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Heffter arrays and related topics

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Introduction

The study of combinatorial arrays has always been present in human history, as many of these structures present some degree of balance and symmetry in their definition. As such, through the years many of these arrays were considered as tokens of good luck, and carved on amulets that could be carried during dangerous voyages or in the everyday life. Nonetheless the existence and applications of these objects often raise interesting mathematical questions, that in many cases are still unsolved and inspired much of the research done in Discrete Mathematics over the course of the last century. In particular, one of the most interesting characteristics of these combinatorial structures is the variety of formulations of the same object under different points of view, and the relationships between the different solutions that the same object codifies. Among the various combinatorial arrays that can be considered we mention the notable classes of Latin squares [31], Hadamard matrices [44] and Room squares [46]. Each of these families of arrays shares its wide variety of applications and connections with Block Designs, Coding Theory, Graph Theory and Finite Geometry.

In this thesis we focus on a class of combinatorial arrays called Heffter arrays, introduced by Archdeacon [6] in 2015 as a link between Combinatorial Designs and Topological Graph Theory. Moreover, the rather general definition of Heffter arrays and their variety of applications led to the introduction of many variants and generalizations.

One of the main themes of the area of Combinatorial Designs is the construction and the arrangement of a collection of objects in subsets satisfying certain regularity conditions. For instance, a $2-(v, k, 1)$ design (also called Steiner system) is a pair $(\mathcal{V}, \mathcal{B})$, where \mathcal{V} is a set of v points and \mathcal{B} is a collection of k -subsets called blocks, with the property that every two distinct points are contained in exactly one block. A $2-(v, 3, 1)$ design is called a Steiner triple system of order v , denoted as $\text{STS}(v)$; it can be easily seen that an $\text{STS}(v)$ must have $v \equiv 1, 3 \pmod{6}$, and it has been shown by Kirkman in 1847 [61] that these conditions are also sufficient.

In 1872 Klein published his famous Erlangen program, where he proposed to use group theory to study the symmetries of geometric structures; in this setting the most elegant geometries are those admitting numerous symmetries or, more formally, having a large group of automorphisms.

A similar approach was then considered for Design Theory; in particular Heffter proposed in 1896 and 1897 his two famous difference problems whose solution would allow the construction for every admissible v of an $\text{STS}(v)$ whose group of automorphisms contains the cyclic group \mathbb{Z}_v . As discussed in Section 1.1 these problems were eventually solved by Pelsesohn forty years later [81]. These highly symmetric structures present two main advantages: first of all, they

can be found by using algebraic tools, hence in general they are more simple to find compared to non-symmetric structures (even if the latter usually outnumber the former); secondly, they are easy to exhibit as one can just construct a suitable set of representatives of the block orbits. We underline that the same concept holds in the more general context of graph decompositions, that have been introduced as a generalization of 2-designs in the second half of the 20th century. Given a subgraph Γ of a graph K a (K, Γ) -design, also called a Γ -decomposition of K , is a partition of the edge set of K into graphs isomorphic to Γ . If K and Γ are complete graphs of order v and k respectively we find again the concept of a $2-(v, k, 1)$ design. The most studied graph decompositions in the literature are those in which K is a complete or a complete multipartite graph, and Γ is a complete graph or a cycle graph, see also Section 2.2.

As said before in this thesis we study Heffter arrays, introduced by Archdeacon [6] as a tool to construct decompositions of the complete graph into cycles, and 2-colorable embeddings of the same complete graph over an orientable surface, such that the cyclic group belongs to the automorphism group of these structures. The naming of these arrays is due to Heffter's first difference problem, as from some Heffter arrays one can construct an explicit solution to the difference problem. A Heffter array is a partially filled matrix whose entries form a half-set of a cyclic group, such that any two rows (respectively, any two columns) have the same number of filled cells, and every row and every column has sum equal to zero in the group. Since their introduction much attention has been devoted to establish existence results for these arrays, obtaining a complete solution for square arrays and completely filled rectangular arrays, while partial existence results have been proven for the rectangular partially filled case.

In Chapter 1 we report the main definition of Heffter arrays, and we give a historical overview of the Design Theory's topics that led to their naming. Then we discuss the main existence results, and we present some of the methods that have been used to construct infinite classes of Heffter arrays. Finally, we mention the main variants and generalizations that have been studied throughout the years, and we briefly discuss in each case the motivations that led to their introduction and their main existence results.

In Chapters 2 and 3 we respectively analyse the applications of Heffter arrays to graph decompositions and to 2-colorable embeddings. In particular in Chapter 2 we give some basic definitions of Graph Theory, and we see some notions about 2-designs and graph decompositions. Finally, we introduce difference families as one of the main tools for constructing these combinatorial structures. In Chapter 3 we give some basic notions of Topological Graph Theory that are necessary to construct the embeddings from Heffter arrays. Finally, we investigate the automorphism group of these embeddings presenting some results that we obtained in [36].

In Chapter 4 we study the class of weak Heffter arrays, that is a variant of Heffter arrays introduced by Archdeacon in [6] in order to construct graph embeddings over non-orientable surfaces. In this chapter we present the results that we obtained in [37] on the existence of these arrays, and on the graph embeddings over non-orientable surfaces that can be constructed from them.

In Chapters 5 and 6 we present the results that we respectively obtained in [69] and [70] on non-zero sum Heffter arrays and on generalized Archdeacon arrays. In the former we construct some classes of non-zero sum Heffter arrays that moreover satisfy a very strict additional property, called global simplicity;

we also compute the explicit formulas for the length of faces of the embeddings that can be constructed from these arrays. In the latter we completely solve the existence problem of globally simple non-zero sum Heffter arrays for cyclic groups, and we give a very general constructive result for non-zero sum Heffter arrays over any (not necessarily abelian) group.

To conclude, in Chapter 7 we study a concept recently introduced in [23] called Heffter space, that connects Heffter arrays with topics of Finite Geometry. We then present some results that have been developed in a work in progress [21], where we construct infinite families of Heffter spaces. Moreover, we show that in some cases these Heffter spaces are in a certain sense the best ones that can be constructed.

Chapter 1

Heffter arrays

As mentioned in the Introduction, Archdeacon [6] defined a class of arrays, that he called Heffter arrays in view of their connection with the first Heffter difference problem [57]. These arrays were introduced as a tool to construct regular graph decompositions into cycles, and their embedding on orientable surfaces, as shown in Chapters 2 and 3 respectively. In this chapter we give a general overview of Heffter arrays and of their main variants and generalizations that have been introduced throughout the years. In Section 1.1 we discuss classical Heffter arrays, together with a historical overview of some combinatorial problems that led to their definition and the main existence results for Heffter arrays. These results have been obtained by means of various construction methods: we present some of them in Section 1.2. Finally, in Section 1.3 we report the main variants and generalizations of Heffter arrays; for each of these classes of arrays we also briefly discuss the motivations that led to their introduction, and their main existence results.

1.1 General introduction

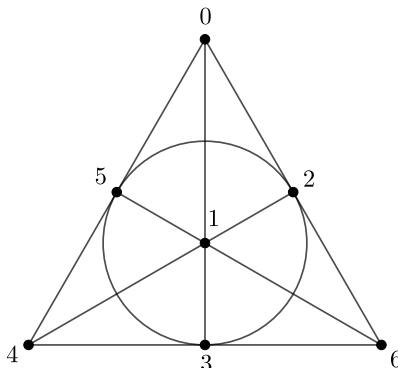
We begin this section by introducing some concepts of Design Theory that are necessary to give a historical overview of the concept and the naming of the Heffter arrays.

A *Steiner triple system* of order v (in short, $\text{STS}(v)$) is a pair $(\mathcal{V}, \mathcal{B})$ given by a set \mathcal{V} of v elements called *points*, together with a family \mathcal{B} of 3-subsets called *blocks* or *triples*, such that every pair of distinct points appears in precisely one block.

Example 1.1.1. The most well-known Steiner triple system is the unique $\text{STS}(7)$ $(\mathcal{V}, \mathcal{B})$ called *Fano plane*, where:

$$\begin{aligned}\mathcal{V} &= \{0, 1, 2, 3, 4, 5, 6\}, \\ \mathcal{B} &= \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{0, 4, 5\}, \{1, 5, 6\}, \{0, 2, 6\}\}.\end{aligned}$$

The Fano plane can then be represented as in the following picture, where the 7 blocks of \mathcal{B} are nothing but the lines of the plane.



While the complete existence result for this kind of designs was established by Kirkman in 1847 [61], scholars introduced various tools and problems in later years in order to find simpler and more practical constructions of Steiner triple systems. In particular Heffter in 1896 proposed the following combinatorial problem, see [57]:

Heffter's first difference problem. Given any $v \equiv 1 \pmod{6}$, $v \geq 7$, is it always possible to partition the set $\{1, 2, \dots, \frac{v-1}{2}\}$ into $\frac{v-1}{6}$ triples $\{a, b, c\}$ such that either $a + b = c$ or $a + b + c = v$?

The existence of such a partition gives rise to a STS(v) that admits \mathbb{Z}_v as group of automorphisms (we discuss more in detail this concept in Section 2.3.3 of this thesis). The affirmative answer to this problem was eventually given forty years later by Pelsesohn [81]. Later works on the same problem were published by Skolem [92, 93], who obtained a simpler solution by using combinatorial sequences that are now known as *Skolem sequences*, see also [90].

In order to proceed with the discussion it is first necessary to introduce some terms and notation that will be used throughout the thesis. Given an abelian group G and a subgroup J , we write J as $\langle s_1, \dots, s_a \rangle$ if J is generated by a subset $\{s_1, \dots, s_a\}$ of G . Moreover, a subset of G is said to be *zero-sum* if the sum of its element is the identity of the group.

In order to show the connection between Heffter's first difference problem and Heffter arrays we also give the following definition:

Definition 1.1.2. Let J be a subgroup of an additive group G . A *half-set* of $G \setminus J$ is a subset L such that L and $-L$ partition $G \setminus J$.

Clearly a half-set of $G \setminus J$ exists if and only if the involutions of G are in J . In the particular case $J = \{0\}$ we simply say that L is a *half-set of G* . More in general we define a half-set of $S \subset G$ to be a subset L such that L and $-L$ partition S .

Example 1.1.3. The following is a half-set of $\mathbb{Z}_{45} \setminus \langle 9 \rangle$:

$$\{-1, 2, 3, -4, 5, -6, -7, 8, -10, 11, 12, -13, 14, -15, -16, 17, -19, 20, 21, -22\}.$$

Note that the condition of the triples given in Heffter's first difference problem can be equivalently restated as $a + b - c = 0$ or $a + b + c \equiv 0 \pmod{v}$, and that the underlying group of the problem is \mathbb{Z}_v , while the set considered by Heffter is $\{1, \dots, \frac{v-1}{2}\}$. Thus we can see that Heffter's first difference problem asks for a partition of a half-set of \mathbb{Z}_v into zero-sum triples.

More in general one could ask for a partition of a half-set into zero-sum subsets, see [71]:

Definition 1.1.4. A *Heffter system* $D(nk, k)$ is a partition of a half-set of $G \setminus J$ having nk elements into zero-sum k -subsets called *blocks*.

We remark that in the original notation given by Archdeacon [6] a Heffter system over a group of order v is written as $D(v, k)$; here we follow the more recent notation given in [23].

Definition 1.1.5. Two Heffter systems D_1 and D_2 are *orthogonal* if they are defined on the same half-set, and for every pair of blocks $B_1 \in D_1$ and $B_2 \in D_2$ it holds $|B_1 \cap B_2| \leq 1$.

Example 1.1.6. The following set L is a half-set of $\mathbb{Z}_{43} \setminus \{0\}$:

$$L = \{-1, -2, -3, -4, 5, 6, 7, -8, -9, 10, -11, 12, \\ -13, 14, 15, 16, 17, 18, 19, 20, 21\}.$$

Then, L can be partitioned into the following two Heffter systems $D(21, 3)$:

$$D_1 = \{\{-1, -9, 10\}, \{-2, -13, 15\}, \{-3, -11, 14\}, \\ \{-4, -8, 12\}, \{5, 17, 21\}, \{6, 18, 19\}, \{7, 16, 20\}\}; \\ D_2 = \{\{-1, -13, 14\}, \{-2, -8, 10\}, \{-3, -9, 12\}, \\ \{-4, -11, 15\}, \{5, 18, 20\}, \{6, 16, 21\}, \{7, 17, 19\}\}.$$

It can be seen that D_1 and D_2 are orthogonal. For instance, if we take $B_1 = \{-1, -9, 10\} \in D_1$ and $B_2 = \{-2, -8, 10\} \in D_2$, $B_1 \cap B_2 = \{10\}$.

The concept of a Heffter array was given by Archdeacon in [6] as an array whose rows and columns respectively form two orthogonal Heffter systems. In particular, if $D_1 = \{R_1, \dots, R_m\}$ and $D_2 = \{C_1, \dots, C_n\}$ are two orthogonal Heffter systems, then for each $i \in \{1, \dots, m\}$ and each $j \in \{1, \dots, n\}$ the (i, j) -th cell of an $m \times n$ Heffter array is filled with the common element of R_i and C_j , if any, and the cell is empty otherwise. The standard definition that is given now of a Heffter array is the following one:

Definition 1.1.7. Let v be a positive integer with $v = 2nk + 1$. A *Heffter array* A over \mathbb{Z}_v , denoted by $H(m, n; h, k)$, is an $m \times n$ partially filled array A such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b) the entries of A form a half-set of \mathbb{Z}_v ;
- (c) every row and column is zero-sum in \mathbb{Z}_v .

It is clear that a necessary condition for the existence of an $H(m, n; h, k)$ is $nk = mh$. Moreover, since the sum must be equal to 0 in each row and each column, property (b) implies that $h, k \geq 3$. For simplicity we say that a quadruple of integers (m, n, h, k) is *admissible* if $mh = nk$ and $3 \leq h \leq n$, $3 \leq k \leq m$. Note that if a Heffter array is square, i.e. $m = n$, then necessarily $h = k$, and we simply write $H(n; k)$. If the array is completely filled, i.e. $n = h$ and $m = k$, then we write $H(m, n)$.

Given an array A its *skeleton* $skel(A)$ is the set its filled cells, while by $\mathcal{E}(A)$ we denote its set of entries. For an array A containing elements in \mathbb{Z}_v we define its *support* to be the set:

$$supp(A) = \begin{cases} x & \text{if } x \in \left\{0, 1, \dots, \left\lfloor \frac{v}{2} \right\rfloor\right\}, \\ -x & \text{otherwise.} \end{cases}$$

Example 1.1.8. In this example we construct the following Heffter array over \mathbb{Z}_{43} starting from the two orthogonal Heffter systems of Example 1.1.6; in particular blocks of D_1 correspond to rows of the array, while the blocks of D_2 are its columns, see Example 1.8 of [79]:

-1	-9	10				
-13		-2	15			
14	-3		-11			
	12	-8	-4			
				5	17	21
				18	19	6
				20	7	16

Definition 1.1.9. An *integer* $H(m, n; h, k)$ is a partially filled $m \times n$ array whose entries form a half-set of $\{-mh, \dots, mh\} \subset \mathbb{Z}$, whose rows and columns are zero-sum, and satisfying property (a) of Definition 1.1.7.

Clearly any integer Heffter array can also be seen as a Heffter array over a cyclic group, while the converse is not true, see for instance the array of Example 1.1.8. A simple counting argument shows that an integer $H(m, n; h, k)$ may exist only for admissible quadruples (m, n, h, k) with $nk \equiv 0, 3 \pmod{4}$.

Example 1.1.10. The following is an integer $H(5; 4)$ over \mathbb{Z}_{41} :

1		18	-9	-10
-6	2		19	-15
-11	-12	3		20
16	-7	-13	4	
	17	-8	-14	5

Since the introduction of Heffter arrays researchers found many results regarding their existence, and in what follows we report the main ones; we remark that all of these statements are derived from a constructive approach. In particular a complete solution for the existence problem has been established when the array is square, and when the array is rectangular and totally filled. As a first step, in [8, 48] it is proved that:

Theorem 1.1.11. *There exists an integer $H(n; k)$ if and only if $3 \leq k \leq n$ and $nk \equiv 0, 3 \pmod{4}$.*

In [29] it is shown the existence of an $H(n; k)$ (that clearly cannot be integer) for $nk \equiv 1, 2 \pmod{4}$, thus proving the following theorem:

Theorem 1.1.12. *There exists an $H(n; k)$ if and only if $3 \leq k \leq n$.*

The existence result on totally filled rectangular arrays is showed in [7]:

Theorem 1.1.13. *There exists an $H(m, n)$ if and only if $m, n \geq 3$. Moreover, there exists an integer $H(m, n)$ if and only if $mn \equiv 0, 3 \pmod{4}$.*

The existence problem of rectangular Heffter arrays having empty cells is still open even if throughout the years a lot of partial results have been found, especially focused on the integer case [76]. Here, we summarize these results. In [76] it has been shown that:

Theorem 1.1.14. *Let (m, n, h, k) be an admissible quadruple, and set $d = \gcd(h, k)$. There exists an integer $H(m, n; h, k)$ in each of the following cases:*

- (1) $d \equiv 0 \pmod{4}$;
- (2) $d \equiv 1 \pmod{4}$ with $d \geq 5$ and $nk \equiv 3 \pmod{4}$;
- (3) $d \equiv 2 \pmod{4}$ and $nk \equiv 0 \pmod{4}$;
- (4) $d \equiv 3 \pmod{4}$ and $nk \equiv 0, 3 \pmod{4}$.

It is then proven in [76] that:

Theorem 1.1.15. *There exists an integer $H(m, n; h, k)$ for any admissible quadruple (m, n, h, k) with $h \equiv 0 \pmod{4}$ and $k \neq 5$ odd.*

To conclude, in a recent article [80] the authors show:

Theorem 1.1.16. *Let (m, n, h, k) be an admissible quadruple with $nk \equiv 0, 3 \pmod{4}$. Then, there exists an integer $H(m, n; h, k)$ whenever $k \geq 7 \cdot \gcd(h, k)$ is odd and $h \neq 3, 5, 6, 10$.*

1.2 Construction methods

In this section we briefly examine the main tools and construction methods that have been employed to build square Heffter arrays and their variants. In general, an $H(n; k)$ is constructed recursively starting from an $H(n; k')$ where $k' < k$ and $k' \equiv k \pmod{4}$, and then by adding suitable zero-sum blocks in the empty cells of the starting array.

First, if $a \leq b$ are two integers by $[a, b]$ we denote the set $\{a, a + 1, \dots, b\}$, and given a set $S = \{a_1, \dots, a_k\}$ and an integer x , by $S + x$ we mean $\{a_1 + x, \dots, a_k + x\}$. A *shiftable (Heffter) array* is an (Heffter) array containing in each row and in each column $k/2$ positive numbers and $k/2$ negative numbers for some positive even integer k . For instance, the Heffter array in Example 1.1.10 is shiftable. If A is a shiftable array by $A \pm x$ we denote the array obtained by adding $+x$ to the positive entries of A and $-x$ to the negative entries of A . It can be seen that the row and column sums of $A \pm x$ are the same as those of A , while $\text{supp}(A \pm x) = \text{supp}(A) + x$. Moreover, $A \pm x$ is shiftable as well.

Example 1.2.1. Let A be the shiftable Heffter array $H(5; 4)$ of Example 1.1.10. As an example of a shift of A , let $x = 10$; then $A \pm x$ is the following array:

11		28	-19	-20
-16	12		29	-25
-21	-22	13		30
26	-17	-23	14	
	27	-18	-24	15

It can be seen that $\text{supp}(A) = [1, 20]$, and that $\text{supp}(A \pm x) = [11, 30] = \text{supp}(A) + 10$.

It has then been shown in [8] that:

Theorem 1.2.2. *There exists a shiftable integer $H(n; k)$ if and only if k is even and $nk \equiv 0 \pmod{4}$.*

Shiftable integer Heffter arrays are then used as a building block for the construction of Heffter arrays having a large number of filled cells in each row and column. The idea, that we show in more detail in Example 1.2.3, is to construct an integer $H(n; k)$, say A , with a small value of k , and a shiftable integer $H(n; k')$, say B , such that the two arrays have disjoint skeletons. One can then construct a Heffter array $H(n; k + k')$ by overlapping A with $B \pm nk$. The resulting array is zero-sum in the integers and its support is $[1, nk] \cup [nk + 1, n(k + k')] = [1, n(k + k')]$, hence it is an $H(n; k + k')$.

To explain in detail this strategy we need to give some further remarks on the structure of the arrays that have been built in the literature. Indeed, many Heffter arrays have been constructed with a diagonal structure in order to apply the recursive construction and for their applications to biembeddings (see Section 3.5). Moreover, this structure is particularly suited for computer searches. Given a square array A of size n , for every $i \in [1, n]$ we define the i -th diagonal D_i as the set of cells:

$$D_i = \{(i, 1), (i + 1, 2), \dots, (i - 1, n)\},$$

where the elements are seen in \mathbb{Z}_n using the reduced residues $[1, n]$. Given $n \geq k \geq 1$ a partially filled $n \times n$ array A is k -diagonal if its nonempty cells are exactly those of k diagonals. Then an $n \times n$ array A is *cyclically k -diagonal* if its filled cells are precisely k consecutive diagonals.

In [48] the authors introduced the following concise notation to describe a diagonal array. We point out that this notation has been used in several subsequent papers including [40, 43], and it is also used in Chapter 7 of this thesis where we report the results that we obtained in [21]. This procedure for filling a sequence of cells on a diagonal is termed *diag* and it has six parameters, as follows.

Let A be an $n \times n$ partially filled array; the procedure $\text{diag}(r, c, s, \Delta_1, \Delta_2, \ell)$ fills the entries

$$A[r + i\Delta_1, c + i\Delta_1] = s + i\Delta_2 \quad \text{for } i \in [0, \ell - 1].$$

The parameters used in the *diag* procedure have the following meaning:

- (1) r denotes the starting row;

- (2) c denotes the starting column;
- (3) s denotes the entry $A[r, c]$;
- (4) Δ_1 denotes the increasing value of the row and column at each step;
- (5) Δ_2 denotes how much the entry is changed at each step;
- (6) ℓ is the length of the chain.

The construction of integer Heffter arrays has been established using some diagonal shiftable Heffter arrays in [8] and the *diag* method introduced above in [48], hence proving Theorem 1.1.11.

Example 1.2.3. In this example we show how to construct an integer $H(11; 9)$ starting from an integer 5-diagonal $H(11; 5)$ that we call A :

10	53				24	-33				-54
-36	-9	44				32	-31			
	-45	-8	52				30	-29		
		-37	-7	43				28	-27	
			-46	-6	51				26	-25
-21				-38	-11	47				23
12	-13				-42	4	39			
	14	-15				-50	3	48		
		16	-17				-41	2	40	
			18	-19				-49	-5	55
35				20	-22				-34	1

As a first step we construct a shiftable and integer $H(11; 4)$ with a diagonal structure, that we denote by B , such that $skel(A) \cap skel(B) = \emptyset$. As shown in Theorem 2.2 of [48], if we choose two pairs of consecutive diagonals that are not filled in A , say $\{D_4, D_5\}$ and $\{D_8, D_9\}$, then the array B constructed by applying the following procedures

$$\begin{aligned} &diag(4, 1, 1, 1, 2, 11), && diag(4, 8, -23, 1, -2, 11), \\ &diag(5, 1, -2, 1, -2, 11), && diag(5, 8, 24, 1, 2, 11), \end{aligned}$$

is an integer and shiftable $H(11; 4)$. In fact, every row except for the 4-th one is filled with $\{-a, a + 1, b, -(b + 1)\}$ for some $a, b \in \mathbb{Z}_{89}$, hence it is zero-sum. It can be immediately seen that the 4-th row is zero-sum as well as it contains $\{1, -2n, -(2n + 1), 4n\}$, where $n = 11$ is the dimension of the array. A similar argument can be done on the columns of the array. Here we explicitly write B :

			38	-39			-16	17		
				40	-41			-18	19	
					42	-43			-20	21
1						44	-23			-22
-2	3						24	-25		
	-4	5						26	-27	
		-6	7						28	-29
-31			-8	9						30
32	-33			-10	11					
	34	-35			-12	13				
		36	-37			-14	15			

We have then that $\text{supp}(A) = [1, 55]$ and $\text{supp}(B) = [1, 44]$; in order to construct an $H(11; 9)$ we consider the array $B \pm 55$ having support $[56, 99]$:

			93	-94			-71	72		
				95	-96			-73	74	
					97	-98			-75	76
56						99	-78			-77
-57	58						79	-80		
	-59	60						81	-82	
		-61	62						83	-84
-86			-63	64						85
87	-88			-65	66					
	89	-90			-67	68				
		91	-92			-69	70			

The array obtained by overlapping A and $B \pm 55$ is an integer 9-diagonal $H(11; 9)$:

10	53		93	-94	24	-33	-71	72		-54
-36	-9	44		95	-96	32	-31	-73	74	
	-45	-8	52		97	-98	30	-29	-75	76
56		-37	-7	43		99	-78	28	-27	-77
-57	58		-46	-6	51		79	-80	26	-25
-21	-59	60		-38	-11	47		81	-82	23
12	-13	-61	62		-42	4	39		83	-84
-86	14	-15	-63	64		-50	3	48		85
87	-88	16	-17	-65	66		-41	2	40	
	89	-90	18	-19	-67	68		-49	-5	55
35		91	-92	20	-22	-69	70		-34	1

In the remainder on this section we discuss some further construction methods that have been used to build rectangular Heffter arrays. The existence of some rectangular Heffter arrays without empty cells has been established in [7]: as for the case of the $H(n; k)$, the provided construction relies on a recursive method that has been divided in 8 different cases depending on the parity of n and k . Without entering into the details of their construction, we remark that some existence results are obtained starting from particular solutions of Heffter's first difference problem by means of suitable Skolem sequences, see [92].

A different construction for rectangular and completely filled Heffter arrays has been recently obtained in [19] in some particular instances of the problem. Given $m, n \geq 3$ odd and coprime integers with $2mn + 1$ prime, let ξ and ϵ respectively be an n -th and an m -th root of unity in \mathbb{Z}_{2mn+1} . Let then $A_{m,n}$ be an $m \times n$ array whose (i, j) -th cell is $\epsilon^{i-1}\xi^{j-1} \pmod{2mn+1}$. Then:

Theorem 1.2.4. $A_{m,n}$ is an $H(m, n)$ over \mathbb{Z}_{2mn+1} .

This result is a corollary of more general results proved in [19], where the author studies the concept of Heffter arrays over a finite field that moreover have rank one in the standard sense of linear algebra (see also Section 1.3.6). Some further existence results on rectangular Heffter arrays with empty cells can be found in [77].

1.3 Main variants and generalizations

The variety of problems and applications related to Heffter arrays paired with the flexibility provided from their definition lead to the study of numerous variants and generalizations, each with its own set of applications and relations with other topics of Discrete Mathematics. In this section we give a brief overview of the main variants and generalization of Heffter arrays that have been introduced throughout the years. Many of these classes of arrays are studied in more detail in other parts of this thesis, hence here we focus on the definitions, the motivation for their introduction, and the main known results.

1.3.1 Weak Heffter arrays

In the same paper that first defined Heffter arrays Archdeacon proposed the notion of a variant of Heffter arrays, called *weak Heffter arrays*. In a weak Heffter array the entries are allowed to have two distinct signs, the *row sign* and the *column sign*; as explained in more detail in Chapter 4 the application to orthogonal cycle decompositions is maintained, while the 2-colorable embedding may be over a non-orientable surface. For many years the only paper that discussed weak Heffter array was [6], with only one example available that is Example 1.3.2 below. In [37], we studied this family of array together with their connection with some variants that have been introduced after [6].

Definition 1.3.1. Let v be a positive integer with $v = 2nk + 1$. A *weak Heffter array* $\text{WH}(m, n; h, k)$ is an $m \times n$ partially filled array A such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b₁) for every $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, there is exactly one cell of A whose element is one of the following: $x, -x, \pm x, \mp x$, where the upper sign on \pm or \mp is the row sign and the lower sign is the column sign;
- (c) every row and column, read with the corresponding signs, is zero-sum in \mathbb{Z}_{2nk+1} .

Example 1.3.2. A weak Heffter array over the group \mathbb{Z}_{25} :

1	-7	-6	12
2	-4	± 10	∓ 8
-3	∓ 11	± 9	5

Clearly the necessary conditions for Heffter arrays on the set of parameters, that are $nk = mh$, $n \geq k \geq 3$ and $m \geq h \geq 3$, are also necessary for weak Heffter arrays. Moreover, any Heffter array is also weak. It is also possible to apply trivial transformations to the elements of a Heffter array in order to obtain a weak Heffter array whose elements have distinct row and column sign. However, the converse is not true; for this reason, in [37] we studied the concept of *strictly weak Heffter arrays*, that are weak Heffter arrays that cannot be obtained from standard Heffter arrays. In this thesis the results on weak Heffter arrays are presented in Chapter 4.

1.3.2 Relative Heffter arrays

As mentioned in the previous section, Heffter arrays have been introduced as a tool to construct graph decompositions of the complete graph (see Section 2.4 for further details). The following class of arrays, introduced in [40], has been defined as a generalization of the classical concept of Heffter arrays in order to consider decompositions of a complete multipartite graph.

Definition 1.3.3. Let t be a positive integer dividing $2nk$, and let $J = \langle \frac{2nk+t}{t} \rangle$ be the subgroup of \mathbb{Z}_{2nk+t} of order t . A *Heffter array A over \mathbb{Z}_{2nk+t} relative to J* , denoted by $H_t(m, n; h, k)$, is an $m \times n$ partially filled array with elements in \mathbb{Z}_{2nk+t} such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b₂) the set of entries of A is a half-set of $\mathbb{Z}_{2nk+t} \setminus J$;
- (c) every row and column is zero-sum in \mathbb{Z}_{2nk+t} .

Note that the original definition of a Heffter array can be recovered by considering $t = 1$. Clearly, the previously established necessary conditions on the set of parameters hold also for this class of arrays. We remark that v is even if and only if t is even, and in this case the subgroup J contains the involution $v/2$, and property (b₂) ensures $h, k \geq 3$.

Example 1.3.4. A relative Heffter array $H_{16}(4; 4)$ over the group \mathbb{Z}_{48} , whose subgroup of order 16 consists on the multiples of 3, see Example 1.4 of [40]:

1	-7	-16	22
23	2	-8	-17
-13	19	4	-10
-11	-14	20	5

We recall that a Heffter array can be equivalently seen as two orthogonal Heffter systems; similarly, relative Heffter arrays are two orthogonal Heffter systems over $\mathbb{Z}_{2nk+t} \setminus \langle \frac{2nk+t}{t} \rangle$, where t is a positive integer dividing $2nk$.

In [40] it was shown that a relative integer Heffter array must satisfy the following necessary conditions:

Proposition 1.3.5. *Suppose that there exists an integer $H_t(n; k)$.*

- (1) *If t divides nk , then*

$$nk = 0 \pmod{4} \text{ or } nk = -t = \pm 1 \pmod{4}.$$

- (2) *If $t = 2nk$, then k must be even.*
- (3) *If $t \neq 2nk$ does not divide nk , then*

$$t + 2nk = 0 \pmod{8}.$$

In [40] it has also been shown that these conditions are not sufficient for the existence of a relative Heffter array, as they do not grant the existence of a relative Heffter system.

Proposition 1.3.6. *For every $n \geq 3$ there does not exist a Heffter system $D(3n; 3)$ over $\mathbb{Z}_{9n} \setminus \langle 3 \rangle$.*

The most complete existence result regarding square relative Heffter arrays is the following:

Theorem 1.3.7. *Let $3 \leq k \leq n$ with $k \neq 5$. There exists an integer $H_k(n; k)$ if and only if one of the following holds:*

- (1) k is odd and $n \equiv 0, 3 \pmod{4}$;
- (2) $k \equiv 2 \pmod{4}$ and n is even;
- (3) $k \equiv 0 \pmod{4}$.

Furthermore, there exists an integer $H_5(n; 5)$ if $n \equiv 3 \pmod{4}$ and it does not exist if $n \equiv 1, 2 \pmod{4}$.

Note that for $k = 5$ the existence problem of integer relative Heffter arrays $H_5(n; 5)$ has been proved only for $n \equiv 3 \pmod{4}$, leaving the case $n \equiv 0 \pmod{4}$ open. For this class in [40] there are only two examples for $n = 8$ and $n = 16$.

Other results on relative Heffter arrays can be found in [43, 75] and in Chapter 7 of this thesis as an intermediate step for the construction of Heffter spaces.

1.3.3 Non-zero sum Heffter arrays

The property of a (relative) Heffter array to be zero-sum provides cycle decompositions of the complete graph and their 2-colorable embedding. The variant of non-zero sum Heffter arrays was introduced in [35] as a way to provide path decompositions of a complete graph. Moreover, these arrays are related to a conjecture due to Alspach regarding the sequentiability of a group, see Section 2.4.

Definition 1.3.8. A *non-zero sum Heffter array* A over \mathbb{Z}_{2nk+1} , denoted by $NH(m, n; h, k)$, is an $m \times n$ partially filled array with elements in \mathbb{Z}_{2nk+1} such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b) the set of entries of A is a half-set of \mathbb{Z}_{2nk+1} ;
- (c₁) every row and column is *not* zero-sum in \mathbb{Z}_{2nk+1} .

For these arrays, the necessary conditions for their existence are $mh = nk$ and $m \geq h \geq 1, n \geq k \geq 1$.

Example 1.3.9. An $NH(3, 4)$ over \mathbb{Z}_{25} :

10	1	-11	12
8	6	-3	-5
-7	2	9	4

Clearly, these arrays are much easier to construct compared to the Heffter arrays. As we will see in Chapter 5, in the same paper [35] that introduced this variant the authors provided a complete result for $NH(n; k)$ and $NH(m, n; n, m)$ that moreover satisfy a very strict additional property. As for Heffter arrays it is possible to extend the definition of non-zero Heffter arrays to the relative case; we discuss this topic more extensively in Chapter 5.

1.3.4 λ -fold Heffter arrays

λ -fold Heffter arrays were introduced in [40, 41] in order to exploit the relationship standing between Heffter arrays and a generalization of the concept of difference families.

Definition 1.3.10. Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, where t divides $\frac{2nk}{\lambda}$. Let J be the subgroup of \mathbb{Z}_v of order t . A λ -fold Heffter array A over \mathbb{Z}_v relative to J , denoted by ${}^\lambda H_t(m, n; h, k)$, is an $m \times n$ partially filled array with elements in \mathbb{Z}_v such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b₃) the multiset $\{\pm x : x \in A\}$ contains each element of $\mathbb{Z}_v \setminus J$ exactly λ times;
- (c) every row and column is zero-sum in \mathbb{Z}_v .

Example 1.3.11. An ${}^2 H_1(5; 3)$ over \mathbb{Z}_{16} , see Example 1.8 of [41]:

3	5	-8		
	2	5	-7	
		3	1	-4
-4			6	-2
1	-7			6

Note that 8 is the involution of \mathbb{Z}_{16} , hence it appears once in the array, while every other element appears twice.

Some necessary conditions regarding this class of arrays have been established in [41]:

Proposition 1.3.12. Suppose that there exists a ${}^\lambda H_t(m, n; h, k)$ and set $v = \frac{2nk}{\lambda} + t$. If either v is odd or v and t are even, then λ has to be a divisor of nk .

Proposition 1.3.13. If $\lambda \equiv 2 \pmod{4}$, $\frac{2nk}{\lambda} + t \equiv 2 \pmod{4}$ and t is odd, then a ${}^\lambda H_t(m, n; h, k)$ cannot exist.

Some further necessary conditions have been obtained in [41] by generalizing the argument of Proposition 1.3.5:

Proposition 1.3.14. Suppose that there exists an integer ${}^\lambda H_t(m, n; h, k)$ with λ odd.

- (1) If t divides $\frac{nk}{\lambda}$, then $\frac{nk}{\lambda} \equiv 0 \pmod{4}$ or $\frac{nk}{\lambda} \equiv -t \equiv \pm 1 \pmod{4}$.
- (2) If $t = \frac{2nk}{\lambda}$, then h and k must be even.
- (3) If $t \neq \frac{2nk}{\lambda}$ does not divide $\frac{nk}{\lambda}$, then $\frac{2nk}{\lambda} + t \equiv 0 \pmod{8}$.

For further existence results regarding this type of arrays, see Section 6.2 of [79] and its references therein.

1.3.5 Simple and globally simple Heffter arrays

Heffter arrays were introduced in order to construct orthogonal cyclic cycle decompositions of complete graphs using difference families, and their embedding on orientable surfaces. As we will see in Section 2.4, in order to construct the cycle decompositions one has to find for each row and each column of the array an ordering of its elements satisfying the following property:

Definition 1.3.15. Given a k -subset T of a group $(G, +)$, let $\omega = (t_1, \dots, t_k)$ be an ordering of its elements, and let $s_i = \sum_1^i t_j$ for each $i \in [1, k]$. Then ω is *simple* if the partial sums s_1, \dots, s_k are pairwise distinct and different from zero, except possibly for s_k . A Heffter array is said to be *simple* if every row and every column admits a simple ordering.

Example 1.3.16. Consider the following $H(3, 8)$ over \mathbb{Z}_{49} :

-13	-1	14	6	10	-8	-11	3
4	-20	-2	17	5	-16	-7	19
9	21	-12	-23	-15	24	18	-22

It can be readily verified that the following are simple orderings for the rows of the array:

$$\begin{aligned}\omega_1 &= (-13, -11, 14, 6, 3, 10, -8, -1), \\ \omega_2 &= (4, -7, 17, 19, 5, -16, -2, -20), \\ \omega_3 &= (9, 18, -23, -22, -15, 24, -12, 21).\end{aligned}$$

For instance, the partial sums (s_1, \dots, s_8) of ω_1 are:

$$(-13, -24, -10, -4, -1, 9, 1, 0),$$

that are distinct and non-zero, except for the last term. It is easy to see that a simple ordering of the columns is provided by reading the elements from top to bottom, hence the array is simple.

Clearly, providing a simple ordering could be a long and tedious task for large arrays containing many element in each row or column, and it is not known if it is granted to exist (see Section 2.4 for further details). It would then be desirable to find arrays where the simple ordering is already known from the construction and in particular it is the most “natural” one: the *natural ordering* of a row of an array is the one given by reading its elements from left to right; similarly, the *natural ordering* of a column is given by reading its elements from top to bottom. For this reason, in [39] the following special class of Heffter arrays has been introduced:

Definition 1.3.17. A Heffter array is *globally simple* if for every row and every column the natural ordering is simple.

Example 1.3.18. A globally simple $H(8; 7)$ over the group \mathbb{Z}_{113} , see Example

1.7 of [39]:

4	35	-45	46		20	-36	-24
48	-5	23	-47	-18		37	-38
-32	-10	-6	31	-41	42		16
33	-34	44	3	11	-43	-14	
	15	-28	-22	7	27	-53	54
-13		29	-30	56	1	12	-55
-49	50		19	-40	-21	2	39
9	-51	-17		25	-26	52	8

For instance, the partial sums of the first row read under the natural ordering are:

$$(4, 39, -6, 40, 60, 24, 0),$$

which are pairwise distinct in \mathbb{Z}_{113} .

As remarked in Section 1.2, the construction methods of Heffter arrays having many filled cells in their rows and columns are mainly focused on the addition of zero-sum blocks, hence most of the already known Heffter arrays are not globally simple. In general globally simple arrays are quite difficult to construct, and there are many open cases left regarding their existence.

In [39] the authors showed that:

Theorem 1.3.19. *Let $3 \leq k \leq 10$. Then there exists an integer globally simple $H(n; k)$ if and only if $n \geq k$ and $nk \equiv 0, 3 \pmod{4}$.*

Later, in [27] it has been proven that:

Theorem 1.3.20. *Let $n \geq k \geq 3$. There exists an integer globally simple $H(n; k)$ when:*

- (1) $k \equiv 0 \pmod{4}$;
- (2) $n \equiv 1 \pmod{4}$ and $k \equiv 3 \pmod{4}$;
- (3) $n \equiv 0 \pmod{4}$, $k \equiv 3 \pmod{4}$ and $n \gg k$.

The following result, proven in [47], shows the existence of globally simple Heffter arrays in the rectangular case where the number of rows is 3 or 5.

Theorem 1.3.21. *There exists a globally simple $H(3, n)$ for every $n \geq 3$ and a globally simple $H(5, n)$ for all $3 \leq n \leq 100$.*

In a recent paper [19] the author studies Heffter arrays over finite fields, in which one finds the classic notion of Heffter arrays whenever the group has prime order. In many of these cases the constructed arrays are globally simple:

Theorem 1.3.22. *If $q = 2mn + 1$ is a prime number, with m, n odd and coprime, then there exists a globally simple $H(m, n)$.*

Globally simple arrays are also interesting from the point of view of their application to graph embeddings: indeed, as discussed in Section 3.4 a crucial role in the construction of the embedding is played by the row and column orderings, that should be chosen to be simple. In the case of globally simple arrays it can be shown that this is equivalent to finding the solution of a tour problem over the filled cells of the array, see Section 3.5 for more details.

1.3.6 Heffter arrays over finite fields

In this section we present a generalization of the concept of a Heffter array recently introduced in [19].

Definition 1.3.23. For a prime power $q = 2nk + 1$, a *Heffter array over \mathbb{F}_q* denoted by $H(m, n; h, k)$ is an $m \times n$ partially filled array with elements in \mathbb{F}_q such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b₅) the entries of A form a half-set of \mathbb{F}_q ;
- (c) every row and column is zero-sum in \mathbb{F}_q .

As for classical Heffter arrays we write $H(n; k)$ if $n = m$, and $H(m, n)$ for the totally filled case. Clearly if q is a prime number a Heffter array over \mathbb{F}_q is a classical Heffter array over \mathbb{Z}_q .

In [19] the author focuses on totally filled Heffter arrays over finite fields that moreover satisfy the restrictive property of being *rank-one* in the standard sense of linear algebra. In particular, it is shown that this requirement is equivalent to finding a half-set L of \mathbb{F}_q admitting two subsets $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ such that $L = \{x \cdot y : x \in X, y \in Y\}$. Then, the array $A_{m,n}$ whose (i, j) -th cell is $x_i \cdot y_j$ is a rank-one Heffter array over \mathbb{F}_q .

Example 1.3.24. Here we construct a rank one Heffter array over \mathbb{F}_{25} , see Example 1.3 of [19]. Let g be a root of the primitive polynomial $x^2 + x + 2$ over \mathbb{F}_5 . Let L be the following half-set of \mathbb{F}_{25} :

$$L = \{1, 3, g, g + 1, g + 4, 2g + 3, 3g, 3g + 1, 3g + 3, 3g + 4, 4g + 2, 4g + 3\}.$$

Define the following subsets $X = \{1, g, g + 4, 3g\}$ and $Y = \{1, 3g + 1, 2g + 3\}$; it can be seen that $L = \{x \cdot y : x \in X, y \in Y\}$. A rank-one Heffter array over \mathbb{F}_{25} is:

1	g	$g + 4$	$3g$
$3g + 1$	$3g + 4$	3	$4g + 2$
$2g + 3$	$g + 1$	$4g + 3$	$3g + 3$

In [19] it is shown that:

Theorem 1.3.25. *Let m, n be two integers such that $q = 2mn + 1$ is a prime power such that there exist two distinct odd primes p, p' dividing m and n , respectively. Then there exists a rank-one $H(m, n)$ over \mathbb{F}_q .*

In Section 3.6 we study the embeddings that can be constructed from class of arrays in the particular case of q being a prime number, following the results that we obtained in [36].

1.3.7 Archdeacon arrays

The following generalization of Heffter arrays has been introduced in [43].

Definition 1.3.26. An *Archdeacon array* A over an abelian group $(G, +)$ is an $m \times n$ partially filled array with elements in G such that:

- (b₄) if $g \in \mathcal{E}(A)$, then $-g \notin \mathcal{E}(A)$;

(c) every row and column is zero-sum in G .

Note that in this definition there is no condition on the number of filled cells in the array and the entries may not be necessarily a half-set of G ; for simplicity, here we assume that an Archdeacon array may not contain empty rows or columns. The classical notion of (relative) Heffter array can be recovered if $G = \mathbb{Z}_v$, $\pm\mathcal{E}(A) = G \setminus J$, where J is a subgroup of \mathbb{Z}_v , and

(a) every row has h filled cells and every column has k filled cells.

These arrays can then be used to construct graph decompositions that are invariant under the action of the group G , see also Section 2.3.

Example 1.3.27. An Archdeacon array over $\mathbb{Z}_{41} \oplus \mathbb{Z}_d$ for every $d \geq 3$:

$(-1, 1)$	$(0, -1)$	$(-14, 0)$	$(9, 0)$	$(6, 0)$
$(2, -1)$	$(-7, 1)$		$(-5, 0)$	$(10, 0)$
$(17, 0)$	$(12, 0)$	$(-16, 0)$		$(-13, 0)$
$(-18, 0)$	$(15, 0)$	$(11, 0)$	$(-8, 0)$	
	$(-20, 0)$	$(19, 0)$	$(4, 0)$	$(-3, 0)$

Since the definition of an Archdeacon array is quite general the known result focus on the property of being globally simple. Given two $m \times n$ partially filled arrays $A = [a_{i,j}]$ and $B = [b_{i,j}]$ respectively defined on two abelian groups G_1 and G_2 , the array $E = A \oplus B = [e_{i,j}]$ is the $m \times n$ partially filled array having entries in $G_1 \oplus G_2$ so defined:

$$e_{i,j} := \begin{cases} (a_{i,j}, b_{i,j}) & \text{if } (i, j) \in \text{skel}(A) \cap \text{skel}(B), \\ (a_{i,j}, 0_{G_1}) & \text{if } (i, j) \in \text{skel}(A) \setminus \text{skel}(B), \\ (0_{G_2}, b_{i,j}) & \text{if } (i, j) \in \text{skel}(B) \setminus \text{skel}(A). \end{cases}$$

In [43] it has been shown that:

Theorem 1.3.28. *Let A, B be $m \times n$ globally simple Archdeacon arrays over two abelian groups G_1 and G_2 respectively, such that:*

- (1) *for each row there exists at least one cell that is filled in both A and B ;*
- (2) *for each column there exists at least one cell that is filled in both A and B ;*
- (3) *each row and each column of A and B is zero-sum in their respective groups.*

Then, the array $A \oplus B$ is a globally simple Archdeacon array over $G_1 \oplus G_2$.

In Chapter 6 we consider a generalization of this concept that merges the notions of λ -fold Heffter arrays, non-zero sum Heffter arrays and Archdeacon arrays.

Chapter 2

Graph decompositions

In this chapter we show the connection standing between Heffter arrays and decompositions of a complete (multipartite) graph whose order is determined by the parameters of the Heffter array. Firstly, in Section 2.1 we define some classical terms of Graph Theory, while in Section 2.2 we discuss the concept of graph decompositions. In Section 2.3 we describe difference families, that are combinatorial structures that can be employed to easily construct graph decompositions admitting a large group of automorphisms. Finally we show the connections of Heffter arrays with difference families, and with graph decompositions.

2.1 Graphs

In this section we recall some elementary definitions about graphs; good references for this topic are [45, 95]. We remark that in this thesis we discuss *simple* graphs.

Definition 2.1.1. A *graph* Γ is a pair (V, E) where V is a set of elements, called *vertices*, and E is a set of unordered pairs of distinct vertices of Γ , called *edges*. Two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ are *isomorphic*, denoted as $\Gamma \cong \Gamma'$, if there exists a bijection, called *graph isomorphism*, $f : V \rightarrow V'$ such that for every $u, v \in V$ it holds $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in E'$. A graph isomorphism from V to itself is called an *automorphism* of Γ .

Given a graph Γ we also denote by $V(\Gamma)$ its set of vertices and by $E(\Gamma)$ its set of edges. The automorphisms of a graph can be intuitively seen as its symmetries; for this reason we say that a graph having a rich group of automorphisms is *highly symmetric*.

The number of vertices of a graph Γ is called *order of* Γ , while the number of its edges is called *size of* Γ . A *subgraph* Γ' of a graph Γ is a graph such that $V(\Gamma') \subset V(\Gamma)$ and $E(\Gamma') \subset E(\Gamma)$. Two vertices u, v are *adjacent* if $e = \{u, v\}$ is an edge of Γ , and in this case u and v are said to be the *endvertices* of e . The *degree* of a vertex v is the number of vertices that are adjacent to v , and a graph is said to be *regular* of degree d (or *d-regular*) if every vertex has degree d . A *walk* is a sequence of vertices (v_1, v_2, \dots, v_n) with v_i adjacent to v_{i+1} for every $i \in \{1, 2, \dots, n-1\}$; a walk is *closed* if $v_1 = v_n$ and it is *open* otherwise.

A graph Γ is said to be *bipartite* if its vertex set can be partitioned into two disjoint sets A and B such that every edge of Γ has one endvertex that belongs

to A , and the other one belongs to B . A simple and very well known result shows that a graph is bipartite if and only if it does not contain any cycle of odd length.

We now define some significant classes of graphs that will be used in the whole thesis. The *path graph* P_n of order n is the graph having $\{v_1, \dots, v_n\}$ as set of vertices, and $\{\{v_i, v_{i+1}\} : i \in \{1, \dots, n-1\}\}$ as set of edges. Note that a path is an open walk over distinct vertices. The *cycle graph* C_n of order n is the graph having $\{v_1, \dots, v_n\}$ as set of vertices, and $\{\{v_i, v_{i+1}\} : i \in \{1, \dots, n-1\}\} \cup \{v_n, v_1\}$ as set of edges. Equivalently, a cycle graph is a closed walk over distinct vertices.

The *complete graph* K_n of order n is the graph having n vertices, such that every pair of distinct vertices is adjacent. Let p, m be any pair of positive integers, and let P_1, \dots, P_m be m mutually disjoint sets of vertices, each one having size p . Then, the *complete multipartite graph* $K_{m \times p}$ is the graph having $\cup_{i=1}^m P_i$ as vertices, and such that two vertices are adjacent if and only if they lie in different sets. The sets P_i are also called the *parts* of $K_{m \times p}$. In the particular case $m = 2$ the graph is bipartite and it is denoted as $K_{p,p}$ instead of $K_{2 \times p}$.

2.1.1 Cayley graphs

The description of some highly symmetric graphs can be simplified by considering graphs whose vertices are elements of a suitable group. In this section we denote by $(G, +)$ a group having neutral element 0 , where for every $x \in G \setminus \{0\}$ by $-x$ we denote its inverse.

Definition 2.1.2. Let $\Omega \subseteq G \setminus \{0\}$, such that $\omega \in \Omega$ implies $-\omega \in \Omega$. The *Cayley graph on G with connection set Ω* , denoted as $\text{Cay}[G : \Omega]$, is the graph (V, E) such that:

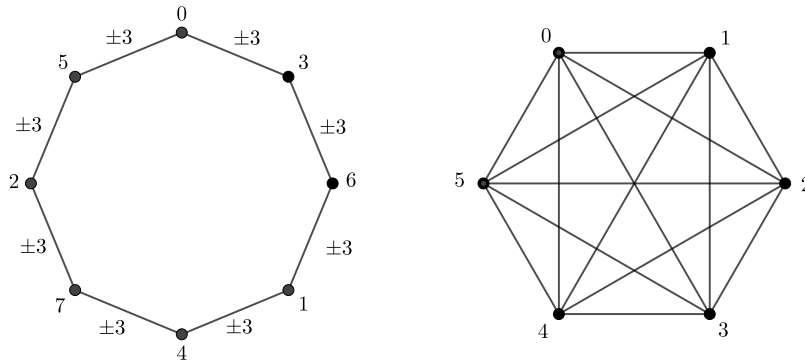
$$V = G, \quad \{u, v\} \in E \Leftrightarrow u - v \in \Omega.$$

It then easily follows that $|\Omega|$ is even if it does not contain involutions, and that $\text{Cay}[G : \Omega]$ is $|\Omega|$ -regular.

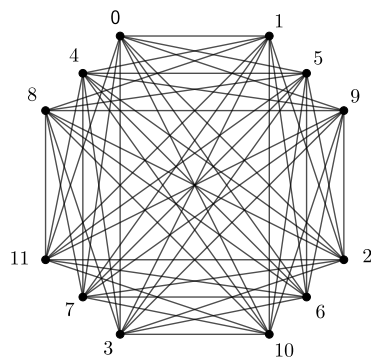
Remark 2.1.3. Some classes of graphs previously defined are particular Cayley graphs, namely:

- (1) the cycle graph C_n is isomorphic to $\text{Cay}[\mathbb{Z}_n : \{\pm a\}]$, where a is a generator of \mathbb{Z}_n ;
- (2) the complete graph K_n is isomorphic to $\text{Cay}[G : G \setminus \{0\}]$, for every group G of order n ;
- (3) the complete multipartite graph $K_{m \times p}$ is isomorphic to $\text{Cay}[G : G \setminus J]$, where G is any group of order mp and J is a subgroup of G having order p .

Example 2.1.4. In what follows we show an example of the cycle graph C_8 , seen as the Cayley graph $\text{Cay}[\mathbb{Z}_8 : \{\pm 3\}]$, and of the complete graph K_6 , seen as $\text{Cay}[\mathbb{Z}_6 : \mathbb{Z}_6 \setminus \{0\}]$ (to avoid confusion in the picture we have shown the differences only for the cycle graph):



In the following figure we show the complete multipartite graph $K_{4 \times 3}$ on 12 vertices, having 4 parts of 3 vertices, seen as the Cayley graph $\text{Cay}[\mathbb{Z}_{12} : \mathbb{Z}_{12} \setminus \{0, 4, 8\}]$:



Assume now that $\Gamma = (V, E)$ is a subgraph of a Cayley graph $\text{Cay}[G : \Omega]$ for some group $(G, +)$ and some connection set Ω . We can then label the vertices of Γ with the elements of the group G and, in order to simplify the exposition, we identify the vertices of Γ as elements of G . Moreover, one can define a natural action of G on the vertices of $\text{Cay}[G : \Omega]$: for every $g \in G$ the *action* of g is the map $\phi_g : x \mapsto x + g$. For simplicity we directly denote by $\Gamma + g$ the image of Γ through ϕ_g . It can be easily seen that ϕ_g is an automorphism of $\text{Cay}[G : \Omega]$, while the graph Γ may be distinct from the graph $\Gamma + g$. In general the set $\{\Gamma + g : g \in G\}$ is said to be the *orbit* of Γ . We develop this concept in the following sections, when dealing with the topics of graph decompositions and difference family.

2.2 Designs and graph decompositions

In this section we consider two important topics of Discrete Mathematics. The first one, called 2-designs, is a class of combinatorial structures whose introduc-

tion can be traced to the 19th century by Plücker [82] during his studies on algebraic curves. Some examples of these structures are Steiner triple systems, that we briefly discussed in Section 1.1. 2-designs have been studied due to their various connections with topics of finite geometry and the design of experiments.

Then we consider a generalization of the concept of a 2-design, called graph decompositions, and some of their related topics.

2.2.1 2-designs

In this subsection we consider a particular class of block designs that has been widely studied in the literature (see for instance [67]):

Definition 2.2.1. A $2-(v, k, \lambda)$ design is a pair $(\mathcal{V}, \mathcal{B})$, where \mathcal{V} is a set of v points and \mathcal{B} is a set of k -subsets of \mathcal{V} , called *blocks*, such that every pair of points is contained in exactly λ blocks. Two designs $(\mathcal{V}, \mathcal{B})$ and $(\mathcal{V}', \mathcal{B}')$ are *isomorphic* if there exists a bijection $f : \mathcal{V} \rightarrow \mathcal{V}'$, called *isomorphism*, such that for every subset B of \mathcal{V} it holds $B \in \mathcal{B}$ if and only if $f(B) \in \mathcal{B}'$. An *automorphism* is an isomorphism of a design with itself.

A $2-(v, k, 1)$ design is also called a *Steiner system*, denoted as $S(2, k, v)$. For $k = 3$ the design is called a *Steiner triple system of order v* , denoted as $\text{STS}(v)$.

One can easily deduce some necessary conditions on the parameters for the existence of a $2-(v, k, \lambda)$ design: clearly $k < v$ (the case $k = v$ leads to trivial block designs); the *replication number* r , that is the number of blocks containing a fixed point, is given by $\frac{\lambda(v-1)}{k-1}$, while the *number of blocks* b is $\frac{\lambda v(v-1)}{k(k-1)}$. Since both r and b have to be integers we have:

$$\lambda(v-1) \equiv 0 \pmod{k-1}, \quad \lambda v(v-1) \equiv 0 \pmod{k(k-1)}.$$

Finally, Fisher's inequality $b \geq v$ implies that $\lambda(v-1) \geq k(k-1)$. A set of parameters satisfying the trivial necessary conditions stated above is said to be *admissible*.

It has been proven in [55] that these conditions are also sufficient for $k = 3, 4, 5$, while in 1972 it has been proven by Wilson [97] that:

Theorem 2.2.2. *The necessary conditions for the existence of a $2-(v, k, \lambda)$ design are also sufficient for sufficiently large values of v .*

This result, while being very important for the solution of the existence problem, gives no bounds on the minimum value of v for which there exists a $2-(v, k, \lambda)$ design for fixed values of k and λ .

Among the various block designs that one can consider, the most noble and difficult to construct are those with $\lambda = 1$, i.e. the Steiner systems. As for classical designs, given a parameter k it is not easy to find a value v for which there exists an $S(2, k, v)$, except for the notable cases where k or $k-1$ is a prime power, $k > 2$ (in this case, the designs are given by projective and affine geometries over finite fields). Indeed, the existence of an $S(2, 15, v)$, where $k = 15$ is the smallest integer such that neither $k = p^n$ or $k = p^n + 1$ for some prime p , remained an open problem until 1972 when Wilson showed in [96] the existence of an $S(2, 15, 76231)$.

A complete existence results can be found for small values of k and in particular for $k = 3$. As already mentioned in Section 1.1 a complete existence result for these structures is due to Kirkman [61].

2.2.2 Graph decompositions

While some pioneering results can be framed in terms of graph decompositions, the subject took off in the second half of the twentieth century as a generalization of the concept of block designs, see for instance [11, 15]. Indeed, $2-(v, k, \lambda)$ block designs can be seen as a particular class of graph decompositions.

Definition 2.2.3. Given a graph K , a *decomposition* of K is a collection $\mathcal{B} = \{B_1, \dots, B_n\}$ of subgraphs (*blocks*) of K whose edges partition $E(K)$. If all blocks are isomorphic to a given graph Γ the decomposition is said to be a (K, Γ) -*design* or a Γ -*decomposition* of K .

Let K be a graph whose vertex set is an additive group G . A (K, Γ) -design \mathcal{B} is said to be G -*regular*, or *regular under G* , if $B + g \in \mathcal{B}$ for every $B \in \mathcal{B}$. A (K, Γ) -design is said to be *cyclic*, *abelian*, ... if it is regular under a group G having the respective property.

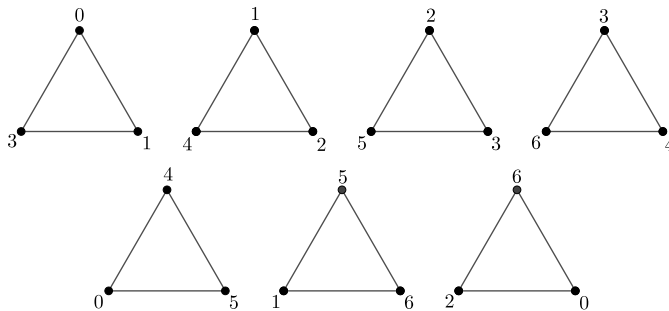
For a graph $K = (V, E)$ having $\Gamma = (V', E')$ as a subgraph, a simple counting argument shows the following necessary conditions for the existence of a (K, Γ) -design:

- (1) $|E'|$ is a divisor of $|E|$;
- (2) if K and Γ are regular of degree d and d' respectively, then d' is a divisor of d .

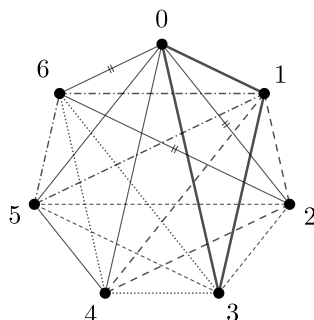
The most studied (K, Γ) -designs in the literature are those where $K = K_v$, and Γ is either a complete graph K_k or a cycle graph C_k for some positive integers v and k , with $v \geq k$.

It is not hard to see that the former case is equivalent to a $S(2, k, v)$ Steiner system $(\mathcal{V}, \mathcal{B})$ where the points of \mathcal{V} are identified with the vertices of K_v and the blocks correspond to the vertex set of the copies of K_k . The latter class of graph decompositions are called k -*cycle systems of order v* , see for instance [14]. We remark that in particular a 3-cycle system of order v is precisely an STS(v).

Example 2.2.4. An example of a Steiner system is the Fano plane given in Section 1.1. The blocks of the Fano plane can then be seen as the following 3-cycles:



These blocks form a (K_7, C_3) -design (or, equivalently, a 3-cycle system of order 7):



The existence of a $S(2, k, v)$ for a set of admissible parameters is in general an unsolved problem; indeed, the best known result in this direction is Theorem 2.2.2. This is not the case for k -cycle systems: in fact, a series of papers published over the course of more than 30 years showed constructively that the necessary conditions for the existence of a k -cycle system of order v are also sufficient:

Theorem 2.2.5. *There exists a k -cycle system of order v if and only if $k \leq v$, v is odd and $v(v-1) \equiv 0 \pmod{2k}$.*

The first construction of k -cycle systems, apart from the case $k = 3$ (that corresponds to a STS, already discussed in the previous section), is the one given by Walecki, and described in [66], where $k = v$. Several other results were then established throughout the years, see for instance Kotzig [63] and Rosa [85, 86], who constructed decompositions that were moreover cyclic. A major breakthrough was then reached in 1978, when it was showed in [9] that if for an even k there exists a k -cycle system for every admissible order $v \in [k, 3k]$, then there exists a k -cycle system for every admissible order. In [59, 84] an analogous result was proven for every odd k . The proof of Theorem 2.2.5 was then completed when a k -cycle system of order $v \in [k, 3k]$ was constructed in [3] for every odd k (see also [17] for an alternative proof) and in [88, 87] for every even k .

In [16, 20], the authors studied cyclic k -cycle systems, showing that:

Theorem 2.2.6. *There exists a cyclic k -cycle system of order v for every $v \equiv 1 \pmod{2k}$.*

We remark that it has been shown in [18, 98] that there are infinitely many pairs (v, k) satisfying the trivial necessary conditions for the existence of a cyclic k -cycle system of order v , for which such a decomposition does not exist. We highlight the fact that as a consequence of a classical result that we show in Section 2.3, namely Corollary 2.3.6, the previous statement implies the following:

Corollary 2.2.7. *There exists a $D(nk; k)$ Heffter system for every $n \geq k \geq 3$.*

We recall the definition of orthogonal decompositions, which will be useful to explain the connection between Heffter arrays and this topic:

Definition 2.2.8. Two graph decompositions of a graph K are said to be *orthogonal* if any block of the first shares at most one edge with any block of the second.

2.3 Difference families

In this section we introduce a tool, called *difference family*, that has been used throughout the literature in order to construct regular graph decompositions and in particular to prove Theorem 2.2.5. We moreover discuss two problems introduced by Heffter, that link Steiner systems with difference families, and direct the attention towards Heffter arrays and their relation with cycle systems, as discussed in Section 2.4.

The tool of difference families was introduced by Anstice in 1852 [4] (see also [1]) as a method to construct various combinatorial structures, like cyclic Steiner systems and Kirkman systems. Difference families can also be used to construct finite projective planes, as done by Kirkman in 1857 [62], Hadamard matrices [94], cycle systems and graph decompositions. Firstly, we give the original definition of a difference family over blocks, and then we consider its generalization over graphs introduced in [22]. We show in Theorem 2.3.5 and in Corollary 2.3.6 the connection between difference families and graph decompositions.

2.3.1 Classical Definition

Let $B = \{b_1, \dots, b_k\}$ be a subset of an additive group G . Its *list of differences* is the multiset $\Delta(B) = \{b_i - b_j : i, j \in [1, k], i \neq j\}$. If $\mathcal{B} = \{B_1, \dots, B_\ell\}$ is a collection of sets, its *list of differences* is the multiset given by the union of the lists of differences of the B_i s, namely $\Delta(\mathcal{B}) = \bigcup \Delta(B_i)$.

Definition 2.3.1. Let \mathcal{B} be a collection of subsets of a group $(G, +)$, and let J be a subgroup of G . \mathcal{B} is a $(G, J, k, 1)$ *difference family over G relative to J* (in short, RDF) if its list of differences covers each element of $G \setminus J$ exactly once, that is $\Delta(\mathcal{B}) = G \setminus J$.

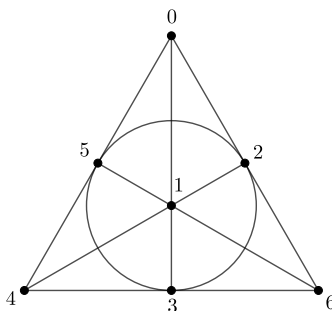
As for (K, Γ) -designs, a difference family is said to be *cyclic, abelian, ...* if G has the respective property. A $(G, \{0\}, k, 1)$ RDF is denoted as $(G, k, 1)$, and it is simply called *difference family* (in short, DF). A $(\mathbb{Z}_v, J, k, 1)$ difference family is usually denoted in the literature as a $(v, J, k, 1)$ RDF. Moreover, a difference family that consists of a single block is called a *difference set*.

As for 2-designs there are some necessary conditions that have to be satisfied by the parameters for the existence of a $(G, J, k, 1)$ difference family, namely:

$$|G| \equiv |J| \pmod{k(k-1)}.$$

Finally, the *development* of a collection of sets \mathcal{B} under the action of G is the multiset $dev(\mathcal{B}) = \{B + g \mid B \in \mathcal{B}, g \in G\}$.

Example 2.3.2. Let $B = \{0, 1, 3\}$ be a subset of \mathbb{Z}_7 . It is immediate to see that $\Delta(B) = \pm\{1, 2, 3\}$, hence $\mathcal{B} = \{B\}$ is a $(7, 3, 1)$ difference set whose development is the Fano plane, as shown in the following figure:



In other words, $dev(\mathcal{B})$ is a cyclic STS(7).

Difference families have been used in many constructions of Steiner systems: for instance, these structures were implicitly employed by Skolem in [92, 93] to construct cyclic STSs.

2.3.2 Difference family over graphs

The concept of a difference family can be generalized to the context of graphs, see [22]. Given a graph $\Gamma = (V, E)$ whose vertices are elements of a group $(G, +)$, we say that its *list of differences* is the following multiset:

$$\Delta(\Gamma) = \{u - v, v - u \mid \{u, v\} \in E\}.$$

We remark that if $x \in G$ is contained in the list of differences, $-x$ is in the list of differences as well. For a collection \mathcal{F} of graphs Γ_i , its *list of differences* is the multiset given by the union of the lists of differences of the Γ_i s, namely $\Delta(\mathcal{F}) = \bigcup \Delta(\Gamma_i)$.

Definition 2.3.3. Let \mathcal{F} be a collection of graphs isomorphic to a given graph Γ with vertices in a group $(G, +)$, and let J be a subgroup of G . \mathcal{F} is called a $(G, J, \Gamma, 1)$ *difference family over G relative to J* (in short, RDF) if its list of differences covers each element of $G \setminus J$ exactly once, that is $\Delta(\mathcal{F}) = G \setminus J$.

Also in this case, a difference family is said to be *cyclic, abelian, ...* if G is cyclic, abelian, As for classical relative difference families, a $(\mathbb{Z}_v, J, \Gamma, 1)$ relative difference family is denoted as $(v, J, \Gamma, 1)$, and if \mathcal{F} consists of a single block Γ it is also called *difference graph*. In the case $J = \{0\}$, the difference family is more simply denoted as $(G, \Gamma, 1)$.

Remark 2.3.4. A $(G, J, K_k, 1)$ RDF is nothing but a classical $(G, J, k, 1)$ RDF.

The effectiveness of this tool is shown in the following well-known result [22].

Theorem 2.3.5. *Let J be a subgroup of order n of an additive group G of order mn . If there exists a $(G, J, \Gamma, 1)$ relative difference family \mathcal{F} , then $dev(\mathcal{F})$ is a $(K_{m \times n}, \Gamma)$ -design regular under G .*

Conversely, every $(K_{m \times n}, \Gamma)$ -design regular under G is the development of a $(G, J, \Gamma, 1)$ relative difference family.

Proof. To prove the first part of the statement, let $\mathcal{F} = \{\Gamma_1, \dots, \Gamma_n\}$ be a $(G, J, \Gamma, 1)$ relative difference family, where Γ_i is isomorphic to Γ for each $i \in \{1, \dots, n\}$. Let $x, y \in G$ be two elements that do not belong to the same coset of J . We show that there exists an index i and an element $t \in G \setminus J$ such that $\Gamma_i + t$ contains the edge $\{x, y\}$. Let $\delta := y - x$, and let i be the index such that $\delta \in \Delta(\Gamma_i)$ (that exists since x and y belong to different cosets of J , and it is unique by the definition of a difference family): since $x \neq y$, we have $\delta \neq 0$. Denote by $\{u, v\}$ the edge of Γ_i having difference $v - u = \delta$, and let $g := u - x$. It can immediately be seen that:

$$v - g = u + \delta - g = x + \delta = y,$$

hence the edge $\{x, y\}$ belongs to the graph $\Gamma_i - g$.

To prove the second part of the statement, let \mathcal{B} be a $(K_{m \times n}, \Gamma)$ -design regular under a group G . By definition for every block $B \in \mathcal{B}$ and $g \in G$ it holds $B + g \in \mathcal{B}$, hence \mathcal{B} can be partitioned into orbits. Let now \mathcal{F} be a collection of blocks such that they pairwise belong to different orbits: it can be easily seen that these are pairwise edge-disjoint graphs, and that the set of differences of \mathcal{F} contains every element of $G \setminus J$ exactly once. We can then conclude that \mathcal{F} is a $(G, J, \Gamma, 1)$ difference family. \square

We highlight the following particular case:

Corollary 2.3.6. *Let G be an additive group of order m . If there exists a $(G, \Gamma, 1)$ difference family \mathcal{F} , then $dev(\mathcal{F})$ is a (K_m, Γ) -design regular under G .*

By generalizing the previous result, we can state the following remark that will be of particular interest when dealing with certain classes of Heffter arrays.

Remark 2.3.7. If \mathcal{F} is a collection of isomorphic copies of a graph Γ with vertices in a group G , then $dev(\mathcal{F})$ is a Γ -decomposition of $\text{Cay}[G : \Delta(\mathcal{F})]$.

We furthermore remark that to construct a decomposition of a Cayley graph $[G : \Omega]$ the important condition is that for a set \mathcal{B} of graphs it holds $\Delta(\mathcal{B}) = \Omega$, and the definition of a difference family can be relaxed to include collections of non-isomorphic graphs.

2.3.3 Heffter's difference problems

As discussed in the previous sections, one of the first classes of block designs whose existence has been extensively studied is the one of Steiner triple systems. While their existence result was completely established by Kirkman in 1847, the constructions that he provided were rather complicated and not easy to follow. In particular, they lacked many desirable features, like symmetries, in order to get simpler representations of these designs. For this reason, researchers tried to construct Steiner triple systems having a large automorphism group: the first and most obvious case to consider was the one of *cyclic* Steiner triple systems. Indeed, at the end of 1800s Heffter introduced in [57, 58] two difference problems, that were later named after him, whose solutions allow the construction of cyclic Steiner triple systems. Both difference problems were solved by Pelsesohn in [81], and later by Skolem in [92, 93] by using a different approach that involves the so-called Skolem sequences.

The first Heffter's difference problem asks for a partition of the set $\{1, \dots, 3n\}$ into triples $\{a, b, c\}$ such that $a + b = c$ or $a + b + c \equiv 0 \pmod{6n + 1}$. Hence:

Proposition 2.3.8. *Let $\{\{a_i, b_i, c_i\} \mid i \in [1, n]\}$ be a solution to Heffter's first difference problem. The set $\{\{0, a_i, a_i + b_i\} \mid i \in [1, n]\}$ is a $(6n + 1, 3, 1)$ difference family whose elements are the base blocks of an STS($6n + 1$).*

Meanwhile, the second Heffter's difference problem asks for a partition of the set $\{1, \dots, 3n + 1\} \setminus \{2n + 1\}$ into triples $\{a, b, c\}$ such that $a + b = c$ or $a + b + c \equiv 0 \pmod{6n + 3}$.

Proposition 2.3.9. *Let $\{\{a_i, b_i, c_i\} \mid i \in [1, n]\}$ be a solution to Heffter's second difference problem. The set $\{\{0, a_i, a_i + b_i\} \mid i \in [1, n]\}$ is a $(6n + 3, 3, 3, 1)$ difference family, and together with $\{0, 2n + 1, 4n + 2\}$ it forms the base blocks of an STS($6n + 3$).*

From the name and the statement of these problems one can easily deduce their relevance in this thesis: indeed, a Heffter system $D(nk, k)$ with $k = 3$ is a solution to the first Heffter's difference problem.

2.4 Connection with Heffter arrays

In this section we discuss the connection between difference families and Heffter arrays. Firstly, we show that from a Heffter system whose blocks admits a simple ordering it is possible to construct a cyclic difference family whose base blocks are cycles. As a consequence we obtain that it is possible to construct orthogonal cyclic cycle systems from a simple Heffter array (see Section 1.3.5).

Let $R = \{r_1, \dots, r_h\}$ be the set of elements of a row of a Heffter array $H(m, n; h, k)$ and let ω_R be an ordering of R . Denote by s_1, s_2, \dots, s_h the partial sums of ω_R . Note that since R is zero-sum, ω_R is simple if and only if it does not contain any proper subsequence that sums to 0. Construct the following walk:

$$W(\omega_R) = (0, s_1, s_2, \dots, s_{h-1}, s_h). \quad (2.4.1)$$

We prove the following simple lemma:

Lemma 2.4.1. *The walk $W(\omega_R)$ of Equation (2.4.1) is*

- (1) *closed with no repeated edges;*
- (2) *a cycle if and only if ω_R is simple.*

Proof. (1) Since R is zero-sum, it follows that the last element of the vertex sequence $s_h = \sum_1^h r_i$ is the vertex 0, hence the walk $W(\omega_R)$ is closed. Assume now that two edges $\{s_i, s_{i+1}\}$ and $\{s_j, s_{j+1}\}$ coincide: this would imply that $s_{i+1} - s_i \in \{\pm(s_{j+1} - s_j)\}$. However, it would follow that

$$s_{i+1} - s_i = r_{i+1} \in \{\pm r_{j+1}\} = \{\pm(s_{j+1} - s_j)\}.$$

By the property (b) of a Heffter array, exactly one between r_{i+1} and $-r_{i+1}$ may appear (precisely once) in the array, that is a contradiction.

(2) By definition of simplicity the s_i s are pairwise distinct and different from 0 for $i \neq h$, thus $W(\omega_R)$ is a cycle; the converse immediately follows. \square

We can then state the main connection between graph decompositions and Heffter arrays.

Proposition 2.4.2. *Let A be an $H(m, n; h, k)$ Heffter array such that for every row and for every column there exists a simple ordering. Then:*

- (1) *there exists a $(\mathbb{Z}_{2nk+1}, C_h, 1)$ difference family, hence a cyclic h -cycle system of order $2nk + 1$;*
- (2) *there exists a $(\mathbb{Z}_{2nk+1}, C_k, 1)$ difference family, hence a cyclic k -cycle system of order $2nk + 1$.*

Moreover, the two cycle systems are orthogonal.

Proof. To prove (1) (respectively (2)) it is sufficient to apply Lemma 2.4.1 to every row (respectively column) of the Heffter array A , considering its simple ordering. For every row $R_i = \{r_1, \dots, r_h\}$, with $i \in [1, m]$, let $B_{R_i} = W(\omega_{R_i})$ be the cycle obtained from R_i using its simple ordering ω_{R_i} . Then, $\Delta(B_{R_i}) = \pm R_i$, hence $\Delta(\bigcup B_{R_i}) = \mathbb{Z}_{2nk+1} \setminus \{0\}$ and $\bigcup B_{R_i}$ is a $(2nk + 1, C_h, 1)$ difference family. The same reasoning can be applied to every column of A .

Let \mathcal{D}_R and \mathcal{D}_C respectively be the two cycle systems of (1) and of (2), and let $B \in \mathcal{D}_R$ and $B' \in \mathcal{D}_C$; by construction, $\Delta(B) = \pm R_i$ for some row R_i , and $\Delta(B') = \pm C_j$ for some column C_j . It is trivial to observe that a necessary condition for two edges to coincide is that they have the same difference; B and B' might share an edge having difference $\pm x$ only whenever $x \in R_i \cap C_j$. Since a row and a column can meet in at most one cell, the statement follows. \square

Some easy considerations derived from the properties of a Heffter array show that an $H(m, n; h, k)$ is always simple for $h, k \leq 5$. More in general the study of the simplicity of a subset of a finite group has been a subject of research before the introduction of Heffter arrays. Indeed, Alspach proposed the following conjecture, studied in [10]:

Conjecture 2.4.3. *For every $T \subseteq \mathbb{Z}_v \setminus \{0\}$ such that $\sum_{t \in T} t \neq 0$ there exists an ordering of the elements of T such that the partial sums are all distinct and non-zero.*

In view of the application to Heffter arrays, in [38] the authors proposed the following conjecture, that is related to the previous one:

Conjecture 2.4.4. *Let $(G, +)$ be an abelian group. Let T be a zero-sum finite subset of $G \setminus \{0\}$ with no zero-sum 2-subset. Then T admits a simple ordering.*

Clearly the validity of Conjecture 2.4.4 for cyclic groups would imply that every Heffter array is simple. It has been proven in [38] that Conjecture 2.4.4 holds for every abelian group if $|T| \leq 10$ and for every subset T if the order of the abelian group G is smaller or equal than 27.

Remark 2.4.5. Since a Heffter array over \mathbb{Z}_v relative to the subgroup J of order t (see Definition 1.3.3) is filled with elements in $\mathbb{Z}_v \setminus J$, it follows that if such an array is simple it is possible to construct cyclic cycle systems of a complete multipartite graph. In particular from a simple $H_t(m, n; h, k)$, there exist a $(K_{v/t \times t}, C_h)$ -design and a $(K_{v/t \times t}, C_k)$ -design, and the two decompositions are orthogonal.

Remark 2.4.6. Assume that there exists a simple array satisfying Conditions (a) and (c) of Definition 1.1.7, and the following condition:

(b_{Cay}) the set $\{\pm x : x \in A\}$ contains each element of $G \setminus S$ exactly once,

where S is a subset of a group $(G, +)$, such that $-S = S$, $0 \in S$. Then, it is possible to construct G -regular $(\text{Cay}[G : G \setminus S], C_h)$ and $(\text{Cay}[G : G \setminus S], C_k)$ -designs, that are moreover orthogonal.

Chapter 3

Biembeddings

In this chapter we show an interesting connection between a classical concept of Topological Graph Theory and Heffter arrays; indeed, this relation is one of the reasons for which Heffter arrays were introduced in the first place by Archdeacon in [6]. In particular, we define the problem of the embedding of a graph on a surface and its face coloring. We show that Heffter arrays can be used to construct 2-face colorable embeddings of the complete (multipartite) graph. Moreover, the construction of these embeddings depends on the concept of orderings introduced in Section 2.4, and the natural orderings play an interesting role in the definition of a tour problem over the filled cells of the Heffter array, proposed in [33]. In Section 3.1 we introduce some notions of Topological Graph Theory, that give the basic definitions regarding the embedding of graph over surfaces. In Section 3.2 we further develop this theory thanks to the concept of current graphs. In Section 3.3 we introduce the problem of the coloring of the faces of an embedding. In Section 3.4 we discuss the relation between Heffter arrays and biembeddings, while in Section 3.5 we study the aforementioned tour problem whose solution allows the construction of a biembedding, showing some results obtained in [68]. Finally in Section 3.6 we study the automorphism group of the biembedding that can be constructed from a Heffter array, contained in a joint work with Costa in [36].

3.1 Graph embeddings

In this section, following Chapter 3 of [54], we introduce the classical definition of a graph embedding over a (closed) surface, showing some tools that have been developed throughout the years to efficiently represent these embeddings. Other references that discuss graph over surfaces are [73, 74]. As we will see it can be proved that to construct an embedding it is sufficient (and it is actually equivalent) to know whether a given edge preserves or reverses the orientation of the surface, and to know a cyclic orientation of the edges that are adjacent to a vertex. We highlight that in this whole chapter we assume for simplicity, unless explicitly stated, that the surfaces that we consider are *closed* and *orientable*.

We recall that an edge e in a graph Γ is an unordered 2-subset of the set of vertices of Γ . In this section we need to introduce the notion of a *directed graph*. Given a set V of vertices a *directed edge* is an ordered pair of distinct vertices.

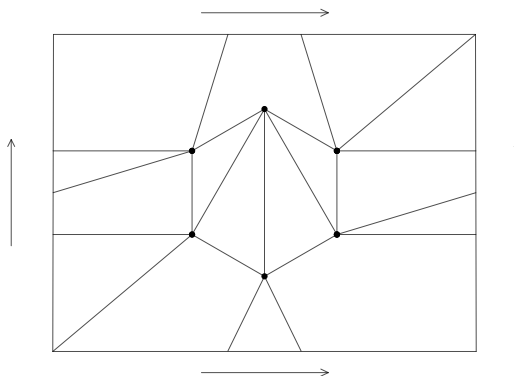
A *directed graph* (also called *digraph*) is a pair of a set of vertices and a set of directed edges. From an unordered edge $e = \{u, v\}$ we can naturally construct two directed edges, that we denote as $e^+ = (u, v)$ and $e^- = (v, u)$. If $\Gamma = (V, E)$ is a graph where every edge is undirected, we denote by $\vec{\Gamma} = (V, D)$ the graph having V as set of vertices and $\{e^+, e^- : e \in E\}$ as its set of edges. We also write $D(\Gamma)$ to denote the set of edges of $\vec{\Gamma}$. We now introduce a topological structure over a graph.

Any graph $\Gamma = (V, E)$ can be represented by a topological space, called *topological representation of Γ* , in the following way: each vertex $v \in V$ is represented by a distinct point and each edge by a distinct arc homeomorphic to the interval $[0, 1]$. To do this, we recall that two topological spaces X and Y are *homeomorphic* if there exists a function $f : X \rightarrow Y$, called *homeomorphism*, that is bijective, continuous, and with continuous inverse. We moreover require that the interiors of the arcs are mutually disjoint, and two arcs intersect in at most one of the points of their boundary. We can define then an embedding of a graph over a surface, see [73]:

Definition 3.1.1. Given a graph Γ and a surface Σ , an *embedding of Γ in Σ* is a continuous injective mapping ψ from the topological representation of Γ to the surface Σ .

Informally speaking an embedding of a graph Γ in Σ is a drawing of Γ over Σ such that no two edges cross each other, as shown in the following example.

Example 3.1.2. In the figure below we show an embedding of the complete graph K_6 over the torus:



The connected components of $\Sigma \setminus \psi(\Gamma)$ are called *ψ -faces*, that we more simply call *faces* if the function ψ is clear from the context. If each face is homeomorphic to an open disc, the embedding ψ is said to be *cellular*. From now on, we implicitly assume that all embeddings are cellular. Two embeddings $\psi : \Gamma \rightarrow \Sigma$ and $\psi' : \Gamma' \rightarrow \Sigma'$ are *isomorphic* whenever there exists a graph isomorphism $\sigma : \Gamma \rightarrow \Gamma'$ such that $\sigma(F)$ is a ψ' -face if and only if F is a ψ -face. For an embedding of a graph $\Gamma = (V, E)$, whose set of faces is F , the *Euler characteristic of Γ* , denoted by $\chi(\Gamma)$, is given by:

$$\chi(\Gamma) = |V| - |E| + |F|.$$

From Example 3.1.2, for instance, we can see that for an embedding of K_6 on the torus we have $|V| = 6$, $|E| = 15$ and $|F| = 9$, hence $\chi(K_6) = 0$.

A simple observation shows that every graph can be embedded in a suitable surface:

Remark 3.1.3. Given a map ψ from a graph Γ to a surface Σ admitting two edges e and e' that cross each other, construct a surface Σ' from Σ as follows: cut two small disks on each side of one of the two edges, say e' , and attach a handle between the two holes. Let now ψ' be the map from Γ to Σ' that reroutes the edge e along new handle: it is easy to see that e and e' are now disjoint in Σ' . If this procedure is repeated until all crossings are eliminated, we get an embedding of Γ on an orientable surface.

Clearly, if a graph can be embedded in a surface Σ , then it can be embedded in every surface Σ_n obtained by attaching n -handles to Σ . Moreover, the classification theorem of closed surfaces states that every connected, closed and orientable surface Σ is homeomorphic to a surface obtained by attaching n -handles to a sphere, for some positive integer $n \geq 0$. The number n of handles is called the *genus of the surface* Σ and is denoted by $\gamma(\Sigma)$.

Given a graph Γ , it is interesting to study which is the minimum integer n such that Γ can be embedded in a surface having genus n . This parameter, called *genus of the graph* and denoted as $\gamma(\Gamma)$, is of much interest in research and in general it is quite hard to determine. It is worth pointing out that the procedure given in Remark 3.1.3 does not give very good upper bounds on the genus of the graph: for instance, any drawing of K_6 on the sphere results in at least three edge crossings, but they can be removed by adding just one handle; indeed, as shown in Example 3.1.2 K_6 can be embedded in the torus.

An important class of graphs that it is worth mentioning is the one of the so-called *planar graphs*, that are those graphs that can be embedded in the euclidean plane. As a first remark it is not hard to see that by means of a stereographic projection a graph is planar if and only if it can be embedded on a sphere, i.e. it has genus 0. A very classical characterization of planar graphs is the following, due to Kuratowski [65]:

Theorem 3.1.4. *A graph is planar if and only if it does not contain a subgraph that is homeomorphic to $K_{3,3}$ or K_5 .*

We conclude with the following important relation between the genus of a graph and its Euler characteristic:

Theorem 3.1.5. *Let Γ be a graph having genus $\gamma(\Gamma) = g$. Let ψ be an embedding of Γ on a surface of genus g and let $\chi(\Gamma)$ be the Euler characteristic of the graph embedding. Then:*

$$\chi(\Gamma) = 2 - 2g.$$

Note that the embedding of K_6 on the torus of Example 3.1.2 has $\chi(K_6) = 0$, hence $g = 1$; indeed the torus has genus 1.

Given a graph embedding on a surface Σ , one can conveniently describe the embedding by means of a schematic called band decomposition, that can be further simplified in the form of a rotation system. In what follows we describe these two structures.

Band decomposition

A band decomposition is a topological structure that allows to simplify the description of an embedding. Informally speaking, a band decomposition of an embedding is obtained by removing from the surface the interior of the faces of the embedding, except for a sufficiently small neighbourhood of the vertices and of the edges of the embedded graph.

A 0 *band* is a topological space homeomorphic to an open disk, while a 1 *band* is a topological space b that is homeomorphic to $[0, 1] \times [0, 1]$ through a map $h : [0, 1] \times [0, 1] \rightarrow b$. The arcs $h([0, 1] \times i)$, for $i = 0, 1$ are called the *ends* of b . A (*reduced*) *band decomposition* is a collection of 0 bands and 1 bands such that:

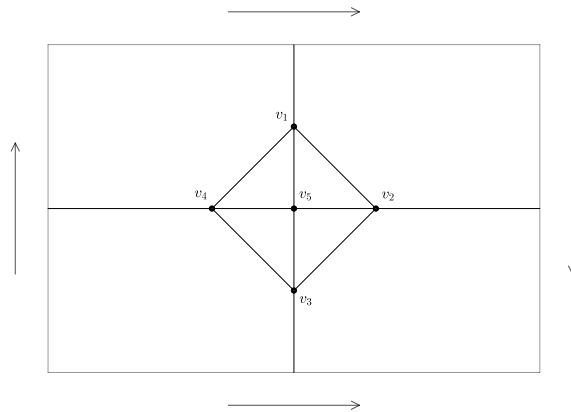
- (1) different bands intersect only along arcs in their boundaries;
- (2) the 0 bands are pairwise disjoint;
- (3) the ends of a 1 band are contained in a 0 band.

A *band decomposition of a graph embedding* $\Gamma \rightarrow \Sigma$ is a band decomposition of Σ such that:

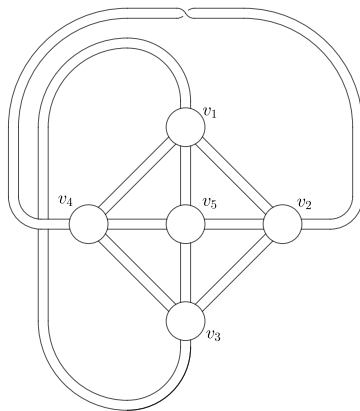
- (1) there exists a bijection between vertices of Γ and 0 bands;
- (2) if $\{u, v\}$ is an edge of Γ , then there is a 1 band having non empty intersection with the 0 bands corresponding to u and v .

Starting from a graph embedding $\Gamma \rightarrow \Sigma$, a band decomposition is obtained from the union of a sufficiently small tubular neighbourhood of the image on Σ of each edge of Γ , and a sufficiently small disk around the image of each vertex.

Example 3.1.6. In this example we show the band decomposition of the following embedding of the complete graph K_5 over the Klein bottle (we considered a non-orientable surface in order to show the possible 1-bands that can be obtained):



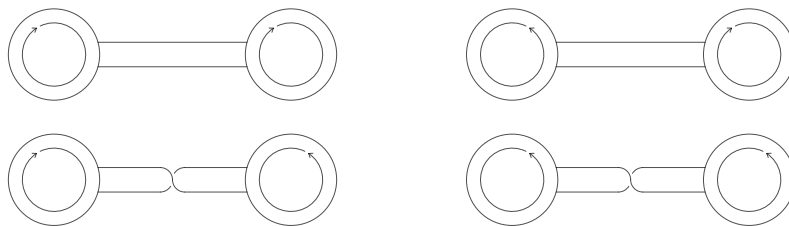
A band decomposition is then shown in the following figure:



Notice that the Klein bottle is a non-orientable surface, and one might think that this can be easily seen by the twisted 1 band corresponding to the edge $\{v_2, v_4\}$. However, this condition turns out to not be sufficient, as in order to identify the orientability of the surface from a band decomposition, one has to verify if the twisted 1 band is a part of a larger orientable part of the surface. In the following part we give a more formal description of this concept.

Orientation and rotation systems

Given a band decomposition of a graph embedding an embedding is said to be *locally oriented* if an orientation is assigned to each 0 band. A 1 band is *orientation preserving* if the directions induced on its ends by adjoining 0 bands are the same as those induced by one of the possible orientation of the 1 band. If this is not the case the 1 band is called *orientation reversing*; the four different possibilities are shown in the following figure, where the 1 bands on the left are orientation preserving, and the 1 bands on the right are orientation reversing:



An edge of the graph embedding is *of type 0* if the corresponding 1 band is order preserving, and *of type 1* otherwise. For instance, assuming that every 0 band has the same local counterclockwise orientation, the 1 band corresponding to the edge $\{v_2, v_4\}$ of Example 3.1.6 is orientation reversing and $\{v_2, v_4\}$ is a type 1 edge, while every other 1 band is orientation preserving and the corresponding edge has type 0.

A subgraph of the graph embedding is *of type 1* if it contains an odd number of edges of type 1, and is *of type 0* otherwise. It can then be proven the following statement:

Theorem 3.1.7. *Let $\Gamma \rightarrow \Sigma$ be a graph embedding. Σ is a non orientable surface if and only if there exists a cycle of type 1 in the graph embedding.*

It then can be clearly seen that the cycle of Example 3.1.6 having as sequence of vertices (v_1, v_4, v_2) is of type 1, hence the surface is non-orientable: indeed, the surface is a Klein bottle. We point out the following operation that preserves the surface and can be used to reverse the orientation of some 0 bands:

- (1) let ρ_v be the orientation of the 0 band corresponding to the vertex v ;
- (2) reverse the orientation ρ_v ;
- (3) change the orientation type of every edge that is incident to v in the embedding.

We now are ready to introduce a key concept in the description of an embedding, see also Example 3.1.10:

Definition 3.1.8. A (*local*) *rotation* of a vertex v is an ordered list, up to a cyclic permutation, of the directed edges originating at v in the graph embedding read with respect to the occurrences of their corresponding 1 bands in the local rotation given by the 0 band corresponding to v . A *rotation system* is an assignment of a rotation to each vertex, and an orientation type to each edge.

The importance of this definition is then clear from the following result:

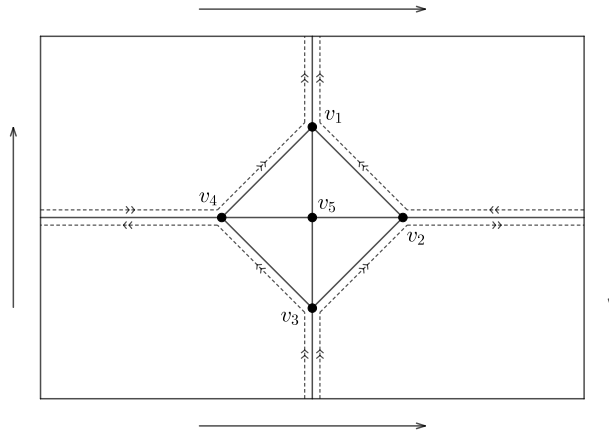
Theorem 3.1.9. *Every rotation system defines (up to isomorphism) a unique locally oriented graph embedding. Conversely, every locally oriented graph embedding defines a rotation system for the graph.*

Moreover, if the graph Γ has no vertex of degree 2, then the faces of the embedding can be reconstructed with the so-called **face tracing algorithm**, see [6, 54]:

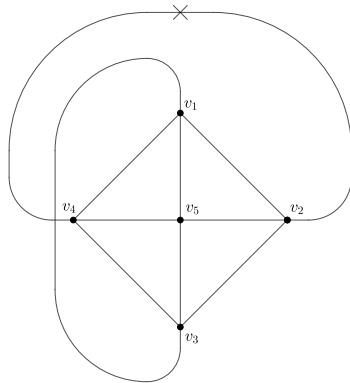
Let v_0 be an initial vertex for a face, and let $e_1 = \{v_0, v_1\}$ be any edge that is incident to v_0 . If ρ_{v_1} is the rotation of v_1 , then the second edge e_2 of the boundary of the face is obtained by $\rho_{v_1}^\mu(e_1)$, where $\mu = +1$ if e_1 is of type 0, and -1 otherwise. Iterate this process, where the edge e_{i+1} is $\rho_{v_i}^\mu(e_i)$, where $\mu = +1$ if the walk (e_1, \dots, e_i) is of type 0, and -1 otherwise. The face is then completely traced when, by iterating the previous procedure, we find again e_1 followed by e_2 .

Example 3.1.10. Consider again the embedding of K_5 on the Klein bottle of Example 3.1.6. It is clear that, for instance, one of the faces of the embedding is the one having the following sequence of vertices (see also the figure below):

$$f = (v_1, v_3, v_2, v_4, v_1, v_3, v_4, v_2).$$



We now verify that the face tracing algorithm applied to a rotation system of K_5 produces the same face, when the starting edge is, for instance, (v_1, v_3) . We begin with the following drawing of K_5 where, as it is standard, an edge of type 1 has the \times symbol drawn on top of it. For simplicity we assume that the rotation assigned to every vertex is counterclockwise: for example, the local rotation assigned to the vertex v_1 is the cyclic permutation $((v_1, v_3), (v_1, v_4), (v_1, v_5), (v_1, v_2))$.



We then apply the face tracing algorithm in the following table, where we give the face in terms of its vertex sequence; we recall that, if $\mu = +1$, then the next edge is given by a counterclockwise rotation, and a clockwise rotation otherwise:

Edge	Edge type	Next vertex	μ	Face
(v_1, v_3)	0	v_3	+1	$(v_1$
(v_3, v_2)	0	v_2	+1	$(v_1, v_3$
(v_2, v_4)	1	v_4	-1	$(v_1, v_3, v_2$
(v_4, v_1)	0	v_1	-1	$(v_1, v_3, v_2, v_4$
(v_1, v_3)	0	v_3	-1	$(v_1, v_3, v_2, v_4, v_1$
(v_3, v_4)	0	v_4	-1	$(v_1, v_3, v_2, v_4, v_1, v_3$
(v_4, v_2)	1	v_2	+1	$(v_1, v_3, v_2, v_4, v_1, v_3, v_4$
(v_2, v_1)	0	v_1	+1	$(v_1, v_3, v_2, v_4, v_1, v_3, v_4, v_2)$

We end the algorithm since the next vertex would be v_1 , and the edges of the following iterations would be again (v_1, v_3) and (v_3, v_2) (equivalently we meet the edge (v_1, v_3) with the same parameter μ).

An embedding can then be equivalently defined in purely combinatorial terms by means of its rotation system, see [52, 91]:

Definition 3.1.11. Let Γ be a graph, a *combinatorial embedding* of Γ is a pair $\Pi = (\Gamma, \rho)$ where $\rho : D(\Gamma) \rightarrow D(\Gamma)$ is a map such that:

- (1) for every $(u, v) \in D(\Gamma)$ there exists $(u, v') \in D(\Gamma)$ such that $\rho((u, v)) = (u, v')$;
- (2) for every vertex $v \in V(\Gamma)$ the restriction ρ_v of ρ is a permutation of maximum length.

3.2 Embedding of Cayley graphs and current graphs

As discussed in the previous section many tools have been developed to simplify the description of the embedding of a graph on an (orientable) surface. Here we focus on a method that has been introduced to describe a particular class of embeddings of Cayley graphs. Indeed, the rotation system by itself might be unpractical to use for depicting embedding of graphs having a large number of vertices: in particular it would be desirable to find a more efficient method to describe these embeddings whenever the graph admits some symmetries, like for Cayley graphs. As we will see such a description can be efficiently achieved using the so-called current graphs, introduced by Youngs in [99] (see also Chapter 4 of [54]). Moreover, the description of the embeddings that arise from Heffter arrays relies on current graphs, and some Heffter arrays have been constructed in the literature using current graphs [8].

Definition 3.2.1. Given a graph $\Gamma = (V, E)$, a *current assignment* of Γ is a function α from the directed edges of $\vec{\Gamma} = (V, D)$ to a group $(G, +)$, such that:

$$\alpha(e^+), \alpha(e^-) \neq 0, \quad \alpha(e^+) = -\alpha(e^-) \quad \text{for every } e \in E.$$

A *current graph* is defined as a triple (Γ, ρ, α) , where Γ is an undirected graph, ρ is a rotation system, and α is a current assignment of Γ .

We now show how one can use a current graph to construct an embedding.

Let F be the set of faces of the embedding associated to the pair (Γ, ρ) , that can be found from the face tracing algorithm. The *derived graph* Γ_α of a current graph (Γ, ρ, α) , where α maps the directed edges of $\vec{\Gamma}$ to a group $(G, +)$, is the graph having as vertex set the cartesian product $F \times G$, and having the following edge set: an edge of Γ_α is for convenience denoted by a pair (e, g) , where e is an edge of Γ and $g \in G$; (e, g) is incident to the vertices (f_1, g) and $(f_2, \alpha(e^+) + g)$, where f_1 and f_2 are the faces in F containing e^+ and e^- respectively.

It is then possible to define in a natural way a rotation system, hence an embedding, for the derived graph Γ_α . Indeed if $f = (e_0, e_1, \dots, e_{n-1})$ is a face of the embedding associated to the rotation system (Γ, ϕ) , then the *derived*

rotation system ρ_α is the cyclic permutation that maps the oriented edge (e_i, g) to the oriented edge (e_{i+1}, g) , for $i = 0, 1, \dots, n-1$, and where indices are read modulo n .

The faces can then be found by considering the rotations e_0, \dots, e_{n-1} of the oriented edges originating at a vertex v of Γ . In view of the application to Heffter arrays and to slightly simplify the exposition, we assume that for every vertex v of the current graph (Γ, ρ, α) Kirchhoff's current law holds at v , namely:

$$\alpha(e_{n-1}^+) + \alpha(e_{n-2}^+) + \dots + \alpha(e_0^+) = 0.$$

The faces can be found by considering the following collection of directed edges (we recall that we use the convention of labeling the directed edges of the derived graph as a pair (e, g) for some $e \in E$ and $g \in G$):

$$(e_0, g), (e_1, \alpha(e_0^+) + g), (e_2, \alpha(e_1^+) + \alpha(e_0^+) + g), \dots, (e_{n-1}, \alpha(e_{n-2}^+) + \dots + \alpha(e_0^+) + g).$$

Remark 3.2.2. If $F = \{f\}$ is a singleton it is then easy to see that the derived graph is the Cayley graph $\text{Cay}[G : \{\pm\alpha(e^+) : e \in E\}]$.

If F is not a singleton the derived graph is the graph having $F \times G$ as vertex set, and as edges those having difference:

$$(\pm(f_2 - f_1), \pm\alpha(e^+)),$$

where e^+ ranges in D , and f_1 and f_2 are the faces in F containing e^+ and e^- respectively. The difference $f_2 - f_1$ can be seen as a word of length 2 in the free group having free generating set $\{\pm f : f \in F\}$.

3.3 Face colorings

In this section we introduce some classical Graph Theory notions that are related to the embedding of a graph over a surface. Some references for this topic are [45, 54].

Definition 3.3.1. Given a graph Γ and a positive integer k , a k -coloring of Γ is a function from the set of vertices of Γ to a set of labels, called *colors* and usually denoted with the integers $\{1, 2, \dots, k\}$, such that adjacent vertices receive distinct colors.

We remark that in the literature these colorings are also called *proper*. Vertices that receive the same color are said to belong to the same *color class*. A graph is k -colorable if it admits a k -coloring. The *chromatic number* of a graph is the minimum integer k such that Γ is k -colorable. We remark that a 2-colorable graph is precisely a bipartite graph (see Section 2.1) and the two parts of the graph are the two color classes.

The topic of graph colorings has been subject of study due to its many applications and open problems. Moreover, one can consider a coloring of the edges of the graph (where edges incident to the same vertex receive distinct colors) or, given a graph embedding, a coloring of the faces of the graph (where faces that have an edge in common receive distinct colors). It can be shown that these problems are related to the classical vertex coloring problem, that is applied to an auxiliary graph constructed from the original one. Here we consider the latter problem of coloring the faces of an embedding.

For a graph embedding $\psi : \Gamma \rightarrow \Sigma$ let F be the set of its faces. We define the *face dual* of the embedding as the graph $\Gamma_F = (V_F, E_F)$ that has $|F|$ vertices, and that admits a bijection $\phi : V_F \rightarrow F$ such that $\{f_1, f_2\} \in E_F$ if and only if $\phi(f_1)$ and $\phi(f_2)$ are faces having at least one edge in common. An embedding $\psi : \Gamma \rightarrow \Sigma$ is *k-face colorable* if the face dual is *k-colorable*.

Many interesting problems have been raised in relation to the face colorability of an embedding. Perhaps the most important result on this topic is the well known Four colors Theorem, proven by Appel and Haken [5] in 1976 using a computer-assisted proof:

Theorem 3.3.2. *Every planar graph is 4-face colorable.*

For a general orientable surface of genus $g > 0$, Heawood [56] conjectured in 1890 that the maximum number of colors needed to color a graph embedded in the surface is:

$$\gamma(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor.$$

The validity of this upper bound was then shown by Ringel and Youngs [83] in 1968. Note that for the planar case we have $g = 0$ and $\gamma(0) = 4$, hence this upper bound is also tight.

Remark 3.3.3. Regarding the context of Heffter arrays, in the following sections we consider graph embeddings that are 2-face colorable, also called *biembeddings*.

It is easy to notice that a necessary condition for the existence of a biembedding of a graph is that the degree of each vertex must be even: indeed, the face dual of a biembedding is 2-colorable, thus it is a bipartite graph, and a rotation around a vertex corresponds to a cycle subgraph on the face dual. Since a graph is bipartite if and only if it does not contain odd cycles (see Section 2.1) we conclude that every vertex has even degree.

3.4 Biembeddings and Heffter arrays

In this section we follow the construction given by Archdeacon in [6] to show that, under suitable assumptions, it is possible to construct biembeddings of cycle systems starting from a Heffter array. This connection has been studied in many articles on Heffter arrays, see for instance [30, 39, 42, 47]. In the literature a lot of attention has been devoted to the construction of embeddings of cycle systems, see for instance [50, 51, 52, 53]. Moreover, the faces of the embedding can be easily determined from some properties of the array. To conclude, the constructed embedding has a quite large group of automorphisms, see also Section 3.6.

Remark 3.4.1. Following [32] we call an embedding arising from a Heffter array *Archdeacon embedding*, or *embedding of Archdeacon type*.

For a vertex u of a directed graph $\vec{\Gamma} = (V, D)$, let $D_u = \{(u, w) \in D\}$ be the set of directed edges originating from u . Given a current graph (Γ, ϕ, α) , where (Γ, ϕ) has only edges of type 0 and α is a current assignment from the directed edges of $\vec{\Gamma} = (V, D)$ to \mathbb{Z}_v for some positive integer v , we define the following properties:

- (1) **Kirchhoff's current law:** for every $u \in V$, $\sum_{e \in D_u} \alpha(e) = 0$.
- (2) **Unique currents:** α is a bijection between D and $\mathbb{Z}_v \setminus \{0\}$.
- (3) **Monofacial embedding:** the embedding constructed from (Γ, ϕ) has a single face.

Archdeacon showed the following:

Theorem 3.4.2. *Let $\Gamma = (V, E)$ be a bipartite graph having one color class with m vertices of degree h and the other with n vertices of degree k , for some $m \geq k$ and $n \geq h$ positive integers. Let (Γ, ϕ, α) be a current graph satisfying Conditions (1), (2), and (3). Then there is an embedding of K_{2nk+1} on an orientable surface such that each edge lies on a face of length h and on a face of length k . Moreover, if each local rotation on (Γ, ϕ) is simple with respect to α , then the faces of K_{2nk+1} are cycles.*

Proof. As shown in the previous section, from a current graph (Γ, ϕ, α) one can construct an embedding of the derived graph. In what follows we show that the current graph (Γ, ϕ, α) as in the statement has K_{2nk+1} as derived graph.

Vertex set: The derived graph of a current graph (Γ, ϕ, α) has as vertex set the cartesian product $F \times G$, where F is the set of faces of the embedding constructed from the rotation system (Γ, ϕ) , and G is the underlying set of the group $(G, +)$ to which the currents belong. Since (Γ, ϕ, α) satisfies Condition (3) the set $F = \{f\}$ is a singleton, and $F \times G \cong G$; in particular $(G, +) = (\mathbb{Z}_{2nk+1}, +)$, hence the vertex set of the derived graph can be identified with the vertex set of K_{2nk+1} .

Edge set: The edge set of the derived graph has elements labelled with the pair (e, g) for every $e \in E$ and $g \in G$: the edge (e, g) has as endvertices (f_1, g) and $(f_2, \alpha(e^+) + g)$, where f_1 and f_2 are the faces containing respectively e^+ and e^- . Again, by Condition (3) the edge (e, g) is incident to the vertices having labels (f, g) and $(f, \alpha(e^+) + g)$. As g varies in \mathbb{Z}_{2nk+1} it is easy to see that $\{(e, g) : g \in \mathbb{Z}_{2nk+1}\}$ is precisely the set of edges in K_{2nk+1} having differences $\pm\alpha(e^+)$: by Condition (2) we deduce that as the edge e varies in E we construct the Cayley graph $\text{Cay}[\mathbb{Z}_{2nk+1} : \mathbb{Z}_{2nk+1} \setminus \{0\}] \cong K_{2nk+1}$, thus achieving the statement (see also Remark 3.2.2).

Face set: We recall that given a rotation e_1, e_2, \dots, e_ℓ of the oriented edges originating at a vertex u of Γ , under the assumption of Condition (1) a face of the embedding is given by

$$(e_1, g), (e_2, \alpha(e_1^+) + g), (e_3, \alpha(e_2^+) + \alpha(e_1^+) + g), \dots, (e_k, \alpha(e_{k-1}^+) + \dots + \alpha(e_1^+) + g). \quad (3.4.1)$$

Since the vertices of the two color classes of Γ have respectively degree k and h , it follows that every face of the embedding has either length k or h (hence $\ell = k$ or $\ell = h$). As Γ is bipartite every edge is incident with one vertex belonging to each color class, thus every edge of the derived graph lies on a face of length k and one of length h . It can be seen that the sequence of vertices relative to the edges of Equation (3.4.1) is given by $S = ((f, s_1), (f, s_2), (f, s_3), \dots, (f, s_k))$,

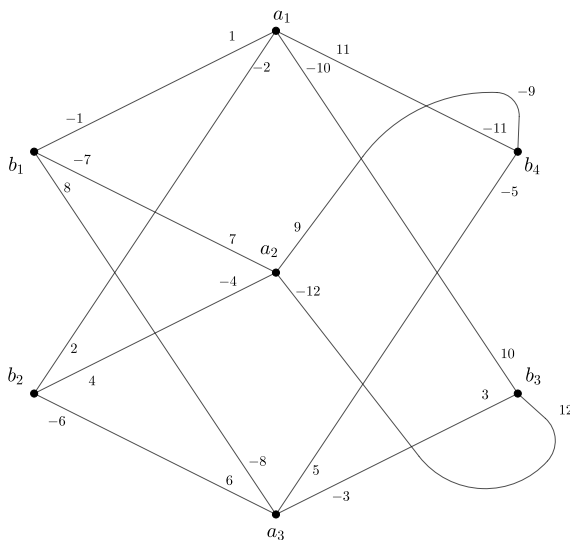
where $s_i = (\sum_{j=1}^i \alpha(e_j^+)) + g$. Hence if the rotation is simple, i.e. the partial sums of $(\alpha(e_1^+), \dots, \alpha(e_k^+))$ are all distinct, the vertices of S are all distinct and the face is isomorphic to a cycle. \square

Remark 3.4.3. From now on if the embedding of the current graph is monofacial, i.e. $F = \{f\}$, we may omit for simplicity the first component of the vertices in the derived graph, as it would always be f . For instance we write a vertex (f, g) of the derived graph simply as g .

Remark 3.4.4. From the proof of the previous theorem it can be easily deduced that, given a graph $\Gamma = (V, E)$ and its directed version $\vec{\Gamma} = (V, D)$, if α is a current assignment between D and a subset $\Omega \subseteq \mathbb{Z}_v \setminus \{0\}$ (for v odd), by the current assignment definition we have that $\Omega = -\Omega$, and the derived graph is isomorphic to $\text{Cay}[\mathbb{Z}_v : \Omega]$. In particular if $\Omega = \mathbb{Z}_v \setminus J$, where J is a subgroup of \mathbb{Z}_v , then the derived graph is a complete multipartite graph (see Remark 2.1.3). It follows from Theorem 3.4.2 that under Conditions (1), (2), and (3), it is possible to construct a biembedding of these derived graphs.

Example 3.4.5. In this example we construct the embedding given in Theorem 3.4.2, starting from a current graph Γ satisfying Conditions (1), (2), and (3) and having the characteristics given in the statement of the theorem. We highlight that this example was given in [6].

Let (Γ, ρ, α) be following current graph, where ρ is the counterclockwise rotation for each vertex, and α maps the directed edges of $\vec{\Gamma}$ to elements of \mathbb{Z}_{25} (following [6] the current assigned to a directed edge originating in a vertex v is the one that is written closer to v):



It can be easily seen that α is a unique-current assignment and that it satisfies Kirchhoff's current law: for instance, if we consider the vertex a_2 we find the directed edges having currents $\{7, 9, -4, -12\}$, and this set is zero-sum in \mathbb{Z}_{25} . Moreover the rotation ρ is monofacial, as the unique face f is given by:

$$(1, 8, 6, 4, -12, 10, 11, -5, -8, -7, -4, 2, -10, 3, 5, -9, 7, -1, -2, -6, -3, 12, 9, -11).$$

Construct now the derived graph Γ_α having $\{f\} \times \mathbb{Z}_{25} \cong \mathbb{Z}_{25}$ as vertex set, and fix any directed edge in $\vec{\Gamma}$, say $e = (b_2, a_1)$. It can be seen that $\alpha(e) = 2$, hence the edges in Γ_α labelled (e, g) are those having endvertices of the form $\{g, g+2\}$, as g varies in \mathbb{Z}_{25} . It then follows that we obtain K_{25} as the derived graph of (Γ, ρ, α) .

Fix now a vertex of Γ , say again a_2 , and let $(9, 7, -4, -12)$ be a local rotation of the edges originating in a_2 . As g ranges in \mathbb{Z}_{25} the faces that are obtained from a_2 have as sequence of vertices

$$(g, 9 + g, 7 + 9 + g, -4 + 7 + 9 + g).$$

It immediately follows that if the local rotation around a vertex is simple, then the faces obtained are isomorphic to cycle graphs.

To explain the relationship between Heffter arrays and the current graph of Theorem 3.4.2 we have to introduce the concept of compatible orderings, see [6].

Definition 3.4.6. Given a partially filled array A , where every row R has an ordering ω_R and every column C has an ordering ω_C , respectively denote by $\omega_r = \Pi_R \omega_R$ and $\omega_c = \Pi_C \omega_C$ the *ordering of the rows* and the *ordering of the columns* of A . Then, the orderings ω_r and ω_c are said to be *compatible* if $\omega_r \circ \omega_c$ is a cycle of length $\mathcal{E}(A)$.

Example 3.4.7. Let A be the following $H(3, 8)$ over \mathbb{Z}_{49} :

-13	-1	14	6	10	-8	-11	3
4	-20	-2	17	5	-16	-7	19
9	21	-12	-23	-15	24	18	-22

Then, consider the following row and column orderings:

$$\begin{aligned} \omega_r &= (-13, -1, 14, 6, 10, -8, -11, 3)(4, -20, -2, 17, 5, -16, -7, 19) \\ &\quad (9, 21, -12, -23, -15, 24, 18, -22) \\ \omega_c &= (-13, 4, 9)(-1, -20, 21)(14, -2, -12)(6, 17, -23) \\ &\quad (10, 5, -15)(-8, -16, 24)(-11, -7, 18)(3, 19, -22). \end{aligned}$$

It can be seen that they are compatible, as:

$$\begin{aligned} \omega_r \circ \omega_c &= (-13, -20, -12, 6, 5, 24, -11, 19, 9, -1, -2, \\ &\quad -23, 10, -16, 18, 3, 4, 21, 14, 17, -15, -8, -7, -22). \end{aligned}$$

Now we are ready to state the following result, obtained by Archdeacon in [6]:

Theorem 3.4.8. *Let H be an $H(m, n; h, k)$. H is equivalent to a current assignment α satisfying Conditions (1) and (2) on a bipartite graph Γ having one color class with m vertices of degree h and the other class with n vertices of degree k . Γ is connected if and only if H is not, up to row and column permutations, fully indecomposable.*

Two compatible orderings ω_r and ω_c of the rows and columns of H are equivalent to a monofacial rotation ρ on Γ . Moreover, if ω_r and ω_c are both simple, then ρ is simple with respect to α .

Proof. Let $H = [h_{i,j}]$ be an $H(m, n; h, k)$ and construct the graph $\Gamma = (A \cup B, E)$ having $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ as a partition of its vertex set, and as edge set $\{\{a_i, b_j\} : h_{i,j} \text{ is a filled cell}\}$. It is immediate to see that Γ is a bipartite graph with color classes A and B , and from the definition of a Heffter array every vertex of A has degree h and every vertex of B has degree k . For the directed version of Γ denoted as $\vec{\Gamma} = (V, D)$ consider the current assignment:

$$\alpha((a_i, b_j)) = h_{i,j}, \quad \alpha((b_j, a_i)) = -h_{i,j}.$$

Since every row and every column of H is zero-sum in \mathbb{Z}_{2nk+1} , α satisfies Condition (1). Moreover, as the elements of H form a half-set of \mathbb{Z}_{2nk+1} , it follows that α is a unique current assignment on Γ .

Assume now that H (after possibly some row or column permutations) is a fully indecomposable matrix, and that the graph Γ is connected: let then $h_{i,j}$ and $h_{i',j'}$ be filled cells of H belonging to different blocks. As Γ is connected there exists a sequence of edges connecting the vertices corresponding to the i -th row and the i' -th row of H : this implies that it is possible, by means of vertical or horizontal moves on the filled cells of H , to reach $h_{i,j}$ from $h_{i',j'}$, that is a contradiction with H fully indecomposable. Equivalently, if Γ is connected then H can not be fully indecomposable.

Let now ω_r and ω_c be compatible orderings of H , and define $\tau(a) = -a$ for every $a \in \mathbb{Z}_{2nk+1} \setminus \{0\}$. Define the following map $\rho : \mathbb{Z}_{2nk+1} \setminus \{0\} \rightarrow \mathbb{Z}_{2nk+1} \setminus \{0\}$ as:

$$\rho(a) = \begin{cases} \omega_r(a) & \text{if } a \in H, \\ \tau \circ \omega_c \circ \tau(a) & \text{if } a \notin H. \end{cases}$$

Note that if a belongs to the i -th row of H , then its orbit through ρ is precisely the i -th row of H . Similarly, if the inverse of a is contained in the j -th column of H , so is the opposite of the orbit of a through ρ . To verify that the embedding is monofacial we have to show that $\rho \circ \tau$ is a cyclic permutation of $\mathbb{Z}_{2nk+1} \setminus \{0\}$. For $a \in H$, $(\rho \circ \tau)^2(a) = \omega_r \circ \omega_c(a)$, and by the compatibility condition of ω_r and ω_c it can be deduced that $(\rho \circ \tau)^2$ acts cyclically on the entries of H . Similarly, the odd powers of $\rho \circ \tau$ act cyclically on the entries of $-H$ obtaining a monofacial embedding of Γ . To conclude it is easy to show that if the orderings are simple the induced rotation is simple. \square

By combining Theorems 3.4.2 and 3.4.8 it follows that from a Heffter array admitting two compatible orderings it is possible to construct an embedding of a complete graph. We remark that in some particular cases, as shown in [42], it is possible to construct from a single Heffter array an exponential number of distinct graph embeddings.

Remark 3.4.9. Since every edge of the embedding constructed from an $H(m, n; h, k)$ Heffter array lies on a face of length h and on a face of length k , the embedding is face 2-colorable: it is sufficient to assign one color to the faces having length h , and the other color to the faces having length k .

Remark 3.4.10. From Remark 3.4.4 it can be seen that the proof of Theorem 3.4.8 can be readily adapted to the case of relative Heffter arrays (see Definition 1.3.3), hence obtaining from an $H_t(m, n; h, k)$ a biembedding of the complete multipartite graph $K_{\frac{2nk+t}{t} \times t}$.

Example 3.4.11. Let (Γ, ρ, α) be the current graph of Example 3.4.5. By following the proof of Theorem 3.4.8 it can be seen that the current graph (Γ, ρ, α) is equivalent to the following Heffter array over \mathbb{Z}_{25} :

$$H = \begin{array}{|c|c|c|c|} \hline 1 & -2 & -10 & 11 \\ \hline -8 & 6 & -3 & 5 \\ \hline 7 & -4 & -12 & 9 \\ \hline \end{array}$$

paired with the following row and column orderings:

$$\begin{aligned} \omega_r &= (1, -2, -10, 11)(-8, 6, -3, 5)(7, -4, -12, 9), \\ \omega_c &= (1, -8, 7)(-2, 6, -4)(-10, -3, -12)(11, 5, 9). \end{aligned}$$

Note that, following the notation of the current graph given in Example 3.4.5, ω_r is the local rotation assigned to the vertices a_1, a_2 and a_3 , while $\tau \circ \omega_c$ is the local orientation assigned to the vertices b_1, b_2, b_3 and b_4 . It can be easily seen that ω_r and ω_c are compatible:

$$\omega_r \circ \omega_c = (1, 6, -12, 11, -8, -4, -10, 5, 7, -2, -3, 9).$$

3.5 A tour problem on a Heffter array

In the previous section it has been shown the connection between Heffter arrays and the biembedding of a complete graph on a surface. A fundamental role in the construction of such an embedding is played by the orderings ω_r and ω_c of the rows and of the columns of the array, that have to be compatible, i.e. $\omega_r \circ \omega_c$ has to be a cyclic permutation on the elements contained in the filled cells of the array. Moreover, it would be desirable to find a pair of compatible orderings that are also simple in order to have faces isomorphic to simple cycles. However, it is not very practical to find these orderings in a Heffter array having either a large size or a large number of filled cells in each row or column. For this reason, the class of globally simple Heffter arrays has been introduced in [39], where by definition the natural ordering of each row and each column is simple. It now remains to find natural compatible orderings in the globally simple Heffter array. As we will see this problem can be seen as a tour problem over the the filled cells of the array, that now is seen as embedded inside a toroidal surface.

It is easy to verify that if an ordering (t_1, \dots, t_k) in a group $(G, +)$ is simple, then the ordering (t_k, \dots, t_1) is simple as well. In terms of natural orderings, this implies that if we read the elements of a row (respectively of a column) from left to right (respectively from top to bottom) or from right to left (respectively from bottom to top), then the orderings are simple as well if the array is globally simple. Hence, when searching for a pair of compatible orderings, we can choose for each row and for each column either the “natural” direction or its inverse. Moreover, if we assume that the array that we are considering is globally simple, we are only interested in knowing which cells of the array are filled and their actual content can be ignored.

In what follows we give a more formal description of these concepts, and show that these are related to a tour problem of a chess-like piece over a holed toroidal chessboard, that is given by the positions of the cells of the array that are filled. This problem was introduced in [33] and studied by myself in [68].

For an $m \times n$ partially filled array A , we recall that by $skel(A)$ we denote its *skeleton*, that is the set of pairs (i, j) such that the (i, j) -th cell of A is filled. Then, a *move function* is a permutation of $skel(A)$, while a *tour* is a move function that is also a cycle of length $|skel(A)|$. The problem of finding a tour over the cells of an array has been a subject of study of discrete mathematics for quite a long time: for instance, the most famous problem in this topic, that is the problem of finding a tour of the classical chess knight over a chessboard, can be traced back to the 9th century AD. The following result due to Schwenk in 1991 [89], completely solves the knight's tour problem over a generalized chessboard:

Theorem 3.5.1. *A knight's tour over the $m \times n$ chessboard, with $m \leq n$, is always possible unless at least one of these conditions are met:*

- (1) mn is odd;
- (2) $m \in \{1, 2, 4\}$;
- (3) $m = 3$ and $n \in \{4, 6, 8\}$.

We now describe the move function which defines the rules for the tour that we consider on the filled cells of a Heffter array. We remark that from now on the indices of the rows and of the columns are read with respect to their residue class modulo respectively the number of rows and columns of the array. For $(i, j) \in skel(A)$ we denote by $s_r((i, j)) = (i, j + k)$ the *row successor* of (i, j) , where $k \geq 1$ is the minimum integer such that $(i, j + k) \in skel(A)$. Similarly the *column successor* $s_c(i, j)$ is $(i + k, j)$, where $k \geq 1$ is the minimum integer such that $(i + k, j) \in skel(A)$. Note that this definition of a row and a column successor allow to see the array as embedded on a toroidal surface. We define the Crazy Knight's move function as:

$$KN_{RC}(i, j) = \begin{cases} s_c(s_r((i, j))) & \text{if } r_i = 1 \text{ and } c_{j'} = 1, \text{ where } s_r((i, j)) = (i, j'); \\ s_c(s_r^{-1}((i, j))) & \text{if } r_i = -1 \text{ and } c_{j'} = 1, \text{ where } s_r^{-1}((i, j)) = (i, j'); \\ s_c^{-1}(s_r((i, j))) & \text{if } r_i = 1 \text{ and } c_{j'} = -1, \text{ where } s_r((i, j)) = (i, j'); \\ s_c^{-1}(s_r^{-1}((i, j))) & \text{if } r_i = -1 \text{ and } c_{j'} = -1, \text{ where } s_r^{-1}((i, j)) = (i, j'). \end{cases}$$

Then, the Crazy Knight's tour problem is defined as follows [33]:

Crazy Knight's tour problem. Given an $m \times n$ array A , determine whether there exist $R \in \{-1, 1\}^m$ and $C \in \{-1, 1\}^n$ such that KN_{RC} is a tour.

Given an array A we denote by $P(A)$ the Crazy Knight's tour problem on A .

Example 3.5.2. Let A be the following 3×8 array, where a filled cell is denoted by the symbol \bullet :

\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet
\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet	\bullet

Then, a solution to $P(A)$ is given by $R = (1, 1, 1)$ and $C = (1, 1, 1, 1, 1, 1, 1, 1)$; here we report the resulting tour, where the (i, j) -th cell is filled with the smallest

integer k such that $(KN_{RC}(1,1))^{k-1} = (i,j)$:

	↓	↓	↓	↓	↓	↓	↓	↓
→	1	10	19	4	13	22	7	16
→	17	2	11	20	5	14	23	8
→	9	18	3	12	21	6	15	24

Note that this solution to $P(A)$ is equivalent to the compatible orderings of in Example 3.4.7.

It is easy to see that the Crazy Knight's move function is equivalent to the composition $\omega_c \circ \omega_r$ whenever ω_r and ω_c are the natural orderings (or their inverses) of the rows and columns of the Heffter array. A solution to $P(A)$ is equivalent to find two compatible orderings ω_r and ω_c , hence a biembedding of a complete (multipartite) graph on a orientable surface, whenever a globally simple (relative) Heffter array having the same set of filled cells can be provided.

It is clear that one should focus on the construction of solutions to the Crazy Knight's tour problem for partially filled matrices that have particular structures or symmetries. Moreover, by keeping in mind the classical constructions of Heffter arrays, two classes of arrays naturally arise for studying this problem: completely filled rectangular arrays and cyclically diagonal square arrays. As the name suggests a *completely filled* $m \times n$ array is an array A without empty cells, namely such that $skel(A) = [1, m] \times [1, n]$. We recall the definition of another class of arrays given in Section 1.2. An $n \times n$ array A is *cyclically k -diagonal* if its filled cells are precisely k consecutive diagonals: since the array is toroidal it is not restrictive to assume that the filled diagonals are the first k ones, namely $skel(A) = \bigcup_{i=1}^k D_i$.

In what follows we report some results obtained in [33] regarding the solution to the Crazy Knight's tour problem for completely filled and cyclically diagonal square matrices. We begin with the following necessary condition, see Theorem 2.7 of [33]:

Proposition 3.5.3. *Let A be an $m \times n$ array with $|skel(A)|$ filled cells. If there exists a solution to $P(A)$, then $|skel(A)| \equiv m + n - 1 \pmod{2}$.*

It has moreover been proven that the necessary conditions are also sufficient in the case of completely filled arrays, indeed (Theorem 3.3 of [33]):

Theorem 3.5.4. *Let A be an $m \times n$ totally filled array. Then there exists a solution to $P(A)$ if and only if m and n are not both even.*

The authors of [33] then focused on cyclically k -diagonal arrays of size n (notice that by Proposition 3.5.3 both n and k must be odd): in particular they studied the case when one of the two vectors R and C is identically equal to 1. This was done in order to obtain equivalent conditions for the existence of a solution to $P(A)$ that moreover rely on a covering problem of a smaller set of cells, as the next statement shows. We remark that in [33] the authors always considered $R \equiv 1$ and $C \in \{-1, 1\}^n$, but due to the natural symmetries of the array the same results hold when $C \equiv 1$ and $R \in \{-1, 1\}^n$. Given $R = (r_1, \dots, r_n) \in \{-1, 1\}^n$, let E be the set of indices $\{e_1, \dots, e_t\}$ with $e_1 < e_2 < \dots < e_t$ such that:

$$r_i = \begin{cases} -1 & \text{if } i \in E, \\ 1 & \text{if } i \notin E. \end{cases} \quad (3.5.1)$$

We then have the following necessary and sufficient conditions, see Lemma 4.7 of [33]:

Proposition 3.5.5. *Let $k \geq 3$ be an odd integer and let A be a cyclically k -diagonal array of odd size $n > k$. If E is a subset of $[1, n]$, then $C = (1, \dots, 1)$ and $R = (r_1, \dots, r_n) \in \{-1, 1\}^n$, where $r_i = -1$ if and only if $i \in E$, are a solution of $P(A)$ if and only if:*

- (1) E covers all the congruence classes modulo $\gcd(n, k - 1)$;
- (2) the orbit of $(1, 1)$ contains the set of cells $\{(e, e) | e \in E\}$.

In particular from Proposition 3.5.5 we can find a solution to $P(A)$ whenever n and $k - 1$ are coprime (see Proposition 4.8 of [33]):

Proposition 3.5.6. *Let $k \geq 3$ be an odd integer and let A be a cyclically k -diagonal array of size $n > k$. If n and $k - 1$ are coprime, then $R = (-1, 1, \dots, 1)$ and $C = (1, \dots, 1)$ are a solution to $P(A)$.*

We moreover report this extension theorem, that highly reduces the spectrum of cases to study:

Theorem 3.5.7. *Let $k \geq 3$ be an odd integer and let A be a cyclically k -diagonal array of size $n > k$. Assume that the vectors $C = (1, \dots, 1)$ and $R \in \{-1, 1\}^n$ are a solution of $P(A)$. Then there exists a solution of $P(A')$ for any cyclically k -diagonal array A' of size $n' = n + \lambda(k - 1)$ for any integer $\lambda \geq 0$.*

Through the aid of a computer, in [33] the authors were able to find solutions to the Crazy Knight's Tour Problem for cyclically k -diagonal arrays of size n for many pairs of odd integers (n, k) . It seems then natural to conjecture that for every pair of odd integer (n, k) , with $n \geq k \geq 3$, there exists a solution to $P(A)$. Moreover, from Theorem 3.5.7 it follows that for any fixed n it would be sufficient to find a solution to $P(A)$ whenever $k \in [\frac{n+1}{2}, n - 2]$.

In [68] we studied some instances of the Crazy Knight's tour problem for cyclically diagonal square matrices. Here we omit for brevity the proof of the following results, as they are quite technical and involved:

Proposition 3.5.8. *Let A be an $n \times n$ cyclically k -diagonal array and let $g = \gcd(n, k - 1)$. If $\gcd(g + 1, k - 1) = 2$, then there exists a solution to $P(A)$.*

By carefully choosing the parameters n and k we get the following corollary:

Corollary 3.5.9. *Let g , m and ℓ be positive integers such that:*

- (1) g and m are odd, ℓ is even, $m > \ell$ and $g, m, \ell > 1$;
- (2) $\gcd(g + 1, \ell) = 2$ and $\gcd(m, \ell) = 1$.

If $n = mg$ and $k = 1 + \ell g$, then there exists a solution to $P(A)$, where A is a cyclically k -diagonal $n \times n$ array.

Another significant case proved in [68] is the following one:

Proposition 3.5.10. *Let m and g be positive odd integers, $m, g \geq 3$. Let A be a cyclically k -diagonal $n \times n$ array, where:*

$$n = mg \quad k = 1 + (m - 1)g.$$

Then there exists a solution to $P(A)$.

We remark that from Corollary 3.5.9 and Proposition 3.5.10 we can construct solutions to the Crazy Knight's Tour Problem for infinite families of cyclically k -diagonal $n \times n$ arrays. Indeed, for instance from Corollary 3.5.9 one can choose $g + 1 = 2p$ for some odd prime p , and any even ℓ that is not divisible by p , while from Proposition 3.5.10 we only need $m, g \geq 3$ to be positive odd integers.

3.6 Automorphisms of the Archdeacon embedding

In this section we study the automorphism group of the embeddings of Archdeacon type, presenting some new results that we obtained in [36]. Informally speaking it can be easily proven that the constructed embeddings have numerous symmetries: in general this holds for derived graphs of current graphs. For this reason, in this context, such symmetries are said to be *trivial*; it is then natural to wonder if there could exist Archdeacon embeddings admitting non-trivial symmetries.

In this section we examine the basic definitions regarding the automorphisms of an embedding; we focus then on the automorphism group of the Archdeacon embeddings, showing that it always contains a cyclic group whose order can be derived from the parameters of the Heffter array from which the embedding is constructed. It has been proven in [32] that the automorphism group is almost always precisely this cyclic group. However there are examples where the Archdeacon embedding has many automorphisms: we then conclude this section by showing the construction of these embeddings.

We recall that two embeddings $\psi : \Gamma \rightarrow \Sigma$ and $\psi' : \Gamma' \rightarrow \Sigma'$ are isomorphic if there exists a graph isomorphism $\sigma : \Gamma \rightarrow \Gamma'$ such that F is a ψ -face if and only if $\sigma(F)$ is a ψ' -face. σ is an *automorphism* of the embedding if $\psi = \psi'$.

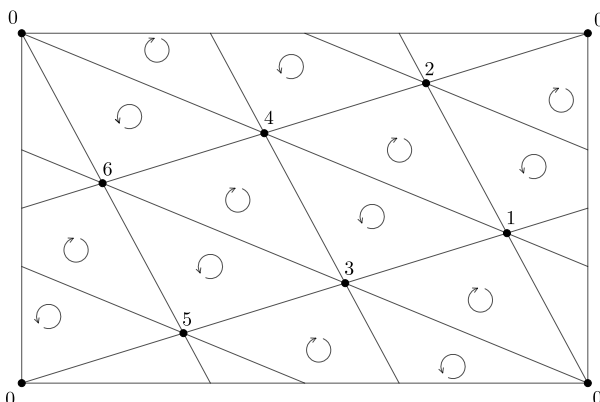
We moreover recall that an embedding can be equivalently seen as the pair $\Pi = (\Gamma, \rho)$, where Γ is a graph and ρ is a rotation of the directed edges of $\bar{\Gamma}$.

Definition 3.6.1. Given $\Gamma = (V, E)$ and $\Gamma' = (V', E')$, we say that two embeddings $\Pi = (\Gamma, \rho)$ and $\Pi' = (\Gamma', \rho')$ are *isomorphic* if there exists a graph isomorphism $\sigma : \Gamma \rightarrow \Gamma'$, called (*embedding*) *isomorphism*, such that for any $(x, y) \in D(\Gamma)$ we have either

- (1) $\sigma \circ \rho(x, y) = \rho' \circ \sigma(x, y)$, in which case σ is called *orientation preserving isomorphism*;
- (2) $\sigma \circ \rho(x, y) = (\rho')^{-1} \circ \sigma(x, y)$, in which case σ is called *orientation reversing isomorphism*.

If $\rho = \rho'$, then σ is an *automorphism* of Π .

Example 3.6.2. Consider the following embedding Π of K_7 on the torus having \mathbb{Z}_7 as vertex set, where we have highlighted the orientation of each face (see [91]):



It can be seen that the bijection $\sigma : x \mapsto x + 1$ is an automorphism of Π , that moreover preserves the orientation the faces of the embedding: for instance $(\sigma(0), \sigma(1), \sigma(3)) = (1, 2, 4)$ has the same clockwise orientation of $(0, 1, 3)$. Hence σ is an orientation preserving automorphism.

Consider now the bijection $\alpha : x \mapsto 3x$. It can be easily verified again that α is an automorphism of Π . However, α does not preserve the orientation of the faces: for instance $(\alpha(1), \alpha(4), \alpha(3)) = (3, 5, 2)$ has clockwise orientation, while $(1, 4, 3)$ has counterclockwise orientation. Hence α is an orientation reversing automorphism.

In what follows we examine embeddings arising from Heffter arrays, hence we identify the vertices of Γ with the elements of a cyclic group.

The set of automorphisms of an embedding $\Pi = (\Gamma, \rho)$ is denoted as $\text{Aut}(\Pi)$, and by $\text{Aut}^+(\Pi)$ we mean its subset of orientation-preserving automorphisms. Similarly, by $\text{Aut}_0(\Pi)$ and $\text{Aut}_0^+(\Pi)$ we respectively denote the subsets of automorphisms in $\text{Aut}(\Pi)$ and in $\text{Aut}^+(\Pi)$ that fix the vertex 0. By $\text{Aut}^-(\Pi)$ and $\text{Aut}_0^-(\Pi)$ we denote the orientation-reversing automorphisms of $\text{Aut}(\Pi)$ and $\text{Aut}_0(\Pi)$ respectively. Now, if \circ denotes the standard composition of functions, it can be easily shown that:

Proposition 3.6.3. *Let Π be an embedding on an orientable surface of a graph $\Gamma = (V, E)$, and let $0 \in V$ be a vertex. Then, $(\text{Aut}(\Pi), \circ)$ and $(\text{Aut}_0(\Pi), \circ)$ are groups. Moreover, $\text{Aut}^+(\Pi)$ and $\text{Aut}_0^+(\Pi)$ are, respectively, normal subgroups of $\text{Aut}(\Pi)$ and $\text{Aut}_0(\Pi)$ having index either 1 or 2.*

Proof. It is trivial to show that $(\text{Aut}(\Pi), \circ)$ and $(\text{Aut}_0(\Pi), \circ)$ are groups. Indeed, it is easy to see that the map $\psi_e : u \mapsto u$ for every vertex u of Γ is an automorphism (that fixes 0) and an identity for $(\text{Aut}(\Pi), \circ)$ (and for $(\text{Aut}_0(\Pi), \circ)$). For every $\psi \in \text{Aut}(\Pi)$ or $\psi \in \text{Aut}_0(\Pi)$, its unique inverse is simply $\eta : v \mapsto \psi^{-1}(v)$. The associativity follows from the associativity of the composition of functions.

Every orientable surface admits precisely two orientations, hence the index of $\text{Aut}^+(\Pi)$ in $\text{Aut}(\Pi)$ can only be 1 or 2, that imply that $\text{Aut}^+(\Pi)$ is normal in $\text{Aut}(\Pi)$. \square

Since in the embeddings of Archdeacon type the vertices of $\Gamma = (V, E)$ are identified with the elements of \mathbb{Z}_v for $v = |V|$, we define for every $g \in \mathbb{Z}_v$ the translation map $\tau_g(u) = u + g$. It is easy to verify that $(\{\tau_g : g \in \mathbb{Z}_v\}, \circ)$ is isomorphic to $(\mathbb{Z}_v, +)$, hence in what follows we directly identify $(\{\tau_g : g \in \mathbb{Z}_v\}, \circ)$

with $(\mathbb{Z}_v, +)$. We then report the following statement proven by Archdeacon in [6], that is a direct consequence of the construction of the embedding as the derived graph of a current graph.

Theorem 3.6.4. *Let H be an $H(m, n; h, k)$ Heffter array, and let $\Pi = (K_{2nk+1}, \rho)$ be an embedding constructed from H . Then \mathbb{Z}_{2nk+1} is a subgroup of $\text{Aut}(\Pi)$.*

It is then interesting to ask whether it is possible to construct embeddings admitting more symmetries. This question is justified by the following result, proven in [32], which determines that for an Archdeacon embedding Π the cyclic group of Theorem 3.6.4 is almost surely the automorphism group of Π .

Theorem 3.6.5. *Let $B \subset [1, m] \times [1, n]$ be the set of cells of an $m \times n$ array that contains exactly $h \geq 3$ and $k \geq 3$ cells in each row and column, respectively, admitting two compatible orderings α_r and α_c . Let A be an $m \times n$ matrix satisfying properties (a) and (b) of Definition 1.1.7 such that:*

- (1) $\text{skel}(A) = B$;
- (2) A contains a half-set of \mathbb{Z}_{2nk+1} .

Let ω_r and ω_c be the compatible orderings that correspond to α_r and α_c , and let Π be the embedding of K_{2nk+1} defined by A . Then the probability that $\text{Aut}(\Pi) \neq \mathbb{Z}_{2nk+1}$ is $\mathcal{O}(\frac{1}{nk})$.

In what follows we report the results given in [36], that prove the existence of an infinite family of examples of Archdeacon embeddings having automorphism group strictly larger than the cyclic group of Theorem 3.6.4, and in particular that is as large as possible. To do this we have to recall the concept of Heffter arrays over finite fields, introduced by Buratti in [19].

We begin with the following remark:

Remark 3.6.6. Let $\Pi = (K_v, \rho)$ be a biembedding of Archdeacon type where $v = 2nk + 1$. Then $\sigma \in \text{Aut}_0(\Pi)$ (respectively $\text{Aut}_0^+(\Pi)$) acts on $\mathbb{Z}_v \setminus \{0\}$ as an element of the dihedral group Dih_{2nk} having $2nk$ elements (respectively the cyclic group \mathbb{Z}_{2nk}).

More precisely, if we set $\rho_0 = (x_1, x_2, \dots, x_{2nk})$ and we read the indices modulo $2nk$ we have that:

- (1) given $\sigma \in \text{Aut}_0^+(\Pi)$

$$\sigma|_{K_v \setminus \{0\}} = \rho_0^\ell \text{ for some } \ell \in \{1, \dots, 2nk\};$$

- (2) given $\sigma \in \text{Aut}_0^-(\Pi)$

$$\sigma(x_j) = x_{\ell-j} \text{ for some } \ell \in \{1, \dots, 2nk\}.$$

In the remainder of this section we consider the variant of Heffter arrays recently introduced in [19] and discussed in Section 1.3.6, that are totally filled rank-one Heffter arrays over finite fields. We are interested in the simplest construction of these arrays, that is also shown in Section 1.2 and that we briefly recall here. Given two odd and coprime integers m, n such that $2mn + 1$ is a prime number, let ξ and ϵ respectively be an n -th and an m -th root of unity.

Then the array $A_{m,n}$ whose (i, j) -th cell is $a_{i,j} := \epsilon^{i-1}\xi^{j-1} \pmod{2mn+1}$ is a globally simple $H(m, n)$.

Moreover, since m and n are odd and coprime, by Theorem 3.5.4 there exists a pair of compatible natural orderings, hence there exists an automorphism $\Pi_{m,n}$ whose faces are simple cycle graphs of length m and n . From now on, the presented results have been obtained in a joint work with Costa [36].

Proposition 3.6.7. *Let $q = 2mn + 1$ be a prime power, with m, n odd and coprime, and let $\Pi_{m,n}$ be an embedding constructed from a pair of natural and compatible orderings of $A_{m,n}$. Then:*

$$\text{Aut}_0^+(\Pi_{m,n}) \cong \mathbb{Z}_{mn}.$$

Proof. We recall that the array $A_{m,n}$ has in position (i, j) the element $\epsilon^{i-1}\xi^{j-1}$, where ξ and ϵ have respectively order n and m in $\mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Given $1 \leq i' \leq m$ and $1 \leq j' \leq n$, we define the map $\lambda_{i',j'} : \mathbb{F}_q \rightarrow \mathbb{F}_q$ such that $\lambda_{i',j'}(z) = a_{i',j'}z = (\epsilon^{i'-1}\xi^{j'-1})z$. Clearly this map is a graph automorphism of K_q that fixes 0. In the following, we will denote for simplicity $a_{i',j'}$ by η .

We want to prove that $\lambda_{i',j'}$ is also an orientation preserving automorphism of $\Pi_{m,n}$, i.e. we need to check that given $(x, y) \in D(K_q)$ we have:

$$\lambda_{i',j'} \circ \rho(x, y) = \rho \circ \lambda_{i',j'}(x, y). \quad (3.6.1)$$

Because of the definition of Archdeacon embedding associated with ω_r and ω_c , we have that

$$\rho \circ \lambda_{i',j'}(x, y) = \rho(\eta x, \eta y) = \rho_{\eta x}(\eta x, \eta y) = \rho_{\eta x}(\eta x, \eta x + (\eta y - \eta x)).$$

Here we have two cases:

$$\rho \circ \lambda_{i',j'}(x, y) = \begin{cases} (\eta x, \eta x - \omega_r(\eta y - \eta x)) & \text{if } (\eta y - \eta x) \in \mathcal{E}(A); \\ (\eta x, \eta y + \omega_c(\eta x - \eta y)) & \text{otherwise.} \end{cases} \quad (3.6.2)$$

By considering the row indices modulo m and the column indices modulo n , and by recalling that $\eta = a_{i',j'} = \epsilon^{i'-1}\xi^{j'-1}$, we have

- (1) $\eta a_{i,j} = a_{i'+i-1, j'+j-1}$;
- (2) $\omega_r(a_{i,j}) = a_{i, j+1}$;
- (3) $\omega_c(a_{i,j}) = a_{i+1, j}$.

Hence, if $z = a_{i,j}$,

$$\eta \omega_r(z) = \eta \omega_r(a_{i,j}) = a_{i'+i-1, j'+j} = \omega_r(a_{i'+i-1, j'+j-1}) = \omega_r(\eta z).$$

Reasoning similarly on the columns of A , we also have that:

$$\eta \omega_c(z) = \omega_c(\eta z).$$

We now notice that $(y - x) \in \mathcal{E}(A)$ if and only if $(\eta y - \eta x) \in \mathcal{E}(A)$. Therefore, for $(y - x) \in \mathcal{E}(A)$ we have

$$\lambda_{i',j'} \circ \rho_x(x, x + (y - x)) = (\eta x, \eta x - \eta \omega_r(y - x)) = (\eta x, \eta x - \omega_r(\eta y - \eta x))$$

and, due to Equation (3.6.2),

$$(\eta x, \eta x - \omega_r(\eta y - \eta x)) = \rho_{\eta x}(\eta x, \eta x + (\eta y - \eta x)) = \rho \circ \lambda_{i', j'}(x, y).$$

Instead, if $(y - x) \notin \mathcal{E}(A)$, again because of Equation (3.6.2) we derive that

$$\lambda_{i', j'} \circ \rho_x(x, x + (y - x)) = (\eta x, \eta x + \eta \omega_c(x - y)) =$$

$$(\eta x, \eta x - \omega_c(\eta x - \eta y)) = \rho_{\eta x}(\eta x, \eta x + (\eta y - \eta x)) = \rho \circ \lambda_{i', j'}(x, y).$$

Hence, in both cases, Equation (3.6.1) is satisfied and $\lambda_{i', j'} \in \text{Aut}_0(\Pi_{m, n})$.

Since for every $\eta = a_{i', j'} = \epsilon^{i'-1} \zeta^{j'-1}$ the associated graph automorphism $\lambda_{i', j'}$ is an automorphism of the embedding $\Pi_{m, n}$, the automorphism group $\text{Aut}_0(\Pi_{m, n})$ contains a subgroup isomorphic to $\mathbb{Z}_m \oplus \mathbb{Z}_n = \mathbb{Z}_{mn}$ as m and n are coprime.

Now we prove that this is exactly the group of the orientation-preserving automorphisms that fix zero. We first recall from Remark 3.6.6 that $\text{Aut}_0^+(\Pi_{m, n})$ is isomorphic to a subgroup of the cyclic group of order $2mn$. Moreover, from the definition of the Π we deduce that for every $x \in \mathbb{F}_q^*$ the edge $\{0, x\}$ belongs to a face F_1 whose length is m , and to a face F_2 whose length is n . Since $m \neq n$, any element $\sigma \in \text{Aut}_0^+(\Pi_{m, n})$ must preserve the face-length. This means that σ is of the form $\rho_0^{2\ell}$ for some $\ell \in \{1, \dots, mn\}$. Therefore we have at most mn elements in $\text{Aut}_0^+(\Pi_{m, n})$.

It follows that $\text{Aut}_0^+(\Pi_{m, n})$ is isomorphic to \mathbb{Z}_{mn} . \square

Now we will show that these embeddings do not admit orientation-reversing automorphisms. We begin by proving a technical lemma:

Lemma 3.6.8. *Let Π be an Archdeacon embedding of K_{2mn+1} such that every edge is on a face whose boundary is an m -cycle and on a face whose boundary is a n -cycle, where $m \neq n$. Then any $\sigma \in \text{Aut}_0^-(\Pi)$ fixes only the vertex 0.*

Proof. Let Π , m and n be as in the statement, and choose any $\sigma \in \text{Aut}_0^-(\Pi)$. Set $\rho_0 = (x_1, x_2, \dots, x_{2mn})$ to be the local rotation around 0. By Remark 3.6.6, we have that $\sigma(x_j) = x_{\ell-j}$ for some $\ell \in \{1, \dots, 2mn\}$.

Assume now that σ fixes a vertex x_i for some $i \in \{1, \dots, 2mn\}$. Hence, the edge $\{0, x_i\}$ is fixed by the action of σ . Let F_1 and F_2 be the faces of length m and n respectively containing $\{0, x_i\}$. Since F_1 and F_2 have different lengths, it follows that both faces are pointwise fixed by the action of σ . In particular σ fixes x_{i+1} . On the other hand we have that $\sigma(x_i) = x_{\ell-i} = x_i$, that implies

$$\sigma(x_{i+1}) = x_{\ell-(i+1)} = x_{i-1} \neq x_{i+1},$$

where the last relation holds because $2mn > 3$. It follows that σ does not have any fixed element other than 0. \square

Proposition 3.6.9. *Let Π be an Archdeacon embedding of K_{2mn+1} such that every edge is on a face whose boundary is an m -cycle and on a face whose boundary is a n -cycle, where m and n are distinct odd integers. Then $\text{Aut}_0^-(\Pi)$ is empty.*

Proof. Let Π , m and n be as in the statement, and let σ be an automorphism in $\text{Aut}_0^-(\Pi)$. By Lemma 3.6.8 it follows that σ has no fixed vertices except 0, hence by Remark 3.6.6 we have $\sigma(x_i) = x_{\ell-i}$ for some $\ell \in \{1, \dots, 2mn\}$. Here ℓ must be odd, since otherwise we would have i such that $\ell - i \equiv i \pmod{2mn}$, and hence $\sigma(x_i) = x_{\ell-i} = x_i$. Given an odd $\ell \in \{1, \dots, 2mn\}$, the equation $\ell - i \equiv i + 1 \pmod{2mn}$ has two solutions in $\{1, \dots, 2mn\}$, thus there exist exactly two indices j and k such that $\sigma(x_j) = x_{j+1}$, $\sigma(x_{j+1}) = x_j$ and $\sigma(x_k) = x_{k+1}$, $\sigma(x_{k+1}) = x_k$.

Now we assume without loss of generality that $m > n$. We also assume that $n > 3$ (the case where $n = 3$ will be considered later).

Let F_1 be the face of length m containing the edge $\{x_j, x_{j+1}\}$. Since σ exchanges x_j and x_{j+1} , and F_1 is the unique face of length m containing these vertices, we have that F_1 is fixed by the action of σ . As m is odd exactly one vertex of F_1 is fixed by σ , and from Lemma 3.6.8 we deduce that this vertex is 0, hence $0 \in F_1$. Moreover, the vertices that are adjacent to 0 in F_1 are exchanged by the action of σ and, since $m > 3$ and F_1 is simple, they must be x_k and x_{k+1} . Similarly, if F_2 is the face of length $n > 3$ containing the edge $\{x_j, x_{j+1}\}$, we obtain again that $0 \in F_2$ and that x_k and x_{k+1} are adjacent to 0 in F_2 as well. We then gain a contradiction by noticing that $\rho_0(x_{k+1}) \neq x_k$ and hence the path $(x_k, 0, x_{k+1})$ can not be contained in two different faces.

Finally, let us suppose $n = 3$. In this case, since $m > n$, we still have that $(x_k, 0, x_{k+1})$ belongs to a face F_1 of length m . Here we have that, considering the face F_2 of length $n = 3$ that contains the edge $\{x_k, x_{k+1}\}$, σ must exchange x_k and x_{k+1} , and thus it fixes the third vertex of F_2 that must be 0. But this means that $(x_k, 0, x_{k+1})$ also belongs to a face F_2 of length $n = 3$. Since this path can not be contained in two different faces we obtain a contradiction also in this case.

It follows that $\text{Aut}_0^-(\Pi)$ is empty. \square

Remark 3.6.10. For every Archdeacon embedding Π and integers m, n that satisfy the hypothesis of Proposition 3.6.9, the following holds:

$$|\text{Aut}_0(\Pi)| = |\text{Aut}_0^+(\Pi)| \leq mn.$$

Indeed any automorphism that fixes zero is in $\text{Aut}_0^+(\Pi)$ and, since each edge belongs to two faces of different lengths, any element of $\text{Aut}_0^+(\Pi)$ is of the form $\rho_0^{2\ell}$ with $\ell \in \{1, \dots, mn\}$.

We are then able to derive the following result, that exactly determines the size of the full automorphism group of this class of Archdeacon embedding.

Theorem 3.6.11. *Let $q = 2mn + 1$ be a prime power, with m, n odd and coprime, and let $\Pi_{m,n}$ be an embedding constructed from a pair of natural and compatible orderings of $A_{m,n}$. Then:*

$$\text{Aut}_0(\Pi_{m,n}) \cong \mathbb{Z}_{mn} \text{ and } |\text{Aut}(\Pi_{m,n})| = \binom{2mn+1}{2}.$$

Moreover, the faces of $\Pi_{m,n}$ are simple cycles of length m and n .

Proof. The statement follows from Proposition 3.6.9. \square

Remark 3.6.12. This theorem shows that an Archdeacon embedding Π over \mathbb{F}_q can have a group $\text{Aut}_0(\Pi)$ whose size reaches the upper bound $mn = \frac{q-1}{2}$ of Remark 3.6.10, and we note that this upper bound can be chosen to be arbitrarily large. Indeed, the embeddings considered here have been obtained using Heffter arrays over the group \mathbb{F}_q , but if $q = p$ is a prime they can be obtained using classical Heffter arrays (over \mathbb{Z}_p). This happens infinitely many times since there are infinitely many primes p of the form $p = 2mn + 1$ where $m, n \geq 3$ are odd and coprime.

To prove this statement, it suffices to fix $n = 3$ and look for primes p of the form $6m + 1$ with m coprime with 3, that is $m \equiv 1, 2 \pmod{3}$. This is equivalent to look for $p \equiv 7, 13 \pmod{18}$, and it is well known from Dirichlet's Theorem that there are infinitely many primes in these congruence classes.

Example 3.6.13. Consider the pair $(m, n) = (3, 5)$. These two numbers are coprime and odd, and since $q = 2mn + 1 = 31$ is prime we can construct the following $H(3, 5)$:

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 8 & 16 \\ \hline 5 & 10 & 20 & 9 & 18 \\ \hline 25 & 19 & 7 & 14 & 28 \\ \hline \end{array}$$

Since 3 and 5 are coprime we can consider the natural orderings of each row from left to right, and of each column from top to bottom. From these orderings and the array A we can then construct an Archdeacon embedding $\Pi_{3,5}$ of the complete graph K_{31} , whose vertices are identified with the elements of \mathbb{Z}_{31} .

Starting from the cell filled with the element 1, we can write the rotation:

$$\begin{aligned} \rho_0 = (1, -2, 10, -20, 7, -14, 8, -16, 18, -5, 25, -19, 2, -4, 20, -9, \\ 14, -28, 16, -1, 5, -10, 19, -7, 4, -8, 9, -18, 28, -25). \end{aligned}$$

What can then be noticed is that ρ_0 is invariant under conjugation by λ_η where λ_η is the multiplication by an element $\eta \in \mathbb{Z}_{31}$ that is contained in A . Consider for example $\eta = 9 \in A$ and define $\lambda_\eta(x) = \eta x$. Then, if $\rho_0 = (x_1, \dots, x_{2mn})$, with $x_1 = 1$, it holds:

$$\begin{aligned} \lambda_\eta \circ \rho_0 = (1, -18, 4, -10, 16, -9, 2, -5, 8, -20)(-2, 28, -8, 19, -1, 14, -4, 25, -16, 7) \\ (5, -28, 20, -19, 18, -14, 10, -25, 9, -7) = \rho_0 \circ \lambda_\eta. \end{aligned}$$

That implies:

$$\lambda_\eta \circ \rho_0(x) = \rho_0 \circ \lambda_\eta(x).$$

and hence we can see that λ_η is an automorphism since

$$\begin{aligned} \lambda_\eta \circ \rho(x, y) = \lambda_\eta \circ \rho(x, x + (y - x)) = (\eta x, \eta x + \eta \rho_0(y - x)) = \\ (\eta x, \eta x + \rho_0(\eta y - \eta x)) = \rho \circ \lambda_\eta(x, y). \end{aligned}$$

The action of η can be also seen directly on every cell of the array A . Let $a_{i,j}$ denote the element of A in the (i, j) -th cell, and assume that $\eta = a_{i',j'}$ for some pair (i', j') . Then:

$$\eta a_{i,j} = a_{i'+i-1, j'+j-1},$$

where the row and column indices are viewed modulo m and n respectively. Let then $\eta A_{3,5}$ denote the array whose (i, j) -th cell is filled by $\eta a_{i,j}$.

For $\eta = a_{i',j'}$, and for any face F given by, say, the j -th column of $A_{3,5}$, we have that ηF is one of the faces obtained from the j -th column of $\eta A_{3,5}$. Moreover, this is also a face F' obtained from the $(j + j' - 1)$ -th column of $A_{3,5}$ consistently with the fact that λ_η maps faces into faces.

Here, setting again $\eta = 9 = 5^1 2^3$, we can see that:

$$\eta \cdot \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 8 & 16 \\ \hline 5 & 10 & 20 & 9 & 18 \\ \hline 25 & 19 & 7 & 14 & 28 \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 9 & 18 & 5 & 10 & 20 \\ \hline 14 & 28 & 25 & 19 & 7 \\ \hline 8 & 16 & 1 & 2 & 4 \\ \hline \end{array}.$$

As an example, we verify that a face of $\Pi_{3,5}$ is mapped into another face by the action of $\eta = 5^1 2^3$. For instance, we pick $F_1 = (0, 2, 12)$ that is obtained from the second column of $A_{3,5}$, and we consider $\eta F_1 = (0, 18, 15)$, that is the development of the second column of $\eta A_{3,5}$. We conclude by recognising that ηF_1 is also obtained by translating the development of the fifth column of $A_{3,5}$, indeed

$$\eta F_1 = (0, 18, 15) = (16, 3, 0) + 15$$

and $(0, 16, 3)$ is the face obtained as the development of the fifth column.

Chapter 4

Weak Heffter arrays

In this chapter we study a variant of the concept of Heffter array defined by Archdeacon in his seminal paper [6]. These arrays, called weak Heffter arrays, were introduced in order to study regular cycle decompositions and biembeddings of graphs over non-necessarily orientable surfaces. Informally speaking, weak Heffter arrays are similar to Heffter arrays, except that some cells are allowed to be read with distinct row and column sign. In the following sections we introduce the definition of weak and strictly weak Heffter arrays, and their generalization to weak Heffter arrays relative to a subgroup of a cyclic group (analogously to the generalization of classical Heffter arrays to relative Heffter arrays), as shown in [6, 37]. Moreover, we determine some necessary conditions, we provide constructions for these classes of arrays, and we investigate the relation between weak Heffter arrays and embeddings over non-necessarily orientable surfaces, that has been considered in [6] in terms of current graphs, and further studied in [37]. Excluding the original definition of weak Heffter arrays and their applications to non-orientable biembeddings, the results contained in this chapter are contained in a joint work with S. Costa and A. Pasotti, see [37].

4.1 Definition

In this section we recall the concept of weak Heffter arrays and we investigate their relation with classical Heffter arrays. Indeed some weak Heffter array can be constructed from a classical one by placing different row and column signs in a suitable set of cells of the array; however not all weak Heffter arrays can be constructed in this way. This induced the introduction in [37] of the concept of strictly weak Heffter arrays. In what follows we report these notions, together with some necessary conditions and existence results.

Remark 4.1.1. In this chapter, given an element x in an additive group a cell of an array can be filled with one of the following: $x, -x, \pm x, \mp x$, where the upper sign on \pm or \mp is the row sign and the lower sign is the column sign.

Definition 4.1.2. A *weak Heffter array* $\text{WH}(m, n; h, k)$ is an $m \times n$ matrix A such that:

- (a) each row contains h filled cells and each column contains k filled cells;

(b_{weak}) for every $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, there is exactly one cell of A whose element is one of $x, -x, \pm x, \mp x$;

(c) every row and column (with the corresponding sign) is zero-sum in \mathbb{Z}_{2nk+1} .

As for classical Heffter arrays if $m = n$ then $h = k$ and we use the notation $\text{WH}(n; k)$ instead of $\text{WH}(n, n; k, k)$. Furthermore a rectangular array with no empty cells $\text{WH}(m, n; n, m)$ is denoted by $\text{WH}(m, n)$.

Definition 4.1.3. An *integer* $\text{WH}(m, n; h, k)$ is a partially filled $m \times n$ array such that for every $\{1, \dots, mh\} \subset \mathbb{Z}$ there exists a unique cell whose element is one of $x, -x, \pm x, \mp x$, whose rows and columns are zero-sum, and satisfying property (a) of Definition 4.1.2.

As for classical Heffter arrays, an integer weak Heffter array can be seen as a weak Heffter array over a cyclic group.

Example 4.1.4. The following is an integer weak Heffter array $\text{WH}(4; 4)$ over \mathbb{Z}_{33} :

1	-2	-5	6
± 3	∓ 4	8	-7
-9	10	13	-14
11	-12	± 16	∓ 15

Clearly, given a Heffter array A if we replace all the elements a of A with $\pm a$ we trivially get a weak Heffter array. It is also easy to see that given a set \mathcal{R} of rows of A if we replace each element a of \mathcal{R} with $\mp a$ we get a weak Heffter array. Obviously a similar reasoning can be done on the columns.

In what follows, given an array A we respectively denote by R_i and C_j the i -th row and the j -th column of A . We point out that in this section, with a little abuse of notation, we identify a row (column) of a (weak) $\text{H}(m, n; h, k)$ with the h -subset (k -subset) of \mathbb{Z}_{2nk+1} whose elements are those of the given row (column).

The following result shows a way to get a weak Heffter array starting from a classical one, having few cells with different row and column signs.

Proposition 4.1.5. *If there exists an $\text{H}(m, n; h, k)$ with a row or a column containing a proper zero-sum subset (modulo $2nk + 1$), then there exists a $\text{WH}(m, n; h, k)$ whose number of cells containing an element with different row and column signs is at most equal to $\max\{\lfloor \frac{h}{2} \rfloor, \lfloor \frac{k}{2} \rfloor\}$.*

Proof. Let A be an array as in the statement. It is not restrictive to reason on a row. So let S be a proper zero-sum subset of a row R of A . Since every row is zero-sum also $R \setminus S$ is zero-sum, hence we can suppose that $|S| \leq \lfloor \frac{h}{2} \rfloor$. Now we replace each element in S by $\mp s$, and we leave unchanged all the other elements in A . Call B the new array. Clearly, $\text{supp}(A) = \text{supp}(B)$ and all the rows of B different from R are nothing but the rows of A , hence their sums are zero. About R , the sum is still zero since we have changed the signs of the elements of a subset of R whose sum is zero. Finally, note that the column signs are not changed, so the columns of B are zero-sum. \square

Example 4.1.6. Consider the following $H(8; 6)$ taken from Example 1.4 of [8]:

-1	5	2	-7	-9	10		
3	-4	-6	8	11	-12		
		-13	17	14	-19	-21	22
		15	-16	-18	20	23	-24
-33	34			-25	29	26	-31
35	-36			27	-28	-30	32
38	-43	-45	46			-37	41
-42	44	47	-48			39	-40

Note that for instance the proper subset $\{-1, -9, 10\}$ of the first row sums to 0. By applying the proof of the previous proposition we have the following $WH(8; 6)$:

± 1	5	2	-7	± 9	∓ 10		
3	-4	-6	8	11	-12		
		-13	17	14	-19	-21	22
		15	-16	-18	20	23	-24
-33	34			-25	29	26	-31
35	-36			27	-28	-30	32
38	-43	-45	46			-37	41
-42	44	47	-48			39	-40

We believe that it is natural to ask if some of the classical Heffter arrays have the property of Proposition 4.1.5. First of all note that since a row (column) of an $H(m, n; h, k)$ does not contain 0 nor 2-subsets of the form $\{x, -x\}$, if the property is satisfied by a row (column) then $h \geq 6$ ($k \geq 6$). We recall that in all the constructions of square integer Heffter arrays, see [8, 48], the authors obtain an $H(n; k + 4)$ starting from an $H(n; k)$ by adding to each row and each column four elements having sum zero. So the condition required by Proposition 4.1.5 is trivially satisfied. Hence we get the following.

Corollary 4.1.7. *For every $n > k \geq 3$, there exists an integer $WH(n; k)$ with exactly ℓ cells containing an element with different row and column signs, where*

- (1) $\ell = 3$ if $k \equiv 3 \pmod{4}$ and $n \equiv 0, 1 \pmod{4}$;
- (2) $\ell = 4$ if $k \equiv 0 \pmod{4}$;
- (3) $\ell = 4$ if $k \equiv 1 \pmod{4}$, $k \neq 5$ and $n \equiv 0, 3 \pmod{4}$;
- (4) $\ell = 4$ if $k \equiv 2 \pmod{4}$, $k \neq 6$ and n is even.

Proof. In each case there exists an integer Heffter array having a row or a column containing a (not necessarily proper) zero-sum subset of size ℓ . Hence the result follows by Proposition 4.1.5. In the following table we summarize where the results on classical Heffter arrays having the above property have been obtained. Note n and k represent congruence classes modulo 4.

$n \setminus k$	0	1	2	3
0	Theorem 2.1 [8]	Theorem 4.3 [48]	Theorem 2.1 [8]	Theorem 3.11 [8]
1	Corollary 2.4 [8]			Theorem 3.5 [8]
2	Theorem 2.1 [8]		Theorem 2.1 [8]	
3	Corollary 2.4 [8]	Theorem 3.3 [48]		

□

Remark 4.1.8. In the cases not considered by the constructions in the previous table, it is not immediate to understand if there exists a row or a column of the array containing a proper zero-sum subset. Hence one should check all the constructions details, but this is not the aim of this thesis.

Remark 4.1.9. Given a Heffter array A let \mathcal{S} be a set of rows and columns of A . For every row $R_i \in \mathcal{S}$, we replace each element $r \in R_i$ with $\mp r$ and for every column $C_j \in \mathcal{S}$, we replace each element $c \in C_j$ with $\pm c$. Note that, if an element x belongs to both a row R_i and a column C_j of \mathcal{S} , we replace it with $\pm(\mp x) = -x$. Then the result is a weak Heffter array. Clearly the set \mathcal{S} may contain no rows or no columns.

Example 4.1.10. Let A be the $H(8;6)$ of Example 4.1.6 and set $\mathcal{S} = \{R_3, R_4, C_6\}$. Following previous remark, starting from A we get the following $WH(8;6)$.

-1	5	2	-7	-9	± 10		
3	-4	-6	8	11	∓ 12		
		± 13	∓ 17	∓ 14	19	± 21	∓ 22
		∓ 15	± 16	± 18	-20	∓ 23	± 24
-33	34			-25	± 29	26	-31
35	-36			27	∓ 28	-30	32
38	-43	-45	46			-37	41
-42	44	47	-48			39	-40

It is really easy to see that one can apply on the same array Proposition 4.1.5 and Remark 4.1.9; for this reason we believe it is not necessary to write all the details, hence we only present this idea in the following example.

Example 4.1.11. Let again A be the $H(8;6)$ of Example 4.1.6. Note that $\{11, 14, -25\}$ is a zero-sum subset of C_5 . Set $\mathcal{S} = \{R_7, C_6\}$. Starting from A we get the following $WH(8;6)$.

-1	5	2	-7	-9	± 10		
3	-4	-6	8	± 11	∓ 12		
		-13	17	± 14	∓ 19	-21	22
		15	-16	-18	± 20	23	-24
-33	34			∓ 25	± 29	26	-31
35	-36			27	∓ 28	-30	32
∓ 38	± 43	± 45	∓ 46			± 37	∓ 41
-42	44	47	-48			39	-40

Clearly, not all weak Heffter arrays can be constructed starting from a classical one. Indeed the one shown in Example 4.1.4 cannot be obtained from a classical $H(3,4)$ by placing different row and column signs in a suitable set of cells. This induces us to introduce the following definition.

Definition 4.1.12. A weak $WH(m, n; h, k)$ is said to be *strictly weak* if it cannot be obtained from a classical $H(m, n; h, k)$ by placing different row and column signs in a suitable set of cells.

In other words, a weak Heffter array $A = (a_{ij})$ is strictly weak if there is no Heffter array $B = (b_{ij})$ with the same parameters such that, up to the sign, $a_{ij} = b_{ij}$ for every pair i, j . Hence it is easy to see that the $\text{WH}(3, 4)$ of Example 4.1.4 is strictly weak. The terminology *strictly weak* may induce the reader to believe it is easier to construct these arrays than the classical ones. However, as shown in what follows, this is not the case. Notably, for some values of n and k , an $\text{H}(n; k)$ exists while a strictly weak $\text{WH}(n; k)$ does not exist:

Remark 4.1.13. By Theorem 1.1.12 an $\text{H}(n; k)$ exists for any $n \geq k \geq 3$. On the other hand we have checked in [37], with the aid of a computer, that no strictly $\text{WH}(n; 3)$ exists when $n = 3, 4$.

Remark 4.1.14. We point out that if there is no weak Heffter array this means there is also no classical or strictly weak Heffter array with the same parameters. While if there is no *strictly* weak Heffter array, this gives no information about the existence of a classical or weak Heffter array with the same parameters.

We start with a preliminary consideration.

Proposition 4.1.15. *A weak Heffter array $\text{WH}(m, n; h, k)$ has either 0 or at least 3 cells containing distinct row and column signs.*

Proof. In this proof given an array A by a_{ij}^R (respectively, a_{ij}^C) we denote the element of A in the position (i, j) with its row (respectively, column) sign. Also, by $\sum A^R$ (respectively, $\sum A^C$) we mean the sum of all the elements in A with their row (respectively, column) sign. Set $I = \{(i, j) : a_{ij}^R \neq a_{ij}^C\}$. If A is a $\text{WH}(m, n; h, k)$, then the sum of all its elements is zero in \mathbb{Z}_{2nk+1} . Hence

$$\sum A^R = \sum A^C \equiv 0 \pmod{2nk+1},$$

which implies

$$2 \sum_{(i,j) \in I} a_{ij}^R \equiv 0 \pmod{2nk+1}.$$

From this, it follows that $|I| \neq 1$ since $0 \notin A$ and \mathbb{Z}_{2nk+1} does not contain the involution. Also $|I| \neq 2$ since A does not contain opposite elements and, again, the involution does not exist. The thesis follows. \square

In the remainder of this section we show some non-existence results for weak Heffter arrays and we compare the existence of classical and strictly weak Heffter arrays for the same given class of parameters.

Remark 4.1.16. A weak relative Heffter array cannot have exactly one cell with different row and column signs since it does not contain 0 and the involution (if it exists). On the other hand, a weak relative Heffter array may have exactly two cells with different row and column signs, as in the following $\text{WH}_{16}(4; 4)$ derived from Example 1.4 of [40]:

± 1	-7	-16	22
± 23	2	-8	-17
-13	19	4	-10
-11	-14	20	5

To present our non-existence results we have to recall that the rows and the columns of a (weak) Heffter array give two Heffter systems. It is then clear that given a subgroup J of \mathbb{Z}_v , the existence of a Heffter system on $\mathbb{Z}_v \setminus J$ is a necessary condition for the existence of a (weak) Heffter array on \mathbb{Z}_v relative to J . This fact leads us to generalize the statement of Lemma 3.3 of [40] and to obtain a non-existence result of weak Heffter arrays for an infinite family of parameters.

Lemma 4.1.17. *There is no Heffter system $D(9n; 3)$ over $\mathbb{Z}_{9n} \setminus \langle 3 \rangle$, for every $n \geq 3$.*

Proof. Assume that D is a Heffter system $D(9n; 3)$ over $\mathbb{Z}_{9n} \setminus \langle 3 \rangle$. If $\{a, b, c\}$ is a triple of D , then we have $\{a, b, c\} \subset \mathbb{Z}_{9n} \setminus \langle 3 \rangle = (\langle 3 \rangle + 1) \cup (\langle 3 \rangle + 2)$. Then, considering that $a + b + c = 0$, we have either $\{a, b, c\} \subset \langle 3 \rangle + 1$ or $\{a, b, c\} \subset \langle 3 \rangle + 2$. Since if we substitute $\{a, b, c\}$ with $\{-a, -b, -c\}$ we still have a Heffter system, it is not restrictive to assume that $\{a, b, c\} \subset \langle 3 \rangle + 1$.

Thus $\langle 3 \rangle + 1$ is partitioned by the triples of D contained in it and then, considering that each of these triples is zero-sum, $\langle 3 \rangle + 1$ should be zero-sum as well. On the other hand we have

$$\sum_{x \in \langle 3 \rangle + 1} x = \left(\sum_{g \in \langle 3 \rangle} g \right) + |\langle 3 \rangle| = \begin{cases} 9\frac{n}{2} + 3n \neq 0 & \text{if } n \text{ is even;} \\ 0 + 3n \neq 0 & \text{if } n \text{ is odd} \end{cases}$$

since, obviously, the sum of the elements of a cyclic group G is the involution or zero according to whether its order is even or odd, respectively. We got a contradiction. \square

Proposition 4.1.18. *There is no $\text{WH}_{3n}(n; 3)$ for every $n \geq 3$.*

Proof. The result immediately follows from the previous lemma. \square

We point out that there are examples where there exists exactly one Heffter system with some given parameters. Hence, also in these cases, it is not possible to construct a Heffter array, neither classical nor weak. Finally, the existence of at least two Heffter systems is not a sufficient condition for the existence of a Heffter array, see Proposition 4.1.21.

We present the following result whose proof relies on the fact that, if we are working in \mathbb{Z} or in \mathbb{Z}_v with v even, the array must contain an even number of odd numbers. This statement generalizes Proposition 3.1 of [40].

Proposition 4.1.19. *Suppose that there exists an integer $\text{WH}_t(n; k)$, or a non-integer $\text{WH}_t(n; k)$ with t even.*

(1) *If t divides nk , then*

$$nk \equiv 0 \pmod{4} \quad \text{or} \quad nk \equiv -t \equiv \pm 1 \pmod{4}.$$

(2) *If $t = 2nk$, then k must be even.*

(3) *If $t \neq 2nk$ does not divide nk , then*

$$t + 2nk \equiv 0 \pmod{8}.$$

Thanks to this proposition we establish a complete solution for the existence problem of a $\text{WH}_t(n; 3)$ for $n \in \{3, 4\}$.

Proposition 4.1.20. *There exists*

- (1) an $\text{H}_t(3; 3)$ if and only if $t = 1, 3, 6$;
- (2) a strictly weak $\text{WH}_t(3; 3)$ if and only if $t = 6$.

Proof. Since t has to divide $2nk = 18$, it may assume only values $1, 2, 3, 6, 9, 18$. We already know that a $\text{WH}_t(3; 3)$ does not exist for $t = 2$, see Proposition 4.1.19(3), for $t = 9$, see Proposition 4.1.18, and for $t = 18$, see Proposition 4.1.19(2). Hence for these values of the parameters not a classical or a strictly weak Heffter array can exist (see Remark 4.1.14).

(1) The existence of an $\text{H}_t(3; 3)$ is known for $t = 1$, see Theorem 1.1.12, and $t = 3$, see Theorem 1.5 of [40]. The following is an $\text{H}_6(3; 3)$:

1	2	-3
5	9	10
-6	-11	-7

(2) For $t = 1$ a strictly $\text{WH}_t(3; 3)$ does not exist by Remark 4.1.13. For $t = 3$ there are only the following four Heffter systems $D(21; 3)$ over $\mathbb{Z}_{21} \setminus \langle 7 \rangle$:

$$\begin{aligned} \mathcal{D}_1 &= \{\{1, 2, -3\}, \{4, 8, 9\}, \{5, 6, 10\}\}, \\ \mathcal{D}_2 &= \{\{1, 4, -5\}, \{2, 8, -10\}, \{3, 6, -9\}\}, \\ \mathcal{D}_3 &= \{\{1, 8, -9\}, \{2, 3, -5\}, \{4, 6, -10\}\}, \\ \mathcal{D}_4 &= \{\{1, 9, -10\}, \{2, 4, -6\}, \{3, 5, -8\}\}. \end{aligned}$$

We have checked, with the aid of a computer, that using them it is not possible to construct a strictly weak Heffter array, in other words that D_i and D_j are not orthogonal for any $i, j \in \{1, 2, 3, 4\}$. The source code is available at:

<https://www.combinatorics.org/ojs/index.php/eljc/article/view/v31i1p8/v31i1p8data>.

Finally, the following is a strictly $\text{WH}_6(3; 3)$:

1	2	-3
5	± 9	± 10
-6	∓ 7	-11

□

Proposition 4.1.21. *There exists*

- (1) an $\text{H}_t(4; 3)$ if and only if $t = 1, 2, 3, 4, 6$;
- (2) a strictly weak $\text{WH}_t(4; 3)$ if and only if $t = 2, 4$.

Proof. Since t has to divide $2nk = 24$, it may assume only values $1, 2, 3, 4, 6, 8, 12, 24$. We already know that a $\text{WH}_t(4; 3)$ does not exist for $t = 12$, see Proposition 4.1.18, and for $t = 24$, see Proposition 4.1.19(2). Furthermore for $t = 8$ there exists exactly one Heffter system $D(32; 3)$ over $\mathbb{Z}_{32} \setminus \langle 4 \rangle$, that is

$$\mathcal{D} = \{\{3, 10, -13\}, \{6, 5, -11\}, \{2, 7, -9\}, \{1, 14, -15\}\}.$$

Hence for all these values of the parameters neither a classical nor a strictly weak Heffter array can exist.

(1) The existence of an $H_t(4; 3)$ is known for $t = 1$, see Theorem 1.1.12, and $t = 3$, see Theorem 1.5 of [40]. The following are an $H_2(4; 3)$, an $H_4(4; 3)$ and an $H_6(4; 3)$:

-1	9	-8	
12	-2		-10
	-7	3	4
-11		5	6

-1	-5	6	
-10	2		8
11		13	4
	3	9	-12

1	-7	6	
-9		11	-2
	3	13	14
8	4		-12

(2) For $t = 1$ a strictly $WH_t(4; 3)$ does not exist by Remark 4.1.13. The following are a strictly $WH_2(4; 3)$ and a strictly $WH_4(4; 3)$:

1	2	-3	
4		∓ 10	6
-5	∓ 9		∓ 12
	-11	-7	∓ 8

1	-4	3	
	10	∓ 2	-8
∓ 12		-5	-11
-13	-6		-9

For $t = 3, 6$ there are respectively 9 and 10 Heffter systems $D(24 + t; 3)$ over $\mathbb{Z}_{24+t} \setminus \langle \frac{24+t}{t} \rangle$, and we have checked using a computer that it is not possible to construct a strictly $WH_t(4; 3)$ starting from them. \square

4.2 An infinite class of strictly weak Heffter arrays

In [40] the authors, after having introduced the concept of *relative* Heffter arrays, have obtained the following almost complete result for the integer square case with $t = k$.

Theorem 4.2.1. *Let $3 \leq k \leq n$ with $k \neq 5$. There exists an integer $H_k(n; k)$ if and only if one of the following holds:*

- (1) k is odd and $n \equiv 0, 3 \pmod{4}$;
- (2) $k \equiv 2 \pmod{4}$ and n is even;
- (3) $k \equiv 0 \pmod{4}$.

Furthermore, there exists an integer $H_5(n; 5)$ if $n \equiv 3 \pmod{4}$ and it does not exist if $n \equiv 1, 2 \pmod{4}$.

Note that for $k = 5$ the existence problem of integer relative Heffter arrays $H_5(n; 5)$ has been proved only for $n \equiv 3 \pmod{4}$, leaving the case $n \equiv 0 \pmod{4}$ open. For this class in [40] there are only two examples for $n = 8$ and $n = 16$.

We then report the result proven in [37] that there exists a strictly weak integer $WH_5(n; 5)$ for every $n \equiv 0 \pmod{4}$, with $n \geq 12$.

The construction we are going to present is based on filling in the cells of a set of consecutive diagonals. As described in Section 1.2, we recall that an array whose filled cells coincide with k consecutive diagonals is said to be *cyclically*

k-diagonal. Our construction is moreover based on the *diag* method to fill the diagonals of the array, that has been introduced in [48] and discussed in Section 1.2.

We remark that in Example 4.2.3 we follow step by step the construction illustrated in the proof of the following theorem. Hopefully it can help the reader to understand the idea developed in the proof.

Theorem 4.2.2. *There exists a strictly weak integer $\text{WH}_5(n; 5)$ for every $n \equiv 0 \pmod{4}$ with $n \geq 12$.*

Proof. We begin by considering the integer cyclically 3-diagonal $\text{H}_3(n; 3)$ with $n \equiv 0 \pmod{4}$ constructed in Proposition 5.3 of [40], so let A be the $n \times n$ array built using the following procedures labeled A to J:

$$\begin{aligned} \text{A} : \text{diag} \left(2, 2, 1, 1, 1, \frac{n-4}{2} \right); & \quad \text{B} : \text{diag} \left(\frac{n+6}{2}, \frac{n+6}{2}, -\frac{n+4}{2}, 1, -1, \frac{n-4}{2} \right); \\ \text{C} : \text{diag} \left(2, 1, -\frac{5n+4}{2}, 2, -1, \frac{n}{4} \right); & \quad \text{D} : \text{diag} \left(3, 2, -\frac{3n+2}{2}, 2, -1, \frac{n-4}{4} \right); \\ \text{E} : \text{diag} \left(1, 2, \frac{3n}{2}, 2, -1, \frac{n}{4} \right); & \quad \text{F} : \text{diag} \left(2, 3, \frac{5n+2}{2}, 2, -1, \frac{n-4}{4} \right); \\ \text{G} : \text{diag} \left(\frac{n+6}{2}, \frac{n+4}{2}, -\frac{5n}{4}, 2, 1, \frac{n}{4} \right); & \quad \text{H} : \text{diag} \left(\frac{n+8}{2}, \frac{n+6}{2}, -\frac{9n}{4}, 2, 1, \frac{n-4}{4} \right); \\ \text{I} : \text{diag} \left(\frac{n+4}{2}, \frac{n+6}{2}, \frac{11n+8}{4}, 2, 1, \frac{n}{4} \right); & \quad \text{J} : \text{diag} \left(\frac{n+6}{2}, \frac{n+8}{2}, \frac{7n+8}{4}, 2, 1, \frac{n-4}{4} \right). \end{aligned}$$

We also fill the following cells of A in an *ad hoc* manner:

$$\begin{aligned} A[1, 1] &= -\frac{n-2}{2}, & A\left[\frac{n}{2}, \frac{n}{2}\right] &= n, & A\left[\frac{n}{2}, \frac{n+2}{2}\right] &= \frac{7n+4}{4}, \\ A\left[\frac{n+2}{2}, \frac{n}{2}\right] &= -\frac{9n+4}{4}, & A\left[\frac{n+2}{2}, \frac{n+2}{2}\right] &= \frac{n+2}{2}, & A\left[\frac{n+2}{2}, \frac{n+4}{2}\right] &= \frac{7n}{4}, \\ A\left[\frac{n+4}{2}, \frac{n+2}{2}\right] &= -\frac{9n+8}{4}, & A\left[\frac{n+4}{2}, \frac{n+4}{2}\right] &= -\frac{n}{2}. \end{aligned}$$

Note that the filled cells of A are exactly those of the diagonals D_1 , D_2 and D_n . Also, it is easy to see that:

$$\begin{aligned} \text{supp}(D_1) &= [1, n] \\ \text{supp}(D_2 \cup D_n) &= [n+1, 3n+1] \setminus \{2n+1\}. \end{aligned}$$

Consider now the matrix B obtained from A by adding $4n+2$ to the positive elements of D_1 and $-(4n+2)$ to the negative elements of D_1 . Since we have only changed the elements in the main diagonal of A , it follows that the total sum of the i -th row of B is equal to the total sum of the i -th column of B , for every $i \in [1, n]$. In particular, their sum is $4n+2$ if $i \in [2, \frac{n}{2}+1]$, and $-(4n+2)$ if $i \in \{1\} \cup [\frac{n}{2}+2, n]$. Clearly now $\text{supp}(D_1) = [4n+3, 5n+2]$, while the support of D_2 and D_n is unchanged. Hence

$$\text{supp}(B) = [n+1, 2n] \cup [2n+2, 3n+1] \cup [4n+3, 5n+2].$$

Consider the following arrays:

$$M_1 = \begin{bmatrix} \pm 1 & 4n+1 \\ -4n & \mp 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \mp \left(\frac{n}{2}+1\right) & -\frac{7n}{2} \\ \mp \frac{7n}{2} + 1 & \pm \left(\frac{n}{2}+2\right) \end{bmatrix}$$

and for any $i \in \{3, 5, 7, \dots, \frac{n}{2}-1\} \cup \{\frac{n}{2}+3, \frac{n}{2}+5, \dots, n-1\}$, define M_i to be:

$$M_i = \begin{bmatrix} i+1 & 4n+1-i \\ 4n+2-i & i \end{bmatrix}.$$

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We now construct the array B obtained from A by adding $4n + 2 = 50$ to the positive elements of D_1 and -50 to the negative ones:

-55	18										-13
-32	51	31									
	-19	52	17								
		-33	53	30							
			-20	54	16						
				-34	62	22					
					-28	57	21				
						-29	-56	35			
							-15	-58	23		
								-27	-59	36	
									-14	-60	24
37										-26	-61

Consider the arrays M_i for $i \in \{1, 2\} \cup \{3, 5\} \cup \{9, 11\}$:

$$M_1 = \begin{bmatrix} \pm 1 & 49 \\ -48 & \mp 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \mp 7 & -42 \\ 43 & \pm 8 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 4 & 46 \\ 47 & 3 \end{bmatrix},$$

$$M_5 = \begin{bmatrix} 6 & 44 \\ 45 & 5 \end{bmatrix}, \quad M_9 = \begin{bmatrix} 10 & 40 \\ 41 & 9 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 12 & 38 \\ 39 & 11 \end{bmatrix}.$$

We then insert the elements of these arrays in B obtaining the new array C :

-55	18					± 1	49				-13
-32	51	31				-48	∓ 2				
	-19	52	17					∓ 4	∓ 46		
		-33	53	30				∓ 47	∓ 3		
			-20	54	16					∓ 6	∓ 44
				-34	62	22				∓ 45	∓ 5
∓ 7	-42				-28	57	21				
43	± 8					-29	-56	35			
		± 10	± 41				-15	-58	23		
		± 40	± 9					-27	-59	36	
				± 12	± 39				-14	-60	24
37				± 38	± 11					-26	-61

Notice that the ordered list of the total sum of the rows and that of the columns of C is:

$$(0, 0, 0, 0, 0, 0, +1, +1, +1, -1, +1, -1).$$

Hence, we need to apply the final readjustment of some elements of the main diagonal of C to gain the zero-sum property on every row and column (see the

elements in bold):

-55	18					± 1	49				-13
-32	51	31				-48	∓ 2				
	-19	52	17					∓ 4	∓ 46		
		-33	53	30				∓ 47	∓ 3		
			-20	54	16					∓ 6	∓ 44
				-34	62	22				∓ 45	∓ 5
∓ 7	-42				-28	56	21				
43	± 8					-29	-57	35			
		± 10	± 41				-15	-59	23		
		± 40	± 9					-27	-58	36	
				± 12	± 39				-14	-61	24
37				± 38	± 11					-26	-60

It can now be seen that the obtained array is a strictly $\text{WH}_5(12; 5)$.

4.3 Weak Heffter arrays and biembeddings

As shown in Section 3.1 an embedding of a graph Γ over a surface Σ is equivalent to a rotation system over Γ , that is an assignment of a rotation to each vertex and an orientation type (either 0 or 1) to each edge. Here, as done for the case of classical Heffter arrays, we construct the embedding as the derived graph of a current assignment on a particular graph Γ .

Before proceeding with the theory developed by Archdeacon in [6] regarding the connection between Heffter arrays and nonorientable embeddings, we discuss the construction of the derived embedding that can be obtained from the current graph on a nonorientable surface (see Section 4.4.5 of [54]). A current graph (Γ, ϕ, β) over a nonorientable surface having currents in a group G must satisfy the following property:

$$\beta(e^-) = \begin{cases} -\beta(e^+) & \text{if } e \text{ is of type 0,} \\ \beta(e^+) & \text{if } e \text{ is of type 1.} \end{cases}$$

The derived graph can be constructed as shown in Section 3.2. The derived embedding can be described by considering an extended version of the face-tracing algorithm: while tracing the faces of Γ following the standard algorithm, we specify at which corners $e_{i-1}e_i$ the corresponding vertex rotation is reversed. For a face f having boundary $e_1^{\epsilon_1} \dots e_n^{\epsilon_n}$, the rotation at the vertex (f, b) for every $b \in G$ is given by:

$$(e_1, b_1)^{\epsilon_1} (e_2, b_2)^{\epsilon_2} \dots (e_n, b_n)^{\epsilon_n},$$

where

- (1) $b_i = b$ if $\epsilon_i = +1$ and the rotation at the vertex incident to the edges $e_{i-1}e_i$ is not reversed, or if $\epsilon_i = -1$ and the rotation at the vertex incident to $e_i e_{i+1}$ is reversed;
- (2) $b_i = b - \beta(e^+)$ otherwise.

The edge type in the derived embedding of an edge (e, b) is defined so that (e, b) has type 0 if and only if both e^+ and e^- appear in the directed boundary walk of the current graph. Finally, the faces of the derived embedding are determined as in the case of an orientable embedding. In general the derived embedding obtained from a current graph over a nonorientable surface might be orientable: however, due to the application to weak Heffter arrays, we remark that if the group G has odd order, then the embedding of the derived graph is nonorientable if and only if the current graph is nonorientable.

As done in Section 3.4, for a graph $\Gamma = (V, E)$ with $|V| = v$ and for its directed version $\vec{\Gamma} = (V, D)$ we consider the following properties that are defined on a current graph (Γ, ϕ, α) , having currents in \mathbb{Z}_v :

- (1) **Kirchhoff's current law:** for every $u \in V$, $\sum_{e \in D_u} \alpha(e) = 0$.
- (2) **Signed-unique currents:** every element $x \in \mathbb{Z}_v \setminus \{0\}$ appears once as a current on a unique directed edge, unless x or $-x$ appears twice on an edge having type 1.
- (3) **Monofacial embedding:** the embedding constructed from (Γ, ϕ) has a single face.

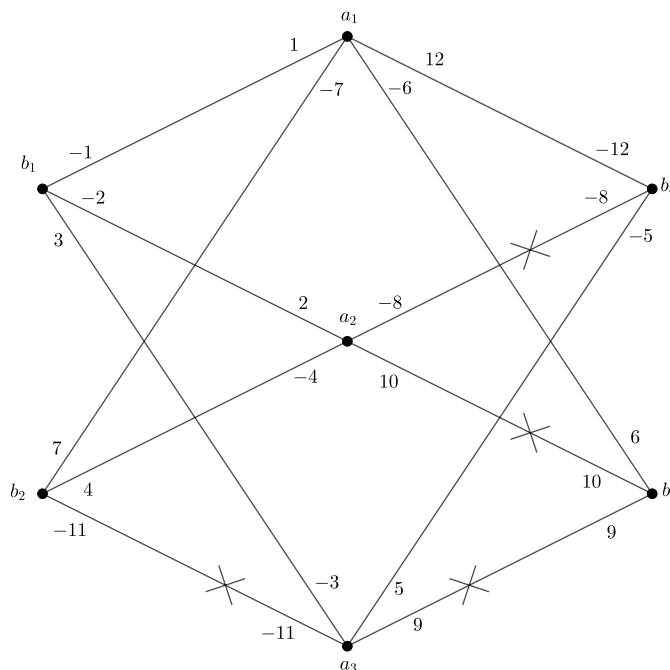
Archdeacon proved in [6] that:

Theorem 4.3.1. *Let $\Gamma = (V, E)$ be a bipartite graph having one color class with m vertices of degree s and the other with n vertices of degree t , for some $m \geq t$ and $n \geq s$ positive integers. Let (Γ, ϕ, α) be a current graph satisfying Conditions (1), (2), and (3). Then there is an embedding of K_{2ms+1} on a surface such that each edge lies on a face of length s and a face of length t . The surface is non-orientable if and only if the embedding (Γ, ϕ) is non-orientable. Moreover, the faces of K_{2ms+1} are simple cycles if each local rotation on (Γ, ϕ) is simple with respect to α .*

Proof. Analogous to the one of Theorem 3.4.2; the non-orientability of the surface follows directly from the non-orientability of the embedding (Γ, ϕ) , since \mathbb{Z}_{2ms+1} is a group of odd order. \square

Example 4.3.2. In this example we consider the embedding of Theorem 4.3.1 starting from a current graph satisfying Conditions (1), (2), and (3) and having the characteristics given in the statement of the theorem. We highlight that the same example was given in [6].

Let (Γ, ρ, α) be following current graph, where ρ is the counterclockwise rotation for each vertex and α maps the directed edges of $\vec{\Gamma}$ to elements of \mathbb{Z}_{25} (following [99], edges of Γ of type 1 are shown with a \times symbol in the middle):



The face-tracing algorithm gives the unique face f of the embedding:

$$(1, 3, -11, 7, 1, -2, -8, -5, -3, -2, -4, 7, \\ -6, 10, -4, -11, 9, 10, -8, -12, -6, 9, 5, -12).$$

By choosing $e_1 = (a_1, b_1)$ (and consequently e_2, e_3, \dots, e_{24}), for every pair of consecutive edges $e_i e_{i+1}$ in f we moreover write the sequence of reversed and not reversed corners that we respectively denote as $+1$ and -1 (informally speaking, the sign of the element of the sequence changes whenever we cross a type 1 edge in f):

$$\mu = (1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, 1, -1, -1, 1, -1, 1, -1, -1, -1, 1, 1, 1).$$

Since the embedding (Γ, ρ, α) is monofacial, the vertex set of the derived graph Γ_α is $\{f\} \times \mathbb{Z}_{25} \cong \mathbb{Z}_{25}$.

We now construct the edges (e, b) of Γ_α for the directed edge $e^+ = (a_1, b_1)$, where $\alpha(e^+) = 1$ (we remark that in this example we write the current assignment as a function on both e^+ and e^- , hence $\epsilon_i = 1$ for every directed edge e_i of f). A first observation shows that this edge is traversed twice in f , and the corresponding elements of the sequence μ have opposite signs: this implies that the corresponding edges of the derived graph have type 1. If we traverse e^+ with not reversed corners, the corresponding edges (e, g) in Γ_α are of the form $(g, g + 1)$ for $g \in \mathbb{Z}_{25}$, while traversing e^+ with reversed corners result in the edges having form $(g, g - 1)$ for $g \in \mathbb{Z}_{25}$.

Consider now the edge $e = \{b_1, a_3\}$ and say that $e^+ = (b_1, a_3)$ with $\alpha(e^+) = 3$: we can see that in f we traverse both e^+ and e^- , hence the corresponding edges in Γ_α have type 0. As done in Example 3.4.5 the corresponding edges of (e^+, g) and (e^-, g) are those of the form $(g, g + 3)$ and $(g, g - 3)$, respectively.

By repeating the same reasoning on every edge of f , one can see that any edge e of the current graph correspond to edges in the derived graph having difference $\pm\alpha(e^+)$, hence the derived graph is K_{25} : a directed edge e^+ is traversed twice in the same direction if and only if it is traversed once with reversed corners and once with not reversed corners.

To conclude we write the faces of the derived graph, that are obtained from a vertex of the current graph. Fix any vertex of Γ , say a_3 , and let $(9, 5, -3, -11)$ be its local rotation. As g ranges in \mathbb{Z}_{25} , the faces obtained from a_3 have as sequence of vertices

$$(g, g + 9, g + 9 + 5, g + 9 + 5 - 3).$$

It is immediate then to see that the faces are isomorphic to cycle graphs if the local rotation is simple, as in this case.

Before considering the analogous of Theorem 3.4.8 we need to modify the concept of compatible orderings given in Section 3.4, in order to adapt it for the context of weak Heffter arrays.

Let A be a $\text{WH}(m, n; h, k)$ and denote by $\Theta(A)$ the $m \times n$ array obtained by deleting from A the elements of type $\pm x$ or $\mp x$. We denote by $\Omega(A)$ the complement of $\Theta(A)$ in A . We note that

$$\mathbb{Z}_{2nk+1} \setminus \{0\} = \pm\mathcal{E}(\Theta(A)) \cup \mathcal{E}(\Omega(A)).$$

Therefore, we can define the map $\lambda : \mathbb{Z}_{2nk+1} \setminus \{0\} \rightarrow \{1, -1\}$ such that

$$\lambda(a) = \begin{cases} 1 & \text{if } a \in \pm\mathcal{E}(\Theta(A)); \\ -1 & \text{if } a \in \mathcal{E}(\Omega(A)). \end{cases}$$

Definition 4.3.3. Let A be a $\text{WH}(m, n; h, k)$ and let us consider the ordering ω_r for the rows and ω_c for the columns. Note that we can assume without loss of generality that the cell $(1, 1)$ is filled with the element a_1 , considered with its row sign. We define recursively

$$a_{i+1} = \begin{cases} -\omega_c^{\mu_i}(\lambda(a_i)a_i) & \text{if } i \text{ is odd,} \\ \omega_r^{\mu_i}(-\lambda(a_i)a_i) & \text{if } i \text{ is even;} \end{cases} \quad (4.3.1)$$

where $\mu_1 = \lambda(a_1)$ and

$$\mu_{i+1} = \begin{cases} \mu_i & \text{if } \lambda(a_{i+1}) = 1; \\ -\mu_i & \text{if } \lambda(a_{i+1}) = -1. \end{cases} \quad (4.3.2)$$

From now on we will use the notation (b_i) to indicate the sequence of elements (b_1, b_2, \dots) .

Proposition 4.3.4. *Given a $\text{WH}(m, n; h, k)$, the following facts are equivalent:*

- (1) *the sequence $((a_i, \mu_i))$ has period $2nk$;*
- (2) *the first $2nk$ elements of $(a_i\mu_i)$ are all distinct;*
- (3) *the sequence $(a_i\mu_i)$ has period $2nk$;*

(4) the sequence $((a_{2i+1}, \mu_{2i+1}))$ has period nk .

Proof. Let A be a $\text{WH}(m, n; h, k)$. Firstly, note that from Equation (4.3.1) it follows that:

$$a_{i-1} = \begin{cases} -\lambda(\omega_r^{-\mu_{i-1}}(a_i))\omega_r^{-\mu_{i-1}}(a_i) & \text{if } i \text{ is odd;} \\ \lambda(\omega_c^{-\mu_{i-1}}(-a_i))\omega_c^{-\mu_{i-1}}(-a_i) & \text{if } i \text{ is even.} \end{cases} \quad (4.3.3)$$

(1) \Rightarrow (2). Suppose by contradiction that (1) holds but (2) does not. This means that there exist i and j with $2nk \geq j > i \geq 1$ (which implies $j - i < 2nk$) such that $a_i\mu_i = a_j\mu_j$. Also, let i and j be indices with this property with minimum difference. Due to the definition of the sequences (a_i) and (μ_i) , if $a_i = a_j$, $\mu_i = \mu_j$ and $i \equiv j \pmod{2}$, then $T|(j - i)$ where $T = 2nk$ is the period of the sequence $((a_i, \mu_i))$. Moreover, because of the definition of (a_i) , if $a_i = -a_j$ we must have that $i \not\equiv j \pmod{2}$. In both cases, if $a_i\mu_i = a_j\mu_j$ and $j - i < 2nk$, then $i \not\equiv j \pmod{2}$. Furthermore, since in each row and each column we have more than one element, $j - i > 1$.

Here we have four possible cases:

- a) $(a_i, \mu_i) = (a_j, \mu_j)$, i is odd and j is even;
- b) $(a_i, \mu_i) = (a_j, \mu_j)$, i is even and j is odd;
- c) $(a_i, \mu_i) = (-a_j, -\mu_j)$, i is odd and j is even;
- d) $(a_i, \mu_i) = (-a_j, -\mu_j)$, i is even and j is odd.

Note that the first two cases can occur when $a_i = a_j \in \mathcal{E}(\Omega(A))$ while the latter two cases when $a_i = -a_j \in \pm\mathcal{E}(\Theta(A))$.

Case a). Assume that $(a_i, \mu_i) = (a_j, \mu_j)$ where i is odd and j is even. Then

$$a_{i+1} = -\omega_c^{\mu_i}(\lambda(a_i)a_i).$$

Since $\mu_{i+1} = \mu_i\lambda(-\omega_c^{\mu_i}(\lambda(a_i)a_i))$, it follows that

$$a_{i+1}\mu_{i+1} = -\omega_c^{\mu_i}(\lambda(a_i)a_i)\mu_i\lambda(-\omega_c^{\mu_i}(\lambda(a_i)a_i)).$$

Moreover, since $a_i \in \mathcal{E}(\Omega(A))$, $\lambda(a_i) = -1$, and hence we have that

$$a_{i+1}\mu_{i+1} = -\omega_c^{\mu_i}(-a_i)\mu_i\lambda(-\omega_c^{\mu_i}(-a_i)).$$

On the other hand, we have that

$$a_{j-1} = \lambda(\omega_c^{-\mu_{j-1}}(-a_j))\omega_c^{-\mu_{j-1}}(-a_j).$$

Also, since $a_j \in \mathcal{E}(\Omega(A))$, $\mu_j = -\mu_{j-1}$ and hence

$$a_{j-1}\mu_{j-1} = -\mu_j\lambda(\omega_c^{\mu_j}(-a_j))\omega_c^{\mu_j}(-a_j).$$

Finally from $(a_j, \mu_j) = (a_i, \mu_i)$ and $\lambda(-\omega_c^{\mu_i}(-a_i)) = \lambda(\omega_c^{\mu_i}(-a_i))$, it follows that:

$$a_{j-1}\mu_{j-1} = -\mu_j\lambda(\omega_c^{\mu_j}(-a_j))\omega_c^{\mu_j}(-a_j) = -\mu_i\lambda(-\omega_c^{\mu_i}(-a_i))\omega_c^{\mu_i}(-a_i) = a_{i+1}\mu_{i+1}.$$

Note that this is a contradiction since we are assuming that i and j are at a minimal distance and $j - i > 1$.

Case b). Assume that $(a_i, \mu_i) = (a_j, \mu_j)$ where i is even and j is odd. Then,

$$a_{i+1} = \omega_r^{\mu_i}(-\lambda(a_i)a_i).$$

Since $\mu_{i+1} = \mu_i \lambda(\omega_r^{\mu_i}(-\lambda(a_i)a_i))$ and $\lambda(a_i) = -1$, we have that

$$a_{i+1}\mu_{i+1} = \mu_i \lambda(\omega_r^{\mu_i}(a_i))\omega_r^{\mu_i}(a_i).$$

On the other hand we have that

$$a_{j-1} = -\lambda(\omega_r^{-\mu_{j-1}}(a_j))\omega_r^{-\mu_{j-1}}(a_j).$$

Also, since $a_j \in \mathcal{E}(\Omega(A))$, $\mu_j = -\mu_{j-1}$ and hence

$$a_{j-1}\mu_{j-1} = \mu_j \lambda(\omega_r^{\mu_j}(a_j))\omega_r^{\mu_j}(a_j).$$

Finally from $(a_j, \mu_j) = (a_i, \mu_i)$, it follows that:

$$a_{j-1}\mu_{j-1} = \mu_j \lambda(\omega_r^{\mu_j}(a_j))\omega_r^{\mu_j}(a_j) = \mu_i \lambda(\omega_r^{\mu_i}(a_i))\omega_r^{\mu_i}(a_i) = a_{i+1}\mu_{i+1}.$$

Note that this is a contradiction since we are assuming that i and j are at a minimal distance and $j - i > 1$.

Case c). Assume that $(a_i, \mu_i) = (-a_j, -\mu_j)$ where i is odd and j is even. Then

$$a_{i+1} = -\omega_c^{\mu_i}(\lambda(a_i)a_i).$$

Since $\mu_{i+1} = \mu_i \lambda(-\omega_c^{\mu_i}(\lambda(a_i)a_i))$ and $\lambda(a_i) = 1$, we have that

$$a_{i+1}\mu_{i+1} = -\omega_c^{\mu_i}(a_i)\mu_i \lambda(-\omega_c^{\mu_i}(a_i)).$$

On the other hand, we have that

$$a_{j-1} = \lambda(\omega_c^{-\mu_{j-1}}(-a_j))\omega_c^{-\mu_{j-1}}(-a_j).$$

Also, since $a_j \in \pm\mathcal{E}(\Theta(A))$, $\mu_j = \mu_{j-1}$ and hence

$$a_{j-1}\mu_{j-1} = \mu_j \lambda(\omega_c^{-\mu_j}(-a_j))\omega_c^{-\mu_j}(-a_j).$$

Finally from $(-a_j, -\mu_j) = (a_i, \mu_i)$ and $\lambda(-\omega_c^{\mu_i}(-a_i)) = \lambda(\omega_c^{\mu_i}(-a_i))$, it follows that:

$$a_{j-1}\mu_{j-1} = \mu_j \lambda(\omega_c^{-\mu_j}(-a_j))\omega_c^{-\mu_j}(-a_j) = -\mu_i \lambda(-\omega_c^{\mu_i}(a_i))\omega_c^{\mu_i}(a_i) = a_{i+1}\mu_{i+1}.$$

Note that this is a contradiction since we are assuming that i and j are at a minimal distance and $j - i > 1$.

Case d). Assume that $(a_i, \mu_i) = (-a_j, -\mu_j)$ where i is even and j is odd. Then,

$$a_{i+1} = \omega_r^{\mu_i}(-\lambda(a_i)a_i).$$

Since $\mu_{i+1} = \mu_i \lambda(\omega_r^{\mu_i}(-\lambda(a_i)a_i))$ and $\lambda(a_i) = 1$, we have that

$$a_{i+1}\mu_{i+1} = \mu_i \lambda(\omega_r^{\mu_i}(-a_i))\omega_r^{\mu_i}(-a_i).$$

On the other hand we have that

$$a_{j-1} = -\lambda(\omega_r^{-\mu_{j-1}}(a_j))\omega_r^{-\mu_{j-1}}(a_j).$$

Also, since $a_j \in \pm\mathcal{E}(\Theta(A))$, $\mu_j = \mu_{j-1}$ and hence

$$a_{j-1}\mu_{j-1} = -\mu_j\lambda(\omega_r^{-\mu_j}(a_j))\omega_r^{-\mu_j}(a_j).$$

Finally from $(-a_j, -\mu_j) = (a_i, \mu_i)$, it follows that:

$$a_{j-1}\mu_{j-1} = -\mu_j\lambda(\omega_r^{-\mu_j}(a_j))\omega_r^{-\mu_j}(a_j) = \mu_i\lambda(\omega_r^{\mu_i}(-a_i))\omega_r^{\mu_i}(-a_i) = a_{i+1}\mu_{i+1}.$$

Note that this is a contradiction since we are assuming that i and j are at a minimal distance and $j - i > 1$.

Therefore $a_i\mu_i = a_j\mu_j$ can hold only assuming $j \equiv i \pmod{2}$ and hence, as noted above, $(a_i, \mu_i) = (a_j, \mu_j)$. It follows that $(a_{i+\ell}, \mu_{i+\ell}) = (a_{j+\ell}, \mu_{j+\ell})$ for any positive integer ℓ , which means that $T|(j-i)$ where $T = 2nk$ is the period of $((a_i, \mu_i))$, but this is in contradiction with the assumption that $2nk \geq j > i \geq 1$. Thus, assuming (1), i.e. $T = 2nk$, the first $2nk$ elements of $(a_i\mu_i)$ are distinct.

(2) \Rightarrow (3). Assuming that (2) holds, the first $2nk$ elements of $(a_i\mu_i)$ are different. Hence the period T' of $(a_i\mu_i)$ is larger than or equal to $2nk$. Consider the element $a_{2nk+1}\mu_{2nk+1}$: since it is a non-zero element of \mathbb{Z}_{2nk+1} , it must be equal to $a_j\mu_j$ for some $j \in [1, 2nk]$. Moreover, due to the previous discussion, $a_{2nk+1}\mu_{2nk+1} = a_j\mu_j$ can occur only if $j \equiv 2nk + 1 \pmod{2}$ and $(a_i, \mu_i) = (a_j, \mu_j)$ which implies that $T|(2nk + 1 - j)$ where T is the period of $((a_i, \mu_i))$. Note that T is a multiple of the period T' of the sequence $(a_i\mu_i)$. It follows that

$$2nk \leq T' \leq T \leq 2nk + 1 - j \leq 2nk.$$

This can occur only if $j = 1$ and if the period of $(a_i\mu_i)$ is exactly $2nk$. Therefore also property (3) holds.

(3) \Rightarrow (1). By hypothesis, $a_1\mu_1 = a_{2nk+1}\mu_{2nk+1}$. Since also $2nk + 1 \equiv 1 \pmod{2}$, we have that $(a_1, \mu_1) = (a_{2nk+1}, \mu_{2nk+1})$ which implies that

$$(a_{1+\ell}, \mu_{1+\ell}) = (a_{2nk+1+\ell}, \mu_{2nk+1+\ell})$$

for any positive integer ℓ and hence the period T of $((a_i, \mu_i))$ is at most $2nk$. On the other hand, since T is a multiple of the period T' of $(a_i\mu_i)$, we have that

$$2nk \geq T \geq T' = 2nk.$$

It follows that T is exactly $2nk$ and hence (1) holds.

(1) \Rightarrow (4). Since the period T of $((a_i, \mu_i))$ is $2nk$, we have that (a_{2i+1}, μ_{2i+1}) has period T' that is a divisor of nk . On the other hand, given $(a_{2i+1}, \mu_{2i+1}) = (a_{2j+1}, \mu_{2j+1})$, since $2i+1 \equiv 2j+1 \pmod{2}$, we have that $(a_{2i+1+\ell}, \mu_{2i+1+\ell}) = (a_{2j+1+\ell}, \mu_{2j+1+\ell})$ for any positive integer ℓ . It follows that $2(j-i)$ is a multiple of the period $T = 2nk$ of $((a_i, \mu_i))$ and hence $(j-i) \geq nk$. Therefore the period of (a_{2i+1}, μ_{2i+1}) is exactly nk .

(4) \Rightarrow (1). Since the period T' of (a_{2i+1}, μ_{2i+1}) is nk , then the period T of (a_i, μ_i) must be a multiple of $2nk$. Moreover, since $(a_{2nk+1}, \mu_{2nk+1}) = (a_1, \mu_1)$, and $1 \equiv 2nk + 1 \pmod{2}$, we have that $(a_{2nk+1+\ell}, \mu_{2nk+1+\ell}) = (a_{1+\ell}, \mu_{1+\ell})$ for any positive integer ℓ . Therefore, the period of (a_i, μ_i) must be exactly $2nk$. \square

Now we can use the sequence $((a_i, \mu_i))$ in order to generalize the definition of compatible orderings given in [6].

Definition 4.3.5. Let A be a $\text{WH}(m, n; h, k)$ and let ω_r and ω_c respectively be an ordering for the rows and an ordering for the columns of A ; ω_r and ω_c are *compatible* whenever the conditions of Proposition 4.3.4 are satisfied.

Remark 4.3.6. If A is an $\text{H}(m, n; h, k)$, then the compatibility conditions of Proposition 4.3.4 are equivalent to the ones given in Section 3.4.

As a generalization of Theorem 3.4.8, Archdeacon showed in [6] the equivalence between the previous construction and weak Heffter arrays:

Theorem 4.3.7. *Let H be a $\text{WH}(m, n; s, t)$. Then H is equivalent to a current assignment α , satisfying Conditions (1) and (2) of Theorem 4.3.1, on a bipartite graph Γ having one color class with m vertices of degree s and the other with n vertices of degree t . Γ is connected if and only if H is not, up to row and column permutations, fully indecomposable.*

Two compatible orderings ω_r and ω_c of the rows and columns of H are equivalent to a monofacial rotation ρ on Γ . Moreover, if ω_r and ω_c are both simple, then ρ is simple with respect to α .

By combining Theorems 4.3.1 and 4.3.7 we can deduce that from a weak Heffter array admitting two compatible orderings (in the sense given in Definition 4.3.5) it is possible to construct an embedding of the complete graph. As done in Section 3.4 we call an embedding arising from a weak Heffter array *Archdeacon embedding* or *embedding of Archdeacon type*.

Example 4.3.8. Let (Γ, ρ, α) be the current graph of Example 4.3.2: it can be seen that the current graph (Γ, ρ, α) is equivalent to the following weak Heffter array over \mathbb{Z}_{25} :

$$H = \begin{array}{|c|c|c|c|} \hline 1 & -7 & -6 & 12 \\ \hline 2 & -4 & \pm 10 & \mp 8 \\ \hline -3 & \mp 11 & \pm 9 & 5 \\ \hline \end{array}$$

paired with the following row and column orderings:

$$\begin{aligned} \omega_r &= (1, -7, -6, 12)(2, -4, 10, -8)(-3, -11, 9, 5), \\ \omega_c &= (1, -3, 2)(-7, -11, -4)(-6, 10, 9)(12, -8, 5). \end{aligned}$$

Note that following the notation of the current graph given in Example 4.3.2, ω_r is the local rotation assigned to the vertices a_1, a_2 and a_3 ; if τ is the map on $\mathbb{Z}_{25} \setminus \{0\}$ defined as $\tau(x) = -x$ if $x \in \Theta(A)$, $\tau(x) = x$ if $x \in \Omega(A)$, then $\tau \circ \omega_c$ is the local orientation assigned to the vertices b_1, b_2, b_3 and b_4 . It can be seen that ω_r and ω_c are compatible, indeed the sequence (a_i, μ_i) of Definition 4.3.3 is:

$$\begin{aligned} &((1, 1), (-11, -1), (1, -1), (-8, 1), (-3, 1), (-4, 1), (-6, 1), \dots \\ &(-4, -1), (9, -1), (-8, -1), (-6, -1), (5, 1)) \end{aligned}$$

that has length 12.

As previously stated the embedding arising from a current graph having type 1 edges may or may not be orientable: in fact, a current graph (Γ, ϕ, α) is non-orientable if and only if it admits a cycle graph having an odd number of type 1 edges as a subgraph. In terms of the weak Heffter array, we then have the following equivalent condition:

Proposition 4.3.9. *The Archdeacon embedding arising from a weak Heffter array A is non-orientable if and only if there exists a sequence $(a_1, \dots, a_{2\ell})$ of filled cells such that:*

- (1) a_{1+2i} and a_{2i} are in the same row for each $i \in [0, \ell - 1]$;
- (2) a_{2i} and a_{2i+1} are in the same column, for each $i \in [1, \ell]$ (where by $a_{2\ell+1}$ we mean a_1);
- (3) $|\{a_1, \dots, a_{2\ell}\} \cap \Omega(A)|$ is odd.

In the following section we construct an infinite family of non-orientable Archdeacon embeddings, starting from a variation of the tour problem proposed in [33] and previously discussed in Section 3.5.

4.4 A tour problem on a weak Heffter array

In this section we discuss the analogous of the Crazy Knight's tour problem, introduced in [33] and studied in Section 3.5, for the case of Archdeacon embeddings constructed from weak Heffter arrays. We recall that the Crazy Knight's tour problem has been introduced as an equivalent problem to the one of finding compatible natural orderings on the filled cells of a partially filled array. For weak Heffter arrays the notion of compatible orderings is generalized in the sense given by Proposition 4.3.4; we show that it is necessary to define a more general formulation of the Crazy Knight's tour problem in order to include the broader definition of compatible orderings. Moreover, we construct an infinite family of weak Heffter arrays together with their relative compatible orderings, showing the existence of an infinite family of non orientable Archdeacon embeddings.

Given a $\text{WH}(m, n; h, k)$, say A , we recall that by r_i (respectively c_j) we denote the orientation of the i -th row (respectively j -th column), precisely $r_i = 1$ if it is from left to right and $r_i = -1$ if it is from right to left (respectively $c_j = 1$ if it is from top to bottom and $c_j = -1$ from bottom to top). Assume that the orientations $\mathcal{R} = (r_1, \dots, r_m)$ and $\mathcal{C} = (c_1, \dots, c_n)$ are fixed, and consider the following tour on two identical copies of the array A , that we denote by A_1 and by A_{-1} . We recall that by $\text{skel}(A)$ we denote the set of non-empty cells of A . For $t \in \{\pm 1\}$ we index the nonempty cells of A_t with the triples (i, j, t) , where $(i, j) \in \text{skel}(A)$. Finally, given a cell (i, j, t) in the array A_t , for $t \in \{\pm 1\}$ we consider the moves:

- 1) $L_{\mathcal{R}}(i, j, t)$ is the cell (i, j', t') where j' is the column index of the filled cell of the row R_i next to (i, j) in the orientation r_i^t and

$$t' = \begin{cases} t & \text{if } (i, j') \in \text{skel}(\Theta(A)) \\ -t & \text{otherwise.} \end{cases}$$

- 2) $L_{\mathcal{C}}(i, j, t)$ is the cell (i', j, t') where i' is the row index of the filled cell of the column C_j next to (i, j) in the orientation c_j^t and

$$t' = \begin{cases} t & \text{if } (i', j) \in \text{skel}(\Theta(A)) \\ -t & \text{otherwise.} \end{cases}$$

Assuming $(1, 1) \in \text{skel}(A)$ and setting

$$t = \begin{cases} 1 & \text{if } (1, 1) \in \text{skel}(\Theta(A)) \\ -1 & \text{otherwise} \end{cases}$$

we consider the list

$$L_{\mathcal{C}, \mathcal{R}} = ((1, 1, t), L_{\mathcal{R}} \circ L_{\mathcal{C}}(1, 1, t), \dots, (L_{\mathcal{R}} \circ L_{\mathcal{C}})^\ell(1, 1, t))$$

where ℓ is the minimum value such that $(L_{\mathcal{R}} \circ L_{\mathcal{C}})^{\ell+1}(1, 1, t) = (1, 1, t)$.

Generalized Crazy Knight's tour problem. Given an $m \times n$ weak Heffter array A , determine whether there exist $R \in \{-1, 1\}^m$ and $C \in \{-1, 1\}^n$ such that the list $L_{\mathcal{C}, \mathcal{R}}$ has length $|\text{skel}(A)|$.

As done for the Crazy Knight's tour problem, we denote by $P(A)$ the latter tour problem.

Example 4.4.1. Let A be again the WH(3, 4) of Example 4.1.4. Since the arrays A_1 and A_{-1} are copies of A , we have that

$$A_1 = A_{-1} = \begin{array}{|c|c|c|c|} \hline 1 & -7 & -6 & 12 \\ \hline 2 & -4 & \pm 10 & \mp 8 \\ \hline -3 & \mp 11 & \pm 9 & 5 \\ \hline \end{array}$$

We recall that a pair of compatible orderings for A is:

$$\begin{aligned} \omega_r &= (1, -7, -6, 12)(2, -4, 10, -8)(-3, -11, 9, 5), \\ \omega_c &= (1, -3, 2)(-7, -11, -4)(-6, 10, 9)(12, -8, 5). \end{aligned}$$

Consider the orientations $\mathcal{C} = (-1, -1, 1, 1)$ and $\mathcal{R} = (1, 1, 1)$ (note that these correspond respectively to ω_c and ω_r). The list $L_{\mathcal{C}, \mathcal{R}}$ is given by:

$$\begin{aligned} &((1, 1, 1), (3, 2, -1), (1, 1, -1), (2, 4, 1), (3, 1, 1), (2, 2, 1), (1, 3, 1), (2, 2, -1), \\ &(3, 3, -1), (2, 4, -1), (1, 3, -1), (3, 4, 1)). \end{aligned}$$

Since this list has length $12 = |\text{skel}(A)|$, it follows that $(\mathcal{C}, \mathcal{R})$ is a solution of $P(A)$.

We can represent the orientations and the tour directly on the arrays A_1 and A_{-1} as follows: here the arrows represent the orientations (\mathcal{C}, \mathcal{R} on A_1 and their opposites on A_{-1}), we have highlighted in grey the cells of $\text{skel}(\Omega(A))$ and the numbers represent the positions of the cells in the tour.

	↑	↑	↓	↓
→	1		7	
→		6	4	
→	5			12

	↓	↓	↑	↑
←	3		11	
←		8	10	
←		2	9	

In what follows we show that, as for the case of the classical Crazy Knight's tour problem, the generalized tour problem is equivalent to the existence of a pair of compatible natural orderings:

Remark 4.4.2. If A is a $\text{WH}(m, n; h, k)$ such that $P(A)$ admits a solution $(\mathcal{C}, \mathcal{R})$, then for each row R (respectively column C), consider ω_R to be the natural ordering if $r_i = 1$ (respectively $c_i = 1$) and its opposite otherwise. Set now $\omega_r = \omega_{R_1} \circ \cdots \circ \omega_{R_m}$ and $\omega_c = \omega_{C_1} \circ \cdots \circ \omega_{C_n}$. We consider now the sequence (a_i) defined as in Equation (4.3.1) starting from the orderings ω_c, ω_r associated with the orientations \mathcal{C} and \mathcal{R} . Because of the definition of $L_{\mathcal{C}, \mathcal{R}}$, the element (i, j, t) in the ℓ -th position of this list is such that the cell (i, j) of A , considered with its row sign (i.e. its upper sign), contains $a_{2\ell+1}$ and $\mu_{2\ell+1} = t$. It follows that, if the list $L_{\mathcal{C}, \mathcal{R}}$ has length $|\text{skel}(A)| = nk$, then the sequence $((a_{2i+1}, \mu_{2i+1}))$ has period nk . As shown in Proposition 4.3.4, this means that ω_r and ω_c are compatible.

As remarked in Example 4.3.2, directed edges in a current graph that are traversed twice in the same direction correspond to edges having type 1 in the derived graph. However, we still need to verify that in the derived graph there exists at least one cycle containing an odd number of edges having type 1. We prove then a condition that is equivalent to the orientability of the embedding, and that allows us to determine if an Archdeacon embedding is orientable only by looking at the compatible orderings of the weak Heffter array.

Theorem 4.4.3. *An Archdeacon embedding of the complete graph K_{2nk+1} is orientable if and only if every edge of K_{2nk+1} is of type 0.*

Proof. Suppose that every edge of K_{2nk+1} is of type 0. It follows that any cycle of K_{2nk+1} contains zero edges of type 1, hence by Theorem 3.1.7 the embedding is orientable.

Suppose that there is an edge of type 1, say $(\bar{x}, \bar{x} + a)$, and denote by ℓ the additive order of a in \mathbb{Z}_{2nk+1} ; as $|\mathbb{Z}_{2nk+1}|$ is odd we have that also ℓ is odd. Since the type of $(x, x + a)$ is independent from the value of x we have that also the edges $(\bar{x} + a, \bar{x} + 2a), (\bar{x} + 2a, \bar{x} + 3a), \dots, (\bar{x} + (\ell - 1)a, \bar{x})$ are of type 1.

It follows that the cycle

$$C := (\bar{x}, \bar{x} + a, \bar{x} + 2a, \dots, \bar{x} + (\ell - 1)a)$$

contains an odd number (i.e. ℓ) of edges of type 1. We deduce from Theorem 3.1.7 that the embedding is non orientable. \square

This principle can be generalized as follows, giving a non-orientability condition that directly refers to the problem $P(A)$.

Proposition 4.4.4. *Let A be a $\text{WH}(m, n; h, k)$, let $(\mathcal{C}, \mathcal{R})$ be a solution of $P(A)$ and let $\Pi = (K_{2nk+1}, \phi)$ be the associated Archdeacon embedding. If there exists (i, j) such that both $(i, j, 1)$ and $(i, j, -1)$ belong to $L_{\mathcal{C}, \mathcal{R}}$, then the embedding Π is non-orientable.*

Proof. Suppose that both $(i, j, 1)$ and $(i, j, -1)$ appear in the list $L_{\mathcal{C}, \mathcal{R}}$, and assume that $(i, j, 1)$ is in the ℓ -th position of this list and $(i, j, -1)$ is in the ℓ' -th one. Let (a_i) be the sequence defined as in Equation (4.3.1) with the orderings associated with the orientations \mathcal{C} and \mathcal{R} . Because of the definition of $L_{\mathcal{C}, \mathcal{R}}$ and since $(i, j, 1)$ is the ℓ -th element of this list, the element in the cell (i, j) of A considered with its row sign (i.e. its upper sign) is $a_{2\ell+1}$. On the other hand, since $(i, j, -1)$ is the ℓ' -th element in the list $L_{\mathcal{C}, \mathcal{R}}$, the element in the cell

(i, j) of A considered with its row sign is also $a_{2\ell'+1}$. Therefore we have at least one repetition in the sequence (a_{2i+1}) and hence there exists at least one edge e having type 1. It follows from Theorem 4.4.3 that the embedding Π is not orientable. \square

Example 4.4.5. Consider again the weak Heffter array A of Example 4.1.4. We have seen in Example 4.4.1 that the orientations $\mathcal{C} = (-1, -1, 1, 1)$ and $\mathcal{R} = (1, 1, 1)$ are a solution of $P(A)$. We recall that the list $L_{\mathcal{C}, \mathcal{R}}$ is given by:

$$\begin{aligned} &((1, 1, 1), (3, 2, -1), (1, 1, -1), (2, 4, 1), (3, 1, 1), (2, 2, 1), (1, 3, 1), (2, 2, -1), \\ &\quad (3, 3, -1), (2, 4, -1), (1, 3, -1), (3, 4, 1)). \end{aligned}$$

Since both $(1, 1, -1)$ and $(1, 1, 1)$ appear in $L_{\mathcal{C}, \mathcal{R}}$, it follows from Proposition 4.4.4 that this solution of $P(A)$ induces a non-orientable Archdeacon embedding of K_{25} .

Remark 4.4.6. As reported in Theorem 1.1.13 an $H(m, n)$ exists if and only if $m, n \geq 3$. In case $m = 3$ and $n \equiv 1 \pmod{8}$, the $H(3, n)$ presented in Theorem 3.2 of [7] contains the elements $-1, -4k - 5, 4k + 6$ (where $n = 8k + 9$) in the first row. Up to reordering the columns we may assume these elements are in the cells $(1, 1), (1, 2)$ and $(1, n)$. Now we can replace these elements with $\pm 1, \pm(4k + 5), \mp(4k + 6)$. Finally, transposing this array, we obtain a $WH(n, 3)$, say A , such that $skel(\Omega(A)) = \{(1, 1), (2, 1), (n, 1)\}$.

Proposition 4.4.7. *Let A be a $WH(n, 3)$ with $n \geq 4$ such that $skel(\Omega(A)) = \{(1, 1), (2, 1), (n, 1)\}$. Then there exists a solution $(\mathcal{C}, \mathcal{R})$ of $P(A)$ such that both $(1, 3, 1)$ and $(1, 3, -1)$ belong to $L_{\mathcal{C}, \mathcal{R}}$.*

Proof. Let us consider the orientations $\mathcal{C} = (1, -1, -1)$ and $\mathcal{R} = (-1, -1, \dots, -1)$. Since $(1, 1) \in skel(\Omega(A))$, the first element of the list $L_{\mathcal{C}, \mathcal{R}}$ is $(1, 1, -1)$ and $L_{\mathcal{R}} \circ L_{\mathcal{C}}(1, 1, -1) = (n, 3, 1)$. Then

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n, 3, 1) = (n - 1, 2, 1),$$

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n - 1, 2, 1) = (n - 2, 1, 1),$$

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n - 2, 1, 1) = (n - 1, 3, 1),$$

and, more in general, for any $\ell \in [1, n - 5]$

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n - \ell, 3, 1) = (n - 1 - \ell, 2, 1),$$

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n - 1 - \ell, 2, 1) = (n - 2 - \ell, 1, 1),$$

$$L_{\mathcal{R}} \circ L_{\mathcal{C}}(n - 2 - \ell, 1, 1) = (n - 1 - \ell, 3, 1).$$

It follows that the list $L_{\mathcal{C}, \mathcal{R}}$ has $(4, 3, 1)$ in position $3n - 10$ and, hence, we have that

$$\begin{aligned} L_{\mathcal{C}, \mathcal{R}} = &((1, 1, -1), (n, 3, 1), (n - 1, 2, 1), (n - 2, 1, 1), (n - 1, 3, 1), \dots, (4, 3, 1), \\ &(3, 2, 1), (2, 1, -1), (1, 3, 1), (n, 2, 1), (n - 1, 1, 1), (n, 2, -1), (1, 3, -1), (2, 1, 1), \\ &\quad (3, 3, 1), (2, 2, 1)). \end{aligned}$$

Here $(2, 2, 1)$ is the last element since $L_{\mathcal{R}} \circ L_{\mathcal{C}}(2, 2, 1) = (1, 1, -1)$. It is also easy to check that the list $L_{\mathcal{C}, \mathcal{R}}$ contains $3n$ distinct elements, and hence $(\mathcal{C}, \mathcal{R})$ is a solution of $P(A)$. Moreover both $(1, 3, 1)$ and $(1, 3, -1)$ belong to $L_{\mathcal{C}, \mathcal{R}}$. \square

Now we apply this solution to the problem $P(A)$ to obtain an infinite family of non-orientable embeddings.

Theorem 4.4.8. *Let $n \equiv 1 \pmod{8}$ be larger than 1. Then there exists a non-orientable embedding of K_{6n+1} of Archdeacon type.*

Proof. Due to Remark 4.4.6 for any $n \equiv 1 \pmod{8}$ with $n \geq 9$ there exists a $\text{WH}(n, 3)$, denoted by A , such that $\text{skel}(\Omega(A)) = \{(1, 1), (2, 1), (n, 1)\}$ and, because of Proposition 4.4.7, there exists a solution $(\mathcal{C}, \mathcal{R})$ of $P(A)$ such that both $(1, 3, 1)$ and $(1, 3, -1)$ belong to $L_{\mathcal{C}, \mathcal{R}}$.

It follows from Proposition 4.4.4 that this solution of $P(A)$ induces a non-orientable embedding of K_{6n+1} of Archdeacon type. \square

Chapter 5

Non-zero sum (relative) Heffter arrays

In this chapter we consider a variant of Heffter arrays called *non-zero sum (relative) Heffter arrays*, that has been recently introduced in [35] and studied in other subsequent papers, see [34, 69, 70]. As we will see these arrays were introduced due to their connection with Alspach's conjecture on partial sums, that is Conjecture 2.4.3 of this thesis. Moreover, as for classical (relative) Heffter arrays they have analogous applications to graph decompositions and face 2-colorable embeddings. While non-zero sum Heffter arrays are easier to construct compared to the classical Heffter arrays (namely being zero-sum), it remains a hard task to find a simple ordering of their rows and columns. For this reason a problem that has been considered since their introduction in [35] and later on studied in [69, 70] is the construction of *globally simple* non-zero sum Heffter arrays. Moreover, their easier construction allows to obtain quite complete existence results in many more general settings, like whenever the array is filled with a suitable subset of a (not necessarily abelian) group, or where the rows and columns of the array have a different number of filled cells. To conclude, we remark that many statements have been proved using both constructive and probabilistic approaches, showing that it is possible to apply different methods to this kind of problems. The results presented in this chapter are contained in a joint work with A. Pasotti, see [69].

5.1 Definition

Following the notation of [34] we give the notion of a non-zero sum (relative) Heffter array as an array filled with elements of an abelian group G , that we always assume to be written in additive notation whose neutral element is denoted as 0. We remark that the original definition of [35] was given for $G = \mathbb{Z}_v$. If the group is not specified it is assumed that the array is filled with the elements of a suitable cyclic group.

Definition 5.1.1. Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, where t divides $\frac{2nk}{\lambda}$. Let G be an abelian group of order v , let J be a subgroup of G of order t . A λ -fold non-zero sum Heffter array A over G relative to J , denoted by ${}^\lambda\text{NH}_t(m, n; h, k)$,

is an $m \times n$ partially filled array with elements in G such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b₃) the multiset $\{\pm x : x \in A\}$ contains each element of $G \setminus J$ exactly λ times;
- (c₁) the sum of the elements in every row and column is different from 0 in G .

As done for Heffter arrays, if $t = 1$ and $\lambda = 1$ we write $\text{NH}(m, n; h, k)$, while if the array is square, i.e. $m = n$ and $h = k$, then we write $\text{NH}_t(n; k)$. Notice that in this case the parameters h and k can be as low as 1 (while $h, k \geq 3$ for Heffter arrays).

Remark 5.1.2. The notion of λ -fold Heffter array was given in [41] as a generalization of the standard notion of (relative) Heffter arrays; in the previous chapters of this thesis we refrained to use this notion in order to simplify the notation and the definitions. However, we need this notion in order to introduce the concept of generalized Archdeacon arrays that is given in Chapter 6.

It is then clear that the existence problem of arrays having non-zero sum is easier than the one of arrays having sum equal to zero. Indeed, in the same article that introduced non-zero sum Heffter arrays the authors established constructively that (Corollary 3.5 and Theorem 4.2 of [35]):

Theorem 5.1.3. *There exists a cyclically k -diagonal globally simple $\text{NH}(n; k)$ if and only if $n \geq k \geq 1$.*

Theorem 5.1.4. *There exists a globally simple $\text{NH}(m, n; n, m)$ if and only if $m, n \geq 1$.*

Using a probabilistic approach the authors of [34] showed that:

Proposition 5.1.5. *Let G be a group of size $v = \frac{2nk}{\lambda} + t$ and let J be a subgroup of G of order t . Then there exists a ${}^\lambda\text{NH}_t(m, n; h, k)$ over G relative to J for every v even, $v \geq 42$ and for every v odd, $v \geq 29$.*

5.2 Constructions of globally simple $\text{NH}_t(n; n)$

In this section we construct examples of completely filled and square non-zero sum Heffter arrays; moreover, in Section 5.3 we compute the face length of the biembedding that can be constructed from these arrays. From now on all the results presented here are contained in [69].

The following lemma plays a fundamental role in showing that the arrays we are going to construct are globally simple. Its proof is trivial and hence it is left to the reader.

Lemma 5.2.1. *Let a, b and g be elements in an abelian group G , and let ℓ be a positive integer. Consider the orderings $\omega = (a, b, a + g, b - g, a + 2g, b - 2g, \dots, b - (\ell - 1)g, a + \ell g)$ and $\nu = (a, b, a + g, b - g, a + 2g, b - 2g, \dots, a + \ell g, b - \ell g)$. Then:*

- (1) *the unordered list of the partial sums of the ordering ω is*

$$S(\omega) = \{k(a + b) : 1 \leq k \leq \ell\} \cup \{a + k(a + b + g) : 0 \leq k \leq \ell\};$$

(2) the unordered list of the partial sums of the ordering ω^{-1} is

$$S(\omega^{-1}) = \{k(a+b+g) : 1 \leq k \leq \ell\} \cup \{a + \ell g + k(a+b) : 0 \leq k \leq \ell\};$$

(3) the unordered list of the partial sums of the ordering ν is

$$S(\nu) = \{k(a+b) : 1 \leq k \leq \ell+1\} \cup \{a + k(a+b+g) : 0 \leq k \leq \ell\};$$

(4) the unordered list of the partial sums of the ordering ν^{-1} is

$$S(\nu^{-1}) = \{k(a+b) : 1 \leq k \leq \ell+1\} \cup \{b - \ell g + k(a+b+g) : 0 \leq k \leq \ell\}.$$

We point out that all the following proofs are constructive, in fact we give direct constructions of a globally simple $\text{NH}_t(n; n)$ for suitable values of n and t , and we show that all the required properties are satisfied.

5.2.1 Globally simple $\text{NH}_2(n; n)$

The arrays described in the following proposition can be found in [35, Theorem 5.5], where a construction of a globally simple $\text{NH}(m, n; n, m)$ for every $m, n \geq 1$ is given. Here we show that for every integer $n = m$ the same array is also a globally simple $\text{NH}_2(n; n)$.

Proposition 5.2.2. *For every integer $n \geq 1$, there exists a globally simple $\text{NH}_2(n; n)$.*

Proof. Let $n \geq 1$ be an integer, and consider the $n \times n$ matrix $A = (a_{i,j})$ with elements in \mathbb{Z}_{2n^2+2} whose (i, j) -th entry is:

$$a_{i,j} = \varepsilon[j + (i-1)n],$$

where $\varepsilon = 1$ if $i \equiv j \pmod{2}$, and $\varepsilon = -1$ otherwise. Since the array is totally filled and, as shown in [35], $\text{supp}(A) = [1, n^2]$, conditions (a) and (b) of Definition 1.3.8 are satisfied.

It now remains to check that the array satisfies condition (c₁) of Definition 1.3.8 with $G = \mathbb{Z}_{2n^2+2}$ and it is globally simple. As proved in [35], the unordered list of the partial sums of the j -th column of A , for $j \in [1, n]$, is $S(C_j) = X_j \cup Y_j$, where:

$$X_j = \left\{ (-1)^j kn : k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right] \right\},$$

$$Y_j = \left\{ (-1)^{j+1}(j + kn) : k \in \left[0, \left\lfloor \frac{n-1}{2} \right\rfloor \right] \right\}.$$

It can be easily seen that X_j and Y_j are disjoint sets modulo $2n^2+2$, containing only non-zero elements.

Now, for every $i \in [1, n]$ the unordered list of partial sums of the i -th row of A is $S(R_i) = X_i \cup Y_i$, where:

$$X_i = \left\{ (-1)^i k : k \in \left[1, \left\lfloor \frac{n}{2} \right\rfloor \right] \right\},$$

$$Y_i = \left\{ (-1)^{i+1}(1 + (i-1)n + k) : k \in \left[0, \left\lfloor \frac{n-1}{2} \right\rfloor \right] \right\}.$$

As before, it can be seen that X_i and Y_i are disjoint sets modulo $2n^2 + 2$, and that no element in these sets is zero.

Hence, for every integer $n \geq 1$ the array is globally simple, so, in particular, every row and column has non-zero sum, proving that also condition (c_1) of Definition 1.3.8 is satisfied. \square

Remark 5.2.3. From the globally simple array A shown in Proposition 5.2.2, the total sum of the elements of each row and of each column can be easily found, depending on the case n even or odd. For every odd integer $n \geq 1$, considering the element of Y_j and the ones of Y_i for $k = \frac{n-1}{2}$, we get that for every $i, j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left(j + \frac{n-1}{2} n \right);$$

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i+1} \left((i-1)n + \frac{n+1}{2} \right).$$

Analogously, for every even integer $n \geq 1$, considering the element of X_j and that of X_i with $k = \frac{n}{2}$, we obtain that for every $i, j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^j \frac{n^2}{2};$$

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^i \frac{n}{2}.$$

Example 5.2.4. Following Proposition 5.2.2, we obtain the $NH_2(10; 10)$ below, filled with elements in \mathbb{Z}_{202} :

1	-2	3	-4	5	-6	7	-8	9	-10
-11	12	-13	14	-15	16	-17	18	-19	20
21	-22	23	-24	25	-26	27	-28	29	-30
-31	32	-33	34	-35	36	-37	38	-39	40
41	-42	43	-44	45	-46	47	-48	49	-50
-51	52	-53	54	-55	56	-57	58	-59	60
61	-62	63	-64	65	-66	67	-68	69	-70
-71	72	-73	74	-75	76	-77	78	-79	80
81	-82	83	-84	85	-86	87	-88	89	-90
-91	92	-93	94	-95	96	-97	98	-99	100

And similarly we obtain the $NH_2(11; 11)$ below, whose elements belong to \mathbb{Z}_{244} :

1	-2	3	-4	5	-6	7	-8	9	-10	11
-12	13	-14	15	-16	17	-18	19	-20	21	-22
23	-24	25	-26	27	-28	29	-30	31	-32	33
-34	35	-36	37	-38	39	-40	41	-42	43	-44
45	-46	47	-48	49	-50	51	-52	53	-54	55
-56	57	-58	59	-60	61	-62	63	-64	65	-66
67	-68	69	-70	71	-72	73	-74	75	-76	77
-78	79	-80	81	-82	83	-84	85	-86	87	-88
89	-90	91	-92	93	-94	95	-96	97	-98	99
-100	101	-102	103	-104	105	-106	107	-108	109	-110
111	-112	113	-114	115	-116	117	-118	119	-120	121

5.2.2 Globally simple $\text{NH}_{2n}(n; n)$

In this subsection we construct globally simple non-zero sum Heffter arrays relative to the subgroup of order $2n$.

Proposition 5.2.5. *For every odd integer $n \geq 1$ there exists a globally simple $\text{NH}_{2n}(n; n)$.*

Proof. Let $n \geq 1$ be odd and consider the $n \times n$ array A with elements in \mathbb{Z}_{2n^2+2n} whose (i, j) -th entry is:

$$a_{i,j} = \begin{cases} i + (n+1)(j-1) & \text{for } i \equiv j \pmod{2}, \\ n^2 + n - i - (n+1)(j-1) & \text{for } i \not\equiv j \pmod{2}. \end{cases}$$

A has no empty cells, thus condition (a) of Definition 1.3.8 is trivially satisfied. It can then be verified that, for $j \in [1, \frac{n-1}{2}]$, the support of the columns is:

$$\text{supp}(C_j \cup C_{n-j+1}) = [1 + (n+1)(j-1), n + (n+1)(j-1)] \cup [n^2 - (n+1)(j-1), n^2 + n - 1 - (n+1)(j-1)]$$

and that

$$\text{supp}\left(C_{\frac{n+1}{2}}\right) = \left[1 + \frac{n^2-1}{2}, n + \frac{n^2-1}{2}\right],$$

from which we obtain that $\text{supp}(A) = [1, n^2+n] \setminus \{n+1, 2(n+1), \dots, n(n+1)\}$.

We now examine the partial sums of the columns of A . For any odd $j \in [1, n]$ the entries of the j -th column $C_j = (c_{i,j})_{i \in [1, n]}$ of A are:

$$c_{i,j} = \begin{cases} i + (n+1)(j-1) & \text{if } i \text{ is odd,} \\ n^2 + n - i + (n+1)(j-1) & \text{if } i \text{ is even.} \end{cases}$$

Define now:

$$X = \left\{k(n^2 + n - 1) : k \in \left[1, \frac{n-1}{2}\right]\right\},$$

$$Y_j = \left\{1 + (n+1)(j-1) + k(n^2 + n + 1) : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 5.2.1(1) with $a = 1 + (n+1)(j-1)$, $b = n^2 + n - 2 - (n+1)(j-1)$, $g = 2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any column C_j with j odd, it results $S(C_j) = X \cup Y_j$. We notice that in \mathbb{Z}_{2n^2+2n} both X and Y_j are sets not containing zero. It remains to check that X and Y_j are disjoint. Assume by contradiction that there exist $k_1 \in [1, \frac{n-1}{2}]$, $k_2 \in [0, \frac{n-1}{2}]$ and an odd j such that:

$$k_1(n^2 + n - 1) \equiv 1 + (n+1)(j-1) + k_2(n^2 + n + 1) \pmod{2n^2 + 2n}.$$

It is easy to see that this would imply:

$$k_1 + k_2 + 1 \equiv 0 \pmod{n+1}.$$

However, by the hypothesis on k_1 and k_2 we have $1 \leq k_1 + k_2 \leq n-1$, so we obtain a contradiction. Then the partial sums of C_j for j odd are pairwise distinct, and non-zero.

It can be checked that for any even $j \in [1, n]$ the entries of the j -th column $C_j = (c_{i,j})_{i \in [1, n]}$ of A are:

$$c_{i,j} = \begin{cases} n^2 + n - i + (n+1)(j-1) & \text{if } i \text{ is odd,} \\ i + (n+1)(j-1) & \text{if } i \text{ is even.} \end{cases}$$

Now consider:

$$X = \left\{ k(n^2 + n + 1) : k \in \left[1, \frac{n-1}{2}\right] \right\},$$

$$Y_j = \left\{ n^2 - 2 - (n+1)(j-2) + k(n^2 + n - 1) : k \in \left[0, \frac{n-1}{2}\right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = n^2 + n - 1 - (n+1)(j-1)$, $b = (n+1)(j-1) + 2$, $g = -2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any column C_j , with j even, we have $S(C_j) = X \cup Y_j$. Reasoning as before, one can see that X and Y_j are disjoint sets not containing zero, hence the partial sums are pairwise distinct and non-zero.

Now we consider the partial sums of the rows. It is easy to see that for any odd $i \in [1, n]$ the elements of the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of the matrix are:

$$r_{i,j} = \begin{cases} i + (n+1)(j-1) & \text{if } j \text{ is odd,} \\ n^2 + n - i - (n+1)(j-1) & \text{if } j \text{ is even.} \end{cases}$$

Set:

$$X = \left\{ k(n^2 - 1) : k \in \left[1, \frac{n-1}{2}\right] \right\},$$

$$Y_i = \left\{ i + k(n^2 + 2n + 1) : k \in \left[0, \frac{n-1}{2}\right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = i$, $b = n^2 - 1 - i$, $g = 2n + 2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any row R_i , i odd, it results $S(R_i) = X \cup Y_i$. Since we are working in \mathbb{Z}_{2n^2+2n} , we can rewrite Y_i as:

$$Y_i = \left\{ i + k(1 - n^2) : k \in \left[0, \frac{n-1}{2}\right] \right\}.$$

Now, we have to show that $S(R_i)$ is a set and that $0 \notin S(R_i)$. We first notice that in \mathbb{Z}_{2n^2+2n} both X and Y_i are sets not containing zero. By way of contradiction, assume that for some $k_1 \in [1, \frac{n-1}{2}]$, $k_2 \in [0, \frac{n-1}{2}]$ and odd i we have:

$$k_1(n^2 - 1) \equiv i + k_2(1 - n^2) \pmod{2n^2 + 2n}.$$

This would imply that:

$$(k_1 + k_2)(n^2 - 1) \equiv (k_1 + k_2)(n-1)(n+1) \equiv i \pmod{2n^2 + 2n}.$$

It would then follow that i is a multiple of $n+1$; however, $i \in [1, n]$. We have then proven that all the partial sums are distinct and non-zero.

Finally it is not hard to see that for any even $i \in [1, n]$ the elements of the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A are defined as:

$$r_{i,j} = \begin{cases} i + (n+1)(j-1) & \text{if } j \text{ is odd,} \\ n^2 + n - i - (n+1)(j-1) & \text{if } j \text{ is even.} \end{cases}$$

Now, let:

$$X = \left\{ k(n^2 + 2n + 1) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_i = \left\{ n^2 + n - i + k(n^2 - 1) : \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = n^2 + n - i$, $b = n + 1 + i$, $g = -2n - 2$ and $\ell = \frac{n-1}{2}$, it can be seen that for any row R_i , i even, we have $S(R_i) = X \cup Y_i$. Since, as before, we are working in \mathbb{Z}_{2n^2+2n} , we can rewrite X as:

$$X = \left\{ k(1 - n^2) : k \in \left[1, \frac{n-1}{2} \right] \right\}.$$

With an argument similar to the previous one we can show that X and Y_i are disjoint sets not containing zero, proving that the rows of A admit a simple natural ordering. This concludes the proof. \square

Remark 5.2.6. Let A be the globally simple array described in Proposition 5.2.5. For every odd $j \in [1, n]$ looking at the element of the corresponding Y_j obtained for $k = \frac{n-1}{2}$ we get

$$\sum_{a_{i,j} \in C_j} a_{i,j} = j(n+1) + \frac{n^3 - 2n - 1}{2} = (n+1) \left(\frac{n^2 - n - 1}{2} + j \right);$$

while for every even $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = -j(n+1) + \frac{n^3 + 2n^2 + 2n + 1}{2} = (n+1) \left(\frac{n^2 + n + 1}{2} - j \right).$$

It can be seen that the previous expressions can be written in a more compact form, obtaining that for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (n+1) \left(\frac{n^2}{2} + (-1)^j \frac{n+1-2j}{2} \right).$$

Finally, for every $i \in [1, n]$, considering the element of the corresponding Y_i with $k = \frac{n-1}{2}$ we get

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{n^2 + n}{2} + (-1)^i \left(\frac{n^3 + 1}{2} - i \right).$$

Example 5.2.7. From Proposition 5.2.5 we obtain the $\text{NH}_{22}(11; 11)$ below, whose elements belong to \mathbb{Z}_{264} :

1	119	25	95	49	71	73	47	97	23	121
130	14	106	38	82	62	58	86	34	110	10
3	117	27	93	51	69	75	45	99	21	123
128	16	104	40	80	64	56	88	32	112	8
5	115	29	91	53	67	77	43	101	19	125
126	18	102	42	78	66	54	90	30	114	6
7	113	31	89	55	65	79	41	103	17	127
124	20	100	44	76	68	52	92	28	116	4
9	111	33	87	57	63	81	39	105	15	129
122	22	98	46	74	70	50	94	26	118	2
11	109	35	85	59	61	83	37	107	13	131

5.2.3 Globally simple $\text{NH}_{n^2}(n; n)$

In this subsection we construct globally simple square non-zero sum Heffter array relative to the subgroup of order n^2 .

Proposition 5.2.8. *For every odd integer $n \geq 1$, there exists a globally simple $\text{NH}_{n^2}(n; n)$.*

Proof. Let $n \geq 1$ be an odd integer, and consider the $n \times n$ matrix A with elements in \mathbb{Z}_{3n^2} whose (i, j) -th entry is:

$$a_{i,j} = \begin{cases} \varepsilon(3n(j-1) + 3(i-1) + 1) & \text{for } j \in \left[1, \frac{n+1}{2}\right], \\ \varepsilon(3nj - 3i + 1) & \text{for } j \in \left[\frac{n+3}{2}, n\right], \end{cases}$$

where $\varepsilon = 1$ if $i \equiv j \pmod{2}$ and $\varepsilon = -1$ otherwise. Condition (a) of Definition 1.3.8 is trivially verified. It is easy to see that for $j \in \left[1, \frac{n-1}{2}\right]$:

$$\text{supp}(C_j \cup C_{n-j+1}) = [1 + 3n(j-1), 3nj - 1] \setminus \{3k : k \in [n(j-1) + 1, nj - 1]\},$$

also,

$$\text{supp}\left(C_{\frac{n+1}{2}}\right) = \left[1 + 3n\frac{n-1}{2}, \frac{3n^2-1}{2}\right] \setminus \left\{3k : k \in \left[\frac{n^2-n+2}{2}, \frac{n^2-1}{2}\right]\right\}.$$

It then follows that condition (b) of Definition 1.3.8 is satisfied.

We now need to check that the property of being globally simple holds. It is easy to see that for any odd $j \in \left[1, \frac{n+1}{2}\right]$ the entries of the j -th column $C_j = (c_{i,j})_{i \in [1, n]}$ of A are:

$$c_{i,j} = (-1)^{i+1}(3n(j-1) + 3i - 2).$$

Define now:

$$X = \left\{-3k : k \in \left[1, \frac{n-1}{2}\right]\right\},$$

$$Y_j = \left\{3n(j-1) + 3k + 1 : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 5.2.1(1), with $a = 3n(j-1) + 1$, $b = -3n(j-1) - 4$, $g = 6$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. It can then be seen that X and Y_j are sets not containing 0. Note also that X and Y_j are trivially disjoint for every odd $j \in \left[1, \frac{n+1}{2}\right]$, as their elements belong to different equivalence classes modulo 3.

Now, for every odd $j \in \left[\frac{n+3}{2}, n\right]$ the elements in the j -th column $C_j = (c_{i,j})_{i \in [1, n]}$ of A are:

$$c_{i,j} = (-1)^{i+1}(3nj - 3i + 1).$$

Define now:

$$X = \left\{3k : k \in \left[1, \frac{n-1}{2}\right]\right\},$$

$$Y_j = \left\{3nj - 2 - 3k : k \in \left[0, \frac{n-1}{2}\right]\right\}.$$

By applying Lemma 5.2.1(1) with $a = 3nj - 2$, $b = -3nj + 5$, $g = -6$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. Clearly, X and Y_j are sets containing only non-zero elements. By following the same argument to the previous one, we have that they are disjoint for every odd $j \in [\frac{n+3}{2}, n]$.

It can be verified that for every even $j \in [1, \frac{n+1}{2}]$ the entries of the j -th column $C_j = (c_{i,j})_{i \in [1,n]}$ of A are:

$$c_{i,j} = (-1)^i(3n(j-1) + 3i - 2).$$

Define:

$$X = \left\{ 3k : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ -3n(j-1) - 1 - 3k : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = -3n(j-1) - 1$, $b = 3n(j-1) + 4$, $g = -6$ and $\ell = \frac{n-1}{2}$ it is easy to see that $S(C_j) = X \cup Y_j$. Moreover, X and Y_j are sets not containing zero that are disjoint for every even $j \in [1, \frac{n+1}{2}]$ since their elements belong to different equivalence classes modulo 3.

Finally, for every even $j \in [\frac{n+3}{2}, n]$ the elements of the j -th column $C_j = (c_{i,j})_{i \in [1,n]}$ of A are:

$$c_{i,j} = (-1)^i(3nj - 3i - 1).$$

Define then:

$$X = \left\{ -3k : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2 - 3nj + 3k : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = -3nj + 2$, $b = 3nj - 5$, $g = 6$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. As before, note that X and Y_j are disjoint sets containing only non-zero elements.

We have then proven that every column of the array admits a simple natural ordering.

Now, in order to check that the same property holds also for the rows we need to distinguish two cases according to the congruence class of n modulo 4.

Case 1: $n \equiv 1 \pmod{4}$.

Let $i \in [1, n]$ be odd, it can then be checked that the entries of the i -th row $R_i = (r_{i,j})_{j \in [1,n]}$ of A are:

$$r_{i,j} = \begin{cases} (-1)^{j+1}(3n(j-1) + 3i - 2) & \text{if } j \leq \frac{n+1}{2}, \\ (-1)^j(3n(n-j) + 3i - 1) & \text{otherwise.} \end{cases}$$

Define:

$$\begin{aligned} X &= \left\{ -3nk : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ Y_i &= \left\{ 3i - 2 + 3nk : k \in \left[0, \frac{n-1}{4} \right] \right\}, \\ Z_i &= \left\{ 3i - 2 + 3n \left(k + \frac{n-1}{4} \right) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ T_i &= \left\{ 6i - 3 + 3n \frac{3n-7}{4} - 3nk : k \in \left[0, \frac{n-5}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 5.2.1(1) to the first $\frac{n+1}{2}$ elements of R_i , with $a = 3i - 2$, $b = -3n - 3i + 2$, $g = 6n$ and $\ell = \frac{n-1}{4}$, we obtain that $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 5.2.1(4) to the remaining $\frac{n-1}{2}$ elements of R_i with $a = 1 - 3i$, $b = -1 + 3n + 3i$, $g = -6n$ and $\ell = \frac{n-5}{4}$, and adding the $\frac{n+1}{2}$ -th partial sum of R_i , that is $3i - 2 + 3n \frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It is easy to see that these are sets not containing zero; it then remains to check that they are pairwise disjoint. Notice that some of these sets are trivially disjoint modulo $3n^2$, since their elements belong to different equivalence classes modulo 3. Hence, it suffices to check that $X \cap T_i = \emptyset$ and $Y_i \cap Z_i = \emptyset$. In the first case assume by contradiction that there exist $k_1 \in [1, \frac{n-1}{4}]$, $k_2 \in [0, \frac{n-5}{4}]$ and i such that:

$$-3nk_1 \equiv 6i - 3 + 3n \frac{3n-7}{4} - 3nk_2 \pmod{3n^2}.$$

This can be rewritten as:

$$6i - 3 + 3n \left(\frac{3n-7}{4} + k_1 - k_2 \right) \equiv 0 \pmod{3n^2}.$$

By the hypothesis on k_1 and k_2 , the left-hand side is strictly positive in \mathbb{Z} and the element in brackets is less than or equal to $n - 2$. This fact, combined with $i \leq n$, proves that the left-hand side in \mathbb{Z} is an integer in $[1, 3n^2 - 3]$, so we get a contradiction. Now, if there exists i such that $Y_i \cap Z_i \neq \emptyset$, we would have:

$$3n \left(k_2 - k_1 + \frac{n-1}{4} \right) \equiv 0 \pmod{3n^2}$$

with $k_1 \in [0, \frac{n-1}{4}]$ and $k_2 \in [1, \frac{n-1}{4}]$. However, the term inside brackets is strictly positive and cannot exceed $n/2 - 1$, thus the equation cannot hold.

It can be checked that for every even $i \in [1, n]$ the elements in the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A are:

$$r_{i,j} = \begin{cases} (-1)^j (3n(j-1) + 3i - 2) & \text{if } j \leq \frac{n+1}{2}, \\ (-1)^{j+1} (3n(n-j) + 3i - 1) & \text{otherwise.} \end{cases}$$

Set:

$$\begin{aligned} X &= \left\{ 3nk : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ Y_i &= \left\{ 2 - 3i - 3nk : k \in \left[0, \frac{n-1}{4} \right] \right\}, \\ Z_i &= \left\{ 2 - 3i - 3n \left(k + \frac{n-1}{4} \right) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ T_i &= \left\{ 3 - 6i + 3n \frac{7-3n}{4} + 3nk : k \in \left[0, \frac{n-5}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 5.2.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = -1 - 3(i-1)$, $b = 1 + 3(i-1) + 3n$, $g = -6n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. By applying Lemma 5.2.1(4) to the remaining $\frac{n-1}{2}$ terms of R_i , with $a = -1 + 3i$, $b = 1 - 3n - 3i$, $g = 6n$ and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ -th partial sum of R_i that is $2 - 3i - 3n \frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It can be seen that these are sets containing only non-zero elements, thus it remains to check that they are pairwise disjoint modulo $3n^2$. Some pairs of these sets are trivially disjoint, as their elements belong to different equivalence classes modulo 3. We then need to verify $Y_i \cap Z_i = \emptyset$ and $X \cap T_i = \emptyset$. In the first case, suppose by contradiction that:

$$3n \left(\frac{n-1}{4} + k_2 - k_1 \right) \equiv 0 \pmod{3n^2}$$

for some $k_1 \in \left[0, \frac{n-1}{4} \right]$ and $k_2 \in \left[1, \frac{n-1}{4} \right]$. However, it can be seen that this cannot hold as $k_2 - k_1 \leq \frac{n-1}{4}$. In the second case, again by contradiction, suppose we have:

$$3nk_1 \equiv 3 - 6i + 3n \frac{7-3n}{4} + 3nk_2 \pmod{3n^2},$$

for some $k_1 \in \left[1, \frac{n-1}{4} \right]$, $k_2 \in \left[0, \frac{n-5}{4} \right]$ and for some even i . This is equivalent to:

$$3 - 6i + 3n \left(\frac{7-3n}{4} + k_2 - k_1 \right) \equiv 0 \pmod{3n^2}.$$

Now, it can be seen that the term inside the brackets is strictly negative in \mathbb{Z} , and is bounded from below by $2 - n$. By noticing that $i < n$, it can be seen that the left-hand side is a negative integer greater than $3 - 3n^2$, so we get a contradiction.

Case 2: $n \equiv 3 \pmod{4}$.

It is not hard to see that, since we are working modulo $3n^2$, for every odd $i \in [1, n]$ the elements of the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A are:

$$r_{i,j} = \begin{cases} (-1)^{j+1} (3n(j-1) + 3i - 2) & \text{if } j \leq \frac{n+1}{2}, \\ (-1)^j (3n(n-j) + 3i - 1) & \text{otherwise.} \end{cases}$$

Define:

$$\begin{aligned} X &= \left\{ -3nk : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\ Y_i &= \left\{ 3i - 2 + 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\ Z &= \left\{ -3n \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\ T_i &= \left\{ 1 - 3i - 3n \frac{3n-5}{4} + 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}. \end{aligned}$$

Now, by applying Lemma 5.2.1(3) to the first $\frac{n+1}{2}$ elements of R_i , with $a = 3i - 2$, $b = -3n - 3i + 2$, $g = 6n$ and $\ell = \frac{n-3}{4}$ we obtain that $X \cup Y_i \subset S(R_i)$. By applying Lemma 5.2.1(2) to the remaining $\frac{n-1}{2}$ elements with $a = 1 - 3i$, $b = -1 + 3n + 3i$, $g = -6n$ and $\ell = \frac{n-3}{4}$, and adding the $\frac{n+1}{2}$ -th partial sum of R_i that is $-3n \frac{n+1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z \cup T_i$. It can then be easily seen that these are sets not containing any zero element, thus it remains to check that they are pairwise disjoint. By considering the same sets written modulo 3, we only need to check that $X \cap Z = \emptyset$ and $Y_i \cap T_i = \emptyset$. In the first case suppose by contradiction that:

$$-3nk_1 \equiv -3n \left(k_2 + \frac{n+1}{4} \right) \pmod{3n^2},$$

for some $k_1 \in [1, \frac{n+1}{4}]$ and $k_2 \in [1, \frac{n-3}{4}]$. Clearly, this cannot hold since $k_2 - k_1 \leq \frac{n-7}{4}$. Suppose now $Y_i \cap T_i \neq \emptyset$, namely that

$$3i - 2 + 3nk_1 \equiv 1 - 3i - 3n \frac{3n-5}{4} + 3nk_2 \pmod{3n^2}$$

for some $k_1, k_2 \in [0, \frac{n-3}{4}]$ and some odd i . Then:

$$3 - 6i - 3n \left(\frac{3n-5}{4} - k_2 + k_1 \right) \equiv 0 \pmod{3n^2}.$$

It can then be seen that in \mathbb{Z} the left-hand side is strictly negative, and that the term in brackets cannot exceed $n - 2$. Since $i \in [1, n]$ the left-hand side is an integer contained in $[-3n^2 + 3, -1]$, hence we have a contradiction.

Recalling that we are working in \mathbb{Z}_{3n^2} one can check that for every even $i \in [1, n]$ the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A contains the elements:

$$r_{i,j} = \begin{cases} (-1)^j (3n(j-1) + 3i - 2) & \text{if } j \leq \frac{n+1}{2}, \\ (-1)^{j+1} (3n(n-j) + 3i - 1) & \text{otherwise.} \end{cases}$$

Define now:

$$\begin{aligned} X &= \left\{ 3nk : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\ Y_i &= \left\{ 2 - 3i - 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\ Z &= \left\{ 3n \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\ T_i &= \left\{ -1 + 3i + 3n \frac{3n-5}{4} - 3nk : k \in \left[0, \frac{n-3}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 5.2.1(3) to the first $\frac{n+1}{2}$ elements of R_i with $a = -1 - 3(i-1)$, $b = 1 + 3(i-1) + 3n$, $g = -6n$ and $\ell = \frac{n-3}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Finally, by applying Lemma 5.2.1(2) to the remaining $\frac{n-1}{2}$ terms, with $a = -1 + 3i$, $b = 1 - 3n - 3i$, $g = 6n$ and $\ell = \frac{n-3}{4}$, and adding the $\frac{n+1}{2}$ -th partial sum of R_i , that is $3n\frac{n+1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z \cup T_i$. We can see that these are sets containing only non-zero elements, so we only need to prove that they are pairwise disjoint. As before, it suffices to verify $X \cap Z = \emptyset$ and $Y_i \cap T_i = \emptyset$. In the first case, by means of contradiction, suppose

$$3nk_1 \equiv 3n \left(k_2 + \frac{n+1}{4} \right) \pmod{3n^2}$$

for some $k_1 \in [1, \frac{n+1}{4}]$ and $k_2 \in [1, \frac{n-3}{4}]$. This cannot hold since $k_2 - k_1 \leq \frac{n-3}{4}$. In the second case, by contradiction, assume that for some $k_1, k_2 \in [0, \frac{n-3}{4}]$ and even i it holds:

$$2 - 3i - 3nk_1 \equiv -1 + 3i + 3n\frac{3n-5}{4} - 3nk_2 \pmod{3n^2}.$$

This is equivalent to:

$$6i - 3 + 3n \left(\frac{3n-5}{4} + k_1 - k_2 \right) \equiv 0 \pmod{3n^2}.$$

It can be seen that the left-hand side is strictly positive in \mathbb{Z} ; since $i \leq n$ and $k_1 - k_2 \leq \frac{n-3}{4}$, the left-hand side is smaller than or equal to $3n^2 - 3$. Thus the previous equation cannot hold, proving that Y_i and T_i are disjoint sets.

We then have shown that for every odd n the rows and the columns of A admit a simple natural ordering. This concludes the proof. \square

Remark 5.2.9. For every $j \in [1, n]$ the sum of the elements of each column of the globally simple array A described in Proposition 5.2.8 is the following:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j-1} \left(3nj - \frac{3n+1}{2} \right).$$

Looking at the columns it is easy to see that for every $j \in [1, n]$ and for every $k \in [0, \lfloor \frac{n-5}{4} \rfloor]$ we have:

$$a_{1+2k,j} + a_{2+2k,j} = -(a_{n-2k,j} + a_{n-2k-1,j}).$$

Hence if $n \equiv 1 \pmod{4}$ the total sum of the j -th column is given by $a_{\frac{n+1}{2},j}$, while for $n \equiv 3 \pmod{4}$ the sum is given by $a_{\frac{n-1}{2},j} + a_{\frac{n+1}{2},j} + a_{\frac{n+3}{2},j}$.

Analogously it can be seen that for every $i \in [1, n]$ and for every $k \in [0, \lfloor \frac{n-5}{4} \rfloor]$ we have:

$$a_{i,1+2k} + a_{i,2+2k} = -(a_{i,n-2k} + a_{i,n-2k-1}).$$

We then obtain that for $n \equiv 1 \pmod{4}$ the total sum of the i -th row is given by $a_{i, \frac{n+1}{2}}$, while for $n \equiv 3 \pmod{4}$ the sum is given by $a_{i, \frac{n-1}{2}} + a_{i, \frac{n+1}{2}} + a_{i, \frac{n+3}{2}}$. Note also that $|a_{i, \frac{n-1}{2}} - a_{i, \frac{n+3}{2}}| = 1$.

Hence, for the rows we have that, if $n \equiv 1 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3i - 2 + 3n \frac{n-1}{2} \right) = a_{i, \frac{n+1}{2}};$$

while, if $n \equiv 3 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3(n+1-i) - 2 + 3n \frac{n-1}{2} \right) = a_{i, \frac{n+1}{2}} + (-1)^i.$$

Example 5.2.10. Following the proof of Proposition 5.2.8 we obtain the $\text{NH}_{121}(11; 11)$ below, filled with elements in \mathbb{Z}_{363} :

1	-34	67	-100	133	-166	-134	101	-68	35	-2
-4	37	-70	103	-136	169	137	-104	71	-38	5
7	-40	73	-106	139	-172	-140	107	-74	41	-8
-10	43	-76	109	-142	175	143	-110	77	-44	11
13	-46	79	-112	145	-178	-146	113	-80	47	-14
-16	49	-82	115	-148	181	149	-116	83	-50	17
19	-52	85	-118	151	-179	-152	119	-86	53	-20
-22	55	-88	121	-154	-176	155	-122	89	-56	23
25	-58	91	-124	157	-173	-158	125	-92	59	-26
-28	61	-94	127	-160	-170	161	-128	95	-62	29
31	-64	97	-130	163	-167	-164	131	-98	65	-32

5.2.4 Globally simple $\text{NH}_{2n^2}(n; n)$

In this subsection we present a construction for globally simple non-zero sum Heffter arrays relative to the subgroup of order $2n^2$.

Proposition 5.2.11. *For every odd integer $n \geq 1$, there exists a globally simple $\text{NH}_{2n^2}(n; n)$.*

Proof. Let $n \geq 1$ be odd, and consider the $n \times n$ matrix A with elements in \mathbb{Z}_{4n^2} whose (i, j) -th entry is:

$$a_{i,j} = \begin{cases} 2n(j-1) + 2i - 1 & \text{for } i \equiv j \pmod{2} \text{ and } j \in \left[1, \frac{n+1}{2}\right], \\ 2n(n-j+1) - 2i + 1 & \text{for } i \not\equiv j \pmod{2} \text{ and } j \in \left[1, \frac{n+1}{2}\right], \\ 2nj - 2i + 1 & \text{for } i \equiv j \pmod{2} \text{ and } j \in \left[\frac{n+3}{2}, n\right], \\ 2n(n-j) + 2i - 1 & \text{for } i \not\equiv j \pmod{2} \text{ and } j \in \left[\frac{n+3}{2}, n\right]. \end{cases}$$

Since A has no empty cells condition (a) of Definition 1.3.8 is trivially satisfied.

For $j \in \left[1, \frac{n-1}{2}\right]$, one can check that $\text{supp}(C_j \cup C_{n-j+1})$ contains exactly all the odd elements of $[1+2n(j-1), 2nj-1] \cup [2n^2+1-2nj, 2n^2-1-2n(j-1)]$, and that $\text{supp}(C_{\frac{n+1}{2}})$ contains exactly all the odd elements of $[n(n-1)+1, n(n+1)-1]$. Since we have to avoid the elements of the subgroup of order $2n^2$ of \mathbb{Z}_{4n^2} , condition (b) of Definition 1.3.8 is satisfied too.

Now we consider the partial sums of the columns of A . By the description of A it immediately follows that for every odd $j \in [1, \frac{n+1}{2}]$ we have that $C_j = (c_{i,j})_{i \in [1,n]}$ contains the following elements:

$$c_{i,j} = \begin{cases} 2n(j-1) + 2i - 1 & \text{if } i \text{ is odd,} \\ 2n(n-j+1) - 2i + 1 & \text{otherwise.} \end{cases}$$

Define now:

$$X = \left\{ k(2n^2 - 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 1 + 2n(j-1) + k(2n^2 + 2) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = 2n(j-1) + 1$, $b = 2n(n-j+1) - 3$, $g = 4$ and $\ell = \frac{n-1}{2}$, we obtain $S(C_j) = X \cup Y_j$. It can be checked that both X and Y_j are sets containing no zero elements, and they are trivially disjoint as their elements belong to different classes modulo 2. Similarly it can be seen that for every odd $j \in [\frac{n+3}{2}, n]$, the entries of the j -th column $C_j = (c_{i,j})_{i \in [1,n]}$ of A are:

$$c_{i,j} = \begin{cases} 2nj - 2i + 1 & \text{if } i \text{ is odd,} \\ 2n(n-j) + 2i - 1 & \text{otherwise.} \end{cases}$$

Set:

$$X = \left\{ k(2n^2 + 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2nj - 1 + k(2n^2 - 2) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = 2nj - 1$, $b = 2n(n-j) + 3$, $g = -4$ and $\ell = \frac{n-1}{2}$, we get $S(C_j) = X \cup Y_j$. It can then be easily verified that these are disjoint sets not containing zero.

Now, for any even $j \in [1, \frac{n+1}{2}]$ the entries of the j -th column $C_j = (c_{i,j})_{i \in [1,n]}$ of A are:

$$c_{i,j} = \begin{cases} 2n(n-j+1) - 2i + 1 & \text{if } i \text{ is odd,} \\ 2n(j-1) + 2i - 1 & \text{otherwise.} \end{cases}$$

Consider:

$$X = \left\{ k(2n^2 + 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2n(n-j+1) - 1 + k(2n^2 - 2) : k \in \left[0, \frac{n-1}{2} \right] \right\}.$$

By applying Lemma 5.2.1(1) with $a = 2n(n-j+1) - 1$, $b = 2n(j-1) + 3$, $g = -4$ and $\ell = \frac{n-1}{2}$ we have $S(C_j) = X \cup Y_j$. It can be seen that X and Y_j are sets containing exclusively non-zero elements, and as before X and Y_j are trivially disjoint.

Similarly, it can be seen that for every even $j \in [\frac{n+3}{2}, n]$ the j -th column $C_j = (c_{i,j})_{i \in [1,n]}$ of A has entries:

$$c_{i,j} = \begin{cases} 2n(n-j) + 2i - 1 & \text{if } i \text{ is odd,} \\ 2nj - 2i + 1 & \text{otherwise.} \end{cases}$$

By applying Lemma 5.2.1(1) with $a = 2n(n-j) + 1$, $b = 2nj - 3$, $g = 4$ and $\ell = \frac{n-1}{2}$, we have $S(C_j) = X \cup Y_j$, where:

$$X = \left\{ k(2n^2 - 2) : k \in \left[1, \frac{n-1}{2} \right] \right\},$$

$$Y_j = \left\{ 2n(n-j) + 1 + k(2n^2 + 2) : k \in \left[0, \frac{n-1}{2} \right] \right\},$$

which can be checked to be disjoint sets not containing zero elements.

It remains to consider the partial sums of every row. We have to split the proof into two cases according to the congruence class of n modulo 4.

Case 1: $n \equiv 1 \pmod{4}$.

For every odd $i \in [1, n]$ the entries of the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A are:

$$r_{i,j} = \begin{cases} 2n(j-1) + 2i - 1 & \text{if } j \leq \frac{n+1}{2} \text{ is odd,} \\ 2n(n-j+1) - 2i + 1 & \text{if } j \leq \frac{n+1}{2} \text{ is even,} \\ 2nj - 2i + 1 & \text{if } j \geq \frac{n+3}{2} \text{ is odd,} \\ 2n(n-j) + 2i - 1 & \text{if } j \geq \frac{n+3}{2} \text{ is even.} \end{cases}$$

Set:

$$X = \left\{ k(2n^2 - 2n) : k \in \left[1, \frac{n-1}{4} \right] \right\},$$

$$Y_i = \left\{ 2i - 1 + k(2n^2 + 2n) : k \in \left[0, \frac{n-1}{4} \right] \right\},$$

$$Z_i = \left\{ -1 + 2i + (2n^2 + 2n) \left(k + \frac{n-1}{4} \right) : k \in \left[1, \frac{n-1}{4} \right] \right\},$$

$$T_i = \left\{ 4i - 2 + \frac{n^3 + 2n^2 - 7n}{2} + k(2n^2 - 2n) : k \in \left[0, \frac{n-5}{4} \right] \right\}.$$

By applying Lemma 5.2.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = 2i - 1$, $b = 2n(n-1) - 2i + 1$, $g = 4n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. Then, by applying Lemma 5.2.1(4) to the remaining elements with $a = 2n^2 - 2i + 1$, $b = 2n + 2i - 1$, $g = -4n$ and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ -th partial sum of R_i , that is $2i - 1 + (2n^2 + 2n)\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. We can see that these are sets containing only non-zero elements. It remains to check that they are pairwise disjoint. By checking these sets modulo 2, it is easy to see that it suffices to verify that $Y_i \cap Z_i = \emptyset$ and $X \cap T_i = \emptyset$. In the first case assuming that there exists an element in the intersection, this would imply that for some $k_1 \in [0, \frac{n-1}{4}]$, $k_2 \in [1, \frac{n-1}{4}]$ and i odd the following holds:

$$2i - 1 + k_1(2n^2 + 2n) \equiv 2i - 1 + (2n^2 + 2n) \left(k_2 + \frac{n-1}{4} \right) \pmod{4n^2},$$

that is:

$$\left(k_2 + \frac{n-1}{4} - k_1 \right) (2n^2 + 2n) \equiv 0 \pmod{4n^2}.$$

Since we are working in \mathbb{Z}_{4n^2} , $2n^2(k_2 + \frac{n-1}{4} - k_1) \in \{0, 2n^2\}$ we get a contradiction by noticing that $k_2 + \frac{n-1}{4} - k_1 \leq \frac{n-1}{2}$.

In the second case, again by contradiction suppose we have:

$$k_1(2n^2 - 2n) \equiv 4i - 2 + \frac{n^3 + 2n^2 - 7n}{2} + k_2(2n^2 - 2n) \pmod{4n^2},$$

for some $k_1 \in [1, \frac{n-1}{4}]$, $k_2 \in [0, \frac{n-5}{4}]$ and some odd i . This can be rewritten as:

$$-2 + 4i + \frac{n^3 + 2n^2 - 7n}{2} + (k_2 - k_1)(2n^2 - 2n) \equiv 0 \pmod{4n^2}.$$

Note that, since $n \equiv 1 \pmod{4}$, we have that $n^3 = (4l+1)n^2 \equiv n^2 \pmod{4n^2}$, hence we can rewrite the previous expression modulo $2n^2$, obtaining:

$$-2 + 4i + \frac{3n^2 - 7n}{2} - 2n(k_2 - k_1) \equiv 0 \pmod{2n^2},$$

which is equivalent to:

$$-2 + 4i + 2n \left(\frac{3n-7}{4} - k_2 + k_1 \right) \equiv 0 \pmod{2n^2}.$$

After noticing that the left-hand side is strictly positive in \mathbb{Z} , we observe that the bracket is bounded from above by $n-2$, thus from $i \leq n$, the left-hand side seen in \mathbb{Z} is less than or equal to $2n^2 - 2$, that is a contradiction.

For every even $i \in [1, n]$ the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A has as entries:

$$r_{i,j} = \begin{cases} 2n(n-j+1) - 2i + 1 & \text{if } j \leq \frac{n+1}{2} \text{ is odd,} \\ 2n(j-1) + 2i - 1 & \text{if } j \leq \frac{n+1}{2} \text{ is even,} \\ 2n(n-j) + 2i - 1 & \text{if } j \geq \frac{n+3}{2} \text{ is odd,} \\ 2nj - 2i + 1 & \text{if } j \geq \frac{n+3}{2} \text{ is even.} \end{cases}$$

Let now:

$$\begin{aligned} X &= \left\{ k(2n^2 + 2n) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ Y_i &= \left\{ 2n^2 + 1 - 2i + k(2n^2 - 2n) : k \in \left[0, \frac{n-1}{4} \right] \right\}, \\ Z_i &= \left\{ 2n^2 + 1 - 2i + (2n^2 - 2n) \left(\frac{n-1}{4} + k \right) : k \in \left[1, \frac{n-1}{4} \right] \right\}, \\ T_i &= \left\{ 2 - 4i + \frac{n^3 + 4n^2 + 7n}{2} + k(2n^2 + 2n) : k \in \left[0, \frac{n-5}{4} \right] \right\}. \end{aligned}$$

By applying Lemma 5.2.1(1) to the first $\frac{n+1}{2}$ elements of R_i with $a = 2n^2 - 2i + 1$, $b = 2n + 2i - 1$, $g = -4n$ and $\ell = \frac{n-1}{4}$, we obtain $X \cup Y_i \subset S(R_i)$. By applying Lemma 5.2.1(4) to the remaining elements with $a = 2i - 1$, $b = 2n(n-1) - 2i + 1$, $g = 4n$ and $\ell = \frac{n-5}{4}$, and by adding the $\frac{n+1}{2}$ -th partial sum of R_i , that is

$2n^2 + 1 - 2i + (2n^2 - 2n)\frac{n-1}{4}$, we obtain $S(R_i) = X \cup Y_i \cup Z_i \cup T_i$. It can be easily seen that these are sets containing only non-zero elements, thus it only remains to check that they have trivial intersection. As before, it suffices to check that $Y_i \cap Z_i = \emptyset$ and that $X \cap T_i = \emptyset$. In the first case, by contradiction, we would have for some $k_1 \in [0, \frac{n-1}{4}]$, $k_2 \in [1, \frac{n-1}{4}]$ and even i :

$$2n^2 + 1 - 2i + k_1(2n^2 - 2n) \equiv 2n^2 + 1 - 2i + (2n^2 - 2n) \left(\frac{n-1}{4} + k_2 \right) \pmod{4n^2},$$

that is:

$$(2n^2 - 2n) \left(\frac{n-1}{4} + k_2 - k_1 \right) \equiv 0 \pmod{4n^2}.$$

We then have that $2n^2(\frac{n-1}{4} + k_2 - k_1) \in \{0, 2n^2\}$ modulo $4n^2$, thus we obtain a contradiction by noticing that $\frac{n-1}{4} + k_2 - k_1$ is strictly positive and bounded by $\frac{n-1}{2}$.

In the second case, again by contradiction, we would have for some $k_1 \in [1, \frac{n-1}{4}]$, $k_2 \in [0, \frac{n-5}{4}]$ and even i that:

$$k_1(2n^2 + 2n) \equiv 2 - 4i + \frac{n^3 + 4n^2 + 7n}{2} + k_2(2n^2 + 2n) \pmod{4n^2}.$$

As before $n^3 \equiv n^2 \pmod{4n^2}$, hence we would have:

$$2 - 4i + \frac{n^2 + 7n}{2} + 2n(k_2 - k_1) \equiv 0 \pmod{2n^2}.$$

It can be seen that the left-hand side is strictly positive in \mathbb{Z} , and as $i \in [1, n]$ and $k_2 - k_1 \in [\frac{1-n}{4}, \frac{n-9}{4}]$, is bounded from below by 2 and from above by $n^2 - n - 2$, hence we have a contradiction.

Case 2: $n \equiv 3 \pmod{4}$.

It can be checked that for every odd $i \in [1, n]$ the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A has entries:

$$r_{i,j} = \begin{cases} 2n(j-1) + 2i - 1 & \text{if } j \leq \frac{n+1}{2} \text{ is odd,} \\ 2n(n-j+1) - 2i + 1 & \text{if } j \leq \frac{n+1}{2} \text{ is even,} \\ 2nj - 2i + 1 & \text{if } j \geq \frac{n+3}{2} \text{ is odd,} \\ 2n(n-j) + 2i - 1 & \text{if } j \geq \frac{n+3}{2} \text{ is even.} \end{cases}$$

Now, define:

$$\begin{aligned} X &= \left\{ k(2n^2 - 2n) : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\ Y_i &= \left\{ 2i - 1 + k(2n^2 + 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\ Z &= \left\{ (2n^2 - 2n) \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\ T_i &= \left\{ 1 - 2i + \frac{n^3 + 2n^2 + 5n}{2} + k(2n^2 + 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}. \end{aligned}$$

As done for the case $n \equiv 1 \pmod{4}$, it is not hard to see that these are disjoint sets not containing zero. Also, applying again Lemma 5.2.1, one can check that $S(R_i) = X \cup Y_i \cup Z \cup T_i$.

For every even $i \in [1, n]$ the entries of the i -th row $R_i = (r_{i,j})_{j \in [1, n]}$ of A are:

$$r_{i,j} = \begin{cases} 2n(n-j+1) - 2i + 1 & \text{if } j \leq \frac{n+1}{2} \text{ is odd,} \\ 2n(j-1) + 2i - 1 & \text{if } j \leq \frac{n+1}{2} \text{ is even,} \\ 2n(n-j) + 2i - 1 & \text{if } j \geq \frac{n+3}{2} \text{ is odd,} \\ 2nj - 2i + 1 & \text{if } j \geq \frac{n+3}{2} \text{ is even.} \end{cases}$$

Let now:

$$\begin{aligned} X &= \left\{ k(2n^2 + 2n) : k \in \left[1, \frac{n+1}{4} \right] \right\}, \\ Y_i &= \left\{ 2n^2 - 2i + 1 + k(2n^2 - 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}, \\ Z &= \left\{ (2n^2 + 2n) \left(k + \frac{n+1}{4} \right) : k \in \left[1, \frac{n-3}{4} \right] \right\}, \\ T_i &= \left\{ 2i - 1 + \frac{n^3 + 4n^2 - 5n}{2} + k(2n^2 - 2n) : k \in \left[0, \frac{n-3}{4} \right] \right\}. \end{aligned}$$

It can then be seen that these are pairwise disjoint sets not containing zero, and, applying Lemma 5.2.1, that $S(R_i) = X \cup Y_i \cup Z \cup T_i$.

We can then conclude that for every odd n the array A is a globally simple $\text{NH}_{2n^2}(n; n)$. \square

Remark 5.2.12. For every odd integer n we write the total sum of every row and column of the array A constructed in the proof of Proposition 5.2.11. In particular if we take the element of the corresponding set Y_j with $k = \frac{n-1}{2}$ for every $j \in [1, n]$, we get:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = n^3 + (-1)^{j-1}(2nj - n^2 - n).$$

Analogously, if $n \equiv 1 \pmod{4}$ then for every $i \in [1, n]$ if we take the element of the corresponding set T_i with $k = \frac{n-5}{4}$, we get:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(2i - n - 1);$$

if $n \equiv 3 \pmod{4}$ then for every $i \in [1, n]$ taking the element of the corresponding set T_i with $k = \frac{n-3}{4}$, we obtain:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1}(n + 1 - 2i).$$

Example 5.2.13. Following the proof of Proposition 5.2.11 we obtain the $\text{NH}_{338}(13; 13)$ below, whose elements belong to \mathbb{Z}_{676} :

1	311	53	259	105	207	157	131	233	79	285	27	337
335	29	283	81	231	133	179	205	107	257	55	309	3
5	307	57	255	109	203	161	135	229	83	281	31	333
331	33	279	85	227	137	175	201	111	253	59	305	7
9	303	61	251	113	199	165	139	225	87	277	35	329
327	37	275	89	223	141	171	197	115	249	63	301	11
13	299	65	247	117	195	169	143	221	91	273	39	325
323	41	271	93	219	145	167	193	119	245	67	297	15
17	295	69	243	121	191	173	147	217	95	269	43	321
319	45	267	97	215	149	163	189	123	241	71	293	19
21	291	73	239	125	187	177	151	213	99	265	47	317
315	49	263	101	211	153	159	185	127	237	75	289	23
25	287	77	235	129	183	181	155	209	103	261	51	313

And similarly we obtain the $NH_{450}(15; 15)$ below, filled with elements in \mathbb{Z}_{900} :

1	419	61	359	121	299	181	239	269	151	329	91	389	31	449
447	33	387	93	327	153	267	213	183	297	123	357	63	417	3
5	415	65	355	125	295	185	235	265	155	325	95	385	35	445
443	37	383	97	323	157	263	217	187	293	127	353	67	413	7
9	411	69	351	129	291	189	231	261	159	321	99	381	39	441
439	41	379	101	319	161	259	221	191	289	131	349	71	409	11
13	407	73	347	133	287	193	227	257	163	317	103	377	43	437
435	45	375	105	315	165	255	225	195	285	135	345	75	405	15
17	403	77	343	137	283	197	223	253	167	313	107	373	47	433
431	49	371	109	311	169	251	229	199	281	139	341	79	401	19
21	399	81	339	141	279	201	219	249	171	309	111	369	51	429
427	53	367	113	307	173	247	233	203	277	143	337	83	397	23
25	395	85	335	145	275	205	215	245	175	305	115	365	55	425
423	57	363	117	303	177	243	237	207	273	147	333	87	393	27
29	391	89	331	149	271	209	211	241	179	301	119	361	59	421

5.2.5 Globally simple $NH_t(n; n)$, t divides n

Here we construct globally simple non-zero sum Heffter arrays of odd order n relative to the subgroup of order t , where t divides n .

Theorem 5.2.14. *For every odd integer $n \geq 1$ and for every divisor t of n , there exists a globally simple $NH_t(n; n)$.*

Proof. Let n and t be as in the statement and set $v = 2n^2 + t$. Let H be the $n \times t$ array with elements in \mathbb{Z}_v whose columns $C_j = (c_{i,j})_{i \in [1,n]}$ are so defined: for j odd

$$c_{i,j} = \begin{cases} (j-1)\frac{v}{t} + i & \text{if } i \text{ is odd,} \\ \frac{jv}{t} - i & \text{otherwise.} \end{cases}$$

for j even

$$c_{i,j} = \begin{cases} -(j-1)\frac{v}{t} - i & \text{if } i \text{ is odd,} \\ -\frac{jv}{t} + i & \text{otherwise.} \end{cases}$$

For $\alpha \in [1, \frac{n}{t} - 1]$ let $H_{\alpha n}$ be the matrix obtained from H by adding $\varepsilon \alpha n$ to the elements of H , where $\varepsilon = 1$ for the elements of H in the positions (i, j) with $i \equiv j \pmod{2}$, $\varepsilon = -1$ for the elements of H in the positions (i, j) with $i \not\equiv j \pmod{2}$. Now let A be $n \times n$ array so constructed:

$$\boxed{H \quad -H_n \quad H_{2n} \quad -H_{3n} \quad H_{4n} \quad \dots \quad H_{(\frac{n}{t}-1)n}}$$

where if $t = n$, $A = H$ is understood. To prove that $\text{supp}(A)$ is the required one it is useful to observe that, since the elements belong to \mathbb{Z}_v , some columns of H can be written also in the following equivalent expression: if j is odd and $j \in [\frac{t+3}{2}, t]$ then $C_j = (c_{i,j})_{i \in [1,n]}$ has elements

$$c_{i,j} = \begin{cases} (j-t-1)\frac{v}{t} + i & \text{if } i \text{ is odd,} \\ (j-t)\frac{v}{t} - i & \text{otherwise.} \end{cases}$$

while if j is even and $j \in [\frac{t+3}{2}, t]$ then $C_j = (c_{i,j})_{i \in [1,n]}$ has as entries

$$c_{i,j} = \begin{cases} (t-j+1)\frac{v}{t} - i & \text{if } i \text{ is odd,} \\ (t-j)\frac{v}{t} + i & \text{otherwise.} \end{cases}$$

Now, it is not hard to see that for any $j \in [0, \frac{t-3}{2}]$ it results

$$\text{supp}(C_{j+1} \cup C_{t-j}) = \left[j\frac{v}{t} + 1, j\frac{v}{t} + n \right] \cup \left[(j+1)\frac{v}{t} - n, (j+1)\frac{v}{t} - 1 \right]$$

also

$$\text{supp}\left(C_{\frac{t+1}{2}}\right) = \left[\frac{t-1}{2} \cdot \frac{v}{t} + 1, \frac{t-1}{2} \cdot \frac{v}{t} + n \right].$$

By the definition of the arrays $H_{\alpha n}$, it follows that

$$\begin{aligned} \text{supp}(H_{\alpha n}) &= \left(\left[j\frac{v}{t} + 1, j\frac{v}{t} + n \right] + \alpha n \right) \cup \left(\left[(j+1)\frac{v}{t} - n, (j+1)\frac{v}{t} - 1 \right] - \alpha n \right) \cup \\ &\cup \left(\left[\frac{t-1}{2} \cdot \frac{v}{t} + 1, \frac{t-1}{2} \cdot \frac{v}{t} + n \right] + \alpha n \right) \end{aligned}$$

where $j \in [0, \frac{t-3}{2}]$. This implies that $\text{supp}(A) = [1, \frac{v-1}{2}] \setminus \left\{ \frac{v}{t}, 2\frac{v}{t}, \dots, \frac{t-1}{2}\frac{v}{t} \right\}$.

Now we consider the partial sums of the columns of A . If for every odd $j \in [1, t]$ we apply Lemma 5.2.1(1) to the j -th column of A , with $a = (j-1)\frac{v}{t} + 1$, $b = j\frac{v}{t} - 2$, $g = 2$ and $\ell = \frac{n-1}{2}$, we get $S(C_j) = X_j \cup Y_j$, where:

$$\begin{aligned} X_j &= \left\{ k \left[(2j-1)\frac{v}{t} - 1 \right] : k \in \left[1, \frac{n-1}{2} \right] \right\}, \\ Y_j &= \left\{ (j-1)\frac{v}{t} + 1 + k \left[(2j-1)\frac{v}{t} + 1 \right] : k \in \left[0, \frac{n-1}{2} \right] \right\}. \end{aligned}$$

It is easy to see that these are sets containing non-zero elements, and that they are disjoint as their elements belong to different classes modulo $\frac{v}{t}$.

Similarly, for every even $j \in [1, t]$ Lemma 5.2.1(1) can be applied to the j -th column of A with $a = -(j-1)\frac{v}{t} - 1$, $b = -j\frac{v}{t} + 2$, $g = -2$ and $\ell = \frac{n-1}{2}$, obtaining $S(C_j) = X_j \cup Y_j$, where:

$$\begin{aligned} X_j &= \left\{ k \left[(1-2j)\frac{v}{t} + 1 \right] : k \in \left[1, \frac{n-1}{2} \right] \right\}, \\ Y_j &= \left\{ -(j-1)\frac{v}{t} - 1 + k \left[(1-2j)\frac{v}{t} - 1 \right] : k \in \left[0, \frac{n-1}{2} \right] \right\}. \end{aligned}$$

With the very same argument it can be shown that X_j and Y_j are disjoint sets not containing zero.

One can directly check that for $\alpha \in [1, \frac{n}{t} - 1]$ and for every $j \in [1, t]$ the partial sums of the j -th column of $H_{\alpha n}$ are given by $X_{j,\alpha} \cup Y_{j,\alpha}$ where:

$$X_{j,\alpha} = X_j \quad \text{and} \quad Y_{j,\alpha} = Y_j + (-1)^{j+1}\alpha n.$$

It can then be seen that the sets $X_{j,\alpha}$ and $Y_{j,\alpha}$ are disjoint and contain only non-zero elements.

Hence each column of A admits a simple natural ordering.

In order to prove that the same property holds also for the rows of A we start by observing that the rows of the array H are the following ones. For every odd $i \in [1, n]$ the i -th row $R_i = (r_{i,j})_{j \in [1, t]}$ has entries

$$r_{i,j} = (-1)^{j+1} \left((j-1) \frac{v}{t} + i \right)$$

and for every even $i \in [1, n]$ the elements of $R_i = (r_{i,j})_{j \in [1, t]}$ are

$$r_{i,j} = (-1)^{j+1} \left(\frac{jv}{t} - i \right).$$

For every odd $i \in [1, n]$ we can apply Lemma 5.2.1(1) with $a = i$, $b = -\frac{v}{t} - i$, $g = 2\frac{v}{t}$ and $\ell = \frac{t-1}{2}$, obtaining $S(R_i) = X \cup Y_i$ where:

$$X = \left\{ -\frac{v}{t}k : k \in \left[1, \frac{t-1}{2} \right] \right\},$$

$$Y_i = \left\{ i + k\frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

Analogously, for every even $i \in [1, n]$ we can apply Lemma 5.2.1(1) with $a = \frac{v}{t} - i$, $b = -2\frac{v}{t} + i$, $g = 2\frac{v}{t}$ and $\ell = \frac{t-1}{2}$, obtaining $S(R_i) = X \cup Y_i$ where:

$$X = \left\{ -\frac{v}{t}k : k \in \left[1, \frac{t-1}{2} \right] \right\},$$

$$Y_i = \left\{ \frac{v}{t} - i + k\frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

It is immediate to see that in both cases X and Y_i are disjoint sets not containing zero.

One can directly check that for $\alpha \in [1, \frac{n}{t} - 1]$ and for every $i \in [1, n]$ the partial sums of the i -th row of $H_{\alpha n}$ are given by $X_\alpha \cup Y_{i,\alpha}$ where:

$$X_\alpha = X \quad \text{and} \quad Y_{i,\alpha} = Y_i + (-1)^{i+1}\alpha n.$$

The sets are disjoint and do not contain zero.

Now, let $\alpha_1, \alpha_2 \in [0, \frac{n}{t} - 1]$ be distinct integers, for any $i \in [1, n]$ and for $j \in \{1, 2\}$ denote by $R_{i,\alpha_j} = (a_{i,1+\alpha_j t}, a_{i,2+\alpha_j t}, \dots, a_{i,t+\alpha_j t})$ the elements in the i -th row of A corresponding to the block $(-1)^\alpha H_{\alpha_j n}$, where $H_0 = H$ is understood. By $S(R_{i,\alpha_j})$ we mean the list of the partial sums of R_i corresponding to the subsequence R_{i,α_j} , that is $\{s_a : a \in [1 + \alpha_j t, t + \alpha_j t]\}$. To prove that the rows of

Admit a simple natural ordering, we have to show that $S(R_{i,\alpha_1}) \cap S(R_{i,\alpha_2}) = \emptyset$ for every $i \in [1, n]$ and for every choice of $\alpha_1, \alpha_2 \in [0, \frac{n}{t} - 1]$ with $\alpha_1 \neq \alpha_2$.

As a first step we notice that it suffices to control the aforementioned property for $|\alpha_1 - \alpha_2| = 1$. In fact, the i -th row of A written modulo $\frac{v}{t}$ is:

$$R_i \equiv (-1)^{i+1} \left(\underbrace{i, -i, i, \dots, i}_{t \text{ terms}}, \underbrace{-(n+i), n+i, \dots, -(n+i)}_{t \text{ terms}}, \right. \\ \left. \underbrace{2n+i, -(2n+i), \dots, 2n+i}_{t \text{ terms}}, \underbrace{-(3n+i), 3n+i, \dots, -(3n+i)}_{t \text{ terms}}, \dots, \right. \\ \left. \underbrace{\left(\frac{n}{t}-1\right)n+i, -\left(\left(\frac{n}{t}-1\right)n+i\right), \dots, \left(\frac{n}{t}-1\right)n+i}_{t \text{ terms}} \right),$$

hence its ordered list of partial sums, considered modulo $\frac{v}{t}$, is given by:

$$(-1)^{i+1} \left(\underbrace{i, 0, i, \dots, i}_{t \text{ terms}}, \underbrace{-n, i, \dots, -n}_{t \text{ terms}}, \underbrace{n+i, -n, \dots, n+i}_{t \text{ terms}}, \underbrace{-2n, n+i, \dots, -2n}_{t \text{ terms}}, \dots, \right. \\ \left. \underbrace{\frac{1}{2}\left(\frac{n}{t}-1\right)n+i, -\frac{1}{2}\left(\frac{n}{t}-1\right)n, \dots, \frac{1}{2}\left(\frac{n}{t}-1\right)n+i}_{t \text{ terms}} \right).$$

In particular it can be seen that for every $\alpha \in [0, \frac{n}{t} - 1]$ the partial sums of $R_{i,\alpha}$ modulo $\frac{v}{t}$ are:

$$\begin{aligned} (-1)^{i+1} \left(i + \frac{\alpha-1}{2} \right) & \quad \text{and } (-1)^i \frac{\alpha+1}{2} n & \quad \text{for } \alpha \text{ odd,} \\ (-1)^{i+1} \left(i + \frac{\alpha}{2} \right) & \quad \text{and } (-1)^i \frac{\alpha}{2} n & \quad \text{for } \alpha \text{ even.} \end{aligned}$$

We can then conclude that for $|\alpha_1 - \alpha_2| > 1$ the partial sums of the i -th row of A corresponding to R_{i,α_1} and R_{i,α_2} are trivially disjoint. In the following for any $i \in [1, n]$ and $\alpha \in [0, \frac{n}{t} - 1]$ set $c = \sum_{j=1}^{\alpha t} a_{i,j}$. Fix now an odd $i \in [1, n]$, let $\alpha \in [0, \frac{n}{t} - 1]$ be an even integer, we have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$X_\alpha = \left\{ c - k \frac{v}{t} : k \in \left[1, \frac{t-1}{2} \right] \right\}, \\ Y_{i,\alpha} = \left\{ c + i + \alpha n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

It is then not hard to see that $S(R_{i,\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + (c + i + \alpha n + \frac{t-1}{2} \frac{v}{t})$, where:

$$X_{\alpha+1} = \left\{ k \frac{v}{t} : k \in \left[1, \frac{t-1}{2} \right] \right\}, \\ Y_{i,\alpha+1} = \left\{ -i - (\alpha+1)n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2} \right] \right\}.$$

We have only to check that $Y_{i,\alpha} \cap (X_{\alpha+1} + (c + i + \alpha n + \frac{t-1}{2} \frac{v}{t})) = \emptyset$. Assume by contradiction that there exist $k_1 \in [0, \frac{t-1}{2}]$, $k_2 \in [1, \frac{t-1}{2}]$ and an odd i such that:

$$c + i + \alpha n + k_1 \frac{v}{t} \equiv k_2 \frac{v}{t} + c + i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This can be rewritten as:

$$\left(\frac{t-1}{2} + k_2 - k_1\right) \frac{v}{t} \equiv 0 \pmod{v}.$$

It can then be easily seen that this equation cannot hold, as $k_2 - k_1 \in \left[\frac{3-t}{2}, \frac{t-1}{2}\right]$.

Now let $\alpha \in [0, \frac{n}{t} - 1]$ be an odd integer; similarly, we have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$X_\alpha = \left\{c + k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha} = \left\{c - i - \alpha n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

It can now be seen that $S(R_{i,\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + (c - i - \alpha n - \frac{t-1}{2} \frac{v}{t})$, where:

$$X_{\alpha+1} = \left\{-k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha+1} = \left\{i + (\alpha+1)n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

As before it is sufficient to check that $Y_{i,\alpha} \cap (X_{\alpha+1} + (c - i - \alpha n - \frac{t-1}{2} \frac{v}{t})) = \emptyset$. Assume by contradiction that there exist $k_1 \in [0, \frac{t-1}{2}]$, $k_2 \in [1, \frac{t-1}{2}]$ and an odd i such that:

$$c - i - \alpha n - k_1 \frac{v}{t} \equiv -k_2 \frac{v}{t} + c - i - \alpha n - \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This can be rewritten as:

$$\left(\frac{t-1}{2} + k_2 - k_1\right) \frac{v}{t} \equiv 0 \pmod{v}.$$

With the very same argument used before, it can be seen that this equation is never verified, obtaining a contradiction.

Assume now that $i \in [1, n]$ is even, and consider any even integer $\alpha \in [0, \frac{n}{t} - 1]$. We have that $S(R_{i,\alpha}) = X_\alpha \cup Y_{i,\alpha}$, where:

$$X_\alpha = \left\{c - k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha} = \left\{c + \frac{v}{t} - i - \alpha n + k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

We have then that $S(R_{\alpha+1}) = (X_{\alpha+1} \cup Y_{i,\alpha+1}) + (c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \frac{v}{t})$, where:

$$X_{\alpha+1} = \left\{k \frac{v}{t} : k \in \left[1, \frac{t-1}{2}\right]\right\},$$

$$Y_{i,\alpha+1} = \left\{-\frac{v}{t} + i + (\alpha+1)n - k \frac{v}{t} : k \in \left[0, \frac{t-1}{2}\right]\right\}.$$

We need to check $Y_{i,\alpha} \cap (X_{\alpha+1} + (c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \frac{v}{t})) = \emptyset$; assume by contradiction that there exist $k_1 \in [0, \frac{t-1}{2}]$, $k_2 \in [1, \frac{t-1}{2}]$ and an even i such that:

$$c + \frac{v}{t} - i - \alpha n + k_1 \frac{v}{t} \equiv k_2 \frac{v}{t} + c + \frac{v}{t} - i - \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \pmod{v}.$$

This is equivalent to:

$$\left(\frac{t-1}{2} + k_2 - k_1\right) \frac{v}{t} \equiv 0 \pmod{v}.$$

It is easy to see that this is never the case.

With the very same argument, it can be shown that for every even i , the partial sums of $R_{i,\alpha}$ and $R_{i,\alpha+1}$ are distinct for every odd α . \square

Remark 5.2.15. From the globally simple array A constructed in Theorem 5.2.14 one can deduce the total sum of each row and of each column. Let $n, t \geq 1$ be odd integers with t dividing n , then for every $j \in [1, t]$ and every $\alpha \in [0, \frac{n}{t} - 1]$ we have:

$$\sum_{a_{i,j} \in C_{j+\alpha t}} a_{i,j} = (-1)^{\alpha+j+1} \left[(j-1) \frac{v}{t} + 1 + \alpha n + \frac{n-1}{2} \left[(2j-1) \frac{v}{t} + 1 \right] \right].$$

We now focus on the rows of A . Consider any odd $i \in [1, n]$, it can be seen that for every $\alpha \in [0, \frac{n}{t} - 1]$ we have:

$$\sum_{j \in [1+\alpha t, t+\alpha t]} a_{i,j} = (-1)^\alpha \left(i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right);$$

hence the sum of the elements in the i -th row of A , for i odd, is:

$$\sum_{\alpha \in [0, \frac{n}{t} - 1]} (-1)^\alpha \left(i + \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right).$$

Thus, it is not hard to see that for every odd $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = i + n \left(\frac{n}{t} - 1 \right) \cdot \frac{1}{2} + \frac{t-1}{2} \cdot \frac{v}{t}.$$

Similarly, it can be seen that for every even $i \in [1, n]$ and for every $\alpha \in [0, \frac{n}{t} - 1]$:

$$\sum_{j \in [1+\alpha t, t+\alpha t]} a_{i,j} = (-1)^\alpha \left(\frac{v}{t} - i - \alpha n + \frac{t-1}{2} \cdot \frac{v}{t} \right).$$

From which we obtain that for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{v}{t} - i - n \left(\frac{n}{t} - 1 \right) \cdot \frac{1}{2} + \frac{t-1}{2} \cdot \frac{v}{t}.$$

Example 5.2.16. Below we have an $\text{NH}_{11}(11; 11)$, say A , whose elements belong to \mathbb{Z}_{253} . Following the notation of the proof of Theorem 5.2.14, we are in the case $t = n$ hence $A = H$.

1	-24	47	-70	93	-116	-114	91	-68	45	-22
21	-44	67	-90	113	117	-94	71	-48	25	-2
3	-26	49	-72	95	-118	-112	89	-66	43	-20
19	-42	65	-88	111	119	-96	73	-50	27	-4
5	-28	51	-74	97	-120	-110	87	-64	41	-18
17	-40	63	-86	109	121	-98	75	-52	29	-6
7	-30	53	-76	99	-122	-108	85	-62	39	-16
15	-38	61	-84	107	123	-100	77	-54	31	-8
9	-32	55	-78	101	-124	-106	83	-60	37	-14
13	-36	59	-82	105	125	-102	79	-56	33	-10
11	-34	57	-80	103	-126	-104	81	-58	35	-12

Example 5.2.17. Here we take $n = 15$, $t = 3, 5$ and we follow the proof of Theorem 5.2.14. An $NH_3(15; 15)$, say A , has elements in \mathbb{Z}_{453} , $\frac{v}{t} = 151$ and we have

$$A = \boxed{H \mid -H_{15} \mid H_{30} \mid -H_{45} \mid H_{60}}$$

1	-152	-150	-16	167	135	31	-182	-120	-46	197	105	61	-212	-90
149	153	-2	-134	-168	17	119	183	-32	-104	-198	47	89	213	-62
3	-154	-148	-18	169	133	33	-184	-118	-48	199	103	63	-214	-88
147	155	-4	-132	-170	19	117	185	-34	-102	-200	49	87	215	-64
5	-156	-146	-20	171	131	35	-186	-116	-50	201	101	65	-216	-86
145	157	-6	-130	-172	21	115	187	-36	-100	-202	51	85	217	-66
7	-158	-144	-22	173	129	37	-188	-114	-52	203	99	67	-218	-84
143	159	-8	-128	-174	23	113	189	-38	-98	-204	53	83	219	-68
9	-160	-142	-24	175	127	39	-190	-112	-54	205	97	69	-220	-82
141	161	-10	-126	-176	25	111	191	-40	-96	-206	55	81	221	-70
11	-162	-140	-26	177	125	41	-192	-110	-56	207	95	71	-222	-80
139	163	-12	-124	-178	27	109	193	-42	-94	-208	57	79	223	-72
13	-164	-138	-28	179	123	43	-194	-108	-58	209	93	73	-224	-78
137	165	-14	-122	-180	29	107	195	-44	-92	-210	59	77	225	-74
15	-166	-136	-30	181	121	45	-196	-106	-60	211	91	75	-226	-76

An $NH_5(15; 15)$, say A , has elements in \mathbb{Z}_{455} , $\frac{v}{t} = 91$ and we have

$$A = \boxed{H \mid -H_{15} \mid H_{30}}$$

1	-92	183	181	-90	-16	107	-198	-166	75	31	-122	213	151	-60
89	-180	-184	93	-2	-74	165	199	-108	17	59	-150	-214	123	-32
3	-94	185	179	-88	-18	109	-200	-164	73	33	-124	215	149	-58
87	-178	-186	95	-4	-72	163	201	-110	19	57	-148	-216	125	-34
5	-96	187	177	-86	-20	111	-202	-162	71	35	-126	217	147	-56
85	-176	-188	97	-6	-70	161	203	-112	21	55	-146	-218	127	-36
7	-98	189	175	-84	-22	113	-204	-160	69	37	-128	219	145	-54
83	-174	-190	99	-8	-68	159	205	-114	23	53	-144	-220	129	-38
9	-100	191	173	-82	-24	115	-206	-158	67	39	-130	221	143	-52
81	-172	-192	101	-10	-66	157	207	-116	25	51	-142	-222	131	-40
11	-102	193	171	-80	-26	117	-208	-156	65	41	-132	223	141	-50
79	-170	-194	103	-12	-64	155	209	-118	27	49	-140	-224	133	-42
13	-104	195	169	-78	-28	119	-210	-154	63	43	-134	225	139	-48
77	-168	-196	105	-14	-62	153	211	-120	29	47	-138	-226	135	-44
15	-106	197	167	-76	-30	121	-212	-152	61	45	-136	227	137	-46

5.2.6 Globally simple $NH_t(n; n)$ for every n prime

In the following we show that a square globally simple non-zero sum Heffter array of odd prime order n relative to the subgroup of order t exists for every n . We start with an easy remark for $n = 2$.

Remark 5.2.18. Note that the existence of a globally simple $NH_t(2; 2)$ for any admissible t is trivial. Below we give an example for each possible case, that is for $t = 1, 2, 4, 8$ (the first array is both an $NH(2; 2)$ and an $NH_2(2; 2)$):

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 5 & 7 \\ \hline \end{array}$$

Now we can present a complete solution whenever n is a prime.

Theorem 5.2.19. *Let n be a prime. There exists a globally simple $\text{NH}_t(n; n)$ for every admissible t .*

Proof. Since n is a prime the admissible values for t are $1, 2, n, 2n, n^2, 2n^2$. For $t = 1$ the result is contained in Theorem 5.5 of [35]. For $t \neq 1$ the result follows by Remark 5.2.18 and by Propositions 5.2.2, 5.2.5, 5.2.8, 5.2.11 and Theorem 5.2.14. \square

5.3 Graph decompositions and biembeddings

In this section we show the connection between non-zero sum (relative) Heffter arrays and graph decompositions and biembeddings; this connection was first explained in [35], and later generalized in [69]. As discussed in Section 2.4 the rows and the columns of a Heffter array give rise to two distinct difference families whose blocks are closed trails, that are cycles whenever the condition of simplicity is met. It can then be similarly derived that from the rows and columns of a non-zero Heffter array one can construct two distinct difference families whose blocks are open trails, that are paths if the chosen orderings are simple. Also, the results of this section are contained in [69].

We recall that for a k -subset T of a group $(G, +)$ and an ordering of its elements $\omega = (t_1, \dots, t_k)$, we say that ω is *simple* if its partial sums are pairwise distinct and different from zero, except possibly for the last one. Let $R = \{r_1, \dots, r_h\}$ be the set of elements of a row of a non-zero (relative) Heffter array $\text{NH}_t(m, n; h, k)$, and let ω_R be an ordering of this set of elements, having partial sums s_1, \dots, s_h . Construct then the following walk:

$$W(\omega_R) = (0, s_1, s_2, \dots, s_h). \quad (5.3.1)$$

Then:

Lemma 5.3.1. *The walk $W(\omega_R)$ of Equation (5.3.1) is*

- (1) *an open trail;*
- (2) *a path if and only if ω_R is simple.*

Proof. Analogous to that of Lemma 2.4.1. \square

We can then deduce the following, see also [35]:

Proposition 5.3.2. *Let A be a $\text{NH}_t(m, n; h, k)$ non-zero sum relative Heffter array such that for every row and for every column there exists a simple ordering. Then:*

- (1) *there exists a $(\mathbb{Z}_{2nk+t}, P_h)$ difference family relative to $\langle \frac{2nk+t}{t} \rangle$, hence a cyclic $(K_{\frac{2nk+t}{t} \times t}, P_h)$ -design \mathcal{D}_R ;*
- (2) *there exists a $(\mathbb{Z}_{2nk+t}, P_k)$ difference family relative to $\langle \frac{2nk+t}{t} \rangle$, hence a cyclic $(K_{\frac{2nk+t}{t}}, P_k)$ -design \mathcal{D}_C .*

Moreover, the two graph decompositions are orthogonal.

We recall that the embeddings arising from Heffter arrays are constructed as the derived graph of a suitable current graph. Given a current graph (Γ, ϕ, α) , where (Γ, ϕ) has only edges of type 0 and α is a current assignment from the directed edges of $\vec{\Gamma} = (V, D)$ to \mathbb{Z}_v for some positive integer v , we define the following properties for a positive integer t such that t divides $\frac{v}{t} - 1$:

- (1) **Excess current:** for every $u \in V$, $\sum_{e \in D_u} \alpha(e) \neq 0$.
- (2) **Unique currents:** α is a bijection between D and $\mathbb{Z}_v \setminus J$, where J is the subgroup of \mathbb{Z}_v having order t .
- (3) **Monofacial embedding:** the embedding constructed from (Γ, ϕ) has a single face.

As done for classical Heffter arrays it can be immediately proven the following:

Theorem 5.3.3. *Let A be an $\text{NH}_t(m, n; h, k)$. Then A is equivalent to a current assignment α , satisfying Conditions (1) and (2), on a bipartite graph Γ having one color class with m vertices of degree h and the other with n vertices of degree k . Γ is connected if and only if A is not, up to row and column permutations, fully indecomposable.*

Two compatible orderings ω_r and ω_c of the rows and columns of A are equivalent to a monofacial rotation ρ on Γ . Moreover, if ω_r and ω_c are both simple, then ρ is simple with respect to α .

From a non-zero relative Heffter array admitting two compatible orderings it is possible to construct an embedding of a complete multipartite graph. The faces of this embedding in general are closed walks, as shown in the construction of the derived embedding for a current graph having excess current [54]. Then:

Proposition 5.3.4. *Let A be an $\text{NH}_t(m, n; h, k)$ admitting a pair of compatible orderings. Then, there exists a biembedding of $K_{\frac{2nk+t}{t} \times t}$ into an orientable surface such that the length of the faces are multiples of h or multiples of k .*

We can then compute the face length of the biembeddings constructed in Section 5.2.

In the next propositions we consider some particular cases where the length of the faces have nice expressions.

We begin by considering the Archdeacon embedding obtained for $t = 2$:

Proposition 5.3.5. *Let $n \geq 1$ be an odd integer, then there exists a cellular biembedding of $K_{(n^2+1) \times 2}$ such that every face has length $4n$ or a multiple of $n \frac{n^2+1}{2}$.*

Proof. Let n be an odd positive integer and A be the $\text{NH}_2(n; n)$ constructed in Proposition 5.2.2; it will be enough to prove that the total sum of every row and of every column has order 4 or a multiple of $\frac{n^2+1}{2}$ modulo $2(n^2 + 1)$. By Remark 5.2.3 for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left(j + \frac{n-1}{2} n \right).$$

Note that for $j = \frac{n+1}{2}$ the previous expression yields $\pm \frac{n^2+1}{2}$, obtaining order 4 modulo $2(n^2+1)$. For $j \neq \frac{n+1}{2}$ it can be seen that the greatest common divisor between the total sum and n^2+1 is either 1 or 2, obtaining that the order is a multiple of $\frac{n^2+1}{2}$.

Now, again by Remark 5.2.3, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i+1} \left((i-1)n + \frac{n+1}{2} \right).$$

Similarly to the previous case, for $i = \frac{n+1}{2}$ we obtain order 4, while for $i \neq \frac{n+1}{2}$ the order is a multiple of $\frac{n^2+1}{2}$. \square

Proposition 5.3.6. *Let n be an odd prime, then there exists a cellular biembedding of $K_{(n+1) \times 2n}$ such that every face has length $4n$ or a multiple of n^2 .*

Proof. Let A be the $\text{NH}_{2n}(n; n)$ described in Proposition 5.2.5. By Remark 5.2.6 we have that for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (n+1) \left(\frac{n^2}{2} + (-1)^j \frac{n+1-2j}{2} \right).$$

As n is prime and $j \in [1, n]$ it can be seen that for $j \neq \frac{n+1}{2}$ the rightmost bracket is coprime with n ; hence the order in $\mathbb{Z}_{2n(n+1)}$ is $4n$, so we obtain an embedding where the length of the corresponding faces is $4n^2$. For $j = \frac{n+1}{2}$, the previous expression reads $n^2(n+1)/2$, obtaining order 4, thus the length of the corresponding faces is $4n$.

We will now prove that for every row of A the order of its total sum in \mathbb{Z}_{2n^2+2n} is either 4 or a multiple of n . For every odd $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{-n^3 + n^2 + n - 1 + 2i}{2}.$$

Firstly, since $n^2 + 2n = -n^2$ in \mathbb{Z}_{2n^2+2n} , for $i = \frac{n+1}{2}$ the previous expression reads $\frac{-n^3-n^2}{2} = \frac{-n^2(n+1)}{2}$, thus obtaining order 4 and corresponding length of the faces $4n$. For $i \neq \frac{n+1}{2}$ the total sum is not a multiple of n , as $\frac{-n^3+n^2}{2}$ is a multiple of n , while $\frac{n-1+2i}{2}$ is not. Thus, for $i \neq \frac{n+1}{2}$, the order of the total sum is a multiple of n .

Finally, for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = \frac{n^3 + n^2 + n + 1 - 2i}{2}.$$

As for the previous case, it can be seen that the order of the total sum of the i -th row of A is a multiple of n for $i \neq \frac{n+1}{2}$, and it is 4 for $i = \frac{n+1}{2}$. \square

Proposition 5.3.7. *Let n be an odd prime, then there exists a cellular biembedding of $K_{3 \times n^2}$ such that every face has length $3n^2$ or $3n^3$.*

Proof. Let A be the $\text{NH}_{n^2}(n; n)$ described in Proposition 5.2.8; we follow Remark 5.2.9 where the total sums of rows and columns are described. In particular, for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j-1} \left(3nj - \frac{3n+1}{2} \right).$$

Now, since $\frac{3n+1}{2}$ is neither a multiple of n , nor a multiple of 3, we obtain that $\sum_{a_{i,j} \in C_j} a_{i,j}$ and $3n^2$ are coprime, thus every face constructed from the columns of A has length $3n^3$.

If $n \equiv 1 \pmod{4}$, for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3i - 2 + 3n \frac{n-1}{2} \right).$$

It can then be seen that this term is coprime with $3n^2$ except when $3i$ is equal to $n+2$ (that implies $n \equiv 1 \pmod{3}$) or $2n+2$ (that may hold if $n \equiv 2 \pmod{3}$). Let \bar{i} denote such an index, note that $3n \sum_{a_{\bar{i},j} \in R_{\bar{i}}} a_{\bar{i},j} = 0 \pmod{3n^2}$.

For every $n \equiv 3 \pmod{4}$ and $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = (-1)^{i-1} \left(3(n+1-i) - 2 + 3n \frac{n-1}{2} \right).$$

Similarly, this term is coprime with $3n^2$ except when $3i$ is equal to $n+1$ (that implies $n \equiv 2 \pmod{3}$) or $2n+1$ (that may hold if $n \equiv 1 \pmod{3}$). Let \bar{i} be such an index, then $3n \sum_{a_{\bar{i},j} \in R_{\bar{i}}} a_{\bar{i},j} = 0 \pmod{3n^2}$.

In any case, we have proven that the length of the faces obtained by the \bar{i} -th row is $3n^2$, while every other face has length $3n^3$. \square

Proposition 5.3.8. *Let n be an odd prime, then there exists a cellular biembedding of $K_{2 \times 2n^2}$ such that every face has length $4n$, $4n^2$ or $4n^3$.*

Proof. Let A be the $\text{NH}_{2n^2}(n; n)$ described in Proposition 5.2.11; we recall that the total sum of the rows and the columns of A is given in Remark 5.2.12. For every $j \in [1, n]$ we have:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = n^3 + (-1)^{j-1} (2nj - n^2 - n).$$

It can be seen that for $j \neq \frac{n+1}{2}$, the order of the sum in \mathbb{Z}_{4n^2} is $4n$. For $j = \frac{n+1}{2}$ the previous expression yields n^3 , which has order 4 in \mathbb{Z}_{4n^2} . Thus the corresponding faces have length $4n^2$ or $4n$.

For every $n \equiv 1 \pmod{4}$ and $i \in [1, n]$ we have:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1} (2i - n - 1).$$

It can be seen that for $i = \frac{n+1}{2}$ the previous expression yields n^3 , thus the corresponding faces have length $4n$. For $i \in [1, n]$, $i \neq \frac{n+1}{2}$, the previous expression gives a number coprime with $4n^2$, obtaining that the faces constructed from these rows have length $4n^3$.

For every $n \equiv 3 \pmod{4}$ and for every $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = n^3 + (-1)^{i-1} (n+1-2i).$$

It is then an easy check to verify that this term is coprime with $4n^2$ for every $i \in [1, n]$, $i \neq \frac{n+1}{2}$, obtaining faces with length $4n^3$, while for $i = \frac{n+1}{2}$ the faces have length $4n$. \square

Proposition 5.3.9. *Let $n \geq 1$ be an odd integer such that $2n+1$ is prime, then there exists a cellular biembedding of $K_{(2n+1) \times n}$ such that the length of every face is a multiple of $n(2n+1)$.*

Proof. Let A be the $\text{NH}_n(n; n)$ obtained from Theorem 5.2.14, we follow Remark 5.2.15 where we reported the total sums of rows and columns of A , keeping in mind that $t = n$ which implies $\alpha = 0$. In particular, for every $j \in [1, n]$:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} \left[(j-1)(2n+1) + 1 + \frac{n-1}{2} [(2j-1)(2n+1) + 1] \right].$$

Since we are in \mathbb{Z}_{2n^2+n} , the previous expression is equivalent to:

$$\sum_{a_{i,j} \in C_j} a_{i,j} = (-1)^{j+1} (-n^2 - n) = (-1)^j n(n+1).$$

We can then conclude by noticing that $n+1$ and $2n+1$ are coprime, thus obtaining that the total sum of the elements in every column has order $2n+1$ in \mathbb{Z}_{2n^2+n} and that the corresponding faces of the Archdeacon embedding have length $n(2n+1)$.

Now, we have for every odd $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = i + \frac{n-1}{2} (2n+1).$$

Since $2n+1$ is prime, it is immediate to see that this expression and $2n+1$ are coprime, thus the length of the corresponding faces is a multiple of $n(2n+1)$. Similarly, for every even $i \in [1, n]$:

$$\sum_{a_{i,j} \in R_i} a_{i,j} = -i + \frac{n+1}{2} (2n+1).$$

As before, it can be seen that this expression is coprime with $2n+1$, obtaining as length face multiples of $n(2n+1)$. \square

Chapter 6

Generalized Archdeacon Arrays

In this chapter we introduce a generalized class of Heffter-type arrays that can be defined in general over any group (not necessarily abelian). In particular we consider partially filled arrays whose sum along rows and columns have to be identically zero or non-zero. This concept, introduced in [70], generalizes the concepts of zero and non-zero sum Heffter arrays. The applications to graph decompositions of Cayley graphs and biembeddings still hold for generalized Archdeacon arrays. We give various existence results for this class of arrays, that can be classified in these main categories: existence results of non-zero sum Heffter arrays in general groups (non-necessarily abelian) and of globally simple zero and non-zero sum Heffter arrays over cyclic groups. The content of this chapter is contained in a joint work with T. Traetta, see [70].

6.1 Definition

In this section we give the notion of a generalized Archdeacon array: as the name suggests these arrays generalize the concept of an Archdeacon array, introduced in [43] and described in Section 1.3.7.

Let G be an additive group not necessarily abelian, and choose a map $\|\cdot\| : G \rightarrow G, a \mapsto \|a\|$ such that $\|a\| = \|-a\| \in \{\pm a\}$, for every $a \in G$. We refer to $\|a\|$ as the *absolute value* of $a \in G$. For a multiset H of G we define the (unordered) list $\|H\| = \{\|h\| : h \in H\}$, while H^+ represents the underlying set of $\|H\|$ minus the zero element (e.g., set $\mathbb{Z}^+ = \mathbb{N} = \{1, 2, \dots\}$). For a set $S \subseteq G$ we denote by $I(S)$ the set of all involutions (i.e., elements of order 2) in S .

Given an $m \times n$ matrix A with entries from G , the *row-weight* of A is the sequence $w_r(A) = (h_1, \dots, h_m)$ where h_i is the number of non-zero entries in the i -th row of A ; the *column-weight* $w_c(A)$ of A is the row-weight of the transpose of A .

Definition 6.1.1. Let $S \subseteq G^+$, $(m, n, \lambda) \in \mathbb{N}^3$, and let $\mathbf{h} = (h_1, \dots, h_m)$ and $\mathbf{k} = (k_1, \dots, k_n)$, with $1 \leq h_i \leq n$ and $1 \leq k_j \leq m$ for each i and j . A *generalized Heffter array* $\text{GHA}_S^\lambda(m, n; \mathbf{h}, \mathbf{k})$ (briefly, GHA) over G is an $m \times n$ matrix H with entries from G , satisfying the following properties:

- (a_{gen}) the i -th row (respectively j -th column) of A contains exactly h_i (respectively k_j) non-zero elements, that is, $(w_r(H), w_c(H)) = (\mathbf{h}, \mathbf{k})$;
- (b_{fold}) the multiset $\pm\mathcal{E}(A)$ contains λ copies of each element of S .

We drop the parameter λ (representing the *multiplicity* of a GHA) when it is equal to 1. We speak of a *uniform* GHA whenever the weight sequences \mathbf{h} and \mathbf{k} are constant, that is $\mathbf{h} = (h, \dots, h)$ and $\mathbf{k} = (k, \dots, k)$; in this case we write $\text{GHA}_S^\lambda(m, n; h, k)$, namely when property (a_{gen}) is nothing but property (a) of the classical Definition 1.1.7.

It follows that for a $\text{GHA}_S^\lambda(m, n; \mathbf{h}, \mathbf{k})$ to exist, we necessarily have that

$$\begin{aligned} \lambda \text{ must be even when } I(S) \neq \emptyset, \text{ and} \\ \lambda|S| - \frac{\lambda}{2}|I(S)| = h_1 + \dots + h_m = k_1 + \dots + k_n. \end{aligned} \quad (6.1.1)$$

Clearly, changing the signs of some entries of a GHA produces another GHA with the same parameters.

We recall the following result by Gale and Ryser [60, Theorem 7.7.4] on the existence of matrices over \mathbb{Z}_2 with given row and column weights.

Theorem 6.1.2. *Let $\mathbf{h} = (h_1, \dots, h_m)$ and $\mathbf{k} = (k_1, \dots, k_n)$ be two sequences of positive integers such that $\sum_{i=1}^m h_i = \sum_{j=1}^n k_j$. There exists an $m \times n$ matrix A over \mathbb{Z}_2 with $w_r(A) = \mathbf{h}$ and $w_c(A) = \mathbf{k}$ if and only if*

$$\sum_{i=1}^m \min(h_i, u) \geq \sum_{j=1}^n k'_j, \text{ for every } 1 \leq u \leq n. \quad (6.1.2)$$

where (k'_1, \dots, k'_n) is the decreasing reordering of \mathbf{k} .

Replacing each non-zero entry of a $\text{GHA}_S^\lambda(m, n; \mathbf{h}, \mathbf{k})$ with 1 produces a matrix over $\mathbb{Z}_2 = \{0, 1\}$ whose row and column weights are \mathbf{h} and \mathbf{k} . Therefore, Condition (6.1.2) provides another necessary condition for the existence of a GHA. We will refer to Conditions (6.1.1) and (6.1.2) as *the necessary conditions* for the existence of a GHA.

As for classical Heffter arrays GHAs can be used to construct orthogonal path or cycle decompositions and biembeddings of Cayley graphs onto orientable surfaces. The structural properties of these decompositions and biembeddings depend on the sum of the entries in each row and column of a GHA, with respect to a given ordering. Indeed, it is needless to recall that the sum of elements in a non abelian group heavily depends on the chosen ordering. Given a group G and a sequence $\mathbf{a} = (a_1, \dots, a_n) \in G^n$, we denote by $s_0(\mathbf{a}) = \sum_{i=1}^n a_i$ the sum of \mathbf{a} .

Given an ordering ω_r and ω_c of the rows and of the columns of a GHA A , we call $\omega = (\omega_r, \omega_c)$ an *ordering* of A , and the pair (A, ω) an *ordered matrix*. Then:

Definition 6.1.3. An ordered $m \times n$ matrix (A, ω) over a group G is

- (1) zero-sum, if $s_0(\omega_{r,i}) = 0 = s_0(\omega_{c,j})$ for every $1 \leq i \leq m$ and $1 \leq j \leq n$;
- (2) non-zero sum, if $s_0(\omega_{r,i}) \neq 0 \neq s_0(\omega_{c,j})$ for every $1 \leq i \leq m$ and $1 \leq j \leq n$.

The matrix A is zero-sum (respectively non-zero sum) if it is so with respect to some ordering.

Note that when G is abelian, the matrix A cannot be at the same time zero-sum and non-zero sum; in this case, the property of being a zero-sum or non-zero sum matrix is independent of the chosen ordering. Furthermore, given an arbitrary group G and considering that a cyclic shift of a sequence \mathbf{a} of G turns it into another sequence whose sum is in the same conjugacy class of $s_0(\mathbf{a})$, it follows that a zero-sum matrix over G with respect to a natural ordering continues to be zero-sum with respect to every natural ordering.

6.1.1 Notation

Given an integer q , an element a of a group G , and two finite sequences $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_t)$ over G and length n and t , respectively, we use the following terminology and notation.

- (1) $\underline{a} = (a, \dots, a)$. The length of \underline{a} will be always clear from the context.
- (2) $(\mathbf{a}, \mathbf{b}) = (a_1, \dots, a_n, b_1, \dots, b_t)$, $q\mathbf{a} = (qa_1, \dots, qa_n)$, and $-\mathbf{a} = (-1)\mathbf{a}$.
- (3) $s_{i,j}(\mathbf{a}) = \sum_{\ell=i}^j a_\ell$ is a *run* of \mathbf{a} , for every $1 \leq i \leq j \leq n$, and we call it *proper* when $(i, j) \neq (1, n)$.
- (4) $s_i(\mathbf{a}) = s_{1,i}(\mathbf{a})$ is the *i -th partial sum* of \mathbf{a} , for $1 \leq i \leq n$. For our convenience, we set $s_0(\mathbf{a}) = s_n(\mathbf{a})$.
- (5) \mathbf{a} is called *simple* if all proper runs of \mathbf{a} are different from zero; this is equivalent to saying that $0 \neq s_i(\mathbf{a}) \neq s_j(\mathbf{a})$ for all $1 \leq i < j \leq n$.
- (6) \mathbf{a} is called a *zero-sum sequence* (respectively *non-zero sum sequence*) if $s_0(\mathbf{a}) = 0$ (respectively $s_0(\mathbf{a}) \neq 0$).
- (7) The *alternated forms* of \mathbf{a} are the sequences $\mathbf{a}^\mp = (-a_1, \dots, (-1)^i a_i, \dots, (-1)^n a_n)$ and $-\mathbf{a}^\mp$.

Remark 6.1.4. If G is abelian, given an integer q and a sequence $\mathbf{a} = (a_1, \dots, a_n) \in G^n$, we have that $s_i(q\mathbf{a}) = qs_i(\mathbf{a})$, hence $s_i(-\mathbf{a}) = -s_i(\mathbf{a})$, for $1 \leq i \leq n$.

Given an $m \times n$ matrix $A = (a_{ij})$ with entries from an arbitrary group G , not necessarily abelian, we use the following terminology and notation. We denote by A_i and A^j the i -th row and the j -th column of A , respectively. The *reduced i -th row* of A , denoted by $A_{(i)}$, is the left-to-right sequence of non-zero elements in A_i . Similarly, the *reduced j -th column* of A , denoted by $A^{(j)}$, is the top-to-bottom sequence of non-zero elements in A^j .

The *row-weight* of A is the sequence $w_r(A) = (w_1(A), \dots, w_m(A))$ where each $w_i(A)$ is the length of $A_{(i)}$, that is, the number of non-zero entries in the i -th row of A . The *column-weight* of A is the sequence $w_c(A) = (w^1(A), \dots, w^n(A))$ where each $w^j(A)$ is the length of $A^{(j)}$, that is, the number of non-zero entries in the j -th column of A ; clearly, $w_c(A) = w_r(A^t)$, where A^t denotes the transpose of A . The *weight* of A is the number $w(A)$ of non-zero entries of A . Clearly,

$$w(A) = \sum_{i=1}^m w_i(A) = \sum_{j=1}^n w^j(A) = |\mathcal{E}(A)|.$$

The *non-zero position matrix* associated to A is the $m \times n$ array $A^* = (a_{ij}^*)$ where $a_{ij}^* = p$ if (i, j) is the p -th element in $skel(A)$ endowed with the

lexicographic order, otherwise $a_{ij}^* = 0$. In other words, a_{ij}^* counts the number of non-zero entries of A from a_{11} up to $a_{ij} \neq 0$, using the lexicographic order over the indices (i, j) .

Given a map $f : G \rightarrow G$, let $f(A) = (f(a_{ij}))$ denote the $m \times n$ matrix obtained by applying f element-wise on A .

Letting $B = (b_{ij}) \in \mathbb{Z}^{m,n}$, the Hadamard product of B and A is the $m \times n$ matrix $B \circ A = (b_{ij}a_{ij})$.

6.2 Near alternating sign matrices

In this section, we construct near alternating sign matrices, which we use in the following to build simple NGHAs.

Definition 6.2.1. An $m \times n$ matrix A with entries from $\{0, \pm 1\}$ is called a *near alternating sign matrix* (NASM) if the non-zero entries of each row and each column alternate.

Letting $\mathbf{h} = w_r(A)$ and $\mathbf{k} = w_c(A)$, we say that A is a $\text{NASM}(m, n; \mathbf{h}, \mathbf{k})$ or simply a $\text{NASM}(m, n)$. Finally, if $\mathbf{h} = (h, \dots, h)$ and $\mathbf{k} = (k, \dots, k)$, we say that A is *uniform* and write $\text{NASM}(m, n; h, k)$.

The name of these matrices is derived from the classical notion of *alternating sign matrices* (ASMs), that are NASMs satisfying the additional property of having row and column sum equal to 1. ASMs have been studied as a generalization of permutation matrices, as they naturally arise from a method proposed in [49] by Dodgson (also known as Lewis Carroll) for the computation of the determinant of a matrix. Moreover, a famous conjecture regarding the number of ASMs of a given size was proposed in [72], and was eventually solved 10 years later in [100].

Example 6.2.2. The following is a $\text{NASM}(3, 3; (2, 3, 1), (1, 3, 2))$:

0	1	-1
1	-1	1
0	1	0

Near alternating sign matrices were first considered in [13], although the terminology was suggested by Brualdi [12]. Here, we are interested in building them with given row-weight and column-weight sequences.

Remark 6.2.3. Since the absolute value of a $\text{NASM}(m, n; \mathbf{h}, \mathbf{k})$ is a $\{0, 1\}$ -matrix, by Theorem 6.1.2 it follows that \mathbf{h} and \mathbf{k} must satisfy condition (6.1.2).

We now define the concept of frame of a NASM, a parameter that will enable us to join two or more suitable NASMs to obtain larger NASMs.

Definition 6.2.4. Let A be a $\text{NASM}(m, n)$. The *frame* of A is the quadruple $\varphi(A) = (A^{\leftarrow}, A^{\rightarrow}, A^{\uparrow}, A^{\downarrow})$ defined as follows:

- (1) $A^{\leftarrow} = (a_1, \dots, a_m)$ where each a_i is the first non-zero entry of the i -th row of A ;
- (2) $A^{\rightarrow} = (a_1, \dots, a_m)$ where each a_i is the last non-zero entry of the i -th row of A ;

(3) $A^\uparrow = (A^t)^\leftarrow$ and $A^\downarrow = (A^t)^\rightarrow$, where A^t is the transpose of A .

The following two lemmas are straightforward.

Lemma 6.2.5. *An $f \times g$ block matrix $B = (A_{ij})$ is a NASM whenever the following types of submatrices of B are all NASM*

$$A_{ij}, \quad [A_{ij} \quad A_{i,j+1}], \quad \begin{bmatrix} A_{ij} \\ A_{i+1,j} \end{bmatrix}.$$

Lemma 6.2.6. *Let A_i be a NASM($m_i, n_i; \mathbf{h}_i, \mathbf{k}_i$), for $i = 1, 2$.*

(1) $[A_1 \quad A_2]$ is a NASM($m_1, n_1 + n_2; \mathbf{h}_1 + \mathbf{h}_2, (\mathbf{k}_1, \mathbf{k}_2)$) if and only if $m_1 = m_2$ and $A_1^\rightarrow = -A_2^\leftarrow$.

(2) $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ is a NASM($m_1 + m_2, n_1; (\mathbf{h}_1, \mathbf{h}_2), \mathbf{k}_1 + \mathbf{k}_2$) if and only if $n_1 = n_2$ and $A_1^\downarrow = -A_2^\uparrow$.

We now show how to build uniform NASMs.

Theorem 6.2.7. *There exists a NASM($m, n; h, k$) if and only if $mh = nk$.*

Proof. If a NASM($m, n; h, k$) exists, then necessarily $mh = nk$. To prove sufficiency, let $f = \gcd(m, k)$ and $g = \gcd(n, h)$. Since $mh = nk$, we have that $\frac{m}{f} \frac{h}{g} = \frac{n}{g} \frac{k}{f}$. Since $\gcd(\frac{m}{f}, \frac{k}{f}) = 1 = \gcd(\frac{h}{g}, \frac{n}{g})$, we can write $\frac{m}{f} = \frac{n}{g} = \ell$, hence $\frac{h}{g} = \frac{k}{f} = d$.

We start by constructing a NASM($\ell, \ell; d, d$). Let $A = (a_{ij})$ be the $\ell \times \ell$ matrix over $\{0, \pm 1\}$ defined as follows:

$$a_{ij} = \begin{cases} (-1)^{i+j} & \text{if } j \leq i \leq \min(\ell, j+d-1), \text{ or} \\ & 1 \leq i \leq j+d-\ell-1 \text{ and } \ell-d \text{ is even,} \\ (-1)^{i+j+1} & \text{if } 1 \leq i \leq j+d-\ell-1 \text{ and } \ell-d \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

One can easily check that each reduced row (respectively reduced column) of A is alternating of length d , that is, A is a NASM($\ell, \ell; d, d$). Furthermore,

$$A^\rightarrow = (-1)^{d+1} A^\leftarrow \quad \text{and} \quad A^\downarrow = (-1)^{d+1} A^\uparrow. \quad (6.2.1)$$

Now we are going to show that the matrix obtained by suitable repetitions of A or $-A$ is a NASM($m, n; h, k$). Let $B = (b_{ij})$ be the $f \times g$ block matrix defined as follows:

$$b_{ij} = \begin{cases} A & \text{if } d \text{ is even,} \\ (-1)^{i+j} A & \text{if } d \text{ is odd,} \end{cases}$$

for $1 \leq i \leq f$ and $1 \leq j \leq g$. Clearly, B is a matrix with $m = f\ell$ rows and $n = g\ell$ columns. Also, its reduced rows (respectively columns) have each length $h = gd$ (respectively $k = fd$). Finally, by Condition (6.2.1) and Lemma 6.2.6, we have that $\pm[A \quad (-1)^d A]$ and $\pm \begin{bmatrix} A \\ (-1)^d A \end{bmatrix}$ are NASMs. It then follows by Lemma 6.2.5 that B is a NASM($m, n; h, k$). \square

Example 6.2.8. In this example we construct a NASM(6, 9; 6, 4) by using the procedure described in the proof of Theorem 6.2.7. We have that:

$$f = \gcd(m, k) = 2, \quad g = \gcd(n, h) = 3,$$

$$\ell = \frac{m}{f} = 3, \quad d = \frac{h}{g} = 2.$$

We can then construct the following NASM(3, 3; 2, 2):

$$A = \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline -1 & 1 & 0 \\ \hline 0 & -1 & 1 \\ \hline \end{array}$$

Since d is even, we build the following array B , that is a NASM(6, 9; 6, 4):

$$B = \begin{array}{|c|c|c|} \hline A & A & A \\ \hline A & A & A \\ \hline \end{array} = \begin{array}{|c|c|c||c|c|c||c|c|c|} \hline 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ \hline -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ \hline 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ \hline 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 \\ \hline -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ \hline 0 & -1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 \\ \hline \end{array}$$

In order to show the complete procedure of the proof of Theorem 6.2.7, we also construct a NASM(12, 16; 4, 3). As before, we compute the parameters:

$$f = \gcd(m, k) = 3, \quad g = \gcd(n, h) = 4,$$

$$\ell = \frac{m}{f} = 4, \quad d = \frac{h}{g} = 1.$$

We take as a NASM(4, 4; 1, 1) the 4×4 identity matrix, denoted by I . Since d is odd, we construct the following block matrix, that is a NASM(12, 16; 4, 3):

$$\begin{array}{|c|c|c|c|} \hline I & -I & I & -I \\ \hline -I & I & -I & I \\ \hline I & -I & I & -I \\ \hline \end{array}$$

We end this section by constructing NASMs whose row and column weights are even.

Theorem 6.2.9. *Let $\mathbf{h} = (h_1, h_1, \dots, h_m, h_m)$ and $\mathbf{k} = (k_1, k_1, \dots, k_n, k_n)$ be sequences of positive integers. There exists a NASM($2m, 2n; 2\mathbf{h}, 2\mathbf{k}$) if and only if Condition (6.1.2) holds.*

Proof. By Remark 6.2.3, we only need to show sufficiency. Since \mathbf{h} and \mathbf{k} satisfy Condition (6.1.2), one can check that the same holds for $\mathbf{h}' = (h_1, \dots, h_m)$ and $\mathbf{k}' = (k_1, \dots, k_n)$. Then, Theorem 6.1.2 guarantees the existence of an $m \times n$ matrix, say A , over \mathbb{Z}_2 such that $w_r(A) = \mathbf{h}'$ and $w_c(A) = \mathbf{k}'$. The array obtained from A by replacing each 0 with $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and each 1 with $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is the desired NASM. \square

6.3 Simple GHAs over a cyclic group

In this section, we build simple GHAs over a cyclic group. We start by showing that the alternated form of an increasing sequence of integers is simple modulo v for every sufficiently large v .

Lemma 6.3.1. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be an increasing sequence of positive integers, and let $\mathbf{b} \in \{\mathbf{a}^\mp, -\mathbf{a}^\mp\}$ and $v > a_n$. Then, all runs of \mathbf{b} are non-zero (mod v).*

Proof. By Remark 6.1.4, it is enough to prove the assertion when $\mathbf{b} = \mathbf{a}^\mp = (-a_1, a_2, -a_3, \dots, (-1)^n a_n)$. We start by showing that the sum $s_0(\mathbf{b})$ of \mathbf{b} is non-zero modulo v . Set $a_0 = 0$, and let $x = a_n$ if n is odd, otherwise set $x = 0$. We notice that $1 \leq -a_{2j-1} + a_{2j} \leq -a_{2j-2} + a_{2j} - 1$, for $1 \leq j \leq \lfloor n/2 \rfloor$. Therefore,

$$\begin{aligned} \lfloor n/2 \rfloor - x \leq s_0(\mathbf{b}) &= \sum_{j=1}^{\lfloor n/2 \rfloor} (-a_{2j-1} + a_{2j}) - x \leq \sum_{j=1}^{\lfloor n/2 \rfloor} (-a_{2j-2} + a_{2j} - 1) - x \\ &= a_{2\lfloor n/2 \rfloor} - x - \lfloor n/2 \rfloor. \end{aligned}$$

In other words,

if n is even, then $x = 0$ and $0 < n/2 \leq s_0(\mathbf{b}) \leq a_n - n/2$;

if n is odd, then $x = a_n$ and $\lfloor n/2 \rfloor - a_n \leq s_0(\mathbf{b}) \leq a_{n-1} - a_n - \lfloor n/2 \rfloor < 0$.

Hence, $s_0(\mathbf{b}) \not\equiv 0 \pmod{v}$. This means that the sums of the alternated forms of an increasing sequence of integers are non-zero modulo v , provided that v is larger than the maximum integer in the sequence.

For every $1 \leq i < j \leq n$, set $\mathbf{a}_{ij} = (a_i, a_{i+1}, \dots, a_j)$ and $\mathbf{b}_{ij} = \mathbf{a}_{ij}^\mp$. Clearly, $s_{ij}(\mathbf{b}) = s_0(\mathbf{b}_{ij})$. Since \mathbf{a}_{ij} is increasing and $v > a_n \geq a_j$, by the first part of the proof it follows that $s_{ij}(\mathbf{b}) \not\equiv 0 \pmod{v}$. \square

The previous construction can be slightly modified to obtain simple zero-sum sequences.

Lemma 6.3.2. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be a sequence of distinct positive integers, such that $a_1 < a_2 < \dots < a_{n-1}$ and let $\mathbf{b} \in \{\mathbf{a}^\mp, -\mathbf{a}^\mp\}$ and $v > a_n$. If $s_0(\mathbf{b}) \equiv 0 \pmod{v}$, then all proper runs of \mathbf{b} are non-zero (mod v).*

Proof. By Remark 6.1.4, it is enough to prove the assertion when $\mathbf{b} = \mathbf{a}^\mp$. Since (a_1, \dots, a_{n-1}) is increasing, by Lemma 6.3.1 we have that all runs of $(-a_1, a_2, -a_3, \dots, (-1)^{n-1} a_{n-1})$ are non-zero modulo v . Considering that by assumption $s_0(\mathbf{b}) \equiv 0 \pmod{v}$, it follows that all proper runs of \mathbf{b} are non-zero. \square

Remark 6.3.3. Sequences satisfying the assumptions of Lemma 6.3.2 can be constructed starting from a balanced sequence, defined in [26, Definition 3.1]. More precisely, an increasing sequence of positive integers $\mathbf{a}' = (a_1, \dots, a_{2n})$ is called *balanced* if there exists $\tau \in [1, n]$ such that $s_0(\mathbf{b}^\mp) = s_0(\mathbf{c}^\mp)$, where $\mathbf{b} = (a_1, \dots, a_{2\tau})$ and $\mathbf{c} = (a_{2\tau+1}, \dots, a_{2n})$. In [26], using a different terminology, it is shown that the sequence

$$\mathbf{a} = (a_1, \dots, a_{2\tau}, a_{2\tau+2}, \dots, a_{2n}, a_{2\tau+1})$$

and its alternated forms \mathbf{a}^\mp and $-\mathbf{a}^\mp$ satisfy the assumptions of Lemma 6.3.2, hence \mathbf{a}^\mp and $-\mathbf{a}^\mp$ are zero-sum and simple modulo $v > a_{2n}$. This result is used in the proof of Theorem 6.3.7.

Lemmas 6.3.1 and 6.3.2 are used in the following, together with near alternating sign matrices, to build simple GHAs.

6.3.1 Non-zero sum and simple GHAs

In this section, we show that NASMs can be used to construct non-zero sum GHAs, that moreover are globally simple.

Theorem 6.3.4. *Assume there is a NASM($m, n; \mathbf{h}, \mathbf{k}$). Then, there exists a non-zero sum and simple NGHA $_S(m, n; \mathbf{h}, \mathbf{k})$ over \mathbb{Z}_v if and only if the necessary Condition (6.1.1) holds, that is, S has no involutions and $|S| = s_0(\mathbf{h}) = s_0(\mathbf{k})$.*

Proof. The necessity of Condition (6.1.1) has been discussed in Section 6.1; therefore, it is enough to prove sufficiency.

Let A be a NASM($m, n; \mathbf{h}, \mathbf{k}$) and $S \subseteq \mathbb{Z}_v^+$ a set satisfying the assumptions. Note that $v \geq 2|S| + 1$ and $|S|$ is the weight of A . Furthermore, set $H = \pi_v(A \circ f(A^*))$ where $f : [0, |S|] \rightarrow [0, \lfloor \frac{v-1}{2} \rfloor]$ is the increasing map fixing 0 such that $S = \pi_v(f[1, |S|])$.

We claim that H is the desired NGHA. First, we notice that $\|\mathcal{E}(H)\| = \pi_v(f(A^*)) = \pi_v(f[1, |S|]) = S$. Now, set $\mathbf{a}_i = f(A^*)_{(i)}$, $\mathbf{b}_i = A_{(i)} \circ \mathbf{a}_i$, for some $i = 1, \dots, m$, and note that $H_{(i)} = \pi_v(\mathbf{b}_i)$. Since A is a NASM, then $A_{(i)}$ is alternating, hence $\mathbf{b}_i \in \{\mathbf{a}_i^\mp, -\mathbf{a}_i^\mp\}$. By the definition of f , we have that \mathbf{a}_i is increasing and its maximum entry is less than v . Therefore, by Lemma 6.3.1, it follows that all runs of \mathbf{b}_i are non-zero modulo v , that is, $H_{(i)}$ is a non-zero sum and simple sequence. Similarly, one can show that each $H^{(j)}$ is non-zero sum and simple. Therefore, H is the desired NGHA. \square

By Theorems 6.2.7 and 6.2.9, Conditions (6.1.1) and (6.1.2) independently guarantee the existence of a NASM($m, n; \mathbf{h}, \mathbf{k}$). Therefore, as a consequence of Theorem 6.3.4 we prove the following:

Theorem 6.3.5. *There exists a non-zero sum and simple NGHA $_S(m, n; \mathbf{h}, \mathbf{k})$ over \mathbb{Z}_v whenever the necessary condition (6.1.1) holds (that is, $S \subseteq [1, \lfloor \frac{v-1}{2} \rfloor]$ and $|S| = s_0(\mathbf{h}) = s_0(\mathbf{k})$), and either*

- (1) \mathbf{h} and \mathbf{k} are constant, or
- (2) m and n are even, $\mathbf{h} = 2 \cdot (h_1, h_1, \dots, h_m, h_m)$, $\mathbf{k} = 2 \cdot (k_1, k_1, \dots, k_n, k_n)$, and the necessary condition (6.1.2) holds.

In particular, Theorems 6.2.7 and 6.3.4 imply the following (constructive) result:

Corollary 6.3.6. *There exists a globally simple NH $_t(m, n; h, k)$ if and only if $m \geq h \geq 2$, $n \geq k \geq 2$, $mh = nk$ and t divides $2nk$.*

We remark that the previous corollary established a constructive results on non-zero sum Heffter arrays that comprehends all the existence results contained in Chapter 5. However, we remark that in Section 5.3 we could easily compute

the face length of some biembeddings that could be constructed from the non-zero sum Heffter arrays. This is not the case for the arrays constructed in Corollary 6.3.6, due to the high generality of the existence result: in fact, it is not even granted that the biembedding exists.

6.3.2 Zero-sum and simple GHAs

In this section we construct GHAs that are globally simple, starting from the non-zero sum GHAs of the previous section.

Theorem 6.3.7. *Let $T = \pi_v(S) \cup \pi_v(S + x)$ where $S \subseteq \mathbb{N}$ and $x, v \in \mathbb{N}$ satisfy the following properties*

- (1) $|S| \equiv 0 \pmod{4}$,
- (2) S is the disjoint union of pairs of consecutive integers,
- (3) $\frac{v}{2} - \max S > x > \max S - \min S$.

If there exists a NASM($m, n; 4\mathbf{h}, 2\mathbf{k}$) of weight $|S|$, then there exists a zero-sum and simple $GHA_T(2m, n; (4\mathbf{h}, 4\mathbf{h}), 4\mathbf{k})$ over \mathbb{Z}_v .

Proof. Let $\mathbf{h} = (h_1, \dots, h_m) \in \mathbb{N}^m$ and let A be a NASM($m, n; \mathbf{h}, 2\mathbf{k}$) of weight $|S| = 4(h_1 + \dots + h_m)$. Since S is the disjoint union of pairs of consecutive integers, we can write $S = \bigcup_{i=1}^m S_i$, with $S_i = \{a_{i1}, \dots, a_{i,4h_i}\}$, such that

- (1) $a_{ij} < a_{pq}$ whenever $(i, j) < (p, q)$ (according to the lexicographic order on $\mathbb{N} \times \mathbb{N}$), and
- (2) $a_{i,2\ell} = a_{i,2\ell-1} + 1$,

for every i, j, ℓ, p, q belonging to the appropriate range of positive integers.

We build the $m \times n$ matrix $B(\epsilon)$, with $\epsilon \in \{0, 1\}$, as follows: for every $i \in [1, m]$ and $j \in [1, 4h_i]$, we replace (following the natural ordering) the j -th non-zero entry of the i -th row of A with $b_{\epsilon, i, j}$, where

$$b_{\epsilon, i, j} = \epsilon x + \begin{cases} a_{ij}, & \text{if } j \in [1, 2h_i]; \\ a_{i, j+1}, & \text{if } j \in [2h_i + 1, 4h_i - 1], \\ a_{i, 2h_i+1}, & \text{if } j = 4h_i. \end{cases} \quad (6.3.1)$$

We claim that $C = \pi_v \begin{pmatrix} A \circ B(0) \\ -A \circ B(1) \end{pmatrix}$ is the desired GHA. First notice that $\|\mathcal{E}(C)\| = \pi_v(\mathcal{E}(B(0))) \cup \pi_v(\mathcal{E}(B(1))) = T$.

We now show that the i -th reduced row of C , that is,

$$C^{(i)} = \pm \begin{cases} \pi_v(A^{(i)} \circ B(0)^{(i)}) & \text{if } 1 \leq i \leq m, \\ \pi_v(A^{(i)} \circ B(1)^{(i)}) & \text{if } m+1 \leq i \leq 2m, \end{cases}$$

is a zero-sum and simple sequence. Let $\mathbf{b} = B(\epsilon)^{(i)} = (b_{\epsilon, i, 1}, \dots, b_{\epsilon, i, 4h_i})$ for some $\epsilon \in \{0, 1\}$ and $i \in [1, m]$. By Condition (1), we have that $(b_{\epsilon, i, 1}, \dots, b_{\epsilon, i, 4h_i-1})$ is increasing, and by Condition (2), one can check that $s_0(\mathbf{b}^\mp) = 0$; also, $v > b_{\epsilon, i, 4h_i}$ (assumption 3). Therefore, by Lemma 6.3.2, all proper runs of

\mathbf{b} are non-zero modulo v . Since $A_{(i)} \circ B(\epsilon)_{(i)} \in \{\mathbf{b}^\mp, -\mathbf{b}^\mp\}$, by Remark 6.1.4, we have that the sequence $C_{(i)}$ is zero-sum and simple.

It is left to show that the j -th reduced column $C^{(j)}$ of C is a zero-sum and simple sequence with respect to some ordering. Letting $B(0)^{(j)} = (b_1, \dots, b_{2k_j})$, by construction we have that

$$C^{(j)} = \pm \pi_v(-b_1, b_2, -b_3, \dots, b_{2k_j}, \\ b_1 + x, -(b_2 + x), b_3 + x, \dots, -(b_{2k_j} + x)).$$

Setting $\mathbf{b} = (b_1, \dots, b_{2k_j}, b_2 + x, b_{2k_j} + x, b_1 + x)$, one can check that $s_0(\mathbf{b}^\mp) = 0$. Since $x > \max S - \min S$ (assumption 3) and in view of Condition (1), removing the last term of \mathbf{b} produces an increasing sequence. Therefore, by Lemma 6.3.2, we have that $\pi_v(\mathbf{b}^\mp)$ and $-\pi_v(\mathbf{b}^\mp)$ are zero-sum and simple sequences. Since one of them is a reordering of $C^{(j)}$, the assertion follows. \square

Remark 6.3.8. When $4\mathbf{h} = \underline{n}$ and $2\mathbf{k} = \underline{m}$, the array C built in Theorem 6.3.7 can be easily rearranged to obtain a zero-sum GHA (with the same list of row and column weights) that is globally simple. Indeed, it is enough to move to the end the $(m+1)$ -th row of C .

In the following, we focus on (the most studied case for classic Heffter arrays, that is) the case where $S = (\mathbb{Z}_v \setminus J)^+$ for some subgroup J of \mathbb{Z}_v .

Corollary 6.3.9. *Let $v = (2d+1)u \equiv u \pmod{16}$, where $u = 1$ or $u \equiv 0 \pmod{4}$, and let $\mathbf{h} = (h_1, \dots, h_m)$ be a partition of $\frac{du}{8}$. If there is a NASM($m, n; 4\mathbf{h}, 2\mathbf{k}$), then there exists a zero-sum and simple $GHA_{(\mathbb{Z}_v \setminus J)^+}(2m, n; (4\mathbf{h}, 4\mathbf{h}), 4\mathbf{k})$, where J is the subgroup of \mathbb{Z}_v of order u .*

Proof. Let $T = S \cup (S+x)$, where $x = \frac{d}{2}$ and $S = [1, x]$ when $u = 1$, otherwise $x = (2d+1)\frac{u}{4}$ and $S = [1, x] \setminus \{i(2d+1) : 1 \leq i \leq \frac{u}{4}\}$. Note that S, x and v satisfy the assumptions of Theorem 6.3.7. Considering that $\pi_v(T) = (\mathbb{Z}_v \setminus J)^+$ and $|S| = \frac{du}{2} = 4(h_1 + \dots + h_m)$, the result follows by Theorem 6.3.7. \square

The existence of NASMs provided by Theorems 6.2.7 and 6.2.9, together with Corollary 6.3.9, implies the following existence result for zero-sum GHAs:

Theorem 6.3.10. *Let $v = (2d+1)u \equiv u \pmod{16}$, where $u = 1$ or $u \equiv 0 \pmod{4}$, and let J be the subgroup of \mathbb{Z}_v of order u . If the necessary conditions (6.1.1) and (6.1.2) hold, then there exist*

- (1) *a zero-sum and simple $GHA_{(\mathbb{Z}_v \setminus J)^+}(4m, 2n; (4\mathbf{h}, 4\mathbf{h}), 4\mathbf{k})$ with*

$$\mathbf{h} = (h_1, h_1, \dots, h_m, h_m) \quad \text{and} \quad \mathbf{k} = (k_1, k_1, \dots, k_n, k_n);$$

- (2) *a zero-sum and simple $GHA_{(\mathbb{Z}_v \setminus J)^+}(2m, n; 4\mathbf{h}, 4\mathbf{k})$.*

In the particular case $m = 2k$ and $n = 4h$, the array that can be build from Theorem 6.3.10.(2) is a $GHA_{(\mathbb{Z}_v \setminus J)^+}(4k, 4h; 4\mathbf{h}, 4\mathbf{k})$; following Remark 6.3.8, by shifting down a suitable row of this array we can obtain a zero-sum GHA (with the same parameters) that is simple according to the natural ordering.

Corollary 6.3.11. *Let $v = 16hk+1$, with $h, k \geq 3$. Then, there exists a globally simple $H(4k, 4h; 4\mathbf{h}, 4\mathbf{k})$.*

6.4 Non-zero sum GHAs over arbitrary groups

In this section, we describe an algorithm that builds a non-zero sum GHA over an arbitrary group G , by replacing the 1's of a suitable $m \times n$ matrix A over \mathbb{Z}_2 with the elements of a multisubset of G . We remark that if the group is not abelian, then the total sum in a row or column of an array depends on the ordering of its elements. In this section, we always assume that the chosen ordering is the natural one and that each row and every column of A contains at least one filled cell.

Consider the following subsets of the skeleton of A :

$$\begin{aligned}\mathcal{R} &= \{(i, j) : a_{ij} \neq 0 = a_{i,j+1} = \cdots = a_{in}\}, \\ \mathcal{C} &= \{(i, j) : a_{ij} \neq 0 = a_{i+1,j} = \cdots = a_{mj}\}, \\ \mathcal{L} &= \mathcal{R} \cap \mathcal{C}.\end{aligned}$$

Clearly, $|\mathcal{R}| = m$, $|\mathcal{C}| = n$ and $1 \leq |\mathcal{L}| \leq \min(m, n)$. We now consider the simple graph $\Phi = (V(\Phi), E(\Phi))$ having $V(\Phi) = \mathcal{R} \cup \mathcal{C}$ as vertex-set, and whose edge-set $E(\Phi)$ is defined as follows:

$$E(\Phi) = \{\{(i, j), (u, v)\} : (i, j) \in \mathcal{R}, (u, v) \in \mathcal{C} \text{ and either } i = u \text{ or } j = v.\} \quad (6.4.1)$$

We start by showing the following.

Lemma 6.4.1. *Φ is a forest, and each one of its connected components contains exactly one vertex of \mathcal{L} .*

Proof. We first show that the smallest vertex (with respect to the lexicographic order) of a path of Φ , say $P = [x_1, \dots, x_\ell]$, is necessarily an end-vertex. Indeed, assume for a contradiction that there is $i \in [2, \ell - 1]$ such that

$$x_i = (u_i, v_i) < x_j = (u_j, v_j), \quad \text{for every } j \in [1, \ell] \setminus \{i\}. \quad (6.4.2)$$

By the definition of the graph Φ , if $x_i \in \mathcal{R}$, then $x_{i-1}, x_{i+1} \in \mathcal{C}$. Hence, by (6.4.1) and (6.4.2), it follows that either $u_i < u_j$ and $v_i = v_j$, or $u_i = u_j$ and $v_i < v_j$, for each $j \in \{i - 1, i + 1\}$. The latter case cannot happen, since $x_i \in \mathcal{R}$ is the last cell of its row with a non-zero entry. Therefore, $u_i < u_j$ and $v_i = v_j$ for each $j \in \{i - 1, i + 1\}$, which however contradicts the fact that $x_{i-1}, x_{i+1} \in \mathcal{C}$. With a similar reasoning when $x_i \in \mathcal{C}$, we obtain another contradiction. Therefore, the smallest vertex of a path of Φ is one of its end-vertices.

This property of Φ implies that it does not contain cycles, hence Φ is a forest. Also, there is no path joining two vertices of \mathcal{L} . Indeed, if $P = [x_1, \dots, x_\ell]$ is such a path, and x_1 is its smallest vertex, then by definition of \mathcal{L} , we would have $x_2 < x_1$, contradicting the minimality of x_1 . Therefore, the vertices of \mathcal{L} belong to different components of Φ . \square

Example 6.4.2. Let $m = 6$, $n = 8$, and:

$$\mathbf{h} = (4, 3, 3, 2, 2, 5) \quad \mathbf{k} = (3, 2, 3, 2, 3, 3, 1, 2).$$

We construct the following 6×8 matrix A with elements in \mathbb{Z}_2 such that $w_r(A) = \mathbf{h}$ and $w_c(A) = \mathbf{k}$:

1	1	1	0	1	0	0	0
1	0	1	0	0	1	0	0
0	1	0	1	1	0	0	0
0	0	1	0	1	0	0	0
0	0	0	0	0	1	0	1
1	0	0	1	0	1	1	1

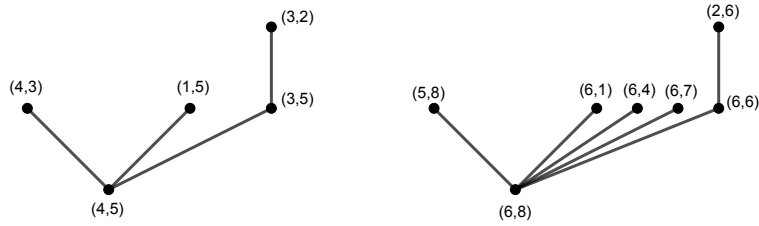
where in bold we have highlighted the cells corresponding to $V(\Phi)$. We then have:

$$\mathcal{R} = \{(1, 5), (2, 6), (3, 5), (4, 5), (5, 8), (6, 8)\},$$

$$\mathcal{C} = \{(6, 1), (3, 2), (4, 3), (6, 4), (4, 5), (6, 6), (6, 7), (6, 8)\},$$

$$\mathcal{L} = \{(4, 5), (6, 8)\}.$$

We conclude with a drawing of Φ :



We are now ready to prove the main result of this section.

Theorem 6.4.3. *Let A be an $m \times n$ array over \mathbb{Z}_2 such that $w_r(A) = \mathbf{h}$ and $w_c(A) = \mathbf{k}$. Let G be an arbitrary group and let $S \subseteq G^+$ such that*

$$|S \setminus I(S)| \geq |\mathcal{R} \cup \mathcal{C}|. \quad (6.4.3)$$

There exists a non-zero sum $NGHA_S^\lambda(m, n; \mathbf{h}, \mathbf{k})$ over G if and only if the necessary condition (6.1.1) holds.

Proof. Let $A = (a_{i,j})$, \mathbf{h} , \mathbf{k} and S be as in the statement. By cyclically shifting the rows (if necessary), we can assume that

$$\text{the last row of } A \text{ has the largest weight.} \quad (6.4.4)$$

Let \mathcal{R} , \mathcal{C} and \mathcal{L} be the sets of cells of A defined above, and let Φ be the forest with vertex-set $V(\Phi) = \mathcal{R} \cup \mathcal{C}$, associated to A . In view of Lemma 6.4.1, for every $z \in \mathcal{L}$, we denote by Φ_z the connected component of Φ containing z and consider a maximal path P_z in Φ_z that contains z as an end-vertex.

Let $T = {}^\lambda(S \setminus I(S)) \cup {}^{\frac{\lambda}{2}}I(S)$. By Condition (6.4.3), there exists a subset F of $S \setminus I(S)$ having the same cardinality as $|V(\mathcal{F})|$. We are going to describe an algorithm that will construct the desired GHA by *properly filling* the cells of A , that is, by replacing the 1's in A with all the elements of T in such a way that the rows and columns of the final matrix have non-zero sums.

1. Filling $\mathcal{B} = \text{skel}(A) \setminus (\mathcal{R} \cup \mathcal{C})$. We arbitrarily fill the cells of A belonging to \mathcal{B} with $T \setminus F$.

2. Filling isolated vertices of Φ . Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of isolated vertices of Φ . For every $(i, j) \in \mathcal{L}'$ and $g \in G$, we say that g is a forbidden element for (i, j) if at least one of the following conditions holds:

- (a) $(i, j) \in \mathcal{R}$ and by filling the cell (i, j) of A with g we have $\sigma_0(A_i) = 0$;
- (b) $(i, j) \in \mathcal{C}$ and by filling the cell (i, j) of A with g we have $\sigma_0(A^j) = 0$.

It follows that each cell in \mathcal{L}' has at most two forbidden elements. Note that $|\mathcal{L}'| \leq |\mathcal{L}| \leq |V(\Phi)|$. If $|\mathcal{L}'| < |\mathcal{L}|$, considering that the elements of F are pairwise distinct in absolute value, it is not difficult to check that there is an injective map $\varphi : \mathcal{L}' \rightarrow F$ such that either $\varphi(i, j)$ or $-\varphi(i, j)$ is not a forbidden element for (i, j) . We use this element to fill the cell (i, j) of A , for every $(i, j) \in \mathcal{L}'$. This procedure guarantees that all rows and columns through a cell of \mathcal{L}' have a non-zero sum. We then remove $\varphi(\mathcal{L}')$ from F .

Now assume that $|\mathcal{L}'| = |\mathcal{L}|$. This means that all components of Φ consist of isolated vertices, that is, $\mathcal{L} = \mathcal{R} = \mathcal{C}$. Therefore, since all cells in the last row of A belong to \mathcal{C} , it follows that the weight of A_m is 1, and by condition (6.4.4), each row of A has weight 1. In this case, 0 is the only forbidden element for each of the isolated vertices. Hence, we can arbitrarily fill the cells of A belonging to \mathcal{L}' with the elements of F , without creating rows or columns whose sum is 0.

3. Filling $W = V(\Phi_z) \setminus V(P_z)$, for every $z \in \mathcal{L} \setminus \mathcal{L}'$. As before, for every $(i, j) \in W$, $g \in G$ is a forbidden element for (i, j) if either condition (a) or (b) (in step 2) is satisfied. Note that only one between (a) and (b) holds, since $W \cap \mathcal{L} = \emptyset$, hence each cell in W has exactly one forbidden element. Since $|F| > |W|$ and all the elements of F are pairwise distinct, there is an injective map $\varphi : W \rightarrow F$ such that $\varphi(i, j)$ is none of the forbidden elements for (i, j) . This guarantees that filling each cell (i, j) of A with $\varphi(i, j)$, for every $(i, j) \in W$ does not produce rows or columns whose sum is zero. We then remove $\varphi(W)$ from F .

4. Filling $V(P_z)$, for every $z \in \mathcal{L} \setminus \mathcal{L}'$. Let $P_z = [x_1, \dots, x_\ell = z]$, where each $x_i = (u_i, v_i)$, and let $F_z = \{f_1, \dots, f_\ell\}$ be any subset of F of size ℓ . If $u_{\ell-1} = u_\ell$ (respectively $v_{\ell-1} = v_\ell$), let $g_\ell \in G$ such that by filling the cell x_ℓ of A with g_ℓ we have $s_0(A^{v_\ell}) = 0$ (respectively $s_0(A_{u_\ell}) = 0$). In other words, g_ℓ is the element that, if used to fill x_ℓ , makes equal to 0 the sum of either the row or the column through x_ℓ . Since $\ell > 1$ and the elements of $F_z = \{f_1, \dots, f_\ell\}$ are distinct in absolute value, we can apply a permutation to F_z so that $g_\ell \notin \{f_\ell, -f_\ell\}$.

Now for every $1 \leq i \leq \ell - 1$ we proceed as follows.

- (1) Let $g_i \in G$ be the unique element in G such that (u_i, v_i) and g_i satisfy either condition (a) or (b) (in step 2);
- (2) Fill cell x_i of A with some element in $\{f_i, -f_i\} \setminus \{g_i\}$;

This iteration guarantees that all rows and columns through the cells of $V(P_z)$ have non-zero sums, except possibly for the row and column through x_ℓ . The

two forbidden elements for x_ℓ are g_ℓ and g'_ℓ . Recall that $g_\ell \notin \{f_\ell, -f_\ell\}$, therefore we can fill x_ℓ with some element in $\{f_\ell, -f_\ell\} \setminus \{g_\ell, g'_\ell\}$ so that the row and column through x_ℓ have a non-zero sum.

It follows that A is a $\text{GHA}_S^\lambda(m, n; \mathbf{h}', \mathbf{k})$ over G , where \mathbf{h}' is a cyclic permutation of \mathbf{h} . Since a cyclic shift of the rows does not change the order of the column sums, by shifting back the rows of A (if necessary), we obtain the desired GHA. \square

By using the trivial upper bound $|\mathcal{RUC}| \leq m+n-1$, we obtain the following:

Theorem 6.4.4. *Let G be an arbitrary group and let $S \subseteq G^+$ such that:*

$$|S \setminus I(S)| \geq m+n-1.$$

Then, there exists a non-zero sum $\text{GHA}_S^\lambda(m, n; \mathbf{h}, \mathbf{k})$ over G if and only if Conditions 6.1.1 and 6.1.2 hold.

This result improves previous results contained in [34], where the existence of a non-zero sum uniform GHA_S^λ is proven, using a probabilistic and non-constructive approach, when $|G| \geq 41$ and $S = (G \setminus J)^+$, where J is a subgroup of G .

Example 6.4.5. Let A be the following 6×8 array having $w_r(A) = \mathbf{h} = (5, 6, 5, 4, 5, 5)$ and $w_c(A) = \mathbf{k} = (6, 3, 4, 6, 4, 3, 2, 2)$ (notice that its graph Φ is isomorphic to the one of the array constructed in Example 6.4.2):

1	1	1	1	1	0	0	0
1	1	1	1	1	1	0	0
1	1	1	1	1	0	0	0
1	0	1	1	1	0	0	0
1	0	0	1	0	1	1	1
1	0	0	1	0	1	1	1

We have $|\mathcal{RUC}| = 12$ and $w(A) = 30$, hence for $S = \{\alpha^j : j \in [1, 12]\} \cup \{\alpha^j \beta : j \in [0, 5]\}$ subset of the dihedral group $\text{Dih}_{25} = \langle \alpha, \beta \mid \alpha^{25} = \beta^2 = (\alpha\beta)^2 = 1 \rangle$, having order 50, we construct a $\text{NGHA}_S^2(6, 8; \mathbf{h}, \mathbf{k})$ by following the proof of Theorem 6.4.4. We begin by defining $F = S \setminus I(S) = \{\alpha^j : j \in [1, 12]\}$, and we fill the array B with the elements of ${}^2S \setminus (F \cup I(S))$:

β	$\alpha\beta$	$\alpha^2\beta$	α^2	1	0	0	0
$\alpha^3\beta$	α	$\alpha^4\beta$	α^4	α^3	1	0	0
α^7	1	α^6	$\alpha^5\beta$	1	0	0	0
α^{10}	0	1	α^5	1	0	0	0
α^{11}	0	0	α^8	0	α^9	α^{12}	α^2
1	0	0	1	0	1	1	1

The graph Φ does not contain isolated vertices, thus we directly move to step 3 of the proof of Theorem 6.4.4 by filling $V(\Phi) \setminus V(P_1 \cup P_2)$, where $P_1 = [(6, 8), (6, 1)]$ and $P_2 = [(4, 5), (3, 5), (3, 2)]$. We can fill each one of these cells with either a

or a^{-1} , for any a in F :

β	$\alpha\beta$	$\alpha^2\beta$	α^2	α^{-6}	0	0	0
$\alpha^3\beta$	α	$\alpha^4\beta$	α^4	α^3	α^7	0	0
α^7	1	α^6	$\alpha^5\beta$	1	0	0	0
α^{10}	0	α^{-4}	α^5	1	0	0	0
α^{11}	0	0	α^8	0	α^9	α^{12}	α^2
1	0	0	α^5	0	α	α^3	1

In step 4 we have $F_1 = \{\alpha^8, \alpha^9\}$ and $F_2 = \{\alpha^{10}, \alpha^{11}, \alpha^{12}\}$. Since no rows or columns of the array have forbidden elements contained in $F_1 \cup F_2$, we can fill the remaining cells as follows:

β	$\alpha\beta$	$\alpha^2\beta$	α^2	α^{-6}	0	0	0
$\alpha^3\beta$	α	$\alpha^4\beta$	α^4	α^3	α^7	0	0
α^7	α^{10}	α^6	$\alpha^5\beta$	α^{-11}	0	0	0
α^{10}	0	α^{-4}	α^5	α^{12}	0	0	0
α^{11}	0	0	α^8	0	α^9	α^{12}	α^2
α^8	0	0	α^5	0	α	α^3	α^9

Since in a cyclic group $I(S) \leq 1$, we have the following result, that generalizes the one proved in [34]:

Corollary 6.4.6. *There exists a $\text{NH}_t^\lambda(m, n; h, k)$ if and only if $m \geq h \geq 2$, $n \geq k \geq 2$, $mh = nk$ and t divides $2nk/\lambda$.*

Chapter 7

Heffter spaces

As showed in Chapter 1 a Heffter array can be seen as two orthogonal Heffter systems. An intriguing question recently raised and studied by Buratti and Pasotti in [23, 24] asks then if and when it is possible to find more Heffter systems defined on the same half-set that are pairwise orthogonal; this collection of Heffter systems is then said to be a Heffter space. In this setting the elements of the half-set can be represented as points in an incidence structure whose parallel classes correspond to the Heffter systems. In Section 7.1 we report the notions of Finite Geometry that are necessary to introduce Heffter spaces in Section 7.2, following [23, 24]. In Section 7.3 we consider relative Heffter spaces, and we present some existence results that we recently obtained in a joint work with M. Buratti, L. Johnson and A. Pasotti in [21], still in progress.

7.1 Linear spaces and partial linear spaces

In this section we introduce some combinatorial structures that are necessary to define Heffter spaces.

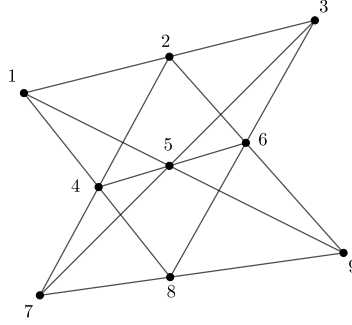
Definition 7.1.1. A *partial linear space* (in short, PLS) is a pair $(\mathcal{V}, \mathcal{B})$ of a set \mathcal{V} of v points and a set \mathcal{B} of blocks, such that any pair of points is contained in at most one block and every block contains at least two points. A PLS is said to be *linear* if every pair of points is contained in exactly one block.

The *degree* of a point p is the number of blocks in the PLS that contain p . A PLS is *k-uniform* (or simply *uniform*) if every block has the same size k , while it is said to be *of degree r* if every point has degree r . A k -uniform PLS of degree r over v points is said to be a $(v, k; r)$ *configuration*. It can be seen that a k -uniform linear space is nothing but an $S(2, v, k)$.

A *parallel class* of a PLS $(\mathcal{V}, \mathcal{B})$ is a set of blocks that partitions \mathcal{V} , and a PLS is said to be *resolvable* if the block set \mathcal{B} can be partitioned into parallel classes; such a partition is said to be a *resolution* of the PLS. A *resolved* PLS is a resolvable PLS, together with one specific resolution of it. Given three integers v , k and r we write $(v, k; r)$ -RPLS to denote a resolved PLS with v points whose resolution consists of r k -uniform parallel classes.

Example 7.1.2. The Fano plane of Example 1.1.1 is a linear space that is not resolvable: indeed, a necessary condition for the resolvability of a k -uniform

PLS over v points is that k must divide v , that is not the case for $k = 3$ and $v = 7$. The *Pappus configuration* represented below is a resolvable $(9, 3; 3)$ configuration:



The unique resolution of the Pappus configuration is the following:

$$\mathcal{P} = \{ \{ \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\} \}, \\ \{ \{1, 4, 8\}, \{2, 6, 9\}, \{3, 5, 7\} \}, \\ \{ \{1, 5, 9\}, \{2, 4, 7\}, \{3, 6, 8\} \} \}.$$

Note that the Pappus configuration can be completed to a resolvable linear space by adding the blocks $\{1, 6, 7\}, \{2, 5, 8\}, \{3, 4, 9\}$.

Example 7.1.3. In this example we examine a notable class of resolvable linear PLS called *Desarguesian affine planes*; we remark that while this definition can be given in general for prime powers, here we consider the prime case. Let then p be an odd prime and consider the following subsets of \mathbb{Z}_p^2 :

- (1) for every pair $(a, b) \in \mathbb{Z}_p^2$, let $B_{a,b} = \{(x, y) \in \mathbb{Z}_p^2 : y = ax + b\}$;
- (2) for every $b \in \mathbb{Z}_p$, let $B_{\infty,b} = \{(x, y) \in \mathbb{Z}_p^2 : x = b\}$.

We then define the *Desarguesian affine plane of order p* as the pair $(\mathcal{V}, \mathcal{B})$, where:

$$\mathcal{V} = \mathbb{Z}_p^2, \quad \mathcal{B} = \left(\bigcup_{(a,b) \in \mathbb{Z}_p^2} B_{a,b} \right) \cup \left(\bigcup_{b \in \mathbb{Z}_p} B_{\infty,b} \right).$$

The parallel classes of \mathcal{B} are:

$$\mathcal{L}_a = \bigcup_{b \in \mathbb{Z}_p} B_{a,b} \text{ for every } a \in \mathbb{Z}_p, \quad \mathcal{L}_\infty = \bigcup_{b \in \mathbb{Z}_p} B_{\infty,b},$$

hence a resolution of \mathcal{B} is given by $\{\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{p-1}, \mathcal{L}_\infty\}$.

7.2 Heffter spaces

In this section we introduce the notion of Heffter spaces, that are a class of combinatorial structures defined in [23]. This concept merges the topics of Finite Geometry presented in the previous section with the theory of Heffter systems and Heffter arrays.

Definition 7.2.1. A $(v, k; r)$ Heffter space over an abelian group G is a $(v, k; r)$ RPLS such that:

- (1) its point-set is a half-set of G ;
- (2) each block is zero-sum.

A Heffter space is *linear* if the RPLS is linear as well.

We remark that the original definition of a Heffter space given in [23] is more general, but the one that is given here is sufficient for the purposes of this thesis. It has been proven in [23] that a $(v, k; r)$ Heffter space is equivalent to r mutually orthogonal Heffter systems $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$ defined over the same half-set, such that each \mathcal{P}_i is a (v, k) Heffter system. Thus a Heffter space of degree 1 is a Heffter system, while a Heffter space of degree 2 is a (square) Heffter array. Clearly the main goal of this topic is to construct Heffter spaces having a large degree.

Example 7.2.2. Consider the following Heffter systems over \mathbb{Z}_{41} :

$$\begin{aligned}\mathcal{P}_1 &= \{\{1, 3, 4, 33\}, \{2, 5, 13, 21\}, \{6, 23, 24, 29\}, \{7, 9, 10, 15\}, \{11, 19, 25, 27\}\}, \\ \mathcal{P}_2 &= \{\{1, 9, 12, 29\}, \{3, 6, 13, 19\}, \{4, 5, 7, 25\}, \{10, 21, 24, 27\}, \{11, 15, 23, 33\}\}, \\ \mathcal{P}_3 &= \{\{1, 6, 7, 27\}, \{2, 4, 11, 24\}, \{3, 5, 10, 23\}, \{9, 19, 21, 33\}, \{13, 15, 25, 29\}\}.\end{aligned}$$

It can be seen that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ are defined over the same half-set of \mathbb{Z}_{41} and pairwise mutually orthogonal, hence $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is a $(20, 4; 3)$ Heffter space.

As remarked in [23] Heffter spaces share an interesting connection with the concept of *additive designs* introduced in [28]; here we directly define this concept on partial linear spaces. A PLS $(\mathcal{V}, \mathcal{B})$ is said to be *G-additive* if there exists an injective map f from \mathcal{V} to an abelian group $(G, +)$ such that $\sum_{x \in B} f(x) = 0$ for every $B \in \mathcal{B}$. It follows that a Heffter space over a group G is a *G-additive* PLS $(\mathcal{V}, \mathcal{B})$, such that $\{f(x) : x \in \mathcal{V}\}$ is a half-set of G .

In [23] the authors showed that if a $(v, k; r)$ Heffter space exists, then:

$$r(k-1) \leq v-1, \tag{7.2.1}$$

where the equality holds if and only if the Heffter space is linear.

It has been proven in [23, 24] using finite fields that it is possible to construct Heffter spaces having arbitrarily large degree:

Theorem 7.2.3. *Given any $k \geq 3$ and any $r > 0$, there are infinitely many values of v for which there exists a $(v, k; r)$ Heffter space.*

This result was obtained by constructing the Heffter space for v sufficiently large with respect to k and r . As remarked in [23] the most interesting and difficult to construct are Heffter spaces close to being linear spaces. For this reason, in [23] the authors introduce a parameter that quantifies how close a Heffter space is to being linear:

Definition 7.2.4. Given a $(v, k; r)$ Heffter space \mathcal{H} its *density* is

$$\delta(\mathcal{H}) = \frac{r(k-1)}{v-1}.$$

The density of a Heffter space \mathcal{H} is a parameter between 0 and 1, where the upper bound is reached if and only if \mathcal{H} is linear.

Remark 7.2.5. The densest Heffter space of [23] has density equal to 0.75, while Nakić [78] with the aid of a computer established the existence of a Heffter space with density $\frac{21}{26} > 0.807$.

The existence of a linear Heffter space is currently an open problem. It has been shown in [23] that such a combinatorial structure may exist under very restrictive conditions.

Proposition 7.2.6. *If there exists a Heffter linear space of degree r over a group G , then $r - 1$ is a multiple of the order of any element $g \in G$.*

In particular, as highlighted in [23]:

Corollary 7.2.7. *A Heffter space over a cyclic group cannot be linear.*

As for Heffter systems and for Heffter arrays, an interesting property that can be studied on Heffter spaces in view of their applications is the simplicity:

Definition 7.2.8. A Heffter space \mathcal{H} is *simple* if every block of \mathcal{H} admits a simple ordering of its elements.

It is easy to see that, as for simple Heffter arrays, it holds:

Proposition 7.2.9. *If there exists a simple $(v, k; r)$ Heffter space over an abelian group G , then there exist r mutually orthogonal G -regular k -cycle systems of order $2v + 1$.*

Clearly, the most interesting problem is that of establishing the maximum number of mutually orthogonal k -cycle systems of order v , denoted by $\mu(k, v)$ in [25]. Firstly, the existence results on simple $H(n; k)$ Heffter arrays obtained in [27, 39, 47] imply that $\mu(k, 2nk + 1) \geq 2$ for the corresponding values of n, k . Further constructions of pairs of orthogonal k -cycle systems are shown in [64]. In [25] the authors give interesting lower bounds on $\mu(k, v)$, proving that:

$$\mu(k, 2nk + 1) \geq \begin{cases} 4n & \text{if } k = 4, \\ \frac{n}{4k - 2} - 1 & \text{if } k \equiv 0 \pmod{4}, k > 4, \\ \frac{n}{24k - 18} - 1 & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

As a consequence of the existence results on simple Heffter spaces presented in [23, 24] we have the following:

Theorem 7.2.10. *Let $2nk + 1$ be a prime power for some integers $3 \leq k \leq n$. Then:*

$$\mu(k, 2nk + 1) \geq \begin{cases} \left\lfloor \frac{n}{k^3} \right\rfloor & \text{if } n \text{ is odd,} \\ \left\lceil \frac{n}{4k^4} \right\rceil & \text{if } nk \text{ is odd.} \end{cases}$$

7.3 Relative Heffter spaces

In this section we consider the notion of a relative Heffter space, that is a Heffter space whose point-set is a half-set of an abelian group G minus a non trivial subgroup. More formally:

Definition 7.3.1. Given an abelian group G and a non trivial subgroup J of G , a $(v, k; r)$ Heffter space over G relative to J is a $(v, k; r)$ RPLS such that:

- (1) its point-set is a half-set of $G \setminus J$;
- (2) each block is zero-sum.

A relative Heffter space is *linear* if the RPLS is linear as well.

It can be proven analogously to the statement given in [23] that a $(v, k; r)$ relative Heffter space over $G \setminus J$ is equivalent to r mutually orthogonal Heffter systems $\{\mathcal{P}_1, \dots, \mathcal{P}_r\}$ defined over the same half-set, such that each \mathcal{P}_i is a (v, k) Heffter system over $G \setminus J$. In particular a relative Heffter space of degree 2 is nothing but a (square) relative Heffter array.

As for classical Heffter spaces, if a relative $(v, k; r)$ Heffter space exists then:

$$r(k - 1) \leq v - 1. \tag{7.3.1}$$

Example 7.3.2. Let $v = 15$ and $k = 3$. Let L be the following half-set of $\mathbb{Z}_{35} \setminus \langle 7 \rangle$:

$$L = \{1, 2, -3, 4, -5, -6, 8, 9, -10, 11, -12, -13, 15, 16, -17\}.$$

We consider the following Heffter systems over L :

$$\begin{aligned} H_1 &= \{\{1, 2, -3\}, \{4, 8, -12\}, \{9, 11, 15\}, \{-5, -13, -17\}, \{-6, -10, 16\}\}, \\ H_2 &= \{\{1, 16, -17\}, \{2, 8, -10\}, \{-3, -12, 15\}, \{4, 9, -13\}, \{-5, -6, 11\}\}, \\ H_3 &= \{\{1, 11, -12\}, \{2, 4, -6\}, \{-3, -13, 16\}, \{-5, -10, 15\}, \{8, 9, -17\}\}, \\ H_4 &= \{\{1, 4, -5\}, \{2, 15, -17\}, \{-3, -6, 9\}, \{8, 11, 16\}, \{-10, -12, -13\}\}, \\ H_5 &= \{\{1, 9, -10\}, \{2, 11, -13\}, \{-3, -5, 8\}, \{4, 15, 16\}, \{-6, -12, -17\}\}. \end{aligned}$$

It can be seen that these Heffter systems are pairwise orthogonal, hence $\mathcal{H} = \{H_1, H_2, H_3, H_4, H_5\}$ is a set of mutually orthogonal Heffter systems, or equivalently a $(15, 3; 5)$ Heffter space over $\mathbb{Z}_{35} \setminus \langle 7 \rangle$. Moreover, from each pair of orthogonal Heffter systems we can construct an $H_5(5; 3)$. For instance we can construct the following relative Heffter array whose rows (columns respectively) represent H_1 (H_2 respectively):

1	2	-3		
	8	-12	4	
		15	9	11
-17			-13	-5
16	-10			-6

It is easy to see that Proposition 7.2.6 holds also in the context of relative Heffter spaces. Hence we have the following:

Corollary 7.3.3. *A relative Heffter space over a cyclic group cannot be linear.*

As for simple Heffter spaces, from a simple relative Heffter space one can construct mutually orthogonal cycle decompositions of a complete multipartite graph:

Proposition 7.3.4. *If there exists a simple $(v, k; r)$ Heffter space over an abelian group G relative to a subgroup J , then there exist r mutually orthogonal G -regular k -cycle decompositions of $K_{|G|/|J| \times |J|}$.*

Following the results obtained in [21] we show a new construction of an infinite family of simple relative Heffter spaces. In order to present these results we first build a new family of globally simple relative square Heffter arrays. Then we show that these arrays contain many pairwise orthogonal Heffter systems apart from the ones given by their rows and columns, that overall form a relative and simple Heffter space.

We show now a construction for an infinite family of relative globally simple $H_n(n; k)$. We remark that for this set of parameters the existence of a relative Heffter array was not known; moreover, from this family of arrays we can construct relative Heffter spaces of degree $r > 2$. For the sake of exposition we represent the Heffter space as being embedded inside of the Heffter array. We begin with the following auxiliary lemmas:

Lemma 7.3.5. *Let $a \in \mathbb{Z}_{n(2k+1)} \setminus \{0\}$, and let J be the subgroup of $\mathbb{Z}_{n(2k+1)}$ of order n . Then, for any $b \in \mathbb{Z}_{n(2k+1)} \setminus \{0\}$, $\{a + b \cdot i(2k+1) : i \in [1, n]\}$ is the coset $a + J$ if and only if $\gcd(b, n) = 1$.*

Lemma 7.3.6. *There exists a zero-sum half-set of \mathbb{Z}_{2k+1} admitting a simple ordering for every integer $k \geq 3$.*

Proof. We divide the proof into cases, depending on the value of k modulo 4.

Case $k \equiv 1 \pmod{4}$. If $k = 5$, it is immediate to check that $L = \{1, -2, 3, 4, 5\}$ is a zero-sum half-set admitting a simple ordering in \mathbb{Z}_{11} . For $k \geq 9$ consider the following half-set of \mathbb{Z}_{2k+1} :

$$L = \{-1, -2, 3\} \cup \left\{ 2i, -(2i+1) \mid i \in \left[2, \frac{k+3}{4} \right] \right\} \cup \left\{ -2i, 2i+1 \mid i \in \left[\frac{k+7}{4}, \frac{k-3}{2} \right] \right\} \cup \{-(k-1), -k\}.$$

It can be easily seen that the sum of the elements in L is 0 modulo $2k+1$: the first bracket sums to zero, while the second and the third bracket respectively add to $-\frac{k-1}{4}$ and $\frac{k-9}{4}$, and the last one sums to $-2k+1$. A simple ordering of L is then:

$$\left(-k, -1, 4, -5, 6, -7, \dots, \frac{k+3}{2}, -\frac{k+5}{2}, -2, -(k-1), k-2, -(k-3), \dots, -\frac{k+7}{2}, 3 \right)$$

Indeed, its partial sums are:

$$\left(-k, -k-1, -k+3, -k-2, -k+4, -k-3, \dots, \frac{-3k+7}{4}, -\frac{5k+3}{4}, -\frac{5k+11}{4}, -\frac{k+3}{4}, \frac{3k-11}{4}, -\frac{k-1}{4}, \dots, -3, 0 \right),$$

that are all distinct.

Case $k \equiv 2 \pmod{4}$. We construct the following half-set of \mathbb{Z}_{2k+1} :

$$L = \left\{ 2i - 1, -2i \mid i \in \left[1, \frac{k+2}{4} \right] \right\} \cup \left\{ -(2i-1), 2i \mid i \in \left[\frac{k+6}{4}, \frac{k}{2} - 1 \right] \right\} \cup \{-(k-1), -k\}.$$

It can be easily verified that L is zero-sum in \mathbb{Z}_{2k+1} ; a simple ordering of L is:

$$\left(-k, 1, -2, 3, -4, \dots, \frac{k}{2}, -\frac{k+2}{2}, -(k-1), k-2, -(k-3), \dots, \frac{k}{2} + 3, -\frac{k+4}{2} \right),$$

with partial sums

$$\left(-k, -k+1, -k-1, -k+2, -k-2, \dots, \frac{-3k+2}{4}, \frac{-5k+2}{4}, \frac{k-6}{4}, \frac{3k-2}{4}, \frac{k-10}{4}, \dots, \frac{k}{2} + 2, 0 \right)$$

that are all distinct.

Case $k \equiv 3 \pmod{4}$. If $k = 3$, then choose $L = \{1, 2, -3\}$, where clearly any ordering of its elements is simple. Assume then that $k \geq 7$, and consider the following half-set L of \mathbb{Z}_{2k+1} :

$$L = \{1\} \cup \left\{ 2i, -(2i+1) \mid i \in \left[1, \frac{k+1}{4} \right] \right\} \cup \left\{ -2i, 2i+1 \mid i \in \left[\frac{k+5}{4}, \frac{k-1}{2} \right] \right\}.$$

Then, a simple ordering for L is:

$$\left(k, 2, -3, 4, -5, \dots, \frac{k+1}{2}, -\frac{k+3}{2}, -(k-1), k-2, -(k-3), \dots, -\frac{k+7}{2}, \frac{k+5}{2}, 1 \right)$$

whose partial sums are:

$$\left(k, k+2, k-1, k+3, k-4, \dots, \frac{5k+5}{4}, \frac{3k-1}{4}, \frac{k-3}{4}, \frac{3k-5}{4}, \frac{-k+7}{4}, \dots, \frac{k+5}{2}, -1, 0 \right).$$

Case $k \equiv 0 \pmod{4}$. Construct the half-set L of \mathbb{Z}_{2k+1} defined as:

$$L = \left\{ 2i - 1, -2i \mid i \in \left[1, \frac{k}{4} \right] \right\} \cup \left\{ -(2i-1), 2i \mid i \in \left[\frac{k}{4} + 1, \frac{k}{2} \right] \right\}.$$

A simple ordering is then the following:

$$\left(k, 1, -2, 3, -4, \dots, -\frac{k}{2}, -(k-1), k-2, -(k-3), \dots, \frac{k}{2} + 2, -\left(\frac{k}{2} + 1\right) \right)$$

since its partial sums are:

$$\left(k, k+1, k-1, k+2, k-2, \dots, \frac{5k}{4} - 1, \frac{3k}{4}, -\frac{k}{4} + 1, \frac{3k}{4} - 1, -\frac{k}{4} + 2, \dots, \frac{k}{2} + 1, 0 \right).$$

□

Example 7.3.7. In this example we construct a simple ordering of a half-set of \mathbb{Z}_{2k+1} for each congruence class of k modulo 4, following the proof of Lemma 7.3.6. In each case, we represent the elements of \mathbb{Z}_{2k+1} using the residues $[-k, k]$.

Let $k = 13 \equiv 1 \pmod{4}$. We consider the following half-set L of $\mathbb{Z}_{2k+1} = \mathbb{Z}_{27}$ (to aid the reader, we partition L as done in the proof of Proposition 7.3.6):

$$L = \{-1, -2, 3\} \cup \{4, -5, 6, -7, 8, -9\} \cup \{-10, 11\} \cup \{-12, -13\}.$$

Then, a simple ordering of L is given by:

$$(-13, -1, 4, -5, 6, -7, 8, -9, -2, -12, 11, -10, 3),$$

having partial sums:

$$(-13, 13, -10, 12, -9, 11, -8, 10, 8, -4, 7, -3, 0).$$

Let $k = 14 \equiv 2 \pmod{4}$. We construct the following half-set of \mathbb{Z}_{29} :

$$L = \{1, -2, 3, -4, 5, -6, 7, -8\} \cup \{-9, 10, -11, 12\} \cup \{-13, -14\}.$$

We then have the following ordering:

$$(-14, 1, -2, 3, -4, 5, -6, 7, -8, -13, 12, -11, 10, -9).$$

Let $k = 15 \equiv 3 \pmod{4}$. The following set L is a zero-sum half-set of \mathbb{Z}_{31} :

$$L = \{1\} \cup \{2, -3, 4, -5, 6, -7, 8, -9\} \cup \{-10, 11, -12, 13, -14, 15\}.$$

Then, a simple ordering of L is:

$$(15, 2, -3, 4, -5, 6, -7, 8, -9, -14, 13, -12, 11, -10, 1).$$

Let $k = 16 \equiv 0 \pmod{4}$. We construct the zero-sum half-set L of \mathbb{Z}_{33} :

$$L = \{1, -2, 3, -4, 5, -6, 7, -8\} \cup \{-9, 10, -11, 12, -13, 14, -15, 16\},$$

We then have the following simple ordering:

$$(16, 1, -2, 3, -4, 5, -6, 7, -8, -15, 14, -13, 12, -11, 10, -9).$$

Proposition 7.3.8. *For every odd integer n and every $3 \leq k \leq n$ there exists a globally simple $H_n(n; k)$.*

Proof. Let $L = \{\ell_1, \dots, \ell_k\}$ be the zero-sum simple half-set of \mathbb{Z}_{2k+1} given in the proof of Lemma 7.3.8, where its elements are labelled so that $\omega = (\ell_1, \dots, \ell_k)$ is a simple ordering of L . Assume first that L is also zero-sum in the integers. For each $i \in \{3, 4, 5\}$, define the following blocks of diagonals:

$\mathcal{D}_3(\{a_1, a_2, a_3\}; m) :$	$\text{diag}(1, m, a_1, 1, 2k+1, n)$ $\text{diag}(1, m+1, a_2, 1, -2(2k+1), n)$ $\text{diag}(1, m+2, a_3, 1, 2k+1, n)$
$\mathcal{D}_4(\{a_1, a_2, a_3, a_4\}; m) :$	$\text{diag}(1, m, a_1, 1, 2k+1, n)$ $\text{diag}(1, m+1, a_2, 1, -(2k+1), n)$ $\text{diag}(1, m+2, a_3, 1, -(2k+1), n)$ $\text{diag}(1, m+3, a_4, 1, 2k+1, n)$

$\mathcal{D}_5(\{a_1, a_2, a_3, a_4, a_5\}; m) :$	$\text{diag}(1, m, a_1, 1, 2k + 1, n)$ $\text{diag}(1, m + 1, a_2, 1, -2(2k + 1), n)$ $\text{diag}(1, m + 2, a_3, 1, 2(2k + 1), n)$ $\text{diag}(1, m + 3, a_4, 1, -2(2k + 1), n)$ $\text{diag}(1, m + 4, a_5, 1, 2k + 1, n)$
---	---

We now construct the desired $H_n(n; k)$, which is dependant upon the congruence class of k modulo 4:

- $k \equiv 0 \pmod{4}$: let $k = 4t$ for some positive integer t . The following t block diagonals (where $i \in [0, t - 1]$) form an $n \times n$ array A :

$$\mathcal{D}_4(\{\ell_{1+4i}, \ell_{2+4i}, \ell_{3+4i}, \ell_{4+4i}\}; 4i + 1).$$

- $k \equiv 1 \pmod{4}$: let $k = 5 + 4t$ for some positive integer t . Construct an $n \times n$ array A from the following block diagonals:

$$\begin{aligned} &\mathcal{D}_4(\{\ell_{1+4i}, \ell_{2+4i}, \ell_{3+4i}, \ell_{4+4i}\}; 4i + 1) \text{ for every } i \in [0, t - 1]; \\ &\mathcal{D}_5(\{\ell_{k-4}, \ell_{k-3}, \ell_{k-2}, \ell_{k-1}, \ell_k\}; 4t + 1). \end{aligned}$$

- $k \equiv 2 \pmod{4}$: let $k = 6 + 4t$ for some positive integer t , and let A be an $n \times n$ array containing the following block diagonals:

$$\begin{aligned} &\mathcal{D}_4(\{\ell_{1+4i}, \ell_{2+4i}, \ell_{3+4i}, \ell_{4+4i}\}; 4i + 1) \text{ for every } i \in [0, t - 1]; \\ &\mathcal{D}_3(\{\ell_{k-5}, \ell_{k-4}, \ell_{k-3}\}; 4t + 1); \\ &\mathcal{D}_3(\{\ell_{k-2}, \ell_{k-1}, \ell_k\}; 4t + 4). \end{aligned}$$

- $k \equiv 3 \pmod{4}$: let $k = 3 + 4t$ for some positive integer t . We can then build an $n \times n$ array A using the following block diagonals:

$$\begin{aligned} &\mathcal{D}_4(\{\ell_{1+4i}, \ell_{2+4i}, \ell_{3+4i}, \ell_{4+4i}\}; 4i + 1) \text{ for every } i \in [0, t - 1]; \\ &\mathcal{D}_3(\{\ell_{k-2}, \ell_{k-1}, \ell_k\}; 4t + 1). \end{aligned}$$

It can be easily seen that if S is a set of size $r \in \{3, 4, 5\}$, then every row and column of $\mathcal{D}_r(S, m)$ has sum $\sum_{s \in S} s$. This implies that each row and each column of A sums to $\sum_{i=1}^k \ell_i \equiv 0 \pmod{2pk + p}$ for each congruence class of k modulo 4. Moreover, by applying Lemma 7.3.5 with $b \in \{1, 2\}$, it can be seen that the array A is filled with the elements $L + J = \{\ell + j : \ell \in L, j \in J\}$; since L is a half-set of \mathbb{Z}_{2k+1} , $L + J$ is a half-set of $\mathbb{Z}_{n(2k+1)}$.

Assume now that $\sum_{\ell \in L} \ell \equiv \lambda \pmod{2nk + n}$ (note that necessarily λ is a multiple of $2k + 1$); it is sufficient to construct the array A as previously shown, where now the elements that have to be placed in the block diagonals are $\ell_1 - \lambda, \ell_2, \dots, \ell_k$. It follows that every row and every column is zero-sum, and the array is filled with $L + J$, that is again a half-set of $\mathbb{Z}_{n(2k+1)}$.

Now, every natural ordering of every row and every column of A modulo $2k + 1$ is a cyclic permutation of the ordering $\omega = (\ell_1, \dots, \ell_k)$, that is simple by hypothesis. Clearly, a cyclic permutation of a simple ordering in a zero-sum set is simple as well, and if the natural ordering of a row (respectively, of a column) is simple modulo \mathbb{Z}_{2k+1} , then it is simple also modulo $\mathbb{Z}_{n(2k+1)}$. Thus A is the desired $H_n(n; k)$. \square

Example 7.3.9. In this example we build an $H_{11}(11; 9)$ by following the construction given in the proof of Proposition 7.3.8. For this set of parameters we have that the group is \mathbb{Z}_{209} and the array is filled with the elements of $\mathbb{Z}_{209} \setminus \langle 19 \rangle$. We begin with the following zero-sum half-set in \mathbb{Z}_{19} :

$$L = \{-1, -2, 3, 4, -5, 6, -7, -8, -9\}.$$

Note that L adds to -19 in the integers: we consider then the set $L' = \{18, -2, 3, 4, -5, 6, -7, -8, -9\}$ in \mathbb{Z}_{209} . A simple ordering of L' is

$$\omega = (-9, 18, 4, -5, 6, -7, -2, -8, 3).$$

We now construct the block diagonals with the following procedures:

$$\begin{array}{l} \mathcal{D}_4(\{-9, 18, 4, -5\}; 1) : \\ \underline{\text{diag}(1, 1, -9, 1, 19, 11)} \\ \text{diag}(1, 2, 18, 1, -19, 11) \\ \text{diag}(1, 3, 4, 1, -19, 11) \\ \underline{\text{diag}(1, 4, -5, 1, 19, 11)} \end{array} \qquad \begin{array}{l} \mathcal{D}_5(\{6, -7, -2, -8, 3\}; 5) : \\ \underline{\text{diag}(1, 5, 6, 1, 19, 11)} \\ \text{diag}(1, 6, -7, 1, -38, 11) \\ \text{diag}(1, 7, -2, 1, 38, 11) \\ \text{diag}(1, 8, -8, 1, -38, 11) \\ \underline{\text{diag}(1, 9, 3, 1, 19, 11)} \end{array}$$

Resulting in the following array for $\mathcal{D}_4(\{-9, 18, 4, -5\}; 1)$:

-9	18	4	-5							
	10	-1	-15	14						
		29	-20	-34	33					
			48	-39	-53	52				
				67	-58	-72	71			
					86	-77	-91	90		
						-104	-96	99	-100	
							-85	94	80	-81
-62								-66	75	61
42	-43								-47	56
37	23	-24								-28

and the following one for $\mathcal{D}_5(\{6, -7, -2, -8, 3\}; 5)$:

				6	-7	-2	-8	3		
					25	-45	36	-46	22	
						44	-83	74	-84	41
60							63	88	-97	87
49	79							82	50	-59
-21	11	98							101	12
-26	17	-27	-92							-89
-70	-64	55	-65	-73						
	-51	-102	93	-103	-54					
		-32	69	-78	68	-35				
			-13	31	-40	30	-16			

We then construct the following globally simple $H_{11}(11; 9)$:

-9	18	4	-5	6	-7	-2	-8	3		
	10	-1	-15	14	25	-45	36	-46	22	
		29	-20	-34	33	44	-83	74	-84	41
60			48	-39	53	52	63	88	-97	87
49	79			67	-58	-72	71	82	50	-59
-21	11	98			86	-77	-91	90	101	12
-26	17	-27	-92			-104	-96	99	-100	-89
-70	-64	55	-65	-73			-85	94	80	-81
-62	-51	-102	93	-103	-54			-66	75	61
42	-43	-32	69	-78	68	-35			-47	56
37	23	-24	-13	31	-40	30	-16			-28

In what follows we show that for some particular values of n the relative Heffter array built in Proposition 7.3.8 contains several pairwise mutually orthogonal Heffter systems.

We first define a class of partial linear spaces. For an odd prime p , let A be a $p \times p$ partially filled array, and let \mathbb{Z}_p^2 be the set given by the pairs of the row and the column index of each cell of A . Define the following subsets of \mathbb{Z}_p^2 :

- (1) for every pair $(a, b) \in \mathbb{Z}_p^2$, let $B_{a,b} := \{(x, y) \in \mathbb{Z}_p^2 : y = ax + b\}$;
- (2) for every $b \in \mathbb{Z}_p$, let $B_{\infty,b} = \{(x, y) \in \mathbb{Z}_p^2 : x = b\}$.

We then define the partial linear space \mathcal{S}_p as the pair

$$\mathcal{S}_p = (\mathcal{V}_p, \mathcal{B}_p), \text{ where } \mathcal{V}_p = \mathbb{Z}_p^2, \quad \mathcal{B}_p = \left(\bigcup_{(a,b) \in \mathbb{Z}_p^2, a \neq 1} B_{a,b} \right) \cup \left(\bigcup_{b \in \mathbb{Z}_p} B_{\infty,b} \right). \quad (7.3.2)$$

Note that for every $b \in \mathbb{Z}_p$ the block set \mathcal{B}_p does not contain $B_{1,b}$.

Remark 7.3.10. The partial linear space \mathcal{S}_p is resolvable having parallel classes:

$$\mathcal{L}_a = \{B_{a,b} : b \in \mathbb{Z}_p\} \quad \text{for } a \in \mathbb{Z}_p \setminus \{1\},$$

$$\mathcal{L}_\infty = \{B_{\infty,b} : b \in \mathbb{Z}_p\}.$$

Equivalently, \mathcal{S}_p can be seen as an incidence structure isomorphic to the Desarguesian affine plane of order p (see Example 7.1.3), with the removal of one of its parallel classes.

Given a block $B \in \mathcal{B}_p$ denote by $\mathcal{E}(B)$ the entries of the filled cells of A that correspond to the block B . For instance, if A is the $H_5(5; 3)$ of Example 7.3.2 and B is the first column of A , then $\mathcal{E}(B) = \{1, 16, -17\}$.

Lemma 7.3.11. *Let p be an odd prime and let $3 \leq k \leq p$ be any integer. Let A be the $H_p(p; k)$ built in Proposition 7.3.8, and let $\mathcal{S}_p = (\mathcal{V}_p, \mathcal{B}_p)$ be the partial linear space defined by Equation (7.3.2). Then for every $B \in \mathcal{B}_p$ it holds:*

$$\sum_{x \in \mathcal{E}(B)} x \equiv 0 \pmod{2pk + p}.$$

Moreover $\{x : x \in \mathcal{E}(B)\}$ admits a simple ordering.

Proof. Let p , k and A be as in the statement, and denote by L the set given by the entries of A modulo $2k+1$; it can be easily seen from the construction in Proposition 7.3.8 that L is a zero-sum half-set of \mathbb{Z}_{2k+1} . Since A is an $H_p(p; k)$, the statement immediately follows for blocks of type $B_{0,b}$ for every $b \in \mathbb{Z}_p$ and for blocks of type $B_{\infty,b}$ for every $b \in \mathbb{Z}_p$. So, from now on we assume that a given block B is of type $B_{a,b}$, for some $a \notin \{0, 1\}$.

In what follows we consider a collection of blocks of the Desarguesian affine plane, that are not blocks of \mathcal{S}_p . Let then $\mathcal{L}_1 = \{B_{1,b} : b \in \mathbb{Z}_p\}$: since $\mathcal{S}_p \cup \mathcal{L}_1$ is the set of blocks of the affine Desarguesian plane of order p , for every $B \in \mathcal{B}_p$ and $B_{1,b} \in \mathcal{L}_1$ it holds $|B \cap B_{1,b}| = 1$. As a consequence, $\sum_{x \in \mathcal{E}(B)} x \equiv \sum_{\ell \in L} \ell \pmod{2k+1}$, hence by the hypothesis on L it holds $\sum_{x \in \mathcal{E}(B)} x \equiv 0 \pmod{2k+1}$. Furthermore $\mathcal{E}(B)$ admits a simple ordering since L does. We now show that $\sum_{x \in \mathcal{E}(B)} x \equiv 0 \pmod{2pk+p}$.

We recall that A is built by filling sets of consecutive diagonals using the procedures $\mathcal{D}_3(\{a_1, a_2, a_3\}; m)$, $\mathcal{D}_4(\{a_1, a_2, a_3, a_4\}; m)$ and $\mathcal{D}_5(\{a_1, a_2, a_3, a_4, a_5\}; m)$ described in the proof of Proposition 7.3.8. In particular, for each $r \in \{3, 4, 5\}$ the application of the procedure \mathcal{D}_r fills cells corresponding to $\bigcup_{i=0}^{r-1} B_{1,m+i}$ for some $m \in \mathbb{Z}_p$; for simplicity, denote by $S_{m,r}$ this set of cells.

Given two blocks $B_{a,b} \in \mathcal{B}_p$ and $B_{1,c} \in \mathcal{L}_1$, let $(u, v) \in \mathbb{Z}_p^2$ be their unique intersection; after some simple algebra it can be seen that $B_{a,b} \cap B_{1,c+1} = (u - \frac{1}{1-a}, v - \frac{a}{1-a})$. We then show that for any fixed $B_{a,b} \in \mathcal{B}$ it holds

$$\sum_{x \in \mathcal{E}(S_{m,r} \cap B_{a,b})} x = \sum_{i=1}^r a_i.$$

The statement follows from the fact that the sum of the entries in the first row of A is equal to 0 $\pmod{2pk+p}$. Now we split the proof into 3 cases according to the values of r .

Case $r = 3$. Assume that $\mathcal{E}(\bigcup_{i=0}^2 B_{1,m+i})$ is filled by applying the procedure $\mathcal{D}_3(\{a_1, a_2, a_3\}; m)$ for some $a_1, a_2, a_3 \in \mathbb{Z}_{2pk+p}$. If $(u, v) = B_{a,b} \cap B_{1,m}$, then:

$$\begin{aligned} B_{a,b} \cap B_{1,m+1} &= \left(u - \frac{1}{1-a}, v - \frac{a}{1-a} \right), \\ B_{a,b} \cap B_{1,m+2} &= \left(u - \frac{2}{1-a}, v - \frac{2a}{1-a} \right). \end{aligned}$$

We have

$$\begin{aligned} \mathcal{E}(B_{a,b} \cap B_{1,m}) &= a_1 + u(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+1}) &= a_2 - 2 \left(u - \frac{1}{1-a} \right) (2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+2}) &= a_3 + \left(u - \frac{2}{1-a} \right) (2k+1). \end{aligned}$$

It can then be easily seen that the sum of the elements of the previous equation is $a_1 + a_2 + a_3$.

Case $r = 4$. Assume that $\mathcal{E}(\bigcup_{i=0}^3 B_{1,m+i})$ is filled by applying the procedure $\mathcal{D}_4(\{a_1, a_2, a_3, a_4\}; m)$ for some $a_1, a_2, a_3, a_4 \in \mathbb{Z}_{2pk+p}$. Let $(u, v) = B_{a,b} \cap$

$B_{1,m}$; as done in the previous case, we have

$$\begin{aligned}\mathcal{E}(B_{a,b} \cap B_{1,m}) &= a_1 + u(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+1}) &= a_2 - \left(u - \frac{1}{1-a}\right)(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+2}) &= a_3 - \left(u - \frac{2}{1-a}\right)(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+3}) &= a_4 + \left(u - \frac{3}{1-a}\right)(2k+1).\end{aligned}$$

It can be easily seen that these elements sum to $a_1 + a_2 + a_3 + a_4$.

Case $r = 5$. Assume that $\mathcal{E}(\bigcup_{i=0}^4 B_{1,m_i})$ is filled by applying the procedure $\mathcal{D}_5(\{a_1, a_2, a_3, a_4, a_5\}; m)$ for some $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Z}_{2pk+p}$. Let $(u, v) = B_{a,b} \cap B_{1,m}$, then

$$\begin{aligned}\mathcal{E}(B_{a,b} \cap B_{1,m}) &= a_1 + u(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+1}) &= a_2 - 2\left(u - \frac{1}{1-a}\right)(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+2}) &= a_3 + 2\left(u - \frac{2}{1-a}\right)(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+3}) &= a_4 - 2\left(u - \frac{3}{1-a}\right)(2k+1), \\ \mathcal{E}(B_{a,b} \cap B_{1,m+4}) &= a_5 + \left(u - \frac{4}{1-a}\right)(2k+1).\end{aligned}$$

It can be seen that these elements sum to $a_1 + a_2 + a_3 + a_4 + a_5$. \square

We prove the following:

Theorem 7.3.12. *Let p be an odd prime, and let $3 \leq k \leq p$ be any integer. Then, there exists a simple $(pk, k; p)$ relative Heffter space over $\mathbb{Z}_{2pk+p} \setminus \langle 2k+1 \rangle$.*

Proof. Let A be the $H_p(p; k)$ built in Proposition 7.3.8, and let $\mathcal{S}_p = (\mathbb{Z}_p^2, \mathcal{B}_p)$ be the partial linear space defined in Equation (7.3.2). By Remark 7.3.10, \mathcal{S}_p is resolvable with parallel classes $\mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_{p-1}, \mathcal{L}_\infty$, and since it is contained in an affine plane of order p every pair of blocks belonging to distinct parallel classes meet in exactly one (possibly empty) cell of A .

From Lemma 7.3.11 we have that each parallel class of \mathcal{B}_p is a simple Heffter system over $\mathbb{Z}_{2pk+p} \setminus \langle 2k+1 \rangle$, and any two distinct parallel classes are orthogonal by Remark 7.3.10, hence

$$\mathcal{H} = \{\{\mathcal{E}(B) : B \in \mathcal{L}_i\} : i \in [2, p-1]\} \cup \{\mathcal{L}_\infty\},$$

is a relative Heffter space over $\mathbb{Z}_{2pk+p} \setminus \langle 2k+1 \rangle$. Moreover, every block $B \in \mathcal{B}_p$ has k elements, thus \mathcal{H} is a $(pk, k; p)$ relative Heffter space. \square

Remark 7.3.13. Given an odd prime p and an integer $3 \leq k \leq p$, the density of the Heffter space \mathcal{H} constructed in Theorem 7.3.12 is:

$$\delta(\mathcal{H}) = \frac{pk-p}{pk-1}.$$

It can then be seen that as p grows towards infinity the constructed Heffter spaces have density approaching $\frac{k-1}{k}$.

Remark 7.3.14. If $k = p$ the density of the Heffter space \mathcal{H} constructed in Theorem 7.3.12 is $\frac{p}{p+1}$, that as p grows becomes arbitrarily close to 1. It follows moreover by the inequality (7.3.1) that a relative Heffter *linear* space $(p^2, p; r)$ has $r = p + 1$, that cannot exist on a cyclic group by Corollary 7.3.3. Hence the cyclic relative Heffter space of Theorem 7.3.12 is the densest that can be achieved on this set of parameters.

Example 7.3.15. Call A the $H_5(5; 3)$ array constructed in Example 7.3.2 whose entries form a half-set of $\mathbb{Z}_{35} \setminus \langle 7 \rangle$:

1	2	-3		
	8	-12	4	
		15	9	11
-17			-13	-5
16	-10			-6

It can be seen that A has been filled with the procedure $\mathcal{D}_3(\{1, 2, -3\}; 1)$. We construct then the incidence structure $\mathcal{S}_5 = (\mathcal{V}_5, \mathcal{B}_5)$ having $\mathcal{V}_5 = \mathbb{Z}_5 \times \mathbb{Z}_5$, whose block set \mathcal{B}_5 is partitioned in the following parallel classes \mathcal{L}_i with $i \in I = \{0, 2, 3, 4, \infty\}$ (for simplicity, we denote an element $(x, y) \in \mathbb{Z}_5 \times \mathbb{Z}_5$ as xy):

$$\begin{aligned} \mathcal{L}_0 &= \{\{00, 10, 20, 30, 40\}, \{01, 11, 21, 31, 41\}, \{02, 12, 22, 32, 42\}, \\ &\quad \{03, 13, 23, 33, 43\}, \{04, 14, 24, 34, 44\}\} \\ \mathcal{L}_2 &= \{\{00, 12, 24, 31, 43\}, \{01, 13, 20, 32, 44\}, \{02, 14, 21, 33, 40\}, \\ &\quad \{03, 10, 22, 34, 41\}, \{04, 11, 23, 30, 42\}\}, \\ \mathcal{L}_3 &= \{\{00, 13, 21, 34, 42\}, \{01, 14, 22, 30, 43\}, \{02, 10, 23, 31, 44\}, \\ &\quad \{03, 11, 24, 32, 40\}, \{04, 12, 20, 33, 41\}\}, \\ \mathcal{L}_4 &= \{\{00, 14, 23, 32, 41\}, \{01, 10, 24, 33, 42\}, \{02, 11, 20, 34, 43\}, \\ &\quad \{03, 12, 21, 30, 44\}, \{04, 13, 22, 31, 40\}\}, \\ \mathcal{L}_\infty &= \{\{00, 01, 02, 03, 04\}, \{10, 11, 12, 13, 14\}, \{20, 21, 22, 23, 24\}, \\ &\quad \{30, 31, 32, 33, 34\}, \{40, 41, 42, 43, 44\}\}. \end{aligned}$$

It can then be seen that for every $i \in I$ and for every $B \in \mathcal{L}_i$ it holds $\sum_{x \in \mathcal{E}(B)} x \equiv 0 \pmod{35}$. In particular the Heffter systems obtained from the parallel classes of \mathcal{B}_5 are precisely the ones shown in Example 7.3.2, namely:

$$\begin{aligned} H_1 &= \bigcup_{B \in \mathcal{L}_\infty} \mathcal{E}(B), & H_2 &= \bigcup_{B \in \mathcal{L}_0} \mathcal{E}(B), & H_3 &= \bigcup_{B \in \mathcal{L}_2} \mathcal{E}(B), \\ H_4 &= \bigcup_{B \in \mathcal{L}_3} \mathcal{E}(B), & H_5 &= \bigcup_{B \in \mathcal{L}_4} \mathcal{E}(B). \end{aligned}$$

Since the constructed Heffter spaces are simple, we have the following decompositions of a complete multipartite graph:

Corollary 7.3.16. *Let p be an odd prime, and let $3 \leq k \leq p$ be any integer. Then, there exist p mutually orthogonal cyclic k -cycle decompositions of $K_{(2k+1) \times p}$.*

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