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A Hilbert–Mumford criterion for polystability for actions of real reductive Lie groups

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Abstract

We study a Hilbert–Mumford criterion for polystablility associated with an action of a real reductive Lie group G on a real submanifold X of a Kähler manifold Z. Suppose the action of a compact Lie group with Lie algebra $\mathfrak{u}$ extends holomorphically to an action of the complexified group $U^{\mathbb{C}}$ and that the U-action on Z is Hamiltonian. If $G \subset U^{\mathbb{C}}$ is compatible, there is a corresponding gradient map $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of the Lie algebra of G. Under some mild restrictions on the G-action on X, we characterize which G-orbits in X intersect $\mu_{\mathfrak{p}}^{-1}(0)$ in terms of the maximal weight functions, which we viewed as a collection of maps defined on the boundary at infinity ($\partial_\infty G/K$) of the symmetric space G/K. We also establish the Hilbert–Mumford criterion for polystability of the action of G on measures.

Keywords Momentum map · Hilbert criterion · Stability

Mathematics Subject Classification 53D20 · 14L24

1 Introduction

The classical Hilbert–Mumford criterion in Geometric Invariant Theory (in projective algebraic geometry) is an explicit numerical criterion for finding the stability of a point in terms of an invariant known as maximal weight function [26]. This criterion has been extended to the non-algebraic Kählerian settings using the theory of Kähler quotients and a version of maximal weight function [10, 22, 27, 28, 30]. For this setting, a Kähler manifold (Z, ω) with a holomorphic action of a complex reductive Lie group $U^{\mathbb{C}}$, where $U^{\mathbb{C}}$ is the complexification of a compact Lie group U with Lie algebra $\mathfrak{u}$ is considered. Assume ω is U-invariant

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and that there is a U-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$. By definition, for any $\xi \in \mathfrak{u}$ and $z \in Z$, $d\mu^\xi = i_{\xi_Z}\omega$, where $\mu^\xi(z) := \mu(z)(\xi)$ and ξ_Z denotes the fundamental vector field induced on Z by the action of U, i.e.,

$$\xi_Z(z) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)z$$

(see, for example, [23] for more details on the momentum map).

We aim to investigate a class of actions of real reductive Lie groups on real submanifolds of Z using gradient map techniques. This setting was recently introduced in [16–18]. More precisely, a subgroup G of $U^{\mathbb{C}}$ is compatible if G is closed and the map $K \times \mathfrak{p} \to G$, $(k, \beta) \mapsto k\exp(\beta)$ is a diffeomorpism where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. The Lie algebra $\mathfrak{u}^{\mathbb{C}}$ of $U^{\mathbb{C}}$ is the direct sum $\mathfrak{u} \oplus \mathrm{i}\mathfrak{u}$. It follows that G is compatible with the Cartan decomposition of $U^{\mathbb{C}} = U\exp(\mathrm{i}\mathfrak{u})$, K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The inclusion $\mathrm{i}\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction, a K-equivariant map $\mu_{\mathrm{i}\mathfrak{p}} : Z \to (\mathrm{i}\mathfrak{p})^*$. One can choose and fix an $\mathrm{Ad}(U^{\mathbb{C}})$-invariant inner product B of Euclidean type on the Lie algebra $\mathfrak{u}^{\mathbb{C}}$, see [10, Section 3.2], [25, Definition 3.2.4] and also [20, Section 2.1] for the analog in the algebraic GIT. Such an inner product will automatically induce a well-defined inner product on any maximal compact subgroup U' of $U^{\mathbb{C}}$.

Let $\langle \cdot, \cdot \rangle$ denote the real part B. Then $\langle \cdot, \cdot \rangle$ is positive definite on $\mathrm{i}\mathfrak{u}$, negative definite on $\mathfrak{u}$, $\langle \mathfrak{u}, \mathrm{i}\mathfrak{u} \rangle = 0$ and finally the multiplication by i satisfies $\langle \mathrm{i}\cdot, \mathrm{i}\cdot \rangle = -\langle \cdot, \cdot \rangle$. We use $-\langle \cdot, \cdot \rangle$ to identify $\mathfrak{u}^*$ with $\mathfrak{u}$ and we think the momentum map μ as a $\mathfrak{u}$-valued map. Hence we replace consideration of $\mu_{\mathrm{i}\mathfrak{p}}$ by that of $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$, where

$$\mu_{\mathfrak{p}}^{\beta}(x) := \langle \mu_{\mathfrak{p}}(x), \beta \rangle := \langle \mathrm{i}\mu(x), \beta \rangle = -\langle \mu(x), -\mathrm{i}\beta \rangle = \mu^{-\mathrm{i}\beta}(x).$$

The map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ is called the *G-gradient map associated with μ*. It is K-equivariant and $\operatorname{grad} \mu_{\mathfrak{p}}^{\beta} = \beta_Z$ for any $\beta \in \mathfrak{p}$. Here the grad is computed with respect to the Riemannian metric induced by the Kähler structure. For a G-stable locally closed real submanifold X of Z, we also denote the restriction $\mu_{\mathfrak{p}}$ to X by$\mu_{\mathfrak{p}} : X \to \mathfrak{p}$. We have $\operatorname{grad} \mu_{\mathfrak{p}} = \beta_X$ for any $\beta \in \mathfrak{p}$, where the gradient is now computed with respect to the induced Riemannian metric on X.

Different notions of stability of points in X can be identified by taking into account the position of their G-orbits with respect to $\mu_{\mathfrak{p}}^{-1}(0)$. A point $x \in X$ is polystable if it's G-orbit intersects the level set $\mu_{\mathfrak{p}}^{-1}(0)$ (i.e., $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$). As pointed out in the introduction of [28] (see also [5]), a set of polystable points plays a critical role in the construction of a good quotient of X by the action of G. The aim of this article is to answer the first part of question 1.1 in [28] for actions of real Lie groups on real submanifolds of a Kähler manifold, generalizing [28]. Following [28], we require a mild technical restriction to be satisfied; namely, the fundamental vector field induced by the action grows at most linearly with respect to the distance function from a given base point. More precisely, we require the following assumption.

Assumption 1.1 X is connected, and there exists a point $x_0 \in X$ and a constant $C > 0$ such that for any $x \in X$ and any $\beta \in \mathfrak{p}$,

$$\| \beta_X(x) \| \leq C \| \beta \| (1 + d_X(x_0, x)), \tag{1}$$

where d_X denotes the geodesic distance between points of X with respect to the induced Riemannian metric on X.

If X is compact or if X is a vector space and the G-action on X is linear then this condition is satisfied. Under this assumption, we construct the maximal weight function

$$\lambda_x : \partial_\infty(G/K) \to \mathbb{R} \cup \{\infty\}$$

for any $x \in X$. It is well-known that G acts on $\partial_\infty(G/K)$ and the G-action on $\partial_\infty(G/K)$ is continuous with respect to the sphere topology [7]. The same idea given in [28] proves that the maximal weight functions are G-equivariant. If $g \in G$, $p \in \partial_\infty(G/K)$ and $x \in X$, then $\lambda_{gx}(p) = \lambda_x(g^{-1}p)$. We then prove that a point $x \in X$ is polystable if and only if $\lambda_x \geq 0$ and for any $p \in \partial_\infty(G/K)$ such that $\lambda_x(p) = 0$ there exists $p' \in \partial_\infty(G/K)$ such that p and p' are connected in the sense of Definition 5.1 below. In the classical case of a group action on a Kähler manifold this characterization is due to Mundet i Riera [28].

The idea of viewing the maximal weights as defining functions on the boundary $\partial_\infty M$ appeared in [22]. They also give a characterization of polystability which they refer to as nice semistability [22, Definition 3.13]. Finally, we prove the polystability criterion for the G-action on measures. Polystable measures are interested in an application to upper bounds for the first eigenvalue of the Laplacian of functions, see, for instance, [3, Section 1.17], [8, 19] and the introduction to [1].

2 Compatible subgroups, parabolic subgroups, and gradient maps

Let U be a compact Lie group and let $U^{\mathbb{C}}$ be the corresponding complex linear algebraic group [11]. The group $U^{\mathbb{C}}$ is reductive and is the universal complexification of U in the sense of [21]. On the Lie algebra level, we have the Cartan decomposition $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ with a corresponding Cartan involution $\theta : \mathfrak{u}^{\mathbb{C}} \longrightarrow \mathfrak{u}^{\mathbb{C}}$ given by $\xi + i\nu \mapsto \xi - i\nu$. We also denote by θ the corresponding involution on $U^{\mathbb{C}}$. The real analytic map $F : U \times i\mathfrak{u} \longrightarrow U^{\mathbb{C}}$, $(u, \xi) \mapsto u\exp(\xi)$ is a diffeomorphism. We refer to the composition $U^{\mathbb{C}} = U\exp(i\mathfrak{u})$ as the Cartan decomposition of $U^{\mathbb{C}}$.

Let $G \subset U^{\mathbb{C}}$ be a closed real subgroup of $U^{\mathbb{C}}$. We say that G is *compatible* with the Cartan decomposition of $U^{\mathbb{C}}$ if $F(K \times \mathfrak{p}) = G$ where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$. The restriction of F to $K \times \mathfrak{p}$ is then a diffeomorphism onto G. It follows that K is a maximal compact subgroup of G and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since K is a retraction of G, it follows that G has only finitely many connected components and $G = KG^o$, where G^o denotes the connected component of G containing e.

Lemma 2.1 *([2, Lemma 7])*

a) If $G \subset U^{\mathbb{C}}$ is a compatible subgroup, and $H \subset G$ is closed and θ-invariant, then H is compatible if and only if H has only finitely many connected components.
b) If $G \subset U^{\mathbb{C}}$ is a connected compatible subgroup, then G_{ss} is compatible.
c) If $G \subset U^{\mathbb{C}}$ is a compatible subgroup and $E \subset \mathfrak{p}$ is any subset, then $G^E = \{g \in G : \mathrm{Ad}(g)(\beta) = \beta,\ \forall\ \beta \in E\}$ is compatible. Indeed, $G^E = K^E \exp(\mathfrak{p}^E)$, where $K^E = K \cap G^E$ and $\mathfrak{p}^E = \{x \in \mathfrak{p} : [x, E] = 0\}$. If $E = \{\beta\}$ then we simply write K^β, $\mathfrak{p}^\beta$ and G^β.

If $\beta \in \mathfrak{p}$ we define,

$$G^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta)\, g \exp(-t\beta) \text{ exists}\},$$

$$R^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta)\, g \exp(-t\beta) = e\}.$$

Then $G^{\beta+}$ is a parabolic subgroup of G with unipotent radical $R^{\beta+}$. $G^{\beta+}$ is the semi-direct product of G^β and $R^{\beta+}$.

Proposition 2.2 *For any $\beta \in \mathfrak{p}$, we have $G = KG^{\beta+}$.*

Proof If G is connected, the result is well-known, see for instance [2, Lemma 9] and [17, Lemma 4.1]. Since $G = KG^o$, it follows that $G = KG^o = K(G^o)^{\beta+} = KG^{\beta+}$, concluding the proof. □

Let (Z, ω) be a Kähler manifold endowed with a holomorphic action $U^{\mathbb{C}} \times Z \longrightarrow Z$. We assume that ω is U-invariant and there exists a U-equivariant momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. We fix an $\mathrm{Ad}(U^{\mathbb{C}})$ inner product B on $\mathfrak{u}^{\mathbb{C}}$ and we denote by $\langle \cdot, \cdot \rangle$ it's real part. Then $\langle \cdot, \cdot \rangle$ is positive definite on $\mathrm{i}\mathfrak{u}$, negative definite on $\mathfrak{u}$, $\langle \mathfrak{u}, \mathrm{i}\mathfrak{u} \rangle = 0$ and finally the multiplication by i satisfies $\langle \mathrm{i}\cdot, \mathrm{i}\cdot \rangle = -\langle \cdot, \cdot \rangle$. We may think of the momentum map as a $\mathfrak{u}$-valued map using $-\langle \cdot, \cdot \rangle$.

Let G be a closed and compatible subgroup of $U^{\mathbb{C}}$. The G-gradient map associated with μ is given by $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ where

$$\mu_{\mathfrak{p}}^{\beta}(x) := \langle \mu_{\mathfrak{p}}(x), \beta \rangle := \langle \mathrm{i}\mu(x), \beta \rangle = -\langle \mu(x), -\mathrm{i}\beta \rangle = \mu^{-\mathrm{i}\beta}(x),$$

for any $\beta \in \mathfrak{p}$. For the rest of the paper, we fix a G-invariant locally closed submanifold X of Z. We also denote the restriction of $\mu_{\mathfrak{p}}$ to X by $\mu_{\mathfrak{p}}$. The map $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ is K-equivariant and $\mathrm{grad}\, \mu_{\mathfrak{p}}^{\beta} = \beta_X$, for any $\beta \in \mathfrak{p}$, where the gradient is computed with respect to the induced Riemannian structure.

Let $\beta \in \mathfrak{p}$ and let $X^\beta = \{z \in X : \beta_X(z) = 0\}$. If $A = \exp(\mathbb{R}\beta)$ we have a Slice Theorem at every point of X [17, Theorem 3.1]. In particular, X^β is a smooth, possibly disconnected, submanifold of X. Since $\mathrm{grad}\, \mu_{\mathfrak{p}}^{\beta} = \beta_X$ it follows that X^β is the set of critical points of $\mu_{\mathfrak{p}}^{\beta}$ that we denote by $\mathrm{Crit}\, \mu_{\mathfrak{p}}^{\beta}$. Moreover, $\mu_{\mathfrak{p}}^{\beta} : X \longrightarrow \mathbb{R}$ is a Morse-Bott function, see for instance [2, Corollary 2.3].

Assume that X is compact. The Slice Theorem implies that the limit $\lim_{t\to+\infty} \exp(t\beta)x$ exists and it lies in X^β for any $x \in X$.

Let $C_1, \dots, C_k$ be the connected components of X^β. Let

$$W_i^\beta := \{x \in X : \lim_{t\to+\infty} \exp(t\beta)x \in C_i\},$$

for $i = 1, \dots, k$. One of the fundamental theorems of Morse theory is the following, see for instance [9].

Theorem 2.3 *W_i^β is an immersed submanifold, which is called the unstable manifold corresponding to C_i, and*

$$\varphi_\infty : W_i^\beta \longrightarrow C_i, \qquad x \mapsto \exp(t\beta)x,$$

is smooth. Moreover, $X = \bigsqcup W_i^\beta$ (disjoint union).

3 Symmetric spaces

Let $G \subset U^{\mathbb{C}}$ be a closed compatible subgroup. Then $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of G and $\mathfrak{p} := \mathfrak{g} \cap \mathrm{i}\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. Let $M = G/K$ and let $\langle \cdot, \cdot \rangle$ be the real part of the fixed $\mathrm{Ad}(U^{\mathbb{C}})$-invariant inner product B of Euclidean type on the Lie algebra $\mathfrak{u}^{\mathbb{C}}$. Then $\langle\cdot,\cdot\rangle$ is positive definite on $\mathrm{i}\mathfrak{u}$, negative definite on $\mathfrak{u}$, $\langle\mathfrak{u},\mathrm{i}\mathfrak{u}\rangle = 0$ and finally the multiplication by i satisfies $\langle \mathrm{i}\cdot,\mathrm{i}\cdot\rangle = -\langle\cdot,\cdot\rangle$. Since $\mathfrak{k}\subset\mathfrak{u}$, respectively $\mathfrak{p}\subset\mathrm{i}\mathfrak{u}$, the formula $\langle \xi_1+\beta_1,\xi_2+\beta_2\rangle := -\langle\xi_1,\xi_2\rangle + \langle\beta_1,\beta_2\rangle$ where $\xi_1,\xi_2\in\mathfrak{k}$ and $\beta_1,\beta_2\in\mathfrak{p}$, defines an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$ and so it induces a G-invariant Riemannian metric of nonpositive curvature on M. Moreover, M is a symmetric space of non-compact type [7]. Let $\pi : G \to M$ be the projection onto the right cosets of G. G acts isometrically on M from left by

$$L_g : M \to M;\quad L_g(hK) := ghK,\quad g,h\in G.$$

A geodesic γ in M is given by $\gamma = g\exp(t\beta)K$, where $g\in G$ and $\beta\in\mathfrak{p}$. For $\beta\in\mathfrak{p}$, we set $\gamma^\beta(t) = \exp(t\beta)K$ and $o := K \in M$.

Since M is a Hadamard manifold there is a natural notion of a boundary at infinity which can be described using geodesics. We refer the reader to [7, 14] for more details. Two unit speed geodesics $\gamma,\gamma' : \mathbb{R}\to M$ are equivalent, denoted by $\gamma\sim\gamma'$, if $\sup_{t>0} d(\gamma(t),\gamma'(t)) < +\infty$.

Definition 3.1 The *Tits boundary* of M denoted by $\partial_\infty M$ is the set of equivalence classes of unit speed geodesics in M.

The map that sends $\beta\in\mathfrak{p}$ to the tangent vector $\dot\gamma^\beta(0)$ produces an isomorphism $\mathfrak{p}\cong T_oM$. Since any geodesic ray in M is equivalent to a unique ray starting from o, the map

$$e : S(\mathfrak{p}) \to \partial_\infty M;$$
$$e(\beta) := [\gamma^\beta]$$

where $S(\mathfrak{p}) := \{\beta\in\mathfrak{p} : \parallel\beta\parallel = 1\}$ is the unit sphere in $\mathfrak{p}$, is a bijection. The *sphere topology* is the topology on $\partial_\infty M$ such that e is a homomorphism. Since G acts by isometries on M, then for every unit speed geodesic γ, $g\gamma$ is also a unit speed geodesic for any $g\in G$. Moreover, if $\gamma\sim\gamma'$ then $g\gamma\sim g\gamma'$. There is a G-action on $\partial_\infty M$ given by:

$$g\cdot[\gamma] = [g\cdot\gamma]$$

and this action also induces a G-action on $S(\mathfrak{p})$ given by

$$g\cdot\beta := e^{-1}(g\cdot e(\beta)) = e^{-1}[g\cdot\gamma^\beta].$$

This action is continuous with respect to the sphere topology on $\partial_\infty M$. The K-action on $\partial_\infty M$ induces the adjoint action of K on $S(\mathfrak{p})$, see for instance [7].

Let H be a compatible subgroup of G, i.e $H := L\exp(\mathfrak{q})$, where $L := H\cap K$ and $\mathfrak{q} = \mathfrak{h}\cap\mathfrak{p}$, where $\mathfrak{h}$ is the Lie algebra of H. It follows that H is a real reductive subgroup of G. The Cartan involution of G induces a Cartan involution of H, L is a maximal compact subgroup of H, and $\mathfrak{h} = \mathfrak{l}\oplus\mathfrak{q}$. The inclusion $M' := H/L \hookrightarrow M = G/K$ is totally geodesic and induces an inclusion $\partial_\infty M' \hookrightarrow \partial_\infty M$.

3.1 The Kempf–Ness function

Given G a real reductive group which acts smoothly on Z; $G = K\exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Let X be a G-invariant locally closed submanifold of Z. As Mundet pointed out in [29], there exists a function $\Phi : X\times G\to\mathbb{R}$, such that

$$\langle\mu_{\mathfrak{p}}(x),\xi\rangle = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0}\Phi(x,\exp(t\xi)),\qquad \xi\in\mathfrak{p},$$

and satisfying the following conditions:

a) For any $x \in X$, the function $\Phi(x, .)$ is smooth on G.
b) The function $\Phi(x, .)$ is left-invariant with respect to K, i.e., $\Phi(x, kg) = \Phi(x, g)$.
c) For any $x \in X$, $v \in \mathfrak{p}$ and $t \in \mathbb{R}$;

$$\frac{d^2}{dt^2}\Phi(x, \exp(tv)) \geq 0.$$

Moreover:

$$\frac{d^2}{dt^2}\Phi(x, \exp(tv)) = 0$$

if and only if $\exp(\mathbb{R}v) \subset G_x$.
d) For any $x \in X$, and any $g, h \in G$;

$$\Phi(x, hg) = \Phi(x, g) + \Phi(gx, h).$$

This equation is called the cocycle condition. Finally, using the cocycle condition, we have

$$\frac{\mathrm{d}}{\mathrm{dt}}\Phi(x, \exp(t\beta)) = \langle \mu_{\mathfrak{p}}(\exp(t\xi)x), \beta\rangle.$$

The function $\Phi : X \times G \to \mathbb{R}$ is called the Kempf-Ness function for (X, G, K). It is just the restriction of the classical Kempf-Ness function $Z \times U^{\mathbb{C}} \longrightarrow \mathbb{R}$ considered in [29, 30] to $X \times G$ [4]. Moreover, if $H \subset G$ is compatible and $Y \subset X$ is a H-stable submanifold of X, then the restriction $\Phi_{|Y \times H}$ is the Kempf-Ness function of the H-gradient map on Y.

Let $x \in X$. By property $b)$, i.e. $\Phi(x, kg) = \Phi(x, g)$, the function $\Phi_x : G \to \mathbb{R}$ given by $\Phi_x(g) := \Phi(x, g^{-1})$ descends to a function on M which we denote by the same symbol. That is

$$\Phi_x : M \longrightarrow \mathbb{R}; \qquad \Phi_x(gK) := \Phi(x, g^{-1}).$$

The cocycle condition $d)$ can be rewritten as

$$\Phi_x(ghK) = \Phi_{g^{-1}x}(hK) + \Phi_x(gK), \tag{2}$$

and it is equivalent to $L_g^*\Phi_x = \Phi_{g^{-1}x} + \Phi_{g^{-1}x}(gK)$, where L_g denotes the action of G on X given above.

Note that

$$-(d\Phi_x)_o(\dot{\gamma}^\beta(0)) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0}\Phi_x(\exp(-t\beta)K) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0}\Phi(x, \exp(t\beta)) = \langle \mu_{\mathfrak{p}}(x), \beta\rangle.$$

Lemma 3.1 *Let $x \in X$ and let $\Phi_x : M \to \mathbb{R}$. Suppose $\gamma(t) = g\exp(t\beta)K$ for $\beta \in \mathfrak{p}$ is a geodesic in M, then $\Phi_x \circ \gamma$ is convex and so,*

$$\lim_{t\to\infty}\frac{d}{dt}(\Phi_x \circ \gamma) = \lim_{t\to\infty}\frac{\Phi_x \circ \gamma}{t}$$

.

Proof That Φ_x is a convex function on M follows from [3, Lemma 2.19]. Let $f(t) = (\Phi_x \circ \gamma)(t)$. Since f is convex,

$$\frac{f(s)}{s} \leq f'(s) \leq \frac{f(t) - f(s)}{t - s} \quad 0 < s < t.$$

Furthermore, the two quantities are increasing in s, while the third in t. Hence,

$$\lim_{s\to\infty}\frac{f(s)}{s}\le\lim_{s\to\infty}f'(s)\le\lim_{t\to\infty}\frac{f(t)-f(s)}{t-s}.$$

Since the last limit is the same with the first limit, we have $\lim_{t\to\infty}f'(t)=\lim_{t\to\infty}\frac{f(t)}{t}$ and the result follows. □

4 Stability and maximal weight function

Let U be a compact Lie group and $U^{\mathbb{C}}$ its complexification. Let (Z,ω) be a Kähler manifold. In this paper, we assume that the complex reductive Lie group $U^{\mathbb{C}}$ acts holomorphically on Z. The Kähler form ω is U-invariant and the U-action on Z is Hamiltonian and so there exists a momentum map $\mu:Z\to\mathfrak{u}$. Let $G\subset U^{\mathbb{C}}$ be a closed compatible subgroup. Then $G=K\exp(\mathfrak{p})$, where $K:=G\cap U$ is a maximal compact subgroup of G and $\mathfrak{p}:=\mathfrak{g}\cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. Suppose $X\subset Z$ is a G-stable locally closed connected real submanifold of Z with the gradient map $\mu_{\mathfrak{p}}:X\to\mathfrak{p}$. From now on, X satisfies Assumption 1.1.

We recall that by G_x and K_x, we denote the stabilizer subgroup of $x\in X$ with respect to the G-action and the K-action respectively, and by $\mathfrak{g}_x$ and $\mathfrak{k}_x$ their respective Lie algebras.

Definition 4.1 Let $x\in X$. Then:

a) x is stable if $G\cdot x\cap\mu_{\mathfrak{p}}^{-1}(0)\neq\emptyset$ and $\mathfrak{g}_x$ is conjugate to a Lie subalgebra of $\mathfrak{k}$.
b) x is polystable if $G\cdot x\cap\mu_{\mathfrak{p}}^{-1}(0)\neq\emptyset$.
c) x is semistable if $\overline{G\cdot x}\cap\mu_{\mathfrak{p}}^{-1}(0)\neq\emptyset$.

We denote by $X^{s}_{\mu_{\mathfrak{p}}}$, $X^{ss}_{\mu_{\mathfrak{p}}}$, $X^{ps}_{\mu_{\mathfrak{p}}}$ the set of stable, respectively semistable, polystable, points. It follows directly from the definitions above that the conditions are G-invariant in the sense that if a point satisfies one of the conditions, then every point in its orbit satisfies the same condition, and for stability, recall that $\mathfrak{g}_{gx}=\mathrm{Ad}(g)(\mathfrak{g}_x)$.

The following well-known result establishes a relation between the Kempf-Ness function and the polystability condition. A proof is given in [5].

Proposition 4.1 *Let $x\in X$ and let $g\in G$. The following conditions are equivalent:*

a) $\mu_{\mathfrak{p}}(gx)=0$.
b) *g is a critical point of $\Phi(x,\cdot)$.*
c) *$g^{-1}K$ is a critical point of Φ_x.*

Proposition 4.2 *Let $x\in X$.*

- *If x is polystable, then G_x is compatible.*
- *If x is stable, then G_x is compact.*

4.0.1. maximal weight function

In this section, we introduce the maximal weight function associated with an element $x\in X$.

For any $t\in\mathbb{R}$, define $\lambda(x,\beta,t)=\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\beta\rangle$.

$$\lambda(x,\beta,t)=\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\beta\rangle=\frac{d}{dt}\Phi(x,\exp(t\beta)),\tag{3}$$

where $\Phi : X \times G \to \mathbb{R}$ is the Kempf-Ness function. By the properties of the Kempf-Ness function,

$$\frac{d}{dt}\lambda(x, \beta, t) = \frac{d^2}{dt^2}\Phi(x, \exp(t\beta)) \geq 0.$$

This means that $\lambda(x, \beta, t)$ is a non decreasing function as a function of t.

The maximal weight of $x \in X$ in the direction of $\beta \in \mathfrak{p}$ is defined in [5] as the numerical value

$$\lambda(x, \beta) = \lim_{t\to\infty} \lambda(x, \beta, t) = \lim_{t\to+\infty} \langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta\rangle \in \mathbb{R} \cup \{\infty\}.$$

Note that

$$\frac{d}{dt}\lambda(x, \beta, t) = \parallel \beta_X(\exp(t\beta)x) \parallel^2,$$

and so

$$\lambda(x, \beta, t) = \langle \mu_{\mathfrak{p}}(x), \beta\rangle + \int_0^t \parallel \beta_X(\exp(s\beta)x) \parallel^2 \mathrm{ds}. \tag{4}$$

Lemma 4.3 *Let $\beta, \beta' \in \mathfrak{p}$. If $\beta \in \mathfrak{g}_x$ and $[\beta, \beta'] = 0$, then*

$$\lim_{t\to+\infty} \frac{d}{dt}\Phi(x, \exp(t(\beta + \beta')) = \lim_{t\to+\infty} \frac{d}{dt}\Phi(x, \exp(t\beta)) + \frac{d}{dt}\Phi(x, \exp(t\beta')),$$

Proof By the cocycle condition, keeping in mind that $[\beta, \beta'] = 0$, we have

$$\begin{aligned}\Phi(x, \exp(t(\beta + \beta')) &= \Phi(x, \exp(t\beta')) + \Phi(\exp(t\beta)x, \exp(t\beta')) \\ &= \Phi(x, \exp(t\beta')) + \Phi(x, \exp(t\beta')),\end{aligned}$$

and so the result follows. □

Let $\gamma : [0, +\infty) \longrightarrow G/K$ be a geodesic ray and let Φ_x be the Kempf-Ness function at x. We define

$$\lambda_x(\gamma) = \lim_{t\to+\infty} \frac{\Phi_x(\gamma(t))}{t}.$$

The results proved in [28, sections 3.2 and 3.3] hold in our setting. Therefore, we have the following result.

Proposition 4.4 *The function $\lambda_x : \partial_\infty M \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$\lambda_x([\gamma]) = \lambda_x(\gamma)$$

is well-defined and G-equivariant, i.e., $\lambda_{gx}(p) = \lambda_x(g^{-1}p)$ for any $g \in G$ and any $p \in \partial_\infty(G/K)$.

We conclude this section by recalling the results that will be needed in the following sections.

By Lemma 3.1 for any $\beta \in S(\mathfrak{p})$, keeping in mind formula (3), we have

$$\begin{aligned}\lambda_x(e(\beta)) &= \lim_{t\to\infty} \frac{d}{dt}\Phi_x(\exp(t\beta)K) = \lim_{t\to\infty} \frac{d}{dt}\Phi(x, \exp(-t\beta)) \\ &= \lim_{t\to+\infty} \langle \mu_{\mathfrak{p}}(\exp(t\beta)x, -\beta\rangle.\end{aligned} \tag{5}$$

Lemma 4.5 *Let* $\beta \in \mathfrak{p} - \{0\}$ *and let* $v = \frac{\beta}{\|\beta\|}$. *Then*

$$\lambda_x(e(v)) = \frac{1}{\| \beta \|} \lim_{t \to +\infty} \frac{d}{dt} \Phi(x, \exp(-t\beta)).$$

Proof $\Phi_x(\exp(tv)) = \Phi_x(\exp\left(\frac{t}{\|\beta\|}\beta\right))$. Then

$$\lambda_x(e(\beta)) = \frac{1}{\| \beta \|} \lim_{t \to +\infty} \frac{d}{dt} \Phi(x, \exp(-t\beta)).$$

□

A proof of the following lemma is given in [5, Lemma 3.5, p. 92], see also [30].

Lemma 4.6 *Let* V *be a subspace of* $\mathfrak{p}$. *The following are equivalent for a point* $x \in X$*:*

a) the map $\Phi(x, \cdot)$ *is linearly properly on* V, *i.e., there exist positive constants* C_1 *and* C_2 *such that*

$$\| v \| \leq C_1 \Phi(x, \exp(v)) + C_2, \ \forall v \in V.$$

b) $\lambda_x, (e(\beta)) > 0$ *for every* $\beta \in S(V)$.

The following theorem is well-known and it gives a numerical criterion for stable points in terms of maximal weights. A proof is given in [5, Theorem 3.7].

Theorem 4.7 *Let* $x \in X$. *Then* x *is stable if and only if* $\lambda_x > 0$ *on* $\partial_\infty M$.

5 Polystability

Definition 5.1 We say that $p, q \in \partial_\infty M$ are connected if there exists a geodesic α in X such that $p = \alpha(\infty)$ and $q = \alpha(-\infty)$.

For any $x \in X$, as in [28], see also [3], let $Z(x) := \{p \in \partial_\infty M : \lambda_x(p) = 0\}$.

Lemma 5.1 *Let* $x \in X$ *be such that* $\mu_\mathfrak{p}(x) = 0$, *then* $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{p}_x$ *and* $Z(x) = e(S(\mathfrak{p}_x)) = \partial_\infty G_x / K_x$.

Proof By Proposition 4.2 if $\mu_\mathfrak{p}(x) = 0$, G_x is a compatible subgroup of G. Hence, $\mathfrak{g}_x = \mathfrak{k}_x \oplus \mathfrak{p}_x$. To prove the second assertion, let $\beta \in S(\mathfrak{p})$. Suppose $e(\beta) \in Z(x)$. This means that $\lambda_x(e(\beta)) = 0$, then the convex function $f(t) := \Phi_x(\exp(t\beta)K)$ satisfies

$$f'(\infty) = \lim_{t \to \infty} \frac{d}{dt} \Phi_x(\exp(t\beta)K) = \lambda_x(e(\beta)) = 0$$

and

$$f'(0) = \left.\frac{d}{dt}\right|_{t=0} \Phi_x(\exp(t\beta)K) = \langle \mu_\mathfrak{p}(x), -\beta \rangle = 0.$$

These imply that f is constant for all $t > 0$, and by the condition (c) of Kempf-Ness function, $\exp(\mathbb{R}\beta) \subset G_x$. Since G_x is compatible, $\beta \in S(\mathfrak{p}_x)$. Conversely, if $\beta \in S(\mathfrak{p}_x)$, then f is linear. Moreover, $f'(0) = 0$. Therefore, $f \equiv 0$ and $e(\beta) \in Z(x)$. □

Let $x \in X$ and $\beta \in S(\mathfrak{p})$. Since $\mathfrak{p} \subset i\mathfrak{u}$, then $i\beta \in \mathfrak{u}$. We define the torus T_β given as

$$T_\beta := \overline{\{\exp(ti\beta) : t \in \mathbb{R}\}} \subseteq U^o,$$

where U^o denotes the connected component of U containing the identity.

Lemma 5.2 *Let $g \in G$. Then* $\dim T_\beta = \dim T_{g\cdot\beta}$.

Proof It is well-known that $G^{\beta+}$ fixes $e(\beta)$, see for instance [14, Proposition 2.17.3, p.102]. Then for any $g \in G$, keeping in mind Proposition 2.2, write $g = kh$, where $k \in K$ and $h \in G^{\beta+}$. Hence $g \cdot \beta = kh \cdot \beta = k \cdot \beta = \mathrm{Ad}(k)(\beta)$ and so

$$\begin{aligned} T_{g\cdot\beta} &= \overline{\{\exp(it\mathrm{Ad}(k)\beta) : t \in \mathbb{R}\}} \\ &= \overline{\{k\exp(it\beta)k^{-1} : t \in \mathbb{R}\}} \\ &= kT_\beta k^{-1}. \end{aligned}$$

Therefore $\dim T_\beta = \dim T_{g\cdot\beta}$. ◻

Lemma 5.3 *Let $x \in X$ and $p, p' \in Z(x)$ be connected. Then there exists $g \in G$ and $\xi \in S(\mathfrak{p})$ such that $\xi \in \mathfrak{p}_y$, where $y = gx$.*

Proof Since $p, p' \in Z(x)$ are connected, then there exists geodesic $\alpha \in M$ such that $\alpha(+\infty) = p \in Z(x)$ and $\alpha(-\infty) = p' \in Z(x)$. Assume $\alpha(t) = g\exp(t\xi)K$, $g \in G$. Then $p = g \cdot e(\xi)$ and $p' = g \cdot e(-\xi)$. By the G-invariant property of the maximal weigh we get

$$\lambda_{g^{-1}x}(e(\xi)) = \lambda_x(g \cdot e(\xi)) = \lambda_x(p) = 0$$

and

$$\lambda_{g^{-1}x}(e(-\xi)) = \lambda_x(g \cdot e(-\xi)) = \lambda_x(p') = 0.$$

Let $y = g^{-1}x$. This means that the convex function $t \mapsto \Phi_y(\exp(t\xi)K)$ has zero derivatives at both $+\infty$ and $-\infty$, and so, it is constant and by property (c) of Kempf-Ness function, $\exp(\mathbb{R}\xi) \subset G_y$, $\xi \in \mathfrak{p}_y$. ◻

Let $X^\beta := \{z \in X : \beta_X(z) = 0\}$. G^β preserves X^β [5, Prop. 2.9] and X^β is the disjoint union of closed submanifold of X [17]. The following result is proved in [5, Proposition 2.10,p.92]

Proposition 5.4 *The restriction $(\mu_{\mathfrak{p}})_{|_{X^\beta}}$ takes value on $\mathfrak{p}^\beta$ and so it coincides with the G^β-gradient map $(\mu_{\mathfrak{p}^\beta})_{|_{X^\beta}}$.*

Corollary 5.1 *If $x \in X^\beta$ is G^β-polystable, then x is G-polystable.*

Theorem 5.5 *A point $x \in X$ is polystable if and only if $\lambda_x \geq 0$ and for any $p \in Z(x)$ there exists $p' \in Z(x)$ such that p and p' are connected.*

Proof Let $x \in X$. If $Z(x) = \emptyset$, $\lambda_x > 0$ and by Theorem 4.7, x is stable and hence polystable. Suppose $Z(x) \neq \emptyset$. Let $p \in Z(x)$. Let $\beta \in S(\mathfrak{p})$ such that $p = e(\beta)$. Suppose $p \in Z(x)$ is chosen such that the of the torus T_β satisfies

$$\dim T_\beta = \max_{\eta \in e^{-1}(Z(x))} \dim T_\eta.$$

By assumption there is a geodesic $\alpha \in M$ such that $\alpha(+\infty) = p \in Z(x)$ and $\alpha(-\infty) = p' \in Z(x)$. Assume $\alpha(t) = g\exp(t\xi)K$, $g \in G$. Then $p = g \cdot e(\xi)$ and $p' = g \cdot e(-\xi)$. By Lemma 5.3, $\xi \in \mathfrak{p}_y$ where $y = g^{-1}x$. Moreover, since $e(\beta) = p = g \cdot e(\xi)$, using Lemma 5.2,

$$\dim T_\xi = \dim T_\beta = \max_{\eta \in e^{-1}(Z(x))} \dim T_\eta .$$

Let $\mathfrak{t}_\xi$ be the Lie algebra of T_ξ. Then $\mathfrak{a} = i\mathfrak{t}_\xi \cap \mathfrak{p}^\xi$ is an Abelian subalgebra contained $\mathfrak{p}^\xi$ different from zero since $\beta \in \mathfrak{a}$. Since $T_\xi = \overline{\exp(i\mathbb{R}\xi)}$ fixes y it follows that $\mathfrak{a} \subseteq \mathfrak{g}_y$

Let Y be the connected component of $X^{\mathfrak{a}}$ containing y. By Lemma 2.1, $(G^{\mathfrak{a}})^o = (K^{\mathfrak{a}})^o \exp(\mathfrak{p}^{\mathfrak{a}})$ is compatible and preserves Y. By Proposition 5.4 we get $(\mu_{\mathfrak{p}})|_Y = \mu_{\mathfrak{p}^{\mathfrak{a}}}$. Hence, if y is $(G^{\mathfrak{a}})^o$-polystable, then it is G-polystable. We split $\mathfrak{p}^{\mathfrak{a}} = \mathrm{span}(\mathfrak{a}) \oplus \mathfrak{p}^{'}$, where $\mathfrak{p}^{'}$ is the orthogonal of $\mathfrak{a}$ and so it is a $K^{\mathfrak{a}}$-invariant splitting..

Claim: $\lambda_y(e(\beta')) > 0$ for all $\beta' \in S(\mathfrak{p}')$. Indeed, we prove this claim by contradiction. Suppose there exists $\beta' \in S(\mathfrak{p}')$ such that $\lambda_y(e(\beta')) = 0$. Hence $[\xi, \beta'] = 0$ by the choice of ξ and β', and they are linearly independent. Let $a > 0$. Since $[\xi, \beta'] = 0$ and $\xi \in \mathfrak{g}_y$, by Lemma 4.3 it follows that

$$\lim_{t \to +\infty} \Phi(y, \exp(t(\xi + a\beta')) = \lim_{t \to +\infty} \Phi(y, \exp(t(\xi)) + a \lim_{t \to +\infty} \Phi(y, \exp(t\beta')).$$

Since $\lambda_y(e(\xi)) = \lambda_y(e(\beta')) = 0$, it follows by Proposition 4.4 that

$$\lim_{t \to +\infty} \Phi(y, \exp(t(\xi + a\beta')) = 0.$$

Applying Lemma 4.5, we have

$$\lambda_y(e(\frac{\xi + a\beta'}{\parallel \xi + a\beta' \parallel})) = 0,$$

and so the vector

$$\frac{\xi + a\beta'}{\parallel \xi + a\beta' \parallel}$$

belongs to $e^{-1}(Z(y))$.

We claim that for some $a > 0$, $\dim T_{\xi + a\beta'} > \dim T_\xi$.

Let $T' = \overline{\exp(\mathbb{R}i\xi + \mathbb{R}i\beta')} \subseteq (U^\xi)^o$ and $T_{\beta'} = \overline{\exp(\mathbb{R}i\beta')}$. Let $U' \subseteq (U^\xi)^o$ be a compact connected subgroup such that the morphism

$$T_\xi \times U' \to (U^\xi)^o, (a, b) \mapsto (ab)$$

is surjective with a finite center. Since $\beta' \notin \mathfrak{a}$, it follows that $i\beta \notin \mathfrak{t}_\xi$. Hence, $T_{\beta'} \subseteq U'$ and the morphism

$$f : T_\xi \times T_{\beta'} \to T', \quad f(a, b) = ab$$

is a finite covering. Let $\{e_1, \ldots, e_n\}$, respectively $\{e_1^{'}, \ldots, e_m^{'}\}$, be a basis of the lattice $\ker \exp \subset \mathfrak{t}_\xi$, respectively $\ker \exp \subset \mathfrak{t}_{\beta'}$. If $i\xi = X_1 e_1 + \cdots + X_n e_n$ and $i\beta' = Y_1 e_1' + \cdots + Y_m e_m'$, then $i(\xi + a\beta') = X_1 e_1 + \cdots + X_n e_n + aY_1 e_1' + \cdots + aY_m e_m'$. Denote by $T'_{\xi + a\beta}$ the closure of $\exp(\mathbb{R}(i(\xi + a\beta'))$. Since f is a covering, $\dim T_{\xi + a\beta'} = \dim T'_{\xi + a\beta'}$. Hence,

$$\dim T_{\xi + a\beta'} = \dim_{\mathbb{Q}}(\mathbb{Q}X_1 + \cdots + \mathbb{Q}X_n + \mathbb{Q}aY_1 + \cdots + \mathbb{Q}aY_m),$$

see for instance [12, $p.61$]. Since $\beta' \neq 0$, $Y_j \neq 0$ for some j. Choose a such that $aY_j \notin \mathbb{Q}X_1 + \cdots + \mathbb{Q}X_n$. Then $\dim T_{\frac{\xi+a\beta'}{\|\xi+a\beta\|}} > \dim T_\xi$ which is a contradiction. Therefore, $\lambda_y > 0$ on $e(S(\mathfrak{p}'))$. By Lemma 4.6, $\Phi(y,\cdot)$ is linearly proper on $\mathfrak{p}'$. This implies that $\Phi(y,\cdot)$ is bounded from below on $\mathfrak{p}'$ and

$$m = \inf_{\alpha\in\mathfrak{p}'}\Phi(y,\exp(\alpha)),$$

is achieved. We claim that

$$m = \inf_{\alpha\in\mathfrak{p}^{\mathfrak{a}}}\Phi(y,\exp(\alpha)).$$

Indeed, let $v \in \mathfrak{p}^{\mathfrak{a}}$. Then $v = v_1 + v_2$, where $v_1 \in \mathfrak{a}$ and $v_2 \in \mathfrak{p}'$. By the cocycle condition, keeping in mind that $[v_1, v_2] = 0$ and $v_1 \in \mathfrak{g}_y$, we get

$$\Phi(y,\exp(v)) = \Phi(y,\exp(v_1)) + \Phi(y,\exp(v_2)).$$

We claim that $\Phi(y,\exp(v_1)) = 0$. Indeed, since $\mathfrak{a} \subset \mathfrak{g}_y$, If $w \in S(\mathfrak{a})$ then by formula (5) we get

$$\lambda_y(e(w)) = \langle \mu_{\mathfrak{p}^{\mathfrak{a}}}(y), -w\rangle \geq 0,$$

for any $w \in S(\mathfrak{a})$. This implies $\lambda_y(e(-w)) = -\lambda_y(e(w))$ and so $\lambda_y(e(w)) = 0$ for any $w \in \mathfrak{a}$.

Let $w \in \mathfrak{a} - \{0\}$ and let $s : \mathbb{R} \longrightarrow \mathbb{R}$ be the function $s(t) = \Phi(y,\exp(tw))$. Since $\exp(tw)y = y$ for any $t \in \mathbb{R}$, it follows that $s(t)$ is a linear function. Therefore, $s(t) = bt$ for some $b \in \mathbb{R}$. On the other hand

$$0 = \lambda_y\left(e\left(\frac{w}{\| w \|}\right)\right) = \lim_{t\to+\infty}\frac{1}{\| w \|}\frac{d}{dt}\Phi(y,\exp(tw)) = b.$$

This proves

$$\inf_{\alpha\in\mathfrak{p}'}\Phi(y,\exp(\alpha)) = \inf_{\alpha\in\mathfrak{p}^{\mathfrak{a}}}\Phi(y,\exp(\alpha)),$$

and so $\Phi_y : (G^{\mathfrak{a}})^o/(K^{\mathfrak{a}})^o \longrightarrow \mathbb{R}$ has a minumum and so a critical point. By Proposition 4.1, it follows that y is $(G^{\mathfrak{a}})^o$ polystable and by Corollary 5.1, y is G-polystable.

Suppose x is polystable. There exists $g \in G$ such that $\mu_{\mathfrak{p}}(gx) = 0$. Let $y = gx$ and fix $\beta \in \mathfrak{p}$. Since the Kempf-Ness function is convex along geodesics,

$$\lambda_y(e(\beta)) = \lim_{t\to\infty}\frac{d}{dt}\Phi_y(\exp(t\beta)K)) \geq \frac{d}{dt}|_{t=0}\Phi_y(\exp(-t\beta)) = \langle\mu_{\mathfrak{p}}(y), -\beta\rangle = 0.$$

This shows that $\lambda_y \geq 0$ on $\partial_\infty M$. By the G-equivariance of the maximal weight it follows that $\lambda_x \geq 0$. By Lemma 5.1, G_y is compatible with Lie algebra $\mathfrak{g}_y = \mathfrak{k}_y \oplus \mathfrak{p}_y$ and $Z(y) = e(S(\mathfrak{p}_y))$. Suppose there exist $p = e(\beta) \in Z(y)$. Then $e(-\beta) \in Z(y)$ also. Furthermore, $e(\beta)$ and $e(-\beta)$ are connected by the geodesic $[\exp(t\beta)]$. This means that the condition of the Theorem holds for $Z(y)$. Now, for $p \in Z(x)$, $g \cdot p \in Z(y)$. Let $q \in Z(y)$ be connected to $g \cdot p$ by a geodesic α. Then the geodesic $g^{-1} \circ \alpha$ connects p to $g^{-1} \cdot q \in Z(x)$. This concludes the proof of the theorem. □

Corollary 5.2 *A point $x \in X$ is polystable if and only if there exist $\beta \in S(\mathfrak{p})$, $y \in G \cdot x$ and $g \in (G^{\beta})^o$ such that $\lambda_x(e(\beta)) = 0$ and $\mu_{\mathfrak{p}}(gy) = 0$.*

6 Measure

Let N be a compact manifold. We denote by $\mathscr{M}(N)$ the vector space of finite signed Borel measures on N. These measures are automatically Radon [15, Thm. 7.8, p. 217]. Denote by $\mathrm{C}(N)$ the space of real continuous function on N. It is a Banach space with the sup–norm. By the Riesz Representation Theorem [15, p.223] $\mathscr{M}(N)$ is the topological dual of $\mathrm{C}(N)$. The induced norm on $\mathscr{M}(N)$ is the following one:

$$||\nu|| := \sup\left\{\int_N f d\nu : f \in \mathrm{C}(M), \sup_M |f| \leq 1\right\}. \tag{6}$$

We endow $\mathscr{M}(N)$ with the weak-$*$ topology as dual of $\mathrm{C}(N)$. Usually, this is simply called the *weak topology* on measures. We use the symbol $\nu_\alpha \rightharpoonup \nu$ to denote the weak convergence of the net $\{\nu_\alpha\}$ to the measure ν. Denote by $\mathscr{P}(N) \subset \mathscr{M}(N)$ the set of Borel probability measures on N. We claim that $\mathscr{P}(N)$ is a compact convex subset of $\mathscr{M}(N)$. Indeed the cone of positive measures is closed and $\mathscr{P}(N)$ is the intersection of this cone with the closed affine hyperplane $\{\nu \in \mathscr{M}(N) : \nu(N) = 1\}$. Hence $\mathscr{P}(N)$ is closed. For a positive measure $|\nu| = \nu$, so $\mathscr{P}(N)$ is contained in the closed unit ball in $\mathscr{M}(N)$, which is compact in the weak topology by the Banach-Alaoglu Theorem [13, p. 425]. Since $\mathrm{C}(N)$ is separable, the weak topology on $\mathscr{P}(N)$ is metrizable [13, p. 426].

If $f : X \longrightarrow Y$ is a measurable map between measurable spaces and ν is a measure on X, the *image measure* $f_\star \nu$ is defined by $f_\star\nu(A) := \nu(f^{-1}(A))$. It satisfies the *change of variables formula*

$$\int_Y u(y)\mathrm{d}(f_\star\nu)(y) = \int_X u(f(x))\mathrm{d}\nu(x). \tag{7}$$

Lemma 6.1 *[3, Lemma 5.5] Let N be a compact manifold. If G is a Lie group acting continuously on N, the map*

$$G \times \mathscr{P}(N) \longrightarrow \mathscr{P}(N), \qquad (g, \nu) \mapsto g_\star \nu, \tag{8}$$

defines a continuous action of G on $\mathscr{P}(N)$ provided with the weak topology.

Let (Z, ω) be a compact connected Kähler manifold. Let U be a compact Lie group and $U^{\mathbb{C}}$ its complexification. As before, we assume that $U^{\mathbb{C}}$ acts holomorphically on Z, and the Kähler form is U-invariant. It is also assumed that there exists a momentum map $\mu : Z \longrightarrow \mathfrak{u}$. If $G \subset U^{\mathbb{C}}$ is closed and compatible we denote by $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ the associated G-gradient map. Finally, If X is a compact connect G-invariant submanifold of Z then $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ is a K-equivariant map such that $\operatorname{grad} \mu_{\mathfrak{p}}^{\beta} = \beta_X$. In [4] the authors introduced an abstract setting for actions of noncompact real reductive Lie groups on topological spaces that admit functions similar to the Kempf-Ness function.

Let $\Phi : X \times G \longrightarrow X$ be the Kempf-Ness function such that

$$\langle \mu_{\mathfrak{p}}(x), \beta\rangle = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Phi(x, \exp(t\beta)),$$

for any $\beta \in \mathfrak{p}$. As before, we have fixed B an $\mathrm{Ad}(U^{\mathbb{C}})$-invariant inner product on $\mathfrak{u}^{\mathbb{C}}$ and $\langle \cdot, \cdot \rangle$ denotes the real part of B restricted on $\mathfrak{g}$.

Proposition 6.2 *[4, Proposition 31] The function*

$$\Phi^{\mathscr{P}} : \mathscr{P}(X) \times G \longrightarrow \mathbb{R}, \qquad \Phi^{\mathscr{P}}(\nu) := \int_X \Phi(x, g) \mathrm{d}\nu(x),$$

is the Kempf-Ness function for $(\mathscr{P}(X), G, K)$ with gradient map

$$\mathscr{F}(\nu) = \int_X \mu_{\mathfrak{p}}(x) \mathrm{d}\nu(x).$$

Definition 6.1 Let $\nu \in \mathscr{P}(X)$. Then

a) ν is called *polystable* if $G \cdot \nu \cap \mathscr{F}^{-1}(0) \neq \emptyset$.
b) ν is called *stable* if it is polystable and G_ν is compact.
c) ν is called *semistable* if $\overline{G \cdot \nu} \cap \mathscr{F}^{-1}(0) \neq \emptyset$.
d) ν is called *unstable* if it is not stable, polystable and semistable.

In [4], see also [3], the authors construct the maximal weight function

$$\lambda_\nu : \partial_\infty(G/K) \longrightarrow \mathbb{R} \cup \{+\infty\},$$

for any $\nu \in \mathscr{P}(X)$ proving that the maximal weight is G-equivariant. The main goal of this section is to show that the Mundet criterion for polystability holds for the G-action on the measure. The same proof of Theorem 5.5 works.

Let $\beta \in \mathfrak{p}$. Then $\mu_{\mathfrak{p}}^{\beta} : X \longrightarrow \mathbb{R}$ is a Morse-Bott function with $\mathrm{Crit}\, \mu_{\mathfrak{p}}^{\beta} = X^\beta$. Let $C_1, \ldots, C_k$ be the connected components of X^β and let

$$W_i^\beta := \{x \in X : \lim_{t \to +\infty} \exp(t\beta)x \in C_i\}.$$

By Theorem 2.3, W_i^β is an immersed submanifold and $X = \bigcup W_i^\beta$ is a disjoint union.

Lemma 6.3 *Let $\nu \in \mathscr{P}(X)$ and let $\beta \in \mathfrak{p}$. If $\beta \in \mathfrak{g}_\nu$ then $\nu(X^\beta) = \nu(X)$ and so $\nu(X - X^\beta) =$ 0*

Proof $\exp(\mathbb{R}\beta)$ fixes pointwise C_i and so $C_i \subseteq W_i^\beta$. Let $L_n = \exp(n\beta)(W_i^\beta)$ for any $n \in \mathbb{N}$. Since $\exp(t\beta)$ fixes ν it follows that $\nu(L_n) = \nu(W_i^\beta)$ for any $n \in \mathbb{N}$. Since W_i^β is $\exp(\mathbb{R}\beta)$-invariant, it follows that $L_{n+1} \subset L_n$ and $C_i = \bigcap_{n=1}^{+\infty} L_n$. Therefore

$$\nu(C_i) = \lim_{n \mapsto +\infty} \nu(L_n) = \nu(W_i^\beta).$$

Hence $\nu(X) = \sum_{i=1}^k \nu(W_i^\beta) = \sum_{i=1}^k \nu(C_i) = \nu(X^\beta)$, concluding the proof. □

Corollary 6.1 *Let $\nu \in \mathscr{P}(Z)$ and let $\beta \in i\mathfrak{u}$. If $\beta \in \mathfrak{u}_\nu^{\mathbb{C}}$ then $i\beta \in \mathfrak{u}_\nu^{\mathbb{C}}$.*

Proof Since $(\mathrm{i}\beta)_Z = J(\beta_Z)$ it follows that $Z^{\mathrm{i}\beta} = Z^\beta$. Let $U \subset Z$. Then $U = (U \cap Z^{\mathrm{i}\beta}) \cup (U - Z^{\mathrm{i}\beta})$ and both set are $\exp(t\mathrm{i}\beta)$-invariant. Therefore,

$$\begin{aligned} \nu(\exp(-t\mathrm{i}\beta)(U)) &= \nu(\exp(-t\mathrm{i}\beta)(U \cap Z^{\mathrm{i}\beta})) + \nu(\exp(-t\mathrm{i}\beta)(U - Z^{\mathrm{i}\beta})) \\ &= \nu(\exp(U \cap Z^{\mathrm{i}\beta})) \\ &= \nu(U), \end{aligned}$$

concluding the proof. □

If $\beta \in \mathfrak{u}$ and $\beta \in \mathfrak{u}_\nu^{\mathbb{C}}$ then it is not true that $\mathrm{i}\beta \in \mathfrak{u}_\nu^{\mathbb{C}}$. Indeed, the volume form ω on the unit sphere S^2 is invariant with respect to SO(3). In particular the Killing field X generated to the one parameter subgroup $t \mapsto \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}$ preserves ω. The vector field $J(X)$ is the gradient of the the height function

$$S^2 \longrightarrow \mathbb{R}, \qquad p \mapsto \langle p, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \rangle$$

where $\langle \cdot, \cdot \rangle$ is the euclidean scalar product. We claim that $J(X)$ does not preserve ω. Indeed, keeping in mind that $(S^2)^{J(X)} = \{e_3, -e_3\}$, if the flow of $J(X)$ fixes the Borel measure ν associated to ω, then by Lemma 6.3 it follows that $1 = \nu(S^2) = \nu(\{e_3\}) + \nu(\{-e_3\})$. A contradiction.

Proposition 6.4 *Let $\nu \in \mathscr{P}(X)$ and let $\mathfrak{a} \in \mathfrak{p}$ be an Abelian subalgebra. Let $\nu \in \mathscr{P}(X)$. If $\mathfrak{a} \subset \mathfrak{g}_\nu$ then $\mathfrak{F}(\nu) \in \mathfrak{p}^{\mathfrak{a}}$.*

Proof By [6, Theorem 1,1] there exists $\beta \in \mathfrak{a}$ such that

$$X^{\mathfrak{a}} = \{p \in X : \gamma_X(p) = 0, \quad \text{for any } \gamma \in \mathfrak{a}\} = X^\beta.$$

By change of variable formula, we get $\mathscr{F}(\nu) = \mathscr{F}(\exp(t\beta)\nu) = \int_X \mu_{\mathfrak{p}}(\exp(t\beta)x)\mathrm{d}\nu(x)$. Taking the limit for $t \mapsto +\infty$ we get

$$\begin{aligned} \mathscr{F}(\nu) &= \sum_{i=1} \int_{W_i^\beta} \mu_{\mathfrak{p}}(x)\mathrm{d}\nu(x) = \sum_{I=1}^{k} \lim_{t\to+\infty} \int_{W_i^\beta} \mu_{\mathfrak{p}}(\exp(t\beta)x)\mathrm{d}\nu(x) \\ &= \sum_{i=1}^{k} \int_{C_i} \mu_{\mathfrak{p}}(x)\mathrm{d}\nu(x), \end{aligned}$$

where $C_1, \ldots, C_k$ are the connected components of $X^{\mathfrak{a}}$. By [5, Proposition 2.10], the image of $(\mu_{\mathfrak{p}})_{|C_i}$ lies in $\mathfrak{p}^{\mathfrak{a}}$ and so the result follows. □

Finally, one can characterize the stability condition in terms of the maximal weight functions, see for instance [4, Theorem13].

Theorem 6.5 *A measure ν is stable if and only if $\lambda_\nu > 0$.*

Theorem 6.6 *A measure ν is polystable if and only if $\lambda_\nu \geq 0$ and for any $p \in Z(\nu)$ there exists $p' \in Z(\nu)$ such that p and p' are connected.*

Proof If $Z(\nu) = \emptyset$ then ν is stable. Otherwise, by Lemma 5.2 and 5.3 there exists $\nu' \in G \cdot \nu$ and $\xi \in S(\mathfrak{p})$ such that $\lambda_{\nu'}(\xi) = 0$, $\beta \in \mathfrak{g}_{\nu'}$

$$\dim T_\xi = \dim T_\beta = \max_{\eta \in e^{-1}(Z(\nu'))} \dim T_\eta.$$

Let $\mathfrak{t}_\xi$ be the Lie algebra of T_ξ. Let $i : X \hookrightarrow Z$ be the inclusion and let $\nu'' = i_\# \nu'$. Since $\xi \in \mathfrak{p}_{\nu'}$ it follows that $\xi \in \mathfrak{p}_{\nu''}$. By Corollary 6.1, $\mathrm{i}\xi \in \mathfrak{u}_{\nu''}^{\mathbb{C}}$ and so T_ξ fixes ν''. Therefore $\mathfrak{a} = \mathrm{i}\mathfrak{t}_\xi \cap \mathfrak{p}^\xi \cap \mathfrak{p}_{\nu'}$ is an Abelian subalgebra contained in $\mathfrak{p}^\xi$ and different from zero since $\xi \in \mathfrak{a}$. From now on, the proof of Theorem 5.5 holds for the G-action on the measure. □

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