirs kept at equal temperatures, $T_\ell = T_r = T$, the process remains reversible, admitting μ_T as the unique stationary probability measure. In the following Theorem we extend the statement to the asymmetric process, for which we prove the existence of a unique reversible probability measure that is in the form of a product measure times a function of the total energy of the system $E(x)$.

Theorem 25 (Reversible measure for ABEP) *The ABEP$(\sigma, 2k)$ with equal reservoir temperatures $T_\ell = T_r = T$ is reversible with respect to the unique stationary probability measure $\mu_T^{\text{ABEP}}(x)\ dx$, with*

$$\mu_T^{\text{ABEP}}(x) = \exp\left\{\frac{e^{-\sigma E(x)}-1}{\sigma T}\right\} \cdot \prod_{i=1}^{L} \frac{(1-e^{-\sigma x_i})^{(2k-1)}}{\Gamma(2k)\sigma^{2k-1}T^{2k}}\ e^{-\sigma x_i(2k(i-1)+1)} \tag{13.4.2}$$

Proof of Theorem 25: We want to use item i) of Theorem 24. To this aim it is enough to compute $(\mu_T^{\text{BEP}} \circ g)(x)$. Indeed,

$$\mu_T^{\text{ABEP}}(x) = ((\mu_T^{\text{BEP}} \det\mathcal{J}(g(x))) \circ g)(x) = \prod_{i=1}^{L} \mu_T^{\text{BEP}}(g_i(x)) \det\mathcal{J}(g(x))$$

$$
\begin{aligned}
&= \prod_{i=1}^{L} \frac{e^{-g_i(x)/T}(g_i(x))^{(2k-1)}}{\Gamma(2k)T^{2k}} \det\mathcal{J}(g(x)) \\
&= \prod_{i=1}^{L} \frac{1}{\Gamma(2k)T^{2k}} \exp\left\{-\frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{\sigma T}\right\} \\
&\times (1-e^{-\sigma x_i})^{(2k-1)} \, \frac{e^{-\sigma(2k-1)E_{i+1}(x)}}{\sigma^{(2k-1)}} \, e^{-\sigma E_i(x)} \\
&= \exp\left\{\frac{e^{-\sigma E(x)}-1}{\sigma T}\right\} \cdot \prod_{i=1}^{L} \frac{(1-e^{-\sigma x_i})^{(2k-1)}}{\sigma^{2k-1}T^{2k}\Gamma(2k)} e^{-(2k\sigma i+T)x_i} e^{(2k\sigma+T-\sigma)x_i} \, .
\end{aligned}
$$

Here, by calling again $z = g(x)$, we have that the determinant of the $N \times N$ Jacobian matrix $\mathcal{J}$ is given by

$$
\det\left(\mathcal{J}(\mathrm{z})\Big|_{\mathrm{z}=\mathrm{g}(\mathrm{x})}\right) = \mathrm{e}^{-\sigma \sum_{\mathrm{i}=1}^{\mathrm{L}} \mathrm{E}_{\mathrm{i}}(\mathrm{x})} \tag{13.4.3}
$$

where equation (13.3.14) has been used. Therefore, (13.4.2) follows.

□

Remark 35 *In Theorem 3.3 of [51] a family of reversible measures has been found for ABEP$(\sigma, 2k)$ with closed boundary. This family is labeled by the temperature T. The measure corresponding to the temperature T (eq. (3.15 - 3.16) of [51]) does not match with μ_T^{ABEP} found in* (13.4.2). *Indeed it differs from it only for the factor in front of the product in* (13.4.2) *that is a function of the total energy $E(x)$. This is due to the fact that, in absence of reservoirs, the total energy is an invariant of the dynamics, and then this term becomes a constant that simplifies with the normalizing factor of the probability measure. In the presence of two thermal reservoirs instead, even in the case of equal temperatures $T_\ell = T_r = T$, the system does not conserve the total energy anymore, and the initial factor in* (13.4.2) *can not be neglected anymore.*

13.5 Duality results

When $T_\ell \neq T_r$ reversibility is lost. Nevertheless there exists a unique stationary measure depending on both temperatures T_ℓ and T_r. However a full characterization of such a measure is a difficult and still open problem, even for the symmetric case. A tool that has proven to be of great help in the study of the properties of the stationary measure is duality (see Section 3.2.1. We will return to the study of steady state in Section 13.6. In the next section we prove two duality relations between the Asymmetric Brownian Energy process and the Symmetric Inclusion process with absorbing boundaries.

13.5.1 Duality between ABEP and SIP

As we have proved in Theorem 12 there is a duality result linking the boundary driven BEP(2k) and the SEP(2k) with absorbing boundaries. Moreover, from Section 3.2.1 it follows that we can discover information about the non-equilibrium steady state of the boundary driven BEP(2k) by studying the SIP(2K) with absorbing boundaries.

This result can be extended also for the asymmetric case. In particular, for the closed boundary case the Inclusion process with closed boundaries has been proved to be dual to the

ABEP (see [51]). This is the first example of duality between an asymmetric system (i.e. bulk-driven) and a symmetric system (with zero current). This is made possible by the fact that the dependence on the asymmetry parameter σ is transferred to the duality function (in the spirit of item *ii*) of Theorem 24. Here we generalize the result to the ABEP with reservoirs defined in (13.2.3). Again, we show that it is dual to the Inclusion process with absorbing boundaries, exactly as its symmetric counterpart. This property will be proven using item *ii*) of Theorem 24 and using equation (13.3.5) that connects ABEP and BEP through the map g. We will prove two different duality relations between the same two processes. The first duality relation is via the so-called *classical duality function* [12], the second is in terms of a duality function that is a product of Laguerre polynomials, i.e. of the type *orthogonal polynomial duality function.* This orthogonal duality result has been proven in [125] only for a symmetric closed boundary system, therefore, we first extend it to a symmetric boundary-driven system (see Theorem 26) and we finally generalize for the asymmetric boundary-driven process (see Theorem 27).

Theorem 26 *(Absorbing duality between open BEP and SIP) The BEP(*$2k$*) with an open boundaries, with generator* $\mathcal{L}^{\mathrm{BEP}}$ *defined in* (13.3.1)-(13.3.4)*, is dual to the SIP(*$2k$*) with absorbing boundaries defined in Definition 3.2.77 with respect to the orthogonal duality function*

$$\mathfrak{D}_T(z,\xi) = (T_\ell - T)^{\xi_0} \cdot \prod_{i=1}^{L} (-T)^{\xi_i} \cdot {}_1F_1\left(\begin{matrix} -\xi_i \\ 2k \end{matrix} \middle| \frac{z_i}{T} \right) \cdot (T_r - T)^{\xi_{L+1}} \ , \tag{13.5.1}$$

for all $T > 0$*. Above we wrote the orthogonal duality function in terms of the* ${}_1F_1$ *hypergeometric function, that, for* $n \in \mathbb{N}$*, is defined (see Section 1.4 of [126]) as* ${}_1F_1\left(\begin{matrix} -n \\ 2k \end{matrix} \middle| x \right) := \sum_{k=0}^{n} \frac{(-x)^k}{k!} \frac{n!}{(n-k)!} \frac{\Gamma(2k)}{\Gamma(2k+n)}$.

Proof of Theorem 26: We have to show that

$$[\mathcal{L}^{\mathrm{BEP}} \mathfrak{D}_T(\cdot, \xi)](z) = [L^{\mathrm{SIP}} \mathfrak{D}_T(z, \cdot)](\xi) \tag{13.5.2}$$

Since both $\mathcal{L}^{\mathrm{BEP}}$ and L^{SIP} of a bulk term and two reservoir terms, it is sufficient to show that the duality relation for generators holds true term by term. The relation for the bulk terms of the generators has been proved in Section 4.2 of [125], where it has been shown that, defining $d(\zeta, k) = (-T)^k {}_1F_1\left(\begin{smallmatrix} -k \\ 2k \end{smallmatrix} \middle| \frac{\zeta}{T} \right)$, for all $i \in \{1, \ldots, N-1\}$,

$$[\mathcal{L}^{\mathrm{BEP}}_{i,i+1} d_T(\cdot, \xi_i) \cdot d(\cdot, \xi_{i+1})](z_i, z_{i+1}) = [L^{\mathrm{SIP}}_{i,i+1} d(z_i, \cdot) \cdot d(z_{i+1}, \cdot)](\xi_i, \xi_{i+1}) \ . \tag{13.5.3}$$

It is remains to show that the duality relation holds for the two boundary terms. i.e. that

$$[\mathcal{L}^{\mathrm{BEP}}_{\mathrm{left}} \mathfrak{D}_T(\cdot, \xi)](z) = [L^{\mathrm{abs}}_{\mathrm{left}} \mathfrak{D}_T(z, \cdot)](\xi) \qquad \text{and} \qquad [\mathcal{L}^{\mathrm{BEP}}_{\mathrm{right}} \mathfrak{D}_T(\cdot, \xi)](z) = [L^{\mathrm{abs}}_{\mathrm{right}} \mathfrak{D}_T(z, \cdot)](\xi) \ . \tag{13.5.4}$$

Being the two relations completely analogous, it is sufficient to prove one of them, we prove it for the left boundary. We note that $\mathcal{L}^{\mathrm{BEP}}_{\mathrm{left}}$ acts only on site one whereas $L^{\mathrm{abs}}_{\mathrm{left}}$ acts only on sites 0 and 1. For this reason it is sufficient to show that, for $d_\ell(k) := (T_\ell - T)^k$,

$$[\mathcal{L}^{\mathrm{BEP}}_{\mathrm{left}} d_\ell(\xi_0) d(\cdot, \xi_1)](z_1) = [L^{\mathrm{abs}}_{\mathrm{left}} d_\ell(\cdot) d(z_1, \cdot)](\xi_0, \xi_1) \tag{13.5.5}$$

At this aim, using the hypergeometric relation satisfied by Laguerre polynomials (see Section 9.12 in [126]), we find that

$$z_1\partial_{z_1}^2 d(z_1,\xi_1) + 2k\partial_{z_1} d(z_1,\xi_1) = \xi_1 d(z_1,\xi_1-1) \tag{13.5.6}$$
$$z_1\partial_{z_1} d(z_1,\xi_1) = \xi_1 d(z_1,\xi_1) + \xi_1 T d(z_1,\xi_1-1)\ . \tag{13.5.7}$$

The above identities allow us to write the action of $\mathcal{L}_{\text{left}}^{\text{BEP}}$ on $d(z_1,\xi_1)$ as an action on the variable ξ_1.

$$\begin{aligned}[\mathcal{L}_{\text{left}}^{\text{BEP}} d_\ell(\xi_0) d(\cdot,\xi_1)](z_1) &= (T_\ell - T)^{\xi_0}\left[T_\ell \xi_1 d\left(z_1,\xi_1-1\right) - \xi_1 d\left(z_1,\xi_1\right) - \xi_1 T d\left(z_1,\xi_1-1\right)\right] = \\ \xi_1 \Big[(T_\ell - T)^{\xi_0+1} d(z_1,\xi_1-1) &- (T_\ell - T)^{\xi_0} d(z_1,\xi_1)\Big] = [L_{\text{left}}^{\text{abs}} d_\ell(\cdot) d(z_1,\cdot)](\xi_0,\xi_1)\end{aligned}$$

that concludes the proof.

□

Remark 36 *The so called orthogonal duality function $\mathfrak{D}_T$ is related to the so-called generalized Laguerre polynomial via a normalizing factor only depending on the variable ξ. More precisely, the generalized Laguerre polynomial of degree n, variable x and parameter $2k$ is defined as follows*

$$\mathfrak{L}_\xi^{(2k-1)}(z) = \frac{\Gamma(2k+\xi)}{\Gamma(2k)\xi!}\, {}_1F_1\left(\begin{matrix}-\xi \\ 2k\end{matrix}\,\middle|\, z\right) \tag{13.5.8}$$

and then the single site duality function d is related to these via the following relation

$$d(\zeta,k) = (-T)^k \cdot \frac{\Gamma(2k)k!}{\Gamma(2k+k)} \cdot \mathfrak{L}_k^{(2k-1)}(\zeta)\ . \tag{13.5.9}$$

Once the duality relation for the symmetric process is proven we can invoke Theorem 24 to extend the result to the ABEP.

Theorem 27 *(Absorbing duality between open ABEP and SIP) The ABEP$(\sigma, 2k)$ with an open boundaries, with generator $\mathcal{L}^{\text{ABEP}}$ defined in* (13.2.3)-(13.2.6)*, is dual to the SIP*$(2k)$ *with absorbing boundaries defined in Definition 3.2.77 with respect to the following duality functions:*

1. **classical duality:**

$$D^\sigma(x,\xi) = T_\ell^{\xi_0} \cdot \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k+\xi_i)} (g_i(x))^{\xi_i} \cdot T_r^{\xi_{L+1}}, \tag{13.5.10}$$

2. **orthogonal duality:**

$$\mathfrak{D}_T^\sigma(x,\xi) = (T_\ell - T)^{\xi_0} \cdot \prod_{i=1}^{L} (-T)^{\xi_i} \cdot {}_1F_1\left(\begin{matrix}-\xi_i \\ 2k\end{matrix}\,\middle|\, \frac{g_i(x)}{T}\right) \cdot (T_r - T)^{\xi_{L+1}}\ , \tag{13.5.11}$$

for all $T > 0$. Here g is the map given in Definition 19.

Proof of Theorem 27: the result is a natural consequence of Theorem 12, Theorem 26 and the second item of Theorem 24.

□

13.6 Applications of duality

Due to irreducibility, the ABEP admits a unique stationary probability measure, that we will also call steady state and we will denote it by μ_{ss}. When $T_\ell = T_r = T$ this is reversible and coincides with the measure μ_T computed in Theorem 25. When $T_\ell \neq T_r$, reversibility is lost and μ_{ss} is no longer easy to compute. We will take advantage of the duality property proven in Section 13.5 to compute some particular observables of μ_{ss}, and more precisely, the one and two-point correlations, with respect to μ_{ss}, of the observables $\{e^{-\sigma E_i(x)},\ i \in \Lambda_L\}$ that are inherently related to the non-local map g. We will informally call these quantities σ-exponential moments or correlations. Inspired by the connection between absorption probabilities and the non-equilibrium steady state reported in Section 3.2.1, we exploit the simplicity of the dual process (SIP(2k) with absorbing boundaries) to compute the σ-exponential moments in terms of the absorption probabilities of the SIP particles.

Proposition 21 *Let μ_{ss} be the stationary measure of ABEP(σ, $2k$) with open boundaries defined in* (13.2.3)-(13.2.6)*, then*

$$\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)}\right] = 1 - 2k\sigma\, T_\ell(L-m+1) + \frac{2k\sigma}{L+1}(T_r - T_\ell)\,\frac{(m+L)(m-L-1)}{2}\ . \tag{13.6.1}$$

Here $m \in \{1, \ldots, L\}$.

Proof of Proposition 21: Let $\delta_i \in \Omega^{\text{dual}}$ the SIP($2k$) configuration with just one particle at site $i \in \Lambda_L$, then

$$D^\sigma(x, \delta_i) = \frac{\Gamma(2k)}{\Gamma(2k+1)} \cdot g_i(x) = \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{2k\sigma} = \frac{e^{-\sigma E_{i+1}(x)}}{2k\sigma}\,\left(1 - e^{-\sigma x_i}\right) \tag{13.6.2}$$

If we initialize the dual SIP($2k$) with one particle at site $i \in \Lambda_L$, the dynamics can be described by a continuous time random walk $\{i(t),\ t \geqslant 0\}$ moving on the lattice $\Lambda_L \cup \Lambda_L^{\text{res}}$ performing nearest-neighbor jumps at rate $2k$ and absorbed at boundary sites 0 and $L+1$. We will denote by $\mathbb{P}_i$ the probability distribution of this process initialized at time 0 from site $i \in \Lambda_L$. Then the stationary expectation of the quantity in the right hand side of (13.6.2) linearly interpolates between T_ℓ and T_r:

$$\begin{aligned}\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_{i+1}(x)}(1 - e^{-\sigma x_i})\right] &= \mathbb{E}_{\mu_{ss}}[D^\sigma(x, \delta_i)] \\ &= \lim_{t\to\infty} \mathbb{P}_i(i_t = 0) D^\sigma(x, \delta_0) + \mathbb{P}_i(i_t = L+1) D^\sigma(x, \delta_{L+1}) \\ &= 2k\sigma\left(T_\ell + (T_r - T_\ell)\frac{i}{L+1}\right)\ . \end{aligned} \tag{13.6.3}$$

We take now the sum from m to L on both sides of equation (13.6.3) to get telescopic cancellation. Since $E_{L+1} = 0$, we get

$$\mathbb{E}_{\mu_{ss}}\left[1 - e^{-\sigma E_m(x)}\right] = \sum_{i=m}^{L}\left(2k\sigma T_\ell + 2k\sigma\,(T_r - T_\ell)\,\frac{i}{L+1}\right) \tag{13.6.4}$$

from which follows the result.

□

In the next proposition we will show how to relate the above observation to gather information on the stationary σ-exponential expectation of the partial energies.

Remark 37 *Notice that the observables* $\{e^{-\sigma E_i(x)},\ i \in \Lambda_L\}$ *are reminiscent of the microscopic Cole-Hopf transform (known as the Gärtner transform that has been defined in [127] for the asymmetric exclusion process). The Cole-Hopf transform has been used in the literature to connect the KPZ equation for random growing interfaces and the stochastic heat equation. As remarked in [112], the first hint that such transform is available relies on the existence of a Markov duality relation.*

In order to compute the stationary two-point correlation of the exponential observables $\{e^{-\sigma E_i(x)},\ i \in \Lambda_L\}$ we use the same strategy used in the proof of Proposition 21 to compute the σ-exponential moments. In this case, though, we initialize the dual system with two (and no longer one) particles.

Proposition 22 *Let* μ_{ss} *be the stationary measure of ABEP(*σ*, 2k) with open boundaries defined in* (13.2.3)-(13.2.6)*, then*

$$\begin{aligned}
&\mathbb{E}_{\mu_{ss}}\left[e^{-\sigma E_m(x)}e^{-\sigma E_n(x)}\right] = 1 - 2k\sigma T_\ell(2L-m-n+2) \\
&+ \frac{2k\sigma}{2(L+1)}(T_r - T_\ell)[m^2+n^2-2L^2-2L-m-n] \\
&+ \frac{(2k\sigma)^2(1-m+L)(1-n+L)}{2(L+1)(1+2k(L+1))}\left[T_\ell^2(N-m+2)(1+\tfrac{2}{k}2(L-n+2))\right. \\
&\left.+T_r^2(L+n)(1+\tfrac{2}{k}2(L+m)) + T_\ell T_r(m(1-2k(n-1))-n+2k(n+L(N+2)))\right] \\
&+ \frac{(2\sigma)^2 2k(1-n+L)}{2(L+1)(1+2k(L+1))}\left[T_\ell^2\left(\frac{2k}{3}(2n^2+2L^2+2nL-n+L)\right.\right. \\
&\left.-(n+L)[4k(L+1)+1]+2L+1+4k(L+1)^2\right) \\
&+T_r^2\left(\frac{2k}{3}(2n^2+2L^2+2nL-n+L)+(n+L)-1\right) \\
&\left.+2T_\ell T_r\left(-\frac{2k}{3}(2n^2+2L^2+2nL-n+L)+(n+L)(2k(L+1)-1)+1\right)\right]
\end{aligned}$$

where $m, n \in \{1, \ldots, L\}$ *with* $m \leqslant n$*.*

Proof of Proposition 22: Let $\xi = \delta_i + \delta_j \in \Omega^{\text{dual}}$ be the dual configuration with two particles at sites $i, j \in \Lambda_L$, $i \neq j$. The duality function evaluated in ξ is then given by

$$D^\sigma(x, \delta_i + \delta_j) = \frac{e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}}{2k\sigma} \cdot \frac{e^{-\sigma E_{j+1}(x)} - e^{-\sigma E_j(x)}}{2k\sigma}\ . \tag{13.6.5}$$

Considering the expectation with respect to the stationary measure:

$$\mathbb{E}_{\mu_{ss}}\left[\left(e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}\right)\left(e^{-\sigma E_{j+1}(x)} - e^{-\sigma E_j(x)}\right)\right] = (2k\sigma)^2 \cdot \mathbb{E}_{\mu_{ss}}[D^\sigma(x, \delta_i+\delta_j)] \tag{13.6.6}$$

$$\begin{aligned}
&= (2k\sigma)^2 \lim_{t\to\infty}\Big\{\mathbb{P}_{i,j}(i_t=0, j_t=0)D^\sigma(x, 2\delta_0) + \mathbb{P}_{i,j}(i_t=L+1, j_t=L+1)D^\sigma(x, 2\delta_{L+1}) + \\
&D^\sigma(x, \delta_0+\delta_{L+1})\left(\mathbb{P}_{i,j}(i_t=0, j_t=L+1) + \mathbb{P}_{i,j}(i_t=L+1, j_t=0)\right)\Big\}
\end{aligned}$$

$$= (2k\sigma)^2 \left\{ T_\ell^2 \frac{[1+2k(L+1-i)](L+1-j)}{(L+1)(1+2k(L+1))} + T_r^2 \frac{i(1+2kj)}{(L+1)(1+2k(L+1))} \right. \tag{13.6.7}$$

$$\left. + T_\ell T_r \frac{[2k(L+1)-1]i + [1+2k(L+1)]j - 4kij}{(L+1)(1+2k(L+1))} \right\} \tag{13.6.8}$$

where $\mathbb{P}_{i,j}$ is the probability distribution associated to two dual SIP$(2k)$ particles $\{(i(t), j(t)),\ t \geqslant 0\}$. On the other hand, if $i = j$ we have:

$$D^\sigma(x, 2\delta_i) = \frac{\left(e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)}\right)^2}{2k(2k+1)\sigma^2} \tag{13.6.9}$$

and considering the expectation with respect to the stationary measure:

$$\mathbb{E}_{\mu_{ss}} \left[\left(e^{-\sigma E_{i+1}(x)} - e^{-\sigma E_i(x)} \right)^2 \right] = \mathbb{E}_{\mu_{ss}} 2k(2k+1)\sigma^2 D^\sigma(x, 2\delta_i) \tag{13.6.10}$$

$$= 2k(2k+1)\sigma^2 \lim_{t\to\infty} \left\{ \mathbb{P}_{i,i}(i_t = 0, i_t = 0) D^\sigma(x, 2\delta_0) + \mathbb{P}_{i,i}(i_t = L+1, i_t = L+1) D^\sigma(x, 2\delta_{L+1}) + \right.$$

$$\left. D^\sigma(x, \delta_0 + \delta_{L+1}) \left(\mathbb{P}_{i,i}(i_t = 0, i_t = L+1) + \mathbb{P}_{i,i}(i_t = L+1, i_t = 0) \right) \right\}$$

$$= 2k(2k+1)\sigma^2 \left\{ T_\ell^2 \frac{2(L+1-i)(2k(L+1-i)+1) - 1}{2(L+1)(2k(L+1)+1)} + \right. \tag{13.6.11}$$

$$\left. T_r^2 \frac{2i(1+2ki) - 1}{2(L+1)(2k(L+1)+1)} + T_\ell T_r \frac{(2k(L+1)-1)i + (2k(L+1)-1)i - 4ki^2 + 1}{(L+1)(2k(L+1)+1)} \right\}$$

$$= 2k(2k+1)\sigma^2 \left\{ T_\ell^2 \frac{4ki^2 + (-8kN - 8k + 2)i + (4kN^2 + 8kN + 4k - 2N - 3)}{2(L+1)(2k(L+1)+1)} + \right. \tag{13.6.12}$$

$$\left. T_r^2 \frac{4ki^2 + 2i - 1}{2(L+1)(2k(L+1)+1)} + T_\ell T_r \frac{-4ki^2 + 2i(2kN + 2k - 1) + 1}{(L+1)(2k(L+1)+1)} \right\}$$

This allows us to gather information on the two-point σ-exponential stationary correlations. To achieve this we take a double sum in equation (13.6.6), one from m to L and one from n to L. By telescopic arguments one then gets

$$\begin{aligned}
&\mathbb{E}_{\mu_{ss}} \left[e^{-\sigma E_m(x)} e^{-\sigma E_n(x)} \right] = \mathbb{E}_{\mu_{ss}} \left[e^{-\sigma E_m(x)} \right] + \mathbb{E}_{\mu_{ss}} \left[e^{-\sigma E_n(x)} \right] - 1 + \\
&(2k\sigma)^2 \sum_{i=m}^{L} \sum_{j=n}^{L} \left\{ T_\ell^2 \mathbb{P}_{i,j}(i_t = 0, j_t = 0) + T_r^2 \mathbb{P}_{i,j}(i_t = L+1, j_t = L+1) \right. \\
&\left. + T_\ell T_r \left[\mathbb{P}_{i,j}(i_t = 0, j_t = L+1) + \mathbb{P}_{i,j}(i_t = L+1, j_t = 0) \right] \right\} \\
&+ (2\sigma)^2 2k \sum_{i=n}^{L} \left\{ T_\ell^2 \mathbb{P}_{i,i}(i_t = 0, i_t = 0) + T_r^2 \mathbb{P}_{i,i}(i_t = L+1, i_t = L+1) \right. \\
&\left. + T_\ell T_r \left[\mathbb{P}_{i,i}(i_t = 0, i_t = L+1) + \mathbb{P}_{i,i}(i_t = L+1, i_t = 0) \right] \right\}
\end{aligned} \tag{13.6.13}$$

where the first two terms on the right hand side have been computed in the previous theorem. To conclude the proof it remains to are plug in the expression above the absorption probabilities of two dual SIP$2k$ particles absorbed at the boundaries 0 and $L+1$. These are harmonic function of the two dimensional Laplacian. They solve a systems of discrete equations with appropriate boundary conditions. We show how to get $p_{i,j} := \mathbb{P}_{i,j}(i_t = 0, j_t = 0)$ for $i, j \in \Lambda_L$ as the others can be found similarly.

$$\begin{cases} 4p_{i,j} = p_{i-1,j} + p_{i+1,j} + p_{i,j-1} + p_{i,j+1} \\ 2p_{i,i} = p_{i-1,i} + p_{i,i+1} \end{cases} \tag{13.6.14}$$

for the first two equations we get that

$$p_{i,j} = Ai + Bj + Cij + D \quad \text{for} \quad i \neq j$$

and

$$p_{i,i} = (A+B)i + Ci^2 + D + \frac{B-A}{2} \ .$$

Three of the unknown can be found using the boundary conditions:

$$\begin{cases} p_{0,0} = D = 1 \\ p_{0,j} = Bj + D = 1 - \frac{j}{L+1} \\ p_{L+1,L+1} = A(L+1) + B(L+1) + C(L+1)^2 + D = 0 \end{cases} \tag{13.6.15}$$

while the last one can be found conditioning on the first jump, i.e.

$$(8k+2)p_{i,i+1} = 2kp_{i-1,i+1} + 2kp_{i,i+2} + (2k+1)p_{i,i} + (2k+1)p_{i+1,i+1} \ .$$

This leads to the following solutions for the four unknown

$$\begin{cases} A = -\frac{2k}{1+2k(L+1)} \\ B = -\frac{1}{L+1} \\ C = \frac{2k}{(1+L)(1+2k(L+1))} \\ D = 1 \end{cases} \tag{13.6.16}$$

Finally we obtain

$$\begin{aligned} &\mathbb{P}_{i,j}(i_t = 0, j_t = 0) = p_{i,j} \\ =&\frac{(L+1-j)(2k(-i+L+1)+1)}{(L+1)(2k(L+1)+1)} - \frac{1}{2(L+1)(2k(L+1)+1)}\mathbb{1}_{\{i=j\}} \end{aligned} \tag{13.6.17}$$

for the absorption probabilities of both particles to the left. Similarly one can get the absorption probabilities of both particles to the right:

$$\begin{aligned} &\mathbb{P}_{i,j}(i_t = L+1, j_t = L+1) = p_{i,j} \\ =&\frac{i(1+2kj)}{(L+1)(2k(L+1)+1)} - \frac{1}{2(L+1)(2k(L+1)+1)}\mathbb{1}_{\{i=j\}} \end{aligned} \tag{13.6.18}$$

and the absorption probability of one particle to the left and one to the right

$$\begin{aligned} &\mathbb{P}_{i,j}(i_t = 0, j_t = L+1) + \mathbb{P}_{i,j}(i_t = L+1, j_t = 0)p_{i,j} \\ =&\frac{(2k(L+1)-1)i + (2k(L+1)-1)j - 4kij}{(L+1)(2k(L+1)+1)} \\ +&\frac{1}{(L+1)(2k(L+1)+1)}\mathbb{1}_{\{i=j\}} \ . \end{aligned} \tag{13.6.19}$$

Substituting these expressions in (13.6.13) we obtain the result.

□

13.7 Perspectives

For what concerns the single species model here introduced, a possible future development is to derive the hydrodynamic limit on the infinite line by exploiting the properties of this map g (19). A similar idea has been used in [127], therefore we aim to adapt this technique to the ABEP here exposed. As natural follow-up question, one may study the hydrodynamic limit on a finite segment, obtaining limiting equations with boundary conditions of different type. Moreover, fluctuations from the hydrodynamic limit can be investigated (see for example the ABC model [33, 34])

Once this single species prototype will be well understood, we aim to generalize it to multi-species set up and extend duality and the scaling limits. This last aim could be a starting point for the derivation of bulk-boundary driven IPS that show multi-component uphill diffusion. This could also be useful for some applications in physics and engineering: for instance by giving a rigorous description of the electron transport induced by thermal gradients (see for instance [128]). Other possible applications of these asymmetric boundary driven transport models are the construction of rigorous models to describe other physical phenomena, like the ones of described in [129, 130, 131].

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