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Linear defects and one-dimensional sectors in three dimensional supersymmetric gauge theories

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Abstract

In this thesis, we study three-dimensional supersymmetric gauge theories with extended supersymmetry. Particular care is given to the ABJM model. It is a $\mathcal{N}=6$ supersymmetric Chern-Simons matter theory, depending on a discrete parameter k, called Chern-Simons level. The model interpolates between the weak coupling regime, for large values of k, and the strong coupling one, when k is small. In the former case, the weak coupling limit of the QFT is reliable. In the latter case, the theory is well described by the gravitational dual, which is M-theory on $AdS_4 \times S^7/Z_k$. We focus on observables constrained by $\mathcal{N} \geq 4$ supersymmetry, amenable of exact results.

We will consider two types of one-dimensional subsectors. The first is a set of local operators, whose correlation functions are captured by a one-dimensional topological theory. The second is a family of loop operators known as *latitude*, depending on an additional real parameter ν.

The first two chapters provide a review of the relevant models and of the methods to study them. Specifically, Chapter 1 contains an introduction to various aspects of supersymmetric gauge theories in three dimensions. In Chapter 2 we review the localization technique, and we apply it to 3d gauge theories. We use the result to explore some remarkable features of the ABJM model. For instance, we discuss its infra-red dualities for Chern-Simon level equal to one with an UV $\mathcal{N}=4$ SYM avatar. In Chapter 3 we present our results for the topological sector. We relate integrated topological correlation functions to derivatives of the mass deformed partition function, confirming a pre-existing conjecture. We apply and test the formula at weak coupling in ABJM.

Chapter 4 is devoted to an introduction to BPS Wilson and vortex loop operators in 3d. We frame the discussion in terms of conformal defects. In chapter 5 we derive a matrix model for the latitude Wilson loop in ABJM, proving an existing conjecture. We also extend the result to all $\mathcal{N} \geq 4$ gauge theories. In Chapter 6 we relate the latitude Wilson loop in ABJM to a novel bound state of Wilson-vortex loop in its dual UV theory (for $k=1$). We realize this dual operator as a one-dimensional quantum mechanics coupled to the bulk.

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The results of this thesis have produced three papers:

1. N. Gorini, L. Griguolo, L. Guerrini, S. Penati, D. Seminara, P. Soresina, *The topological line of ABJ(M) theory*, arXiv:2012.11613, JHEP 2106 (2021) 091, doi 10.1007/JHEP06(2021)091;

2. L. Griguolo, L. Guerrini, and I. Yaakov, *Localization and Duality for ABJM Latitude Wilson Loops*, arXiv:2104.04533, JHEP 2108 (2021) 001 , doi 10.1007/JHEP08(2021)001;

3. L. Guerrini, S. Penati, and I. Yaakov, *Generating functions for Higgs/Coulomb branch operator from 1d-3d cohomological equivalence*, arXiv:2112.13816 .

The first two papers were published on the Journal of High Energy Physics. The third one appeared as a preprint on arXiv, and was accepted for pubblication in JHEP on the 5th of April 2022. As one of the authors of the mentioned article, I have the permission to use it or a portion of it in my thesis or dissertation.The main reason is that since 2014 all JHEP papers are published under a CC-BY licence and copyright remains to the authors.

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Introduction

Quantum Field Theory (QFT) provides a universal language to describe a large variety of natural phenomena, ranging from particle physics to condensed matter, and, in certain cases, even gravity. When the coupling constants are small, one can extract the physical information from perturbation theory. This method is not reliable in the strong coupling regime. For instance, we do not have an analytic control on quantum electrodynamics in three dimensions, apart from specific limits. Another remarkable example is the problem of the low energy dynamics and confinement of Yang-Mills theory in four dimensions. A more systematic technology to examine the strongly coupled QFTs is one of the most urgent problems in theoretical physics.

This objective is ambitious and somewhat vague. Then, it is convenient to start with some concrete and universal questions, which provide deep insight. Wilson's idea of renormalization group (RG) guides us through this task [1]. It provides deep insight into QFT and highlights some paradigmatic universal questions. The great intuition is that every QFT must be defined at a characteristic length scale. One can move in the space of QFTs by performing a series of scale transformations. The process modifies the parameter of the QFT and generates the *RG flow* in the space of QFTs. When a theory reaches a fixed point for the RG flow, the theory is scale-invariant. Often the theory exhibits the more constraining conformal symmetry and is called conformal field theory (CFT)[1].

A fruitful approach is to start from CFTs and explore the whole QFT space by turning on various deformations. This program discovers rich and stimulating aspects. CFTs might be strongly coupled or even non-Lagrangian. RG flows give rise to unexpected dualities: it might happen that two theories with different degrees of freedom flow to the same theory. If we stay at the fixed point, one can exploit the conformal symmetry. It turns out that a CFT is specified by a list of numbers, the CFT data, satisfying certain constraints. The implementation of the constraints is the *confomal bootstrap* program, which does not rely on the specific realization of the theory [3]. Moreover, one should also keep into account the global properties of theory, which does not enter local correlation functions. These features are encoded in correlation functions of extended operators like Wilson lines [4].

There is a class of interacting QFTs whose strongly coupled regime is partially accessible. These are the supersymmetric theories [5]. They enjoy a larger symmetry, generated by fermionic generators. In this way, they evade in a non-trivial way the Coleman-Mandula theorem [6]. The physical consequence is that they have the same number of bosonic and

[1]See here for a counterexample [2]

fermionic degrees of freedom. These theories have also a deep relation with string theory, providing solid and testable examples of AdS/CFT [7].

Supersymmetry is not realized in nature at experimentally achievable energy scales. Nevertheless, it is a powerful tool to simplify computations in field theories. At the perturbative level, the net effect is the cancellations of certain divergence in Feynman diagrams. These simplifications are encoded in non-renormalization theorems, which allows us to probe the strong coupling regime [8–15]. In this way, the strong coupling regime of supersymmetric QFTs was studied in three and four dimensions. One of the greatest successes was the proposal for a mechanism for confinement and chiral symmetry breaking in 4d gauge theories. Another example is the discovery of a rich web of dualities in 3d supersymmetric gauge theories.

A more systematic way to exploit the fermionic symmetry is the machinery of *supersymmetric localization*. Supersymmetric localization was originally introduced to study supersymmetric quantum mechanics [16] and later extended to various cohomological topological field theories [17–19]. Further developments concern the computation of the partition function of 4d $\mathcal{N} = 2$ SYM in Ω-background [20]. The idea is that the fermionic symmetry allows us to deform the path integral until the theory becomes effectively free. A renewed push to apply this method in supersymmetric QFTs came with the seminal paper [21], where the method was applied successfully to 4d $\mathcal{N} = 2$ theories on the four sphere. A similar result was obtained for three-dimensional $\mathcal{N} = 2$ gauge theories [22–24]. After this, an impressive number of exact results were achieved, providing insight into dualities, defects, and AdS/CFT. For a thorough collection of results see [25].

All these considerations have a natural meeting point in superconformal theories with extended supersymmetry. For instance, 3d $\mathcal{N} \geq 4$ and 4d $\mathcal{N} \geq 2$ theories contain a special lower-dimensional subsector closed under operator product expansion. When we restrict to these subsectors, the bootstrap problem becomes much simpler. For instance, in the 3d case a 1d sector is mapped to a 1d topological quantum field theory [26, 27]. In the 4d one, a 2d sector is captured by a chiral algebra [28] (see also a similar construction in 6d [29]). This property comes from the invariance under certain supercharges. From this perspective, the idea appeared originally in [30]. The existence of a preserved supercharge makes the subsectors amenable to localization. In 3d a localization scheme was proposed [31–33] on S^3, and generalized to more general manifolds [34]. See also [35] for similar results in 4d. This is somewhat similar to $\mathcal{N} = 4$ SYM, where a 2d subsector is described by the zero instanton sector of 2d Yang-Mills [36–39].

We will investigate the 3d $\mathcal{N} \geq 4$ topological sector. We will prove a *cohomological equivalence* between SUSY-preserving deformations of the 3d theory and topological operators with dimension 1. This implies that derivatives of the mass deformed partition function with respect to the mass compute correlation functions of topological operators, as conjectured in [40, 41]. In Lagrangian theories, the mass deformed partition function is accessible with localization [23]. However, the cohomological equivalence is independent of the specific realization of the theories.

We will apply the formula to the ABJM model [42, 43]. It is a Chern-Simons Matter theory in three spacetime dimensions with gauge group $U(N_1)_k \times U(N_2)_{-k}$, Chern-Simons levels k and $-k$, and $\mathcal{N} = 6$ supersymmetry. It is dual to M-theory on $AdS4 \times S^7/Z_k$ or type IIA string theory on $AdS_4 \times \mathbb{C}P^3$, depending on the particular range of the Chern-Simons level. We compare our formula against a genuine two-loop computation and find perfect agreement.

Local correlation functions do not exhaust the possible observables. QFTs contain also extended operators which probe their global and topological properties [44]. The prototypical example is the Wilson loop operator [45]. It measures the non-abelian phase of a massive quark moving in a gauge field. This phase is nothing but the non-abelian generalization of the Aharonov-Bohm phase in QED. According to the paradigm of [46], extended operators serve as order parameters for new phases associated with higher form symmetries. Indeed, Wilson loops were introduced as a probe to detect confinement in Yang-Mills theory

In supersymmetric theories, a compelling class of observables are BPS line operators [47, 48]. BPS Wilson loops are standard observables in the context of AdS/CFT. Moreover, they are one of the simplest example of conformal defect [49–51]. Besides Wilson loops, there are also classes of operators which cannot be represented as an integral of local fields of the bulk theory over a certain submanifold, like the 't Hooft loops[52]. In certain cases, like supersymmetric theories, these operators are defined by altering the boundary condition of the path integral to keep into account certain singularities. Examples are BPS 't Hooft and "dyonic" loops [53] and surface defects in 4d [54, 55], and supersymmetric vortex loop in 3d [56–58]. These extended operators may also arise by coupling local degrees of freedom localized on the submanifold supporting the operator to the bulk theory [59–61]. From this perspective, BPS Wilson and vortex loops are the second linear subsectors we will discuss.

We concentrate again on theories with $\mathcal{N} \geq 4$ supersymmetry with a particular focus on ABJM. This model has two families of circular Wilson loops. The first one is the bosonic Wilson loops [62–64], originally introduced in [65] $\mathcal{N} = 2$ theories and preserve one-sixth of the supercharges. Later, a family of Wilson loops preserving half of the supersymmetry was introduced in [66]. They are known as fermionic Wilson loops because the matter fermions appear in the (super)connection. A cohomological equivalence relates the expectation value of the fermionic Wilson loop to that of a precise combination of bosonic loops. It was shown that their expectation values are captured by an interacting matrix model [22], extensively studied in several works [67–69].

In this thesis, we will focus on the deformation of both circular Wilson loops introduced in [70, 71], known as latitude. Similar constructions exist in 4d [72, 73] and 5d [74]. These Wilson loops depend on an additional parameter ν. Physically, it is related to the bremsstrahlung function of the theory, which encodes the physical information on the energy emitted by a slowly moving electric probe [75–78]. The latitude loops were extensively studied both at weak and strong coupling [79–84]. These efforts lead to the conjecture that its expectation value is captured by a deformed matrix model [79]. The proposal passed tests up to three loops in perturbation theory and to 1-loop at strong coupling. We will propose a localization scheme based on a different supercharge to derive the matrix model from a localization argument. We will also extend the derivation to all $\mathcal{N} \geq 4$ theories (see here for a discussion on the latitude in $\mathcal{N} = 4$ Chern-Simons matter theory [85]).

We compare our proposal against other existing localization schemes. The latitude supercharge coincides with the $\mathcal{N} = 2$ supercharge for $\nu = 1$ and with the $\mathcal{N} = 4$ supercharge of the topological sector for $\nu \to 0$. A related question concerns the existence of a latitude-like loop in other 3d $\mathcal{N} \geq 4$ theories. For theories without Chern-Simons term, we identify both Wilson and vortex loops. Then, one can also investigate their behavior under 3d dualities, like mirror symmetry [86], Aharony dualities [87], and Giveon-Kutasov duality [88]. Studying the mapping of line operators enriches our understanding of the duality and the loop operators. A similar problem was studied for S-duality [89] and AGT-duality [90, 91]. In 3d, Wilson and vortex loops are exchanged under mirror symmetry [61, 92, 93].

We tackle this problem for the following infra-red duality of $\mathcal{N} = 8$ theories. When $N_1 = N_2$ and k is equal to 1 or 2, the supersymmetry of the ABJM model is enhanced to $\mathcal{N} = 8$ [94, 95]. Specifically, ABJM at level 1 with $N_{1,2} = N$ is a dual description of the IR limit of $\mathcal{N} = 8$ super-Yang-Mills (SYM) with gauge group $U(N)$[42]. $\mathcal{N} = 8$ SYM is also dual to a $\mathcal{N} = 4$ SYM theory coupled to an adjoint and a fundamental hyper [96]. Therefore, the latter theory is also IR dual to ABJM at $k = 1$. We conjecture that the (bosonic) latitude loop is realized in this theory as a novel mixed Wilson-vortex loop. We describe the dual loop as a one-dimensional supersymmetric quantum mechanics, coupled to the bulk theory.

The structure of the thesis is the following one. In chapter 1 we present an overview of 3d gauge theories with $\mathcal{N} \geq 2$ supersymmetry. Along the way, we highlight dynamical aspects, which appear in different flavors along the rest of the work. In chapter 2 we introduce the localization technique in supersymmetric theories. We review some key points related to 3d supersymmetric gauge theories, including discussions on the different supergravity backgrounds, AdS/CFT correspondence, and RG flows. In chapter 3 we focus on the topological sector. We show our proof for topological correlation functions. Then, we apply it to the ABJM model. In chapter 4 we introduce BPS line operators. We frame the discussion in the context of conformal defects. Then, we discuss some relevant and universal observables of these systems. In chapter 5 we discuss the localization of the latitude BPS Wilson loop in ABJM. We study its preserved superalgebra in detail. Inspired by this, we propose our localization scheme, based on the cohomological localization of [97]. In chapter 6 we study latitude-like loop operators in 3d $\mathcal{N} = 4$ SYM theory dual to $\mathcal{N} = 8$ SYM. We compare them to the latitude loop in ABJM at $k = 1$. A careful symmetry analysis leads to the conjecture of the duality with the mixed Wilson-vortex operator. We match their expectation value as strong evidence. Various conventions and technical details are summarized in four appendices.

Chapter 1

From supersymmetry to conformal symmetry in 3d

In this chapter, we review the construction of 3d theories with extended supersymmetry. They will be the main character of this thesis. Three-dimensional QFTs exhibit rich dynamics at low energies and describe interesting condensed-matter systems. Their supersymmetric counterparts are an exciting playground to study dynamical aspects beyond the perturbative regime. Moreover, supersymmetric Chern-Simons theories are related to string theory and M-theory through the AdS/CFT correspondence. To explain these features, along the way we will recall some notions of (super)conformal field theories.

The structure is the following. First of all, in Section 1.1 we will introduce supersymmetry and supersymmetric Quantum Field Theories using the $\mathcal{N}=2$ superspace formalism. Section 1.2 contains a review of conformal and superconformal field theories. Then, in Section 1.3 we will discuss some dynamics of these theories. Some emphasis will be given to the emergence of infrared dualities. In section 1.4 we will focus on the construction of Chern-Simons matter theory with extended supersymmetry. Their defining property, i.e. the Chern-Simons term, is one of the main novelties of three-dimensional physics. These theories are a Lagrangian example of SCFTs in three dimensions. We will conclude the chapter explaining their role in the AdS/CFT correspondence in Section 1.5.

1.1 Supersymmetric theories in 3d

Supersymmetry is the most general spacetime symmetry for a reasonable QFT. Physically, it means that every bosonic particle has a fermionic partner. This symmetry is implemented by fermionic generators Q^r_α called supercharges. Here $\alpha = 1, 2$ is a spinor index, $r = 1, \ldots, \mathcal{N}$ labels the number of supercharges, which in Minkowskian signature are Majorana spinors.[1] If $\mathcal{N} > 1$, the supersymmetry is said extended. We can derive the supersymmetry algebra imposing the closure of the Jacobi identity. If we start from the usual Poincaré

[1] Conventions are summarized in Appendix A.

algebra of translation P_μ and Lorentz rotation $M_{\mu\nu}$, the SUSY algebra reads

$$\{Q_{\alpha r}, Q_{\beta s}\} = 2\delta_{rs}\gamma^\mu_{\alpha\beta}P_\mu + \epsilon_{\alpha\beta}Z_{rs}\,, \tag{1.1}$$

$$[P_\mu, Q_{\alpha r}] = 0\,, \quad [M_{\mu\nu}, Q_{\alpha r}] = \frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{\beta r}\,, \tag{1.2}$$

where $\gamma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. Z^{rs} is a real antisymmetric matrix and it can exist only for extended supersymmetry. Its eigenvalues are called central charges. Indeed, they commute with all the other generators. We introduce the R-symmetry group as the space of internal symmetries acting on the supercharges. We denote their generators by R_{rs} and define their action as

$$[R_{rs}, Q_{\alpha t}] = i\left(\delta_{rt}Q_{\alpha s} - \delta_{st}Q_{\alpha r}\right)\,. \tag{1.3}$$

Consistency with the SUSY algebra fixes the R-symmetry group to be $SO(N)$. Supercharges transform in the fundamental of the R-symmetry group. We will provide examples of supersymmetric theories in the rest of the chapter.

1.1.1 3d $\mathcal{N} = 2$ theories

In this section, we introduce the first examples of supersymmetric theories. We will focus on the $\mathcal{N} = 2$ formalism, which will be the building blocks for theories with more supersymmetries.[2]

The algebra has four fermionic generators, organized into a complex spinor Q_α and its conjugate $\bar{Q}_\alpha$.[3] They close the algebra

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = 2(P^\mu\gamma_\mu)_{\alpha\beta} + 2i\epsilon_{\alpha\beta}Z\,, \tag{1.4}$$

where Z is the central charge. The R-symmetry is $U(1)$, and we give charge -1 to Q_α.

We organize fields in representations of the supersymmetry algebra by using the $\mathcal{N} = 2$ superspace [12, 98, 99]. It extends the usual (Euclidean) space by two fermionic directions θ_α and $\bar{\theta}_\alpha$, generated by Q_α and $\bar{Q}_\alpha$. One can think of the three-dimensional $\mathcal{N} = 2$ superspace as the dimensional reduction of the 4d $\mathcal{N} = 1$ superspace. The supercharges have a convenient realization as a differential operator

$$Q_\alpha = \frac{\partial}{\partial\theta^a} + i\left(\gamma^\mu\bar{\theta}\right)_\alpha\partial_\mu\,, \quad \bar{Q}_\alpha = \frac{\partial}{\partial\bar{\theta}^a} + i\left(\gamma^\mu\theta\right)_\alpha\partial_\mu \tag{1.5}$$

They represent the SUSY algebra with $Z = 0$. Functions on the superspace $f(x^\mu, \theta_\alpha, \bar{\theta}_\alpha)$ are called superfields and encode supersymmetric multiplets. We recover the usual fields expanding the superfield in the fermionic variables around $\theta_\alpha = 0$, $\bar{\theta}_\alpha = 0$. We impose

[2] The reason for which we do not discuss $\mathcal{N} = 1$ is that many of the techniques we will exploit do not apply in that case.

[3] The relation with Eq (1.1) is

$$Q_\alpha = \frac{1}{\sqrt{2}}\left(Q_{\alpha 1} + iQ_{\alpha 2}\right)\,, \quad \bar{Q}_\alpha = \frac{1}{\sqrt{2}}\left(Q_{\alpha 1} - iQ_{\alpha 2}\right)\,.$$

The $U(1) \simeq SO(2)$R-symmetry generator is $R = R_{21}$.

suitable constraints to obtain the irreducible representations of the supersymmetry algebra. To see this, we introduce the two following super-derivatives

$$D_\alpha = \frac{\partial}{\partial \theta^a} - i\left(\gamma^\mu \bar{\theta}\right)_\alpha \partial_\mu , \quad \bar{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}^a} + i\left(\gamma^\mu \theta\right)_\alpha \partial_\mu . \tag{1.6}$$

It is not hard to check that they commute with the supercharges, and that

$$\{D_\alpha, \bar{D}^\beta\} = -2i(\gamma^\mu)_\alpha{}^\beta \partial_\mu , \qquad \{D_\alpha, D^\beta\} = \{\bar{D}_\alpha, \bar{D}^\beta\} = 0 . \tag{1.7}$$

The simplest multiplet is the chiral multiplet. It is represented by a scalar superfield Φ, satisfying the constraint $\bar{D}_\alpha \Phi = 0$. Φ is called chiral superfield and admits the following expansion

$$\Phi(x, \theta, \bar{\theta}) = \phi + \sqrt{2}\theta\psi + \theta^2 F , \tag{1.8}$$

where ϕ is a complex scalar, ψ a Dirac fermion, and F is a complex auxiliary field.[4] They are the field components of the chiral superfield. The SUSY transformations are obtained by acting with $\delta \equiv \varepsilon Q + \bar{\varepsilon}\bar{Q}$

$$\begin{aligned} \delta\phi &= \sqrt{2}\varepsilon\psi \\ \delta\psi &= \sqrt{2}\varepsilon F - \sqrt{2}i\gamma^\mu \bar{\varepsilon}\, \partial_\mu \phi \\ \delta F &= -\sqrt{2}i\bar{\varepsilon}\gamma^\mu \partial_\mu \psi \end{aligned} \tag{1.9}$$

Similarly, we define the anti-chiral superfield as a scalar superfield $\bar{\Phi}$ such that $D_\alpha \bar{\Phi} = 0$. It has a similar expansion

$$\bar{\Phi}(x, \theta, \bar{\theta}) = \bar{\phi} - \sqrt{2}\bar{\theta}\bar{\psi} - \bar{\theta}^2 \bar{F} . \tag{1.10}$$

The SUSY transformations are

$$\begin{aligned} \delta\bar{\phi} &= -\sqrt{2}\bar{\varepsilon}\bar{\psi} \\ \delta\psi &= \sqrt{2}\bar{\varepsilon}F + \sqrt{2}i\gamma^\mu \varepsilon\, \partial_\mu \bar{\phi} \\ \delta F &= -\sqrt{2}i\varepsilon\gamma^\mu \partial_\mu \bar{\psi} \end{aligned} \tag{1.11}$$

We use them to write the first supersymmetric actions. Given N chiral multiplets, the following action is manifestly supersymmetric

$$S = \int d^3x\, d^4\theta K(\Phi_i, \bar{\Phi}_i) . \tag{1.12}$$

K is a real function called Kähler potential. The name comes from the fact that the action leads to a supersymmetric sigma model with a Kähler manifold as target space. We will

[4] This expansion holds in a coordinate system $y^\mu, \theta^a, \bar{\theta}^a$, where $y^\mu = x^\mu + i\bar{\theta}\gamma^\mu\theta$. In this way, the constraint is solved by any superfield $\Phi = \Phi(y, \theta)$.

discuss only the flat target space, which reproduces the canonical kinetic terms

$$S = \int d^3x \left(\partial_\mu \bar{\phi}_i \partial_\mu \phi_i - i\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i - \bar{F}_i F_i\right) . \tag{1.13}$$

We can add interactions by considering a superpotential, defined by the integral of a holomorphic function W over half of the superspace

$$S_{\text{sup}} = \int d^3x\, d^2\theta\, W(\Phi_i) + c.c. = \int d^3x \left(\frac{\partial W(\phi_i)}{\partial \Phi_i} F_i + \frac{\partial^2 W(\phi_i)}{\partial \Phi_i \partial \Phi_j} \psi_i \psi_j + c.c\right) \tag{1.14}$$

An example is the quadratic superpotential $m\Phi^2$, which introduces a complex mass m for the fields in Φ. A quartic superpotential generates a classically marginal sextic superpotential. The interactions preserve R-symmetry if we assign R-charge 2 to W.

We move to gauge interactions. The vector multiplet is realized by a scalar superfield, endowed with the constraint $V = V^\dagger$. We supplement it with gauge invariance

$$e^{-2V} \to e^{i\bar{\Omega}} e^{-2V} e^{-i\Omega} \tag{1.15}$$

where Ω and $\bar{\Omega}$ are a chiral superfield and an anti-chiral superfield. We use the invariance to fix the Wess-Zumino gauge, defined by the following expansion for V

$$V(x, \theta, \tilde{\theta}) = \theta \gamma^\mu \bar{\theta} A_\mu - i\bar{\theta}\theta\sigma + i\bar{\theta}^2 \theta \lambda - i\theta^2 \bar{\theta}\bar{\lambda} - \frac{1}{2}\theta^2 \bar{\theta}^2 D\,. \tag{1.16}$$

In the abelian case, we see that this choice fixes just the combination $i(\Omega - \bar{\Omega})$. The residual freedom to choose the opposite combination produces the usual gauge invariance $\delta A_\mu = i\partial_\mu(\omega + \bar{\omega})$, where ω and $\bar{\omega}$ are the bottom component of Ω and $\bar{\Omega}$. One can verify that a SUSY transformation on V does not preserve the Wess-Zumino gauge. However, we can always perform a gauge transformation to restore it. From now on, δ will always indicate such a combination of transformations. For the gauge multiplet, we have

$$\delta A_\mu = -i\left(\varepsilon \gamma^\mu \bar{\lambda} + \bar{\varepsilon}\gamma^\mu \lambda\right) , \tag{1.17}$$

$$\delta\sigma = -\varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda\,, \tag{1.18}$$

$$\delta\lambda = \left(iD - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} - i\gamma^\mu D_\mu \sigma\right)\varepsilon\,, \tag{1.19}$$

$$\delta\bar{\lambda} = \left(iD - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} + i\gamma^\mu D_\mu \sigma\right)\bar{\varepsilon}\,, \tag{1.20}$$

$$\delta D = \varepsilon\gamma^\mu D_\mu \bar{\lambda} - \bar{\varepsilon}\gamma^\mu D_\mu \lambda - [\varepsilon\bar{\lambda}, \sigma] - [\bar{\varepsilon}\lambda, \sigma]\,. \tag{1.21}$$

We write a Lagrangian in terms of the gaugino superfields, defined by

$$W_\alpha = -\frac{1}{4}\bar{D}^2\left(e^{-2V} D_\alpha e^{2V}\right) , \qquad \bar{W}_\alpha = -\frac{1}{4}D^2\left(e^{-2V}\bar{D}_\alpha e^{2V}\right) \tag{1.22}$$

It is easy to see that it transforms covariantly under gauge transformations, like the field strength in standard gauge theory. Since W_α and $\bar{W}_\alpha$ are respectively a chiral and an antichiral superfield, a supersymmetric gauge invariant action is

$$S = \frac{1}{4g^2}\int d^3x\, d^2\theta\, \mathrm{Tr}\, W^2 + c.c\,. \tag{1.23}$$

The integration over the fermionic coordinates yields the supers Yang-Mills action

$$S_{\mathrm{SYM}} = \frac{1}{g^2}\int d^3x\, \mathrm{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\sigma D_\mu\sigma - i\tilde{\lambda}\gamma^\mu D_\mu\lambda + i\tilde{\lambda}[\sigma,\lambda] + \frac{1}{2}D^2\right)\,. \tag{1.24}$$

We can couple matter fields to gauge multiplets supersymmetrically by demanding invariance under the local transformation

$$\Phi \to e^{i\Lambda}\Phi\,, \qquad \bar{\Phi} \to \bar{\Phi}e^{-i\Lambda} \tag{1.25}$$

where Φ and $\bar{\Phi}$ transform in some representation R of the gauge group G. The action reads

$$\begin{aligned} S &= \int d^3x\, d^4\theta\, \bar{\Phi}\, e^{-2V}\Phi = \\ &= \int d^3x\left(D^\mu\bar{\phi}D_\mu\phi + \bar{\phi}(\sigma^2 + D^2)\phi - i\bar{\psi}\gamma^\mu D_\mu\psi - i\bar{\psi}\sigma\psi + \sqrt{2}i\left(\bar{\phi}\lambda\psi + \bar{\psi}\bar{\lambda}\phi\right) - \bar{F}F\right)\,. \end{aligned} \tag{1.26}$$

This action is invariant under the following SUSY transformations, which includes the gauge transformation restoring the Wess-Zumino gauge

$$\begin{aligned} &\delta\bar{\phi} = -\sqrt{2}\bar{\varepsilon}\bar{\psi} \\ &\delta\psi = \sqrt{2}\bar{\varepsilon}F + \sqrt{2}i\gamma^\mu\varepsilon\, D_\mu\bar{\phi} + \sqrt{2}i\sigma\phi\bar{\varepsilon} \\ &\delta F = -\sqrt{2}i\varepsilon\gamma^\mu D_\mu\bar{\psi} + \sqrt{2}i\sigma\bar{\varepsilon}\psi + \sqrt{2}i\bar{\varepsilon}\bar{\lambda}\phi \end{aligned} \tag{1.27}$$

The gauge transformation modifies the supersymmetry algebra. Indeed, the commutators of two supercharges will include also gauge transformations. We will see examples in the next chapter.

We finally discuss the coupling to background fields preserving supersymmetry. We suppose to have a weekly gauged flavor symmetry G_F. We interpret the extra terms in the Lagrangian as background gauge fields. We call real mass deformation the Lagrangian obtained by the interaction term evaluated on the *rigid background* $\sigma = m$ and with all the other fields vanishing. Then all the supersymmetric variations of the background fields are zero, and $\mathcal{N} = 2$ supersymmetry is preserved. The theory is modified by the following term

$$\delta\mathcal{L}_{\mathrm{mass}} = q^2m^2\bar{\phi}\phi - iqm\bar{\psi}\psi \tag{1.28}$$

where q is the charge of the multiplet under the global symmetry. Similarly, if we have an abelian vector V, we can build a supersymmetric BF-term. It is convenient to introduce the linear multiplet. It is represented by a superfield Σ such that $D^2\Sigma = \bar{D}^2\Sigma = 0$. This multiplet hosts abelian conserved current j_μ: the constraint implies $\partial_\mu j^\mu = 0$. A special feature of three dimensional gauge theories is that $\Sigma = \epsilon^{\alpha\beta}\bar{D}_\alpha D_\beta V$ defines a linear multiplet. It contains the topological conserved current $J_T^\mu \propto \epsilon_{\mu\nu\rho}F^{\nu\rho}$. The BF-term is given by the coupling of Σ to a background vector multiplet $\tilde{V}$

$$S_{\mathrm{BF}} = \frac{i}{\pi}\int d^3 d^2\theta d^2\bar{\theta}\Sigma\tilde{V} = \frac{i}{2\pi}\int d^3x \left(\epsilon^{\mu}\nu\rho A_{\mu}\partial_{\nu}\tilde{A}_{\rho} + iD\tilde{\sigma} + i\sigma\tilde{D} + \bar{\lambda}\tilde{\lambda} + \bar{\tilde{\lambda}}\lambda\right). \quad (1.29)$$

The background $\tilde{\sigma} = \zeta$ preserves supersymmetry. The BF coupling evaluated in this rigid limit yields the FI term

$$S_{\mathrm{FI}} = -\frac{\zeta}{2\pi}\int d^3x\, D \quad (1.30)$$

The FI-term can be introduced for every $U(1)$ factor in the gauge group. We finally stress that both real masses and FI parameters enter in the supersymmetry transformations and algebra. Because of this, they both contribute to the central charge Z.

1.1.2 3d $\mathcal{N} = 4$ theories

We introduce 3d $\mathcal{N} = 4$ theory. Instead of introducing an adapted superspace formulation, we recycle the previous work. We decompose the $\mathcal{N} = 4$ multiplets in the $\mathcal{N} = 2$ language and use the corresponding superfields to write the most general action compatible with $\mathcal{N} = 4$ supersymmetry. The manifest $\mathcal{N} = 2$ invariance and the invariance under $SO(4)$ R-symmetry ensure the full $\mathcal{N} = 4$ supersymmetry invariance. The $\mathcal{N} = 4$ algebra reads

$$\{Q_{\alpha a\dot{a}}, Q^{\beta}{}_{b\dot{b}}\} = 4\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}(\gamma^{\mu})_{\alpha}{}^{\beta}P_{\mu} \quad (1.31)$$

where the indices $a, \dot{a} = 1, 2$ are $SU(2)_H \times SU(2)_C \simeq SO(4)$ R-symmetry indices.

The relevant multiplets are the vector multiplet and the hypermultiplet. The $\mathcal{N} = 4$ vector is made by an $\mathcal{N} = 2$ vector and an adjoint chiral Φ. We can write an $\mathcal{N} = 4$ SYM action as

$$S^{\mathcal{N}=4}_{\mathrm{SYM}} = S^{\mathcal{N}=2}_{\mathrm{SYM}} + S^{\mathcal{N}=2}_{\mathrm{matt}}[\Phi]. \quad (1.32)$$

We stress that the relative coefficient is fixed by requiring $SU(2)_H \times SU(2)_C$ R-symmetry. We express the action in field components making the $\mathcal{N} = 4$ manifest. The fields are organized in a multiplet $\mathcal{V} = \left(A_{\mu}, \lambda_{a\dot{a}}, \Phi_{\dot{a}\dot{b}}, D_{ab}\right)$, containing the gauge field A_{μ}, the gauginos $\lambda_{\alpha,a\dot{a}}$, the dimension-one scalar fields $\Phi_{\dot{a}\dot{b}}$ and the dimension-two scalar fields D_{ab}, transforming in the $(\mathbf{1},\mathbf{1})$, $(\mathbf{2},\mathbf{2})$, $(\mathbf{1},\mathbf{3})$ and $(\mathbf{3},\mathbf{1})$ of $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, respectively. Then, the action reads

$$\begin{aligned} S^{\mathcal{N}=4}_{\mathrm{SYM}} =& \frac{1}{g^2_{\mathrm{YM}}}\int d^3x\sqrt{g}\,\mathrm{Tr}\Big[F^{\mu\nu}F_{\mu\nu} - \mathcal{D}^{\mu}\Phi^{\dot{a}\dot{b}}\mathcal{D}_{\mu}\Phi_{\dot{a}\dot{b}} + i\lambda^{a\dot{a}}\gamma^{\mu}\mathcal{D}_{\mu}\lambda_{a\dot{a}} - D^{ab}D_{ab} + \\ & - i\lambda^{a\dot{a}}[\lambda_a{}^{\dot{b}}, \Phi_{\dot{a}\dot{b}}] - \frac{1}{4}[\Phi^{\dot{a}}{}_{\dot{b}}, \Phi^{\dot{c}}{}_{\dot{d}}]\,[\Phi^{\dot{b}}{}_{\dot{a}}, \Phi^{\dot{d}}{}_{\dot{c}}]\Big]. \end{aligned} \quad (1.33)$$

The hypermultiplet is made by two $\mathcal{N} = 2$ chirals Q and $\tilde{Q}$, transforming in conjugate representations R and $\bar{R}$ of the gauge group. The action is just given by the $\mathcal{N} = 2$ action supplemented by the gauge-invariant superpotential term $W = \tilde{Q}\Phi Q$. Again, the coefficient of the extra term is fixed by supersymmetry. The physical consequence is the absence of wave function renormalization for matter fields. We move to fields components, making manifest $\mathcal{N}=4$ supersymmetry. They are $\mathcal{H}=\left(q_a, \tilde{q}^a, \psi_{\dot{a}}, \tilde{\psi}_{\dot{a}}\right)$, where q_a and $\psi_{\alpha,\dot{a}}$ are scalar and fermion fields transforming in the R of G, and in the $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ of $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, respectively. Similarly, $\tilde{q}^a$ and $\tilde{\psi}^{\dot{a}}$ are scalar and fermion fields transforming in the $\bar{R}$ of G, and in the $(\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2})$ of the R-symmetry group. The action is

$$\begin{aligned} S_{\text{hyper}} = \int d^3x\sqrt{g}\Big[& \mathcal{D}^\mu \tilde{q}^a \mathcal{D}_\mu q_a - i\tilde{\psi}^{\dot{a}}\gamma^\mu \mathcal{D}_\mu \psi_{\dot{a}} + i\tilde{q}^a D_a{}^b q_b - \frac{1}{2}\tilde{q}^a \Phi^{\dot{a}\dot{b}}\Phi_{\dot{a}\dot{b}} q_a + \\ & - i\tilde{\psi}^{\dot{a}}\Phi_{\dot{a}}{}^{\dot{b}}\psi_{\dot{b}} + i\left(\tilde{q}^a \lambda_a{}^{\dot{b}}\psi_{\dot{b}} + \tilde{\psi}^{\dot{a}}\lambda^b{}_{\dot{a}} q_b\right)\Big]. \end{aligned} \tag{1.34}$$

We observe that one could also define a version of these multiplets, but with Higgs and Coulomb charges exchanged. In practice, one has just to switch dotted and undotted indices and vice versa. The corresponding multiplets are called twisted vectors and twisted hypermultiplets. The type of Lagrangians is the same as before, but with the indices exchanged. In the abelian case, one can also couple a vector multiplet with a twisted one. If we denote the field of the twisted vector with a tilde, we get

$$S_{\text{SBF}} = \frac{i}{4\pi}\int d^3x \left(\epsilon^{\mu\nu\rho}F_{\mu\nu}\tilde{A}_\rho - \tilde{\lambda}^{a\dot{a}}\lambda_{a\dot{a}} - \tilde{\Phi}^{ab}D_{ab} - \Phi^{\dot{a}\dot{b}}\tilde{D}_{\dot{a}\dot{b}}\right). \tag{1.35}$$

The Lagrangian is the supersymmetric version of the BF term. One can use it to introduce the FI deformations, similarly to (1.30). The FI parameters will be ζ^{ab} and transform in the $(\mathbf{3},\mathbf{1})$ of the R-symmetry group. Similarly, we can add real mass deformations $m_{\dot{a}\dot{b}}$ in the Cartan of the flavor group, transforming in the triplet of $SU(2)_C$. More details are given in the explicit construction on S^3 in Appendix B.

1.2 Aspects of conformal symmetry

In certain cases, like at many fixed points of the RG flow, QFTs turn out to be invariant under the conformal group. These QFTs are called conformal field theories (CFT). Conformal transformations are the diffeomorphisms that can be modded out by a Weyl transformation. In other words, the conformal group is the set of coordinates transformations $x^\mu \to x'^\mu$ under which the metric transforms as [5]

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \tag{1.36}$$

These transformations extend the Poincaré group by the dilations

$$x^\mu \to \lambda x^\mu \tag{1.37}$$

and the special conformal transformations (SCT)

$$x^\mu \to \frac{x^\mu + b^\mu x^2}{1 + 2x\cdot b + b^2x^2}. \tag{1.38}$$

[5]Because of the broadness of the topic, we will not review it in detail. Instead, we will recall only the notions useful for the rest of the manuscript. For a thorough presentation we refer to [100] and reference therein.

The former is parametrized by a scalar λ, the latter by a vector b^μ. The set of the conformal transformations constitutes the group $SO(3,2)$ in the Minkowskian signature. The corresponding algebra is expressed in terms of the Poincaré generators, the dilatation generator D, and the SCT generator K^μ. The algebra is

$$[M_{\mu\nu}, P_\rho] = i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right)\,, \quad [M_{\mu\nu}, K_\rho] = i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right)\,, \tag{1.39}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right)\,, \tag{1.40}$$

$$[D, P^\mu] = P^\mu\,, \quad [D, K^\mu] = -K^\mu\,, \quad [K_\mu, P_\nu] = -2iM_{\mu\nu} + 2\eta_{\mu\nu}D\,. \tag{1.41}$$

Local operators are organized in representations of the conformal algebra. They are obtained by diagonalizing the generators D and $M_{\mu\nu}$ at the origin. The eigenvalue of D is the *scalign dimension* of the operator. P^μ and K^μ act respectively as raising and lowering operator w.r.t. to D. Since unitarity requires a spectrum bounded from below, there will be operators annihilated by $[K^\mu, O(0)] = 0$. These are called primary operators. All the other operators are obtained acting iteratively with P_μ on some primary. These operators are all called descendants. Then, every multiplet is characterized by the dimension and the spin of his primary state.

It is possible to show that physical operators correspond to unitary representations. In the Euclidean signature, Hermitian conjugation acts as $t \to -t$. Then, in radial quantization, Hermitian conjugation is generated by the inversion. This implies the following action of the conjugation

$$D^\dagger = D\,, \quad P^\dagger_\mu = K_\mu\,, \quad K^\dagger_\mu = P_\mu\,. \tag{1.42}$$

From this, the requirement of unitarity leads to the following bounds for the dimension of the primary operator with dimension Δ and Lorentz spin j [101]

$$\Delta \geq 0\,, \quad \text{if} \quad j = 0 \tag{1.43}$$

$$\Delta \geq 1\,, \quad \text{if} \quad j = \frac{1}{2} \tag{1.44}$$

$$\Delta \geq j + 1\,, \quad \text{if} \quad j > 1 \tag{1.45}$$

A multiplet saturating the above bound is short, otherwise, it is long. The dimension of short multiplets is fixed by the bound. This relation prevents it from acquiring an anomalous dimensions. For instance, any QFT possesses a stress tensor, namely a conserved two indices symmetric traceless tensor. Conservation implies the presence of the null state

$$P^\mu T_{\mu\nu} = 0\,. \tag{1.46}$$

This is due to the fact that the dimension of $T_{\mu\nu}$ is $\Delta = 3$. Similarly, a conserved current j_μ has the null state $P^\mu j_\mu = 0$. It follows that $\Delta_{j_\mu} = 2$ [6].

Conformal symmetry imposes severe constraints on the correlation functions of local operators [102]. For instance, all the two point functions of operators are completely fixed.

[6]The argument can be easily extended to a QFT in dimension d. In this case we find $\Delta_{T_{\mu\nu}} = d$, $\Delta_{j_\mu} = d - 1$.

In the case of scalar operators

$$\langle O_a(x_1)O_b(x_2)\rangle = \frac{C_{ab}}{|x_{12}|^{2\Delta}}\,, \tag{1.47}$$

where Δ is the dimension of both operators O_1 and O_2 and $x_ij \equiv |x_i - x_j|$. If the dimension of the operators is different, the correlator vanishes. Usually, the constants C_{ab} can be tuned to δ_{ab} by choosing an appropriate basis for the operators. When the normalization of an operator comes from a Ward identity, the constants C_{ab} become physical. For instance, the normalization of the stress tensor is fixed by the following Ward identity

$$\sqrt{\frac{2}{3}}\frac{1}{4\pi}\langle\partial_\mu T^\mu{}_\nu(x)O(x_1)\dots O(x_n)\rangle = -\sum_i \langle\delta(x-x_i)O(x_1)\dots\partial_\nu O(x_i)\dots O(x_n)\rangle \tag{1.48}$$

Conserved currents are normalized by analogous Ward identity. Then, the coefficients of their two-point functions become physical. For the stress tensor we have [7]

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{c_T}{24\pi^2}\mathcal{I}_{\mu\nu,\rho\sigma}(x)\frac{1}{x^6} \tag{1.49}$$

where the tensor $\mathcal{I}_{\mu\nu,\rho\sigma}(x)$ is fixed by conformal invariance

$$\mathcal{I}_{\mu\nu,\rho\sigma}(x) = \frac{1}{2}\left[I_{\mu\sigma}(x)I_{\nu\rho}(x) + I_{\mu\rho}(x)I_{\nu\sigma}(x)\right] - \frac{1}{3}\delta_{\mu\nu}\delta_{\rho\sigma}\,, \tag{1.50}$$

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\,. \tag{1.51}$$

The constant c_T is called central charge and provides a universal parameter of the CFT. Similarly, in presence of a global symmetry with conserved current j^a_μ we have

$$\langle j^a_\mu(x)j^b_\nu(0)\rangle = I_{\mu\nu}(x)\delta^{ab}\frac{c_j}{4\pi x^4} \tag{1.52}$$

c_j is the flavor central charge. We will provide a way to compute exactly both c_j and c_T in supersymmetric theories.

Similarly, three point functions are fixed up to a physical constant

$$\langle O_a(x_1)O_b(x_2)O_c(x_3)\rangle = \frac{f_{abc}}{|x_{12}|^{\Delta_a+\Delta_b-\Delta_c}|x_{13}|^{\Delta_a+\Delta_c-\Delta_b}|x_{23}|^{\Delta_b+\Delta_c-\Delta_a}}\,, \tag{1.53}$$

Then, conformal symmetry fixes the kinematics of all two and three points functions. Higher point functions will exhibit a non-trivial kinematical dependence.

A crucial property of CFT is the existence of a convergent operator product expansion (OPE) in correlation functions. If two operators are close enough we can replace their product with an infinite sum of operators

$$O_a(x)O_b(0) = \sum_c f_{abc}C_{\Delta_c,s_c}(x,\partial)O_c(0) \tag{1.54}$$

[7] Normalization is chosen so that $c_T = 1$ for free fields in $d = 3$. This explains the presence of the different normalization from [102].

where Δ_c and s_c are the dimension and the spin of the operator O_c and f_{abc} are the constants of 1.53. One can reconstruct all the correlation functions from the knowledge of all the dimensions of the primaries Δ_O and all their three-point functions f_{abc}.

We conclude by pointing out that four point functions in CFT have non-trivial kinematics. In principle, one can compute them by applying repeatetly the OPE. However, the result must not depend on the order in which we do this. Such a property is called crossing symmetry. It turns out that this condition and unitarity are severe constraints that allow the extraction of precise bounds on the CFT data, like the central charge c_T or the flavor central charges c_j. The implementation of this idea, which does not rely on the existence of any Lagrangian, is the *conformal bootstrap program* [3, 103].

1.2.1 Superconformal symmetry

If the low energy limit of a supersymmetric theory exhibits conformal symmetry, the whole set of symmetries is recast into a structure called superconformal algebra. By definition, superconformal algebras are superalgebras that contain both the standard SUSY algebra and the conformal algebra as subalgebras. Theories invariant under a superconformal algebra are called *superconformal quantum field theories* (SCFT).

A superconformal algebra has a rigid structure [101]. First of all, it must contain the conformal algebra $\mathfrak{so}(d,2)$ as a bosonic subalgebra. The supersymmetry generators Q, which are spinors of $\mathfrak{so}(d)$ and satisfy $\{Q_\alpha, Q_\alpha\} \sim P_\mu$, must satisfy $[D, Q_\alpha] \sim \frac{1}{2} Q_\alpha$, i.e. $\Delta_Q = \frac{1}{2}$. Consistency with the conformal algebra imposes the existence of additional odd spinorial generators S_α, called superconformal supercharges. They must satisfy $[S_\alpha, S_\alpha] \sim K_\mu$ and $\Delta_S = -\frac{1}{2}$. A novelty of the superconformal algebra is that the anticommutator $\{Q, S\}$ must contain bosonic R-symmetry generators. Then, unlike theories with only Poincaré supersymmetry, the R-symmetry must be a symmetry of the theory.

The outlined structure is restrictive. Since Q_α and S_α must recombine in a spinorial representation of $\mathfrak{so}(d,2)$, there cannot be superconformal algebras in $d > 6$. In 3d, the correct superalgebra turns out to be $\mathfrak{osp}(\mathcal{N}|4)$ [104, 105]. Its bosonic part is $\mathfrak{so}(\mathcal{N}) \oplus \mathfrak{sp}(4)$. We identify $\mathfrak{sp}(4) \sim \mathfrak{so}(3,2)$ as the conformal algebra and $\mathfrak{so}(\mathcal{N})$ as the R-symmetry algebra.

The odd-odd part of the superconformal algebra reads

$$\{Q_{\alpha r}, Q_{\beta s}\} = 2\delta_{rs} P_{\alpha\beta}\,, \quad \{S^\alpha{}_r, S^\beta{}_s\} = 2\delta_{rs} K_{\alpha\beta}\,, \tag{1.55}$$

$$\{Q_{\alpha r}, S^\beta{}_s\} = 2\left[\delta_{rs}\left(M_\alpha{}^\beta + \delta_\alpha^\beta D\right) - i\delta_\alpha^\beta R_{rs}\right]\,. \tag{1.56}$$

The even-odd part is given by

$$
\begin{aligned}
[K^\mu, Q_{\alpha r}] &= (\gamma_\mu)_\alpha{}^\beta S_{\beta r}\,, & [P_\mu, S_\alpha] &= (\gamma_\mu)_\alpha{}^\beta Q_{\beta r}\,, & (1.57)\\
[M_{\mu\nu}, Q_{\alpha r}] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{\beta r}\,, & [M_{\mu\nu}, Q_{\alpha r}] &= -\frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_{\beta r}\,, & (1.58)\\
[D, Q_{\alpha r}] &= \frac{1}{2}Q_{\alpha r}\,, & [D, S^\alpha{}_r] &= -\frac{1}{2}S^\alpha{}_r\,, & (1.59)\\
[R_{rs}, Q_{\alpha t}] &= i\left(\delta_{rt}Q_{\alpha s} - \delta_{st}Q_{\alpha r}\right)\,, & [R_{rs}, S^\alpha{}_t] &= i\left(\delta_{rt}S^\alpha{}_s - \delta_{st}S^\alpha{}_r\right)\,, & (1.60)\\
& & & & (1.61)
\end{aligned}
$$

where R_{rs} are the $SO(\mathcal{N})$ R-symmetry generators, obeying the algebra

$$
[R_{rs}, R_{ut}] = i\left(\delta_{ru}R_{st} + \delta_{st}R_{ru} - \delta_{rt}R_{su} - \delta_{su}R_{rt}\right)\,. \tag{1.62}
$$

The radial quantization induces the following conjugation properties for the odd generators $(Q_{\alpha r})^\dagger = -iS^\alpha_r$, $(S^\alpha_r)^\dagger = -iQ_{\alpha r}$. For the R-symmetry we get $R^\dagger_{rs} = R_{rs}$.

Let us explore the constraints of the larger symmetry on superconformal theories. As in the conformal case, operators are organized in multiplets of the superconformal algebra. In the supersymmetric case, the basic raising and lowering operators are the Poincaré supercharges $Q_{\alpha i}$ and the superconformal ones $S_{\alpha i}$. Moreover, a state will be labelled also by the R-symmetry Dynkin labels. We define the superconformal primary of a multiplet as the R-symmetry highest weight state annihilated by all the $S_{\alpha,i}$

$$
S_{\alpha,i}\,|\Delta, j, r\rangle_{\text{h.w.}} = 0\,. \tag{1.63}
$$

where $j \in \frac{\mathbb{N}}{2}$ is the Lorentz spin, $r = [r_1, \ldots, r_{\frac{\mathcal{N}}{2}}]$ is a collective notation for the Dynkin labels of the $\mathfrak{so}(\mathcal{N})$ R-symmetry representation. We can reconstruct the multiplet by acting on the superconformal primary (SCP) with all the $Q_{\alpha i}$. Such states are called superconformal descendants.

A novel feature of unitary SCFTs is the appearance of a new type of shortening condition. They are called BPS shortening conditions and are associated with states satisfying conditions like

$$
Q_{i,\alpha}\,|\Delta, j, r\rangle_{\text{h.w.}} = 0\,. \tag{1.64}
$$

It implies the vanishing of the norm of some superconformal decendants. In turn, this leads to superconformal bounds for the SCP. It comes from the fact that $\{Q, S\} \sim D - M_{\mu\nu} + R_{ij}$ and $S^\dagger = Q$. When the norm of a superconformal decendant vanishes, the bracket of such anticommutator is zero. One can translate this fact into a bound for the dimension Δ of the SCP. For $\mathfrak{osp}(\mathcal{N}|4)$ we distinguish two types of shortening conditions:

- A- type unitarity bound

  $$
  \Delta \geq h_1 + j + 1\,, \quad j \geq 0 \tag{1.65}
  $$

  where h_1 is the first Dynkin label in the orthonormal basis of [104]. These multiplets are usually long and become short when the bound is saturated.

- B-type unitarity bound

$$\Delta \geq h_1\,, \quad j = 0 \tag{1.66}$$

These multiplets are always short. If they are isolated they are absolutely protected.

The bootstrap method can be applied also to superconformal theories [106–110]. A special feature of superconformal field theories with extended supersymmetry is the existence of protected subsectors, which are closed under OPE [28]. These subsectors provide *a solvable truncation* of the bootstrap equation. This was originally shown in 4d with $\mathcal{N} \geq 2$, where the bootstrap problem of 2d subsectors can be reduced to a meromorphic CFT in 2d. A related construction on a line in 3d $\mathcal{N} \geq 4$ leads to a topological field theory [26, 27]. We will discuss this point in Chapter 3.

1.3 Dynamical aspects of 3d gauge theories

In this section, we discuss some aspects of the dynamics of 3d supersymmetric theories. The goal is to describe simple infra-red dualities in 3d supersymmetric gauge theories. An IR duality means that two different theories, with different degrees of freedom and interactions, flow to the same RG flow fixed point in the IR.

In this direction, a crucial point is to understand global symmetries. Their matching is a necessary condition for a duality. We distinguish between R-symmetry, which is a part of the superconformal algebra in the IR, and flavor symmetries, which depend on the details of the theory. Concerning R-symmetry, we stress that the UV R-symmetry does not need to coincide with the IR R-symmetry. For instance, in $\mathcal{N} = 2$ theory, we have an $U(1)_R$ symmetry that can mix with all the $U(1)$ flavor symmetry in the UV. We will see that mixing between flavor and R-symmetry happens also in $\mathcal{N} \geq 4$ theories.

A peculiar global symmetry of 3d gauge theories is the one generated by the topological current $j^\mu_T \propto \epsilon^{\mu\nu\rho} F_{\nu\rho}$. This current is conserved for every abelian factor in the gauge group. The name topological comes from the fact that conservation is ensured by Bianchi identity $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} \equiv 0$. We will call this symmetry the topological symmetry $U(1)_T$. Further flavor symmetries are present when we add matter and they will depend on the specific model. The topological symmetry does not come from the Lagrangian. This type of symmetry is ubiquitous in 3d and plays a central role in dualities. Nevertheless, there are degrees of freedom charged under these symmetries. The simplest example is the *dual photon*. It is a compact scalar field γ dual to a free vector field $F_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\gamma$.

Monopole operators are operators charged under $U(1)_T$ [111, 112]. They are local disorder operators defined by a singularity for the gauge fields. For abelian gauge theories, the prescription is

$$(\star F)_\mu = b\,\frac{x_\mu}{x^3}\,, \tag{1.67}$$

where b is an integer. In other words, we are prescribing a non-zero magnetic flux around the insertion point. The construction is the same in non abelian theory provided that b is a cocharacter. One can make the operator supersymmetric by prescribing a singular profile for the scalar σ such that $\partial_\mu\sigma = (\star F)_\mu$. BPS monopoles can be embedded in theories with higher supersymmetry by choosing a specifc $\mathcal{N}=2$ subalgebra.

Many aspects can be deduced by looking at the structure of the vacua[10–15]. At the classical level, the space of vacua coincides with the set of space-time independent solutions of the equations of motion. Quantum corrections can heavily modify its shape. The space of vacua is called *moduli space of vacua* and usually exhibits a complicated form. It is a set constituted by different branches, distinguished by the vev of the scalar fields. In 3d, moduli spaces have peculiar aspects. For instance, even abelian theories can exhibit non-trivial IR dynamics. Moreover, some parameters are not controlled by holomorphy. These include the real masses and FI-parameter, which come from background gauge fields, and the Chern-Simons level, which cannot be continuously varied.

Let us describe moduli space in some detail. $\mathcal{N}=2$ have a Coulomb branch, defined as the vacua satisfying the equations

$$\mathrm{Tr}([A_\mu, A_\nu])^2 = 0\,, \qquad \mathrm{Tr}([A_\mu, \sigma])^2 = 0\,. \tag{1.68}$$

They restrict σ and A_μ to take value in the Cartan subalgebra of the gauge group. Then, gauge symmetry is spontaneously broken to its maximal torus. The Coulomb branch is parametrized by the expectation value of the scalars σ_i ($i=1,\ldots,r$ with r rank of the gauge group) and the dual photons γ_i. These fields constitute the bottom component of a chiral multiplet $\Phi_i \sim \sigma_i + i\gamma_i$. Since γ_i is compact, the variables $Y_i \sim e^{\frac{\Phi\cdot\alpha_i}{g^2}}$ are a good set of coordinates for the Coulomb branch, where α_i are simple roots. The structure of the Coulomb branch can be drastically modified by strong coupling effects, such as instantons. The presence of a Chern-Simons term lifts the Coulomb branch.

In presence of chiral multiplets, there can be a branch parametrized by the vev of gauge-invariant operators of matter multiplets. This branch is called the Higgs branch. For instance, given N_f chirals in a representation R Q_i and N_f chirals $\tilde{Q}^j$ in the conjugate representation $\bar{R}$, the expectation valued of the mesons $M^j{}_i = \tilde{Q}^j Q_i$ are good coordinates. There can also be mixed branches, parametrized by both matter and vector multiplet scalars.

Intersection points of two different branches are often a signal for non-trivial SCFTs. If two theories have moduli spaces that have the same type of intersection points, they are possible candidates for an IR duality. For instance, let us describe the moduli space of $\mathcal{N}=2$ SQED, which is $U(1)$ gauge theory coupled to a charge 1 and a charge -1 chirals. Its Coulomb branch is parametrized by a chiral $V = e^{\Phi/g^2}$, with $\Phi = \sigma + i\gamma$. There is also a Higgs branch described by $M = \tilde{Q}Q$. The classical Coulomb branch looks like a cylinder, with the radius proportional to g. Strong coupling effects modify its form. At the origin $\Phi = 0$, the topological symmetry $U(1)_T$ acts trivially. This implies that the radius of γ shrinks to zero. Then, the Coulomb branch splits into two regions. At the end of the day, we have three cones with intersecting tips. Two of them describe the Coulomb branch, described by the chiral fields $V_\pm \sim e^{\pm\Phi/g^2}$. The third is the Higgs branch, parametrized by M. It is not hard to see that the same picture emerges for the theory of three chiral $V_\pm$ and M, with superpotential $W = V_+V_-M$. This theory has the same IR fixed point of

SQED. We will show a quantitative check of the duality in the next chapter.

We can uplift the duality to $\mathcal{N}=4$ SQED (i.e. $\mathcal{N}=4$ vector coupled to a hyper) by adding an adjoint chiral Φ with a superpotential $W=\tilde{Q}\Phi Q$. In the dual theory, it corresponds to a superpotential $W=V_+V_-M+M\Phi$. The deformation is just a mass term that leaves V_+ and V_- with no superpotential. Then the dual theory of $\mathcal{N}=4$ SQED is a free (twisted) hyper. Even if this an IR duality, the duality becomes exact along the RG flow triggered by $W=\tilde{Q}\Phi Q$ and connecting the fixed point of $\mathcal{N}=2$ SQED to the one of $\mathcal{N}=4$ SQED [113]. Indeed, the relevant deformation coincides at the UV fixed point of the corresponding RG flow.

The latter case is the simplest example of a family of dualities, known as *mirror symmetry* [86] [8]. The duality was originally conjectured for $\mathcal{N}=4$ Kronheimer gauge theories, and then extended in several directions. $\mathcal{N}=4$ theories have a highly constrained vacua structure. Indeed, we can invariantly split the moduli space in a Coulomb branch and a Higgs branch. They are both Hyper-Kähler manifold. It turns out that the metric of the Higgs branch is not affected by quantum effects. On the contrary, the metric of the Coulomb branch receives 1-loop perturbative corrections and non-perturbative effects. The main features of mirror symmetry are

- The $SU(2)_C$ and $SU(2)_H$ R-symmetry groups are exchanged.
- Higgs and Coulomb branch are exchanged.
- Mass terms and FI parameters are exchanged.

which is seen as an $\mathcal{N}=4$ SYM theory coupled to an adjoint hyper. The dual theory is given by the same theory with an additional fundamental hypermultiplet.[9]

1.4 Supersymmetric Chern-Simons theory

This section is devoted to $\mathcal{N}\geq 2$ supersymmetric Chern-Simons matter theories. They provide an exciting example of superconformal field theories in 3d. The new ingredient is the Chern-Simons term [117]. We can think of it as an alternative kinetic term for the gauge field. The corresponding Lagrangian is

$$\mathcal{L}_{\text{CS}}=\frac{i}{4\pi}\operatorname{Tr}_{\text{CS}}\left[\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}A_\mu A_\nu A_\lambda\right)\right]\equiv\frac{k}{4\pi}CS(A)\,. \tag{1.69}$$

This theory exhibits several peculiarities. First of all, the action is not gauge invariant. However, the actual condition for a gauge-invariant theory is that $e^{iS_{\text{CS}}}$ is independent of the gauge. This implies the quantization condition for the quadratic form defined by $\operatorname{Tr}_{\text{CS}}$. For instance, for the $U(N)$ gauge group, $\operatorname{Tr}_{\text{CS}}=k\operatorname{Tr}_{\text{F}}$, where $\operatorname{Tr}_{\text{F}}$ is the trace in the fundamental representation. The quantization condition becomes $k\in\mathbb{Z}$. We will mostly

[8] Other examples of interesting IR dualities we do not discuss are Aharony dualities [87] and Giveon-Kutasov dualities [88].

[9] A simple explanation of the duality can be given by using a brane construction of the two theories in type IIB string theory. In this setup, mirror symmetry corresponds to S-duality of type IIB string theory [114–116].

focus on $U(N)$ gauge groups. Even though k is not a continuous coupling constant, it is still possible to think the theory at large k weakly coupled and attack the theory with perturbation theory.

The second property of this theory is that it is topological. At the classical level, this is a mere consequence of the metric independence of the Lagrangian. At the quantum level, the situation is more subtle. The regularization breaks diffeomorphism invariance. It is shown in [117] that it can be recovered by adding a suitable gravitational Chern-Simons counterterm. The latter depends in a controlled way on the chosen trivialization of the tangent bundle of the manifold. This choice is parametrized by an integer, known as the *framing*.

Since the theory is topological, there are no local degrees of freedom. However, there are still interesting loop operators, such as Wilson loops. They are defined as the trace of the holonomy of the gauge field along the curve γ

$$W = \mathrm{Tr}_R\, P e^{i \oint_\gamma A} \,. \tag{1.70}$$

where R is a representation of the gauge group. We will discuss extended operators in more detail in Chapter 4. Remarkably, in [117] it was shown how to compute exactly both the partition function and Wilson loops correlators *exactly*.

The Chern-Simons term can be coupled to matter, giving rise to systems with local dynamics [118]. In this case, one can tune the various parameters to have a Lagrangians conformal at the quantum level. The dynamics of these theories are interesting and related to condensed matter systems. From now on, we will focus on the supersymmetric completion of such theories.

1.4.1 $\mathcal{N}=2$ off-shell Chern-Simons terms

We introduce an $\mathcal{N}=2$ Chern-Simons term. In this case the superspace approach is not so convenient. However, since we have already introduced the vector multiplet and his off-shell transformation, one can build directly the Supersymmetric completion of the CS term (1.69) [99]. Indeed, the SUSY variation of the CS term gives

$$\delta \mathcal{L}_{\mathrm{CS}} \propto \epsilon^{\mu\nu\rho} F_{\mu\nu} \delta A_\rho \,. \tag{1.71}$$

We plug into the explicit variations, and after some algebra we show that the variation is δ-exact up to boundary terms

$$\delta \mathcal{L}_{\mathrm{CS}} = \delta \left(-2iD\sigma - \bar{\lambda}\lambda\right) + \partial_\mu K^\mu, \tag{1.72}$$

for some field dependent K^μ, whose explicit form is irrelevant for us. Then, the $\mathcal{N}=2$ supersymmetric CS term is given by

$$\mathcal{L}_{\mathrm{CS}}^{\mathcal{N}=2} = \frac{i}{4\pi} \mathrm{Tr}_{\mathrm{CS}} \left[\epsilon^{\mu\nu\lambda} \left(A_\mu \partial_\nu A_\lambda + \frac{2i}{3} A_\mu A_\nu A_\lambda \right) + 2iD\sigma + \bar{\lambda}\lambda \right] . \tag{1.73}$$

Notice that here all the fields except A_μ are auxiliary. When we integrate them out we recover the pure CS term.

The coupling to the matter goes as in Section 1.1.1. It is possible to integrate out auxiliary fields of the vector multiplet. For instance, for the $U(N)$ theory with N_f chirals in the representation R_i of the gauge group, $i = 1, \dots, N_f$ $\sigma = -\frac{4\pi}{k}\left(\bar{\phi}_i T^a \phi_i\right) T^a$ and similarly for the other vector multiplet fields. At the end of the day, we are left with [65]

$$S^{\mathcal{N}=2} = \int d^3x \Big[\frac{k}{4\pi} CS(A) + D^\mu \bar{\phi}_i D_\mu \phi_i - i\bar{\psi}_i \gamma^\mu D_\mu \psi_i - \frac{4\pi}{k} (\bar{\phi}_i T^a \phi_i)(\bar{\psi}_j T^a \psi_j) + \tag{1.74}$$

$$- \frac{8\pi}{k} (\bar{\psi}_i T^a \phi_i)(\bar{\phi}_j T^a \psi_j) - \frac{16\pi^2}{k^2} (\bar{\phi}^i T^a \phi_i)(\bar{\phi}^i T^b \phi_i)(\bar{\phi}^i T^a T^b \phi_i) \,. \Big] \tag{1.75}$$

If we limit to flat sigma models, the theory is conformal at the quantum level. An intuitive motivation is the integrality of CS level: the only possible effect on CS coupling is just a 1-loop integer shift [65]. Quantum corrections can affect only the Kähler potential, and they are absorbed by rescaling the matter superfields. In conclusion the action is superconformal at the quantum level. It is possible to introduce superpotential deformations. If they do not mix with flavor symmetries they give rise to a conformal manifold, otherwise conformal invariance is broken [10] [121, 122].

A special point in the conformal manifold exhibits $\mathcal{N} = 3$ supersymmetry. We can build such a theory in the $\mathcal{N} = 2$ language. We start with the $\mathcal{N} = 4$ field content and we add to the $\mathcal{N} = 2$ Chern-Simons term a superpotential $W = \mathrm{Tr}\left(\Phi^2\right)$, where Φ is the adjoint chiral in the $\mathcal{N} = 4$ vector. The following action yields a Chern Simons term invariant under $\mathcal{N} = 3$ supersymmetry [65]

$$S = S^{\mathcal{N}=2}_{\mathrm{CS}} - \frac{k}{4\pi} \int d^3x d^2\theta \,\mathrm{Tr}\,\Phi^2 + c.c. \tag{1.76}$$

If we add matter hypermultiplets as explained in Section 1.1.2, the total superpotential becomes

$$W = \int d^2\theta \left(-\frac{k}{4\pi} \mathrm{Tr}\,\Phi^2 + \tilde{Q}\Phi Q + c.c. \right) . \tag{1.77}$$

If we integrate out the adjoint chiral Φ, we can treat the theory as a $\mathcal{N} = 2$ theory with the superpotential

$$W = \frac{2\pi}{k} (\tilde{Q} T^a Q)(\tilde{Q} T^a Q) \tag{1.78}$$

Then we can think of the $\mathcal{N} = 3$ theory as a $\mathcal{N} = 2$ Chern-Simons term, coupled to the $\mathcal{N} = 2$ chirals Q, $\tilde{Q}$ and the above superpotential. We observe that the $SO(3)$ R-symmetry does not allow for any wave function renormalization. Then the $\mathcal{N} = 3$ theory is exactly superconformal, and matter fields have standard dimensions.

We remark that the $\mathcal{N} = 3$ Chern-Simons matter theory admits an off-shell realization. In terms of the $\mathcal{N} = 4$ vector multiplet this is realized by breaking the $SU(2)_H \times SU(2)_C$ to

[10] The conformal manifold is the space of possible conformal field theory modulo global symmetries. For examples in 3d $\mathcal{N} = 2$ Chern-Simons matter theory see [119, 120]

its diagonal $SU(2)$ subgroup by a term $\Phi^{\dot{a}\dot{b}}D^{ab}\epsilon_{a\dot{a}}\epsilon_{b\dot{b}}$. It is hard to have a similar coupling preserving the full $SU(2)_H \times SU(2)_C$ explains why there is no off-shell $\mathcal{N} \geq 4$ Chern-Simons term [11]. Nevertheless, we can write these theories giving up the off-shell closure. We devote the rest of the chapter to introduce such theories. They are crucial for the AdS/CFT correspondence [124].

1.4.2 ABJM model and $\mathcal{N} \geq 4$ Chern-Simons theories

The ABJM model is an $\mathcal{N} = 6$ Chern-Simons matter theory [42]. It is a special case of the $\mathcal{N} = 3$ Chern-Simons theory discussed above. Consider an $\mathcal{N} = 3$ theory with gauge group $U(N) \times U(N)$. The matter content is made by two bifundamental chirals A_1, A_2 and two anti-bifundamentals chirals B_1 and B_2. We add two $\mathcal{N} = 3$ Chern-Simons term, one for each gauge group factor, with level k and $-k$ respectively.

The total superpotential is

$$W = \frac{k}{8\pi}\,\mathrm{Tr}\left(\Phi_{(1)}^2 - \Phi_{(2)}^2\right) + \mathrm{Tr}\big(B_i\Phi_{(1)}A_i\big) + \mathrm{Tr}\big(A_i\Phi_{(2)}B_i\big)\,, \tag{1.79}$$

where $\Phi_{(1,2)}$ are the (auxiliary) adjoint chiral in the $\mathcal{N} = 4$ vector. When we integrate out $\Phi_{(1,2)}$, we obtain

$$W = \frac{4\pi}{k}\,\mathrm{Tr}(A_1B_1A_2B_2 - A_1B_2A_2B_1) \tag{1.80}$$

Let us give a closer look at the global symmetry of the theory. There is a $SU(2) \times SU(2)$ flavor symmetry which rotates the A's and the B's separately. Indeed, we can make the symmetry manifest in the superpotential

$$W = \frac{2\pi}{k}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\,\mathrm{Tr}\left(A_aB_{\dot{a}}A_bB_{\dot{b}}\right)\,. \tag{1.81}$$

There is another global R-symmetry $SU(2)_R$, which rotates A_1 and $\bar{B}_1$ (as well as A_2 and $\bar{B}_2$) together. The $SU(2)_R$ symmetry does not commute with the flavor symmetry $SU(2) \times SU(2)$. All these $SU(2)$ factors are combined in a larger group, which turns out to be $SU(4) \simeq SO(6)$. The fact that $SU(2)_R$ acts on the supercharges implies that also $SU(4)$ does the same. Then the global symmetry is an R-symmetry, and our model has (at least) $\mathcal{N} = 6$ superconformal symmetry. In conclusion, the global symmetry is $SU(4) \times U(1)_T$, where $U(1)_T$ is the topological symmetry. There is also a discrete symmetry given by the usual parity combined with the exchange of the two gauge groups. It acts as charge conjugation on the charged fields [12].

We remark that an $\mathcal{N} = 3$ subalgebra is realized off-shell. After the integration of the auxiliary fields, the action can be written in a manifestly $\mathcal{N} = 6$ invariant way. For instance, the bottom components of the superfields are organized in four scalars. They transform in the fundamental of $SU(4)$ $C_I \equiv (A_1, A_2, \bar{B}_1, \bar{B}_2)$.[13] Similarly, the matter fermions transform in the fundamental of $SU(4)$ and are collectively denoted by ψ_I. The

[11] See here for a possible off-shell formulation [123]. In any case, the coupling to matter is not discussed.
[12] See the next section for a discussion of the case $k = 1, 2$.
[13] We are denoting the bottom components of the chiral multiplet with the superfield itself.

action involves an intricated potential, detailed in Appendix A.

The idea that Chern-Simons theories with extended supersymmetry arise from specific lower supersymmetric theories was originally introduced in [125] and further generalized in [126, 127]. We briefly summarize some of these results. Let us start with the compact symplectic group $Sp(2n)$, endowed with an antisymmetric tensor ω_{AB}, where $A, B = 1, \ldots, 2n$. We consider Chern-Simons theory with gauge group G a subgroup of $Sp(2n)$. The generators of G are denoted by $(t_m)^A{}_B$. In this notation, hypermultiplets are $(q_a^A, \psi_{\dot{a}}^A)$ with the reality conditions

$$\bar{q}_A^a = (q_a^A)^\dagger = \epsilon^{ab}\omega_{AB}q_b^B \,, \qquad \bar{\psi}_A^{\dot{a}} = (\psi_{\dot{a}}^A)^\dagger = \epsilon^{\dot{a}\dot{b}}\omega_{AB}\psi_{\dot{b}}^B \tag{1.82}$$

where $a, \dot{a}$ are $SU(2)_H \times SU(2)_C$ R-symmetry indices. Since the gauge group is a subgroup of $Sp(2n)$, it will act in a pseudo-real representation R. The multiplet is commonly called half-hypermultiplet in the representation R. When R can be decomposed in a complex representation R_0 and its complex conjugate, we can split the half-hyper it in two chiral multiplets, denoted by q_a and $\bar{q}_a$ in section 1.1.2

The strategy of [125] consists in writing the most general $\mathcal{N} = 1$ action, including a superpotential and then imposing $\mathcal{N} = 4$ invariance. The procedure leads to a severe constraint for the generators t^m and the Chern-Simons level, called *fundamental identity*

$$k_{mn}t^m_{(AB}t^n_{C)D} = 0 \,. \tag{1.83}$$

where k_{mn} defines the quadratic form of $\mathrm{Tr}_{\mathrm{CS}}$. This identity is equivalent to the Jacobi identity for a superalgebra which extends the gauge symmetry algebra. The corresponding supergroup are $U(N_1|N_2)$ and $Osp(N|M)$. The consequence is that the gauge group of the $\mathcal{N} = 4$ theories can be either $U(N_1) \times U(N_2)$ and $O(N) \times Sp(M)$, with opposite Chern-Simons level for the two factors. The hypers transform in the bifundamental of the gauge group.

One can also add twisted hypermultiplets. They have the same content of standard hypers, but the role of the R-symmetry indices is exchanged $(\tilde{q}_{\dot{a}}^A, \tilde{\psi}_a^A)$. They transform under the same gauge group as the hypers, but in an eventually different representation. Its generators are denoted by $(\tilde{t}^m)^A{}_B$. We assume that all the generators satisfy the fundamental identity. A supersymmetric Chern-Simons matter theory coupled both to hypers and twisted hypers requires an additional coupling between the hypers and the twisted hypers. This possibility is explored in detail in [126]. Their result allows us to go beyond the classification in terms of superalgebras by alternating hypers and twisted hypers between either the $U(N_1) \times U(N_2)$ or $O(N) \times Sp(M)$ factors. These theories can be classified in terms of quiver diagrams. See also [127] for a further generalization.

Imposing realities properties of the gauge group generators implies a further supersymmetry enhancement. As shown in [128], if $t^m = \tilde{t}^m$ the supersymmetry is automatically enhanced to $\mathcal{N} = 5$. In addition to this, if the matter (twisted) hypers are full (twisted) hypers, SUSY is further enhanced to $\mathcal{N} = 6$. In this way the ABJ family with gauge group $U(N_1)_k \times U(N_2)_{-k}$ or $SO(2)_{2k} \times Sp(2N)_{-k}$ where built.[43, 128] [14] For the case $U(N_1)_k \times U(N_2)_{-k}$, it should hold $|N_1 - N_2| < k$. Otherwise, the CS level is shifted by the 1-loop corrections, and the $\mathcal{N} = 6$ supersymmetry is broken at the quantum level. Similar arguments restrict also the CS level of the $SO(2)_{2k} \times Sp(2N)_{-k}$ theories. In the following, we will refer to all these theories as ABJM, specifying the model only if necessary.

1.5 ABJM: AdS/CFT correspondence and dualities

One of the main motivations to study CS theories with extended supersymmetry comes from the AdS/CFT correspondence [7]. Even if holography is not directly discussed, we find it convenient to review the relationship between the ABJM theory and M-theory. AdS/CFT states the duality between a gravitational theory living in AdS in $d+1$ dimension and a CFT (without gravity) living in d dimensions.

It was originally conjectured by Maldacena [7], who proposed the duality between $\mathcal{N} = 4$ Super Yang-Mills theory in 4d and type IIB string theory on the background $AdS_5 \times S^5$. However, the analog duality for 3d SCFTs remained an open problem for a long time. It is convenient to recall some features of the 4d case. One of the crucial points comes from the dual nature of D-branes. D-branes are extended objects in string theory, labeled by their spatial dimensions (a D1-brane is a string, a D2 is a membrane, and so on). D-branes in type IIB string theory admits two interpretations [131]

- Open string perspective: they are a submanifold where open strings can end. The view holds if the string coupling $g_s N \ll 1$, where N is the number of coincident D-branes.

- Closed string perspective: they are solitons in the low energy limit of string theory, i.e. type IIB supergravity. This holds if $g_s N \gg 1$.

The two visions are complementary and their interplay originates the duality. Let us focus on D3-branes. If one examines the low energy excitations in the first case one identifies 4d $SU(N)$ $\mathcal{N} = 4$ SYM as the low energy limit of open strings attached to the D3-branes. In the second perspective, the low energy limit includes type IIB superstring theory on the background $AdS_5 \times S_5$ [15]. The first evidence of the duality comes from matching the global symmetries. The bosonic part of the global symmetry of $\mathcal{N} = 4$ SYM, which is $SO(4,2) \times SO(6)$, is dual to the isometries of the manifold $AdS_5 \times S_5$. The matching can be extended to the full superconformal group. Let us stress a crucial feature of AdS/CFT. The strong coupling limit of one side of the duality is exchanged to the weak coupling limit of the other side.

The extension of the argument to AdS_4/CFT_3 duality exhibits additional difficulties. This is related to the fact that the duality should involve M-theory. M-theory is an eleven-dimensional theory and is defined as the strong coupling limit of type IIA string theory

[14] See also [129, 130] for more exotic possibilities, which involve additional $U(1)$ factor and/or quotient by discrete groups.

[15] The low energy of both perspectives includes also a decoupled massless closed string sector propagating in 10d Minkowski spacetime

[132]. It is hard to study because of the absence of a dimensionless coupling constant. Despite the lack of a microscopic description, we know that its low energy limit is eleven-dimensional supergravity. We also know that it supports two types of extended objects: M2 and M5 branes.

To understand the difficulties of M-theory, we take N D2-branes in type IIA string theory. The low energy limit of this configuration is the maximally supersymmetric $U(N)$ SYM in 3d. The Lagrangian has only a manifest $SO(7)$ R-symmetry, which reflects the symmetry of a hypothetical transverse compact manifold. Moreover, the SYM coupling is dimensionful and increases at low energy. In the dual picture, it corresponds to the increase of the 11th dimension. Then, when the SYM coupling becomes infinite, D2 branes are lifted to M2-branes in M-theory. The R-symmetry should consequently enhance to $SO(8)$.

There are two difficulties. From the string theory point of view, we cannot take the two limits described above for a stack of M2-branes because of the lack of a dimensionless coupling. On the field theory side, there is no theory with a manifest $\mathcal{N} = 8$ superconformal symmetry. Supersymmetric Chern-Simons matter theories were proposed as the natural candidate to play this role [124]. However, under some assumptions, it was not possible to write a Chern-Simons matter theory with manifest $\mathcal{N} = 8$ supersymmetry. The first proposal for a theory of multiple M2-branes was formulated in [133–135]. The authors proposed a theory based on an unusual algebraic structure called a three-algebra [136].

The problems were solved in the seminal paper [42], where the $U(N)_k \times U(N)_{-k}$ ABJM model was shown to be dual to M-theory on the background $AdS_4 \times S^7/Z_k$. The gravitational side was built by starting with a complicated brane construction, whose description goes beyond our purposes. Since k enters the geometry of the dual, there is no coupling constant in the M-theory dual. The puzzle of the $\mathcal{N} = 8$ SUSY was solved by proposing that for $k = 1, 2$ the ABJM model undergoes a SUSY enhancement, driven by two additional currents charged under the topological symmetry $U(1)_T$. Because of this and being the symmetry not manifest, it was conjectured that these operators are realized as (BPS) monopole operators. Indeed, such dimension 2 conformal primaries, charged under $U(1)_T$, and transforming as Lorentz vector were shown to exist [94, 95]. In conclusion, the ABJM model $U(N)_k \times U(N)_{-k}$ at $k = 1, 2$ is dual to the IR fixed point $\mathcal{N} = 8$ SYM in 3d.

Let us highlight some features. For k large enough, we recover the type IIA string theory picture. To see this, we think of S^7 as an S^1 fibration over $\mathbb{C}P^3$. The orbifold Z_k acts on the S^1, reducing its radius by a factor k. From the supergravity solution, the radius turns out to be of order $(Nk)^{\frac{1}{6}}/k$. Then, it shrinks to zero when $N \ll k^5$. In this limit, the M-theory background is well approximated by type IIA superstrings living on $AdS_4 \times \mathbb{C}P^3$. In this case, the 't Hooft coupling is given by $\lambda = \frac{N}{k}$, interpolating between the weak and strong coupling. When $\lambda \ll 1$ the string theory side is efficiently described by type IIA supergravity, but the QFT is strongly coupled. For $\lambda \gg 1$ the QFT is accessible via perturbation theory, but the gravitational theory is strongly coupled. Another interesting limit is when the S^1 radius is very large, i.e. $N \gg k^5$. In this case, the gravitational side is well approximated by eleven-dimensional supergravity and it is called the M-theory limit. In this case, the QFT is always strongly coupled.

Let us stress that we have identified three theories that are conjectured to flow to the same IR fixed point [94, 95]. The first one is $\mathcal{N} = 8$ SYM, which is dual to $\mathcal{N} = 4$ SYM coupled to a fundamental and an adjoint hyper. According AdS/CFT correspondence, also ABJM at $k = 1$ flows to the same fixed point. As a further check, we argue that the moduli space of $\mathcal{N} = 8$ SYM coincides with the ABJM one. In the former case one finds $\mathcal{M}_{\text{SYM}} = (\mathbb{R}^7 \times S^1)^N/S_N$, where the $\mathbb{R}^7$ factors come from the scalars, S^1 is the dual photon, and S_N is the permutation group, i.e. the Weyl group of $U(N)$. In the IR limit, the radius of the dual photon becomes infinite, and the moduli space becomes [137]

$$\mathcal{M}_{\text{SYM}} = \frac{\mathbb{C}^{4N}}{S_N}\,. \tag{1.84}$$

For ABJM, in the abelian case, it is not hard to see that the superpotential vanishes. Then, the moduli space is parametrized by the free C_I. We can gauge fix the vector fields to zero. Keeping into account the residual constant gauge transformations generated by $\Lambda = 2\pi n/k$, with $n \in \mathbb{Z}$, we obtain that the moduli space is $\mathbb{C}^4/Z_k$. Up to the permutation symmetry, the non-abelian case is reduced to N copies of the abelian case. Therefore the moduli space is [42]

$$\mathcal{M}_{\text{ABJM}} = \frac{(\mathbb{C}^4/Z_k)^N}{S_N}\,. \tag{1.85}$$

For $k = 1$ it coincides with $\mathcal{M}_{\text{SYM}}$.

In the following chapters, we will provide quantitative tests related to the features described in this chapter thanks to the localization technique, opening a window on the non-perturbative regime.

Chapter 2

Localization in three-dimensional gauge theories

This chapter provides an introduction to the localization technique. It is a powerful method that, in favorable circumstances, allows us to reduce the path integral to a more tractable form. The basic argument was developed in the math literature [138–140]. Then, localization found physical applications in topological theories. Later, the method was extended to supersymmetric theories in 4d in the seminal paper [21]. Then, many exciting results were obtained in supersymmetric gauge theories in different dimensions. In this chapter, we focus on applications to the three-dimensional gauge theories [22].

The main idea of localization is that if a theory has a fermionic symmetry satisfying certain assumptions, the path integral can be deformed to a free theory without altering it [1]. Then, a 1-loop computation in this free theory captures the value of supersymmetric observables in the interacting theory.

Let us provide an intuitive motivation for localization. We consider an integral over a supermanifold with bosonic coordinates x^μ, and ψ^μ [142]

$$Z = \int d^n x\, d^n \psi\, e^{-S[x,\psi]}\,. \tag{2.1}$$

If there is a $U(1)$ bosonic symmetry generating the group G, we can choose a coordinate system where the symmetry acts as $\frac{\partial}{\partial x^n}$. It follows that $S[x,\psi]$ does not depend on x^n. If the symmetry has no fixed point, one can use the symmetry to perform the integral along x^n and reduce it to

$$Z = \mathrm{Vol}\, U(1) \int d^{n-1} x\, d^n \psi\, e^{-S[x,\psi]}\,. \tag{2.2}$$

We repeat the argument for a fermionic symmetry associated with a fermionic group G. In this case, Z is proportional to

$$\mathrm{Vol}\, G = \int d\psi^n\,, \tag{2.3}$$

where ψ^n is the fermionic coordinate associated with the symmetry G. The Grassmann

[1] There are also examples that do not rely on supersymmetry [19, 141]. Nevertheless, we will always assume its existence.

integration over ψ^n makes Z vanish. The conclusion is too drastic. There is a loophole that leads to a non-vanishing result. In the derivation, we assumed the absence of fixed points. Around a fixed point, the integral does not need to be zero. Then, the correct conclusion is that only the fixed points of the fermionic symmetry contribute. This mechanism yields the localization of the integral.

In the following, we will make this argument more precise, extending it to QFTs. We discuss localization for 3d gauge theories, including the construction of supersymmetric QFTs in curved spacetime. We will focus on the round sphere S^3 and squashed sphere case. We will also discuss some applications to the physics of 3d gauge theories related to RG flows, dualities, and AdS/CFT.

The structure of the chapter is the following. We begin with an overview of the localization argument in geometry and QFT in Section 2.1. Then, in Section 2.2 we will review the methods to place supersymmetric QFTs on different curved compact backgrounds and discuss their applications to 3d gauge theories. In Section 2.3, I will present the localization for S^3 with several applications. Finally, we extend the results to the squashed sphere in Section 2.4.

2.1 Localization argument

This section aims to review the localization argument. The underlying idea is that in some examples, the asymptotic expansion of integrals of the form turns out to be *exact*. We begin our discussion with a finite-dimensional paradigmatic example and then extend it to the infinite-dimensional case.

2.1.1 The finite dimensional case

We want to compute an integral of a $2n$ differential polyform ω on a $2n$ dimensional manifold M, equipped with a Riemannian metric g. It is possible to reformulate this integral in a supersymmetric way by replacing the differential basis dx^μ with $2n$ fermionic coordinates ψ^μ.

$$\omega = \sum_{l=1}^{2n} \frac{1}{(2l)!}\omega^{(l)}_{\mu_1\dots\mu_l} dx^{\mu_1}\wedge\dots\wedge dx^{\mu_l} \to \omega(x,\psi) = \sum_{l=1}^{2n}\frac{1}{(2l)!}\omega^{(l)}_{\mu_1\dots\mu_l}\psi^{\mu_1}\dots\psi^{\mu_l} \tag{2.4}$$

The integral to compute is

$$I = \int_M \omega \equiv \int d^{2n}x d^{2n}\psi\ \omega(x,\psi)\,, \tag{2.5}$$

where ψ^μ transforms under change of coordinates as dx^μ to make the definition coordinate independent. Let us assume the existence of a compact $U(1)$ action on M generated by a Killing vector $V = V^\mu\partial_\mu$. We use it to introduce a fermionic transformation δ on M closing to this $U(1)$

$$\delta x^\mu = \psi^\mu\,, \qquad \delta\psi^\mu = V^\mu\,. \tag{2.6}$$

We limit to consider δ-invariant differential forms, namely we assume $\delta\omega = 0$. We take the auxiliary quantity

$$I(t) = \int d^{2n}x d^{2n}\psi\, \omega(x, \psi) e^{-t\delta W}\,, \tag{2.7}$$

where $t \in \mathbb{R}$ and W is a $U(1)$ invariant function, called *localizing term*. Under our hypotheses, one can show that the derivative of $I(t)$ w.r.t. to t is zero. This implies that

$$I = I(0) \equiv \lim_{t\to+\infty} I(t)\,. \tag{2.8}$$

To perform the computation, we need an explicit form of W. A convenient choice is

$$W = \delta\psi^\mu g_{\mu\nu}\psi^\nu = V^\mu g_{\mu\nu}\psi^\nu\,. \tag{2.9}$$

We calculate

$$\delta W = \psi^\lambda\psi^\nu\partial_\lambda V_\nu + V^\mu V_\mu \tag{2.10}$$

One can also show that $\delta^2 W = 0$ provided that V is a Killing vector for the metric. Since the bosonic part of δV is just the norm of V, for t very large the integral is dominated by the fixed points of the $U(1)$ symmetry generated by V. We denote the set of fixed points as M_V. We say that the integral localizes on M_V.

We perform the integral explicitly under the assumption that V vanishes on a set of discrete points x_k. Then, we can compute the contribution from every point separately. Let us focus on a specific fixed point. We can always choose a set of coordinates such that the origin coincides with the fixed point. When $t \to \infty$, we scale both x^μ and ψ^μ by a factor $t^{-\frac{1}{2}}$. Then, the contribution of the fixed point is fully captured by the quadratic term in the exponent

$$\int d^{2n}x d^{2n}\psi\, \omega(0) \exp\left[-\frac{1}{2}\left(H_\mu \nu x^\mu x^\nu + S_{\mu\nu}\psi^\mu\psi^\nu\right)\right] \sim \omega(0)\frac{\text{Pfaff}\, S}{\det H} \tag{2.11}$$

At this order, one can consistently take the linearized transformation δ_ℓ of (2.6):

$$\delta_\ell x^\mu = \psi^\mu\,, \qquad \delta_\ell\psi^\mu = x^\nu\partial_\nu V^\mu(0) \tag{2.12}$$

At the linearized level order, $\delta^2 W = 0$ implies that $H_{\mu\nu} = S_{\mu\lambda}\partial_\nu V^\lambda(0)$. Implementing this relation in the ratio of the determinants (2.11), we are left with the much simpler determinant of the matrix representing the linearized action of the Killing vector around the fixed point $\partial_\mu V^\nu(0)$. Finally, we sum over the fixed points. The final result reads

$$\int_M \omega = (-2\pi)^n \sum_{p\in M_V} \frac{\omega(p)}{\det \partial_\mu V^\nu(p)}\,. \tag{2.13}$$

This is the Atiyah-Bott-Berline-Vergne localization formula [139, 140].

The formula can be generalized to the case of a non-isolated fixed point. We skip technical details. The final result reads

$$\int_M \omega = \int_{M_V} \frac{i^*\omega}{e(N_{M_V})}\,, \tag{2.14}$$

where $i^*\omega$ is the pullback of ω on the locus, and $e(N_{M_V})$ is the equivariant Euler class of the normal bundle of M_V in M. We can interpret the result in physical terms. Since the localization locus is continuous, we can split the "modes" in those along with the tangent bundle to the locus and in the normal ones. The integration of the normal modes is Gaussian and can be performed by using some additional technology. It yields to a 1-loop determinant denoted by $e(N_{M_V})$. The tangent modes behave as massless modes and are harder to integrate. For this reason, we are left with a finite-dimensional integral over them. It is possible to verify that the formula collapses to eq (2.13) if M_V is made by discrete points. Indeed, the integral over M_V becomes a sum, $i^*\omega$ yields $\omega(p)$, where $p \in M_V$, and $e(N_{M_V})$ reduces to the determinant of the infinitesimal action of V around the fixed point.

For a more detailed discussion, we refer to [143]. We stress that the fact that an integral simplifies to a lower-dimensional one will be the typical situation in QFT. All the formulas we will derive are infinite-dimensional versions of (2.13).

2.1.2 Localization in QFT

We extend the localization argument to the QFT case. The main novelty is that now we have to deal with infinite dimensional spaces whose coordinates are the quantum fields. The standard integral becomes a path integral

$$Z = \int D\Phi\, e^{-S[\Phi]}\,. \tag{2.15}$$

where Φ is a collective notation for all the fields. We assume the existence of a supersymmetry transformation δ such that $\delta S = 0$. We require that δ closes on a compact bosonic symmetry $U(1)$. We also assume that the fields behave nicely enough at the boundary in field space and that there is no supersymmetry anomaly. These hypotheses ensure the vanishing of the expectation value of any δ-exact observable

$$\langle \delta O[\Phi]\rangle = \int D\Phi\, \delta\left(O[\Phi]e^{-S[\Phi]}\right) \equiv 0\,. \tag{2.16}$$

Therefore, observables differing by δ-exact terms lead to the same result. Given this, we consider the auxiliary quantity $Z(t)$, defined as

$$Z(t) = \int D\Phi\, e^{-S[\Phi]-t\delta W[\Phi]}\,. \tag{2.17}$$

Assuming the invariance of the functional W under the action of δ^2, we get

$$\frac{dZ(t)}{dt} = \int D\Phi\, \delta\left(W[\Phi]e^{-S[\Phi]-t\delta W[\Phi]}\right) = 0\,. \tag{2.18}$$

Then, $Z(t)$ is independent of t, and for $t = 0$ it coincides with Z. We exploit the t-independence to compute Z for a very large value of t. In such a limit, the saddle points of the functional δW control the path integral. When $t \to \infty$ the approximation is exact. This is nothing but the infinite-dimensional analog of eq (2.8).

To perform computation, we select an explicit form of W. There is a standard choice given, by

$$W = \int d^d x \sum_{\{a\}} \psi_{\{a\}} (\delta \psi_{\{a\}})^\dagger \,, \tag{2.19}$$

where ψ indicates a generic fermion and the sum over the index $\{a\}$ indicates that we must sum over all possible fermionic fields contracted with the corresponding δ variation. In the Euclidean signature, there is no natural reality condition for the fields. By conjugation, we mean a choice of a contour in the space of the complexified fields. We always assume the existence of a contour which makes the path integral convergent. In this way, the bosonic part of δW is semi-positive definite

$$\delta W\Big|_{\text{bos}} = \int d^d x \sum_{\{a\}} \left|\delta \psi_{\{a\}}\right|^2 \geq 0 \,. \tag{2.20}$$

For large t the path integral is dominated by the configuration $\delta\psi_{\{a\}} = 0$. The space of fields satisfying this condition is called *localization locus* or *moduli space*. We collectively denote this space as Φ_0. If Φ_0 is a finite-dimensional configuration, the path integral is reduced to a *matrix model*, i.e. a finite-dimensional integral over a matrix space. This is often the case. There are also examples where the moduli space is infinite dimensional and the path integral localizes to a lower-dimensional QFT.

We perform the computation by expanding all the fields around the locus

$$\Phi = \Phi_0 + \frac{1}{\sqrt{t}} \delta\Phi \,. \tag{2.21}$$

For $t \to \infty$, only the contribution from Gaussian fluctuations is not suppressed. This is conceptually similar to a semiclassical analysis. The final result reads

$$Z = \int D\Phi_0 \, e^{-S[\Phi_0]} \, \mathrm{Sdet} \, \frac{\delta^2 W}{\delta^2 \Phi} \,. \tag{2.22}$$

The term $e^{-S[\Phi_0]}$ is the contribution of the classical action evaluated on the locus, and it is called the classical contribution. The result includes the super determinant arising from the Gaussian integral over the fluctuations, which is just the 1-loop contribution in the semiclassical analysis. It is straightforward to extend this argument to all the δ-closed operators $O[\Phi]$. The result must include an additional classical contribution $O[\Phi_0]$ in the final formula.

We conclude with some final remarks. The localization argument requires the supersymmetry δ to be off-shell realized. If this is not the case, the t derivative of $Z(t)$ could contain a contact term proportional to the equation of motion, which spoils the t independence. It might also happen that the same theory admits two or more localizing supercharges.

In principle, the localization result depends on the choice of the localizing supercharge. However, two results obtained from different supercharge must agree. A final comment concerns a technical but crucial point. If the QFT is in flat space, the path integral will be IR divergent. A solution is to place the theory on a compact space, like the d dimensional sphere S^d, preserving supersymmetry. In the next section, we will deal systematically with this issue.

2.2 Supersymmetric theory on curved manifolds

In this section, we introduce supersymmetric QFTs on curved manifolds, following [144–146]. Given a QFT in flat space, It is always possible to couple the theory to gravity, to fix a background metric $g_{\mu\nu}$ and then decouple gravity by taking the rigid limit $G_N \to 0$ with $g_{\mu\nu}$ constant. The resulting QFT lives on the background defined by the metric. However, this procedure breaks supersymmetry: the coupling $g_{\mu\nu}T^{\mu\nu}$ is not supersymmetric.

To make the technical problem clearer, let us step back and perform a SUSY variation on a SUSY QFT on $\mathbb{R}^d$ with action S.

$$\delta S = \int d^d x J^\mu \partial_\mu \varepsilon \tag{2.23}$$

where J^μ is the supersymmetry current, and ε is the infinitesimal parameter. Then, if ε is constant $\delta S = 0$. One can naively generalize the condition to an arbitrary manifold M by taking the minimal coupling to a background metric. The condition becomes $\nabla_\mu \varepsilon = 0$. If this equation was the end of the story, only a few manifolds could support supersymmetric theories. A remarkable example is the Witten index, which corresponds to the partition function on the d-dimensional torus [147]. A more general procedure is the *topological twisting* [18]. Under some hypotheses, it allows us to define supersymmetric theories on a large class of manifolds. The theory obtained in this way is topological, as the stress tensor is δ-exact.

A practical way to study examples beyond those already mentioned is to expand in powers of the curvature coupling. Supposing that the manifold posseses a characteristic curvature scale r, for instance, the radius for a sphere S^d, we can expand in inverse power of r both the supersymmetry transformations and the Lagrangian around the minimally coupled theory

$$\delta = \delta^{(0)}\Big|_{\eta\to g,\partial\to\nabla} + \sum_{n\geq 1} \frac{1}{r^n}\delta^{(n)}\,, \quad \mathcal{L} = \mathcal{L}^{(0)}\Big|_{\eta\to g,\partial\to\nabla} + \sum_{n\geq 1} \frac{1}{r^n}\mathcal{L}^{(n)} \tag{2.24}$$

where $\delta^{(0)}$ and $\mathcal{L}^{(0)}$ denotes respectively the minimally coupled variations and Lagrangian. Imposing $\delta\mathcal{L} = \nabla_\mu(\dots)$ and the closure for δ as a function of r ensures that the Lagrangian $\mathcal{L}$ defines a SUSY theory on the curved manifold M. Even if the procedure is lengthy and there is no guarantee of success, it leads to a definite answer. Indeed, since r is dimensionful, there is only a finite number of relevant corrections to consider. Therefore, either we get a consistent supersymmetric theory, or we do not.

An alternative approach was developed in [144]. It involves the coupling to supergravity (SUGRA). Building on the intuition of the rigid limit of gravity, we make the procedure supersymmetric considering the whole stress tensor multiplet. In supersymmetric theories, the stress tensor $T_{\mu\nu}$ sits in a supermultiplet, containing additional bosonic and fermionic fields. Similarly, the metric is a part of a multiplet, which includes its spin 3/2 superpartner $\psi_{\mu\alpha}$, and additional bosonic and fermionic fields. Coupling to SUGRA means to add an interaction term among these two multiplets

$$\Delta\mathcal{L} = T^{\mu\nu} g_{\mu\nu} + S^{\mu\alpha}\psi_{\mu\alpha} + \dots \tag{2.25}$$

where $S^{\mu\alpha}$ is the supercurrent associated with the supercharge Q_α and the dots include all the linear couplings of the additional fields in the multiplet and eventually higher-order terms. We can make the supergravity dynamical by adding the proper supersymmetrized Hilbert-Einstein action.

A crucial insight of [144] is to use off-shell supergravity. We take the rigid limit $G_N \to 0$ with fixed $g_{\mu\nu}$ of the SUGRA theory without integrating out the auxiliary fields. Rigid supersymmetry is imposed by requiring that all the fermions and their variations in the background multiplet vanish. In particular, the variation of the gravitino looks like $\delta\psi_{\mu\alpha} = \nabla_\mu\varepsilon + \dots$, where the dots include other auxiliary bosonic fields. The equations$\delta\psi_{\mu\alpha} = 0$ are differential equations which generalize the conditions $\nabla_\mu\varepsilon = 0$. We will call them generalized Killing spinor equations, and the solutions (generalized) Killing spinors. We think of it as an equation for the spinor ε and the background auxiliary fields. Then, the theory on the curved space is defined both by the value of the background metric and by the bosonic auxiliary fields. Once we determine a solution for the generalized Killing spinor equation, the theory and its SUSY transformations are those of the SUGRA theory evaluated on such a rigid background.

The approach based on off-shell SUGRA has the advantage of being independent of the details of the theory. The $1/r$ corrections in the Lagrangian and the supersymmetry variations are interpreted as supergravity background fields. In the following, we will concentrate on two explicit supersymmetric backgrounds: the round sphere S^3, and the squashed sphere. The latter is a manifold with the sphere topology but with a deformed metric with fewer isometries.

2.2.1 Supersymmetry on S^3

We construct 3d $\mathcal{N} = 2$ theories on S^3, following [148]. We build the background performing an explicit expansion in $\frac{1}{r}$ around flat space. We start with the vector multiplet. In practice, we deform the SUSY variations of Eq (1.17) by terms proportional $\nabla_\mu\varepsilon$ and $\nabla_\mu\bar{\varepsilon}$. Because of dimensional analysis, we realize that both δA_μ and $\delta\sigma$ cannot change. For $\delta\lambda$ and $\delta\bar{\lambda}$ we also require that they depend respectively only on ε and $\bar{\varepsilon}$, in analogy with the flat space supersymmetry. Then, we can deform them by

$$\sigma\gamma^\mu\nabla_\mu\varepsilon \subset \delta\lambda\,, \quad \sigma\gamma^\mu\nabla_\mu\bar{\varepsilon} \subset \delta\bar{\lambda}\,. \tag{2.26}$$

Similarly, we admit the terms $\lambda\gamma\nabla_\mu\epsilon$ and $\bar{\lambda}\gamma\nabla_\mu\bar{\epsilon}$ in δD. All these terms are defined up to an overall coefficient. We fix these parameters requiring the closure of the supersymmetry algebra. However, the term $\delta^2 D$ contains an extra term proportional to

$$\sigma\left(\varepsilon\gamma^\rho\gamma^\mu\nabla_\rho\nabla_\mu\bar{\varepsilon} - \bar{\varepsilon}\gamma^\rho\gamma^\mu\nabla_\rho\nabla_\mu\varepsilon\right) . \tag{2.27}$$

It can be set to zero requiring that ε and $\bar{\varepsilon}$ are Killing spinors of S^3. That is, they satisfy

$$\nabla_\mu\varepsilon = \pm\frac{i}{2r}\gamma_\mu\varepsilon\,, \quad \nabla_\mu\bar{\varepsilon} = \pm\frac{i}{2r}\gamma_\mu\bar{\varepsilon}\,. \tag{2.28}$$

We conventionally choose the equation with the plus sign, which has two constant solutions in the left invariant frame. At the end of the day, we get the following supersymmetry variations

$$\delta A_\mu = -i\left(\varepsilon\gamma^\mu\bar{\lambda} + \bar{\varepsilon}\gamma^\mu\lambda\right) , \tag{2.29}$$

$$\delta\sigma = -\varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda\,, \tag{2.30}$$

$$\delta\lambda = \left[i\left(D - i\frac{\sigma}{r}\right) - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} - i\gamma^\mu D_\mu\sigma\right]\varepsilon\,, \tag{2.31}$$

$$\delta\bar{\lambda} = \left[-i\left(D - i\frac{\sigma}{r}\right) - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} + i\gamma^\mu D_\mu\sigma\right]\bar{\varepsilon}\,, \tag{2.32}$$

$$\delta D = \varepsilon\gamma^\mu D_\mu\bar{\lambda} - \bar{\varepsilon}\gamma^\mu D_\mu\lambda - [\varepsilon\bar{\lambda}, \sigma] - [\bar{\varepsilon}\lambda, \sigma] . \tag{2.33}$$

They close the supersymmetry algebra $\mathfrak{osp}(2|2)_\ell \oplus \mathfrak{su}(2)_r$, where the subscripts ℓ and r refer to the two isometry group of S^3 $SU(2)_\ell \times SU(2)_r$. We observe that the algebra is not conformal.

We can write two actions invariant under these transformations. The first one is the Yang-Mills Lagrangian

$$\mathcal{L}_{\text{SYM}} = \text{Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}D^\mu\sigma D_\mu\sigma + \frac{1}{2}\left(D + \frac{\sigma}{r}\right)^2 - i\tilde{\lambda}\gamma^\mu D_\mu\lambda + i\tilde{\lambda}[\sigma,\lambda] - \frac{1}{2r}\bar{\lambda}\lambda\right) . \tag{2.34}$$

The second is the Chern-Simons term

$$\mathcal{L}_{\text{CS}} = \frac{i}{4\pi}\,\text{Tr}_{\text{CS}}\left[\epsilon^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda\right) + 2iD\sigma + \bar{\lambda}\lambda\right] . \tag{2.35}$$

While the former action breaks scale invariance, the latter is conformally invariant. Indeed, one can check that the action is invariant under the full superconformal algebra, provided that we include the solutions to (2.28) with the opposite signs [2]. With these two additional Killing spinors, the transformations (2.29) close the $\mathcal{N} = 2$ superconformal algebra $\mathfrak{osp}(2|4)$.

We can make analogous considerations for the chiral fields. We consider a chiral multi-

[2]These two Killing spinors are constant in the right invariant frame. It is possible to recast the S^3 Killing spinors in conformal Killing spinors. See also Appendix C for more details.

plet with R-charge Δ. The supersymmetry transformations are [23, 24]

$$\begin{aligned}
\delta\phi &= \sqrt{2}\varepsilon\psi \\
\delta\psi &= \sqrt{2}\varepsilon F - \sqrt{2}i\gamma^\mu\bar{\varepsilon}\, D_\mu\phi + 2i\sigma\phi\bar{\varepsilon} - \sqrt{2}\frac{\Delta}{r}\bar{\varepsilon}\phi \\
\delta F &= -\sqrt{2}i\bar{\varepsilon}\gamma^\mu D\mu\psi - \sqrt{2}i\sigma\bar{\epsilon}\psi + 2i\bar{\varepsilon}\bar{\lambda}\phi - \frac{\sqrt{2}}{r}\left(\Delta - \frac{1}{2}\right)\bar{\varepsilon}\psi\,.
\end{aligned} \tag{2.36}$$

Provided that eq (2.28) holds, the transformations closes the $\mathfrak{osp}(2|2)_\ell \oplus \mathfrak{su}(2)_r$

$$\delta_\varepsilon^2 = 0\,, \quad \delta_{\bar{\varepsilon}}^2 = 0\,, \quad \{\delta_\varepsilon, \delta_{\bar{\varepsilon}}\} = -v^\mu D_\mu + \frac{1}{r}\Delta_f\,. \tag{2.37}$$

where $v^\mu = \bar{\varepsilon}\gamma^\mu\varepsilon$ generates the infinitesimal $SU(2)_\ell$ isometry, and Δ_f indicates the R-charge of the field (Δ for ϕ, $\Delta - \frac{1}{2}$ for ψ, $\Delta - 1$ for F). The invariant Lagrangian is

$$\begin{aligned}
\mathcal{L} = D^\mu\bar{\phi}D_\mu\phi + \bar{\phi}\left(\sigma^2 + \frac{i(2\Delta-1)}{r}\sigma + \frac{\Delta(\Delta-2)}{r^2}\right)\phi - i\bar{\psi}\gamma^\mu D_\mu\psi + \\
- i\bar{\psi}\left[\sigma + \frac{i}{r}\left(\Delta - \frac{1}{2}\right)\right]\psi + i\sqrt{2}\left(\bar{\phi}\lambda\psi + \bar{\psi}\bar{\lambda}\phi\right) - \bar{F}F
\end{aligned} \tag{2.38}$$

For $\Delta = \frac{1}{2}$ the action enjoys the larger superconformal invariance. In this case, the action can be obtained from the flat space action by performing a Weyl transformation. The extra terms in the action are just conformal masses.

Finally, as in flat space (see Section 2.1), one can add real masses and FI terms. The real mass is obtained by turning on a $U(1)_{\text{fl}}$ background gauge field with $\sigma = irD \equiv \frac{m}{r}$ and all other components set to zero. In this way, the variations of the background gauginos vanish and supersymmetry is preserved. The real mass term modifies the action for a chiral, with $U(1)_{\text{fl}}$ charge q by a term

$$\mathcal{L}_{\text{mass}} = \frac{q^2m^2 + 2i\,(\Delta-1)\,qm}{r^2}\bar{\phi}\phi - i\frac{qm}{r}\bar{\psi}\psi\,. \tag{2.39}$$

Similarly, the FI-term is

$$\mathcal{L}_{\text{FI}} = \zeta\frac{1}{2\pi r}\left(-D + i\frac{\sigma}{r}\right)\,. \tag{2.40}$$

As in flat space, this is the rigid limit of the BF-coupling with the rigid background $\sigma = irD \equiv \frac{\zeta}{r}$. We stress that these deformations modify the supersymmetry algebra discussed for the flat space.

2.2.2 Supersymmetry on general backgrounds

We will focus on theory admitting an $U(1)_R$ symmetry. In this case, the stress tensor sits in the $\mathcal{R}$ multiplet, whose bottom component is the R-symmetry current $j_\mu^{(R)}$ [145, 149]. It contains also the supersymmetry currents $S_{\mu\alpha}$ and $\bar{S}_{\mu\alpha}$, the central charge current $j_\mu(Z)$ and a conserved string current $i\epsilon_{\mu\nu\rho}\partial^\rho J^{(Z)}$ (and the stress tensor itself). The multiplet couples to the new minimal supergravity multiplet $\mathcal{G} = (g_{\mu\nu},\, A_\mu^{(R)}\,, B_{\mu\nu}\,, C_\mu\,, \psi_{\mu\alpha},\, \bar{\psi}_{\mu\alpha})$, where $g_{\mu\nu}$ is the metric, $A_\mu^{(R)}$ and C_μ are gauge fields, and $B_{\mu\nu}$ is a 2-form. We will often consider the Hodge dual of the field strength of C_μ and $B_{\mu\nu}$, respectively $V^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu C_\rho$ and $H = \frac{i}{2}\epsilon^{\mu\nu\rho}\partial_\mu B_{\nu\rho}$. The linearized coupling is

$$\Delta\mathcal{L} = T^{\mu\nu}g_{\mu\nu} + \frac{1}{2}S^{\mu\alpha}\psi_{\mu\alpha} + \frac{1}{2}\bar{S}^{\mu\alpha}\bar{\psi}_{\mu\alpha} + j^{(R)}_\mu\left(A^{(R)\mu} - \frac{3}{2}V^\mu\right) + j^{(Z)}_\mu C_\mu + J^{(Z)}H\,. \quad (2.41)$$

Building on the properties of the multiplet and imposing $\delta\psi_{\mu\alpha} = 0$ and $\delta\bar{\psi}_{\mu\alpha} = 0$, the generalized Killing spinor equations are:

$$\left(\nabla_\mu - iA^{(R)}_\mu\right)\varepsilon = -\frac{1}{2}H\gamma_\mu\varepsilon - iV_\mu\varepsilon - \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\varepsilon \quad (2.42)$$

$$\left(\nabla_\mu + iA^{(R)}_\mu\right)\bar{\varepsilon} = -\frac{1}{2}H\gamma_\mu\bar{\varepsilon} + iV_\mu\bar{\varepsilon} + \frac{1}{2}\epsilon_{\mu\nu\rho}V^\nu\gamma^\rho\bar{\varepsilon} \quad (2.43)$$

We describe the case where both equations have one solution ε and $\bar{\varepsilon}$, with opposite R-charge 1 and -1, respectively. Moreover, the existence of a solution depends closely on the property of the manifolds M [145]. In this case, it is equivalent to the existence of an almost contact metric structure (known also as a transversally holomorphic fibration), which is an odd-dimensional analog of a complex structure [150]. Conversely, if a manifold admits a transversally holomorphic fibration and a real Killing vector K^μ, a solution to (2.42) exists.

Let us describe the solution by choosing a system of adapted coordinates $(\psi, z, \bar{z})$ where $K = \partial_\psi$ and with metric

$$ds^2 = \Omega(z,\bar{z})^2\left(d\psi + a\right)^2 + c(z,\bar{z})^2 dz d\bar{z}\,, \quad (2.44)$$

with $a = a_z(z,\bar{z})dz + a_{\bar{z}}(z,\bar{z})d\bar{z}$. For our purposes, it is enough to consider the case $\Omega(z,\bar{z}) = 1$, which implies $K^\mu K_\mu = 1$. Then, one can check that $\zeta = (1,0)$ and $\bar{\zeta} = (0,1)$ solve the equations for the background fields

$$H = iF_a\,, \quad V^\mu = 0\,, \quad A^{(R)} = F_a e_3 + \frac{1}{2}\omega^{(2d)}_{12}\,, \quad (2.45)$$

where $F_a = 2i\left(\partial_{\bar{z}}a_z - \partial_z a_{\bar{z}}\right)$ and $\omega^{(2d)}_{12}$ is the spin connection of the 2d metric $c(z,\bar{z})^2 dz d\bar{z}$. We stress that K^μ does not need to come from a single isometry. We will consider a case where K^μ comes as a linear combination of two commuting Killing vectors with incommensurate coefficients. In this case, the orbits of K^μ are not closed.

The squashed sphhere

The explicit background we build is the squashed sphere. Other interesting manifolds we do not discuss in the thesis are those related to the superconformal index [151, 152], the topologically twisted index [153], and the Lens space [154–157]. The background we discuss was introduced in [158]. Related configurations are examined in [159–165]. It is a deformation of the round sphere which breaks the isometry group to $U(1)\times U(1)$. It can be understood as an ellipsoid geometry, arising from the manifold defined by the embedding in $\mathbb{R}^4$

$$\frac{{x_1}^2+{x_2}^2}{\ell^2}+\frac{{x_3}^2+{x_4}^2}{\tilde{\ell}^2}=1\,. \tag{2.46}$$

For $\ell=\tilde{\ell}=r$ we recover the standard sphere. More generally, we will call 'squashed sphere" the manifold with the S^3 topology and metric

$$ds^2=f(\theta)^2d\theta^2+\ell^2\sin^2\theta d\varphi^2+\tilde{\ell}^2\cos^2\theta d\tau^2\,. \tag{2.47}$$

The function $f(\theta)$ must be nowhere-vanishing and satisfy $f(0)=\ell$, $f(\pi/2)=\tilde{\ell}$ to avoid conical singularities. For $f=\sqrt{\tilde{\ell}^2\sin^2\theta+\ell^2\cos^2\theta}$, the squashed sphere coincides with the ellipsoid.

The metric has two isometries ∂_φ and ∂_τ. The Killing vector with the unitary norm is

$$K=\frac{1}{\ell}\partial_\varphi+\frac{1}{\tilde{\ell}}\partial_\tau\,. \tag{2.48}$$

We can find a system of local coordinates for the metric

$$x=\int_{\theta_0}^{\theta}dt\frac{f(t)}{\cos t\sin t}\,,\quad y=-\tilde{\ell}\tau+\ell\varphi\,,\quad \psi=\tilde{\ell}\tau\cos^2\theta+\tilde{\ell}\varphi\sin^2\theta \tag{2.49}$$

where $K=\partial_\psi$ and the metric is in the form of Eq (2.44). We identify

$$c=\sin\theta\cos\theta\,,\qquad a=2\left(\tilde{\ell}\tau-\ell\varphi\right)\sin\theta\cos\theta d\theta\,. \tag{2.50}$$

As anticipated, the orbits of K are in general not closed unless $b=$ is rational[3]. Given this geometry, we can use the solution to determine the other background fields

$$H=\frac{i}{f}\,,\quad V^\mu=0\,,\quad A^{(R)}=\frac{1}{2}\left(1-\frac{\ell}{f}\right)d\varphi+\frac{1}{2}\left(1-\frac{\tilde{\ell}}{f}\right)d\tau\,, \tag{2.51}$$

The Killing spinors are

$$\varepsilon=e^{i\frac{\varphi+\tau}{2}}\begin{pmatrix}1\\0\end{pmatrix}\qquad\bar{\varepsilon}=e^{-i\frac{\varphi+\tau}{2}}\begin{pmatrix}1\\0\end{pmatrix} \tag{2.52}$$

We can write the relevant supersymmetric Lagrangian on the squashed sphere. For the vector multiplet, we have the usual $\mathcal{N}=2$ Chern-Simons term. The Yang-Mills Lagrangian gets modified as follows

$$\mathcal{L}_{\rm SYM}={\rm Tr}\left(\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}D^\mu\sigma D_\mu\sigma-\frac{1}{2}\left(D+H\sigma\right)^2-i\tilde{\lambda}\gamma^\mu D_\mu\lambda+i\tilde{\lambda}[\sigma,\lambda]\,+\frac{i}{2}H\bar{\lambda}\lambda\right)\,. \tag{2.53}$$

where $D_\mu=\nabla_\mu-iA_\mu-i\Delta A_\mu^{(R)}$. The SUSY variations which leave the actions invariant

[3]Two special orbits are at $\theta=0$ and $\theta=\pi/2$. They are both closed circles regardless of b..

are

$$
\begin{aligned}
\delta A_\mu &= -i\left(\varepsilon\gamma^\mu\bar{\lambda} + \bar{\varepsilon}\gamma^\mu\lambda\right)\,, \\
\delta\sigma &= -\varepsilon\bar{\lambda} + \bar{\varepsilon}\lambda\,, \\
\delta\lambda &= \left(i(D+H\sigma) - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} - i\gamma^\mu D_\mu\sigma\right)\varepsilon\,, \\
\delta\bar{\lambda} &= \left(-i(D+H\sigma) - \frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho F_{\mu\nu} + i\gamma^\mu D_\mu\sigma\right)\bar{\varepsilon}\,, \\
\delta D &= D_\mu\left(\varepsilon\gamma^\mu\lambda - \bar{\varepsilon}\gamma^\mu\lambda\right) - [\varepsilon\bar{\lambda},\sigma] - [\bar{\varepsilon}\lambda,\sigma] - H(\varepsilon\bar{\lambda} - \bar{\varepsilon}\lambda)\,.
\end{aligned}
\tag{2.54}
$$

For a chiral multiplet with R-charge Δ we have

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\varepsilon\psi \\
\delta\psi &= \sqrt{2}\varepsilon F - \sqrt{2}i\gamma^\mu\bar{\varepsilon}\,D_\mu\phi + 2i\sigma\phi\bar{\varepsilon} + i\sqrt{2}\Delta H\bar{\varepsilon}\phi \\
\delta F &= -\sqrt{2}iD_\mu\left(\bar{\varepsilon}\gamma^\mu\psi\right) - \sqrt{2}i\sigma\bar{\epsilon}\psi + 2i\bar{\varepsilon}\bar{\lambda}\phi - i\sqrt{2}\left(\Delta-2\right)H\bar{\varepsilon}\psi\,.
\end{aligned}
\tag{2.55}
$$

Provided that generalized Killing spinor equations hold, these transformations close the algebra $\mathfrak{su}(1|1)$

$$
\delta_\varepsilon^2 = 0\,,\quad \delta_{\bar{\varepsilon}}^2 = 0\,,\quad \{\delta_\varepsilon,\delta_{\bar{\varepsilon}}\} = -2i(\mathcal{L}'_K + \varepsilon\bar{\varepsilon}\,\Delta H)\,,
\tag{2.56}
$$

where $K^\mu = \bar{\varepsilon}\gamma^\mu\varepsilon$, and Δ is the R-charge and $\mathcal{L}'_K$ is a modified Lie derivative

$$
\mathcal{L}'_K = \mathcal{L}_K - i\Delta K^\mu A_\mu^{(R)}\,,
\tag{2.57}
$$

with $\mathcal{L}_K$ the standard Lie derivative. The supersymmetric Lagrangian is

$$
\begin{aligned}
\mathcal{L} = D^\mu\bar{\phi}D_\mu\phi + \bar{\phi}\left(\sigma^2 + \frac{\Delta}{4}\hat{R} + \Delta\left(\Delta-\frac{1}{2}\right)H^2 + 2H\left(\Delta-\frac{1}{2}\right)\sigma + D\right)\phi + \\
- i\bar{\psi}\gamma^\mu D_\mu\psi - i\bar{\psi}\left[\sigma + H\left(\Delta-\frac{1}{2}\right)\right]\psi + i\sqrt{2}\left(\bar{\phi}\lambda\psi + \bar{\psi}\bar{\lambda}\phi\right) - \bar{F}F\,,
\end{aligned}
\tag{2.58}
$$

where $\hat{R}$ is the Ricci scalar of the metric. One can also add FI-terms and real masses, choosing suitable background gauge fields like for the round sphere.

2.3 Localization on S^3

We are in business to apply localization to 3d gauge theories on S^3 [22–24]. We choose the localizing supercharge to be $\varepsilon = (1,0)$ and $\bar{\varepsilon} = (0,1)$, denoted by $\delta = \delta_\varepsilon + \delta_{\bar{\varepsilon}}$. It turns out that the SYM Lagrangian and the matter Lagrangian are both δ-exact

$$
\delta_\varepsilon\delta_{\bar{\varepsilon}}\left(\frac{\bar{\lambda}\lambda}{2} + i\sigma D\right) = \bar{\varepsilon}\varepsilon\mathcal{L}_{\text{SYM}}\,,\qquad \delta_\varepsilon\delta_{\bar{\varepsilon}}\left(\frac{\bar{\psi}\psi}{2} + i\bar{\phi}\sigma\phi - \frac{\Delta-1}{r}\right).= \bar{\varepsilon}\varepsilon\mathcal{L}_{\text{mat}}
\tag{2.59}
$$

Then, they will both serve as localizing terms.

We start the discussion from the vector multiplet. To compute the path integral, we need to gauge fix it. Then, we add the ghosts c, $\bar{c}$, and b, with the action

$$S_{\text{ghost}} = \int d^3x\sqrt{g}\,\text{Tr}(D^\mu\bar{c}D^\mu c + b\nabla^\mu A_\mu)\,, \tag{2.60}$$

which imposes the gauge condition $\nabla_\mu A^\mu = 0$. To apply localization, the localizing term has to be exact w.r.t. the fermionic symmetry $\delta' = \delta + \delta_{\text{BRST}}$, where δ_{BRST} is generates an infinitesimal BRST transformation. The correct prescription is to deform the path integral by a term $\delta'(\bar{c}\nabla^\mu A_\mu)$. It is not hard to check that it reproduces the localizing term supplemented by the ghost action plus a term $\bar{c}\nabla^\mu\delta A_\mu$. The latter does not affect either the localization locus nor the 1-loop determinants. In conclusion, we can use S_{SYM} as localizing term, but we must include the 1-loop contribution of the ghost action. With a small abuse of notation, we will write δ instead of δ'.

We can examine the localization locus. Since the bosonic part of the SYM action is a sum of quadratic terms, the minima of the action are given by

$$F_{\mu\nu} = 0 \qquad D_\mu\sigma = 0\,, \qquad D + i\frac{\sigma}{r} = 0\,. \tag{2.61}$$

Since S^3 is simply connected, $F_{\mu\nu} = 0$ implies $A_\mu = 0$. The other equations are solved by $\sigma = \frac{\sigma_0}{r}$ and $D = -i\frac{\sigma_0}{r^2}$. Similarly, one can see that there are no fermionic zero modes. Since the Yang-Mills action is δ-exact, the only classical contribution comes from the Chern-Simons term. It reads

$$e^{-S_{\text{CS}}}\Big|_{\text{locus}} = e^{-\,\text{Tr}_{\text{CS}}\,\sigma_0^2}\,. \tag{2.62}$$

Then we move to the fluctuations. We expand the locus by performing the following change of variable in the path integral

$$\sigma = \frac{\sigma_0}{r} + \frac{\sigma'}{\sqrt{t}}\,, \qquad D = -i\frac{\sigma_0}{r^2} + \frac{D'}{\sqrt{t}}\,, \qquad \Phi = \frac{\Phi'}{\sqrt{t}}\,, \tag{2.63}$$

where Φ denotes all the fields except for σ and D. Expanding up to the quadratic order, the localizing term reads

$$\begin{aligned} tS_{\text{SYM}} = \int d^3x\sqrt{g}\bigg(&\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu} + \frac{1}{2}\partial^\mu\sigma'\partial_\mu\sigma' - \frac{1}{2r}[A'_\mu,\sigma_0]^2 - \frac{1}{2}\left(D' + \frac{i}{r} - \sigma'\right)^2 \quad + \\ &- i\bar{\lambda}'\gamma^\mu\nabla_\mu\lambda' - \frac{1}{2r}\bar{\lambda}'\lambda' + \frac{i}{r}\bar{\lambda}'[\sigma_0,\bar{\lambda}']\bigg)\,, \end{aligned} \tag{2.64}$$

where $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ is the abelianized field strength. We decompose the gauge field as $A'_\mu = B_\mu + \partial_\mu\beta$, where B_μ is divergenceless, i.e. $\nabla^\mu B_\mu = 0$. Since this term is quadratic, we compute the 1-loop determinant. Integrating out b' yields a delta functional of $\delta(\Box\beta) = \det^{-\frac{1}{2}}(\Box)\delta(\beta)$, where we define $\Box = \nabla^\mu\partial_\mu$ acting on zero-form. This factor cancels against the determinants coming from the integrations over σ' and the ghosts.

Then, we exploit the residual gauge invariance to fix σ_0 in the Cartan subalgebra $\mathfrak{t}$ of the gauge group. The change of variable causes the presence of an additional Vandermonde determinant $\Delta(\sigma_0) = \prod_{\alpha>0}(\pi\alpha(\sigma_0))^2$, where α are the roots of the gauge Lie algebra $\mathfrak{g}$, and the product is over the positive roots. We expand the fields on a Cartan basis of $\mathfrak{g}$. For instance

$$B_\mu = h_\mu + X^\alpha_\mu E_\alpha\,. \tag{2.65}$$

where h_μ is an element of the Cartan $\mathfrak{t}$ and E_α is a root vector[4]. Plugging the decomposition into the action, we find a bosonic contribution

$$\sum_\alpha \int d^3x\sqrt{g}\left[\frac{1}{2}B^\mu_{-\alpha}\left(-\nabla^2+\frac{1}{r^2}\alpha(\sigma_0)^2\right)B_{\mu\alpha}+\lambda'_{-\alpha}\left(-i\gamma^\mu\nabla_\mu+\frac{i}{r}\alpha(\sigma_0)-\frac{1}{2r}\right)\lambda'_\alpha\right], \tag{2.66}$$

where we included only the terms which couples to σ_0. The eigenvalues of the vector Laplacian ∇^2 acting on divergenceless vector fields are $(j+2)^2$ with $j=0,1,\dots$ and degeneracy $2(j+1)(j+3)$. Those of $i\gamma^\mu\nabla_\mu$ are $\pm\left(j+\frac{3}{2}\right)$ both with degeneracies $(j+1)(j+2)$. After some manipulations, the majority of the eigenvalues cancels and the 1-loop determinant is

$$Z_{\text{vec}}(\sigma_0) = \prod_\alpha \prod_{j=1}^{+\infty}\frac{(j+i\alpha(\sigma_0))^{j+1}}{(j+i\alpha(\sigma_0))^{j-1}}\,. \tag{2.67}$$

The product is divergent. After the zeta-function regularization, we get

$$Z_{\text{vec}}(\sigma_0) = \prod_\alpha \frac{2\sinh\pi\alpha(\sigma_0)}{\pi\alpha(\sigma_0)}\,. \tag{2.68}$$

The denominator is cancelled against the Vandermonde determinant. From now on, we will include such a term in $Z_{\text{vec}}(\sigma_0)=\prod_\alpha 2\sinh\pi\alpha(\sigma_0)$.

We turn to the localization of the chiral multiplet. For simplicity, we work on the locus of the gauge multiplet. Then, the localizing term reduces to:

$$\begin{aligned}\mathcal{L}_{\text{mat}} =&\partial^\mu\bar\phi\partial_\mu\phi+\frac{\bar\phi\left(\sigma_0^2+2i(\Delta-1)\sigma_0+\Delta(2-\Delta)\right)\phi}{r^2}-i\bar\psi\gamma^\mu\nabla_\mu\psi+\\ &-\frac{i\bar\psi\left(\sigma_0+i\left(\Delta-\frac{1}{2}\right)\right)\psi}{r}-\bar F F\,.\end{aligned} \tag{2.69}$$

The localization locus is $\phi=0$, $\psi=0$, and $F=0$. Since the matter action is δ-exact, there is no classical contribution. To calculate the 1-loop determinant, we expand the modes on the weight space of the representation R. Recalling that the eigenvalues of the Laplacian $\Box$ on the scalars are $j(j+2)$, $j=0,1,\dots$ and with degeneracy $2(j+1)^2$, we find after some manipulations

$$Z_{\text{chi}}(\sigma_0) = \prod_{\rho\in R}\prod_{j=0}^{+\infty}\frac{(j+i\rho(\sigma_0)+2-\Delta)^{j+1}}{(j-i\rho(\sigma_0)+\Delta)^{j+1}}\,, \tag{2.70}$$

where the first product runs over the weights ρ of R. We can regularize the divergent product over j using the double sine function

$$s_b(x) = \prod_{m,n\geq 0}\frac{\left(m+\frac{1}{2}b\right)+\left(n+\frac{1}{2}b^{-1}\right)-ix}{\left(m+\frac{1}{2}b\right)+\left(n+\frac{1}{2}b^{-1}\right)+ix}\,. \tag{2.71}$$

[4] We normalize the root $\mathrm{Tr}(E_\alpha E_\beta)=\delta_{\alpha+\beta,0}$. We also recall that $[\sigma_0,E_\alpha]=\alpha(\sigma_0)$.

Then, the 1-loop determinant is given by[5]

$$Z_{\text{chi}}(\sigma_0) = \prod_{\rho \in R} s_{b=1}\left(i(1-\Delta) - \rho(\sigma_0)\right) . \tag{2.72}$$

Combining all the results, we write the partition function of an $\mathcal{N} = 2$ theory with gauge group G, coupled with an arbitrary number of chiral multiplets with R-charge Δ_i

$$Z = \frac{1}{|\mathcal{W}|} \int d\sigma_0 e^{-i\pi \operatorname{Tr}_{\text{CS}} \sigma_0^2} \prod_{\alpha} 2\sinh \pi\alpha(\sigma_0) \prod_{i} \prod_{\rho_i} s_{b=1}\left(i(1-\Delta_i) - \rho(\sigma_0)\right) , \tag{2.73}$$

where i counts the different chiral multiplets. Localization reduces the partition function to a matrix model. This form of the partition is convenient to explore different regimes of 3d gauge theories, including non-perturbative phenomena.

We also discuss the coupling to background flavor symmetries, i.e. the effect of real masses and FI-terms. The inclusion in our computation is straightforward if we treat the real mass m as a background gauge fields. We account for this in the computation of the matter 1-loop determinant by expanding the modes on a weight basis of the direct sum representation of of $G \times F$, where F is the flavor symmetry. That is, if ρ and ω form a weight basis of the representations of the gauge group and the flavor group respectively, we consider a weight basis (ρ, ω) of $G \times F$, whose action on $\sigma_0 + m$ is $\rho(\sigma_0) + \omega(m)$. The net effect is a shift in the 1-loop determinant, which becomes

$$Z_{\text{chi}}(\sigma_0) = \prod_{\rho, \omega} s_{b=1}\left(i(1-\Delta) - \rho(\sigma_0) - \omega(m)\right) . \tag{2.74}$$

The FI-parameter yields a classical contribution which alters the integrand function by a multiplicative factor

$$e^{-S_{\text{FI}}}\Big|_{\text{locus}} = e^{-2\pi i \zeta \operatorname{Tr} \sigma_0} . \tag{2.75}$$

We can consider also superpotential terms. The above result implies that it does not produce any classical contributions. However, its inclusion constraints both the R-charges of the fields and eventual flavor symmetries.

Finally, it is convenient to specialize the result for theory with $\mathcal{N} \geq 3$ SUSY. Here, the matter is organized in hyper multiplets, which can be seen as two chirals transforming in conjugate representations with R-charge 1/2. Using the identity

$$s_b\left(\frac{ib}{2} + x\right) s_b\left(\frac{ib}{2} - x\right) = \frac{1}{\cosh \pi b x} \tag{2.76}$$

[5] The product over can be equivalently regularized using the function

$$\ell(z) = -z\left(1 - e^{2\pi i z}\right) \log z + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\text{Li}_2(e^{2\pi i z})\right) - \frac{i\pi}{12} .$$

The result is $e^{\ell(1-\Delta+i\sigma_0)}$. See the Appendix A of [166] for more details.

The 1-loop determinant factor for the hyper becomes

$$Z_{\text{hyper}} = \prod_{\rho} \frac{1}{\cosh \pi\rho(\sigma_0)} \,. \tag{2.77}$$

Similarly, we compute the 1-loop determinant for the $\mathcal{N}=4$ vector. We need to calculate the 1-loop determinant of the adjoint chiral with $\Delta = 1$. It is not hard to see that its contribution is just 1. For instance, Φ has the right quantum numbers to build a δ-exact mass term for it [167]

$$t \int d^2\theta d^3x \Phi^2 \,. \tag{2.78}$$

Using this localizing term, we directly obtain the 1-loop determinant. Then, the result of the $\mathcal{N}=4$ vector is the same as the $\mathcal{N}=2$ vector.

We apply localization to compute observables in ABJM theory. We write down the matrix model for the $U(N_1)_k \times U(N_2)_{-k}$ model. From the $\mathcal{N}=2$ we have two vector multiplets. Since the root of $U(N)$ are $\alpha(\sigma_0) = \lambda_i - \lambda_j$ where $\sigma_0 = \text{diag}(\lambda_1, \dots, \lambda_N)$, for every gauge group factor we have a 1-loop determinant

$$Z_{\text{vec}} = \prod_{i\neq j}^{N_1} 2\sinh \pi(\lambda_i - \lambda_j) \prod_{i\neq j}^{N_2} 2\sinh \pi(\mu_i - \mu_j) \tag{2.79}$$

Since the matter fields can be organized in one hyper and one twisted hyper, transforming in the (anti-)bifundamental, the corresponding1-loop determinants are cosh thanks to the identity (2.76). The final result is

$$Z^{\text{tot}}_{\text{mat}} = \prod_{i,\,j} \frac{1}{\left(2\cosh \pi(\lambda_i - \mu_j)\right)^2} \tag{2.80}$$

where we used the explicit expression of the weight in the (anti-)bifundamental. Considering also the classical contribution, the ABJM matrix model is

$$Z_{\text{ABJM}} = \frac{1}{N_1! N_2!} \int d\lambda_i d\mu_j e^{ik\pi\left(\sum_i^{N_1} \lambda_i^2 - \sum_j^{N_2} \mu_j^2\right)} \frac{\prod_{i\neq j}^{N_1} 2\sinh \pi(\lambda_i - \lambda_j) \prod_{i\neq j}^{N_2} 2\sinh \pi(\mu_i - \mu_j)}{\prod_{i,\,j} \left(2\cosh \pi(\lambda_i - \mu_j)\right)^2} \,. \tag{2.81}$$

which leads to a highly non-trivial function of N_1, N_2, and k.

2.3.1 Applications

We review some applications of the localization formulas whose ideas are relevant to contextualize the original part. In all the examples, we will use results coming from localization to extract non-perturbative information on the SCFT.

F-maximization

Localization provides a window on the non-perturbative sector of the SCFTs. If an exactly conformal theory admits a Lagrangian (like the ABJM model), we can map the theory on

S^3 with a Weyl transformation (see Appendix C). However, SCFTs might arise as fixed points of non-conformal Lagrangian, like for SQED. In the latter case, we can place the non-conformal Lagrangians on S^3, by coupling the R-multiplet of the theory to new minimal supergravity and taking the rigid limit of Section 2.1.

There is an ambiguity in the choice of the IR R-symmetry. A priori, the R-symmetry current can mix with any abelian flavor symmetry. In other words, for a given R-symmetry current $j_\mu^{(R)}$, the current $j_\mu^{(R)}+\sum_I t_I j_\mu^{(I)}$, with $j_\mu^{(I)}$ is a flavor current, is still an R-symmetry [6]. This choice does affect the localization computation, as the 1-loop determinants depend on the R-charge assignment. Then, the localization computation reproduces the SCFT on $\mathbb{R}^3$ if and only if we select the R-symmetry which is a part of the superconformal algebra. It corresponds to a unique choice of t_I.

We define the free energy $F = -\log|Z(t_I)| = -\mathrm{Re}(\log Z(t_I))$. F-maximization is a procedure that allows to solve this issue and determine the correct R-symmetry. The statement is that the function $F(t_I)$ is locally maximized by the IR value of the coefficients t_I [23, 168, 169]. The idea behind this statement is that derivatives of F w.r.t. t_I are related to certain integrated correlators of operators in the multiplet of the flavor currents. The first derivative must vanish because the one-point functions are always zero in a CFT. A careful analysis of second derivatives, including the possible contact terms, leads to F-maximization [170, 171]. As a byproduct, a formula for the flavor central charge of the flavor currents was derived

$$\tau_{IJ} = -\frac{2}{\pi^2}\frac{1}{Z}\frac{\partial^2 F}{\partial t_I \partial t_J}\bigg|_{t_I=t_I^*} . \tag{2.82}$$

The relation computes exactly the flavor central charge defined as c_j in Eq (1.52). Notice that in a unitary theory $\tau_{IJ} > 0$, and so F is maximized.

We use F-maximization to give strong evidence for the IR duality of $\mathcal{N}=2$ SQED and the model with three chirals of Section 1.3. We recall that it is equivalent to a theory of three chirals X, Y, and Z with superpotential $W = XYZ$. The partition function is given by the integral

$$Z(\Delta) = \int_{\infty}^{+\infty} d\sigma\, e^{\ell(1-\Delta-i\sigma)}\, e^{\ell(1-\Delta+i\sigma)} = e^{2\ell(\Delta)+\ell(1-2\Delta)} . \tag{2.83}$$

The r.h.s. is interpreted as the partition function of a theory with two chirals of dimension $2-\Delta$ and one of dimension 2Δ. Since $Z(\Delta)$ has a maximum for $\Delta = \frac{1}{3}$, we find that all the dual chirals have dimension $2/3$, as expected on the basis of permutation symmetry of X, Y and Z.

A test of $\mathcal{N}=8$ dualities

We probe the duality among the ABJM model and the $\mathcal{N}=4$ SYM UV theory coupled to an adjoint and a fundamental hyper by matching their partition function. Since the latter

[6] At the level of the partition function, it amounts to observing that an imaginary shift m of the R-charge Δ of a chiral multiplet can be interpreted as giving a real mass for the chiral. It also implies that the partition function is holomorphic in $\Delta + im$. A similar consideration holds for FI parameters and the topological current.

theory is the S-dual of $\mathcal{N}=8$ SYM, the duality constitutes evidence that for ABJM at $k=1$ enjoys enhanced supersymmetry. We stress that we cannot use localization directly in $\mathcal{N}=8$ SYM, since the UV R-symmetry is different from the IR one. This phenomenon is signaled by the presence in the of certain monopoles operators [172] which, assuming that the UV R-symmetry coincides with the IR-one, would violate unitarity. At the level of the localized partition function, the non-convergence of the matrix integral highlights the problem [173]. The $\mathcal{N}=4$ SYM model is not affected by these problems. Indeed, since the additional hyper yields a cosh in the denominator, the integral is convergent, and localization provides a trustable result.

The derivation of the result requires two identities. The first one is the Fourier transform of the cosh

$$\int d\lambda \frac{e^{2\pi i\lambda\eta}}{\cosh\pi\lambda} = \frac{1}{\cosh\pi\eta}\,. \tag{2.84}$$

Notice that the l.h.s. is the partition function of $\mathcal{N}=4$ SQED with an FI-parameter η. Then, the physical interpretation of the identity is that this model is dual to a free hyper with real mass η. It is nothing but the statement of abelian mirror symmetry. The second is an identity for hyperbolic functions known as the Cauchy determinant formula

$$\frac{\prod_{i<j}\left[2\sinh\pi(\lambda_i-\lambda_j)\right]\left[2\sinh\pi(\mu-\mu_j)\right]}{\prod_{i,j}2\cosh\left(\lambda_i-\mu_j\right)} = \sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\prod_i\frac{1}{2\cosh\left(\lambda_i-\mu_{\sigma(i)}\right)} \tag{2.85}$$

where the sum is over the permutation σ of N elements (S_N is the set of permutations).

We start with the matrix model of the UV theory. We add a real mass deformation ω for the adjoint hyper and an FI-term η [7]. The partition function of the model is

$$Z_{\mathrm{SYM}} = \frac{1}{N!}\int d\lambda_i \frac{\prod_{i<j}\left[2\sinh\pi(\lambda_i-\lambda_j)^2\right]^2 e^{2\pi i\eta\sum_i\lambda_i}}{\prod_{i,j}2\cosh\pi(\lambda_i-\lambda_j+\omega)\prod_i 2\cosh\pi\lambda_i} \tag{2.86}$$

We can use Cauchy determinant identity to get

$$Z_{\mathrm{SYM}} = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\int d\lambda_i \frac{\epsilon^{2\pi i\eta\sum_i\lambda_i}}{\prod_{i,j}2\cosh\pi(\lambda_i-\lambda_{\sigma(i)}+\omega)\prod_i 2\cosh\pi\lambda_i} \tag{2.87}$$

We want to compare the latter expression with ABJM with $k=1$. We follow the proof of [167].

We take the partition function deformed by a real mass ξ and an FI-parameter ζ [8]

$$Z_{\mathrm{ABJM}} = \frac{1}{(N!)^2}\int d\lambda_i d\mu_j\, e^{ik\pi\left(\sum_i\lambda_i^2-\sum_j\mu_j^2\right)+2\pi i\zeta\left(\sum_i\lambda_i+\sum_j\mu_j\right)}\times$$
$$\times\frac{\prod_{i\neq j}^{N_1}2\sinh\pi(\lambda_i-\lambda_j)\prod_{i\neq j}^{N_2}2\sinh\pi(\mu_i-\mu_j)}{\prod_{i,j}\left(2\cosh\pi(\lambda_i-\mu_j)\right)^2}\,. \tag{2.88}$$

[7] A real mass for the fundamental hyper corresponds just to a shift of the integration variables.

[8] In principle, one could also add another FI-term and another real mass. However, only the diagonal sum of the two $U(1)$ gauge factors acts non-trivially and can be coupled to a background gauge field. A second mass parameter does not affect the physics. It can always be reabsorbed by a shift of the integration variables.

The partition function can be first simplified using twice eq. (2.85). Then, rewriting the two cosh factors as Fourier transform using eq (2.84), we obtain a Gaussian matrix model. Performing all the possible Gaussian integrations, we end up with

$$Z_{\text{ABJM}} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\epsilon(\sigma)} \int d\lambda_i \frac{\epsilon^{2\pi i(\xi+2\zeta)\sum_i \lambda_i}}{\prod_{i,j} 2\cosh \pi(\lambda_i - \lambda_{\sigma(i)} - \xi + 2\zeta) \prod_i 2\cosh \pi\lambda_i} \tag{2.89}$$

This expression coincides with that of Z_{SYM} if we identify

$$\eta = \xi + 2\zeta\,, \qquad \omega = \xi - 2\zeta\,. \tag{2.90}$$

We stress that, unlike standard mirror symmetry, real masses and FI parameters are not simply exchanged, but rather mixed. This intuition will be crucial to decorate the duality with BPS line operators. In conclusion, setting all the deformations to zero, we showed that $Z_{SYM} = Z_{ABJM}$ for $k = 1$ [9].

ABJM and the M-theory limit

We compute the free energy F in the large $N = N_1 = N_2$ limit, keeping k fixed. This regime is the M-theory limit. We exploit the Fermi gas technique [68] (see also [67, 175] for similar results with alternative methods). This method relies on the following representation of the matrix model

$$Z(N,k) = \frac{1}{N!} \sum_{\sigma \in S_N} \int \frac{d^N x}{(2\pi k)^N} \prod_{i=1}^{N} \rho(x_i, x_{\sigma_i})\,, \tag{2.91}$$

with the function ρ

$$\rho(x_1, x_2) = \frac{1}{2\pi k} \frac{1}{2\cosh\frac{x_1}{2}} \frac{1}{2\cosh\frac{x_2}{2}} \frac{1}{2\cosh\frac{x_1 - x_2}{2k}}\,. \tag{2.92}$$

This expression is just the generalization of (2.85) keeping k generic [10].

We interpret the result as the partition function of a one-dimensional ideal Fermi gas of N particles, with canonical density matrix $\rho(x_1, x_2)$. Then, we can study the free energy F of ABJM by exploiting methods from statistical mechanics. As usual, it is more convenient to study the grand canonical problem. Then we introduce the grand canonical potential $\mathcal{J}(\mu, k)$

$$\mathcal{J}(\mu, k) = \log \Xi(\mu, k)\,, \qquad \Xi(\mu, k) = \sum_{N}^{+\infty} Z(N,k) e^{N\mu} \tag{2.93}$$

where μ is the chemical potential. We can define an Hamiltonian operator from the relations $\rho = e^{-H}$. We are thinking of x as the canonical position operator, with p its canonically conjugate momentum. In this way, we fix $[x,p] = 2\pi i k$ and $\rho(x_1, x_2) = \langle x_1|\rho|x_2\rangle$. Then, ρ can be expressed as

$$\rho = e^{-\frac{1}{2}U(x)} e^{-T(p)} e^{-\frac{1}{2}U(x)}\,, \tag{2.94}$$

[9] See also [174] for tests involving the superconformal index.

[10] To the best of our knowledge, there is no interpretation as a UV theory for $k \neq 1$.

with

$$U(x) = \log\left(2\cosh\frac{x}{2}\right), \qquad T(p) = \log\left(2\cosh\frac{p}{2}\right).$$

We can define a Hamiltonian operator from the relations $\rho = e^{-H}$. In doing so, we are thinking of x as the canonical position operator, with p its canonically conjugate momentum. In this way, we have $[x,p] = 2\pi i k$ and $\rho(x_1,x_2) = \langle x_1|\rho|x_2\rangle$. Then, ρ can be expressed as

$$H(x,p) = U + T \simeq \frac{|x| + |p|}{2}. \tag{2.95}$$

We use this to compute the Fermi surface, whose area is proportional to the number of particles $\langle N\rangle$. From standard considerations one gets $\langle N\rangle = \frac{8\mu^2}{4\pi k}$. At large N, the relation with the free energy is rather simple

$$F(N,k) = \mathcal{J}(\mu_*,k) - \mu_* N, \quad \langle N(\mu,k)\rangle = \frac{\partial\mathcal{J}(\mu,k)}{\partial\mu} \simeq \frac{8\mu^2}{4\pi k}, \tag{2.96}$$

where μ^* is defined by inverting the second equation and is a function of k and N. Thus, we obtain the leading N behavior of the free energy

$$F = -\frac{\pi\sqrt{2}}{3}k^{\frac{1}{2}}N^{\frac{3}{2}} + O\left(\frac{1}{N}\right). \tag{2.97}$$

The fact that F scales with $N^{\frac{3}{2}}$ beautifully agrees with the M-theory prediction. This is a highly non-trivial test of AdS_4/CFT_3, as in string theory $F \sim N^2$. The $N^{\frac{3}{2}}$ is the footprint of M-theory! Remarkably, the Fermi gas method allows us to compute systematically all the perturbative $\frac{1}{N}$ corrections and to constraint the non-perturbative ones. The Fermi gas method applies for $\mathcal{N} \geq 3$ Chern-Simons matter theory (for $\mathcal{N} = 2$ the gas would be interacting). However, much less is known for other models (see [176] for a precise review).

2.4 Localization on the squashed sphere

We finally review the localization on the squashed sphere S^3_b [158]. Again, both the SYM and the matter action are δ exact

$$\delta_\varepsilon\delta_{\bar\varepsilon}\left(\frac{\bar\lambda\lambda}{2} + i\sigma D\right) = \bar\varepsilon\varepsilon\mathcal{L}_{\text{SYM}}, \qquad \delta_\varepsilon\delta_{\bar\varepsilon}\left(\frac{\bar\psi\psi}{2} + i\bar\phi\sigma\phi + iH(\Delta-1)\right), = \bar\varepsilon\varepsilon\mathcal{L}_{\text{mat}} \tag{2.98}$$

Then if we localize w.r.t. to $\delta = \delta_\varepsilon + \delta_{\bar\varepsilon}$, the SYM and the matter actions serve as localizing terms. From Eq (2.85), the locus is manifestly given by

$$\sigma = -\frac{D}{H} = \frac{\sigma_0}{r} \tag{2.99}$$

where in this section $r = \sqrt{\ell\tilde\ell}$. Then, the final result will be a matrix model with some b-deformed 1-loop determinants.

We begin with the chiral multiplet with an arbitrary R-charge Δ and in the representation R of the gauge group. As for the round sphere, we can first plug in the locus in the localizing term and then localize. Repeating the same steps, we are left with the following determinant to compute

$$Z_{\text{chir}}(\sigma_0) = \prod_\rho \left(\frac{\mathcal{O}_{\text{F}}(\sigma_0)}{\mathcal{O}_{\text{B}}(\sigma_0)}\right)^{\frac{1}{2}} \tag{2.100}$$

where $\mathcal{O}_{\text{B}}(\sigma_0)$, $\mathcal{O}_{\text{F}}(\sigma_0)$ are the bosonic and fermionic differential operator in the localizing term whose explicitly expression can be read from (2.85), and ρ are the weight of the representation R. In the case of the round sphere, we were able to expand all the modes. In the final result, supersymmetry causes a massive cancellation between bosonic and fermionic modes. The analog computation for the squashed sphere is tedious and involved. However, one can considerably simplify the problem by implementing the cancellation among paired modes *a priori*. It can be done by arguing that unpaired modes are related to the solution to a simpler first-order differential equation. Only these modes contribute to the determinant. We skip the technical details to write the result directly

$$\begin{aligned} Z_{\text{chir}}(\sigma_0) &= \prod_\rho \prod_{m,n} \frac{mb + nb^{-1} + i\rho(\sigma_0) + \frac{Q}{2}(2-\Delta)}{mb + nb^{-1} - i\rho(\sigma_0) + \frac{Q}{2}\Delta} = \\ &= \prod_\rho s_b\left(\frac{iQ}{2}(1-\Delta) - \rho(\sigma_0)\right) , \end{aligned} \tag{2.101}$$

where $Q = b + b^{-1}$. We stress that the results do not depend on the specific metric, but just on the isometry K on which the localizing supercharge squares. This can be understood as follows: we are computing the 1-loop determinant of a differential operator. Since the computation can be done with index theorem [58], which relies only on the infinitesimal action of the Killing vector K around the fixed points, the final result cannot depend on the metric, but only on the form of the isometry. Similarly, one calculates the 1-loop determinant for the vector multiplet

$$Z_{\text{vec}}(\sigma_0) = \prod_{\alpha>0} 4 \sinh \pi b \alpha(\sigma_0) \sinh \frac{\pi}{b}\alpha(\sigma_0) . \tag{2.102}$$

For $b = 1$ we recover the result of the round sphere. Finally, there is also a classical contribution from the Chern-Simons term

$$e^{-S_{\text{CS}}} = e^{-i\pi \operatorname{Tr}_{\text{CS}} \sigma_0^2} . \tag{2.103}$$

We described the Coulomb branch localization for the round and the squashed sphere. An equivalent scheme was proposed in [177, 178]. It is based on a different localizing term. The partition function localizes on a product of vortex and anti-vortex partition functions on $\mathbb{R}^2 \times S^1$. It is convenient to make clear the factorization in holomorphic blocks of the partition function [179, 180].

The free energy of a theory on the squashed sphere is an interesting observable. For instance, it provides a way to compute the central charge of $\mathcal{N} = 2$ SCFTs [145]. If we take the squashing parameter to be small, we can look at the operator which deforms the round sphere background. It turns out that it contains the R-symmetry current. Therefore, the second derivative of F on the squashed sphere with respect to b evaluated at $b = 1$ will be related to the integrated correlator of the two-point function of the R-symmetry current. Supersymmetry relates the overall coefficient of this two-points function to the central charge of the theory. A precise analysis leads to the exact relation

$$c_T = \frac{2}{\pi^2} \mathrm{Re} \left(\frac{\partial^2 F}{\partial b^2} \right) \Big|_{b=1} . \tag{2.104}$$

The full derivation would require a study of counterterms, as explained for F-maximization.

A final comment is in order. In the case of theories with $\mathcal{N} \geq 4$ SUSY on S^3_b, we are not able to recast the two determinants of the chirals in a cosh, unlike for $\mathcal{N} \geq 4$ theories on the round sphere. However, one can still add a real mass. It shifts $\rho(\sigma_0)$ by an opportune factor. One can tune the mass deformation under which the two chirals have the same charge and show that the determinants form a single cosh factor.

In this chapter, we reviewed the localization technique and some of its applications to 3d gauge theories. They rely on a $\mathcal{N} = 2$ supercharge. In the next chapters, we will focus on alternative localization schemes, based on different supercharges. The new localization methods will allow us to analyze more observables and aspects of supersymmetric gauge theories.

Chapter 3

Topological sector in $\mathcal{N} \geq 4$ theories

In this chapter, we investigate the first one-dimensional subsector of the thesis. The correlation functions of local operators are among the most basic observables in a Quantum Field Theory. In CFTs, their structure is highly constrained and boils down to the knowledge of the conformal data. For 3d theories with $\mathcal{N} \geq 4$ superconformal symmetry, some CFT data are encoded in a collection of operators which, when arranged along a straight line, are governed by a one-dimensional topological field theory [26–28] (see also [181] for the construction with $\mathcal{N} = 6$ supersymmetry). They are obtained by introducing an explicit coordinate dependence in the operators, preserving a fraction of the supersymmetry[1]. We call the set of these operators the *topological sector*, and it is the first one-dimensional subsector we will discuss.

The operators in the topological sector come in two types: Higgs branch operators and Coulomb branch operators, depending on whether they are associated with hypermultiplet or vector multiplet excitations. In [31–33], the authors showed that the numerical constants associated with the topological sector, i.e. the data of the 1d Topological Quantum Field Theory (TQFT), can be computed, in this case, by using supersymmetric localization [2].

In [40, 41], the generating function for specific *integrated* correlators in the topological sector was further conjectured to coincide with the three sphere partition function in the presence of certain supersymmetry preserving deformations. For the Higgs branch operators, the conjecture takes the form

$$\left\langle \left(\int d\varphi_1\, J^{A_1}(\varphi_1)\right)^{k_1} \dots \left(\int d\varphi_n\, J^{A_n}(\varphi_n)\right)^{k_n} \right\rangle = \left(-\frac{1}{4\pi r^2}\right)^n \frac{1}{Z} \frac{\partial^k Z[m_{A_1},\dots,m_{A_n}]}{\partial m_{A_1}^{k_1}\dots\partial m_{A_n}^{k_n}}\bigg|_{m_{A_1},m_{A_2},\dots=0} \tag{3.1}$$

where $k = k_1 + \cdots + k_n$ with $k_1, \dots, k_n \in \mathbb{N}$, the J^{A_i} are specific gauge-invariant Higgs branch operators, and $Z[m_{A_1}, ...]$ is the three sphere partition function deformed by supersymmetric mass terms. It can also be viewed as the generating function of a different set of local operators integrated over the entire three sphere. The conjecture was proven for the subclass

[1] Similar ideas appeared also in different dimensions [29].

[2] A different approach to study the Coulomb branch is discussed in [182–184].

of theories in which localization applies.

In principle, localization applies to any Lagrangian field theory with enough supersymmetry. However, certain technical challenges remain for models whose supersymmetry is not realized off-shell [185]. This is the case for 3d $\mathcal{N} \geq 4$ gauge theories with Chern-Simons terms, of the type considered in e.g. [125, 126], including the ABJM model and its generalizations [42, 43, 128]. Localization also cannot be directly applied to non-Lagrangian field theories. For these reasons, it would be very useful to prove the conjecture without resorting to localization for the fundamental fields. We address this problem in this chapter.

As an application, we investigate the topological sector in ABJM at weak coupling. We provide a perturbative computation for two, three, and four-point functions up to 1-loops. For the two-point function, we extend the computation up to two loops. Comparing the result with the matrix model prediction gives further perturbative evidence of the conjecture.

The organization of the chapter is the following. In section 3.1 we give an overview of the construction of the topological sector for $\mathcal{N} \geq 4$ supersymmetric gauge theories in three dimensions, and we explain how to compute them with localization. In section 3.2 we establish the cohomological equivalence at the level of linear coupling to the background fields. For a particular example involving free hypermultiplets, we prove the validity of the equivalence also at the non-linear level in m. In Section 3.3 we discuss the particular case of ABJM at weak coupling and give a perturbative proof of the conjecture for the ABJM model up to two loops. This chapter is based on [186, 187].

3.1 Topological operators: an overview

In this section we review the construction of topological operators in 3d $\mathcal{N} \geq 4$ supersymmetric field theories [26–28]. These are a particular subset of operators whose correlation functions, when localized on a 1d submanifold, are constant.

We begin by discussing the case of $\mathcal{N} \geq 4$ superconformal field theories (SCFTs). In 3d the $\mathcal{N} = 4$ superconformal group is Osp(4|4). The corresponding $\mathfrak{osp}(4|4)$ superalgebra contains as its maximal bosonic subalgebra the 3d conformal algebra $\mathfrak{sp}(4)$ and the R-symmetry algebra $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, whose generators are denoted by ${R_a}^b$ and ${\bar{R}_{\dot{a}}}^{\dot{b}}$, respectively. The odd sector contains Poincaré supercharges $Q_{\alpha,a\dot{a}}$ and superconformal charges $S_{\alpha,a\dot{a}}$.

The essential ingredient for constructing a set of topological operators closed under OPE, is the existence of a nilpotent supercharge $\mathcal{Q}$, which can be obtained as a linear combination of the $Q_{\alpha,a\dot{a}}$ and $S_{\alpha,a\dot{a}}$ generators [26, 28]. This supercharge can be used to define a $\mathcal{Q}$-exact R-twisted translation $\hat{P} \sim P+R$, where P is the ordinary translation along a 1d submanifold. It then follows that given a $\mathcal{Q}$-cohomology class of operators localized at the origin, they can be translated away from the origin by acting with $\hat{P}$, without affecting the $\mathcal{Q}$-cohomology. As a consequence, correlation functions of $\mathcal{Q}$-closed operators placed on the 1d submanifold turn out to be piecewise constant, depending at most on the order of the insertions. We call this set of operators the *topological sector*.

To be concrete, in 3d euclidean flat space we take the 1d submanifold to be the line

$x_1 = x_2 = 0$ and focus on the construction of topological operators in the Higgs branch first. The maximal superconformal algebra preserved by the line is a central extension of the algebra $\mathfrak{su}(1,1|2)$, whose bosonic subalgebra includes the $\mathfrak{so}(2,1)$ conformal algebra and the $\mathfrak{su}(2)$ R-symmetry algebra that we identify with $\mathfrak{su}(2)_H$. The bosonic generators are P_3, K_3, D, ${R_a}^b$, and the fermionic ones are $Q_{1,a\dot{2}}$, $Q_{2,a\dot{1}}$, $S_{2,a\dot{1}}$, and $S_{1,a\dot{2}}$. The central element is $Z = -M_{12} + R_{\dot{1}\dot{2}}$.

If we consider the two supercharges

$$Q_1^H \equiv Q_{11\dot{2}} + \frac{1}{2r}{S^2}_{2\dot{2}}\,, \qquad Q_2^H \equiv Q_{21\dot{1}} + \frac{1}{2r}{S^1}_{2\dot{1}}\,, \tag{3.2}$$

with r being an arbitrary length parameter, from the $\mathfrak{su}(1,1|2)$ algebra it is easy to see that they are both nilpotent operators. Moreover, their anticommutator reads

$$\{Q_1^H, Q_2^H\} = -M_{12} + R_{\dot{1}\dot{2}} = Z \tag{3.3}$$

where M_{12} is the generator of rotations in the plane orthogonal to the line. It then follows that since Z must vanish on the $Q_{1,2}^H$ cohomology classes, operators belonging to these classes are inserted along the fixed point locus of M_{12} and have zero $R_{\dot{1}\dot{2}}$ charge.

We now perform a topological twist by combining the $\{P_3, K_3, D\}$ generators of the $\mathfrak{so}(2,1)$ conformal algebra along the line and the $\mathfrak{su}(2)_H$ R-symmetry generators. They are given by

$$\hat{P} = P_3 + \frac{i}{2r}{R_2}^1\,, \qquad \hat{K} = K_3 - 2ir{R_1}^2\,, \qquad \hat{D} = D - {R_1}^1\,. \tag{3.4}$$

One can think of them as the generators of a new one-dimensional conformal algebra along the x_3-line. It turns out that the twisted generators are all $Q_{1,2}^H$-exact. Therefore, we can first construct operators localized at the origin from $Q_{1,2}^H$-closed, gauge invariant operators of the 3d theory. Explicitly, the cohomology of the two supercharges contain local operators $O_{a_1,\ldots,a_n}$ with the following properties [26]: they are Lorentz scalars, transform in the $(\mathbf{n+1}, \mathbf{1})$ of $\mathfrak{su}(2)_H \oplus \mathfrak{su}(2)_C$, and have conformal dimension $\Delta = n/2$. We then move them along the line by applying the twisted translation generator $\hat{P}$. The corresponding twisted translated operator at position $\vec{x} = (0,\,0,\,s)$ is given by

$$O(s) = e^{is\hat{P}} O_{1,\ldots,1}(0) e^{-is\hat{P}} = O_{a_1,\ldots,a_n}(\vec{x})\big|_{\vec{x}=(0,0,s)} u^{a_1}\ldots u^{a_n}\,, \qquad u^a = \left(1, \frac{s}{2r}\right)\,. \tag{3.5}$$

These operators are still $Q_{1,2}^H$-closed and form the Higgs topological sector of the $\mathcal{N} \geq 4$ SCFT on the line.

Topological operators on S^3

To make contact with localization it is convenient to consider Lagrangian theories defined on S^3. The crucial point is that $Q_{1,2}^H$ defined in (3.2) where now r is the S^3 radius, belong to the Poincaré subalgebra $\mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r$ introduced in [31]. Before entering in the details, we remark that this property allows us to define the topological sector away from the SCFT fixed point. For instance, we can introduce a suitable Yang-Mills term in the action, real mass terms, and FI-deformations (see Appendix B for the details). These theories can flow to interesting $\mathcal{N} \geq 4$ SCFTs. Then, one can use localization to access CFT data from the UV description.

Let us describe the algebra. The bosonic part of the algebra is $\mathfrak{so}(4) \oplus \mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$, where $\mathfrak{so}(4)$ is the isometry algebra of S^3 and $\mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$ is the residual R-symmetry, whose generators are linear combinations of $R_H = \frac{1}{2} h_a{}^b R_b{}^a$ and $R_C = \frac{1}{2} \bar{h}^{\dot{a}}{}_{\dot{b}} \bar{R}^{\dot{b}}{}_{\dot{a}}$, where $h_a{}^b$ and $\bar{h}^{\dot{b}}{}_{\dot{a}}$ are respectively $\mathfrak{su}(2)_H$ and $\mathfrak{su}(2)_C$ matrices, normalized such that $h_a{}^c h_c{}^b = \delta_a^b$ and $\bar{h}^{\dot{b}}{}_{\dot{c}} \bar{h}^{\dot{c}}{}_{\dot{a}} = \delta_{\dot{a}}^{\dot{b}}$. The explicit choice of these matrices is equivalent to select the preserved $U(1)_H \times U(1)_C$ R-symmetry. In this chapter, we take $h_a{}^b = -(\tau_2)_a{}^b$ and $\bar{h}^{\dot{a}}{}_{\dot{b}} = -(\tau_3)^{\dot{a}}{}_{\dot{b}}$. The cohomology algebra on S^3 reads

$$\{Q_1^H, Q_2^H\} = \frac{4i}{r} (P_\tau + R_C + ir\zeta) \,, \tag{3.6}$$

where P_τ is the translation along the great circle S^1_τ on S^3 parametrized by τ (see the metric in eq. (C.1)). We have included the central extension ζ associated to the presence of a FI term [31]. Since Higgs branch topological operators are annihilated by $Q_{1,2}^H$, they are placed on the fixed locus of P_τ, namely the S^1_φ at $\theta = \pi/2$ where τ shrinks, and can carry only Higgs branch R-symmetry.

An equivalent way to proceed is to consider the one-parameter family of supercharges $Q_\beta^H = Q_1^H + \beta Q_2^H$, β being a phase, which satisfy the following algebra

$$\{Q_\beta^H, \tilde{Q}_\beta^H\} = P_\varphi + R_H + irm \tag{3.7}$$

where $\tilde{Q}_\beta^H$ is a suitable linear combination of the supercharges whose explicit expression is not relevant for the discussion, and P_φ is the translation along the great circle S^1_φ. Here we include the central extension m associated to the presence of the real mass term [31]. On S^3, we define the twisted translation

$$\hat{P}_\varphi^H = P_\varphi + R_H \tag{3.8}$$

As in flat space, we can use the twisted translation to move operators in the cohomology of $Q_{1,2}^H$ away from the point $\varphi = 0$ along S^1_φ. Using the same arguments as in flat space, we deduce that correlators of twisted-translated operators will be topological up to a mild φ-dependence $e^{-mr\varphi}$.

We now want to use this construction to define the topological sector in standard (non-conformal) $\mathcal{N} \geq 4$ SYM theories with vector multiplets coupled to hypermultiplets (see appendix B for conventions on the multiplets). The main difference with the case in flat space is that we cannot rely on any argument based on representation theory of the superconformal algebra. That technology is available only at the fixed point. Nevertheless, if q_a and $\tilde{q}_a$ are the scalar fields of the hypermultiplet, a direct inspection shows that q_1 and $\tilde{q}_1$ are annihilated by $Q_{1,2}^H$ at $\theta = \pi/2$, $\varphi = 0$. Acting with $\hat{P}_\varphi^H$, we obtain the corresponding twisted translated operators

$$Q(\varphi) = q_1(\varphi)\cos\frac{\varphi}{2} + q_2(\varphi)\sin\frac{\varphi}{2} \equiv u^a q_a, \quad \tilde{Q}(\varphi) = \tilde{q}_1(\varphi)\cos\frac{\varphi}{2} + \tilde{q}_2(\varphi)\sin\frac{\varphi}{2} \equiv u^a \tilde{q}_a\,, \tag{3.9}$$

where $u^a = (\cos\varphi/2, \sin\varphi/2)$. Topological observables will be gauge invariant polynomials of $Q(\varphi)$ and $\tilde{Q}(\varphi)$. Correlators of twisted operators will depend on φ only through $e^{-mr\varphi}$.

For superconformal theories, one can obtain the same result by performing a Weyl transformation [31]. If we assume that there are no conformal anomalies at quantum level, correlators of twisted operators computed on a line embedded in $\mathbb{R}^3$ and on the great circle $S^1 \subset S^3$ coincide. In other words, setting $s = \tan\frac{\varphi}{2}$ we have that

$$\langle \mathcal{O}(s_1)\dots\mathcal{O}(s_k)\rangle_{\mathbb{R}^3} = \langle \mathcal{O}_{S^3}(\varphi_1)\dots\mathcal{O}_{S^3}(\varphi_k)\rangle_{S^3} \tag{3.10}$$

where $\mathcal{O}(s)$ is an operator of dimension Δ on the line and the $\mathcal{O}_{S^3}(\tau)$ is its counterpart on the circle obtained by contracting the S^3 operator localized on S^1 with polarization vectors $\bar{u}^a_{S^3}$ on the great circle. The effect of the Weyl transormation is the relation $\mathcal{O}_{S^3}(\varphi) = \Lambda^{-\Delta}\mathcal{O}(\varphi)$, with $\Lambda = \cos^2\frac{\varphi}{2}$ being the conformal factor on S^1. From this relation and (3.10) one can infer how the polarization vectors get mapped from the line to the great circle

$$\bar{u}^a_{S^3} = \Lambda^{\frac{1}{2}}\bar{u}^a_{\mathbb{R}^3} = \left(\cos\frac{\varphi}{2}, 0, \sin\frac{\varphi}{2}\right) \tag{3.11}$$

which agrees with the general result below Eq (3.9).

We can apply the same procedure to construct the topological sector of the Coulomb branch. In this case we introduce the two nilpotent supercharges

$$Q^C_1 = \frac{1}{2}\left(Q_{11\dot{2}} + iQ_{12\dot{2}} + Q_{11\dot{1}} + iQ_{12\dot{1}} + \frac{i}{2r}S_{11\dot{2}} - \frac{1}{2r}S_{12\dot{2}} - \frac{i}{2r}S_{11\dot{1}} + \frac{1}{2r}S_{12\dot{1}}\right) \tag{3.12}$$

$$Q^C_2 = \frac{1}{2}\left(Q_{21\dot{1}} - iQ_{22\dot{1}} + Q_{21\dot{2}} - iQ_{22\dot{2}} + \frac{i}{2r}S_{21\dot{1}} + \frac{1}{2r}S_{22\dot{1}} - \frac{i}{2r}S_{21\dot{2}} - \frac{1}{2r}S_{22\dot{2}}\right) \tag{3.13}$$

satisfying the following algebra

$$\{Q^C_1, Q^C_2\} = \frac{4i}{r}\left(P_\tau + R_H + irm\right)\,. \tag{3.14}$$

Since this has the same structure of the algebra in (3.6) except for the replacement $\zeta \to m$ and $C \to H$, the rest of the construction works similarly to the Higgs branch. In particular, Coulomb branch operators in the $Q^C_{1,2}$ cohomologies are annihilated by P_τ and R_H and live on S^1_φ at $\theta = \pi/2$ where τ shrinks.

As before, we can consider the family of cohomological supercharges $Q^C_\beta = Q^C_1 + \beta Q^C_2$ satisfying

$$\{Q^C_\beta, \tilde{Q}^C_\beta\} = P_\varphi + R_C + ir\zeta \tag{3.15}$$

It follows that operators in the Q^C_β-cohomology can be translated with the twisted translating generator $\hat{P}_\varphi = P_\varphi + R_C$ along S^1_φ and they depend on φ only through a factor $e^{-r\zeta\varphi}$.

It is easy to realize that in the Q^C_β-cohomology there is only one operator built from local fields in the vector multiplet, which is

$$\Phi(\varphi) = \Phi_{\dot{a}\dot{b}} v^{\dot{a}} v^{\dot{b}} \Big|_{\theta=\pi/2}, \quad v^{\dot{a}} = \frac{1}{\sqrt{2}}(e^{i\varphi/2}, e^{-i\varphi/2}) \tag{3.16}$$

where $\Phi_{\dot{a}\dot{b}}$ are the triplet of dynamical scalars. Coulomb branch topological operators are given by gauge invariant polynomials of (3.16). We note that the cohomology contains also non-trivial monopole operators. However, they will not enter our derivation.

Since both Q^H_β and Q^C_β are part of a Poincaré subalgebra on S^3, they are potential localizing supercharges. Indeed, one can show that the $\mathcal{N} = 4$ Yang-Mills Lagrangian is both Q^H_β and Q^C_β exact. Then we can still use S_{SYM} as localizing term. It follows that the localization computation for the vector multiplet is left unchanged w.r.t. the $\mathcal{N} = 2$ case. Indeed the locus is parametrized by a constant matrix σ_0 in the Cartan subalgebra of the gauge group.

$$\Phi_{\dot{1}\dot{2}} = irD^{11} = irD^{22} = \frac{1}{r}\sigma_0 \,, \tag{3.17}$$

If we are localizing Coulomb branch operators w.r.t. Q^C_β, one can simply replace the correlators with their value on the locus. The matter action will be quadratic on the locus and can be integrated out, producing the usual 1-loop determinat factor in the matrix model.

We cannot carry out the same procedure in presence of Higgs branch operators. The matter fields insertions make the integration of the matter action not straightforward. However, since the localized matter theory is free, one can still compute correlators. It was shown that the result is described by a 1d free theory with action

$$S_\sigma[Q, \tilde{Q}] = -4\pi r \int d\varphi \left[\tilde{Q}\partial_\varphi Q + \tilde{Q}(\sigma_0 + mr)Q - 2\pi i \,\mathrm{Tr}_\zeta\, \sigma_0\right] , \tag{3.18}$$

where we kept into account possible FI-parameters and real masses. The main point is that the fields Q and $\tilde{Q}$ are identified with the twisted operators defined in (3.9). Therefore one can evaluate correlation functions of twisted fields performing Wick contractions in the 1d theory. The final result is obtained performing the resulting matrix integral. Assuming that the theory on S^3 flows to a given SCFT in the deep IR, this machinery computes some physical CFT data of that SCFT.

3.2 An exact formula for integrated correlators

The localization scheme for the topological sector was proposed in [31–33] for $\mathcal{N} = 4$ SYM theories on S^3 coupled to hypermultiplets. However, the localization requires off-shell closed supersymmetry and cannot be straightforwardly applied to $\mathcal{N} \geq 4$ Chern-Simons matter theory [125, 126], including ABJM [42, 43, 128]. For integrated correlators of dimension-one Higgs branch operators, an alternative formula was proposed in [40, 41]. There, it was conjectured that the 1d integrated correlators of Higgs branch operators are captured by the derivatives of the mass deformed partition function with respect to the mass parameters.

We would like to prove the conjecture in general. The key idea is that Higgs branch operators sit in the bottom component of a conserved current multiplet. A mass deformation is realized, at the linear level, by coupling this multiplet to a background vector multiplet, and taking a rigid limit for the latter. The conjecture can then be restated as follows: the 1d integrated Higgs branch operators are Q^H_β-cohomologous to the mass deformation, meaning that the resulting generating functions coincide.

While the Higgs branch conjecture applies only to a specific class of operators, its validity is more general. It also should not depend on any specific Lagrangian realization of the theory. In the following, we will extend the formula for the generating function to Coulomb branch operators and propose a proof of the conjecture for both cases. We will also discuss the nonlinear coupling in the specific case of hypermultiplets.

3.2.1 Coulomb branch operators

We begin by discussing twisted Coulomb branch operators. We focus on a specific operator of the type of (3.16). In this section, $\Phi_{\dot{a}\dot{b}}$ is the bottom component of the current multiplet associated with the $U(1)$ topological current $j_\mu \propto \epsilon_{\mu\nu\rho}F^{\nu\rho}$. This multiplet also contains the fermions $\lambda_{a\dot{a}}$ and the auxiliary fields D_{ab} (see appendix B for more details). For each $U(1)$ factor of the gauge group, there is a multiplet of this type. One can couple this multiplet to an abelian background twisted vector multiplet $\tilde{\mathcal{V}}_{\text{back}}$. The relevant coupling is simply an $\mathcal{N} = 4$ mixed Chern-Simons term. Taking the supersymmetry preserving rigid limit (B.13) for the background multiplet yields the Fayet-Iliopoulos (FI) term on S^3

$$S_{FI} = i\zeta \int_{S^3} d^3x\sqrt{g}\left(h_a{}^b D_b{}^a - \frac{1}{r}\bar{h}^{\dot{a}}{}_{\dot{b}}\Phi^{\dot{b}}{}_{\dot{a}}\right) . \tag{3.19}$$

We now show how to relate the integrated correlators of topological Coulomb branch operators of the type appearing in eq. (3.16), and derivatives of this term w.r.t. to the FI parameter ζ.

We can infer the precise formula for the equality of generating functions by building on the localization result of [31]. After localization, the effect of the FI term is the multiplication of the matrix integrand function by a term $\exp\left(-8\pi^2 ir\zeta\sigma\right)$, where σ is the integration variable. One may also express σ in terms of the twisted field $\Phi(\varphi)$ defined in eq. (3.16) integrated over S^1_φ. Therefore, following localization, derivatives of the FI deformed partition function compute the simpler 1d integral of twisted Coulomb branch operators[3] via the following formula

$$\langle \int_{S^1_{\varphi_1}} d\varphi_1\, \Phi(\varphi_1) \cdots \int_{S^1_{\varphi_n}} d\varphi_n\, \Phi(\varphi_n) \rangle = \frac{1}{(4\pi ir^2)^n}\frac{\partial^n}{\partial\zeta^n}Z[\zeta]\,, \tag{3.20}$$

[3]The set of Coulomb branch operators also contains monopole operators. In principle, these can appear in our integrated correlators. However, this is not the case, as can be shown using an argument based on the topological symmetry. This symmetry is just a shift invariance for the dual photon γ. Therefore, only derivatives of γ can appear in any effective Lagrangian. Since the FI-term is a mass term, no coupling to γ is included in the deformation. Therefore, the dual photon does not appear at all in the FI-term and monopole operators do not appear in our formulae.

where $Z[\zeta]$ is the FI-deformed partition function. The equality can be argued on the basis of a Ward identity, which holds in any 3d $\mathcal{N} = 4$ theory, regardless of any localization procedures or Lagrangian realization of the theory. The Ward identity is the result of the following cohomological equivalence

$$S_{FI} = -4\pi i r^2 \int_{S^1_\varphi} d\varphi\, \Phi_{\dot a \dot b} v^{\dot a} v^{\dot b} + \delta_\xi \left[\int_{S^3} d^3x \sqrt{g}\, \tilde\eta^{a\dot a} \lambda_{a\dot a} \right] . \tag{3.21}$$

where δ_ξ stands for a variation generated by Q^C_β with parameter $\xi_{a\dot a}$. Let us assume that the last term on the rhs is also δ_ξ closed. Then the difference between the expressions leading, after taking expectation values, to (3.20) is δ_ξ exact, while preserving supersymmetry. It is well known that such expressions vanish after taking expectation values, thereby establishing (3.20). In the following, we will show that a spinor $\tilde\eta_{a\dot a}$ exists such that (3.21) holds.

The presence of both 3d and 1d terms in (3.21) may be surprising. However, a *mild singularity* for $\tilde\eta_{a\dot a}$ at $\theta = \pi/2$ can yield such a contribution. Since $\delta_\xi \lambda_{a\dot a} = i\gamma^\mu \xi_a{}^{\dot b} \nabla_\mu \Phi_{\dot b \dot a} + \cdots$ (see Eq (B.3b)) involves a derivative, integration by parts can give rise to a boundary term expressed as a delta function at $\theta = \pi/2$

$$-\frac{2i}{r} \int_{S^3} d^3x \sqrt{g}\, \delta\left(\theta - \pi/2\right) \Phi_{\dot a \dot b} v^{\dot a} v^{\dot b} \subset \delta_\xi \left[\int_{S^3} d^3x \sqrt{g}\, \tilde\eta^{a\dot a} \lambda_{a\dot a} \right] . \tag{3.22}$$

In this way, cohomological equivalence reduces the 3d functional to a 1d one, in the spirit of [31, 36, 37, 74, 188].

Let us thus discuss the boundary contribution in detail. Integrating by parts the expression $\tilde\eta^{a\dot a} \delta_\xi \lambda_{a\dot a} \sim \tilde\eta^{a\dot a} \gamma^\mu \xi_a{}^{\dot b} \nabla_\mu \Phi_{\dot b \dot a}$, we obtain the following total derivative

$$-i \int_{S^3} \left[\sqrt{g}\, \nabla_\mu \left(w^\mu_{\dot a \dot b} \Phi^{\dot a \dot b} \right) \right] , \tag{3.23}$$

where

$$w^\mu_{\dot a \dot b} \equiv \tilde\eta^a{}_{\dot a} \gamma^\mu \xi_{a \dot b} . \tag{3.24}$$

Let us assume that $w^\mu_{\dot a \dot b}$ has a singularity at $\theta = \pi/2$ such that $w^\theta_{\dot a \dot b}$ has a pole $(\theta - \pi/2)^{-1}$. We excise a small solid torus T, which can be taken to be ∂_φ and ∂_τ invariant, around the circle at $\theta = \pi/2$. The size of the torus is determined by a cutoff θ_0, such that the torus extends along $\theta \in (\theta_0, \pi/2]\, , \varphi \in [0, 2\pi)\, , \tau \in [0, 2\pi)$. In the limit where the size of the torus shrinks to zero, the boundary contribution can be written as

$$-i \int_{S^3} \left[\sqrt{g}\, \nabla_\theta \left(w^\theta_{\dot a \dot b} \Phi^{\dot a \dot b} \right) \right] = -2\pi i\, r^3 \oint_\varphi \left[\Phi^{\dot a \dot b} \Big|_{\theta = \pi/2} \lim_{\theta_0 \to \pi/2} \left(\left(\pi/2 - \theta_0\right) w^\theta_{\dot a \dot b} \Big|_{\theta = \theta_0} \right) \right] , \tag{3.25}$$

where we have used the fact that $\Phi^{\dot a \dot b}$ is non-singular. The factor of 2π comes from integrating over τ, whereas r^3 comes from $\sqrt{g}$. In order for the above to reproduce the delta function term in (3.22), we need to require

$$w^{\theta}_{\dot{a}\dot{b}} \sim \frac{2}{r} \frac{v_{\dot{a}} v_{\dot{b}}}{\theta - \pi/2} \,. \tag{3.26}$$

In addition, the remaining terms from $\int_{S^3} d^3x \sqrt{g}\, \tilde{\eta}^{a\dot{a}} \delta_\xi \lambda_{a\dot{a}}$ must reproduce the contributions in S_{FI} proportional respectively to D_{ab} and $\Phi_{\dot{a}\dot{b}}$, implying the following set of equations

$$w^{\dot{a}\dot{b}}_{\mu}\, \epsilon_{\dot{a}\dot{b}} = 0 \,, \tag{3.27}$$

$$\tilde{\eta}_{(a}{}^{\dot{b}} \xi_{b)\dot{b}} - i h_{ab} = 0 \,, \tag{3.28}$$

$$\nabla_\mu w^\mu_{\dot{a}\dot{b}} - 2\tilde{\eta}^{a}_{(\dot{a}} \xi'_{|a|\dot{b})} + \frac{1}{r} \bar{h}_{\dot{a}\dot{b}} = 0 \,. \tag{3.29}$$

Finally, we must demand that the δ_ξ-exact term is itself invariant under the action of δ_ξ.

The set of equations (3.26-3.29) is an overconstrained system of nine linear algebraic equations and three linear differential equations for the eight unknowns in $\tilde{\eta}_{a\dot{a}}$, further constrained by the demand that $\tilde{\eta}_{a\dot{a}}$ is annihilated by δ^2_ξ. Nevertheless, one can show that a solution exists[4]. We conclude that the cohomological equivalence in eq. (3.21) holds.

3.2.2 Higgs branch operators

We now move to twisted Higgs branch operators. Dimension-one scalar operators can be seen as the bottom components of $\mathcal{N} = 4$ linear multiplets $\Sigma^A = (J^A_{ab}, \chi^A_{a\dot{a}}, j^A_\mu, K^A_{\dot{a}\dot{b}})$.[5] The dimension-one scalars J^A_{ab} are in the $(\mathbf{3}, \mathbf{1})$ of the R-symmetry group, $\chi^A_{a\dot{a}}$ are the fermion partners of dimension 3/2 in the $(\mathbf{2}, \mathbf{2})$ of the R-symmetry group, j^A_μ are the flavor conserved currents, and $K^A_{\dot{a}\dot{b}}$ are dimension-two scalars in the $(\mathbf{1}, \mathbf{3})$ of the R-symmetry group.

We couple Σ^A to a background vector multiplet $\mathcal{V}^A_{\text{back}}$ fixed to a rigid supersymmetric configuration, to produce the real mass deformation. This amounts to modifying the action by the following terms

$$S_{\text{mass}} = m_A \int_{S^3} d^3x \sqrt{g} \left(-i h^{ab} J^A_{ab} + \bar{h}^{\dot{a}\dot{b}} K^A_{\dot{a}\dot{b}} \right) + O\left(m^2\right) \,. \tag{3.30}$$

The terms of order m^2 are additional terms needed to preserve supersymmetry.

According to [31], for theories without Chern-Simons terms localization on S^3 shows that Higgs branch operators are captured by a 1d theory with a quadratic action $S_\sigma[Q\,, \tilde{Q}]$, coupled to the standard matrix model. The 1d fundamental degrees of freedom Q, $\tilde{Q}$ are the twisted operators defined in eq. (3.9). We can introduce a mass deformation at the level of the 1d theory by adding the following term in the 1d action of Eq (3.18)

$$S_\sigma[Q\,, \tilde{Q}] \to S_\sigma[Q\,, \tilde{Q}] - 4\pi r^2 m^A \int_{-\pi}^{\pi} d\varphi\, \tilde{Q}(\varphi) T^A Q(\varphi) \,. \tag{3.31}$$

[4]The solution space is actually three dimensional, with any choice of solution being singular at $\theta = \pi/2$. The explicit form for the solutions is very complicated and not extremely enlightening, so we avoid writing them explicitly.

[5]Here A is an index which runs from 1 to the rank of the flavor symmetry Lie algebra.

where T^A are the generators of the flavor symmetry in the proper representation. The operators $J^A(\varphi) \equiv -\tilde{Q}(\varphi)T^A Q(\varphi)$ are the twisted operators $J^A(\varphi) = J^A_{ab}u^a u^b$. Localization can be used to show that the following equality holds

$$\left\langle \left(\int d\varphi_1\, J^{A_1}(\varphi_1)\right)^{k_1} \cdots \left(\int d\varphi_n\, J^{A_n}(\varphi_n)\right)^{k_n} \right\rangle = \left(-\frac{1}{4\pi r^2}\right)^n \frac{1}{Z}\frac{\partial^k Z[m_{A_1},\ldots,m_{A_n}]}{\partial m_{A_1}^{k_1}\ldots\partial m_{A_n}^{k_n}}\Bigg|_{m_{A_1},m_{A_2},\cdots=0} \tag{3.32}$$

where $k = k_1 + \cdots + k_n$ with $k_1,\ldots,k_n \in \mathbb{N}$. It has been conjectured that this identity should be true even in theories where the localization argument cannot be carried though. We now show that this is the case, relying on a cohomological argument, in analogy with the Coulomb branch case.

We introduce a generating function for a $\mathfrak{u}(1)$ flavor current

$$Z_{\text{flavor}}[m] = \left\langle \exp\left(m\int_{S^3} d^3x\sqrt{g}\left(-\frac{i}{r}h^{ab}J_{ab} + \bar{h}^{\dot{a}\dot{b}}K_{\dot{a}\dot{b}}\right) + O(m^2)\right)\right\rangle, \tag{3.33}$$

and, in analogy with eq. (3.31), a 1d generating function for integrated Higgs branch operators

$$Z_{\text{J}}[m] = \left\langle \exp\left(-4\pi r^2 m \oint_{S^1_\varphi} J(\varphi)\right)\right\rangle. \tag{3.34}$$

The cohomological argument states that mass derivatives of $Z_{\text{flavor}}[m]$ and $Z_{\text{J}}[m]$ are in the same Q^H_β cohomology class, that is

$$\frac{\partial}{\partial m}\left(S_{\text{mass}}[m] - 4\pi r^2 m \oint_{S^1_\varphi} J(\varphi)\right) = \{Q^H_\beta, \ldots\}, \tag{3.35}$$

where $S_{\text{mass}}[m]$ is the action appearing in the exponent of (3.33). Establishing this identity is sufficient in order to show that eq. (3.32) holds.

At linear order, showing that identity (3.35) holds amounts to showing that a spinor $\tilde{\xi}_{a\dot{a}}$ exists such that

$$\delta_\xi \int_{S^3}\left[\sqrt{g}\tilde{\xi}_{a\dot{a}}\chi^{a\dot{a}}\right] = \int_{S^3}\sqrt{g}\left[-\frac{i}{r}h^{ab}J_{ab} + \bar{h}^{\dot{a}\dot{b}}K_{\dot{a}\dot{b}} + 2r^2\frac{1}{\sqrt{g}}\delta\left(\theta-\pi/2\right)J_{ab}u^a u^b\right], \tag{3.36}$$

where δ_ξ indicates the variation generated by Q^H_β with parameter $\xi_{a\dot{a}}$. The derivation goes along the same lines as that of the Coulomb branch operators. Again, we require the appearance of a singularity in $\tilde{\xi}_{a\dot{a}}$ in order to obtain a delta function localizing at $\theta = \pi/2$. Defining $v^\mu{}_{ab} \equiv \tilde{\xi}_a{}^{\dot{b}}\gamma^\mu\xi_{b\dot{b}}$, we need to require the following asymptotic behavior

$$v^{\theta(ab)} \sim 2r^2\left(\pi/2-\theta\right)^{-1}u^a u^b\,. \tag{3.37}$$

This condition must be supplemented by the following conditions

$$v^{\mu}{}_{a}{}^{a} = 0\,, \tag{3.38}$$

$$\tilde{\xi}^{a(\dot{a}}\xi_{a}{}^{\dot{b})} = -\bar{h}^{\dot{a}\dot{b}}\,, \tag{3.39}$$

$$\nabla_{\mu} v^{\mu(ab)} - 2\tilde{\xi}^{(a|\dot{a}|}\xi'^{b)}{}_{\dot{a}} = -\frac{i}{r} h^{ab}\,, \tag{3.40}$$

which arise requiring that all the linear terms in $S_{\text{mass}}[m]$ get correctly reproduced. Indeed, we find a solution. This is parametrized by three regular function of θ and φ: h_1, h_2, and h_3. Its explicit form reads

$$\begin{aligned}
\tilde{\xi}_{1,1\dot{1}} &= e^{-i\tau} h_1(\theta,\varphi) \\
\tilde{\xi}_{2,1\dot{1}} &= \frac{1}{\beta}\left(h_2(\theta,\varphi) + \frac{1}{\sqrt{\sin\theta\cos\varphi+1}}\right) \\
\tilde{\xi}_{1,1\dot{2}} &= h_2(\theta,\varphi) \\
\tilde{\xi}_{2,1\dot{2}} &= \frac{e^{i\tau}\tan\theta\sin\varphi}{\sqrt{\sin\theta\cos\varphi+1}} - \beta e^{i\tau} h_1(\theta,\varphi) \\
\tilde{\xi}_{1,2\dot{1}} &= e^{-i\tau} h_3(\theta,\varphi) \\
\tilde{\xi}_{2,2\dot{1}} &= -e^{i\tau}\sec\theta\left(e^{-i\tau}\sin\theta\sin\varphi\; h_3(\theta,\varphi) + e^{-i\tau} h_1(\theta,\varphi)(\sin\theta\cos\varphi+1)\right) \\
\tilde{\xi}_{1,2\dot{2}} &= \beta\left(-e^{i\tau}\right)\sec\theta\left(e^{-i\tau}\sin\theta\sin\varphi\, h_3(\theta,\varphi) + e^{-i\tau} h_1(\theta,\varphi)(\sin\theta\cos\varphi+1)\right) \\
\tilde{\xi}_{2,2\dot{2}} &= -e^{i\tau}\left(\beta h_3(\theta,\varphi) + \sec(\theta)\sqrt{\sin(\theta)\cos(\varphi)+1}\right)\,.
\end{aligned} \tag{3.41}$$

This solution is also invariant under the action of δ_{ξ}^2. Therefore, we conclude that eq. (3.35) is verified at linear level.

Unlike the Coulomb branch operators, the Higgs branch coupling involves quadratic terms in m. In the next section, we show that the cohomological equivalence persists beyond the linear order, at least in the simple example of the current multiplet built from a hypermultiplet. Presumably, one may argue that this is always true, based on the supersymmetry and gauge symmetry preserving nature of the complete non-linear coupling (see e.g. [150]).

3.2.3 Beyond the linearized analysis: an explicit example

For a generic theory we consider the current multiplet built from a hypermultiplet, coupled to an abelian background vector multiplet of the form (B.10). We consider a $\mathcal{N}=4$ gauge theory for a single hypermultiplet $\mathcal{H} = \left(q_a, \tilde{q}^a, \psi_{\dot{a}}, \tilde{\psi}_{\dot{a}}\right)$ with a weakly gauged $U(1)$ flavor symmetry. We construct the conserved current multiplet associated to the invariance of action (B.5) under the global $U(1)$ symmetry

$$q_a \to e^{i\alpha} q_a\,, \qquad \psi_{\dot{a}} \to e^{i\alpha}\psi_{\dot{a}}\,, \tag{3.42}$$

$$\tilde{q}_a \to e^{-i\alpha}\tilde{q}_a\,, \qquad \tilde{\psi}_{\dot{a}} \to e^{-i\alpha}\tilde{\psi}_{\dot{a}}\,. \tag{3.43}$$

The corresponding Noether current is $j_\mu = i\tilde{q}^a \nabla_\mu q_a - i\nabla_\mu \tilde{q}^a\, q_a - \tilde{\psi}^{\dot{b}}\gamma_\mu \psi_{\dot{b}}$ and it sits in a current multiplet $\Sigma = \left(J_{ab},\, \chi_{a\dot{a}},\, j_\mu,\, K_{\dot{a}\dot{b}}\right)$.

At linearized level, the multiplet is defined as the variation of the action w.r.t. the gauge multiplet $\Sigma \sim \frac{\delta S_{\text{hyper}}}{\delta \mathcal{V}}$. Therefore, its components in terms of the hypermultiplet components are determined by comparing the linearized coupling

$$S_{\text{lin}} = \int d^3x \sqrt{g} \left[A^\mu j_\mu + iD^{ab} J_{ab} + \Phi^{\dot{a}\dot{b}} K_{\dot{a}\dot{b}} + \lambda^{a\dot{a}} \chi_{a\dot{a}}\right] , \tag{3.44}$$

with the terms in S_{Hyper} linear in the vector field components. This allows to find the bottom component

$$J_{ab} = -\tilde{q}_{(a} q_{b)} \,. \tag{3.45}$$

In order to build the fully non-linear expressions of the higher components, we first observe that S_{lin} is SUSY invariant under the following variations of the current multiplet components

$$\delta_\xi J_{ab} = -i\xi_{(a}{}^{\dot{a}} \chi_{b)\dot{a}} \,, \tag{3.46}$$

$$\delta_\xi \chi_{a\dot{a}} = \xi_a{}^{\dot{b}} K_{\dot{a}\dot{b}} + \gamma^\mu \xi^b{}_{\dot{a}} \nabla_\mu J_{ab} + 2\xi'^b{}_{\dot{a}} J_{ab} + \frac{i}{2}\gamma^\mu \xi_{a\dot{a}} j_\mu \tag{3.47}$$

$$\delta_\xi K_{\dot{a}\dot{b}} = -i\nabla_\mu \left(\xi^a{}_{(\dot{a}} \gamma^\mu \chi_{|a|\dot{b})}\right) - 2i\xi'^a{}_{(\dot{a}} \chi_{|a|\dot{b})} \,, \tag{3.48}$$

$$\delta_\xi j_\mu = i\epsilon_{\mu\nu\lambda} \xi^{a\dot{a}} \gamma^\nu \nabla^\lambda \chi_{a\dot{a}} - 2\xi'^{a\dot{a}} \gamma_\mu \chi_{a\dot{a}} \,. \tag{3.49}$$

Now, applying δ_ξ to the bottom component (3.45) and using the explicit variations (B.4), a comparison with (3.46) allows to determine the $\chi_{a\dot{a}}$ fermion

$$\chi_{a\dot{a}} = -i\left(\tilde{q}_a \psi_{\dot{a}} + \tilde{\psi}_{\dot{a}} q_a\right) \,. \tag{3.50}$$

Comparing its variation obtained from (B.4) with variation (3.47) we find the non-linear expressions for j_μ and $K_{\dot{a}\dot{b}}$. Finally, it is easy to check that variations (B.4) imply that this multiplet correctly closes the algebra in (3.46 - 3.49). In terms of the hypermultiplet components, choosing the background (B.10) they are given by

$$\begin{aligned} J_{ab} &= -\tilde{q}_{(a} q_{b)}, & \chi_{a\dot{a}} &= -i\left(\tilde{q}_a \psi_{\dot{a}} + \tilde{\psi}_{\dot{a}} q_a\right) \\ K_{\dot{a}\dot{b}} &= i\tilde{\psi}_{(\dot{a}} \psi_{\dot{b})} - m\bar{h}_{\dot{a}\dot{b}}\, \tilde{q}^b q_b & j_\mu &= i\tilde{q}^a \nabla_\mu q_a - i\nabla_\mu \tilde{q}^a\, q_a - \tilde{\psi}^{\dot{b}} \gamma_\mu \psi_{\dot{b}} \end{aligned} \tag{3.51}$$

We note that some of the components depend on the background fields. In particular, we are interested in the m dependence of $K_{\dot{a}\dot{b}}$, thus we rewrite

$$K_{\dot{a}\dot{b}} = K^{\text{lin}}_{\dot{a}\dot{b}} - m\bar{h}_{\dot{a}\dot{b}}\, \tilde{q}^b q_b \,, \tag{3.52}$$

where $K^{\text{lin}}_{\dot{a}\dot{b}} \equiv i\tilde{\psi}_{(\dot{a}} \psi_{\dot{b})}$ is the $m = 0$ piece.

The hypermultiplet action can be read from (B.5) by substituting the (B.10) background

$$\begin{aligned} S_{\text{hyper}}[m] = \int d^3x\sqrt{g}\Big[& \nabla^\mu \tilde{q}^a \nabla_\mu q_a - i\tilde{\psi}^{\dot{a}}\gamma^\mu\nabla_\mu\psi_{\dot{a}} + \frac{3}{4r^2}\tilde{q}^a q_a \\ & - i\frac{m}{r}\tilde{q}^a h_a{}^b q_b - im\tilde{\psi}^{\dot{a}}\,\bar{h}_{\dot{a}}{}^{\dot{b}}\,\psi_{\dot{b}} - \frac{m^2}{2}\tilde{q}^a\,\bar{h}^{\dot{a}\dot{b}}\bar{h}_{\dot{a}\dot{b}}\,q_a\Big] \\ \equiv S_{\text{hyper}}[0] + mS_{\text{lin}} + \frac{1}{2}m^2\frac{d^2S}{dm^2}\,. \end{aligned} \tag{3.53}$$

where we have defined

$$\begin{aligned} S_{\text{lin}} &= -i\int d^3x\sqrt{g}\,(\frac{1}{r}\tilde{q}^a h_a{}^b q_b + \tilde{\psi}^{\dot{a}}\,\bar{h}_{\dot{a}}{}^{\dot{b}}\,\psi_{\dot{b}}) = \int d^3x\sqrt{g}\,(-\frac{i}{r}h^{ab}J_{ab} + \bar{h}^{\dot{a}\dot{b}}K^{\text{lin}}_{\dot{a}\dot{b}}) \\ \frac{d^2S}{dm^2} &= -\int d^3x\sqrt{g}\,\tilde{q}^a\,\bar{h}^{\dot{a}\dot{b}}\bar{h}_{\dot{a}\dot{b}}\,q_a\,. \end{aligned} \tag{3.54}$$

Here, we are interested in the full non-linear coupling.

The m-dependent term in $K_{\dot{a}\dot{b}}$, eq. (3.52), enters the variation of $\chi_{a\dot{a}}$ (see eq. (3.47)) and consequently affects the cohomological equivalence (3.36) by one extra term linear in m. Here we prove that this extra piece reproduces exactly the quadratic coupling in the 3d action (3.53).

To this end, we observe that retaining the non-linear terms in m, the cohomological equivalence (3.35) is equivalent to

$$S_{\text{lin}} + m\frac{d^2S}{dm^2} - 4\pi r^2\oint_{S^1_\varphi} J(\varphi) = \delta\left[\int d^3x\sqrt{g}\,\tilde{\xi}^{a\dot{a}}\,\chi_{a\dot{a}}\right]. \tag{3.55}$$

where S_{lin} and $\frac{d^2S}{dm^2}$ are given in (3.54). The equivalence at linear order has been already established in the previous section. We then focus only on terms in (3.55) proportional to m. Using variations (3.47), the splitting in (3.52) and eq. (3.39) it is easy to see that

$$\begin{aligned} -\int d^3x\sqrt{g}\,\tilde{\xi}^{a\dot{a}}\,\delta\chi_{a\dot{a}} &\to -\int d^3x\sqrt{g}\,\tilde{\xi}^{a\dot{a}}\,\xi_a{}^{\dot{b}}K_{\dot{a}\dot{b}} \\ &\to m\int d^3x\sqrt{g}\,\tilde{\xi}^{a\dot{a}}\,\xi_a{}^{\dot{b}}\bar{h}_{\dot{a}\dot{b}}\,\tilde{q}^b q_b = -m\int d^3x\sqrt{g}\,\bar{h}^{\dot{a}\dot{b}}\bar{h}_{\dot{a}\dot{b}}\,\tilde{q}^b q_b \end{aligned} \tag{3.56}$$

and this coincides with $\frac{d^2S}{dm^2}$ in (3.54). We have thus shown that at least in this case the cohomological equivalence persists beyond the linear order.

While the linear coupling is universal and does not depend on the specific theory, the second order term might depend on details of the theory, such as the off-shell closure of the supercharge used for the cohomological equivalence. For this, reason the computation does not apply to all possible theories. However, we believe that supersymmetry and gauge invariance control the non-linear terms, and that a mechanism similar to the one discussed in this example will ensure the validity of (3.35) regardless of the specific theory.

A strong indication of this comes from 3d $\mathcal{N}=4$ mirror symmetry, which exchanges topological and ordinary flavor symmetries, and the associated FI and mass terms [86].

In the present context, (3.30) and its non-linear completion is realized in the mirror dual theory as a FI coupling, as in (3.19), which requires no such completion.[6] The remaining equations, and the analysis, only require substituting Coulomb for Higgs everywhere, and implementing the index shuffle brought on by the outer automorphism of the R-symmetry group.

3.3 Perturbative results in ABJM

As an application, we discuss the weak coupling limit of the topological sector in ABJM. As a byproduct, we aim to confirm that the correlators of topological operators do not depend on the position and the Ward identity (3.32). For these reasons, we study two-, three- and four-point functions. We focus only on connected correlation functions. While three- and four-point correlators are evaluated up to one loop, we push the calculation for the two-point function up to two loops to provide a check of (3.32) at a non-trivial perturbative order. Correlators are computed on the straight line and later mapped to the great circle in S^3, to allow for a comparison with localization results discussed in section 3.3.2.

We will focus on the operator with the lowest dimension. They are the superconformal primary of a short B multiplet (see Section 1.2.1). The explicit superconformal primary reads

$$O_I{}^J(x) = \mathrm{Tr}\big(C_I(x)\bar{C}^J(x)\big) - \frac{1}{4}\delta_I^J\,\mathrm{Tr}\big(C_K(x)\bar{C}^K(x)\big)\,. \tag{3.57}$$

The multiplet contains the energy momentum tensor [41, 181]. For instance, the two-point function is related to the central charge c_T

$$\langle O_I{}^J(x) O_K{}^L(x)\rangle = \frac{c_T}{16}\left(\delta_I^L\delta_K^J - \frac{1}{4}\delta_I^J\delta_K^L\right)\frac{1}{16\pi^2 x^2}\,. \tag{3.58}$$

This fact suggests that one can relate derivatives of the mass-deformed ABJM matrix model to CFT data of the stress tensor. If we consider ABJM in the $\mathcal{N}=2$ language, we can identify three $U(1)$ factors which commutes with the manifest $\mathcal{N}=2$ $U(1)_R$ R-symmetry. Using the decomposition of the matter scalar $C_I = (A_1, A_2, \bar{B}_1, \bar{B}_2)$ as bottom components of chiral multiplets, we turn on the following $\mathcal{N}=2$ real masses $(m_+/2, m_-/2, -m_+/2, -m_-/2)$ for the corresponding chiral multiplet. Then, the Ward identities (3.32) allows us to compute CFT data from the derivatives of the mass-deformed ABJM matrix model. For instance, we obtain a simple formula for the central charge, defined in Eq (1.49)

$$c_T = -\frac{64}{\pi^2}\frac{\partial^2 \log Z}{\partial^2 m_\pm}\bigg|_{m_\pm=0}\,. \tag{3.59}$$

We stress that the formula does not require to use of the squashing background or F-maximization.

To perform the twist we select the $\mathfrak{su}(1,1|2)$ algebra preserved by the $\frac{1}{2}$-BPS Wilson line extended along the x_3 direction, and whose bosonic part contains the R-symmetry group ro-

[6] We would like to thank the anonymous referee for pointing this out.

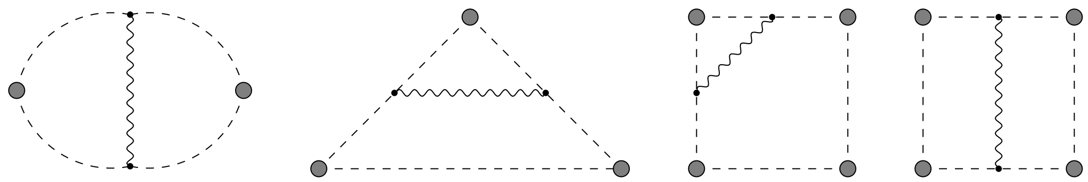

Figure 3.1: Topologies of one-loop diagrams contributing to the correlators.

tating C_2 and C_4 [7]. The corresponding superconformal primary is $O(0) = \mathrm{Tr}\left(C_2(0)\bar{C}^4(0)\right)$. The corresponding twisted operator is [8]

$$\mathcal{O}(s) = \mathrm{Tr}\big(C_I(s)\bar{C}^J(s)\big)\bar{U}^I(s)V_J(s)\,, \qquad \bar{U}^I(s) = (0,1,0,s)\,, \quad V_J(s) = (0,-s,0,1) \quad (3.60)$$

where s denotes the position $\vec{x} = (0,0,s)$.

3.3.1 Correlators on the line

The perturbative evaluation of n-point correlation functions relies on the expansion of the Euclidean path integral

$$\langle O(s_1)\cdots O(s_n)\rangle = \int O(s_1)\cdots O(s_n)\, e^{-S} \quad (3.61)$$

in powers of the coupling constants N_1/k and N_2/k. Here S is the ABJ(M) action defined in Eq (A.17).

Performing all possible contractions and using the scalar propagator in (A.23), for tree-level connected correlators we obtain

$$\langle \mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = -\frac{N_1N_2}{(4\pi)^2} \quad (3.62)$$

$$\langle \mathcal{O}(t)\mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = 0 \quad (3.63)$$

$$\langle \mathcal{O}(z)\mathcal{O}(t)\mathcal{O}(s)\mathcal{O}(0)\rangle^{(0)} = 2\frac{N_1N_2}{(4\pi)^4} \quad (3.64)$$

where z, t, s are the positions on the x_3 line. In the non-vanishing cases, the worldline dependence at the denominator encoded in the propagators is canceled by an analogous numerator coming from the contraction of the polarization vectors.

One-loop corrections to two-, three- and four-point functions are drawn in figure 3.1. It is easy to realize that they all vanish due to geometrical reasons. All the contributions are proportional to one Levi-Civita tensor $\varepsilon_{\mu\nu\rho}$ coming from the gauge propagator (see eq. (A.25)), which is contracted with spacetime derivatives coming from either internal vertices or the gauge propagator. It is a matter of the fact that such structures eventually vanish when projected on the line.

[7]The $\frac{1}{2}$-BPS Wilson line is discussed in the next chapter.
[8]In this section we fix $r = \frac{1}{2}$.

The first non-trivial information comes at two loops. We restrict the evaluation to the two-point function, whose diagrams at this order are given in figures 3.2(a)-3.2(j).

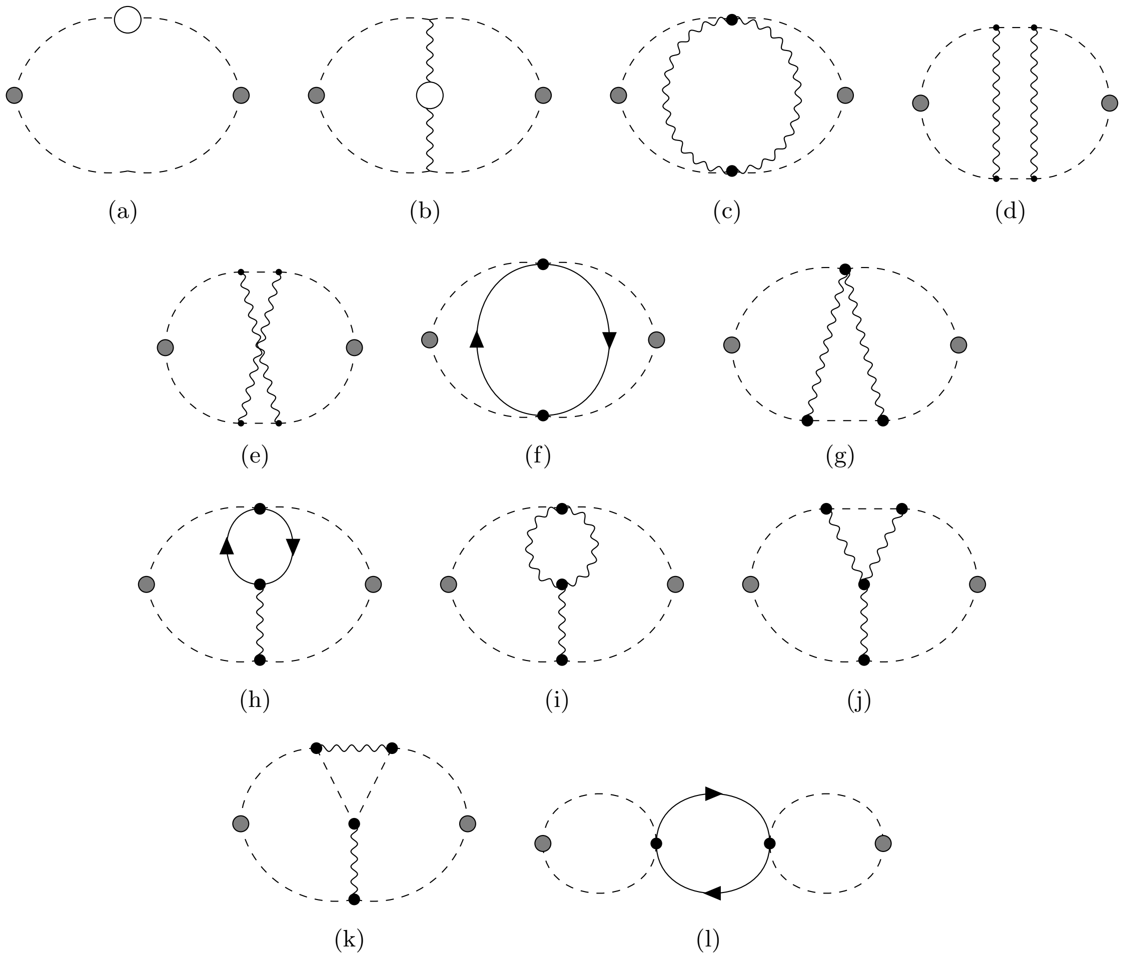

Figure 3.2: Two-loop diagrams for the two-point function. In (a) the white circle is the two-loop correction to the scalar propagator, while in (b) the circle is the one-loop correction to the gauge field propagator. Diagrams (h), (i), (j) and (k) sum up to provide the vertex correction.

The corresponding algebraic expressions, including the combinatorial and color factors, are listed in Appendix D. We evaluate the corresponding integrals by Fourier transforming to momentum space. Potential UV divergences are regularized within the DRED scheme [189, 190]. This amounts to first performing the tensor algebra strictly in three dimensions to reduce the integrals to a linear combination of scalar integrals and then analytically continue the resulting integrals to $d = 3 - 2\epsilon$ dimensions. As usual, we also introduce a dimensionful parameter μ to correct the scale dimensions of the couplings when they are promoted to d dimensions.

Applying *Mathematica* routines[9] based on the uniqueness method the momentum integrals can be analytically evaluated, leading to the results for every single diagram listed below.

[9]We are grateful to Marco Bianchi for sharing with us his routines.

Starting from the first diagram in fig. 3.2(a) we have

$$(3.2(a)) = -\frac{\Gamma^3\left(\frac{1}{2}-\epsilon\right)}{4^3\pi^{\frac{9}{2}-3\epsilon}}\,\frac{1}{2}\mathcal{C}(N_1,N_2)\,|\mu s|^{8\epsilon} \tag{3.65}$$

where $\mathcal{C}(N_1, N_2)$ is the two-loop correction to the scalar propagator computed in [79]. Its expansion at small ϵ is given in eq. (D.1). Therefore, neglecting terms which go to zero in the $\epsilon \to 0$ limit, the contribution of this diagram reads

$$\begin{aligned}(3.2(a)) = \frac{N_1N_2}{k^2}\,\frac{1}{128\pi^2}\Big[&-\left(N_1^2+N_2^2+4N_1N_2-6\right)\frac{1}{\epsilon}\\ &+\left(N_1^2+N_2^2-2\right)\left(\pi^2-2(3+\log 2)\right)+4(N_1N_2-1)\left(\pi^2-2(11+\log 2)\right)\Big]\,|\mu s|^{8\epsilon}\end{aligned}$$

For the rest of the diagrams, neglecting terms that vanish for $\epsilon \to 0$, we obtain

$$\begin{aligned}
(3.2(b)) &= -\frac{N_1N_2}{k^2}\,(N_1N_2-1)\,\frac{(\pi^2-12)}{16\pi^2}\,|\mu s|^{8\epsilon}\\
(3.2(c)) &= \frac{N_1N_2}{k^2}\,(N_1^2+N_2^2-4N_1N_2+2)\left(\frac{1}{128\pi^2}\frac{1}{\epsilon}+\frac{1+\log 2}{64\pi^2}\right)|\mu s|^{8\epsilon}\\
(3.2(d)) &= 0\\
(3.2(e)) &= -\frac{N_1N_2}{k^2}\,(N_1N_2-1)\,\frac{(5\pi^2-48)}{96\pi^2}\,|\mu s|^{8\epsilon}\\
(3.2(f)) &= \frac{N_1N_2}{k^2}\,(N_1N_2-1)\left(\frac{1}{16\pi^2}\frac{1}{\epsilon}+\frac{1+\log 2}{8\pi^2}\right)|\mu s|^{8\epsilon}\\
(3.2(g)) &= -\frac{N_1N_2}{k^2}\,(N_1^2+N_2^2-4N_1N_2+2)\,\frac{(\pi^2-12)}{128\pi^2}\,|\mu s|^{8\epsilon}\\
(3.2(h)) &= 0\\
(3.2(i)) &= 0\\
(3.2(j)) &= \frac{N_1N_2}{k^2}\,(N_1N_2-1)\,\frac{(\pi^2-12)}{48\pi^2}\,|\mu s|^{8\epsilon}\\
(3.2(k)) &= \frac{N_1N_2}{k^2}\,(N_1^2+N_2^2-2)\,\frac{(\pi^2-12)}{192\pi^2}\,|\mu s|^{8\epsilon}\\
(3.2(l)) &= -\frac{N_1N_2}{k^2}\,(N_1-N_2)^2\,\frac{1}{64}\,|\mu s|^{8\epsilon}
\end{aligned}$$

Summing all the contributions, it is easy to realize that the ϵ-poles cancel exactly. We can then safely take the $\epsilon \to 0$ limit and the final result for the two-point function, up to two loops reads

$$\langle\mathcal{O}(s)\mathcal{O}(0)\rangle^{(2)} = -\frac{N_1N_2}{(4\pi)^2}\left(1-\frac{\pi^2}{6k^2}(N_1^2+N_2^2-2)\right) \tag{3.66}$$

We note that for dimensional reasons all the diagrammatic contributions have a dependence on the position of the form $|\mu s|^{8\epsilon}$. In principle, expanding $|\mu s|^{8\epsilon} \sim (1 + 8\epsilon\log|\mu s| + \cdots)$ might produce dangerous, finite $\log|\mu s|$ terms that would spoil the topological nature of the operators at quantum level. However, this does not happen, thanks to the complete cancellation of the $1/\epsilon$ poles. Therefore, the BPS nature that the operators possess in the parent three-dimensional theory nicely protects the correlators on the line, which then turns out to be topological also at the quantum level.

The topological operators in (3.60) are related to the superprimaries (3.57) of the stress-energy tensor localized on the line. It follows that if we project the two-point function (3.58) on the line by setting $\vec{x} = (0, 0, s)$ and multiplying both sides by $\bar{U}^I(s)V_J(s)\bar{U}^K(0)V_L(0)$, we obtain an explicit expression for the central charge in terms of the topological correlator

$$c_T = -64\,(2\pi)^2\,\langle \mathcal{O}(s)\,\mathcal{O}(0)\rangle = -64\,\left\langle \int_{-\pi}^{\pi} d\tau_1 \mathcal{O}(\tau_1)\,\int_{-\pi}^{\pi} d\tau_2 \mathcal{O}(\tau_2)\right\rangle \tag{3.67}$$

where the second equality has been obtained by taking into account the translational invariance of the correlator and the identity $\langle \mathcal{O}(s)\,\mathcal{O}(0)\rangle_{\mathbb{R}^3} = \langle \mathcal{O}(s)\,\mathcal{O}(0)\rangle_{S^3}$ discussed around Eq (3.11).

Inserting in (3.67) the perturbative result (3.66) for the two-point function, we obtain the expansion of the ABJ(M) central charge at second order in the couplings and at generic, finite values of the group ranks

$$c_T = 16 N_1 N_2 \left(1 - \frac{\pi^2}{6k^2}(N_1^2 + N_2^2 - 2)\; + O\left(\frac{1}{k^3}\right)\right) \tag{3.68}$$

We note that at tree level it reproduces correctly the central for a free theory of $4(N_1 \times N_2)$ chiral multiplets, in our conventions, while for $N_1 = N_2 = 2$, we correctly recover the two-loop approximation of c_T in eq. (5.20) of [108].

3.3.2 The Matrix Model expansion at weak coupling

We are almost ready to test identity (3.32). The last ingredient that we need is the weak coupling expansion of the mass-deformed Matrix Model of ABJ(M) on S^3 [23, 24, 167] and its second derivatives with respect to the masses, to be compared with the perturbative result (3.66) for the topological two-point function.

To this end we consider the mass-deformed Matrix Model of the ABJ(M) theory [23, 24, 167]

$$Z = \frac{1}{(N!)^2}\int d\lambda\, d\mu\, \frac{e^{i\pi k \sum_i(\lambda_i^2-\mu_i^2)}\prod_{i<j} 16\sinh^2\left[\pi\left(\lambda_i - \lambda_j\right)\right]\sinh^2\left[\pi\left(\mu_i - \mu_j\right)\right]}{\prod_{i,j} 4\cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_+}{2}\right]\cosh\left[\pi(\lambda_i - \mu_j) + \frac{\pi m_-}{2}\right]} \tag{3.69}$$

where the mass assignment is the one recalled around Eq (1.49) [41]. Taking derivatives with respect to m_- (the same result would rise taking derivatives with respect to m_+) we immediately find

$$\frac{\partial^2}{\partial m_-^2}\log Z[S^3, m_\pm]\bigg|_{m_\pm=0} = \frac{Z''}{Z} - \left(\frac{Z'}{Z}\right)^2 \tag{3.70}$$

where Z is the undeformed MM, whereas its derivatives are given by

$$Z' = -\frac{1}{(N!)^2} \int d\lambda\, d\mu\, e^{i\pi k \sum_i (\lambda_i^2 - \mu_i^2)}\, Z_{1-\text{loop}}(\lambda_i,\, \mu_j) \sum_{i,j} \tanh \pi(\lambda_i - \mu_j) \tag{3.71}$$

$$Z'' = \frac{1}{(N!)^2} \int d\lambda\, d\mu\, e^{i\pi k \sum_i (\lambda_i^2 - \mu_i^2)}\, Z_{1-\text{loop}}(\lambda_i,\, \mu_j) \tag{3.72}$$
$$\times \frac{\pi^2}{4} \left(\left(\sum_{i,j} \tanh(\pi(\lambda_i - \mu_j)) \right)^2 - \sum_{i,j} \frac{1}{\cosh^2(\pi\,(\lambda_i - \mu_j))} \right)$$

with

$$Z_{1-\text{loop}}(\lambda_i,\, \mu_j) = \frac{\prod_{i<j} 16 \sinh^2 [\pi\,(\lambda_i - \lambda_j)] \sinh^2 [\pi\,(\mu_i - \mu_j)]}{\prod_{i,j} 4 \cosh(\pi\,(\lambda_i - \mu_j)) \cosh(\pi\,(\lambda_i - \mu_j))} \tag{3.73}$$

Since the integrand in Z' is odd under $\lambda \leftrightarrow \mu$ exchange, it vanishes once integrated. Thus we only need to compute contribution (3.72). Performing the following change of variables

$$x_i = \pi\sqrt{k}\lambda_i\,, \qquad y_j = \pi\sqrt{k}\mu_j\,, \qquad g_s = \frac{1}{\sqrt{k}} \tag{3.74}$$

the relevant quantities become

$$Z = \int dX\, dY e^{\frac{i}{\pi} \sum_i (x_i^2 - y_i^2)} f(x,y) \tag{3.75}$$
$$Z'' = \int dX\, dY e^{\frac{i}{\pi} \sum_i (x_i^2 - y_i^2)} f(x,y)\ \frac{\pi^2}{4} \left(\left(\sum_{i,j} \tanh(g_s(x_i - y_j)) \right)^2 - \sum_{i,j} \frac{1}{\cosh^2(g_s\,(x_i - y_j))} \right)$$

where dX, dY are the Haar measures and

$$f(x,y) = \prod_{i<j} \frac{\sinh^2(g_s(x_i - x_j))}{g_s^2(x_i - x_j)^2} \frac{\sinh^2(g_s(y_i - y_j))}{g_s^2(y_i - y_j)^2} \frac{1}{\prod_{i,j} \cosh^2(g_s(x_i - y_j))} \tag{3.76}$$

In order to compute Z and Z'', we find it convenient to canonically normalize them as $Z'' \to Z''/Z_0 \equiv \mathcal{Z}''$, $Z \to Z/Z_0 \equiv \mathcal{Z}$ where

$$Z_0 \equiv \int dX dY e^{\frac{i}{\pi} \sum_i (x_i^2 - y_i^2)} \tag{3.77}$$

is the free partition function. By perturbatively expanding the integrands in (3.75) up to $g_s^4 \sim \frac{1}{k^2}$, i.e. at two loops, and evaluating the normalized Gaussian matrix integrals, we obtain

$$
\begin{aligned}
\mathcal{Z}'' = -\frac{\pi^2}{4} N_1 N_2 \Big[1 + g_s^2 \, \frac{i\pi}{6} \, (N_2 - N_1) \, \left(1 - (N_2 - N_1)^2\right) \\
- g_s^4 \, \frac{\pi^2}{72} \Big(-24 + 16 N_2^2 - 12 N_1 (N_2 - N_1) + N_2^4 + 6 N_2^2 N_1^2 + 2 N_2 N_1^3 - N_1^4 \\
+ (N_2 - N_1)^6 \Big) + O(g_s^6) \Big]
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{\mathcal{Z}} = 1 - g_s^2 \, \frac{i\pi}{6} \, (N_2 - N_1) \left(1 - (N_2 - N_1)^2\right) \\
- g_s^4 \, \frac{\pi^2}{72} \Big(-2(N_2^2 - N_1^2) + 8 N_2 N_1 - 5 N_2^4 + 2 N_2 N_1 (N_2 - N_1)(8 N_2 - 7 N_1) - 3 N_1^4 \\
+ (N_2 - N_1)^6 \Big) + O(g_s^6)
\end{aligned}
\tag{3.78}
$$

If we now substitute back $g_s \to \frac{1}{\sqrt{k}}$, the final result reads

$$
\frac{1}{\pi^2} \frac{\partial^2}{\partial m_-^2} \log Z[S^3, m_\pm] \Big|_{m_\pm = 0} = \frac{1}{\pi^2} \frac{\mathcal{Z}''}{\mathcal{Z}} = -\frac{N_1 N_2}{4} \left(1 - \frac{\pi^2}{6k^2} (N_1^2 + N_2^2 - 2) \; + O\left(\frac{1}{k^3}\right) \right)
\tag{3.79}
$$

It is then easy to see that this expression coincides with the perturbative result (3.66) integrated twice on the great circle. We have thus checked identity (3.32) at the perturbative level.

Just to complete the picture, the central charge in (3.68) indeed satisfies the identity

$$
c_T = -\frac{64}{\pi^2} \frac{\partial^2}{\partial m_-^2} \log Z[S^3, m_\pm] \Big|_{m_\pm = 0}
\tag{3.80}
$$

in agreement with the general finding of [170].

From the general structure of the partition function in (3.69) it is easy to see that all the odd-order mass derivatives evaluated at $m_\pm = 0$ vanish identically due to symmetry reasons. Therefore, odd topological correlators should vanish at any order in loops, in agreement with our findings of section 3.3. An even number of derivatives are instead non-vanishing and can be used to obtain predictions for even topological correlators at weak coupling.

Here we consider the simplest case beyond the two-point function, that is the connected four-point function. Using the prescription in (3.32) for the two-point function in an obvious way, we can write

$$
\langle \mathcal{O}(\tau_1) \mathcal{O}(\tau_2) \mathcal{O}(\tau_3) \mathcal{O}(0) \rangle = \frac{1}{\pi^4 (2\pi)^4} \frac{\partial^4 \log Z}{\partial m_-^4} \Big|_{m_- = 0} = \frac{1}{\pi^4 (2\pi)^4} \left(\frac{\mathcal{Z}''''}{\mathcal{Z}} - 3 \left(\frac{\mathcal{Z}''}{\mathcal{Z}} \right)^2 \right)
\tag{3.81}
$$

where we have used $Z' = 0$ and normalized everything by Z_0, eq. (3.77). The second term can be easily recognized to be three times the square of the two-point function (3.70), thus this expression computes correctly the connected correlation function.

Evaluating explicitly $\mathcal{Z}''''$ at order g_s^4 and using (3.78), we obtain a two-loop prediction for the four-point topological correlator

$$\langle \mathcal{O}(\tau_1)\mathcal{O}(\tau_2)\mathcal{O}(\tau_3)\mathcal{O}(0)\rangle = 2\frac{N_1 N_2}{(4\pi)^4} - \frac{N_1 N_2(N_1^2 + N_2^2 - 2)}{192\pi^2\ k^2} + O\left(\frac{1}{k^3}\right) \tag{3.82}$$

We note that up to one loop it agrees with our perturbative result (3.64), whereas the $\frac{1}{k^2}$ term is a new non-trivial result that should be checked against a genuine two-loop calculation.

In this first part of the thesis, we focussed on the local properties of the SCFT. The main achievements are the proof and non-trivial tests of the formula (3.32). In the next part, we will concentrate on different types of one-dimensional subsectors, i.e. line operators, which modifies also the vacuum of the theory.

Chapter 4

BPS line operators and superconformal defects

In the second part of the thesis, we focus on a different type 1d subsectors, those associated with extended line defects. A defect is a modification induced by a system extended on a submanifold that breaks translational symmetries. If the submanifold is one-dimensional, we have a line defect. Other well-known defects are boundaries, interfaces, domain walls, and surface operators. A concrete example of a line defect is an impurity in condensed matter. In this thesis, we mainly adopt the QFT point of view, and we treat line defects as operators. Nevertheless, we will use both terminology interchangeably.

The prototypical example of extended operator in gauge theory is Wilson loops [45]. They are interesting observables, defined as the trace of the holonomy of the gauge connection. Sometimes they are the only available, like in topological theories (see the discussion for pure Chern-Simons in Sec. 1.4). Physically, a Wilson loop measures the effect of the insertion of a heavy electrically charged probe. Another famous application relates the expectation value of the rectangular Wilson loop to the static potential of the theory. Wilson loops do not exhaust the examples of line operators. Another class of line defects can be specified by certain singularities along the path, like 't Hooft loops in 4d [52] and vortex loops in 3d [191]. All these defects probe aspects of the theory related to the global features not accessible by local correlation functions. They are natural candidates to detect different phases of the theory, for instance, associated with global higher form symmetries [46].

The presence of a defect causes the breaking of the translational symmetries in the directions orthogonal to the defects. Eventually, other symmetries can be broken. In CFTs an interesting class of extended operators is those preserving the conformal group along the submanifold where the defect lives. These defects are called *conformal defects* [49–51]. The larger symmetry allows us to constraint their dynamics, similarly to CFTs. This fact provides a scheme to organize our computations and select universal observables.

In supersymmetric theories, we will concentrate on defects preserving a fraction of the supersymmetry. Perhaps the most standard examples are BPS Wilson loops [47, 48], but there are also supersymmetric boundaries [192] and other extended BPS operators [53, 54, 56–58]. An important motivation to study BPS Wilson loops comes from AdS/CFT correspondence, whose standard dictionary relates them to the fundamental string solution in the dual theory. We can analyze BPS extended operators with the various methods of supersymmetric theories.

Often maximally supersymmetric defects in SCTFs turn out to preserve superconformal symmetry along their worldvolume. In this case, we can use the computational power of supersymmetry to access the physics of the conformal defect [75–78]. This will be one of the main goals of the rest of the thesis. The other objective will be to relate some of the defects to the web of dualities sketched in Chapter 1.

In this chapter, we limit to review the basic ingredients. In Section 4.1 we introduce the main characters, i.e. BPS line defects in ABJM. Then, in Section 4.2 we review universal features of conformal defects, introducing relevant and universal observables. In section 4.3 we adapt the discussion to superconformal defects and reinterpret BPS Wilson loops in this direction. In section 4.4 we end the chapter by discussing an interesting connection with the cusp anomalous dimension and the radiation problem.

4.1 BPS loop operators in 3d gauge theories

In this section, we describe supersymmetric line operators in 3d supersymmetric gauge theories. We focus on BPS Wilson loops and BPS vortex loops. After the definition, we discuss how to compute them via localization.

4.1.1 BPS Wilson loops

Wilson loops are crucial observables in gauge theories, but they do not preserve any supersymmetry. We can build a BPS version of these operators adding extra scalar fields in the connection. Their SUSY variation will compensate the one of the vector field [1]. In three dimensions, the first example of a supersymmetric Wilson loop was introduced in $\mathcal{N} = 2$ theories [65]. The operator is given by

$$W = \mathrm{Tr}_R \, P e^{-i \int dt (\dot{x}^\mu A_\mu + i |\dot{x}| \sigma)} \,, \tag{4.1}$$

where the trace is taken in a generic representation R of the gauge group. It deviates from the standard definition because of the presence of σ in the connection. The operator is locally $\frac{1}{2}$-BPS, provided that

$$\frac{\dot{x}^\mu \gamma_\mu}{|\dot{x}|} \varepsilon = -\varepsilon \,, \tag{4.2}$$

and similarly for $\bar{\varepsilon}$. If the path is a straight line, the solution is globally $\frac{1}{2}$-BPS. In SCFTs, there are two extra solutions associated with superconformal supercharges. Moreover, in that case the line can be conformally mapped to a circle [2]. This loop operator exists in any $\mathcal{N} \geq 2$ Chern-Simons matter theory.

[1] In 4d $\mathcal{N} = 4$ SYM the procedure corresponds to consider the holonomy of a massive W boson, inserted in the theory by a suitable Higgs mechanism. In 3d, a similar interpretation is available for fermionic Wilson loops, introduced below [193].

[2] Large conformal transformations will not preserve the expectation value of the BPS Wilson line as pointed out in [194] in the four-dimensional case.

In Chern-Simon matter theories with $\mathcal{N} \geq 4$ supersymmetry, the scalar σ is integrated out. However, we can define a BPS operator by replacing σ with its on-shell value. For concreteness, we will focus on the ABJM case (see [70] for a recent review). The Wilson loop reads

$$W = \mathrm{Tr}_R \, P \exp\left(-i \int dt \left(\dot{x}^\mu A_\mu - \frac{2\pi i}{k} |\dot{x}| M_I{}^J C_J \bar{C}^I\right)\right) . \tag{4.3}$$

where $M_I{}^J = \mathrm{diag}(1, -1, 1, -1)$ and the curve is either a straight line or a circle. Since the gauge group of these theories is given by a product of two groups, we can define this type of Wilson loop for each group factor. According the convention of Appendix A, Eq (4.3) defines a Wilson loop for the $U(N_1)$ factor. The construction for $U(N_2)$ is analog, up to deform the $U(N_2)$ holonomy with the bilinear $\frac{2\pi i}{k}|\dot{x}| M_I{}^J \bar{C}^I C_J$, which transforms in the adjoint of $U(N_2)$.

The last observation is somewhat puzzling. According to the AdS/CFT dictionary, there should be a $\frac{1}{2}$-BPS dual to the fundamental string. Such an operator should also preserve $SU(3)$ R-symmetry, while the one of Eq (4.3) preserves only $SU(2) \times SU(2)$. The puzzle is solved in [66]. The idea is to define the $\frac{1}{2}$-BPS Wilson loop as the holonomy of a superconnection $\mathcal{L}$ of a supergroup $U(N_1|N_2)$, where the gauge group $U(N_1) \times U(N_2)$ is naturally embedded. The operator reads

$$W_{\mathrm{F}} = \mathrm{STr}_R \left(P \exp\left(-i \int dt \mathcal{L}(t)\right)\right) , \tag{4.4}$$

where STr is the supertrace. The superconnection $L(\tau)$ is a $U(N_1|N_2)$ supermatrix

$$\mathcal{L} \;=\; \begin{pmatrix} \mathcal{A} & i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I \bar{\psi}^I \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I \bar{\eta}^I & \hat{\mathcal{A}} \end{pmatrix} , \quad \text{with} \quad \begin{cases} \mathcal{A} \equiv A_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}| \mathcal{M}_J{}^I C_I \bar{C}^J \\ \\ \hat{\mathcal{A}} \equiv \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}| \mathcal{M}_J{}^I \bar{C}^J C_I \end{cases} , \tag{4.5}$$

We observe that it involves the fermion fields ψ^I and $\bar{\psi}_I$ in the off-diagonal positions and a standard bosonic connection on the diagonal. For this reason, we will call the BPS Wilson loops of this type *fermionic Wilson loops* and the standard ones of 4.3 bosonic.

One has also to weaken the BPS conditions. The usual BPS equations require the SUSY variation of the Wilson loop connection to be zero. In this case, the loop operator is supersymmetric regardless of the presence of a trace. However, the gauge invariance of the fermionic operator allows us to consider a less stringent condition in which the variation is a (super-)gauge transformation, namely

$$\delta L = \mathcal{D}_t G(t) = \partial_t G(t) + i[\mathcal{L}, G(t)] , \tag{4.6}$$

where G is a suitable supermatrix, whose explicit form is not relevant for the thesis[3].

[3] Since G might be not periodic, for the circular path we also need to modify the definition of the trace by a suitable twist matrix, which ensures gauge invariance. Recently, this matrix was reinterpreted as a

Solving Eq (4.6) determines for specific path and couplings ${\mathcal{M}_I}^J$ and η^I_α. A pretty large family of them is identified in [70]. For instance, The 1/2-BPS circular Wilson loop is given by

$$
{\mathcal{M}_I}^J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \eta_I^\alpha \equiv n_I \eta^\alpha = \frac{e^{\frac{i\tau}{2}}}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}_I (1, -ie^{-i\tau})^\alpha ,
$$
$$
\bar{\eta}^I_\alpha \equiv \bar{n}^I \bar{\eta}_\alpha = i(\eta_I^\alpha)^\dagger . \tag{4.7}
$$

This fermionic Wilson loop is related to the bosonic ones by a cohomological equivalence

$$
W_{\mathrm{F}} = \frac{W_{\mathrm{B}} + \hat{W}_{\mathrm{B}}}{2} + \delta V , \tag{4.8}
$$

where δ refers to the supercharge used to localize the $\mathcal{N} = 2$ theories, W_{B} $\hat{W}_{\mathrm{B}}$ are the bosonic Wilson loop respectively for $U(N_1)$ and $U(N_2)$, and V is a function whose explicit form is not needed [4]. Similar constructions exist in Chern-Simons matter theory with $\mathcal{N} = 4$ supersymmetry [200, 201].

Localization for BPS Wilson loops

We can compute BPS Wilson loops in 3d SCFTs via localization on S^3. In Chern-Simons matter theory, their expectation value is an interesting *interpolating function* between the weak coupling $k \gg 1$ and the strong coupling $k \ll 1$. Moreover, the result has applications in 3d dualities and holography. Then, we review the story for the circular Wilson loops in ABJM.

We start with is the definition of the (bosonic) BPS Wilson loop on S^3 by mimicing the flat space one

$$
W = \mathrm{Tr}_R \, P e^{-i \int dt (\dot{x}^\mu A_\mu + i|\dot{x}|\sigma)} , \tag{4.9}
$$

Supposing that t is the arc length, we need to solve the equation $\dot{x}^\mu \gamma_\mu \varepsilon = -\varepsilon$. We solve it by choosing a basis of Killing spinors such that ε is constant. It follows that $\dot{x}^\mu \gamma_\mu$ must be constant. This requires $\dot{x}^\mu$ to be parallel to a constant combination of the vielbein e^μ_i. Without loss of generality, we choose it to be e^μ_3. Then ε is just the eigenvector of $\gamma_3 = \sigma_3$ with eigenvalue -1. The integral curves of $\dot{x}^\mu$ are great circles, and the operator is $\frac{1}{2}$-BPS. If the theory is superconformal, there are two additional solutions. Then, the loop preserves superconformal supersymmetry along its worldline. Since the Wilson loop preserves the localizing supercharge, we can apply localization directly [22]. The expectation value is

background connection [195]. These subtleties are not relevant for this thesis.

[4] To be more precise, there exists a family of fermionic Wilson loops depending on continuous parameters [196–198]. For a certain specific value of these, we recover the combination of Wilson loops appearing in the r.h.s. of the cohomological equivalence (4.8). For a different value, there is a supersymmetry enhancement to the $\frac{1}{2}$-BPS Wilson loop. It is possible to argue that this Wilson loops can be thought of as an exactly marginal deformation of the $\frac{1}{2}$-BPS Wilson loop [199].

given by the matrix model average of the character $\mathrm{Tr}_R\, e^{2\pi\sigma_0}$

$$\langle W\rangle = \frac{1}{Z}\int d\sigma_0\, e^{-i\pi\,\mathrm{Tr}_{\mathrm{CS}}\,\sigma_0}\,\mathrm{Tr}_R\left(e^{2\pi\sigma_0}\right) Z_{\mathrm{vec}}(\sigma_0) Z_{\mathrm{mat}}(\sigma_0)\,. \tag{4.10}$$

We now focus on the specific ABJM model. One can exploit the cohomological equivalence of (4.8) to compute the $\frac{1}{2}$-BPS fermionic Wilson loop [5]. For the sake of simplicity, we limit to the fundamental representation, which is dual to the fundamental string in type IIA string theory. The expectation value of the Wilson loop is

$$\langle W\rangle = \frac{1}{2}\langle\sum_i^{N_1} e^{2\pi\lambda_i} + \sum_j^{N_2} e^{2\pi\mu_j}\rangle_{\mathrm{MM}}\,. \tag{4.11}$$

where $\langle\rangle_{\mathrm{MM}}$ indicates the average on the ABJM matrix model of Eq (2.81). The generalization to an arbitrary representation is straightforward. The Wilson loop has been computed to all perturbative order in $1/N$ using the Fermi gas method, and the result is expressed as an Airy function [68]. The result has passed several checks at weak and strong coupling [67, 203–205].

The situation at weak coupling is rather involved. Due to the Chern-Simon term, an additional framing factor for the loop has to be specified. Let us take for a moment a standard Wilson loop in a Chern-Simons theory without matter along a loop Γ. The computation of its expectation value requires the point-splitting regularization. The latter is encoded in a deformed path Γ_f. This ambiguity is formally encoded in the Gauss linking integral $\mathrm{link}(\Gamma,\Gamma_f)$, which is topological invariant. Its integer value f is the framing of the loop. The choice of the framing affects the phase of the expectation value [117]. For instance, for the $SU(N)$ theory the vev of a Wilson loop changes as follows [206]

$$\langle W_\Gamma\rangle_f = e^{\frac{i\pi N}{k}f}\langle W_\Gamma\rangle_{f=0}\,. \tag{4.12}$$

A similar phenomenon happens also for BPS Wilson loops in Chern-Simons matter theory [22, 207]. From the computation of the expectation value of the circular Wilson loop in pure Chern-Simons theory with localization, one deduces that $f = -1$ is implicitly selected. The interpretation is that regularization has to be compatible with supersymmetry. Point splitting regularization requires that the nearby loop come from the Hopf fibration. This fact explains why we have linking number -1 for the Wilson loop. Perturbative computations are usually done at framing zero. Precision tests must keep framing into account. Moreover, at the quantum level, the cohomological equivalence is realized at framing -1.

4.1.2 BPS vortex loops

Wilson loops do not exhaust all the possible line defects. Another compelling class of codimension two operators is vortex loops. They are disorder operators defined by a vortex-like singularity for the gauge field along a one-dimensional curve in spacetime. Vortex operators were originally defined in pure Chern-Simons theory, where they are equivalent

[5] See also an alternative approach which does not require cohomological equivalence [202]

to Wilson loops [191]. Given a loop γ and an element β of the gauge algebra, we require that for any small loop linking γ the holonomy of the gauge connection A_μ approach β. The parameter β is called vorticity and can be rotated in a Cartan subalgebra of the gauge group.

For concreteness, we take the curve to be along the x_3 axis. We define polar coordinates r, φ in the orthogonal plane. The vortex loop is defined by a singular gauge field $A \sim \beta d\varphi$, where $\sim$ means up to regular terms. The corresponding field strength is

$$F_{12} = 2\pi\beta\, \delta^2(x_1, x_2)\,, \tag{4.13}$$

where $\delta^2(x_1, x_2)$ is the delta function whose support is the x_3-line. The expectation value of the operator is given by the path integral evaluated on such singular configurations. This definition mimics that of surface operators in four dimensions [54]. See also [208, 209] for the analog construction in 2d in the same context.

We want this prescription to be compatible with supersymmetry [56–58]. This is equivalent to setting to zero the variation of the gauginos for some supercharges. Let us focus on $\mathcal{N} = 2$ gauge theories. We make the operator supersymmetric by turning on an imaginary value for D

$$D = 2\pi\beta\, \delta^2(x_1, x_2) = 2F_{12}\,. \tag{4.14}$$

If we set all the other other fields to zero, the vortex operator preserves preserves two supercharges, as one can see from the variations of the gauginos in Eq (2.29). [6]

It is possible to perform localization in the singular background for a background flavor symmetry. First of all, we place the loop on a great circle of S^3 parametrized by a coordinate φ. We need to keep into account that the modes have a modified periodicity. For instance, for a scalar mode $\phi(\theta, \varphi, \tau)$ in a chiral multiplet transforming in a weight ω of the flavor symmetry group we have

$$\phi(\theta, \varphi + 2\pi, \tau) = e^{2\pi i\omega(\beta)}\phi(\theta, \varphi, \tau)\,. \tag{4.15}$$

One can show that this modifies the 1-loop determinant as follows

$$Z^{\text{vort}}_{\text{chi}}(\sigma_0) = \prod_{\rho,\omega} s_{b=1}(i(1-\Delta) - \rho(\sigma_0) + i\omega(\beta))\,. \tag{4.16}$$

The same result applies to gauge vortex operators in abelian theories.

An alternative prescription is proposed in [61]. The vortex loop is represented by adding one-dimensional local degrees of freedom on the curve supporting the loop operators. Integrating out the 1d d.o.f. reproduces the vortex singularity for the bulk fields. One advantage of this definition is that the mixed 1d/3d theory can be read from certain brane configurations in type IIB string theory. This is a great advantage to test dualities with a

[6] Vortex lines can be generalized by allowing for singularities in the matter fields of the theory, depending on additional parameters. In ABJM there are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$ of this type. For instance, the maximally supersymmetric vortex line is characterized by a singular behavior for $C_1 \sim \alpha/\sqrt{z}$, where z is the complex coordinate in the plane orthogonal to the vortex line.

string theory incarnation.

We can compute the expectation value of the vortex loop with localization from the alternative definition. We can define the system on S^3 by turning on background fields to make the 3d/1d theory invariant under the supersymmetry algebra of the Vortex loop, following the idea of Section 2.2. The vev of the vortex loop corresponds to the partition function of the mixed 3d/1d system. The partition function of the SQM admits an index interpretation. It was computed with localization in [210, 211]. The coupling between the bulk and the SQM does not affect the reciprocal saddle points and hence the reciprocal localization procedure. The coupling displays a role because SQM flavor symmetries are gauged by the bulk vector. Therefore, the 1d masses must coincide with the bulk vector moduli. Then, we express the result as a matrix model

$$\langle V \rangle = \frac{1}{Z_{3d}} \int d\sigma_0 Z^{3d}_{\text{1-loop}} \mathcal{I}(\sigma_0) \,, \tag{4.17}$$

where Z_{3d} is the partition function of the bulk theory, $Z^{3d}_{\text{1-loop}}$ is the 1-loop determinant of the bulk theory, and $\mathcal{I}(\sigma_0)$ is the index of the 1d theory. We will provide a concrete computation using (4.17) in Section 6.1.2. We remark that there is also a proposal for the expectation value of non-abelian gauge vortex loops based on the first definition in [58]. Recently, progresses to clarify the relation among the two computations appeared in [212].

4.2 Conformal defects

We consider the insertion of operators that breaks the conformal group $SO(d+1,1)$ to $SO(p+1,1) \times SO(q)$, with $p+q=d$, following [51]. This pattern of symmetry breaking corresponds to the insertion of an extended operator of dimension p or codimension $q=d-p$. The defect preserves the worldvolume conformal symmetry and the transverse Lorentz group. A typical situation is a p dimensional flat defect in $\mathbb{R}^d$. We shall discuss how the paradigm of CFTs gets modified by the insertion of conformal defects. The knowledge of the dimensions of all bulk operators and all their three-point functions will no longer exhaust the possible CFT data. First of all, the restricted symmetry group allows for a non-vanishing one-point function. Moreover, the defect supports local excitations $\hat{O}_i$, whose conformal weights $\hat{\Delta}_i$ are a priori independent of the bulk theory.

For the sake of simplicity, let us concentrate on the flat space setup. The discussion can be applied without conceptual modification to any conformal flat space. We will split the coordinates $x^\mu = (x^i, x^a)$, where x^i indicates the transverse coordinates $i = 1, \dots, q$ and x^a the coordinates along the defects $a = 1, \dots, q$. The defect is placed at the position $x_i = 0$. We will consider correlation functions like

$$\langle O_1(x_1) \dots O_n(x_n) \hat{O}_1(x_1^{a_1}) \dots \hat{O}_k(x_k^{a_k}) \rangle_{\mathcal{D}} = \frac{1}{\langle \mathcal{D} \rangle} \langle O_1(x_1) \dots O_n(x_n) \hat{O}_1(x_1^{a_1}) \dots \hat{O}_k(x_k^{a_k}) \mathcal{D} \rangle \,. \tag{4.18}$$

where $\mathcal{D}$ indicates the presence of the defect. We also introduce the notation $\langle \rangle_{\mathcal{D}}$ for correlators in the normalized defect vacuum.

In presence of a defect, the bulk one-point functions are different from zero. For scalar operators, it is not hard to show that

$$\langle O(x)\rangle_{\mathcal{D}} = \frac{h_O}{|x^i|^{\Delta_O}}\,. \tag{4.19}$$

where Δ_O is the dimension of O and $|x^i|$ its distance from the defect. We stress that h_O is a new dynamical parameter that characterizes the defect CFT. A full-fledged discussion of spinning correlators would require the embedding formalism. Here, we limit to the one-point function for the stress tensor. One can show that (see e.g. [53])

$$\langle T^{\mu\nu}(x)\rangle_{\mathcal{D}} = \frac{h_T}{|x^i|^d} H^{\mu\nu}\,. \tag{4.20}$$

$H^{\mu\nu}$ is a symmetric traceless tensor

$$H^{ij} = -\left(\frac{p+1}{d}\delta^{ij} - \frac{x^i x^j}{|x^i|^2}\right)\,, \quad H^{ab} = \frac{q-1}{d}\delta^{ab}\,, \quad H^{ia} = 0 \tag{4.21}$$

The stress tensor multiplet exists in any local theory. Therefore, h_T is a universal quantity. Higher spinning operators can have a non-vanishing one-point function if one manages to build a tensor in the appropriate representation.

The second novelty is the possibility of defect excitations, represented by operators $\hat{O}(x^a)$ living on the defects. They are constrained in the usual way by the defect conformal group $SO(p+1,1)$. Therefore, they form a (peculiar) theory called the defect conformal field theory (dCFT). The dCFT satisfies the axioms of a standard CFT (see section 1.2). In particular, it will be determined by the defect spectrum $\Delta_{\hat{O}}$ and by the defect three-point functions $\hat{c}_{\hat{O}_i,\hat{O}_j,\hat{O}_k}$. The transverse Lorentz group $SO(q)$ acts as a flavor group. The presence of localized excitations leads to a bulk-to-defect OPE. For a scalar bulk operator of dimension Δ, it reads

$$O(x^a, x^i) \sim b_{O\hat{O}} |x_i|^{\hat{\Delta}-\Delta} \hat{O}(x_a) + \cdots \tag{4.22}$$

where $\hat{\Delta}$ is the dimension of the operator $\hat{O}(x_a)$. The OPE can be understood as follows. If we bring the bulk operator close to the defect, it becomes indistinguishable from defect excitations. In this sense, a non-vanishing one-point function is a non-trivial OPE among the bulk operator and the defect identity. We stress that $b_{O\hat{O}}$ is a dynamical parameters, analogously to the coefficient of three-point functions in standard CFTs. Then, we have a set of novel CFT data specified by h_O, $\Delta_{\hat{O}}$, $b_{O\hat{O}}$, $\hat{c}_{\hat{O}_i,\hat{O}_j,\hat{O}_k}$.

Finally, the two-point functions in the bulk get modified non-trivially. They are no longer completely fixed by conformal invariance. They will depend on a function of two cross-ratios. We do not need to go through it in detail. We limit to observe that the two point functions must obey a crossing symmetry. One can start using the OPE in the bulk and then in the bulk-to-defect channel. Alternatively, one can make the bulk-to-defect OPE, and then a defect OPE. This crossing symmetry is a powerful constraint for the dCFT data, similarly to the bootstrap philosophy. For this reason, this approach is called defect conformal bootstrap [213].

4.2.1 The displacement operator in dCFTs

In this section, we analyze the consequences of broken symmetries in CFTs with conformal defects. The net effect will be a modification of the Ward identity by certain defect operators, which we aim to describe. Here we will focus on the most relevant one, namely the *displacement operator*. It is an operator defined for every dCFT. Physically, it regulates the energy exchanges with the bulk.

The displacement operator D^i is related to the broken transverse translations P^i. It is defined by the Ward identity

$$\partial_\mu T^{\mu i}(x) = -\delta_{\mathcal{D}}(x^i)\mathsf{D}^i(x^a)\,, \tag{4.23}$$

with $\delta_{\mathcal{D}}(x^a)$ is a delta function supported on the defect. We stress that the definition holds inside correlation functions. It describes the breaking of translational invariance due to the discontinuity induced by the defect. The displacement parametrizes the discontinuity. In other words, the bulk exchanges energy with the defect. An integral version of the Ward identity is

$$[P_i, \mathcal{D}] = \int_{\mathcal{D}} d^p x\, \mathsf{D}^i(x^a)\mathcal{D} \tag{4.24}$$

where P^i is defined as the corresponding generator in absence of the defect.

Ward identities for longitudinal translation and scale invariance are left unchanged [7]

$$\partial_\mu T^{\mu a}(x) = 0\,, \qquad T_\mu{}^\mu(x) = 0\,. \tag{4.25}$$

We observe that the displacement is defined up to primary defect operators. We fix them by requiring that $T^{\mu\nu}$ is a conformal primary near the defect. In turn, this implies that the displacement is a defect primary, whose dimension is fixed by the Ward identity (4.23) to be $\Delta_{\mathsf{D}} = p+1$. Being the normalization of the displacement fixed by the Ward identity, the coefficient c_{D} of its two-point function is physical

$$\langle \mathsf{D}^i(x^a)\mathsf{D}^i(y^b)\rangle_{\mathcal{D}} = \delta^{ij}\frac{c_{\mathsf{D}}}{|x-y|^{2\Delta_{\mathsf{D}}}}\,. \tag{4.26}$$

This situation is similar to the two-point function of the stress tensor.

So far the discussion was rather abstract. We can make it more concrete if we assume that the expectation value of the defect admits a Lagrangian representation. In this case, we can couple the theory to a background metric and derive explicitly the (defect) Ward identities. The defect is described by the embedding function $X^\mu = X^\mu(\sigma^a)$, where σ^a defines a set of local coordinates for the submanifold $\mathcal{D}$. Ward identities follow by imposing diffeomorphism invariance of the system. We recall that a diffeomorphism generated by a vector field ξ^μ acts infinitesimally on the embedding function as $\delta_\xi X^\mu = \xi^\mu$ and on the

[7] In principle, one can also consider the breaking of the transverse rotations. In this case, we would have another broken Ward identity of the form $T^{[ij]}(x) = \delta_{\mathcal{D}}(x^i)\lambda^{[ij]}(x^a)$. We do not consider this possibility.

metric as $\delta g_{\mu\nu} = -\nabla_{(\mu}\xi_{\nu)}$. We take ξ^μ to be a conformal Killing vector

$$\nabla_{(\mu}\xi_{\nu)} = -\frac{1}{d}\nabla_\lambda \xi^\lambda g_{\mu\nu} = -\sigma g_{\mu\nu}\,. \tag{4.27}$$

Since its action can be undone by a Weyl transformation $\delta_\sigma g_{\mu\nu} = \sigma g_{\mu\nu}$, $\delta_\xi + \delta_\sigma$ leaves the metric invariant. However, the Weyl transformation does not compensate the variation of the embedding function $X^\mu(\sigma^a)$ in the normal directions. If we implement the transformations in the correlator $\langle \mathcal{X}\rangle_{\mathcal{D}}$, where $\mathcal{X}$ is an arbitrary product of bulk operators, we find [51]

$$(\delta_\xi + \delta_\sigma)\langle \mathcal{X}\rangle_{\mathcal{D}} = \int \xi^i \langle \mathsf{D}_i(x^a)\mathcal{X}\rangle\,. \tag{4.28}$$

The meaning of the equation is that the displacement implements the action of broken symmetries. If the diffeomorphism is associated with a broken translation, one recovers (4.23) and gets an explicit expression for D_i.

Example

Let us take a free theory of scalar theories. We use the following normalization

$$S = \frac{1}{d-2)\Omega_{d-1}}\int d^dx\,\frac{1}{2}\partial_\mu\phi\partial^\mu\phi\,, \qquad \langle\phi(x)\phi(0)\rangle = \frac{1}{|x|^{2\Delta}} \tag{4.29}$$

where Ω_{d-1} is the volume of the sphere S^{d-1}. We insert the planar p-dimensional defect

$$O_{\mathcal{D}} = \exp\left(\lambda \int_{\mathcal{D}} d^px\,\phi(x)\right)\,. \tag{4.30}$$

The defect is conformal if $p = \Delta = \frac{d}{2} - 1$. A relevant case is $d = 4$ and $p = 1$, where the defect is a scalar Wilson line. The equations of motion are modified by the defect

$$\Box\phi = -\lambda(d-2)\Omega_{d-1}\delta_{\mathcal{D}}\,. \tag{4.31}$$

One can also consider the conservation law for the (improved) stress tensor

$$\partial_\mu T^{\mu\nu} = \frac{1}{(d-2)\Omega_{d-1}}\partial^\nu\phi\Box\phi\,. \tag{4.32}$$

Combining the two equations we read the displacement multiplet

$$\mathsf{D}_i = \lambda\,\partial_i\phi\,. \tag{4.33}$$

It is easy to compute

$$\langle \mathsf{D}_i(x^a)\mathsf{D}_i(y^b)\rangle_{\mathcal{D}} = \lambda^2\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\langle\phi(x)\phi(y)\rangle\Big|_{x_\perp=y_\perp=0}\,, \tag{4.34}$$

and verify the consistence with 4.26, provided that $c_{\mathsf{D}} = 2\lambda^2\Delta$.

4.3 Superconformal defects

From now on, we will focus on SCFTs. In this case, the defect breaks also a fraction of the supersymmetries. We are interested in conformal defects that preserve the worldvolume superconformal group, i.e. a supergroup containing the defect conformal group as a part of the bosonic subgroup. The exposition follows [199]. We restrict to defects preserving a fraction of the supercharges and a subgroup of the original R-symmetry. It follows that the defect excitations are organized in superconformal multiplets of the defect subalgebra. Multiplets are labeled by the quantum numbers of the subalgebra. If the codimension q is greater than 2, multiplets are identified also by their representation under the transverse rotation group. This group does not exist for $q = 1$. If $q = 2$ the group of the transverse rotation inexorably mixes with the R-symmetry one. Then, multiplets carry a charge under the mixed $U(1)$. The discussion is conceptually the same for standard SCFTs, including unitarity bounds. We will concentrate on universal multiplets associated with broken symmetries, like displacement.

Following the same logic of the displacement, let us introduce other relevant defect excitations related to the breaking of global symmetries. We expect the existence of an operators localized on the defect for every broken generator of the bulk superconformal symmetry. This is indeed the case. Every operator comes with a corresponding Ward identity. For a broken supercurrent S^μ_α we have

$$\partial_\mu S^\mu_\alpha = \delta_{\mathcal{D}}(x^i)\mathsf{S}_\alpha(x^a)\,, \qquad (\gamma^\mu)_\alpha{}^\beta S_{\mu\alpha} = 0\,. \tag{4.35}$$

S_α is a fermionic defect primary of dimension $\Delta_{\mathsf{S}} = p + \frac{1}{2}$. It implements the action of the broken supercharge on the defect

$$[Q_\alpha, \mathcal{D}] = \int_{\mathcal{D}} d^p x\, \mathsf{S}_\alpha(x^a)\mathcal{D}\,, \tag{4.36}$$

where Q_α is any broken supercharge. S_α will transform under the preserved defect superconformal group and the transverse rotation group. The last option we consider is the breaking of zero form symmetry current j^μ[8]. The related Ward identity is

$$\partial_\mu j^\mu = \delta_{\mathcal{D}}(x)\mathsf{J}(x^a)\,. \tag{4.37}$$

which leads to a scalar primary with dimension Δ_{J}. We also write the finite version of the Ward identity for the corresponding charge F

$$[F, \mathcal{D}] = \int_{\mathcal{D}} d^p x\, \mathsf{J}_F(x^a)\mathcal{D}\,. \tag{4.38}$$

This discussion is valid regardless of the specific origin of the symmetry, and so both for R-symmetry and flavor current.

The defect primaries fall in a multiplet structure. Moreover, they have general structural

[8] It is also possible for the defect to break a higher form symmetry. We do not discuss this issue here.

properties. They come from algebraic considerations, inherited from the full superconformal algebra without defect. For instance, one can consider the action of the preserved supercharges on the Ward identity for the displacement

$$[Q,[P_i,\mathcal{D}]] = \int_{\mathcal{D}} d^p x\,[Q,\mathsf{D}^i(x^a)]\,\mathcal{D} \tag{4.39}$$

The Jacobi identity on the l.h.s. gives two contributions. The first one involves $[Q,\mathcal{D}] = 0$. The second one $[Q,P_i] = 0$. We conclude the r.h.s. of (4.39) must vanish, and so

$$[Q,\mathsf{D}^i(x^a)] \equiv 0\,, \tag{4.40}$$

for any preserved supercharge. Then, the displacement operator $\mathsf{D}^i(x^a)$ is always a top component of a defect superconformal multiplet, which we will call displacement multiplet. The argument can be easily extended to any broken symmetry commuting with all the preserved supercharges. For instance, this is the case of broken flavor symmetries. The corresponding defect scalar $\mathsf{J}_F(x^a)$ will be the top component of a broken current multiplet. As $\Delta_{\mathsf{J}_F} = p$, the operator gives rise to a marginal supersymmetric deformation of the defect.

We apply this formalism to BPS Wilson lines in ABJM. In particular, the goal is the computation of the coefficient c_{D} using localization. In the following, we will explain how BPS line operators fit in the defect description.

4.3.1 Superconformal Wilson lines in ABJM

BPS Wilson lines in supersymmetric theories are examples of conformal defects with a Lagrangian description. In this case, we can give a rather explicit construction of the defect CFT and the corresponding multiplets, including the displacement. Let us introduce the notation

$$W_{t_1,t_2} = e^{i\int_{t_1}^{t_2} dt\,\mathcal{L}(t)}\,, \tag{4.41}$$

where $\mathcal{L}(t)$ is the connection of the Wilson loop. We indicate the endpoint with t_i and t_f. For a straight line we set $t_i = -\infty$, $t_f = +\infty$, for the circular loop $t_i = t_f$. We define the insertion of an operator in the dCFT as

$$W\left[O_1(t_1)\ldots O_n(t_n)\right] \equiv \mathrm{Tr}\,P\left[W_{t_i,t_1}O(t_1)W_{t_1,t_2}O(t_2)\ldots O_n(t_n)W_{t_n,t_f}\right]\,. \tag{4.42}$$

We observe that the local operators $O_i(t_i)$ are untraced. Therefore, they must transform in the same representation of the gauge group. Their correlation functions are defined as

$$\langle O_1(t_1)\ldots O_n(t_n)\rangle_W = \frac{\langle W\left[O_1(t_1)\ldots O_n(t_n)\right]\rangle}{\langle W\rangle}\,. \tag{4.43}$$

In order to build the generator associated with broken symmetries, one can use the relation

$$\delta\log W = -i\int_{t_i}^{t_f} dt\langle\delta\mathcal{L}\rangle_W\,. \tag{4.44}$$

If we choose the δ corresponding to a broken symmetry, we obtain an explicit representation of the associated defect operator. The defect theory might have a holographic interpretation as the fluctuations of the dual fundamental string. For an example in four dimensions see [214].

We specialize the discussion to ABJM, where we have two different types of superconformal Wilson lines.

Bosonic Wilson line

We take the bosonic $\frac{1}{6}$-BPS Wilson line of 4.3 along the x_3 line. The defect preserves two Poincaré supercharges Q and $\bar{Q}$ and two superconformal supercharges. They span the superconformal algebra $\mathfrak{su}(1,1|1)$, which contain the 1d conformal algebra $\mathfrak{sl}(2)$. Since the codimension is two, the superalgebra contains a mixed $\mathfrak{u}(1)$ factor, which is a combination of the transverse rotation generator and an abelian R-symmetry factor. Moreover, because of the specific form of $M_I{}^J$, there is also a global $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, which rotates C_1, C_3 and C_2, C_4 [78].

Using the relation (4.24) for $P_\mathrm{B} = P_1 - iP_2$, one can compute the corresponding components of the displacement operator D_+ and find [78]

$$\mathsf{D}_+ = i(F_{13} - iF_{23}) - \frac{2\pi}{k} D \left(M_I{}^J C_J \bar{C}^I \right) \tag{4.45}$$

where $D = D_1 - iD_2$ and are complex combinations of the covariant derivatives $F_{\mu\nu}$ is the field strength. A similar computation for $\bar{P}_\mathrm{B} = P_1 + iP_2$ leads to the other component D_- of the displacement. Using the superconformal algebra of ABJM, it is possible to argue that it is the top component of a short multiplet. It is possible to identify two broken supercharges Q_B, $\bar{Q}_\mathrm{B}$ which induce two conformal primaries F and $\bar{\mathsf{F}}$ through the usual mechanism. They close the algebra

$$\{Q, Q_\mathrm{B}\} = 2iP_\mathrm{B}\,, \qquad \{\bar{Q}, Q_\mathrm{B}\} = 0\,, \tag{4.46}$$

$$\{Q, \bar{Q}_\mathrm{B}\} = 0\,, \qquad \{\bar{Q}, \bar{Q}_\mathrm{B}\} = 2i\bar{P}_\mathrm{B}\,. \tag{4.47}$$

It is not hard to show that

$$\bar{Q}\mathsf{F} = 0\,, \qquad Q\mathsf{F} = 2\mathsf{D}_+\,. \tag{4.48}$$

In turn, this implies that F is the primary of a short multiplet whose top component is the displacement. A similar story holds for $\bar{\mathsf{F}}$ and D_-. Notice that the quantum numbers are given by the corresponding broken generators. Finally, in a similar fashion, one could derive the defect multiplets associated to the R-symmetry currents.

Fermionic Wilson line

We recall the defintion of the fermionic Wilson line B.14, characterized by the $U(N_1|N_2)$ superconnection. The $\frac{1}{2}$-BPS Wilson loop along the x_3 line is defined by the coupling

$$\mathcal{M}_I{}^J = \mathrm{diag}(-1,\,1,\,1,\,1)\,, \quad \eta_I^\alpha = \sqrt{2}\delta_I^1(1,\,0)^\alpha\,, \quad \bar{\eta}_\alpha^I = i\sqrt{2}\delta_1^I \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\alpha \tag{4.49}$$

The story for the half BPS fermionic line is conceptually similar but technically more complicated since the preserved supercharges do not annihilate the superconnection. The fermionic Wilson loop preserves the naive superalgebra $\mathfrak{su}(1,1|3)$, whose bosonic part is the usual $\mathfrak{sl}(2)$, the $\mathfrak{su}(3)$ R-symmetry algebra which rotates C_2, C_3 and C_4, and a $\mathfrak{u}(1)$ factor, which is the usual combination of R-symmetry and transverse rotation [77, 215]. This specific combination preserves the fermionic couplings of the connection. Even if the fermionic couplings seem to break the transverse rotation group, this is not the case. Indeed, one can compensate the transformation by a gauge transformation [199]. In conclusion, there is a further preserved $\mathfrak{u}(1)$ factor, which commutes with $\mathfrak{su}(1,1|3)$.

The derivation of the displacement multiplet requires additional work. It is essentially fixed by the action of the preserved supercharge on the broken generators [215]. It includes also the defect operators coming from the broken R-symmetry currents, and the broken supercharges. Its superconformal primary is given by

$$\mathsf{F} = i\sqrt{\frac{2\pi}{k}}\begin{pmatrix} 0 & C_1 \\ 0 & 0 \end{pmatrix}\,. \tag{4.50}$$

Unlike the bosonic Wilson line, it does not correspond to any broken symmetry. It is possible to check the multiplet structure by acting with the preserved supercharges. Again, we find a similar story for the other component of the displacement. The displacement supermultiplet admits also a nice interpretation as fluctuations of the fundamental string solution [215].

4.4 From the displacement to the Brehmsstrahlung

In this section, we provide a physical interpretation for the two-point function of the displacement operator of conformal Wilson line operators. Let us introduce an apparently unrelated observable, namely the *cusp anomalous dimension*. Let us consider a Wilson line with a cusp like in the figure 4.4. The expectation value exhibit a UV divergence, depending on the cusp angle φ, and on the coupling constant of the theory λ [216][9]. It is possible to show [217]

$$\langle W_{\mathrm{cusp}}\rangle \propto e^{-\Gamma_{\mathrm{cusp}}(\varphi,\lambda)\log\left(\frac{L}{\epsilon}\right)} \tag{4.51}$$

[9] In the case of the smooth path, the divergences are absorbed in a counterterm proportional to the length of the curve, which reads as a mass term for the heavy probe.

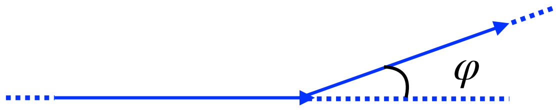

Figure 4.1: The picture is a cartoon of a cusped line with angle φ.

where ϵ is the UV cutoff, and L is the IR one. $\Gamma_{\text{cusp}}(\varphi, \lambda)$ is the cusp anomalous dimension and is an interesting observable of the gauge theory. For instance, it characterizes the IR divergences that arise when we scatter massive colored particles [218, 219]. It enters in the IR divergences of massless particles [220]. It is also related to the static quark potential.[10]

Here we will focus on its small angle limit, where

$$\Gamma_{\text{cusp}}(\lambda, \varphi) = -B(\lambda)\varphi^2 + O(\varphi^4)\,. \tag{4.52}$$

The function of the coupling $B(\lambda)$ is the *Bremsstrahlung function*. In CFTs, the Bremsstrahlung is in turn related to two quantities, the two-point function of the displacement operator c_{D} and the energy emitted by a heavy moving quark. The latter is governed by the following equation (assuming small velocities)

$$\Delta E = A \int dt |\dot{v}|^2 \tag{4.53}$$

for some constant A. We stress that this is relation valid in any CFT and for every one-dimensional defect.

The connection with the displacement comes from an explicit small φ expansion of the cusp anomalous dimension

$$\Gamma_{\text{cusp}}(\lambda, \varphi) \sim -\frac{1}{2}\varphi^2 \int dt \langle \mathsf{D}(t)\mathsf{D}(0)\rangle_W\,. \tag{4.54}$$

We used the fact that the first derivative is vanishing because of the conformal invariance along the loop. Integrating the explicit form of the point function of the displacement leads to [75]

$$c_{\mathsf{D}} = 12B \tag{4.55}$$

It is convenient to think of the line operator as the worldline of a heavy particle. We perform a small time-dependent displacement δx of the path. The probability for the Wilson loop to absorb a given quantum of energy is captured by the two-point function of the displacement operator, which is induced by δx. If we compare this quantity with the

[10] The idea is that in CFTs the plane to cylinder map brings the cusped Wilson loop to a quark anti-quark configuration. The two quarks extend along two $\mathbb{R}$-lines separated by an angle $\pi - \varphi$. Therefore, the cusp anomalous dimension gives the static quark potential associated with this configuration [75].

emitted radiation, we get

$$\Delta E = 2\pi B \int dt |\dot{v}|^2 = \pi \frac{c_{\mathsf{D}}}{6} \int dt |\dot{v}|^2 \,. \tag{4.56}$$

This formula provides a nice physical interpretation for the coefficient of the two-point function of the displacement operator. For a more thorough discussion on the radiation problem we refer to [221] and the references therein.

In the next chapter, we will describe how to relate the bremsstrahlung function to a localization computation.

Chapter 5

Localization for ABJM Latitude Wilson Loops

In this chapter, we study a generalization of the circular BPS Wilson loops (both bosonic and fermionic), which preserve less supersymmetry. These Wilson loops are called *latitude* Wilson loops [71], and depend on a real parameter $\nu \in (0, 1]$. The latitude loops are related to the physics of conformal defects: the ν derivative of its absolute value is proportional to the Bremsstrahlung function. For this reason, the latitude was studied both at weak and strong coupling[80–84]. The various computations lead to a conjecture for a matrix model that computes the vev of the latitude. The matrix integral is a ν-depending deformation of the ABJM matrix model [79] similar to the one coming from squashing.

We aim to put the proposal on more solid grounds, deriving the matrix model from a localization argument. It is a non-trivial task. The reason is that the latitude preserves a ν-dependent supercharge, which is not part of any $\mathcal{N} = 2$ subalgebra. Therefore, a new localization scheme is necessary. We attempt to fill the gap in this chapter. Along the way, we show that the latitude supercharge lives in an $\mathcal{N} = 4$ subalgebra. We use this property to generalize the definition of the latitude to generalized Gaiotto-Witten theories [85].

Somewhat surprisingly, the main obstruction we encounter to preforming these localization computations in ABJM is the difficulty of realizing a single generic supersymmetry transformation off-shell. We will then attempt to overcome the problems with off-shell closure using a cohomological version of Chern-Simons theory introduced by Källén in [97]. With some mild assumptions, we will reproduce the matrix model for ABJM put forward in [79], and provide the generalizations for matrix models of GW theories.

The chapter has the following structure. In the first section, we review the construction of the latitude and its relation with the bremsstrahlung. We also derive the preserved algebra and extend the (bosonic) latitude to the generalized GW theories. In the second section, we uplift the previous results to S^3. Finally, we describe our localization computation. This chapter is based on [185].

5.1 The ABJM model and the latitude loop

We describe in this section a class of Wilson loops in $U(N_1)_k \times U(N_2)_{-k}$ ABJM theory that preserve a certain fraction of the original $\mathcal{N}=6$ supersymmetry. They generalize the construction of the $\frac{1}{6}$ and $\frac{1}{2}$ BPS Wilson loops introduced in Chapter 4 .

We start by recalling the general class of Wilson operators introduced in [70]. They feature a parametric dependence on an α-angle[1] that governs the couplings to matter in the internal R-symmetry space and a geometric angle $\theta_0 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ that fixes the contour to be a latitude circle on the unit sphere

$$\Gamma: \quad x^\mu = (\cos\theta_0 \cos\tau, \cos\theta_0 \sin\tau, \sin\theta_0)\,, \qquad \tau \in [0, 2\pi)\,. \tag{5.1}$$

As discussed in [71], these operators can be constructed in such a way that they depend only on the effective "latitude parameter"

$$\nu \equiv \sin 2\alpha \cos\theta_0\,, \qquad 0 \le \nu \le 1\,. \tag{5.2}$$

The bosonic latitude Wilson loops corresponding to the two gauge groups are explicitly given by

$$\begin{aligned} W_B(\nu, R) &= \frac{1}{\dim(R)} \operatorname{Tr}_R \mathrm{P} \exp\left\{-i \oint_\Gamma d\tau \left(A_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J{}^I C_I \bar{C}^J\right)\right\}, \\ \hat{W}_B(\nu, \hat{R}) &= \frac{1}{\dim(\hat{R})} \operatorname{Tr}_{\hat{R}} \mathrm{P} \exp\left\{-i \oint_\Gamma d\tau \left(\hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k} |\dot{x}| M_J{}^I \bar{C}^J C_I\right)\right\}. \end{aligned} \tag{5.3}$$

where the matrix describing the coupling to the $(C_I, \bar{C}^I)$ scalars reads

$$M_J^{\ I} = \begin{pmatrix} -\nu & e^{-i\tau}\sqrt{1-\nu^2} & 0 & 0 \\ e^{i\tau}\sqrt{1-\nu^2} & \nu & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

The traces in (5.3) are taken over generic representations $R, \hat{R}$ of $U(N_1)$ and $U(N_2)$, respectively. The overall constants have been chosen in order to normalize the tree level expectation values $\langle W_B \rangle^{(0)}$ and $\langle \hat{W}_B \rangle^{(0)}$ to one.

The fermionic latitude Wilson loop is instead defined for a representation[2] $\mathbf{R}$ of the superalgebra $U(N_1|N_2)$

$$W_F(\nu, \mathbf{R}) = \mathcal{R}\, \mathrm{STr}_{\mathbf{R}} \left[\mathrm{P} \exp\left(-i \oint_\Gamma \mathcal{L}(\tau) d\tau\right) \begin{pmatrix} e^{-\frac{i\pi\nu}{2}} \mathbf{1}_{N_1} & 0 \\ 0 & e^{\frac{i\pi\nu}{2}} \mathbf{1}_{N_2} \end{pmatrix}\right], \tag{5.5}$$

[1] The α–angle can be freely chosen in the interval $[0, \frac{\pi}{2}]$, see [70].

[2] Special combinations of representations of $U(N_1) \times U(N_2)$ are also representations of $U(N_1|N_2)$: we will consider only the fundamental representation, that actually belongs to these particular cases

where $\mathcal{L}$ is the $U(N_1|N_2)$ superconnection

$$\mathcal{L} = \begin{pmatrix} \mathcal{A} & i\sqrt{\frac{2\pi}{k}}|\dot{x}|\eta_I\bar{\psi}^I \\ -i\sqrt{\frac{2\pi}{k}}|\dot{x}|\psi_I\bar{\eta}^I & \hat{\mathcal{A}} \end{pmatrix}, \quad \text{with} \quad \begin{cases} \mathcal{A} \equiv A_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|\mathcal{M}_J{}^I C_I \bar{C}^J \\ \\ \hat{\mathcal{A}} \equiv \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{k}|\dot{x}|\mathcal{M}_J{}^I \bar{C}^J C_I \end{cases}, \tag{5.6}$$

and

$$\mathcal{M}_I{}^J = \begin{pmatrix} -\nu & e^{-i\tau}\sqrt{1-\nu^2} & 0 & 0 \\ e^{i\tau}\sqrt{1-\nu^2} & \nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \eta_I^\alpha \equiv n_I \eta^\alpha = \frac{e^{\frac{i\nu\tau}{2}}}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\nu} \\ -\sqrt{1-\nu}e^{i\tau} \\ 0 \\ 0 \end{pmatrix}_I (1, -ie^{-i\tau})^\alpha ,$$

$$\bar{\eta}_\alpha^I \equiv \bar{n}^I \bar{\eta}_\alpha = i(\eta_I^\alpha)^\dagger . \tag{5.7}$$

We observe that for $\nu \to 1$ we recover the more supersymmetric loops (4.3) and (4.6), defined in the previous chapter.

The generalized prescription (5.5) that requires taking the supertrace of the superholonomy times a constant "twist" matrix was originally proposed to assure invariance under super gauge transformations [70]. The overall constant in (5.5) can be chosen again to normalize the expectation value to 1. In the following discussion, the Wilson loops are understood to be in the fundamental representation. For generic values of the parameters, the bosonic latitude operators in (5.3) preserve 1/12 of the original $\mathcal{N} = 6$ supercharges, whereas the fermionic one in (5.5) is $\frac{1}{6}$-BPS. The explicit expressions of the preserved supercharges will be given in the next section.

A remarkable, and maybe unexpected, feature of these constructions is the following: at the classical level the fermionic latitude Wilson loop (5.5) is cohomologically equivalent to the following linear combination of bosonic latitudes

$$W_F(\nu) = \mathcal{R}\left[N_1\, e^{-\frac{i\pi\nu}{2}}\, W_B(\nu) + N_2\, e^{\frac{i\pi\nu}{2}}\, \hat{W}_B(\nu)\right] + \mathcal{Q}(\nu)(\text{something})\,. \tag{5.8}$$

In the above formula, $\mathcal{Q}(\nu)$ is the linear combination of superPoincaré and superconformal charges preserved by both bosonic and fermionic Wilson loops [71]. Assuming that this relation holds at the quantum level, the vacuum expectation value $\langle W_F(\nu)\rangle$ of the fermionic operator is equal to the one for a bosonic operator. However, the problem of understanding how the classical cohomological equivalence gets implemented at the quantum level is strictly interconnected with the problem of understanding framing [207]. In the circular case it holds at framing one, as originally proposed in [22], while for latitudes a generic non-integer framing ν should be assumed (see [71] for an extensive discussion of this point).

The exact computation of the 1/6 BPS circular Wilson loop can be performed by supersymmetric localization [21]. Using this procedure, the ABJM partition function on the three-sphere can be reduced to the matrix model integral, as described in section 2.3. For generic ν the situation is a little bit different: even though the latitude bosonic Wilson loops are BPS operators, the standard localization arguments of [22] cannot be directly applied [71] to their computation. On the other hand, one could expect that a matrix model representation for their vacuum expectation value exists. A conjectured matrix model for the latitude Wilson loops has been therefore proposed in [79], that turned out to be compatible with all the available data points at weak and strong coupling. We briefly describe its genesis. The idea is to start from the matrix model average (2.81) computing the expectation value of 1/6–BPS Wilson loops and try to deform it by introducing a suitable dependence on the ν parameter. The structure of the θ–Bremsstrahlung function [222, 223] suggests that the latitude Wilson loop should be computed by inserting the operator $\mathrm{Tr}\, e^{2\pi\sqrt{\nu}\lambda}$ into a matrix model which is symmetric under the inversion $\nu \leftrightarrow 1/\nu$. Such an argument is reminiscent of the one proposed in [76], but somehow with a reverse logic. In that case, for the ABJM theory a supersymmetric Wilson loop on a squashed sphere was considered, whose matrix model is invariant under the inversion of the squashing parameter [158]. This was used to argue that the derivative of this Wilson loop expectation value with respect to the squashing parameter b, evaluated at $b = 1$, could be traded with the derivative of the multiply wound 1/6–BPS Wilson loop with respect to the winding number m (see [71, 78] for a detailed discussion of these points). Using also some perturbative inputs, the following matrix model average for the expectation value of a bosonic latitude Wilson loop has been proposed

$$\langle W_B(\nu)\rangle = \left\langle \frac{1}{N_1} \sum_{1\leq i\leq N_1} e^{2\pi\sqrt{\nu}\,\lambda_i} \right\rangle , \tag{5.9}$$

where the average is evaluated and normalized using the matrix model partition function

$$\begin{aligned} Z = {} & \frac{1}{N_1!N_2!} \int \prod_{a=1}^{N_1} d\lambda_a\, e^{i\pi k\lambda_a^2} \prod_{b=1}^{N_2} d\mu_b\, e^{-i\pi k\mu_b^2} \qquad\qquad (5.10) \\ & \times \frac{\prod_{a<b}^{N_1} 2\sinh\sqrt{\nu}\pi(\lambda_a-\lambda_b) 2\sinh\frac{\pi(\lambda_a-\lambda_b)}{\sqrt{\nu}} \prod_{a<b}^{N_2} 2\sinh\sqrt{\nu}\pi(\mu_a-\mu_b) 2\sinh\frac{\pi(\mu_a-\mu_b)}{\sqrt{\nu}}}{\prod_{a=1}^{N_1}\prod_{b=1}^{N_2} 2\cosh\sqrt{\nu}\pi(\lambda_a-\mu_b) 2\cosh\frac{\pi(\lambda_a-\mu_b)}{\sqrt{\nu}}} . \end{aligned}$$

Similarly, $\langle\hat{W}_B(\nu)\rangle$ corresponds to the insertion of $\frac{1}{N_2}\sum_{1\leq i\leq N_2} e^{2\pi\sqrt{\nu}\,\mu_i}$. According to the discussion above, such a matrix model should arise from a suitable localization of the ABJ(M) theory. We stress that this is the simplest non-trivial deformation of the matrix model (2.81) reducing to the usual expression at $\nu = 1$, and whose measure is symmetric under $\nu \leftrightarrow 1/\nu$, the specific dependence on ν in the hyperbolic functions and in the operator insertion has been fixed via comparison with the perturbative results [79]. Three-loop weak-coupling computations are consistent with this conjecture: the proposed matrix model is amenable of a Fermi gas reformulation [68] and strong-coupling results can be obtained that coincide with previous and subsequent string-theory calculations at leading and subleading order [80–84].

Analyzing this matrix model, we immediately realize that the ν-dependence cannot be reabsorbed by a simple redefinition of the coupling constant: we expect obviously that the deformation could affect the observable that we average to evaluate the Wilson loop. However to be consistent with a localization result, when we replace the observable with the identity the matrix integral over the deformed measure must still give the partition function of ABJ(M) on S^3. This implies that the dependence of the partition function on ν must become trivial, a peculiar feature that should rely on the intimate structure of the matrix model itself.

Let us first consider the case of equal rank. The crucial property that streamlines this miraculous independence is the Cauchy determinant identity

$$\frac{\prod_{a<b}^{N} \sinh r\,\pi\,(\lambda_a-\lambda_b)\,\sinh r\,\pi\,(\mu_a-\mu_b)}{\prod_{a=1}^{N}\prod_{b=1}^{N}\cosh\,r\,\pi\,(\lambda_a-\mu_b)} = \sum_{\sigma\in S_N}(-1)^{\sigma}\prod_{a=1}^{N}\frac{1}{\cosh\,r\,\pi\,(\lambda_a-\mu_{\sigma(a)})}\,. \tag{5.11}$$

Here the final sum is over all the permutations of the N eigenvalues and r is an arbitrary parameter. Let us consider the case of equal rank: Splitting the integrand of (5.10) into two combinations of hyperbolic functions with arguments containing the factors $\sqrt{\nu}$ and $1/\sqrt{\nu}$ respectively, and applying the Cauchy identity separately with an appropriate choice of r, after few algebraic steps described in [79, 167], we end up with

$$Z=\frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\sigma}\int\frac{dy^N}{(2\pi k)^N}\prod_{a=1}^{N}\left(2\cosh\frac{y_a}{2}\right)^{-1}\left(2\cosh\frac{y_a-y_{\sigma(a)}}{2k}\right)^{-1}, \tag{5.12}$$

where $N=N_1=N_2$. Eq. 5.12 is an alternative form of the ordinary ABJM paritition function, i.e. with $\nu=1$, derived in [167]. While the precise relation among y_a and λ_a, μ_a is not straightforward, for $k=1$ the variables y_a may be physically thought of as representing the vevs of scalars in the vector multiplet of a dual $\mathcal{N}=4$ gauge theory, which is described in Section 2.4. We thus see that the dependence on ν drops completely and the partition function has the same expression as for ABJM theory.

For different ranks of the gauge groups the starting point (5.11) must be replaced by a generalization of the Cauchy determinant identity discussed in [224–226]. Then, if we repeat the steps leading from (5.11) to (5.12), we can isolate and evaluate the $\nu-$dependent part of the partition function

$$\exp\left(\frac{\pi i}{12k}\left(\nu+\frac{1}{\nu}\right)\left((N_1-N_2)^3-(N_1-N_2)\right)\right)\,. \tag{5.13}$$

The only effect of the deformed measure consists in altering the original phase of the ABJ partition function obtained in [67, 227] by a trivial $\nu-$dependent multiplicative factor.

When we insert the Wilson loop in the $\nu-$deformed matrix model we get a $\nu-$dependent result. This can be shown by repeating the steps leading to 5.12, as shown in [79]. For future reference, we write the result for the case $N_1 = N_2 = N$ and $k = 1$:

$$\langle W_B(\nu)\rangle = \frac{1}{Z\,N(N!)}\sum_{\rho\in S_N}(-1)^\rho \int d\sigma^N \sum_{c=1}^{N} \frac{e^{i\pi\nu}e^{2\pi\sigma_c}}{\prod_{j=1}^N 2\cosh\pi\sigma_j \prod_{k=1}^N 2\cosh\pi(\sigma_k-\sigma_{\rho(k)}+i\nu\delta_k^c)}\,. \tag{5.14}$$

This equation makes manifest the fact that the $\nu-$dependence does not disappear.

5.1.1 Relation with the bremsstrahlung function

In this section we relate the latitude Wilson loop to the bremsstrahlung function. Its definition requires a cusped line operator. A cusp of angle φ in the BPS Wilson lines breaks all the preserved supersymmetry, and the expectation value of the Wilson operator develops a divergence. The coefficient of the divergence is the cusp anomalous dimension introduced in Section 4.4. In addition to this, it is possible to introduce an additional internal angle ϑ, which deforms the scalar coupling and, for the fermionic cusp, also the fermionic ones. We define the two cusp anomalous dimensions $\Gamma_{\frac{1}{6}}$ and $\Gamma_{\frac{1}{2}}$ corresponding respectively to the bosonic and fermionic line as [222]

$$\langle W_{\rm B}^{\rm cusp}\rangle \sim \Gamma_{\frac{1}{6}}(\varphi,\vartheta)\log\frac{L}{\epsilon} \qquad \langle W_{\rm F}^{\rm cusp}\rangle \sim \Gamma_{\frac{1}{2}}(\varphi,\vartheta)\log\frac{L}{\epsilon} \tag{5.15}$$

where L and ϵ are the IR and UV regulators.

We define the bremsstrahlung function as the small angles limit of the cusp anomalous dimensions. However, since the fermionic cusp is BPS for $\varphi^2 = \vartheta^2$ we have

$$\Gamma_{\frac{1}{2}}(\varphi,\vartheta) = (\varphi^2-\vartheta^2)B_{\frac{1}{2}}(N,k) \tag{5.16}$$

where $B_{\frac{1}{2}}(N,k)$ is the corresponding bremsstrahlung. Since the bosonic cusp is never BPS we have a bremsstrahlung for each angle, i.e. two functions $B^{\varphi}_{\frac{1}{6}}(N,k)$ and $B^{\vartheta}_{\frac{1}{6}}(N,k)$ such that

$$\Gamma_{\frac{1}{6}}(\varphi,\vartheta) = \vartheta^2 B^{\vartheta}_{\frac{1}{6}}(N,k) - \varphi^2 B^{\varphi}_{\frac{1}{6}}(N,k) \tag{5.17}$$

Nevertheless, it was shown in [78] that $B^{\varphi}_{\frac{1}{6}} = 2B^{\vartheta}_{\frac{1}{6}}$.

Assuming that the defect two-point functions are the same for the straight line and the circular case and using the universal relation between the Bremsstrahlung and the displacement of eq (4.55), one can relate the Bremsstrahlung functions to the latitude deformations of the circular Wilson loops. Specifically, for the latitude bosonic and fermionic Wilson loops[3] [71, 77, 79]

$$B^{\vartheta}_{\frac{1}{6}}(N,k) = \frac{1}{4\pi^2}\frac{\partial}{\partial\nu}\log|\langle W_{\rm B}(\nu)\rangle|\Big|_{\nu=1} \qquad B_{\frac{1}{2}}(N,k) = \frac{1}{4\pi^2}\frac{\partial}{\partial\nu}\log|\langle W_{\rm F}(\nu)\rangle|\Big|_{\nu=1} \tag{5.18}$$

[3] An alternative derivation was proposed in [76] for $B^{\varphi}_{\frac{1}{6}}$, based on the relation with the one point function of the stress tensor in presence of the Wilson line. The latter is accessible by using the squashing deformation. All the existing results are in agreement.

The relation with the Bremsstrahlung provides a strong motivation to put on more solid grounds the proposal for the matrix model of 5.10 [4].

5.1.2 The latitude algebra

The presence of a BPS defect in a superconformal theory breaks the superconformal algebra to a subalgebra. Below, we detail the symmetry algebra left unbroken by the ABJM latitude Wilson loop in its fermionic and bosonic versions. Our supersymmetry conventions for ABJM are collected in Appendix A.

The supercharges preserved by the fermionic latitude are given by [71]

$$\begin{aligned}
\bar\theta_1^{13} &= e^{-i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_1 + e^{i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_2 & \bar\theta_1^{14} &= e^{-i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_3 + e^{i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_4\,,\\
\bar\theta_2^{23} &= -ie^{-i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_1 - ie^{i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_2 & \bar\theta_2^{24} &= -ie^{-i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_3 - ie^{i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_4\,,\\
\bar\epsilon_1^{13} &= ie^{i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_1 - ie^{-i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_2 & \bar\epsilon_1^{14} &= ie^{i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_3 - ie^{-i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_4\,,\\
\bar\epsilon_2^{23} &= e^{-i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_2 - e^{i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_1 & \bar\epsilon_2^{24} &= e^{-i\frac{\theta_0}{2}}\sqrt{1-\nu}\,\omega_4 - e^{i\frac{\theta_0}{2}}\sqrt{1+\nu}\,\omega_3\,,
\end{aligned} \tag{5.19}$$

where ω_i, $i=1,\dots,4$ are bosonic parameters and we have taken the $\mathbb{R}^3$ conformal Killing spinor to be $\bar\Theta^{IJ}_\alpha = \bar\theta^{IJ}_\alpha - x^\mu(\gamma_\mu)_\alpha{}^\beta\bar\epsilon^{IJ}_\beta$. We define the corresponding operators Q_i, $i = 1,\dots,4$, turning on the corresponding ω_i and setting the others to zero. We also perform a rescaling on the supercharges by a factor $2\sqrt{\nu\cos\theta_0}$. In the following sections, we will define the action of the supercharges 5.19 in several slightly different contexts, where we will use different convenient normalizations. The Q_i action on a generic operator $\mathcal{O}$ is defined as:

$$\delta\mathcal{O} \equiv [\bar\theta^{IJ}_\alpha\bar Q^\alpha_{IJ} + \bar\epsilon^{IJ}_\alpha\bar S^\alpha_{IJ}, \mathcal{O}]\,, \tag{5.20}$$

where $\bar Q^\alpha_{IJ}$ and $\bar S^\alpha_{IJ}$ are the operators generating the $\mathfrak{osp}(6|4)$ superconformal algebra.

The couplings of the Wilson loop $\mathcal{M}_I{}^J$ and η^α_I preserve an $SU(2)$ R-symmetry, and it turns out to be convenient to recast the conserved supercharges into two $SU(2)$ doublets

$$\mathcal{Q}^-_a = \begin{pmatrix} Q_1 \\ Q_3 \end{pmatrix}\,, \qquad \mathcal{Q}^+_a = \begin{pmatrix} Q_2 \\ Q_4 \end{pmatrix}\,. \tag{5.21}$$

From the ABJM superconformal algebra, we get the following anti-commutation relations

$$\{\mathcal{Q}^-_a, \mathcal{Q}^-_b\} = 0\,, \qquad \{\mathcal{Q}^+_a, \mathcal{Q}^+_b\} = 0\,, \tag{5.22}$$

$$\{\mathcal{Q}^-_a, \mathcal{Q}^+_b\} = \epsilon_{ab}\,(\mathcal{T} - Z) - 2L_{ab}\,, \tag{5.23}$$

[4] We stress that the above relations are rigorously proved only for $N_1 = N_2 = N$. It is not fully clear if they survive for different ranks of the gauge groups, or in less supersymmetric theory.

where we have introduced the conserved bosonic generators

$$\mathcal{T} = \frac{i}{\cos\theta_0}\left(K_3 - P_3 + 2\sin\theta_0 D\right) , \tag{5.24}$$

$$Z = \frac{1}{\nu}\left(2iM^{12} + J_2{}^2 - J_1{}^1\right) , \tag{5.25}$$

$$L_a{}^b = \begin{pmatrix} J_3{}^3 + \frac{1}{2}J_1{}^1 + \frac{1}{2}J_2{}^2 & J_3{}^4 \\ J_4{}^3 & J_4{}^4 + \frac{1}{2}J_1{}^1 + \frac{1}{2}J_2{}^2 \end{pmatrix} = \begin{pmatrix} L_z & L_- \\ L_+ & -L_z \end{pmatrix} . \tag{5.26}$$

$L_a{}^b$ are the generators of the $\mathfrak{su}(2)$ R-symmetries preserved by the loop. Their action on the conserved supercharges is given by

$$[L_a{}^b, \mathcal{Q}_c^-] = \frac{1}{2}\delta_a^b \mathcal{Q}_c^- - \delta_c^b \mathcal{Q}_a^- , \tag{5.27}$$

$$[L_a{}^b, \mathcal{Q}_c^+] = \frac{1}{2}\delta_a^b \mathcal{Q}_c^+ - \delta_c^b \mathcal{Q}_a^+ . \tag{5.28}$$

$\mathcal{T}$ generates a $U(1)$ symmetry followed by a dilatation. Its action on the supercharges is

$$[\mathcal{T}, \mathcal{Q}_a^-] = -\mathcal{Q}_a^- , \qquad [\mathcal{T}, \mathcal{Q}_a^+] = \mathcal{Q}_a^+ . \tag{5.29}$$

The presence of the dilatation reflects the possibility of moving the loops at different heights of the S^2. Finally, Z is a central element of the subalgebra. It generates a $U(1)$ symmetry which mixes a Lorentz rotation with a specific R-symmetry. The precise combination of the two is fixed by the invariance of the off-diagonal terms in $\mathcal{M}_I{}^J$ (or equivalently $M_I{}^J$). The role of Z will be crucial in the following, especially when we will describe loop operators as mixed 3d/1d systems. The complete superalgebra is a central extension of $\mathfrak{su}(1|2)$, which is isomorphic to $\mathfrak{osp}(2|2)$.

In the rest of the paper, we will be interested mostly in the bosonic latitude. The bosonic latitude preserves only the subset of the supercharges: those generated by Q_2 and Q_3. The resulting $\mathfrak{su}(1|1)$ subalgebra of the full latitude algebra is given by

$$\{Q_2, Q_3\} = \mathcal{T} + 2L_z - Z , \tag{5.30}$$

$$[\mathcal{T}, Q_2] = Q_2 , \qquad [\mathcal{T}, Q_3] = -Q_3 , \tag{5.31}$$

$$[L_z, Q_2] = -\frac{1}{2}Q_2 , \qquad [L_z, Q_3] = \frac{1}{2}Q_3 . \tag{5.32}$$

From now on, we will refer to the bosonic latitude as "the latitude". We will also restrict our analysis to Q_2 and Q_3. Moreover, we will take the limit $\theta_0 \to 0$. In this way we avoid any issues related to the presence of the non compact symmetry D in the latitude algebra. Latitude loops at other values of θ_0 are related to the one at $\theta_0 = 0$ by the action of the conformal group. We can therefore restrict ourselved to examining the expectation value of the $\theta_0 = 0$ loop without loss of generality.

The limit $\nu \to 1$

In the limit $\nu \to 1$, the latitude coincides with the well-known 1/6-BPS circular Wilson loops. In this limit there are also two additional supercharges preserved by the latitude. All of the supercharges preserved by the latitude in this limit can be parameterized as follows

$$\bar{\theta}^{13}_1 = \omega_2 \,, \qquad \bar{\theta}^{24}_2 = -i\omega_3 \,, \tag{5.33a}$$
$$\bar{\theta}^{13}_2 = i\omega_5 \,, \qquad \bar{\theta}^{24}_1 = \omega_6 \,, \tag{5.33b}$$

with the relations

$$\bar{\epsilon}^{IJ}_\alpha = i M_K{}^I (\tau_3)_\alpha{}^\beta \bar{\theta}^{KJ}_\beta \,. \tag{5.34}$$

where now $M_K{}^I = \mathrm{diag}(-1, 1, -1, 1)$.

It is interesting to show how the supersymmetry enhancement in the limit $\nu \to 1$ modifies the latitude superalgebra. Before providing the explicit derivation, we can make an educated guess based on some known results. First, we note that the 1/6-BPS circular Wilson loop is related to the 1/6-BPS Wilson line by a conformal transformation [222]. On the other hand, the superalgebra preserved by the Wilson line is $\mathfrak{su}(1,1|1)$, which is a superconformal algebra on the line [78]. Therefore, we expect to find the same superalgebra for the circular Wilson loop, realized in terms of different generators of the ABJM supersymmetry algebra[5].

Let us verify that this is indeed the case. We introduce operators associated to the supersymmetries 5.33. They are denoted Q_2, Q_3, Q_5 and Q_6, corresponding to the parameters ω_2, ω_3, ω_5, and ω_6. We choose the notation so as to make Q_2 and Q_3 the limit as $\nu \to 1$ of Q_2 and Q_3 of the previous section, up to an overall factor of 1/2. The explicit form of the supercharges is

$$Q_2 = \frac{1}{\sqrt{2}} \left(\bar{Q}_{2,13} - i \bar{S}_{2,13} \right) \,, \qquad Q_3 = \frac{1}{\sqrt{2}} \left(\bar{S}_{1,24} + i \bar{Q}_{1,24} \right) \,, \tag{5.35a}$$
$$Q_5 = \frac{1}{\sqrt{2}} \left(\bar{S}_{1,13} - i \bar{Q}_{1,13} \right) \,, \qquad Q_6 = \frac{1}{\sqrt{2}} \left(\bar{Q}_{2,24} + i \bar{S}_{2,24} \right) \,. \tag{5.35b}$$

It is also convenient to introduce the following space time generators

$$M = -iM_{12} \,, \qquad P = \frac{1}{2} \left(P_+ + K_+ \right) \,, \qquad K = \frac{1}{2} \left(P_- + K_- \right) \,, \tag{5.36}$$

where $P_\pm = P_1 \pm i P_2$ and $K_\pm = K_1 \pm i K_2$. There is also a mixed $\mathfrak{u}(1)$ charge

$$J = \frac{1}{2} \left(i P_3 - i K_3 - 2 \left(J_1{}^1 + J_3{}^3 \right) \right) \,. \tag{5.37}$$

[5] This is the same situation as for the maximally supersymmetric Wilson loop in $\mathcal{N} = 4$ SYM in 4d [73]. The maximally supersymmetric circular Wilson loop and the maximally supersymmetric Wilson line are related by a conformal map, therefore they preserve the same algebra. However, the algebra is realized with different bulk generators.

The odd-odd part of the enhanced superalgebra is given by

$$\{Q_2, Q_3\} = 2(M - J) , \qquad \{Q_5, Q_6\} = 2(M + J) , \tag{5.38a}$$
$$\{Q_2, Q_6\} = -2P , \qquad \{Q_3, Q_5\} = 2K , \tag{5.38b}$$

The bosonic subalgebra is

$$[P, K] = 2M \qquad [M, P] = P , \qquad [M, K] = -K , \tag{5.39}$$

which is indeed the expected $\mathfrak{su}(1,1)$ 1d conformal algebra. Finally, the mixed commutators are given by:

$$[M, Q_2] = \frac{1}{2}Q_2 , \quad [M, Q_3] = -\frac{1}{2}Q_3 , \quad [M, Q_5] = -\frac{1}{2}Q_5 , \quad [M, Q_6] = \frac{1}{2}Q_6 , \tag{5.40a}$$
$$[K, Q_2] = Q_5 , \qquad [K, Q_6] = Q_3 , \qquad [P, Q_5] = Q_2 , \qquad [P, Q_3] = Q_6 , \tag{5.40b}$$
$$[J, Q_2] = \frac{1}{2}Q_2 , \qquad [J, Q_3] = -\frac{1}{2}Q_3 , \qquad [J, Q_5] = \frac{1}{2}Q_5 , \qquad [J, Q_6] = -\frac{1}{2}Q_6 . \tag{5.40c}$$

Note that Q_2 and Q_6 behaves as Poincaré supercharges on the loop, while Q_3 and Q_5 behave like the superconformal ones.

In summary, our analysis shows that the latitude preserves a specific 1d Poincaré type superalgebra. In the limit $\nu \to 1$, this superalgebra enhances to the conformal superalgebra $\mathfrak{su}(1,1|1)$ preserved by the standard 1/6-BPS Gaiotto-Yin Wilson loop.

Embedding in $\mathcal{N} = 4$

The 3d $\mathcal{N}$-extended superconformal algebra is isomorphic to $\mathfrak{osp}(\mathcal{N}|4)$. The algebra is generated by $\mathcal{N}$ quadruplets of spinors in the vector representation of $SO(\mathcal{N})_R$. For the ABJM model, the manifest supersymmetry is $\mathcal{N} = 6$. It is useful in this case to break up the $SO(6)_R$ index , or rather $\mathrm{Spin}(6)_R \simeq SU(4)_R$, into a doublet of anti-symmetric $SU(4)_R$ indices denoted by $\{I, J, K, \ldots\}$. The transformation between the two notations is performed using the 6d Euclidean Clifford algebra and its matrices Γ_{aIJ}

$$\Gamma_1 \equiv i\tau_2 \otimes \mathbb{1} , \quad \Gamma_2 \equiv \tau_2 \otimes \tau_3 , \quad \Gamma_3 \equiv \tau_2 \otimes \tau_1 ,$$
$$\Gamma_4 \equiv i\tau_1 \otimes \tau_2 , \quad \Gamma_5 \equiv \mathbb{1} \otimes \tau_2 , \quad \Gamma_6 \equiv i\tau_3 \otimes \tau_2 ,$$
$$\tilde{\Gamma}_a^{IJ} \equiv \left(\Gamma_a^\dagger\right)^{IJ} ,$$

satisfying

$$\Gamma_{aIJ} = -\Gamma_{aJI}, \quad \Gamma_a\tilde{\Gamma}_b + \Gamma_b\tilde{\Gamma}_a = 2\delta_I{}^J\delta_{ab}, \quad \tilde{\Gamma}_a\Gamma_b + \tilde{\Gamma}_b\Gamma_a = 2\delta^I{}_J\delta_{ab} .$$

Note that $(\Gamma^a)_{IJ}(\Gamma_a)_{KL} = -2\varepsilon_{IJKL}$, so that the usual $SO(6)$ invariant inner product is replaced by contraction of pairs of indices using the ε symbol. A generic $\mathcal{N} = 6$ superconformal transformation is then specified by a tensor

$$\bar{\Theta}_i^{IJ}, \quad I, J \in \{1 \dots 4\}, i \in \{1 \dots 4\},$$
$$\bar{\Theta}_i^{IJ} = -\bar{\Theta}_i^{JI}.$$

The index i can be contracted with a basis for the conformal Killing spinors, thus yielding a Killing spinor in our previous notation $\bar{\Theta}_\alpha^{IJ}$.

One may show that the fermionic latitude supercharges, parameterized by ω_i, can be embedded into an $\mathcal{N} = 4$ subalgebra of the $\mathcal{N} = 6$ supersymmetry algebra of ABJM. Contracting the latitude supercharges with the 6d Γ matrices, we get

$$\bar{\Theta}_\alpha^{IJ}(\omega)\,\Gamma_{aIJ} = 0, \quad a \in \{5, 6\}.$$

If we define the $U(1)$ generator

$$U^I{}_J \equiv \exp\left(t\left(\tilde{\Gamma}_5\Gamma_6 - \tilde{\Gamma}_6\Gamma_5\right)\right), \quad U^\dagger = U^{-1},$$

then

$$U\bar{\Theta}_\alpha(\omega)\,U^t = \bar{\Theta}_\alpha(\omega), \quad U^*\mathcal{M}U^t = \mathcal{M}.$$

U therefore generates a residual $SO(2)$ R-symmetry commuting with all of the latitude supercharges.

In the limit $\nu \to 1$, the supercharges $Q_{2,3}$ satisfy the stronger condition

$$\lim_{\nu\to 1}\bar{\Theta}_\alpha^{IJ}(\omega_{2,3})\,\Gamma_{aIJ} = 0, \quad a \in \{3 \dots 6\},$$

and so form a part of an $\mathcal{N} = 2$ subalgebra. In this limit, the latitude Wilson loop is invariant under an $SO(4)$ subgroup of the R symmetry. There are also 2 additional supercharges preserved by the bosonic latitude loop only in the limit $\nu \to 1$, which were denoted $Q_{5,6}$. These lie in the same $\mathcal{N} = 2$ subalgebra as $Q_{2,3}$.

5.1.3 Gaiotto-Witten Theories

We would like to demonstrate that bosonic latitude type Wilson loops exist in Chern-Simons-matter theories of Gaiotto-Witten (GW) type [125]. GW theories are generic Chern-Simons-matter theories preserving $\mathcal{N} = 4$ supersymmetry. They can be formulated starting from $\mathcal{N} = 1$ superfields, such that $\mathcal{N} = 4$ supersymmetry is realized on-shell after all auxiliary fields have been integrated out. The ABJM family of models are GW theories of a generalized type introduced in [126], with special properties allowing an enhancement to $\mathcal{N} = 6$ [128]. We define a latitude type loop as a bosonic Wilson loop which preserves $Q_{2,3}$ inside the $\mathcal{N} = 4$ algebra. The notation in this section is that of [126], which differs from the rest of the chapter.

We recall that GW theories are specified by a gauge group G, which is a subgroup of $Sp(2n)$ for some n, an invariant quadratic form k^{mn} on the Lie algebra $\mathfrak{g}$, and a $2n$-dimensional representation of G constructed using the subset of $Sp(2n)$ generators $(t^m)^A{}_B$, such that $t_{[AC]} \equiv \omega_{[AB}t^{B}{}_{C]} = 0$, where ω_{AB} is the invariant symplectic form of $Sp\,(2n)$ and square brackets denote anti-symmetrization. In order to yield an $\mathcal{N} = 4$ supersymmetric theory, the representation matrices must satisfy the following *fundamental identity* [125]

$$k_{mn}t^{m}_{(AB}t^{n}_{C)D} = 0. \tag{5.41}$$

We will consider a generalization of GW theories described in [126]. The generalization introduces an additional set of representation matrices, $\left(\tilde{t}^{m}\right)^{A}{}_{B}$, satisfying the same conditions as $(t^{m})^{A}{}_{B}$. A generalized GW theory contains the following fields: a G connection A^{m}_{μ}, and scalar/spinor pairs $q^{A}_{\alpha}, \psi^{A}_{\dot{\alpha}}$ and $\tilde{q}^{A}_{\dot{\alpha}}, \tilde{\psi}^{A}_{\alpha}$ valued in the first and second representation, respectively. $\alpha, \dot{\alpha}$ denote $SU\,(2)_{l} \times SU\,(2)_{r}$ R-symmetry indices. The pair $q^{A}_{\alpha}, \psi^{A}_{\dot{\alpha}}$ form an $\mathcal{N} = 4$ on-shell hypermultiplet, while $\tilde{q}^{A}_{\dot{\alpha}}, \tilde{\psi}^{A}_{\alpha}$ form an on-shell twisted hypermultiplet. The hypermultiplets and twisted hypermultiplets can be in different representations. Following [126], we nevertheless denote all representation indices as $A, B, \ldots$. The Killing spinors are denoted $\eta_{\alpha\dot{\alpha}}$.

The relevant supersymmetry transformations are [126]

$$\delta q^{A}_{\alpha} = i\eta_{\alpha}{}^{\dot{\alpha}}\psi^{A}_{\dot{\alpha}}, \qquad \delta\tilde{q}^{A}_{\dot{\alpha}} = -i\eta^{\alpha}{}_{\dot{\alpha}}\tilde{\psi}^{A}_{\alpha},$$
$$\delta A^{m}_{\mu} = 2\pi i\eta^{\alpha\dot{\alpha}}\gamma_{\mu}\left(j^{m}_{\alpha\dot{\alpha}} - \tilde{j}^{m}_{\dot{\alpha}\alpha}\right).$$

The composite quantities $j^{m}_{\alpha\dot{\alpha}}, \tilde{j}^{m}_{\dot{\alpha}\alpha}$ are defined as

$$j^{m}_{\alpha\dot{\alpha}} \equiv q^{A}_{\alpha}t^{m}_{AB}\psi^{B}_{\dot{\alpha}}, \qquad \tilde{j}^{m}_{\dot{\alpha}\alpha} \equiv \tilde{q}^{A}_{\dot{\alpha}}\tilde{t}^{m}_{AB}\tilde{\psi}^{A}_{\alpha}.$$

We also define the moment maps

$$\mu^{m}_{\alpha\beta} \equiv q^{A}_{\alpha}t^{m}_{AB}q^{B}_{\beta}, \qquad \tilde{\mu}^{m}_{\dot{\alpha}\dot{\beta}} \equiv \tilde{q}^{A}_{\dot{\alpha}}\tilde{t}^{m}_{AB}\tilde{q}^{B}_{\dot{\beta}}.$$

Note that[6]

$$\delta\mu^{m}_{\alpha\beta} = 2i\eta_{(\alpha}{}^{\dot{\alpha}}j^{m}_{\beta)\dot{\alpha}}, \qquad \delta\tilde{\mu}^{m}_{\dot{\alpha}\dot{\beta}} = -2i\eta^{\alpha}{}_{(\dot{\alpha}}\tilde{j}^{m}_{\dot{\beta})\alpha}.$$

Define a generic bosonic Wilson loop

$$\mathcal{W} \equiv \mathcal{P}\mathrm{tr}_{R}\exp\oint_{\ell}\left(A_{\mu}\dot{\ell}^{\mu} + 2\pi\left|\dot{\ell}\right|\left(C^{\alpha\beta}\mu_{\alpha\beta} + \tilde{C}^{\dot{\alpha}\dot{\beta}}\tilde{\mu}_{\dot{\alpha}\dot{\beta}}\right)\right),$$

where $C^{\alpha\beta}, \tilde{C}^{\dot{\alpha}\dot{\beta}}$ are symmetric c-number matrices. The supersymmetry variation of $\mathcal{W}$ is proportional to

$$2\pi i\eta^{\alpha\dot{\alpha}}\gamma_{\mu}\dot{\ell}^{\mu}\left(j^{m}_{\alpha\dot{\alpha}} - \tilde{j}^{m}_{\dot{\alpha}\alpha}\right) + 2\pi i\left|\dot{\ell}\right|\left(C^{\alpha\beta}\eta_{\alpha}{}^{\dot{\alpha}}j_{\beta\dot{\alpha}} - \tilde{C}^{\dot{\alpha}\dot{\beta}}\eta^{\alpha}{}_{\dot{\alpha}}\tilde{j}_{\dot{\beta}\alpha}\right).$$

The conditions for unbroken supersymmetry generated by a spinor $\eta_{\alpha\dot{\alpha}}$ are

$$\eta_{\alpha\dot{\alpha}}\gamma_{\mu}\dot{\ell}^{\mu} = -\left|\dot{\ell}\right|C_{\alpha}{}^{\beta}\eta_{\beta\dot{\alpha}} = -\left|\dot{\ell}\right|\tilde{C}^{\dot{\beta}}_{\dot{\alpha}}\eta_{\alpha\dot{\beta}}.$$

[6]Symmetrized indices are defined as follows $X_{(a}Y_{b)} \equiv \frac{1}{2}\left(X_{a}Y_{b} + X_{b}Y_{a}\right)$.

Let ℓ^μ be the unit circle in the x_1, x_2 plane with angular coordinate τ, and let $\eta_{\alpha\dot{\alpha}}$ represent either of the supercharges $Q_{2,3}$ at $\theta_0 = 0$. One can show that the following matrices solve the supersymmetry equations

$$C^{\alpha\beta} = \begin{pmatrix} -i\sqrt{1-\nu^2}e^{i\tau} & \nu \\ \nu & -i\sqrt{1-\nu^2}e^{-i\tau} \end{pmatrix}, \qquad \tilde{C}^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

making the associated loop operator a BPS latitude loop. Latitude loops at generic θ_0 can be obtained by a conformal transformation.

Off-shell closure for GW theories

Off-shell closure for GW theories can be achieved in a straightforwards fashion for any given $\mathcal{N} = 2$ subalgebra. It can also be achieved for an $\mathcal{N} = 3$ subalgebra by using off-shell $\mathcal{N} = 3$ multiplets and the corresponding Chern-Simons and kinetic terms [228, 229]. In the case of ABJM, the relevant couplings can also be derived in harmonic superspace [230]. Additionally, supersymmetries preserved by the various topological twists of the $\mathcal{N} = 4$ superalgebra can be closed off-shell by introducing appropriate auxiliary fields [231].

An $\mathcal{N} = 4$ superfield construction of GW theories was described in [232]. The construction requires a consistency condition, involving both the vector multiplets and the hypermultiplets, which appears to prevent the construction of an $\mathcal{N} = 4$ off-shell supersymmetric action [232]. The consistency condition is equivalent to the fundamental identity in Eq 5.41. There is then no way to close the entire $\mathcal{N} = 4$ supersymmetry algebra of a GW type theory offshell using superfields. Unfortunately, we do not even know of a construction for closing a *single* generic $\mathcal{N} = 4$ supersymmetry off-shell in the GW class of theories. Specifically, the latitude supercharges do not fall into any of the categories admitting off-shell closure described above.

Let us describe the situation for the latitude supercharges in more detail. Supersymmetry transformations of off-shell hypermultiplets are linear in the hypermultiplet fields. Fields belonging to vector multiplets, as well as those coming from supergravity, may also appear in the hypermultiplet transformations following partial gauge fixing. We will show that the square of the latitude supercharge contains a gauge transformation with a parameter which is quadratic in the hypermultiplet scalars. In order for the latitude supersymmetry to close off-shell, this parameter must presumably be the vev of a scalar in an appropriate vector multiplet. Such a scalar appears in vector multiplets starting from $\mathcal{N} = 2$, with multiple scalars appearing starting at $\mathcal{N} = 3$. We will also show that the square of the latitude supercharge on the three sphere contains a translation by a Killing vector v. It is easy to show, however, that v cannot be generated by the square of an $\mathcal{N} = 2$ supercharge unless $\nu = 1$. We are left, therefore, with the option of trying to close the latitude supercharges off-shell in an ad hoc manner. We explore this option in Section 5.2.2.

5.2 The latitude matrix model

In this section, we attempt to derive the matrix model for the bosonic latitude Wilson loop by supersymmetric localization. In supersymmetric localization, one first deforms a supersymmetric Euclidean action S by adding a term $t\delta V$, where δ is a supersymmetry transformation, V a positive semi-definite fermionic functional, and t a positive number. Localization onto the moduli space, defined as the space of field configurations where δV vanishes, occurs in the limit $t \to \infty$, under some assumptions detailed in Section 2.1.

A somewhat subtle point concern off-shell closure. If the supersymmetry generated by δ is closed off-shell, without using the equations of motion coming from S, then δ^2 is a bosonic symmetry transformation. In this case, one can usually arrange for V to be δ^2 invariant. Unfortunately, off-shell closure of even a single supersymmetry transformation can be tricky. We will deal with this difficulty in the context of the ABJM model in an ad hoc manner.

In Section 5.2.1, we exhibit the superalgebra generated by the latitude supercharges on the three sphere using the ABJM notation. In Section 5.2.2, we perform localization in the ABJM model and derive a matrix model for the expectation value of the bosonic latitude Wilson loop. This calculation almost proves the conjectured form of the matrix model put forth in [79]. The only caveat is our assumption that a certain procedure can be used to close the latitude supersymmetry transformation off-shell.

5.2.1 The ABJM latitude loop on the three sphere

We exhibit the superalgebra generated by the bosonic latitude supercharges on the three sphere. The ABJM model, and the bosonic latitude Wilson loop, can be mapped to the three sphere using the change of coordinates in Appendix C. In order to avoid producing a conformal or Weyl transformation on S^3 in the square of the latitude supercharges, we restrict ourselves to $\theta_0 = 0$. In the coordinate system introduced in Appendix C, the latitude loop then sits on the great circle at $\theta = 0$ parameterized by the value of τ which itself coincides with the affine parameter used in the flat space notation. The latitude loop operator is defined by the same expression as in Eq 5.3 but with the opposite orientation.

We define a latitude supercharge $Q_{\mathrm{L}} \equiv Q_2 + Q_3$. The corresponding $\mathcal{N} = 6$ tensor, in the flat space conformal Killing spinor basis specified in Appendix C, is

$$\bar{\Theta}_1^{\mathbb{R}^3} = \begin{pmatrix} 0 & 0 & \sqrt{\nu+1} & \sqrt{1-\nu} \\ 0 & 0 & 0 & 0 \\ -\sqrt{\nu+1} & 0 & 0 & 0 \\ -\sqrt{1-\nu} & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\Theta}_2^{\mathbb{R}^3} = \begin{pmatrix} 0 & 0 & \sqrt{\nu+1} & \sqrt{1-\nu} \\ 0 & 0 & 0 & 0 \\ -\sqrt{\nu+1} & 0 & 0 & 0 \\ -\sqrt{1-\nu} & 0 & 0 & 0 \end{pmatrix},$$

$$\bar{\Theta}_3^{\mathbb{R}^3} = \begin{pmatrix} 0 & 0 & -i\sqrt{\nu+1} & i\sqrt{1-\nu} \\ 0 & 0 & 0 & 0 \\ i\sqrt{\nu+1} & 0 & 0 & 0 \\ -i\sqrt{1-\nu} & 0 & 0 & 0 \end{pmatrix}, \quad \bar{\Theta}_4^{\mathbb{R}^3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1-\nu} & -\sqrt{\nu+1} \\ 0 & -\sqrt{1-\nu} & 0 & 0 \\ 0 & \sqrt{\nu+1} & 0 & 0 \end{pmatrix}.$$

The change of coordinates described in Appendix C transforms this tensor such that the relevant supercharge on S^3 is given by the S^3 tensor

$$\bar{\Theta}_i^{IJ} \equiv R_i{}^j \bar{\Theta}_j^{\mathbb{R}^3,IJ}.$$

Let $\epsilon^{(i)}$ be the basis for the conformal Killing spinors defined in Appendix C. The corresponding spinor used in the transformation of a field is $\bar{\Theta}_i^{IJ}\epsilon^{(i)}$. We define the latitude supercharge to act using the rescaled spinor

$$\bar{\Theta}_{\text{lat}}^{IJ} \equiv \frac{1}{4\nu^{1/4}} \bar{\Theta}_i^{IJ}\epsilon^{(i)}. \tag{5.42}$$

The action of the latitude supercharges generate a subalgebra of the S^3 superconformal algebra. Specifically, the square of the latitude supercharge generates the following transformations

1. A diffeomorphism by a Killing vector

$$\frac{i}{2}\bar{\Theta}_{\text{lat}}^{IJ}\gamma^\mu \bar{\Theta}_{\text{lat}}^{KL}\varepsilon_{IJKL}\partial_\mu = -\sqrt{\nu}\partial_\varphi + \frac{1}{\sqrt{\nu}}\partial_\tau.$$

2. An R-symmetry transformation acting on $SU(4)_R$ indices by the matrix

$$\mathcal{R}_I{}^J = 2i\bar{\Theta}_{\text{lat}}^{KL}\bar{\Theta}_{\text{lat}}^{MJ}\varepsilon_{KLIM} = \frac{i}{2}\begin{pmatrix} -\nu^{-1/2} & 0 & 0 & 0 \\ 0 & \nu^{-1/2} & 0 & 0 \\ 0 & 0 & -\nu^{1/2} & 0 \\ 0 & 0 & 0 & \nu^{1/2} \end{pmatrix}.$$

3. No Weyl transformation. Note that this is true only when the parameter θ_0 is set to

0.

4. A gauge transformation given by the gauge parameters

$$\Phi_1 = i_v A - \frac{2\pi i}{k\sqrt{\nu}} \tilde{M}_J{}^I C_I \bar{C}^J, \quad \Phi_2 = -i_v \hat{A} + \frac{2\pi i}{k\sqrt{\nu}} \tilde{M}_J{}^I \bar{C}^J C_I \,, \tag{5.43}$$

$$\tilde{M}_J{}^I = \begin{pmatrix} -\nu & e^{-i\tau}\cos\theta\sqrt{1-\nu^2} & 0 & 0 \\ e^{i\tau}\cos\theta\sqrt{1-\nu^2} & \nu & 0 & 0 \\ 0 & 0 & -1 & e^{-i\varphi}\sin\theta\sqrt{1-\nu^2} \\ 0 & 0 & -e^{i\varphi}\sin\theta\sqrt{1-\nu^2} & 1 \end{pmatrix} \tag{5.44}$$

Note that $\tilde{M} = M$ at $\theta = 0$, i.e. on the latitude loop worldvolume.

5.2.2 Localization in the ABJM model

In this section, we perform localization of the bosonic latitude Wilson loop in ABJM. We first describe a procedure which we believe may be utilized in order to obtain off-shell closure and to localize theories of Gaiotto-Witten type, including the ABJM model. This procedure is only necessary in the presence of operators like the latitude loop, which do not preserve supercharges belonging to an $\mathcal{N} \leq 3$ subalgebra. We will show that, assuming the procedure works, we are lead to the matrix model for the latitude Wilson loop conjectured in [79].

Off-shell closure

Given the constraints on off-shell closure in GW type theories reviewed in 5.1.3, we are left with the task of closing the latitude supersymmetry off-shell in an ad hoc manner. By this we mean utilizing auxiliary fields which do not extend to a full spacetime supersymmetry algebra. Our first task is to close off-shell the transformation of the connection using a vector multiplet. An appropriate multiplet is discussed by Källén in [97]. This is a sort of cohomological multiplet, of the type often employed in topological field theory.

The cohomological vector multiplet contains a connection A_μ, a one form Ψ_μ, and a scalar Φ. The cohomological supersymmetry transformation is

$$\delta A = \Psi, \quad \delta\Psi = \mathcal{L}_v A + d_A \Phi, \quad \delta\Phi = 0\,.$$

Other multiplets are needed in order to construct actions. Specifically, we introduce a projection multiplet with a fermion α and a scalar $\tilde{D}$

$$\delta\alpha = \tilde{D}, \quad \delta\tilde{D} = \mathcal{L}_v \alpha + G_\Phi \alpha\,.$$

A field X in a cohomological multiplet satisfies

$$\delta^2 X = \mathcal{L}_v X + G_\Phi X,$$

where G_Φ is a gauge transformation with parameter Φ, and $\mathcal{L}_v$ is the Lie derivative with parameter v.

One can write down a supersymmetric Chern-Simons term for the cohomological multiplet [97]

$$S_{\text{CS}} = \frac{ik}{4\pi} \int_{S^3} \left(\text{CS}(A) - \kappa \wedge \Psi \wedge \Psi - 2d\kappa \wedge \Psi\alpha + (\Phi + i_v A) \left(\kappa \wedge d\kappa \left(\Phi + i_v A - 2\tilde{D} \right) - 2\kappa \wedge F \right) \right), \tag{5.45}$$

where $CS(A)$ is the usual Chern-Simons density for A. The field Φ transforms inhomogeneously under gauge transformations, thus ensuring the gauge invariance of $\exp(-S_{\text{CS}})$.

Källén has shown that these multiplets can be derived by twisting the fields in an $\mathcal{N} = 2$ vector multiplet $\left\{ A_\mu, \sigma, D \middle| \lambda_\alpha, \tilde{\lambda}_\alpha \right) \right\}$, such that, for instance,

$$\Phi = i\sigma + v^\mu A_\mu.$$

However, the validity of the multiplets, and the construction of the Chern-Simons action, is more general. In fact, as argued in [97], all that is required is for v to be a Killing vector, and that there exists a contact form κ, i.e. a one form such that $\kappa \wedge d\kappa$ is a volume form, for which v is the Reeb vector

$$i_v \kappa = 1, \qquad i_v d\kappa = 0.$$

In our case, we identify v with the parameter for the diffeomorphism in the square of the latitude supercharge. An appropriate κ can be obtained by lowering the index on v and replacing $\nu \to \nu^{-1}$. This requires $\nu \neq 0$, and agrees with Källén's $\mathcal{N} = 2$ twist when ν is equal to 1.

Källén's construction has been extended to $\mathcal{N} = 2$ Chern-Simons-matter theories in [233]. The authors of [233] also exhibit an invariant action based on cohomological matter and vector multiplets which is on-shell equivalent to the ABJM model. Unfortunately, both the action and the map between the ABJM fields and the twisted fields in [233] depend on details of the particular contact form being utilized. This contact form is the one associated to moving along the fiber of S^3, viewed as a circle bundle over S^2. More generally, the authors of [233] discuss the contact form associated with the fiber of a general Seifert manifold, which is a fibration over a Riemann surface. Our v is not of this type since, for $\nu \neq 1$, it mixes the vector field along the fiber with one associated to an action on the base. This is possible only because the base, in this case S^2, admits a continuous isometry.[7]

In [233], the square of the cohomological transformation includes an additional central

[7] For $\nu \in \mathbb{Q}_+$, v still generates a compact isometry acting freely on S^3.

generator whose eigenvalue is denote by Δ

$$\delta^2 X = \mathcal{L}_v X + G_\Phi X + \Delta X \,.$$

In principle then, there is enough freedom in the construction of [233] to accommodate the latitude superalgebra given in Section 5.2.1. However, one would have to verify that an action exists, of the type given in [233], which is supersymmetric for the values of v and Δ implied by the latitude superalgebra, and which is still on-shell equivalent to ABJM. While we expect that this is possible, we have not shown it explicitly.

The matrix model

We will proceed under the assumption that the latitude supersymmetry can be closed off-shell using the cohomological multiplets of [97, 233] described in the previous section. The connection appearing in the cohomological multiplet will be identified with the one appearing in the ABJM action. There are therefore two independent adjoint valued scalars appearing in the square of the supersymmetry transformation, one for each gauge group factor. According to our off-shell closure conjecture, the corresponding cohomological fields are identified with $\Phi_{1,2}$ in Eq 5.43.

The cohomological supersymmetry transformations in [97] make it clear that the moduli space associated with the vector multiplet is given by solutions to the following equations

$$F_{\mu\nu} = 0, \quad D_\mu \Phi = 0, \quad \tilde{D} = 0.$$

On S^3, the solutions are gauge equivalent to $A_\mu = 0$ and $\Phi = i\sigma$ where σ is an arbitrary spacetime independent adjoint valued parameter [22]. In this gauge multiplet background, the fermion variations in [233] imply that a chiral multiplet has no moduli whatsoever, as long as the parameter Δ is non-zero.[8] While this conclusion regarding the chiral moduli depends, in principle, on the contact form being used, we expect it to hold in general. Therefore, the moduli space is given by constant profiles for some scalars $\sigma^{(1,2)}$ and the result is a matrix model. The classical contributions to the matrix models, coming from the Chern-Simons terms, can now be read off from Eq 5.45 and coincide with the ones in 5.10.

Localization also yields an effective action for the moduli coming from one loop determinants. These determinants are straightforward to calculate in the cohomological formalism using the equivariant index theorem for transversely elliptic operators [21].[9] We will follow the application of this theorem to the squashed three sphere partition function appearing in [58]. This is convenient because both the moduli space and the bosonic symmetry obtained from the square of the latitude supercharge coincide with those of the squashed sphere, provided we identify $b = \sqrt{\nu}$. Since these are the only ingredients appearing in the equivariant index theorem, apart from the implications coming from the topology of the

[8] In the $\mathcal{N} = 2$ superalgebra, Δ corresponds to the $U(1)_R$ charge of the chiral multiplet. Equivalently, for a superconformal theory Δ is the conformal dimension of the dynamical scalar in the multiplet.

[9] Mathematical background for the equivariant index theorem can be found in [234, 235], while reviews of the application to supersymmetric localization appear in e.g. [236, 237].

manifold which are also the same, the one loop determinants coincide. The cohomological vector and projection multiplets therefore yield the following determinants [58, 158]

$$Z_{\text{vector}}\left(\sigma\right)=\prod_{\alpha>0}\sinh\left(\pi\sqrt{\nu}\alpha\left(\sigma\right)\right)\sinh\left(\pi\frac{\alpha\left(\sigma\right)}{\sqrt{\nu}}\right). \tag{5.46}$$

We must now evaluate the effective action for the moduli coming from matter fields. The values for the the $SU\left(4\right)_R$ symmetry transformations appearing the the square of the latitude supercharge for the four scalars C_I are $\pm i\nu^{\pm1/2}/2$. These should correspond to the parameters Δ for chiral multiplets in [233]. However, as is clear from [238], two of the C_I are the lowest components of chiral multiplets, while the other two are the lowest components of anti-chiral multiplets. In order to use directly the results in [58] for the one loop determinant of a chiral multiplet, we must therefore consider the quantum numbers of two C_I and two $\bar{C}^I$. Of course, the $\bar{C}^I$ fields are also in the complex conjugate gauge representation. The $\mathcal{N}=4$ version of the R-symmetry transformations (6.1) makes it clear that $C_{1,2}$ and $C_{3,4}$ are scalar doublets inside an ordinary and a twisted hypermultiplet respectively. Moreover, we can identify C_1 and C_3 as having the correct R-symmetry charges to be the lowest components of chiral superfields in the limit $\nu\to1$. That means that $\bar{C}_2$ and $\bar{C}_4$ are the the lowest components of the remaining chirals. Assuming that this holds also at $\nu\neq1$, this identification assigns flavor symmetry charges $-i\nu^{\pm1/2}/2$, with multiplicity 2, to the dynamical chiral fields. Note that there is no adjoint valued chiral multiplet in this calculation, as it has been integrated out to produce the ABJM superpotential [233].

Let

$$Q=\nu^{1/2}+\nu^{-1/2}.$$

The one loop determinant coming from the index theorem in [58], for the collection of fields with the quantum numbers discussed above, is[10]

$$Z_{\text{scalar}}\left(\sigma\right)=\prod_{\omega\in\left(\Box,\bar{\Box}\right)}s_{\sqrt{\nu}}\left(i\frac{Q}{2}\pm\omega\left(\sigma^{(1,2)}\right)-i\frac{\nu^{\pm1/2}}{2}\right) \tag{5.47}$$

$$=\prod_{\omega\in\left(\Box,\bar{\Box}\right)}\frac{1}{2\cosh\left(\pi\sqrt{\nu}\omega\left(\sigma^{(1,2)}\right)\right)2\cosh\left(\frac{\pi}{\sqrt{\nu}}\omega\left(\sigma^{(1,2)}\right)\right)}. \tag{5.48}$$

where the notation is meant to imply that we multiply the s_b functions for all values of $\pm$ corresponding to the different scalars. In the second line, we have used the following special function identities

$$s_b\left(-x\right)=s_b\left(x\right)^{-1},\quad \frac{s_b\left(\frac{ib}{2}+x\right)}{s_b\left(x-\frac{ib}{2}\right)}=\frac{1}{2\cosh\left(\pi bx\right)}.$$

The latter identity has recently played a role in the IR formula for correlation functions of Higgs and Coulomb branch operators in $\mathcal{N}=4$ SCFTs proposed in [239].

We can easily generalize the calculation to the Gaiotto-Witten type theories discussed

[10] s_b is the double sine function defined in (2.71) [158].

in Section 5.1.3. To every gauge group factor in such a theory we associate a moduli space given by an adjoint valued scalar σ, and a one loop determinant given by Eq (5.46). A hypermultiplet in a representation R contributes

$$Z_{\text{hyper}}\left(\sigma\right)=\prod_{\omega\in R}\frac{1}{2\cosh\left(\pi\sqrt{\nu}\omega\left(\sigma\right)\right)},$$

while a twisted hypermultiplet contributes

$$Z_{\text{twisted hyper}}\left(\sigma\right)=\prod_{\omega\in R}\frac{1}{2\cosh\left(\frac{\pi}{\sqrt{\nu}}\omega\left(\sigma\right)\right)}.$$

This result coincides with the form for the matrix models for some Chern-Simons-matter theories of this type conjectured in [85].

The unnormalized expectation value of the bosonic latitude loop corresponds simply to an insertion of

$$\frac{1}{N_1}\sum_{\rho\in R}\exp\left(2\pi\sqrt{\nu}\rho\left(\sigma\right)\right)$$

in the matrix model. The various factors of $2,\pi,\sqrt{\nu}$ can be deduced by comparing $\Phi_{1,2}$ at $\theta=0$ with the original bilinear expressions appearing in the bosonic loop 5.3. The complete computation of the expectation value of the bosonic latitude loop in ABJM therefore coincides with Eq 5.9 and the conjecture put forth in [79]. We emphasize that this result holds assuming that the off-shell closure procedure goes through.

In the next chapter we will tackle the problem in $\mathcal{N}=4$ theories without Chern-Simons term, which are not affected by these subtleties.

Chapter 6

Dualities for ABJM Latitude Wilson Loops

In the last chapter, we explore dualities between $\mathcal{N} = 8$ theories with line defects. As discussed in the first two chapters, there is strong evidence that the $U(N)$ ABJM model at $k = 1$ is dual to $\mathcal{N} = 8$ SYM, which is, in turn, dual to a 3d $\mathcal{N} = 4$ theory. Much less is known about the mapping of line operators in the duality. It is natural to start tackling the problem for the BPS case. For them, one can quantitatively test duality proposals with localization. Because localization is not directly reliable for $\mathcal{N} = 8$ SYM, we will compare ABJM at $k = 1$ with its $\mathcal{N} = 4$ dual.

Our perspective is somewhat similar to [61], where the authors studied the problem for mirror symmetry. Mirror symmetry eschanges Wilson and vortex loop [1]. The proposal relies on realizing loop operators as a mixed 3d/1d system, derived from specific brane constructions in type IIB string theory.

ABJM has a rich spectrum of line operators. We will limit to consider bosonic Wilson loops. Since the latitude algebra can be embedded in an $\mathcal{N} = 4$ Poincaré superalgebra, we ask the question for the family of latitude loops. Then, we explore the dual latitude type loops in *standard* $\mathcal{N} = 4$ theories: those without Chern-Simons terms. As a first step, in Section 6.1, we define latitude supercharges in standard theories. We show that the latitude supercharge is related to the supercharges preserving the topological sector of [31] and studied in Chapter 3. We then classify generic loop operators invariant under the latitude supercharge in theories without Chern-Simons terms. In Section 6.1.1, we exhibit the latitude superalgebra and its action on the supersymmetric quantum mechanics on a loop operator worldvolume.

In Section 6.1.2, we consider an $\mathcal{N} = 4$ gauge theory in the same universality class as ABJM. We use the form of the latitude supersymmetry algebra to conjecturally identify the loop operator representing the bosonic latitude in this IR dual theory. We then perform localization in the dual model and derive a matrix model expression for the expectation value of the dual loop operator. The matrix model for the dual theories looks somewhat

[1] The fact that a vortex loop is mapped into a Wilson loop can be readily checked in the abelian case. There, the vortex loop is equivalent to a complex real mass deformation and the Wilson loop to a complex FI term.

different from the one derived in ABJM but yields the same result for the expectation value of the loop. This fact constitutes our primary evidence for the validity of the identification of the dual loop operator.

6.1 The bosonic latitude supercharges in standard $\mathcal{N}=4$ theories

The latitude supercharges lie in an $\mathcal{N}=4$ subalgebra of the $\mathcal{N}=6$ supersymmetry algebra of ABJM. In this section, we study some properties of the bosonic latitude supercharges when these are realized on the three sphere in standard $\mathcal{N}=4$ theories. This is to be contrasted with the realization of these supercharges in the GW type theories of section 5.1.3 and in ABJM. We will use the $\mathcal{N}=4$ supersymmetry conventions of [31]. These are summarized in Appendix B.

We begin by defining an $\mathcal{N}=4$ latitude spinor for the latitude supercharge on S^3, and the analogous spinors for $Q_{2,3}$[2]

$$\left(\xi^L_\nu\right)_{\alpha a\dot{a}} \equiv \frac{i}{8}\bar{\Theta}^{IJ}_i\left(\nu\right)\Gamma^p_{IJ}(\bar{\sigma}_p)_{a\dot{a}}\epsilon^{(i)}_\alpha\,,$$

where

$$\bar{\sigma}_i \equiv i\tau_i\,,\quad i\in\{1,2,3\}\;,\qquad \bar{\sigma}_4\equiv \mathbb{1}_2\,.$$

Recall that $\epsilon^{(i)}_\alpha$ is a basis for the conformal Killing spinors on S^3, and Γ^p_{IJ} are the 6d Euclidean gamma matrices. Exchanging the indices a and $\dot{a}$ would produce a mirror latitude supercharge.

The spinor ξ^L_ν sits inside a linear space of spinors whose action on the fields generates a Poincaré subalgebra of the $\mathfrak{osp}\left(4|4\right)$ superconformal algebra on S^3, of the type discussed in [31]. This Poincaré subalgebra is characterized by the absence of Weyl transformations, and of diffeomorphisms associated with conformal Killing vectors, which generically appear in the anticommutator of conformal supercharges. As shown in [31], a Poincaré subalgebra of this type can be obtained by demanding that a Killing spinor ξ satisfy

$$\nabla_\mu\xi_{a\dot{a}} = \gamma_\mu\xi'_{a\dot{a}}\,,\qquad \xi'_{a\dot{a}} = \frac{i}{2r}h_a{}^b\xi_{b\dot{b}}\bar{h}^{\dot{b}}_{\dot{a}}\,,$$

where

$$h_a{}^b\in\mathfrak{su}\left(2\right)_H\;,\quad h_{\dot{a}}{}^{\dot{b}}\in\mathfrak{su}\left(2\right)_C\;.$$

For the latitude spinors, both h and $\bar{h}$ can be taken to be τ_3. Note that the Killing spinors associated with $Q_{2,3}$ are separately inside the Poincaré subalgebra defined by $h,\bar{h}$, while those of $Q_{1,4}$ are not.

The bosonic symmetries which appear in the square of the transformation using $\delta_{\xi^L_\nu}$ are

1. A diffeomorphism with Killing vector $v=-\nu\,\partial_\phi+\partial_\tau$;

[2] Note the difference in normalization of the spinor, by a factor of $\nu^{1/4}$, from the ABJM rescaled spinor in Eq (5.42)

2. R-symmetry transformations $-\nu R_C$ and R_H, where $R_{C,H}$ act on doublets of $SU(2)_C$ and $SU(2)_H$ as matrices

$$R_{C\dot{a}}{}^{\dot{b}} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_{Ha}{}^{b} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{6.1}$$

3. and a gauge transformation with parameter $\Lambda = \frac{1}{2}\left(\xi^L_\nu\right)^{c\dot{a}}\left(\xi^L_\nu\right)_c{}^{\dot{b}}\Phi_{\dot{a}\dot{b}} - v^\mu A_\mu$ with

$$\left(\xi^L_\nu\right)^{c\dot{a}}\left(\xi^L_\nu\right)_c{}^{\dot{b}} = \frac{1}{2}\begin{pmatrix} -e^{i\varphi}\sqrt{1-\nu^2}\sin\theta & -i \\ -i & e^{-i\varphi}\sqrt{1-\nu^2}\sin\theta \end{pmatrix}.$$

Relationship to topological quantum mechanics

Superconformal theories with 16 supercharges, in 3 and 4 dimensions, admit special local operator algebras whose correlators enjoy enhanced spacetime symmetry [27, 28]. In [31], the authors define such a set of "Higgs branch operators" in any standard 3d $\mathcal{N}=4$ gauge theory. The Higgs branch operators are non-trivial elements of the cohomology of a supercharge $\mathcal{Q}^H_\beta$, which is itself a combination of Poincaré and conformal supercharges. When placed along a line, correlators of Higgs branch operators are position independent, though they may still depend on operator ordering.

The authors of [31] go on to define an analogous cohomology on S^3, where $\mathcal{Q}^H_\beta$ is part of a specific Poincaré subalgebra of the full $\mathcal{N}=4$ superconformal algebra. This fact allows them to deform the $\mathcal{N}=4$ gauge theory by an appropriate $\mathcal{Q}^H_\beta$-exact Yang-Mills term, and to perform a localization computation which captures the expectation values of the operators. In order to preserve $\mathcal{Q}^H_\beta$, Higgs branch operators must be placed along a great circle in S^3. The result of the localization computation can be interpreted as a one dimensional topological field theory living on the circle, i.e. a topological quantum mechanics.

In the notation of [31], the supercharge

$$\mathcal{Q}^H_\beta \equiv \mathcal{Q}^H_1 + \beta\mathcal{Q}^H_2\,,$$

on S^3 has the following properties

$$\left(\mathcal{Q}^H_1\right)^2 = \left(\mathcal{Q}^H_2\right)^2 = 0\,, \qquad \left(\mathcal{Q}^H_\beta\right)^2 = 4i\beta\left(P_\tau + R_C\right),$$

where P_τ is a translation along the τ direction and R_C is an R symmetry transformation inside $SU(2)_C \subset SO(4)_R$.

$\mathcal{Q}^H_\beta$ is represented on S^3 by a spinor $\xi^H_{\beta a\dot{a}}$ which, after re-scaling, can be written in our notation as

$$
\begin{aligned}
\xi^H_{\beta 11} &= -\frac{1}{\sqrt{2}} R_i{}^4 e^{\Omega/2} \epsilon^{(i)}_{\mathbb{R}^3} ,\\
\xi^H_{\beta 12} &= \frac{\beta}{\sqrt{2}} R_i{}^3 e^{\Omega/2} \epsilon^{(i)}_{\mathbb{R}^3} ,\\
\xi^H_{\beta 21} &= -\frac{1}{\sqrt{2}} R_i{}^2 e^{\Omega/2} \epsilon^{(i)}_{\mathbb{R}^3} ,\\
\xi^H_{\beta 22} &= -\frac{\beta}{\sqrt{2}} R_i{}^1 e^{\Omega/2} \epsilon^{(i)}_{\mathbb{R}^3} .
\end{aligned}
$$

The mirror spinor $\xi^C_{\beta a\dot{a}} \equiv \xi^H_{\beta \dot{a} a}$ generates the mirror supercharge $\mathcal{Q}^C$ used to define the Coulomb branch version of the cohomology in [32, 33]. The square of the transformation $\delta_{\xi^C_\beta}$ gives

1. A translation with Killing vector $v_C = \partial_\tau$;
2. An R-symmetry transformation acting on doublets of $SU(2)_H$ as the matrix

$$
R_{H a}{}^b = \frac{i\beta}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,
$$

   with R_C vanishing;

3. and a gauge transformation with parameter $\Lambda_C = \frac{1}{2} \left(\xi^C_\beta\right)^{c\dot{a}} \left(\xi^C_\beta\right)_c{}^{\dot{b}} \Phi_{\dot{a}\dot{b}} - v^\mu_C A_\mu$.

In fact, we can show that $\mathcal{Q}^C$ is related to the latitude supercharge in the limit $\nu \to 0$ and $\beta \to 1$.[3] We do this by exhibiting a *global* $SO(4)_R$ rotation, defined by matrices

$$
(W_1)_a{}^b \in SU(2)_l , \quad (W_2)_{\dot{a}}{}^{\dot{b}} \in SU(2)_r ,
$$

such that

$$
\left(\xi^L_\nu\right)_{\alpha a\dot{a}} = (W_1)_a{}^b (W_2)_{\dot{a}}{}^{\dot{b}} \left(\xi^C_\beta\right)_{\alpha b\dot{b}} .
$$

The explicit matrices are given by

$$
W_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \quad W_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} .
$$

Hence, the latitude supercharge at generic ν, or its mirror dual, *interpolates* between the ordinary $\mathcal{N} = 2$ supercharges used to compute the partition functions in [22] and the supercharges used to define the Higgs and Coulomb branch cohomologies in [31–33]. This is the second connection between the latitude loop and the SQM used in [31, 32]. It is not clear whether there is a physical interpretation for it.

[3]We chose an embedding of the latitude in the UV $\mathcal{N} = 4$ algebra which relates the latitude supercharge to the mirror supercharge $\mathcal{Q}^C$. Taking the mirror embedding would yield a supercharge related to $\mathcal{Q}^H$.

orbit $\dot{\ell}$	$\theta = 0$	$\theta = \pi/2$	θ generic
$\nu = \pm 1$	closed + maximal	closed + maximal	closed + maximal
$\nu = 0$	closed + maximal	point	closed
ν generic	closed + maximal	closed + maximal	non-compact

Table 6.1: The type of curve represented by ℓ as a function of the fixed angle θ. A "maximal" closed curve is a great circle on S^3.

Latitude loops in standard $\mathcal{N} = 4$ theories

In this section, we exhibit supersymmetric loops of bosonic latitude type in standard $\mathcal{N} = 4$ gauge theories. 3d $\mathcal{N} = 4$ gauge theories of this kind admit more than one type of supersymmetric loop observable. Some of these observables are of Wilson loop type, while others are defect operators, which are sometimes called vortex loops [57, 58].[4] Both types of loop observable can be tuned to preserve the bosonic latitude supercharges. We will define prototypical operators of both types, and tabulate the supercharges they preserve.

The worldvolume of a loop operator on S^3, invariant under the latitude supercharge, must itself be invariant under the translation present in the square of the supercharge. Let $\ell^\mu\left(\tau_0\right)$ be coordinates on the loop with affine parameter τ_0. ℓ^μ is the orbit under the action of the vector field $\dot{\ell}$ which must coincide with $-\nu\,\partial_\phi + \partial_\tau$ restricted to some θ. Whether or not ℓ is a closed curve depends on the value of θ where the loop is placed, as well as ν. The types of orbits are detailed in Table 6.1. In addition, there may be closed orbits at special values of ν and $\cos\theta$.

Wilson loops

In a standard $\mathcal{N} = 4$ gauge theory, a supersymmetric Wilson loop can be constructed using only vector multiplet fields. For an ordinary $\mathcal{N} = 4$ vector multiplet, the relevant fields are the connection A_μ and the $SU\left(2\right)_C$ triplet of scalars $\Phi_{\dot{a}\dot{b}}$. We take the following ansatz for a supersymmetric Wilson loop along a contour with coordinates ℓ^μ

$$\mathcal{W} = \frac{1}{\dim\left(\mathfrak{R}\right)}\mathrm{tr}_{\mathfrak{R}}\mathcal{P}\exp\oint_\ell\left(\dot{\ell}^\mu A_\mu + \left|\dot{\ell}\right| M^{\dot{a}\dot{b}}\Phi_{\dot{a}\dot{b}}\right)\,,$$

where $M^{\dot{a}\dot{b}}$ is a coordinate dependent, but field independent, symmetric matrix. One can show that the most general such loop preserving the latitude supercharge runs along $\dot{\ell}^\mu\partial_\mu = -\nu\partial_\varphi + \partial_\tau$ with fixed θ, and has

$$M^{\dot{a}\dot{b}} = M^{\dot{a}\dot{b}}_{\text{Wilson}} \equiv \left|\dot{\ell}\right|^{-1}\begin{pmatrix} \frac{1}{2}e^{i\varphi}\sqrt{1-\nu^2}\sin\theta & \frac{i}{2} \\ \frac{i}{2} & -\frac{1}{2}e^{-i\varphi}\sqrt{1-\nu^2}\sin\theta \end{pmatrix}\,, \tag{6.2}$$

[4] Vortex loops appeared early on in the context of bosonic Chern-Simons theory [117, 191]. Supersymmetric surface operators, which are the four dimensional analogue of vortex loops, were considered in [54, 55] (see [240] for a review).

	$\theta = 0$		$\theta = \pi/2$		θ generic	
supercharges preserved:	total	Poincaré	total	Poincaré	total	Poincaré
$\nu = 1$	8	$Q_{2,3} + 2$	8	$Q_{2,3} + 2$	4	$Q_{2,3}$
$\nu = 0$	8	$Q_{2,3} + 2$	N/A	N/A	4	2
ν generic	8	$Q_{2,3} + 2$	4	2	N/A	N/A

Table 6.2: Supercharges preserved by a generic $\mathcal{N} = 4$ latitude Wilson loop: total number preserved and number preserved inside the Poincaré subalgebra defined by the latitude supercharge, with $Q_{2,3}$ singled out.

which is a familiar expression. Note that at $\theta = 0$ the Wilson loop is ν-independent.

The total number of supercharges preserved by $\mathcal{W}$ are listed in Table 6.2. A subset of these are within the Poincaré subalgebra containing the latitude supercharge. Their number and type depend on the value of θ where the loop is placed and on ν.

Vortex loops

The supersymmetric vortex loops we consider are defect operators associated to singular classical BPS configurations embedded in a vector multiplet, of the type described in e.g. [57, 58]. In these configurations, the field strength F is taken to be proportional to a delta function supported on the loop worldvolume, and an imaginary auxiliary field is turned on to preserve supersymmetry. The effects of the connection associated with F can be felt by charged local operators away from the loop worldvolume, while the profile for the auxiliary field cannot. Nevertheless, this profile should be considered part of the configuration describing the defect for some applications [57].

A BPS defect embedded in a vector multiplet is a singular fixed point of the gaugino transformations. A typical 1/2 BPS abelian vortex loop defect of charge q in an $\mathcal{N} = 2$ gauge theory solves the BPS equation for a vector multiplet in the following way [57]

$$F = 2\pi q \delta_\ell, \quad D = -2\pi q i \star (\delta_\ell \wedge d\ell) \ ,$$

where $d\ell$ is the one form dual to $\dot{\ell}$, and δ_ℓ is the Poincaré dual to the loop worldvolume. To get non-abelian vortex loops, one replaces the number q with an element of the Lie algebra $\mathfrak{g}$.[5] Supersymmetry requires that the loop worldvolume be the integral curve of the vector field one gets from squaring the supercharge. In the case of $\mathcal{N} = 2$ supersymmetry, this worldvolume is always a maximal circle on S^3.

The $\mathcal{N} = 4$ version of the supersymmetric vortex loop is entirely analogous. We set F

[5]The number q, or the Lie algebra element generalizing it, are subject to discrete identifications arising from large gauge transformations [54].

	$\theta = 0$		$\theta = \pi/2$		θ generic	
supercharges preserved:	total	Poincaré	total	Poincaré	total	Poincaré
$\nu = 1$	8	$Q_{2,3} + 2$	8	$Q_{2,3} + 2$	4	$Q_{2,3}$
$\nu = 0$	4	$Q_{2,3}$	N/A	N/A	4	$Q_{2,3}$
ν generic	4	$Q_{2,3}$	8	$Q_{2,3} + 2$	N/A	N/A

Table 6.3: Supercharges preserved by a generic $\mathcal{N} = 4$ latitude vortex loop: total number preserved and number preserved inside the Poincaré subalgebra defined by the latitude supercharge, with $Q_{2,3}$ singled out.

as above, and take the triplet of auxiliary fields in the $\mathcal{N} = 4$ vector multiplet to be

$$D^{ab} = -2\pi q \star (\delta_\ell \wedge d\ell)\, M^{ab}_{\text{vortex}} \,, \tag{6.3}$$

$$M^{ab}_{\text{vortex}} \equiv \begin{pmatrix} -\frac{i}{2} e^{i\tau} \sqrt{1-\nu^2} \cos\theta & \frac{\nu}{2} \\ \frac{\nu}{2} & -\frac{i}{2} e^{-i\tau} \sqrt{1-\nu^2} \cos\theta \end{pmatrix} . \tag{6.4}$$

The resulting singular background solves the BPS equation from [31], i.e. the variation of the gaugino, specialized to the latitude supercharge

$$\delta_{\xi^L_\nu} \lambda_{a\dot{b}} = -\frac{i}{2} \varepsilon^{\mu\mu'\rho} \gamma_\rho \xi^L_{\nu a\dot{b}} F_{\mu\mu'} - D_a{}^c \xi^L_{\nu c\dot{b}} = 0 \,, \quad \forall a, \dot{b} \,.$$

All of the vortex loops defined by the above configuration preserve $Q_{2,3}$ separately, as well as the latitude supercharge. Additional supercharges are given in Table 6.3.

In a twisted $\mathcal{N} = 4$ vector multiplet, the roles of the dotted and undotted indices, and the matrices M_{Wilson} and M_{vortex}, are exchanged. The relationship between the Wilson and vortex loops is basically mirror symmetry [57, 61].

6.1.1 Supersymmetric Quantum Mechanics

An alternative definition of BPS line operators is provided by coupling a 1d supersymmetric quantum mechanics (SQM) supported on the defect to the 3d bulk theory. The coupling can be realized by gauging the flavor symmetries of the 1d system with the 3d vector multiplet. For instance, in [61], a large class of 1/2-BPS loop operators has been described in this way. However, in that case, the 1d theory always preserves 4 supercharges, whereas our SQM leaves unbroken an $\mathcal{N} = 2$ SUSY algebra.

To be concrete, we introduce the action of the latitude supercharges

$$\delta = \epsilon\, Q_2 + \bar{\epsilon}\, Q_3 \,. \tag{6.5}$$

The latitude superalgebra in the $\mathcal{N}=4$ language reads as

$$\{Q_2, Q_3\} = \nu F + \tilde{H}\,, \tag{6.6}$$

where $F \equiv R_C - P_\phi$ and $\tilde{H} \equiv P_\tau - R_H$ [6]. The only non-vanishing commutators are

$$[R_C, Q_2] = -Q_2\,, \qquad [R_C, Q_3] = Q_3\,. \tag{6.7}$$

We want to interpret this algebra as a centrally extended $\mathcal{N}=2$ 1d Supersymmetry algebra. Since $\tilde{H}$ generates the translations along the loop, it plays the role of the Hamiltonian of the SQM. Furthermore, as P_ϕ acts only on the normal bundle in S^3 with respect to the loop worldvolume, νF behaves as a central extension of the algebra and it generates an $U(1)$ flavor symmetry for the SQM. The reader can find a complete description of the $\mathcal{N}=2$ and $\mathcal{N}=4$ SQM in [210]. For now, we limit ourselves to point out that our generator νF should be identified with their generator J_-, which is an R-symmetry from the $\mathcal{N}=4$ point of view, but it is a flavor symmetry from the $\mathcal{N}=2$ perspective.

While the outlined procedure is straightforward for a line operator, we might need to turn on background fields on a curved manifold. This usually occurs when we place supersymmetric theories on curved spaces. Moreover, the determination of the backgrounds fields becomes relevant when we will define the refined Witten index of the SQM. The action of Q_2 and Q_3 allows us to organize the 3d $\mathcal{N}=4$ vector multiplet into 1d $\mathcal{N}=2$ multiplets. Thus, we can read the action of the flavor symmetries looking at the action of δ^2 on the reduced multiplets.

On general grounds, the degrees of freedom of the 3d $\mathcal{N}=4$ vector multiplet are recast into one $\mathcal{N}=2$ 1d vector multiplet, two $\mathcal{N}=2$ 1d chiral multiplets, and two $\mathcal{N}=2$ 1d Fermi multiplets. The explicit decomposition is spelled out in Appendix E.

Let us discuss how ν affects the 1d algebra. On a hand, a chiral multiplet has charge 1 under νF. On the other hand, ν never acts on the 1d vector multiplet. It follows that νF is a flavor symmetry of the $\mathcal{N}=2$ algebra. Also the loops described in [61] exhibit an analogous flavor symmetry, generated by J_-. Thus, we identify νF with J_-. However, for them, ν can take only integer values. In that case, ν can always be reabsorbed into a redefinition of the 1d Killing spinors by a factor $e^{i\nu\tau}$. Thus, the $\mathcal{N}=2$ 1d vector and $\mathcal{N}=2$ chiral constitute an $\mathcal{N}=4$ vector multiplet and the full $\mathcal{N}=4$ 1d SUSY is restored. For latitude loops, the non integer value of ν prevents this kind of enhancement and breaks down the Supersymmetry to $\mathcal{N}=2$.

Then, we claim that if we want to describe the latitude as 3d/1d defect system, we need to gauge the SQM with a vector given by the embedding described in E. In addition, a background flavor symmetry for the generator νF must be turned on. We will provide later the explicit action for the SQM.

The considerations above hold for $\nu \neq 1$. In fact, as described in Section 5.1.2, when $\nu \to 1$ the loop exhibits supersymmetry enhancement. In particular, two extra super-

[6] Up to a ν rescaling, the generators $\tilde{H}$ and F can be identified respectively with the central charge Z and the combination $\mathcal{T} + 2L_z$ as $\theta_0 \to 0$.

charges, Q_5 and Q_6, annihilate the loop. The algebra spanned by Q_2, Q_3, Q_5, and Q_6 is not a part of any three-dimensional Poincaré subalgebra, and the resulting 1d superalgebra is a superconformal algebra, rather than a 1d $\mathcal{N} = 4$ Poincaré algebra as in [61]. Therefore, when $\nu \to 1$ the supersymmetric Quantum Mechanics describing the loop becomes a superconformal Quantum Mechanics. We present the analysis of the 1d superalgebra, repeating the same steps as for generic ν. Details regarding the embedding are given in Appendix *E*.

Thus, let us define the action of the supercharges:

$$\delta = \epsilon\, Q_2 + \bar{\epsilon}\, Q_3 + \rho\, Q_5 + \bar{\rho}\, Q_6 \,. \tag{6.8}$$

It is convenient to recast the four parameters ϵ, $\bar{\epsilon}$, ρ, $\bar{\rho}$ into two "superconformal" parameters:

$$\zeta = e^{-\frac{i}{2}\tau}\rho + e^{\frac{i}{2}\tau}\epsilon\,, \qquad \bar{\zeta} = e^{-\frac{i}{2}\tau}\bar{\rho} + e^{\frac{i}{2}\tau}\bar{\epsilon} \tag{6.9}$$

These 1d spinors are indeed anti-periodic. Even though this choice is somehow unconventional, it reproduces the expected results[7].

A first hint of the realization of an underlying superconformal symmetry comes from the space-time symmetries of δ^2. In fact, the most general diffeomorphism contained in δ^2 generates the 1d conformal group $SL(2,\mathbb{R})$. Besides, δ^2 contains a non zero dilatation, as well as some R-symmetries and a gauge transformation. Finally, in the appendix, we provide some concrete examples of the action of the supercharges on the SQM multiplets. They turn out to agree with the algebra found for the latitude Wilson loop at $\nu = 1$ of section 5.1.2. Then, we conclude that our dual operator reproduces the expected symmetry enhancement, namely the $\mathcal{N} = 2$ ν-dependent Poincaré SUSY algebra "flows" to a conformal $\mathcal{N} = 2$ superconformal algebra as $\nu \to 1$.

6.1.2 Localization in a dual theory

ABJM at Chern-Simons level 1 with gauge group $U(N) \times U(N)$ is a dual infra-red description of $U(N)$ $\mathcal{N} = 8$ super-Yang-Mills [42]. There exists an additional $\mathcal{N} = 4$ gauge theory with no Chern-Simons terms within the same universality class. This theory has the following $\mathcal{N} = 4$ content: a $U(N)$ vector multiplet, one adjoint hypermultiplet, and one fundamental hypermultiplet [42]. We will call this theory "the UV theory". An alternative to localization in ABJM, in the presence of the latitude Wilson loop, is localization in the UV theory with an insertion of the dual loop operator. In this section, we carry out this localization and the identification of the dual loop operator.

The localizing term

The Yang-Mills action S_{YM} as defined in [31], Eq B.7, with appropriate matrices $h, \bar{h}$, is closed under $Q_{2,3}$ and under the latitude supercharge. We can show that S_{YM} is exact, i.e. it is the variation under the latitude supercharge of an appropriate fermionic functional. It can therefore be used as a localizing term.

[7] Anti-periodic Killing spinors have made an appearance in the context of the hyperbolic index, aka the supersymmetric Renyi entropy [241], and the 4d superconformal index [242]

S_{YM} was shown in [31] to be exact using the auxiliary spinor

$$\xi^H_{-\beta a\dot{a}} \equiv \xi^H_{\beta a\dot{b}} (\tau_3)^{\dot{b}}{}_{b} \,.$$

We define the analogous spinor for the latitude

$$\left(\tilde{\xi}^L_\nu\right)_{a\dot{a}\alpha} \equiv \left(\xi^L_\nu\right)_{a\dot{b}\alpha} \tau_3{}^{\dot{b}}{}_{\dot{a}} \,.$$

We can now try to form the following localizing term

$$S_{\mathrm{localizing}} \equiv \delta_{\xi^L_\nu} \delta_{\tilde{\xi}^L_\nu} \left(\frac{1}{2g_{YM}} \tau_3^{ab} \tau_3^{\dot{a}\dot{b}} \int d^3x \sqrt{g} \mathrm{Tr} \left(\lambda_{a\dot{a}} \lambda_{b\dot{b}} - 2 D_{ab} \Phi_{\dot{a}\dot{b}} \right) \right) . \tag{6.10}$$

Somewhat surprisingly, $S_{\mathrm{localizing}}$ is indeed proportional to S_{YM}.

The localization locus for S_{YM} was worked out in [31]. It consists of the usual spacetime independent vev for an adjoint valued scalar field, which in this case is $\Phi_{\dot{1}\dot{2}}$.[8] The one loop determinant, which gives the effective action for this modulus, is identical to the one derived in the $\mathcal{N}=2$ formalism in [22]

$$\sigma \equiv \langle \Phi_{\dot{1}\dot{2}} \rangle \,, \quad Z_{\mathrm{vector}}\left(\sigma\right) = \prod_{\alpha>0} 4 \sinh^2\left(\pi \alpha\left(\sigma\right)\right) = \prod_{i<j}^{N} 4 \sinh^2\left(\pi\left(\sigma_i - \sigma_j\right)\right) \,.$$

At this point, it may seem strange that the effective action for σ coming from the vector multiplet fields does not depend on ν. For instance, we could have used the index theorem to derive the one loop determinant in the cohomological formalism, as we did in 5.2.2 for ABJM, the results of which surely depend on ν. The resolution is that, from the $\mathcal{N}=2$ perspective, the $\mathcal{N}=4$ vector multiplet contains an additional dynamical adjoint chiral multiplet, whose lowest component is given by $\Phi_{\dot{1}\dot{1}}$, which was not present in 5.2.2.[9] As can be seen from the action of the square of the latitude supercharge on $\Phi_{\dot{1}\dot{1}}$, this additional multiplet has the correct quantum numbers to correct the total one loop determinant to the original $\mathcal{N}=2$ expression via the special function identity

$$\begin{aligned}\prod_{\alpha>0} 4 \sinh^2\left(\pi\alpha\left(\tilde{\sigma}\right)\sqrt{\nu}\right) = \prod_{\alpha>0} s_{\sqrt{\nu}} \left(i\frac{Q}{2} - \alpha\left(\tilde{\sigma}\right) - i\sqrt{\nu} \right) s_{\sqrt{\nu}} \left(i\frac{Q}{2} + \alpha\left(\tilde{\sigma}\right) - i\sqrt{\nu} \right) \times \\ \times \sinh\left(\pi\sqrt{\nu}\alpha\left(\tilde{\sigma}\right)\right) \sinh\left(\pi\frac{\alpha\left(\tilde{\sigma}\right)}{\sqrt{\nu}}\right) ,\end{aligned}$$

which again appears in [239].[10]

As argued in [31], the hypermultiplets in a theory of this type do not require localization since their action is quadratic when evaluated in the background of the localized vector

[8] The auxiliary field D_{12} also has a vev.

[9] More accurately, the adjoint chiral multiplet in the ABJM model had the right quantum numbers to form an invariant δ-exact mass term and could therefore be integrated out, or ignored in the localization calculation. In the $\mathcal{N}=2$ formalism, this is the situation for a chiral multiplet with R-charge 1.

[10] In this expression, $\tilde{\sigma} \equiv \langle \Phi_{\dot{1}\dot{2}} \rangle / \sqrt{\nu}$, so that the argument of the sinh function coincides with the one derived above. This has been done in order to match the normalization of the supercharge used in the cohomological localization of ABJM in Section 5.2.2.

multiplet. The fields can therefore be integrated out exactly at one loop without adding any deformation term. The resulting one loop determinants coincide with the ones derived in [22] using the $\mathcal{N}=2$ formalism and are, in particular, ν-independent[11]

$$Z_{\text{adjoint hyper}}(\sigma)=\prod_{i,j}^{N}\frac{1}{2\cosh\left(\pi\left(\sigma_i-\sigma_j\right)\right)},\quad Z_{\text{fund. hyper}}(\sigma)=\prod_{i}^{N}\frac{1}{2\cosh\left(\pi\sigma_i\right)}.$$

In fact, the entire matrix model is the same as the one derived for the UV theory in [167]. This is an analogue of the result obtained with the Higgs branch supercharge, equivalently at $\nu\to 0$, in [31]. We have shown that it holds for arbitrary values of ν.

A UV avatar for the latitude

At first blush, it seems that the Wilson loop described in 6.1, placed at $\theta=0$, could serve as a UV avatar for the bosonic latitude loop in ABJM, i.e. the two operators would be identified at the IR fixed point. However, after localization the expectation value of this loop, in the fundamental representation, is simply

$$\left\langle\sum_i e^{2\pi\sigma_i}\right\rangle_{\text{KWY}}, \tag{6.11}$$

where KWY indicates the original matrix model for the UV theory described in [167]. In particular, the expectation value is ν-independent. Moreover, this Wilson loop preserves far more supersymmetry than does the ABJM bosonic latitude loop.

Another option is to identify the latitude loop with a vortex loop preserving the latitude supercharge at $\theta=0$. In fact, insertion of a vortex loop alters the one loop determinants used in the matrix model in a way which is qualitatively similar to the expressions Eq 5.14, c.f. [57]. However, the identification of the latitude loop with a vortex loop would imply, for instance, that the additional supercharges preserved by the loop at $\nu\to 1$, denoted in 5.1.2 as $Q_{5,6}$, sit in the same Poincaré subalgebra as $Q_{2,3}$, which turns out not to be correct.

In order to identify a UV avatar for the latitude loop we must therefore search for a different BPS loop operator. A clue comes from the comparison of the partition function of the UV theory to ABJM, carried out in [167]. It implies that the relationship between the theories involves a specific $SL(2,\mathbb{Z})$ duality transformation. This transformation is also visible in the original type IIB brane construction in [42]. The transformation is not merely mirror symmetry, i.e. a transformation using the S element of the type IIB S-duality group, but rather an action which involves the T generator as well, of the type studied in e.g. [96]. In particular, FI and mass terms are not exchanged by the duality, but rather mixed.

Under mirror symmetry, Wilson loops are mapped to vortex loops [57, 61], and it seems reasonable to expect that the full action of $SL(2,\mathbb{Z})$ maps Wilson loops to combined vortex-Wilson loop operators. Such mixed operators can presumably be defined in a variety of

[11] From the point of view of the equivariant index theorem of Section 5.2.2, the ν independence here is a simple consequence of having only hypermultiplets and no twisted-hypermultiplets, and of the rescaling of the Killing spinor used in this model.

ways, but we are not aware of any previous attempts to do so. We will argue that this possibility is realized for the bosonic latitude loop of ABJM and its avatar in the UV theory. As preliminary evidence, we note that the supercharges preserved by a combination of Wilson and vortex loops of the UV theory exactly match the supercharges preserved by the latitude loop, even when taking into account the additional supercharges present in the limit $\nu \to 1$. In this statement, the supercharges preserved by the combined line operator are assumed to be the ones in the overlap of those preserved by the constituent lines. Note that the fermionic latitude loop in ABJM *does* preserve the right amount of supersymmetry, at $\nu \to 1$, to be mapped to a pure Wilson loop. The fermionic latitude at generic ν presumably maps to an as-yet-unidentified loop operator of vortex-Wilson type, in the same cohomology class as the bosonic latitude, but preserving more of the supersymmetry.

We will characterize the specific BPS operator dual to the bosonic latitude in the next section. We will make frequent use of the results for 1/2 BPS Wilson and vortex operators produced in [61], whose derivation was aided by the type IIB String Theory construction of the relevant quiver gauge theories [114]. It should be noted, however, that the relevance of supersymmetric Wilson and vortex operators, and presumably of any mixed versions, goes beyond the class of theories which can be engineered in type IIB [93]. This conclusion follows from the applicability of mirror symmetry, with both the S and T generators, in more general classes of quiver gauge theories [243]. More generally, mixed operators should be relevant even to non-supersymmetric theories in the context of the $SL(2,\mathbb{Z})$ action on conformal field theories with abelian symmetry introduced by Witten in [244].

Latitude loops from SQM

According to the authors of [61], we can think of Wilson and vortex loop insertions in 3d $\mathcal{N}=4$ theories in two useful ways

1. as type IIB 1-branes ending on a Hanany-Witten type setup of D3, NS5, and D5 branes engineering the theory [114], and on other branes away from the main setup;

2. or as the coupling of the 3d fields to a supersymmetric quantum mechanics (SQM) living on the loop worldvolume.

In order to recover the effect of the loop operator from the brane description, one starts by reading off the worldvolume SQM. This SQM is the effective theory living on the 1-branes, which have one compact direction. The SQM couples to the bulk theory using gauge and superpotential terms. One then integrates out the quantum mechanical degrees of freedom to obtain a deformation of the bulk theory localized on the loop.

It is often useful to preform localization before integrating out the SQM. This involves a computation of the supersymmetric index (Witten index) of the supersymmetric quantum mechanics with certain deformations. The complete picture is useful, for example, because the action of 3d mirror symmetry can be identified with type IIB S-duality, whose action on all of the branes in the setup is known. We refer the reader to very interesting analysis in reference [61] for more details.

In order to take advantage of the brane description, we need a Hanany-Witten type setup for the UV thoery, and a 1-brane which describes the operator. The former was described in [42]. The authors of [42] described two different setups, one which engineers the ABJM model and another which engineers the UV theory. The two setups are related by S-duality of type IIB string theory. Unfortunately, we do not have a description of the latitude loop as a 1-brane which attaches to these brane setups. We will therefore make an educated guess about the SQM governing the bosonic latitude loop, based on the symmetry algebra and the corresponding theories for Wilson and vortex loops presented in [61]. We then show that integrating out the SQM degrees of freedom reproduces the correct matrix model for the expectation value of the bosonic latitude loop. While we have good reason to expect that the SQM we present is the correct one, we would like to stress that we have no evidence for the realization of this theory on the worldvolume of a compact type IIB 1-brane. We nevertheless make some comments below.

Brane setup

The type IIB setup engineering the ABJM model at level 1 includes [42]

- A stack of N coincident D3-branes along the directions 0126 with the 6 direction compactified to a circle.
- One NS5-brane spanning 012345 and situated at some point in the 6 direction.
- One $(1,1)$ brane spanning $012\,[3,7]_\theta\,[4,8]_\theta\,[5,9]_\theta$ and situated at a different point in the 6 direction. A $(1,1)$ brane is a bound state of an NS5-brane and a D5-brane. The subscript θ indicates that the brane is rotated by an angle θ in the relevant plane. For level 1 we have $\theta = \pi/4$.

Performing an S-duality transformation, and shifting the type IIB axion, brings the ABJM setup to the following one engineering the UV theory [42]

- A stack of N coincident D3-branes along the directions 0126 with the 6 direction compactified to a circle.
- One D5-brane spanning 012345 and situated at some point in the 6 direction.
- One NS5-brane spanning $012\,[3,7]_\theta\,[4,8]_\theta\,[5,9]_\theta$ and situated at a different point in the 6 direction.
- A constant value for the type IIB axion $\chi = 2\pi$.

A Wilson loop in a gauge theory living on D3-branes can be engineered by adding fundamental strings that end on the D3-branes. For a Wilson loop in the fundamental representation, a single string suffices. A vortex loop can be engineered by including D1-branes instead of fundamental strings. Both 1-branes have one compact direction and must end on a 5-brane which is situated away from the main setup [61]. It seems reasonable to expect that the latitude loop can be engineered in a similar fashion. A 1-brane realizing the latitude loop in the UV theory should be a combination of those realizing the Wilson and vortex loops, for instance it could be a bound state. Such bound states indeed exist in type IIB string theory [245]. However, in order to derive the worldvolume SQM theory on such a compact brane one must analyze the boundary conditions imposed by the branes in the compact direction, which is beyond the scope of this paper. We will content ourselves with finding a supersymmetric quantum mechanics with the desired properties.

The latitude supercharges at generic ν can be shown to be linear combinations of the supercharges preserved by the 1/2 BPS Wilson loop treated in [61], while the mirror supercharges are linear combinations of the those preserved by the 1/2 BPS vortex. Our starting point for constructing a SQM for the bosonic latitude will be the worldvolume theory for the vortex loop, engineered by a single D1-brane, as described in [61]. We will deform this theory as necessary to accommodate the properties of the latitude.

The latitude worldvolume theory

The setup engineering 1/2 BPS vortex loops in 3d $\mathcal{N}=4$ theories described in [61] has a D1-brane ending on a D3 brane stack on the one end and on an NS5'-brane on the other. A single D1-brane carries a $U(1)$ gauge field on its worldvolume. In the absence of any other branes, the worldvolume theory has 2d $\mathcal{N}=(8,8)$ supersymmetry. The compact direction causes the worldvolume theory on the D1 to be, effectively, an $\mathcal{N}=4$ gauged supersymmetric quantum mechanics. The S generator of Type IIB S-duality exchanges this D1-brane with a fundamental string ending on the D3 brane stack, which is the standard description for a supersymmetric Wilson loop.

According to the discussion in section 3 of [61], the worldvolume theory of a D1-brane ending on a D3-brane stack has the following matter content, in terms of 1d $\mathcal{N}=4$ multiplets

- a $U(1)$ vector multiplet;
- N charge 1 chiral multiplets. Their $U(N)$ flavor symmetry is gauged by the bulk $U(N)$ gauge symmetry living on the D3 branes where they sit in the fundamental representation;
- N charge -1 chiral multiplets. Their $U(N)$ flavor symmetry is gauged by the bulk $U(N)$ gauge symmetry living on the D3 branes where they sit in the anti-fundamental representation.

The setup for the latitude loop at generic ν must, by definition, preserve only the supercharges $Q_{2,3}$. Hence, we are looking for a ν-dependent deformation of the D1-brane worldvolume theory giving rise to a 1d $\mathcal{N}=2$ $U(1)$ gauge theory with matter. Moreover, as argued in Section 6.1.1, the theory must be conformal in the limit $\nu\to 1$. We have already seen that ν appears in the latitude superalgebra as the coefficient of a central term associated to a flavor symmetry. Such a term indeed breaks conformal invariance. The only continuous $\mathcal{N}=2$ preserving global symmetry in the D1 theory is an $\mathcal{N}=4$ R-symmetry which commutes with an $\mathcal{N}=2$ subalgebra. The generator of this R-symmetry was called

J_- in [61, 210], and was indeed identified with a combination of bulk charges of the type that make up F. Weakly gauging J_- in the D1-brane worldvolume theory, with a background gauge field with holonomy $\exp(2\pi i\nu)$, breaks $\mathcal{N}=4$ to $\mathcal{N}=2$. The full $\mathcal{N}=4$ supersymmetry is recovered in the limit $\nu \to 1$.[12]

An additional part of the worldvolume action for the bosonic latitude must break half of the supersymmetry, but not conformal invariance, even at $\nu \to 1$. A minimal guess is that the worldvolume theory has, in addition to the usual minimally coupled action, a level 1 1d Chern-Simons term for the $U(1)$ gauge field. We will see below that postulating this action leads to the correct result for the expectation value of the latitude loop.[13]

Including a topological term for the worldvolume gauge field in order to produce the "electric" part of a defect operator is familiar from the construction of generic surface operators in 4d $\mathcal{N}=4$ gauge theories described in [54]. The relevant electric parameter of the surface operator is η, which is defined as the coefficient of a term measuring the first Chern class of an abelian bulk gauge field restricted to the surface operator worldvolume [240]. When the surface operator is realized by coupling a 4d theory, with gauge group $SU(2)$ broken near the defect to $U(1)$, to a 2d GLSM with $SU(2)$ flavor symmetry, η is realized as the theta parameter of the dynamical $U(1)$ gauge field in the worldvolume theory. Indeed, the construction of the same surface operator in type IIA string theory using additional D2-branes, described in [246], may be related by T duality to our sought-after type IIB setup.

The index

We now evaluate the 1d supersymmetric index for the latitude worldvolume theory, following [210] and [61]. The index can be deduced from the computations of the example in Appendix B.1 of [61]. We must make a few changes to this example. In the notation of Appendix B.1, the changes are as follows.

1. We set $k \to 1$ so that we describe a single D1-brane.

2. We set $M = N$ and $\vec{m} = \vec{\sigma}$, since we have only one D3-brane segment whose bulk gauge field gauges both chirals. Due to the symmetry in this setup, we set $r_+ = r_- = 1$ and $q_+ = q_- = 1/4$.

3. We add a localized level 1 Chern-Simons term. This is simply an insertion of $\exp(-2\pi i u)$ into the matrix model.

4. We set z, the J_- flavor fugacity, to ν in order to account for the latitude superalgebra.

[12] Actually, as explained in [61], $\mathcal{N}=4$ supersymmetry is recovered for any integer value of ν by using a different Killing spinor in the SQM.

[13] It is tempting to ascribe the Chern-Simons term in a D1-brane worldvolume theory to the presence of a non-vanishing type IIB axion in the quiver engineering the UV theory, but we were not able to verify this intuition.

The resulting index is, up to an overall factor [61]

$$\mathcal{I} = \sum_{c=1}^{N} \left(e^{i\pi\nu} e^{2\pi\sigma_c} \prod_{i<j}^{N} \frac{\sinh\left(\pi\left(\sigma_i - \sigma_j + i\nu\left({\delta^c}_i - {\delta^c}_j\right)\right)\right)}{\sinh\left(\pi\left(\sigma_i - \sigma_j\right)\right)} \prod_{i,j}^{N} \frac{\cosh\left(\pi\left(\sigma_i - \sigma_j\right)\right)}{\cosh\left(\pi\left(\sigma_i - \sigma_j + i\nu{\delta^c}_i\right)\right)} \right).$$

Note that $\mathcal{I}$ is not equal to the product of factors one would get from inserting the 1/2-BPS Wilson and vortex loops separately. Instead, $\mathcal{I}$ represents a mixed loop.

Although not incorporated in [61], a natural way of normalizing $\mathcal{I}$ is to divide it by the value of the Witten index for the decoupled worldvolume theory. By this we mean the theory with both ν and the mass parameters σ_i set to 0.[14] For our $\mathcal{I}$, the result is simply N. We adopt this normalization below. There is also an option of multiplying the answer by a term $\mathcal{W}^{fl}$ corresponding to a finite counterterm: a "flavor" Wilson loop [61].

The latitude loop expectation value

We are now in a position to compute the expectation value for the UV avatar to the bosonic latitude loop in the UV theory. It is given by a coupled 3d-1d calculation, equation 5.48 of reference [61]

$$\langle \text{UV avatar loop} \rangle = \frac{1}{|\mathcal{W}|} \int \prod_{i=1}^{N} d\sigma_i \, Z_{\text{vector}} \, Z_{\text{adjoint hyper}} \, Z_{\text{fund. hyper}} \, \mathcal{I} \, .$$

The form of $\mathcal{I}$ means that it almost completely cancels the bulk term $Z_{\text{vector}} \, Z_{\text{adjoint hyper}}$, and replaces it with shifted terms. The resulting integral expression is

$$\begin{aligned}\langle \text{UV avatar loop} \rangle = \frac{1}{N\,(N!)} \int \prod_{i=1}^{N} d\sigma_i \sum_{c=1}^{N} \Bigg(& e^{i\pi\nu} e^{2\pi\sigma_c} \prod_{i<j}^{N} 2\sinh\left(\pi\left(\sigma_i - \sigma_j + i\nu\left({\delta^c}_i - {\delta^c}_j\right)\right)\right) \\ & \times \prod_{i<j}^{N} 2\sinh\left(\pi\left(\sigma_i - \sigma_j\right)\right) \prod_{i,j}^{N} \frac{1}{2\cosh\left(\pi\left(\sigma_i - \sigma_j + i\nu{\delta^c}_i\right)\right)} \prod_{i}^{N} \frac{1}{2\cosh\left(\pi\sigma_i\right)} \Bigg).\end{aligned}$$

We can now attempt to identify the UV avatar loop with the bosonic latitude. We denote the expectation value of the avatar loop, normalized by the partition function of the UV theory, as $W_{\text{UV}}(\nu)$. Using the Cauchy determinant formula 5.11 with $\lambda_i = \sigma_i + i\nu\delta_i^c$ and $\mu_j = \sigma_j$, we can write the integral expression for $W_{\text{UV}}(\nu)$ as

$$\langle W_{\text{UV}}(\nu) \rangle = \frac{1}{Z\,N(N!)} \sum_{\rho \in S_N} (-1)^{\rho} \int d\sigma^N \sum_{c=1}^{N} \frac{e^{i\pi\nu} e^{2\pi\sigma_c}}{\prod_{j=1}^{N} 2\cosh\pi\sigma_j \prod_{k=1}^{N} 2\cosh\pi(\sigma_k - \sigma_{\rho(k)} + i\nu\delta_k^c)} . \tag{6.12}$$

Taking into account the fact that the partition functions of ABJM and the UV theory coincide at any ν, the expression above is precisely what we found for $W_B(\nu)$ in Eq 5.14.

We have shown that the expectation value of the UV avatar loop in the UV theory matches that of the bosonic latitude in ABJM. Together with the matching of the super-

[14] It is not entirely clear to us why this is the correct limit.

symmetry algebra at all values of ν, this constitutes the evidence we have for the correct identification of the UV avatar for the bosonic latitude loop and for the use of the particular SQM used to define it.

Chapter 7

Conclusions

In this thesis, we studied some aspects of 3d $\mathcal{N} \geq 4$ theories. There are two “philosophical” links between the results. The first is that in both cases we extracted physical information from lower dimensional subsectors. The second is that they are all based on the property of a different supercharge w.r.t. the standard $\mathcal{N} = 2$ one.

Chapters 1 and 2 are devoted to reviewing practical notions on 3d gauge theories and known results for them. Chapter 4 focusses on the localization technique and its application. It also summarizes known results on BPS line operators in 3d. We framed some results in the context of conformal defects.

In chapter 3 we focussed on a family of operators, whose correlation functions restricted on a circle form a one-dimensional topological theory. We built a cohomological equivalence between integrated operators arising from supersymmetric deformations and integrated topological operators. We showed that derivatives of the real mass deformed partition function with respect to the mass compute correlation functions of Higgs branch operators. Similarly, FI deformations are related to Coulomb branch operators. The results apply to any 3d $\mathcal{N} \geq 4$ theory, regardless of their details, like off-shell closed supersymmetry or a Lagrangian description. Then, we applied the conjecture in ABJM. We computed the two-point correlation function at two loops in perturbation theory. We found perfect agreement with the second derivative of the mass-deformed partition function of ABJM theory, evaluated at weak coupling directly from its matrix model representation.

In chapter 5 and 6 we have examined several aspects of latitude Wilson loops in the ABJM model: a family of BPS loop operators parameterized by a real number ν. In chapter 5 we showed that the supercharges preserved by the latitude loop sit inside $\mathcal{N} = 4$ supersymmetry algebra for every value of ν. Consequently, latitude type BPS Wilson loops can be defined in more general Chern-Simons-matter theories of Gaiotto-Witten type (GW). We derived the matrix model conjectured in [79] for the bosonic latitude Wilson loops in ABJM using supersymmetric localization and showed how to extend the computation to the GW theories. In this step, we made some mild assumptions on the off-shell closure of the supercharge.

In chapter 6 we defined latitude type Wilson and vortex loop operators in theories without Chern-Simons terms. In doing so, we observed that at $\nu = 0$ the latitude supercharge is of the same type as the one preserving the topological sector. From symmetry arguments, we showed that this theory does not possess either a vortex or a Wilson loop operator which could serve as a dual to the latitude Wilson loop in ABJM. Instead, we argued that the dual operator must be of mixed Wilson-vortex type. Finally, we examined a $\mathcal{N}=4$ theory which is known to be IR dual to the ABJM model. We identified a BPS loop operator dual to the bosonic latitude Wilson loop. It is given by a novel bound state of Wilson and vortex loops, defined using a worldvolume supersymmetric quantum mechanics. We argued this from symmetry considerations. We also tested our proposal matching the localization results for both sides of the duality.

This thesis suggests some open problems whose investigation would be desirable. One of these is the need for a procedure for closing a single generic supercharge off-shell in GW-type theories. The procedure would allow investigating the topological sector directly in GW theories, including ABJM. We are not aware of any general work in this direction.

Another issue is the need for a general definition of mixed Wilson-vortex type loop operators, including their BPS versions, in gauge theories in three dimensions. Such a definition would include a study of the moduli space of such operators, of the type carried out for the analogous loop operators in four dimensions by Kapustin [53]. In the supersymmetric case, it would be interesting to examine the duality properties of mixed loops, under both mirror symmetry [86] and Aharony duality [87]. In this direction, it would be useful to identify the extended objects in string/M-theory which are responsible for mixed loops to generalize the considerations of [183].

It would also be nice to understand if there is a deeper link between the latitude loop and the topological sector. In this direction, a natural generalization of the present investigation would concern the construction of topological operators inserted into the $\frac{1}{2}$-BPS Wilson line. This work is in progress, and the situation seems rather different from the analog four-dimensional situation [247]. One could also study the topological sector outside the $\frac{1}{2}$-BPS Wilson line. The configuration is potentially accessible both by (defect) conformal bootstrap [213] and localization methods.

Appendix A

ABJM conventions

We work in euclidean space with coordinates $x^\mu = (x^1, x^2, x^3)$ and metric $\delta_{\mu\nu}$ [1]. Gamma matrices satisfying the usual Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$, are chosen to be the Pauli matrices

$$(\gamma^\mu)_\alpha^{\ \beta} \equiv (\sigma^\mu)_\alpha^{\ \beta} \qquad \mu = 1,2,3 \tag{A.1}$$

Standard relations which are useful for perturbative calculations are

$$\gamma^\mu\gamma^\nu = \delta^{\mu\nu} + i\varepsilon^{\mu\nu\rho}\gamma_\rho \tag{A.2}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho = \delta^{\mu\nu}\gamma^\rho - \delta^{\mu\rho}\gamma^\nu + \delta^{\nu\rho}\gamma^\mu + i\varepsilon^{\mu\nu\rho} \tag{A.3}$$

Moreover, we define $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$.

Spinor indices are raised and lowered according to the following rules

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$$

with $\varepsilon^{12} = -\varepsilon_{12} = 1$. Consequently, we define $(\gamma^\mu)_{\alpha\beta} \equiv \varepsilon_{\beta\gamma}(\gamma^\mu)_\alpha^{\ \gamma} = (-\sigma^3, i\mathbb{1}, \sigma^1)$ and $(\gamma^\mu)^{\alpha\beta} \equiv \varepsilon^{\alpha\gamma}(\gamma^\mu)_\gamma^{\ \beta} = (\sigma^3, i\mathbb{1}, -\sigma^1)$. They satisfy $(\gamma^\mu)_{\alpha\beta} = (\gamma^\mu)_{\beta\alpha}$ and $(\gamma^\mu)^{\alpha\beta} = (\gamma^\mu)^{\beta\alpha}$.

The $\mathcal{N} = 6$ superconformal algebra is $\mathfrak{osp}(6|4)$. The bosonic part of the algebra is given by $\mathfrak{so}(3,2) \oplus \mathfrak{so}(6)$. $\mathfrak{so}(3,2)$ is the conformal algebra in three dimensions and $\mathfrak{so}(6)$ is the R-symmetry algebra. The non-trivial commutators are given by

$$[M^{\mu\nu}, M^{\rho\sigma}] = \delta^{\sigma[\mu}M^{\nu]\rho} + \delta^{\rho[\nu}M^{\mu]\sigma}\,, \tag{A.4}$$

$$[P^\mu, M^{\nu\rho}] = \delta^{\mu[\nu}P^{\rho]}\,, \qquad [K^\mu, M^{\nu\rho}] = \delta^{\mu[\nu}K^{\rho]}\,, \tag{A.5}$$

$$[P^\mu, K^\nu] = 2\delta^{\mu\nu}D + 2M^{\mu\nu}\,, \qquad [D, P^\mu] = P^\mu, \qquad [D, K^\mu] = -K^\mu\,. \tag{A.6}$$

Exploiting the isomorphism $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$, we represent the R-symmetry generators as matrices $J_I^{\ J}$ transforming in the adjoint of $\mathfrak{su}(4)$

$$\left[J_I^{\ J}, J_K^{\ L}\right] = \delta_I^L J_K^{\ J} - \delta_K^J J_I^{\ L}\,. \tag{A.7}$$

[1] In this thesis, we do not distinguish flat space tangent indices from the S^3 ones. It should be clear from the context the distinction. We specify only in case of possible confusion (see appendix C).

We take the odd generators $\bar{Q}_{IJ,\alpha}$ and $\bar{S}_{IJ,\alpha}$ as spacetime spinors transforming in the antisymmetric representation of $\mathfrak{su}(4)$. The odd-odd commutation relations are

$$\begin{aligned}
\{\bar{Q}_{IJ,\alpha}, \bar{Q}^{\beta}_{KL}\} &= 2\varepsilon_{IJKL}(\gamma^{\mu})_{\alpha}{}^{\beta}P_{\mu}\,, &\text{(A.8)}\\
\{\bar{S}_{IJ,\alpha}, \bar{S}_{IJ\beta}\} &= 2\varepsilon_{IJKL}(\gamma^{\mu})_{\alpha}{}^{\beta}K_{\mu}\,, &\text{(A.9)}\\
\{\bar{Q}_{IJ,\alpha}, \bar{S}_{KL,\beta}\} &= \varepsilon_{IJKL}\left((\gamma^{\mu\nu})_{\alpha}{}^{\beta}M_{\mu\nu} + 2\delta_{\alpha}^{\beta}D\right) + 2\delta_{\alpha}^{\beta}\varepsilon_{IJMN}\left(\delta_K^N J_L{}^M - \delta_L^N J_K{}^M\right)\,, &\text{(A.10)}
\end{aligned}$$

where ε_{IJKL} is the Euclidean 4d Levi-Civita symbol ($\varepsilon_{1234} = \varepsilon^{1234} = 1$). The mixed commutators are given by

$$\begin{aligned}
[D, \bar{Q}_{IJ,\alpha}] &= \frac{1}{2}\bar{Q}_{IJ,\alpha}\,, & [D, \bar{S}_{IJ,\alpha}] &= -\frac{1}{2}\bar{S}_{IJ,\alpha}\,, &\text{(A.11)}\\
[M^{\mu\nu}, \bar{Q}_{IJ,\alpha}] &= -\frac{1}{2}(\gamma^{\mu\nu})_{\alpha}{}^{\beta}\bar{Q}_{IJ,\beta}\,, & [M^{\mu\nu}, \bar{S}_{IJ,\alpha}] &= -\frac{1}{2}(\gamma^{\mu\nu})_{\alpha}{}^{\beta}\bar{S}_{IJ,\beta}\,, &\text{(A.12)}\\
[K^{\mu}, \bar{Q}_{IJ,\alpha}] &= (\gamma^{\mu})_{\alpha}{}^{\beta}\bar{S}_{IJ,\beta}\,, & [P^{\mu}, \bar{S}_{IJ,\alpha}] &= (\gamma^{\mu})_{\alpha}{}^{\beta}\bar{Q}_{IJ,\beta}\,, &\text{(A.13)}
\end{aligned}$$

and

$$\begin{aligned}
[J_I{}^J, \bar{Q}_{KL,\alpha}] &= \frac{1}{2}\delta_I^J\bar{Q}_{KL,\alpha} - \delta_K^J\bar{Q}_{IL,\alpha} - \delta_L^J\bar{Q}_{KI,\alpha}\,, &\text{(A.14)}\\
[J_I{}^J, \bar{S}_{KL,\alpha}] &= \frac{1}{2}\delta_I^J\bar{S}_{KL,\alpha} - \delta_K^J\bar{S}_{IL,\alpha} - \delta_L^J\bar{S}_{KI,\alpha}\,. &\text{(A.15)}
\end{aligned}$$

Finally, we also write explicitly the action of $J_I{}^J$on the (anti-)fundamental representation

$$[J_I{}^J, O_K] = \frac{1}{4}\delta_I^J O_K - \delta_K^J O_I\,, \qquad [J_I{}^J, O^K] = \delta_I^K O_J - \frac{1}{4}\delta_I^J O^K\,. \tag{A.16}$$

The ABJM theories form a class of Lagrangian $\mathcal{N} = 6$ superconformal theories. They are Chern-Simons matter theories with gauge group $U(N_1)_k \times U(N_2)_{-k}$. In the paper, we follow the conventions of [66]. We denote the two gauge fields with A_μ and $\hat{A}_\mu$, respectively for $U(N_1)$ and $U(N_2)$. The matter scalar fields C_I, $\bar{C}^I$ $I = 1, \ldots, 4$, transforming respectively in the bifundamental and antibifundamental of the gauge group. The lower index I defines the fundamental of the $SU(4)$ R-symmetry group. Finally, the matter spinor fields $\bar{\psi}^I$ and ψ_I transform respectively in the bifundamental and antibifundamental of the gauge group.

The flat space action is given by

$$S = S_{\text{CS}} + S_{\text{mat}} + S_{\text{int}}\,, \tag{A.17}$$

where

$$\begin{aligned}
S_{\text{CS}} &= -i\frac{k}{4\pi}\int d^3x\; \varepsilon^{\mu\nu\rho}\left[\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho\right) - \text{Tr}\left(\hat{A}_\mu\partial_\nu\hat{A}_\rho + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right)\right]\,,\\
S_{\text{mat}} &= \int d^3x\;\; \text{Tr}\left[D_\mu C_I D^\mu \bar{C}^I + i\bar{\psi}^I\gamma^\mu D_\mu\psi_I\right]\,.
\end{aligned} \tag{A.18}$$

The covariant derivative is defined as

$$D_\mu C_I = \partial_\mu C_I + i A_\mu C_I - i C_I \hat{A}_\mu \,, \qquad D_\mu \bar{C}^I = \partial_\mu \bar{C}^I + i \hat{A}_\mu \bar{C}^I - i \bar{C}^I A_\mu \,. \tag{A.19}$$

$S_{\rm int}$ contains the superpotential

$$\begin{aligned} S_{\rm int} \equiv S_{\rm 6pt} + S_{\rm 4pt} = &-\frac{4\pi^2}{3k^2}\int d^3x \ {\rm Tr}\Big[C_I\bar{C}^I C_J\bar{C}^J C_K\bar{C}^K + \bar{C}^I C_I\bar{C}^J C_J\bar{C}^K C_K + \\ &+ 4\, C_I\bar{C}^J C_K\bar{C}^I C_J\bar{C}^K - 6\, C_I\bar{C}^J C_J\bar{C}^I C_K\bar{C}^K\Big] + \\ &-\frac{2\pi i}{k}\int d^3x \ {\rm Tr}\Big[\bar{C}^I C_I\Psi_J\bar{\Psi}^J - C_I\bar{C}^I\bar{\Psi}^J\Psi_J + 2\, C_I\bar{C}^J\bar{\Psi}^I\Psi_J + \\ &- 2\,\bar{C}^I C_J\Psi_I\bar{\Psi}^J - \epsilon_{IJKL}\bar{C}^I\bar{\Psi}^J\bar{C}^K\bar{\Psi}^L + \epsilon^{IJKL}C_I\Psi_J C_K\Psi_L\Big]\,, \end{aligned} \tag{A.20}$$

where $S_{\rm 6pt}$ and $S_{\rm 4pt}$ contains respectively the sextic potential and the the Yukawa interactions.

The ABJM action is invariant under the following SUSY transformations

$$\delta A_\mu = \frac{4\pi i}{k}\bar{\Theta}^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(C_I\psi_{J\beta} + \frac{1}{2}\varepsilon_{IJKL}\bar{\psi}^K_\beta\bar{C}^L\right), \tag{A.21a}$$

$$\delta\hat{A}_\mu = \frac{4\pi i}{k}\bar{\Theta}^{IJ,\alpha}(\gamma_\mu)_\alpha{}^\beta\left(\psi_{J\beta}C_I + \frac{1}{2}\varepsilon_{IJKL}\bar{C}^L\bar{\psi}^K_\beta\right), \tag{A.21b}$$

$$\delta C_K = \bar{\Theta}^{IJ,\alpha}\ \varepsilon_{IJKL}\ \bar{\psi}^L_\alpha\,, \tag{A.21c}$$

$$\delta\bar{C}^K = 2\bar{\Theta}^{KL,\alpha}\ \psi_{L,\alpha}\,, \tag{A.21d}$$

$$\begin{aligned}\delta\bar{\psi}^{K,\beta} = &-2i\bar{\Theta}^{KL,\alpha}(\gamma^\mu)_\alpha{}^\beta D_\mu C_L - \frac{4\pi i}{k}\bar{\Theta}^{KL,\beta}(C_L\bar{C}^M C_M - C_M\bar{C}^M C_L) + \\ &-\frac{8\pi i}{k}\bar{\Theta}^{IJ,\beta}C_I\bar{C}^K C_J - 2i\bar{\epsilon}^{KL,\beta}C_L\,,\end{aligned} \tag{A.21e}$$

$$\delta\psi^\beta_K = -i\bar{\Theta}^{IJ,\alpha}\varepsilon_{IJKL}(\gamma^\mu)_\alpha{}^\beta D_\mu\bar{C}_L + \frac{2\pi i}{k}\bar{\Theta}^{IJ,\beta}\varepsilon_{IJKL}(\bar{C}^L C_M\bar{C}^M - \bar{C}^M C_M\bar{C}^L) \tag{A.21f}$$

$$+\frac{4\pi i}{k}\bar{\Theta}^{IJ,\beta}\varepsilon_{IJML}\bar{C}^M C_K\bar{C}^L - i\bar{\epsilon}^{IJ,\beta}\varepsilon_{IJKL}\bar{C}^L\,. \tag{A.21g}$$

The flat space Killing spinors are taken to be

$$\bar{\Theta}^{IJ} = \bar{\theta}^{IJ} - x_\mu\gamma^\mu\bar{\epsilon}^{IJ}\,. \tag{A.22}$$

With these conventions, Eq A.21 closes the $\mathfrak{osp}(6|4)$ superconformal algebra on-shell.

To perform perturbative computations, we list the propagators at tree and loop orders, as needed for the two-loop calculations, are:

- Scalar propagator

$$\langle (C_I)_i{}^{\hat{j}}(x)\ (\bar{C}^J)_{\hat{k}}{}^l(y)\rangle^{(0)} = \delta^J_I\delta^l_i\delta^{\hat{j}}_{\hat{k}}\ \frac{\Gamma(\frac{1}{2}-\epsilon)}{4\pi^{\frac{3}{2}-\epsilon}}\frac{1}{|x-y|^{1-2\epsilon}} \tag{A.23}$$

$$\langle (C_I)_i{}^{\hat{j}}(x)\ (\bar{C}^J)_{\hat{k}}{}^l(y)\rangle^{(1)} = 0 \tag{A.24}$$

- Vector propagators in Landau gauge

$$\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(0)} = \delta_i^l \delta_k^j\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{\pi^{\frac{1}{2}-\epsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}}$$
$$\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(0)} = -\,\delta_{\hat{i}}^{\hat{l}} \delta_{\hat{k}}^{\hat{j}}\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{\pi^{\frac{1}{2}-\epsilon}}\ \varepsilon_{\mu\nu\rho}\ \frac{(x-y)^\rho}{|x-y|^{3-2\epsilon}} \tag{A.25}$$

$$\langle (A_\mu)_i{}^j(x)\ (A_\nu)_k{}^l(y)\rangle^{(1)} = \delta_i^l \delta_k^j\ \frac{N_2}{k}\,\frac{\Gamma^2(\frac{1}{2}-\epsilon)}{\pi^{1-2\epsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\epsilon}} - \partial_\mu\partial_\nu\frac{|x-y|^{4\epsilon}}{4\epsilon(1+2\epsilon)}\right)$$
$$\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)\ (\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle^{(1)} = \delta_i^l \delta_k^j\ \frac{N_1}{k}\,\frac{\Gamma^2(\frac{1}{2}-\epsilon)}{\pi^{1-2\epsilon}}\left(\frac{\delta_{\mu\nu}}{|x-y|^{2-4\epsilon}} - \partial_\mu\partial_\nu\frac{|x-y|^{4\epsilon}}{4\epsilon(1+2\epsilon)}\right) \tag{A.26}$$

- Fermion propagator

$$\langle (\psi_{\alpha I})_{\hat{i}}{}^j(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}{}_k(y)\rangle^{(0)} = \delta_I^J \delta_{\hat{i}}^{\hat{l}} \delta_k^j\ i\,\frac{\Gamma(\frac{3}{2}-\epsilon)}{2\pi^{\frac{3}{2}-\epsilon}}\ (\gamma^\mu)_\alpha{}^\beta\ \frac{(x-y)_\mu}{|x-y|^{3-2\epsilon}} \tag{A.27}$$

$$\langle (\psi_{\alpha I})_{\hat{i}}{}^j(x)\ (\bar{\psi}^{J\beta})^{\hat{l}}{}_k(y)\rangle^{(1)} = -\delta_I^J \delta_{\hat{i}}^{\hat{l}} \delta_k^j\ \delta_\alpha{}^\beta\ \left(\frac{N_1-N_2}{k}\right)\ i\,\frac{\Gamma^2(\frac{1}{2}-\epsilon)}{8\pi^{2-2\epsilon}}\ \frac{1}{|x-y|^{2-4\epsilon}} \tag{A.28}$$

We note that in the ABJ(M) limit, $N_1 = N_2$, the one-loop correction to the fermionic propagator vanishes.

The vertices entering the perturbative calculations of Section **??** can be easily read from terms (A.18) and (A.20) of the action.

Appendix B

3d $\mathcal{N} = 4$ theories on S^3

We review some elements of the 3d $\mathcal{N} = 4$ theories on S^3, following [33]. We provide the relevant multiplets, their supersymmetry transformations and their actions. Fields are labeled by Lorentz spin and gauge group G and the R-symmetry $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$ representations.[1] We are interested in Lagrangian theories involving vector multiplets and hypermultiplets. The vector multiplet component fields are

$$\mathcal{V} = \left(A_\mu,\, \lambda_{\alpha,a\dot{a}},\, \Phi_{\dot{a}\dot{b}},\, D_{ab}\right) . \tag{B.1}$$

They transform in the adjoint representations of the gauge group G and in the following representations of the R-symmetry group

- A_μ is the vector field transforming in the $(\mathbf{1}, \mathbf{1})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$;
- $\lambda_{\alpha,a\dot{a}}$ (the gaugino) is a spinor transforming in the $(\mathbf{2}, \mathbf{2})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$;
- $\Phi_{\dot{a}\dot{b}}$ is a scalar field transforming in the $(\mathbf{3}, \mathbf{1})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$;
- D_{ab} is a scalar field transforming in the $(\mathbf{1}, \mathbf{3})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$.

The hypermultiplet $\mathcal{H}$ transforms in a unitary representation $\mathcal{R}$ of the gauge group G and its field components are

$$\mathcal{H} = \left(q_a, \tilde{q}^a, \psi_{\dot{a}}, \tilde{\psi}_{\dot{a}}\right) , \tag{B.2}$$

where

- q_a are scalar field transforming in the $(\mathbf{1}, \mathbf{2})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$ and in the $\mathcal{R}$ of G;
- $\tilde{q}^a$ are scalar fields transforming $(\mathbf{1}, \bar{\mathbf{2}})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$ and in the $\bar{\mathcal{R}}$ of G;
- $\psi_{\dot{a}}$ are spinor fields transforming in the $(\mathbf{2}, \mathbf{1})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$ and in the $\mathcal{R}$ of G;
- $\tilde{\psi}^{\dot{a}}$ are spinor fields transforming in the $(\bar{\mathbf{2}}, \mathbf{1})$ of $\mathfrak{su}(2)_C \oplus \mathfrak{su}(2)_H$ and in the $\bar{\mathcal{R}}$ of G.

[1] We denote with $\alpha = 1, 2$ Lorentz spinor indices, with $a, \dot{a} = 1, 2$ respectively $\mathfrak{su}(2)_C$, $\mathfrak{su}(2)_H$ indices.

The supersymmetry transformations for the vector multiplet are given by

$$\delta_\xi A_\mu = \frac{i}{2}\xi^{a\dot{b}}\gamma_\mu\lambda_{a\dot{b}}\,, \tag{B.3a}$$

$$\begin{aligned}\delta_\xi\lambda_{a\dot{b}} = &-\frac{i}{2}\epsilon^{\mu\nu\rho}\gamma_\rho\xi_{a\dot{b}}F_{\mu\nu} - D_a{}^c\xi_{c\dot{b}} - i\gamma^\mu\xi_a{}^{\dot{c}}\mathcal{D}_\mu\Phi_{\dot{c}\dot{b}} + 2i\Phi_{\dot{b}}{}^{\dot{c}}\xi'_{a\dot{c}}+\\ &+\frac{i}{2}\xi_{a\dot{d}}[\Phi_{\dot{b}}{}^{\dot{c}},\Phi_{\dot{c}}{}^{\dot{d}}]\,,\end{aligned} \tag{B.3b}$$

$$\delta_\xi\Phi_{\dot{a}\dot{b}} = \xi^c{}_{(\dot{a}}\lambda_{|c|\dot{b})}\,, \tag{B.3c}$$

$$\delta_\xi D_{ab} = -i\mathcal{D}_\mu(\xi_{(a}{}^{\dot{c}}\gamma^\mu\lambda_{b)\dot{c}}) - 2i\xi'_{(a}{}^{\dot{c}}\lambda_{b)\dot{c}} + i[\xi_{(a}{}^{\dot{c}}\lambda^{\dot{d}}_{b)},\Phi_{\dot{c}\dot{d}}]\,. \tag{B.3d}$$

For the hypermultiplet, we have

$$\delta_\xi q^a = \xi^{a\dot{b}}\psi_{\dot{b}}\,, \qquad \delta_\xi\psi_{\dot{a}} = i\gamma_\mu\xi_{a\dot{a}}\mathcal{D}_\mu q^a + i\xi'_{a\dot{a}}q^a - i\xi_{a\dot{c}}\Phi^{\dot{c}}{}_{\dot{a}}q^a\,, \tag{B.4a}$$

$$\delta_\xi\tilde{q}^a = \xi^{a\dot{b}}\tilde{\psi}_{\dot{b}}\,, \qquad \delta_\xi\tilde{\psi}_{\dot{a}} = i\gamma_\mu\xi_{a\dot{a}}\mathcal{D}_\mu\tilde{q}^a + i\xi'_{a\dot{a}}\tilde{q}^a - i\xi_{a\dot{c}}\Phi^{\dot{c}}{}_{\dot{a}}\tilde{q}^a\,. \tag{B.4b}$$

When the SUSY parameter $\xi_{\alpha,a\dot{a}}$ satisfies the S^3 conformal Killing spinor equation, these transformations realize the whole superconformal algebra $\mathfrak{osp}(4|4)$.

The following is an invariant Lagrangian for the hypermultiplet coupled to the vector multiplet, which is derived from the flat space expression by covariantizing derivatives and by adding specific conformal masses

$$\begin{aligned}S_{\text{hyper}} = \int d^3x\sqrt{g}\Bigg[&\mathcal{D}^\mu\tilde{q}^a\mathcal{D}_\mu q_a - i\tilde{\psi}^{\dot{a}}\gamma^\mu\mathcal{D}_\mu\psi_{\dot{a}} + \frac{3}{4r^2}\tilde{q}^a q_a + i\tilde{q}^a D_a{}^b q_b - \frac{1}{2}\tilde{q}^a\Phi^{\dot{a}\dot{b}}\Phi_{\dot{a}\dot{b}}q_a+\\ &- i\tilde{\psi}^{\dot{a}}\Phi_{\dot{a}}{}^{\dot{b}}\psi_{\dot{b}} + i\left(\tilde{q}^a\lambda_a{}^{\dot{b}}\psi_{\dot{b}} + \tilde{\psi}^{\dot{a}}\lambda^b{}_{\dot{a}}q_b\right)\Bigg]\,.\end{aligned} \tag{B.5}$$

To the best of our knowledge, there is no vector multiplet action invariant under the off-shell $\mathcal{N}=4$ superconformal symmetry. However, if the conformal Killing spinors are further restricted to obey the condition [33]

$$\xi'_{a\dot{a}} \equiv \frac{1}{3}\gamma^\mu\nabla_\mu\xi_{a\dot{a}} = \frac{i}{2r}h_a{}^b\xi_{b\dot{b}}\bar{h}^{\dot{b}}{}_{\dot{a}}\,, \tag{B.6}$$

then the following action turns out to be closed under the transformations B.3

$$\begin{aligned}S_{\text{YM}} = &\frac{1}{g^2_{\text{YM}}}\int d^3x\sqrt{g}\,\text{Tr}\Bigg[F^{\mu\nu}F_{\mu\nu} - \mathcal{D}^\mu\Phi^{\dot{a}\dot{b}}\mathcal{D}_\mu\Phi_{\dot{a}\dot{b}} + i\lambda^{a\dot{a}}\gamma^\mu\mathcal{D}_\mu\lambda_{a\dot{a}} - D^{ab}D_{ab}+\\ &- i\lambda^{a\dot{a}}[\lambda_a{}^{\dot{b}},\Phi_{\dot{a}\dot{b}}] - \frac{1}{4}[\Phi^{\dot{a}}{}_{\dot{b}},\Phi^{\dot{c}}{}_{\dot{d}}]\,[\Phi^{\dot{b}}{}_{\dot{a}},\Phi^{\dot{d}}{}_{\dot{c}}] - \frac{1}{2r}h^{ab}\bar{h}^{\dot{a}\dot{b}}\lambda_{a\dot{a}}\lambda_{b\dot{b}}+\\ &+\frac{1}{r}\left(h_a{}^bD_b{}^a\right)\left(\bar{h}^{\dot{a}}_{\dot{b}}\Phi^{\dot{b}}{}_{\dot{a}}\right) - \frac{1}{r^2}\Phi^{\dot{a}\dot{b}}\Phi_{\dot{a}\dot{b}}\Bigg]\,.\end{aligned} \tag{B.7}$$

where $h_a{}^b$ and $\bar{h}^{\dot{b}}{}_{\dot{a}}$ are respectively $\mathfrak{su}(2)_C$ and $\mathfrak{su}(2)_H$ matrices, normalized such that $h_a{}^c h_c{}^b = \delta^b_a$ and $\bar{h}^{\dot{b}}{}_{\dot{c}}\bar{h}^{\dot{c}}{}_{\dot{a}} = \delta^{\dot{b}}_{\dot{a}}$. The condition B.6 selects the half of the conformal Killing spinors generating the Poincaré subalgebra $\mathfrak{su}(2|1)_\ell \oplus \mathfrak{su}(2|1)_r$. Indeed, one can check that these supercharges generate the isometry group of S^3 $\mathfrak{su}(2)_\ell \oplus \mathfrak{su}(2)_r$, as well as a $\mathfrak{u}(1)_\ell \oplus \mathfrak{u}(1)_r$ R-symmetry, specified by the choice of $h_a{}^b$ and $\bar{h}^{\dot{b}}{}_{\dot{a}}$.

Background flavor symmetries

We review the relevant couplings to background flavor symmetries.

A current multiplet $\Sigma = (J_{ab},\, \chi_{a\dot{a}},\, j_\mu\,, K_{\dot{a}\dot{b}})$ couples to a background vector field $\mathcal{V}_{\text{back}} = \left(A_\mu,\, \lambda_{a\dot{a}},\, \Phi_{\dot{a}\dot{b}},\, D_{ab}\right)$ as

$$\int_{S^3} d^3x\sqrt{g}\left(A_\mu j^\mu + \mathbb{i} D^{ab}J_{ab} + \Phi^{\dot{a}\dot{b}}K_{\dot{a}\dot{b}} + \lambda^{a\dot{a}}\chi_{a\dot{a}} + O\left(\mathcal{V}^2\right)\right) . \tag{B.8}$$

Imposing supersymmetry invariance of this action and exploiting the SUSY transformations of the vector components [31], we can read the linearized variation of the $\chi_{a\dot{a}}$ fermion

$$\delta_\xi \chi_{a\dot{a}} = \frac{\mathbb{i}}{2}\gamma_\mu \xi_{a\dot{a}} j^\mu + 2\xi'^b{}_{\dot{a}} J_{ab} + \gamma^\mu \xi^b{}_{\dot{a}} \nabla_\mu J_{ab} + \xi_a{}^{\dot{b}} K_{\dot{a}\dot{b}}. \tag{B.9}$$

A rigid supersymmetry preserving background is obtained imposing $\lambda_{a\dot{a}} = 0$ and $\delta\lambda_{a\dot{a}} = 0$. We take the following solution

$$\Phi_{\dot{a}\dot{b}} = m\,\bar{h}_{\dot{a}\dot{b}}\,, \quad D_{ab} = -\frac{m}{r}h_{ab}\,, \quad A_\mu = 0\,, \quad \lambda_{a\dot{a}} = 0\,. \tag{B.10}$$

where the parameter m is the real mass. If we restore the adjoint index A, the background reproduces the real mass deformation of eq. (3.30).

A similar construction can be applied to obtain the FI-term (3.19). Because of the Bianchi identities, for each $U(1)$ factor of the gauge group there is an associated conserved current $j_\mu \sim \epsilon_{\mu\nu\rho}F^{\nu\rho}$. We will refer to this symmetry as the topological symmetry. In supersymmetric theories, the topological current sits in a linear multiplet built from vector multiplet components. Such a multiplet can be coupled to an abelian twisted vector multiplet background $\tilde{\mathcal{V}}_{\text{back}} = \left(\tilde{A}_\mu,\, \tilde{\lambda}_{\alpha,a\dot{a}},\, \tilde{\Phi}_{ab},\, \tilde{D}_{\dot{a}\dot{b}}\right)$. We also recall the SUSY variation for the twisted gauginos in the abelian case

$$\delta_\xi \tilde{\lambda}_{a\dot{b}} = -\frac{\mathbb{i}}{2}\epsilon^{\mu\nu\rho}\gamma_\rho \xi_{a\dot{b}} \tilde{F}_{\mu\nu} - D_{\dot{b}}{}^{\dot{c}}\xi_{a\dot{c}} - \mathbb{i}\gamma^\mu \xi_b{}^{\dot{b}}\nabla_\mu \tilde{\Phi}_{ab} + 2\mathbb{i}\tilde{\Phi}^b_a \xi'_{b\dot{b}}\,. \tag{B.11}$$

The coupling, originally found in [248], can be derived by constructing the supersymmetric completion of the BF term. We obtain

$$S_{\text{SBF}} = \frac{\mathbb{i}}{4\pi}\int_{S^3} d^3x\sqrt{g}\,\left(\epsilon^{\mu\nu\rho}F_{\mu\nu}\tilde{A}_\rho - \tilde{\lambda}^{a\dot{a}}\lambda_{a\dot{a}} - \tilde{\Phi}^{ab}D_{ab} - \Phi^{\dot{a}\dot{b}}\tilde{D}_{\dot{a}\dot{b}}\right) . \tag{B.12}$$

This is a topological action which is invariant under the full superconformal algebra. However, the choice of the background $\tilde{\lambda}_{a\dot{a}} = 0, \delta\tilde{\lambda}_{a\dot{a}} = 0$ breaks conformal invariance and selects the Poincaré subalgebra. The explicit rigid background that reproduces (3.19) is

$$\tilde{A}_\mu = 0\,, \quad \tilde{\lambda}_{a\dot{a}} = 0\,, \quad \tilde{\Phi}_{ab} = -4\pi\zeta\, h_{ab}\,, \quad \tilde{D}_{\dot{a}\dot{b}} = \frac{4\pi\zeta}{r}\bar{h}_{\dot{a}\dot{b}}\,. \tag{B.13}$$

Closure of the Supersymmetry algebra

Here we describe explicitly the closure of the 3d $\mathcal{N}=4$ SUSY algebra following [33]. This requires evaluating the action of the bosonic generator $\{\delta_\xi,\delta_{\tilde{\xi}}\}$ on the supermultiplets, denoted generically by $\mathcal{B}$. The action is given by

$$\{\delta_\xi,\delta_{\tilde{\xi}}\}\,\mathcal{B}=\left(\hat{\mathcal{K}}_{\xi,\tilde{\xi}}+\mathcal{G}_\Lambda+\text{e.o.m.}\right)\mathcal{B}\,. \tag{B.14}$$

$\mathcal{G}_\Lambda$ is a gauge transformation with parameter Λ defined as

$$\Lambda=i\left(\tilde{\xi}^{c}{}_{\dot{a}}\xi_{c\dot{b}}\right)\Phi^{\dot{a}\dot{b}}-i(\tilde{\xi}^{a\dot{a}}\gamma^\mu\xi_{a\dot{a}})A_\mu, \tag{B.15}$$

and $\hat{\mathcal{K}}_{\xi,\tilde{\xi}}$ are the representation of bosonic symmetries on the field space. Their explicit form is given by

$$\hat{\mathcal{K}}_{\xi,\tilde{\xi}}=\hat{\mathcal{L}}_v+\hat{R}_C+\hat{R}_H+\hat{\rho}\Delta\,, \tag{B.16}$$

where

- $\hat{\mathcal{L}}_v$ is the Lie derivative along the vector $v^\mu=i\tilde{\xi}^{a\dot{a}}\gamma^\mu\xi_{a\dot{a}}$;
- $\hat{R}_{C/H}$ is an $\mathfrak{su}(2)_{C/H}$ transformation, acting as

$$\begin{aligned}\bar{R}_{\dot{a}\dot{b}}=&i\left(\tilde{\xi}^{c}{}_{(\dot{a}}\xi'_{|a|\dot{b})}+\xi^{c}{}_{(\dot{a}}\tilde{\xi}'_{|c|\dot{b})}\right) &\text{(B.17)}\\ R_{ab}=&i\left(\tilde{\xi}_{(a}{}^{\dot{c}}\xi'_{b)\dot{c}}+\xi_{(a}{}^{\dot{c}}\tilde{\xi}'_{b)\dot{c}}\right)\,, &\text{(B.18)}\end{aligned}$$

  according to the rule $(\hat{R}_Hq)_a=R_{ab}\,q^b$ (the same for $\hat{R}_C$);
- $\hat{\rho}$ is the dilatation parameter

$$\hat{\rho}=i\left(\tilde{\xi}^{a\dot{b}}\xi'_{a\dot{b}}+\xi^{a\dot{b}}\tilde{\xi}'_{a\dot{b}}\right)\,, \tag{B.19}$$

  and Δ represents the dimension of the fields, and takes the values $\Delta[\mathcal{V}]=(0,3/2,1,2)$ and $\Delta[\mathcal{H}]=(1/2,1)$.

The term denoted by e.o.m. stands for equation of motion. We include it for those multiplets whose closure is realized only on-shell. This is indeed the case for the hypermultiplet:

$$\begin{aligned}\{\delta_\xi,\delta_{\tilde{\xi}}\}\,\psi_{\dot{a}}&=\left(\hat{\mathcal{B}}_{\xi,\tilde{\xi}}+\mathcal{G}_\Lambda\right)\psi_{\dot{a}}+\tilde{\xi}^{a\dot{b}}[\xi_{a\dot{a}}(\text{e.o.m.}(\psi))_{\dot{b}}]+\xi^{a\dot{b}}[\tilde{\xi}_{a\dot{a}}(\text{e.o.m.}(\psi))_{\dot{b}}]\\ \{\delta_\xi,\delta_{\tilde{\xi}}\}\,\tilde{\psi}_a&=\left(\hat{\mathcal{B}}_{\xi,\tilde{\xi}}+\mathcal{G}_\Lambda\right)\tilde{\psi}_{\dot{a}}-\tilde{\xi}^{a\dot{b}}[\xi_{a\dot{a}}(\text{e.o.m.}(\tilde{\psi}))_{\dot{b}}]-\xi^{a\dot{b}}[\tilde{\xi}_{a\dot{a}}(\text{e.o.m.}(\tilde{\psi}))_{\dot{b}}]\end{aligned} \tag{B.20}$$

where

$$\begin{aligned}(\text{e.o.m.}(\psi))_{\dot{b}}=&-i\left[\gamma^\mu\mathcal{D}_\mu\psi_{\dot{a}}+\Phi_{\dot{a}}{}^{\dot{b}}\psi_{\dot{b}}+\lambda_{a\dot{a}}q^a\right]\\ (\text{e.o.m.}(\tilde{\psi}))_{\dot{b}}=&i\left[\gamma^\mu\mathcal{D}_\mu\tilde{\psi}_{\dot{a}}-\Phi_{\dot{a}}{}^{\dot{b}}\tilde{\psi}_{\dot{b}}-\tilde{q}^a\lambda_{a\dot{a}}\right]\end{aligned} \tag{B.21}$$

Off-shell closure for hypermultiplets

The 3d $\mathcal{N}=4$ supersymmetry algebra admits two inequivalent off-shell vector multiplets: ordinary and twisted. The two multiplets are related to each other by the outer automorphism of the R-symmetry group $SU(2)_l \times SU(2)_r$ which exchanges the two $SU(2)$ factors. There are also two types of hypermultiplets, ordinary and twisted. These do not sit in any off-shell multiplet with a finite number of fields. Nevertheless, a single supersymmetry in an $\mathcal{N}=4$ theory incorporating regular vector multiplets and hypermultiplets can sometimes be closed off-shell by adding appropriate auxiliary fields [31]. The same is true for the twisted multiplets.

Let $\xi_{\alpha a\dot{a}}$ be the $\mathcal{N}=4$ conformal Killing spinor associated with a supersymmetry transformation δ. Off-shell closure of δ on a hypermultiplet can be achieved by finding another spinor $\chi_{\alpha a\dot{a}}$ satisfying [31]

$$\xi^{\alpha c}{}_{\dot{a}}\xi_{\beta c\dot{b}} = \chi^{\alpha c}{}_{\dot{b}}\chi_{\beta c\dot{a}}\,, \quad \xi_a{}^{\dot{c}}\chi_{b\dot{c}} = 0\,, \quad \xi_{(a}{}^{\dot{c}}\gamma^\mu\nabla_\mu\xi_{b)\dot{c}} = \frac{3i}{2}\chi_{(a}{}^{\dot{c}}\gamma^\mu\nabla_\mu\chi_{b)\dot{c}}\,. \tag{B.22}$$

In order to close off-shell the hypermultiplet transformations, one should add auxiliary fields $G_a, \tilde{G}_a$. One must then make the following modification to the supersymmetry transformations

$$\begin{aligned}\delta\psi_{\dot{a}} \to \delta\psi_{\dot{a}} + i\chi^a{}_{\dot{a}}G_a\,, \quad \delta G^a = i\chi^{a\dot{a}}\Psi^{\text{eom}}_{\dot{a}}\,,\\ \delta\tilde{\psi}_{\dot{a}} \to \delta\tilde{\psi}_{\dot{a}} + i\chi^a{}_{\dot{a}}\tilde{G}_a\,, \quad \delta\tilde{G}^a = -i\chi^{a\dot{a}}\tilde{\Psi}^{\text{eom}}_{\dot{a}}\,,\end{aligned}$$

where $\Psi^{\text{eom}}_{\dot{a}}, \tilde{\Psi}^{\text{eom}}_{\dot{a}}$ are proportional to the fermion equations of motion [31].

We would like to comment on some of the differences in performing localization using the the latitude supercharge versus the supercharge employed in [31], the latter being equivalent to the limit $\nu \to 0$. An interesting difference between ξ^L_ν and $\xi^{C,H}_\beta$ is the existence, or lack thereof, of good solutions to equation B.22. Specifically, the existence, or lack thereof, of a *Killing spinor* satisfying B.22. Equivalently, that the co-kernel of the supersymmetry transformation by ξ on a hypermultiplet scalar, which yields a subset of the hypermultiplet fermions, is spanned by contraction with some other Killing spinors $\chi_{\alpha a\dot{a}}$. $\chi_{\alpha a\dot{a}}$ can then be used to define a canonical orthogonal subspace for the fermions: a subspace whose fields transform into an auxiliary field. For ξ^H_β, a solution is given by

$$\chi = \xi^H_{-\beta}\,.$$

This solution is used in [31] to close the hypermultiplet transformations off-shell. For ξ^L_ν, however, a solution does not exist unless ν is 0 or 1. This presumably makes closing the algebra a more complicated, or even impossible, task. We therefore conclude that $\xi^{C,H}_\beta$ are less generic elements of the Poincaré subalgebra than $\xi^L_{\nu\neq 0,1}$.

Appendix C

Geometry conventions

In this appendix we denote flat space tangent indices by a, $b = 1, 2, 3$. They s are raised/lowered by the metric δ_{ab}. We use Greek indices μ, $\nu = 1, 2, 3$. for S^3 tangent space indices. The gamma matrices are chosen again to be the Pauli matrices.

Flat space spinor algebra

We identify the flat space coordinates with those of the tangent space: x_a. The following is a basis for the conformal Killing spinors on $\mathbb{R}^3$

$$\epsilon^{(1)}_{\mathbb{R}^3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad \epsilon^{(2)}_{\mathbb{R}^3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \epsilon^{(3)}_{\mathbb{R}^3} = -x_a\gamma^a \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \epsilon^{(4)}_{\mathbb{R}^3} = -x_a\gamma^a \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

These satisfy

$$\partial_a \epsilon^{(i)}_{\mathbb{R}^3} = \frac{1}{3}\gamma_a\gamma^b\partial_b\epsilon^{(i)}_{\mathbb{R}^3} .$$

The conformal group is generated by the vectors

$$w^{\mu,ij} \equiv \epsilon^{(i)}_{\mathbb{R}^3}\gamma^\mu\epsilon^{(j)}_{\mathbb{R}^3} ,$$

acting as infinitesimal diffeomorphisms. Suppressing the vector index, the subsets

$$P^m \equiv \tau^m{}_{ij} w^{ij}, \quad M^{mn} \equiv \varepsilon^{mnp}\tau_{p_{ij}} w^{i,j+2} ,$$

generate the isometry algebra of translations and rotations, which is the algebra of the 3d Euclidean Poincaré group. The remaining combinations yield conformal Killing transformations.

Geometry of S^3

We will use toroidal coordinates and the following metric on the round unit radius three sphere

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 + \cos^2\theta d\tau^2\,, \tag{C.1}$$

$$\theta \in [0, \pi/2)\,, \quad \varphi \in [0, 2\pi)\,, \quad \tau \in [0, 2\pi)\ . \tag{C.2}$$

Two maximal circles are located at $\theta = 0, \pi/2$.

We choose the following vielbein

$$\begin{aligned} e^1 &= \sin(\varphi+\tau)\, d\theta + \cos\theta\sin\theta\cos(\varphi+\tau)\, d\varphi - \cos\theta\sin\theta\cos(\varphi+\tau)\, d\tau\,, \\ e^2 &= -\cos(\varphi+\tau)\, d\theta + \cos\theta\sin\theta\sin(\varphi+\tau)\, d\varphi - \cos\theta\sin\theta\sin(\varphi+\tau)\, d\tau\,, \\ e^3 &= \sin^2\theta d\varphi + \cos^2\theta d\tau\,. \end{aligned}$$

In this frame, the spin connection is given by

$$\omega_{abc} = \varepsilon_{abc}\,.$$

The spin covariant derivative is

$$\begin{aligned} \nabla_\mu\epsilon &\equiv \partial_\mu\epsilon + \frac{1}{8}\omega_\mu{}^{ab}\left[\gamma_a, \gamma_b\right]\epsilon \\ &= \partial_\mu\epsilon + \frac{\mathbb{i}}{2}\gamma_\mu\epsilon\,. \end{aligned}$$

A basis for the conformal Killing spinors on S^3 is given by

$$\epsilon^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \epsilon^{(3)} = \begin{pmatrix} e^{-\mathbb{i}\tau}\cos\theta \\ -e^{\mathbb{i}\varphi}\sin\theta \end{pmatrix}, \epsilon^{(4)} = \begin{pmatrix} e^{-\mathbb{i}\varphi}\sin\theta \\ e^{\mathbb{i}\tau}\cos\theta \end{pmatrix}.$$

These satisfy

$$\nabla_\mu\epsilon^{(i)} = \frac{1}{3}\gamma_\mu\gamma^\nu\nabla_\nu\epsilon^{(i)}\,.$$

Defining

$$\eta^{(i)} \equiv \frac{1}{3}\gamma^\mu\nabla_\mu\epsilon^{(i)}\,, \tag{C.3}$$

this becomes the conformal Killing spinor equation

$$\nabla_\mu\epsilon^{(i)} = \gamma_\mu\eta^{(i)}\,.$$

The Lie algebra of the conformal group is generated by the action of the vectors

$$v^{\mu,ij} = \epsilon^{(i)}\gamma^\mu\epsilon^{(j)}\,,$$

acting as infinitesimal diffeomorphisms. The subsets

$$J_l^{\mu a} \equiv \tau^a{}_{ij} v^{ij}\,, \quad J_r^{\mu a} \equiv \tau^a{}_{ij} v^{i+2,j+2}\,, \tag{C.4}$$

generate the isometry algebra, which is isomorphic to $so\,(4) \simeq su_l\,(2) \oplus su_r\,(2)$. The remaining combinations yield conformal Killing transformations. Our choice of vielbein was motivated by

$$e_\mu{}^a = -\frac{1}{2} J^a_{l,\mu}\,.$$

Change of coordinates from flat space

Define the function

$$\exp\Omega \equiv 1 + \sin\theta\cos\varphi\,.$$

The round S^3, in toroidal coordinates, is related to $\mathbb{R}^3$ by the following change of coordinates

$$\begin{aligned} x_1 &\to e^{-\Omega}\cos\theta\cos\tau\,,\\ x_2 &\to e^{-\Omega}\cos\theta\sin\tau\,,\\ x_3 &\to e^{-\Omega}\sin\theta\sin\varphi\,, \end{aligned}$$

followed by a Weyl transformation with parameter Ω.[1]

Let B be the change of variables matrix. The flat metric and the flat vielbein $e_\mu{}^a = \delta_\mu{}^a$ transform as

$$g \to e^{2\Omega} B^T g_{\mathbb{R}^3} B\,, \quad e \to e' \equiv e^{\Omega} B^T e_{\mathbb{R}^3}\,.$$

The frame rotation matrix $F \in SO\,(3)$ is defined as

$$F_a{}^b \equiv e_\mu^{S^3 b} g^{\mu\nu}_{S^3} e'_{\nu a}\,.$$

Define the spinor bilinears

$$A^{ij}_{S^3} \equiv \epsilon^{(i)}\epsilon^{(j)}\,, \quad A^{ij}_{\mathbb{R}^3} \equiv \epsilon^{(i)}_{\mathbb{R}^3}\epsilon^{(j)}_{\mathbb{R}^3}\,.$$

There exists a numerical matrix $R^i{}_j$, unique up to sign, which relates the spinors on $\mathbb{R}^3$ and on S^3 with their chosen coordinate systems, such that

$$\begin{aligned} A^{ij}_{S^3} &= R^i{}_k R^j{}_l A^{kl}_{\mathbb{R}^3}\,,\\ v^{ij}_a &= F_a{}^b R^i{}_k R^j{}_l w^{kl}_b\,. \end{aligned}$$

Given R, we can associate the S^3 spinor $\epsilon^{(i)}$ with

$$e^{\Omega/2} R^i{}_j \epsilon^{(j)}_{\mathbb{R}^3}\,.$$

[1] In order to compare our conventions to those of [31], one should take $r = 1/2$ in [31], and also rescale $\exp\Omega_{\text{here}} = 2\exp\Omega_{\text{there}}$.

One may check that with the current choice of basis for the spinors,

$$R = \begin{pmatrix} -\frac{1}{2} - \frac{\mathrm{i}}{2} & 0 & -\frac{1}{2} + \frac{\mathrm{i}}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} + \frac{\mathrm{i}}{2} \\ 0 & \frac{1}{2} - \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} + \frac{\mathrm{i}}{2} \\ -\frac{1}{2} - \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} - \frac{\mathrm{i}}{2} & 0 \end{pmatrix} .$$

The matrix R satisfies

$$R^{\dagger} R = \mathbb{1}_4 \, .$$

One could derive R by lifting the $SO\,(3)$ frame rotation F to $SU\,(2)$, and acting on the spinor indices.

Appendix D

Two-loop integrals

In this appendix we list the integrals corresponding to the two-loop diagrams in figures 3.2(a)-3.2(l), dressed by their color factors.

Diagram 3.2(a) contains the two-loop correction to the scalar propagator. This has been computed in [79] and reads

$$\begin{aligned}
\mathcal{C}(N_1,N_2) =& \frac{N_1 N_2}{k^2}\left(N_1^2+N_2^2-4N_1N_2+2\right)\left(\frac{\pi}{3\epsilon}+2\pi+O\left(\epsilon\right)\right) \\
&+\frac{N_1 N_2}{k^2}\left(N_1^2+N_2^2-2\right)\left(-\frac{4\pi}{3\epsilon}+\pi\left(\pi^2-8\right)+O\left(\epsilon\right)\right) \\
&+\frac{N_1 N_2}{k^2}\left(N_1N_2-1\right)\left(-\frac{8\pi}{3\epsilon}+4\pi(\pi^2-20\pi)+O\left(\epsilon\right)\right)
\end{aligned} \tag{D.1}$$

To compute the contributions of the other diagrams it is sufficient to rely on Feynman rules listed in appendix A, together with the product of polarization vectors. Explicitly, we find

$$\begin{aligned}
(3.2(b)) =& -s^2\,\frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)}{32\,\pi^{7-6\epsilon}}\,\frac{N_1^2N_2^2}{k^2}\int d^dx d^dy \frac{x^\mu y^\nu}{(x^2)^{\frac{3}{2}-\epsilon}(y^2)^{\frac{3}{2}-\epsilon}\left((x-s)^2\right)^{\frac{1}{2}-\epsilon}\left((y-s)^2\right)^{\frac{1}{2}-\epsilon}} \\
&\times\left[\frac{\delta_{\mu\nu}}{\left[(x-y)^2\right]^{1-2\epsilon}}-\partial_\mu\partial_\nu\frac{\left[(x-y)^2\right]^{2\epsilon}}{4\epsilon(1+2\epsilon)}\right]
\end{aligned} \tag{D.2}$$

$$\begin{aligned}
(3.2(c)) =& s^2\,\frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{7-6\epsilon}}\frac{N_1N_2}{k^2}\left((N_1-N_2)^2-2N_1N_2+2\right)\times \\
&\int\frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\epsilon}(y^2)^{\frac{1}{2}-\epsilon}\left((x-y)^2\right)^{2-2\epsilon}\left((x-s)^2\right)^{\frac{1}{2}-\epsilon}\left((y-s)^2\right)^{\frac{1}{2}-\epsilon}}
\end{aligned} \tag{D.3}$$

$$
\begin{aligned}
(3.2(\mathrm{d})) = s^2\, & \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{256\,\pi^{10-8\epsilon}}\,\frac{N_1 N_2}{k^2}\,(N_1-N_2)^2\ \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau}\ \times \\
\int d^dx d^dy d^dz d^dw\, & \frac{(x-y)^\eta (z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}\,((z-w)^2)^{\frac{3}{2}-\epsilon}\,((x-s)^2)^{\frac{1}{2}-\epsilon}\,((y-s)^2)^{\frac{1}{2}-\epsilon}\,(z^2)^{\frac{1}{2}-\epsilon}(w^2)^{\frac{1}{2}-\epsilon}} \\
& \times\ \partial^\mu\partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}}\ \partial^\nu\partial^\sigma \frac{1}{((y-w)^2)^{\frac{1}{2}-\epsilon}}
\end{aligned}
\tag{D.4}
$$

$$
\begin{aligned}
(3.2(\mathrm{e})) = -s^2\, & \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{10-8\epsilon}}\,\frac{N_1 N_2}{k^2}\,(N_1N_2-1)\ \varepsilon_{\mu\nu\eta}\varepsilon_{\rho\sigma\tau}\ \times \\
\int d^dx d^dy d^dz d^dw\, & \frac{(x-y)^\eta (z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}\,((z-w)^2)^{\frac{3}{2}-\epsilon}\,((x-s)^2)^{\frac{1}{2}-\epsilon}\,((w-s)^2)^{\frac{1}{2}-\epsilon}\,(y^2)^{\frac{1}{2}-\epsilon}(z^2)^{\frac{1}{2}-\epsilon}} \\
& \times\ \partial^\mu\partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}}\ \partial^\nu\partial^\sigma \frac{1}{((y-w)^2)^{\frac{1}{2}-\epsilon}}
\end{aligned}
\tag{D.5}
$$

$$
\begin{aligned}
(3.2(\mathrm{f})) = s^2\, & \frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{16\,\pi^{7-6\epsilon}}\,\frac{N_1 N_2}{k^2}(N_1N_2-1)\ \times \\
& \int \frac{d^dx d^dy}{(x^2)^{\frac{1}{2}-\epsilon}(y^2)^{\frac{1}{2}-\epsilon}\,((x-y)^2)^{2-2\epsilon}\,((x-s)^2)^{\frac{1}{2}-\epsilon}\,((y-s)^2)^{\frac{1}{2}-\epsilon}}
\end{aligned}
\tag{D.6}
$$

We note that in the large N_1, N_2 approximation we obtain (3.2(f)) $= -4$(3.2(c)), in agreement with the results in [249].

$$
\begin{aligned}
(3.2(\mathrm{g})) = -s^2\, & \frac{\Gamma^5\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{\frac{17}{2}-7\epsilon}}\,\frac{N_1 N_2}{k^2}(N_1^2+N_2^2-4N_1N_2+2)\ \varepsilon_{\mu\rho\sigma}\varepsilon_{\mu\nu\eta}\ \times \\
\int d^dx d^dy d^dz\, & \frac{(x-z)^\sigma}{((x-z)^2)^{\frac{3}{2}-\epsilon}}\,\frac{(x-y)^\eta}{((x-y)^2)^{\frac{3}{2}-\epsilon}}\ \times \\
\partial^\rho & \frac{1}{((y-z)^2)^{\frac{1}{2}-\epsilon}}\ \partial^\nu \frac{1}{((y-s)^2)^{\frac{1}{2}-\epsilon}}\ \frac{1}{(x^2)^{\frac{1}{2}-\epsilon}(z^2)^{\frac{1}{2}-\epsilon}((x-s)^2)^{\frac{1}{2}-\epsilon}}
\end{aligned}
\tag{D.7}
$$

$$
(3.2(\mathrm{h})) = 0 \qquad\qquad (3.2(\mathrm{i})) = 0 \tag{D.8}
$$

$$
\begin{aligned}
(3.2(\mathrm{j})) = s^2\, & \frac{\Gamma^5\left(\frac{1}{2}-\epsilon\right)\Gamma^3\left(\frac{3}{2}-\epsilon\right)}{128\,\pi^{10-8\epsilon}}\,\frac{N_1 N_2}{k^2}(N_1^2+N_2^2-2)\ \varepsilon_{\rho\nu\tau}\varepsilon_{\rho\eta\sigma}\varepsilon_{\nu\mu\varphi}\varepsilon_{\tau\chi\xi}\ \times \\
\int d^dx d^dy d^dz d^dw\, & \frac{(x-z)^\varphi (y-z)^\xi (w-z)^\sigma}{((x-z)^2)^{\frac{3}{2}-\epsilon}\,((y-z)^2)^{\frac{3}{2}-\epsilon}\,((w-z)^2)^{\frac{3}{2}-\epsilon}} \\
\times\ \partial^\eta & \frac{1}{((w-s)^2)^{\frac{1}{2}-\epsilon}}\ \partial^\chi \frac{1}{((x-y)^2)^{\frac{1}{2}-\epsilon}}\ \partial^\mu \frac{1}{((x-s)^2)^{\frac{1}{2}-\epsilon}}\ \frac{1}{(y^2)^{\frac{1}{2}-\epsilon}(w^2)^{\frac{1}{2}-\epsilon}}
\end{aligned}
\tag{D.9}
$$

$$
\begin{aligned}
(3.2(\mathrm{k})) = s^2\, & \frac{\Gamma^6\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{256\,\pi^{10-8\epsilon}}\ \frac{N_1N_2}{k^2}(N_1N_2-2)\ \varepsilon_{\mu\nu\epsilon}\varepsilon_{\rho\sigma\tau}\ \times \\
& \int d^dx d^dy d^dz d^dw \frac{(x-y)^\epsilon(z-w)^\tau}{((x-y)^2)^{\frac{3}{2}-\epsilon}\,((z-w)^2)^{\frac{3}{2}-\epsilon}}\ \frac{1}{((w-s)^2)^{\frac{1}{2}-\epsilon}} \\
& \times\ \partial^\rho \frac{1}{((x-z)^2)^{\frac{1}{2}-\epsilon}}\,\partial^\nu \frac{1}{((y-z)^2)^{\frac{1}{2}-\epsilon}}\partial^\sigma \frac{1}{(w^2)^{\frac{1}{2}-\epsilon}} \\
& \qquad\qquad \times\ \left[\partial^\mu \frac{1}{((x-s)^2)^{\frac{1}{2}-\epsilon}}\ \frac{1}{(y^2)^{\frac{1}{2}-\epsilon}} - \partial^\mu \frac{1}{(x^2)^{\frac{1}{2}-\epsilon}}\ \frac{1}{((y-s)^2)^{\frac{1}{2}-\epsilon}}\right]
\end{aligned}
\tag{D.10}
$$

$$
\begin{aligned}
(3.2(\mathrm{l})) = -s^2\, & \frac{\Gamma^4\left(\frac{1}{2}-\epsilon\right)\Gamma^2\left(\frac{3}{2}-\epsilon\right)}{32\,\pi^{7-6\epsilon}}\,\frac{N_1N_2}{k^2}(N_1-N_2)^2 \\
& \qquad \times \int \frac{d^dx d^dy}{((x-s)^2)^{1-2\epsilon}\,((x-y)^2)^{2-2\epsilon}\,(y^2)^{1-2\epsilon}}
\end{aligned}
\tag{D.11}
$$

Appendix E

Details on the SQM

Below we describe how the SQM is gauged by the bulk vector multiplet. Our strategy goes as follows. First, we determine the 3d submultiplet of the vector multiplet generated by the action of Q_2 and Q_3. This can be dimensionally reduced on the curve supporting the defect. Finally, we compare our result $\mathcal{N}=2$ SQM, and we read which symmetries are turned on. Since the authors of [210] define the SQM with a Lorentzian time, while our coordinate τ is a Euclidean time, we introduce a *real time* $t=-i\tau$. Therefore, we also need to set[1]

$$A_\tau = -iA_t\,, \qquad D_\tau = -iD_t\,. \tag{E.1}$$

Moreover, to avoid coordinate singularities as $\theta \to 0$, we use frame indices, which are well defined on the whole S^3.

On general grounds, the 3d $\mathcal{N}=4$ vector multiplet is decomposed as follows:

- 1 $\mathcal{N}=2$ vector multiplet

$$\mathbb{V} = (v_t,\, \sigma,\, \lambda,\, \bar{\lambda},\, D)\,. \tag{E.2}$$

- 2 $\mathcal{N}=2$ chiral multiplets [2]

$$\Phi_i = (\phi_i, \psi_i)\,. \tag{E.3}$$

- 2 $\mathcal{N}=2$ Fermi multiplets

$$\mathbb{F} = (\eta, F)\,, \qquad \tilde{\mathbb{F}} = (\tilde{\eta}, \tilde{F})\,. \tag{E.4}$$

In fact, if we write 3d $\mathcal{N}=4$ vector multiplet as a $\mathcal{N}=2$ vector multiplet plus a $\mathcal{N}=2$ chiral, we can reduce it to a 2d $\mathcal{N}=(2,2)$ vector plus an $\mathcal{N}=(2,2)$ chiral. From an $\mathcal{N}=(0,2)$ point of view, the vector multiplet contains a vector one plus a chiral one. Similarly, the $\mathcal{N}=(2,2)$ chiral multiplet is decomposed into a $\mathcal{N}=(0,2)$ chiral one plus a Fermi multiplet [250]. Since the $\mathcal{N}=2$ SQM is the dimensional reduction of the 2d $\mathcal{N}=(0,2)$, a last dimensional reduction leads to the above decomposition.

Let us provide the explicit SQM structure. Since on the circle $A_\tau = A_3$, it is natural to build the 1d vector multiplet $\mathbb{V}$ acting on it with $\delta = \epsilon\, Q_2 + \bar{\epsilon}\, Q_3$. The multiplet turns out

[1] In our convention $D_\tau = \partial_\tau - iA_\tau$ and $D_t = \partial_t - iA_t$.

[2] $i=1,2$ labels the multiplet. The corresponding anti-chirals are denoted by $\tilde{\Phi}_i = (\tilde{\phi}_i, \tilde{\psi}_i)$.

to be given by

$$v_\tau = iA_3\,, \tag{E.5a}$$
$$\sigma = -\Phi_{\dot{1}\dot{2}}\,, \tag{E.5b}$$
$$\lambda = \left(\frac{1+i}{2}\right)\left(\sqrt{1-\nu}\lambda_{2,1\dot{1}} + i\epsilon^{-i\tau}\sqrt{1+\nu}\lambda_{2,2\dot{1}}\right)\,, \tag{E.5c}$$
$$\bar{\lambda} = \left(\frac{1+i}{2}\right)\left(\sqrt{1-\nu}\,\lambda_{1,2\dot{2}} + i\epsilon^{i\tau}\sqrt{1+\nu}\lambda_{1,1\dot{2}}\right)\,, \tag{E.5d}$$
$$D = --\,iM^{ab}_{\text{vortex}}D_{ab} - i\nu\Phi_{\dot{1}\dot{2}} + F_{12} + \frac{1}{2}[\Phi_{\dot{2}}{}^{\dot{c}}, \Phi_{\dot{c}\dot{1}}]\,. \tag{E.5e}$$

Notice that the auxiliary fields in D are related to the vortex loop defined in Eq 6.4. Similarly, σ coincides the scalar part of the Wilson loop connection in Eq 6.2, up to an i factor. The corresponding supersymmetry transformations are:

$$\delta v_\tau = \frac{i}{2}\epsilon\bar{\lambda} + \frac{i}{2}\bar{\epsilon}\lambda\,, \tag{E.6a}$$
$$\delta\sigma = -\delta v_\tau\,, \tag{E.6b}$$
$$\delta\lambda = \epsilon\left(D_t\sigma + iD\right)\,, \tag{E.6c}$$
$$\delta\bar{\lambda} = \bar{\epsilon}\left(D_t\sigma - iD\right)\,, \tag{E.6d}$$
$$\delta D = \frac{1}{2}\epsilon D_t^+\bar{\lambda} - \frac{1}{2}\bar{\epsilon}D_t^+\lambda\,, \tag{E.6e}$$

where $D_t^+ = D_t - i[\sigma,\ ]$. The crucial point of this decomposition is that $\delta^2\mathbb{V}$ *does not contain* ν. For instance:

$$\delta^2\sigma = i\epsilon\bar{\epsilon}D_\tau\sigma\,. \tag{E.7}$$

The operator δ^2 contains a gauge transformation generated by

$$\Lambda = i(\sigma + v_t)\,. \tag{E.8}$$

The other two scalar fields are recast as lowest components of a chiral and an anti-chiral multiplet Φ_1 and $\tilde{\Phi}_1$. Let us first consider the chiral one

$$\phi_1 = \Phi_{\dot{1}\dot{1}}\,, \tag{E.9a}$$
$$\psi_1 = \left(\frac{1+i}{2}\right)\left(\epsilon^{i\tau}\sqrt{1+\nu}\lambda_{1,1\dot{1}} - i\sqrt{1-\nu}\lambda_{1,2\dot{1}}\right)\,. \tag{E.9b}$$

The supersymmetry transformations are

$$\delta\phi_1 = -\epsilon\psi_1\,, \tag{E.10a}$$
$$\delta\psi_1 = \bar{\epsilon}\left(iD_t^+\phi_1 + i\nu\phi_1\right)\,. \tag{E.10b}$$

It is easy to deduce from the action of δ^2 the presence of a background symmetry proportional to ν

$$\delta^2\phi_1 = \epsilon\bar{\epsilon}(D_\tau^+\phi_1 - i\nu\phi_1)\,, \qquad \delta^2\psi_1 = -\epsilon\bar{\epsilon}(D_\tau^+\,\psi_1 - i\nu\,\psi_1)\,. \tag{E.11}$$

The result is understood giving charge -1 to Φ_1 under the $U(1)$ flavor symmetry generated by νF.

Similarly, we obtain the antichiral multiplet:

$$\tilde{\phi}_1 = \Phi_{\dot{2}\dot{2}}\,, \tag{E.12a}$$

$$\tilde{\psi}_1 = \left(\frac{1+i}{2}\right)\left(-i\sqrt{1-\nu}\lambda_{2,1\dot{2}} + \epsilon^{-i\tau}\sqrt{1+\nu}\lambda_{2,2\dot{2}}\right)\,. \tag{E.12b}$$

The supersymmetry variations are given by

$$\delta\tilde{\phi}_1 = -\bar{\epsilon}\tilde{\psi}_1\,, \tag{E.13a}$$

$$\delta\tilde{\psi}_1 = \bar{\epsilon}\left(iD_t^+\tilde{\phi}_1 - i\nu\tilde{\phi}_1\right)\,. \tag{E.13b}$$

We can interpret these variations assigning charge 1 to $\tilde{\Phi}_1$ under to νF. Since ν is not integer, the ν-background cannot be reabsorbed in any redefinition of the Killing spinors. This fact prevents $\mathbb{V}$ and Φ_1 to constitute an $\mathcal{N}=4$ vector multiplet.

The other chiral multiplets come from the components of the orthogonal components of the gauge field. In particular, the first one is given by

$$\phi_2 = A_1 + iA_2\,, \tag{E.14a}$$

$$\psi_2 = -\left(\frac{1+i}{2}\right)\left(\sqrt{1-\nu}\lambda_{2,2\dot{2}} + i\epsilon^{i\tau}\sqrt{1+\nu}\lambda_{2,1\dot{2}}\right)\,. \tag{E.14b}$$

The variations are given by

$$\delta\phi_2 = -\epsilon\psi_2\,, \tag{E.15a}$$

$$\delta\psi_2 = -i\bar{\epsilon}(iD_t^+\phi_2 + 2i\phi_2) + \epsilon\, i\,\epsilon^{i\tau}\sqrt{1-\nu^2}\,\Phi_{\dot{2}\dot{2}}\,. \tag{E.15b}$$

The remaining component of the gauge field becomes the lowest component of an anti-chiral field

$$\tilde{\phi}_2 = A_1 - iA_2\,, \tag{E.16a}$$

$$\tilde{\psi}_2 = \left(\frac{1+i}{2}\right)\left(\sqrt{1-\nu}\lambda_{1,1\dot{1}} + i\epsilon^{-i\tau}\sqrt{\nu+1}\lambda_{1,2\dot{1}}\right)\,. \tag{E.16b}$$

The variations are given by[3]

$$\delta\tilde{\phi}_2 = -\bar{\epsilon}\tilde{\psi}_2\,, \tag{E.17a}$$

$$\delta\tilde{\psi}_2 = \epsilon(iD_t^+\tilde{\phi}_2 + 2i\tilde{\phi}_2) + \bar{\epsilon}\, i\,\epsilon^{-i\tau}\sqrt{1-\nu^2}\,\Phi_{\dot{1}\dot{1}}\,. \tag{E.17b}$$

The remaining fermionic degrees of freedom are recast in two Fermi fields. At this stage,

[3]The non-chiral parts of the transformations in Eq E.15b and in Eq E.17b do not appear in the 1d algebra of [210]. While these terms are due to a gauge transformations for Q_2^2 and Q_3^2 for the bulk theory, we do not have a clear interpretation from the worldvolume perspective. However, we claim that they do not spoil our arguments for the computation of the index.

we can choose them to be any non singular linear combinations of $\lambda_{a\dot{a}}$. In other words, the only constraint we impose is that the change of variables from the 3d degrees of freedom to the 1d ones is invertible. Our choice is given by

$$\eta = e^{-i\tau}\,\lambda_{2,2\dot{1}}\,, \tag{E.18a}$$

$$F = \frac{1+i}{2}\left(ie^{-i\tau}\sqrt{1-\nu}D_{22} + \sqrt{1+\nu}\left(iD_\tau\Phi_{\dot{1}\dot{2}} - \Phi_{\dot{1}\dot{2}} - D_{12} - iF_{12} + \frac{i}{2}[\Phi_{\dot{1}\dot{1}}, \Phi_{\dot{2}\dot{2}}]\right)\right)\,, \tag{E.18b}$$

and

$$\tilde{\eta} = e^{i\tau}\,\lambda_{1,1\dot{2}}\,, \tag{E.19a}$$

$$\tilde{F} = \frac{1-i}{2}\left(e^{i\tau}\sqrt{1-\nu}D_{11} + \sqrt{1+\nu}\left(iD_{12} - F_{12} - D_\tau\Phi_{\dot{1}\dot{2}} + i\Phi_{\dot{1}\dot{2}} + \frac{1}{2}[\Phi_{\dot{1}\dot{1}}, \Phi_{\dot{2}\dot{2}}]\right)\right)\,. \tag{E.19b}$$

The variations are [4]

$$\delta\eta = \epsilon F - \bar{\epsilon}\frac{1-i}{2}\sqrt{1-\nu}\,e^{-i\tau}\,[\phi_2, \phi_1]\,, \tag{E.20a}$$

$$\delta\tilde{\eta} = \bar{\epsilon}\tilde{F} + \epsilon\frac{1-i}{2}\sqrt{1-\nu}\,e^{i\tau}\,[\tilde{\phi}_2, \tilde{\phi}_1]\,, \tag{E.20b}$$

$$\delta F = \bar{\epsilon}\left(-iD_t^+\eta - \frac{1-i}{2}\sqrt{1-\nu}\,e^{-i\tau}\,\left([\psi_2, \phi_1] + [\phi_2, \psi_1]\right)\right)\,. \tag{E.20c}$$

$$\delta\tilde{F} = \epsilon\left(-iD_t^+\tilde{\eta} + \frac{1-i}{2}\sqrt{1-\nu}\,e^{i\tau}\,\left([\tilde{\psi}_2, \tilde{\phi}_1] + [\tilde{\phi}_2, \tilde{\psi}_1]\right)\right)\,. \tag{E.20d}$$

Embedding at $\nu = 1$

We study in detail the properties of the algebra generated by Q_2, Q_3, Q_5 and Q_6. We define an operator acting on the fields as

$$\delta = \epsilon\,Q_2 + \bar{\epsilon}\,Q_3 + \rho\,Q_5 + \bar{\rho}\,Q_6\,. \tag{E.21}$$

In particular, we discuss the emergence of the conformal symmetry at $\nu = 1$. We compute the action of δ^2 using the variations in Eq B, restricted to the circle $\theta = 0$. We find that δ squares to:

- A diffeomorphism generated by

$$v^a_{\nu=1} = \zeta\bar{\zeta}(0,0,1)\,, \tag{E.22}$$

where ζ and $\bar{\zeta}$ are anti-periodic 1d spinors defined in Eq 6.9. The vector $v^a_{\nu=1}$ can be expanded on the subgroup of Diff $\left(S^1\right)$ generated by L_{-1}, L_0, L_1, where $L_m = \epsilon^{im\tau}\partial_\tau$.

[4] These 1d transformations are chosen to reproduce the expected δ^2 on η and $\tilde{\eta}$. We eliminated parts related to orthogonal derivatives, which cancel among themselves as we take the double variations.

The generators L_m close the Witt algebra

$$[L_m, L_n] = i(m-n)L_{m+n}\,. \tag{E.23}$$

Thus, iL_0, L_{-1}, L_1 constitute a representation of the generators M, P and K introduced in Section 5.1.2.

- A dilatation, given by

$$\hat{\rho} = i\left(e^{i\tau}\epsilon\bar{\rho} - e^{-i\tau}\rho\bar{\epsilon}\right)\,. \tag{E.24}$$

This corresponds to a Weyl transformation contained in the commutators $\{Q_2, Q_6\}$ and $\{Q_3, Q_5\}$, which, from the point of view of the bulk theory, corresponds to the Weyl transformation stored in $K_1 \pm iK_2$.

- A gauge transformation

$$\Lambda = -\bar{\zeta}\zeta(A_3 + i\Phi_{1\dot{2}})\,, \tag{E.25}$$

- The following R-symmetry transformations

$$R_H = \frac{i}{2}\begin{pmatrix} \bar{\rho}\rho - \bar{\epsilon}\epsilon & 0 \\ 0 & \bar{\epsilon}\epsilon - \bar{\rho}\rho \end{pmatrix}\,, \qquad R_C = \frac{i}{2}\begin{pmatrix} \bar{\epsilon}\epsilon - \bar{\rho}\rho & 0 \\ 0 & \bar{\rho}\rho - \bar{\epsilon}\epsilon \end{pmatrix}\,. \tag{E.26}$$

In order to explore the superconformal algebra, we represent the δ-action on the vector multiplet and on the chiral one. We will denote the 1d component fields in the same way as ν generic, even tough the embedding is slightly different from the limit $\nu \to 1$ of the one described in the previous section[5]. Moreover, all the fermions are now taken anti-periodic.

At $\nu = 1$, the embedding for the 1d vector multiplet turns out to be

$$v_\tau = iA_3 = A_t\,, \tag{E.27}$$

$$\sigma = -\Phi_{1\dot{2}}\,, \tag{E.28}$$

$$\lambda = i\frac{1+i}{\sqrt{2}}\,e^{-\frac{i}{2}\tau}\,\lambda_{2,21}\,, \tag{E.29}$$

$$\bar{\lambda} = i\frac{1+i}{\sqrt{2}}\,e^{\frac{i}{2}\tau}\,\lambda_{1,1\dot{2}}\,, \tag{E.30}$$

$$D = F_{12} - iD_{12} + \frac{1}{2}[\Phi_{\dot{2}}{}^{\dot{c}}, \Phi_{\dot{c}1}]\,. \tag{E.31}$$

[5]For instance, the embedding proposed at $\nu-$generic becomes singular for the Fermi fields as $\nu \to 1$.

The supersymmetry transformations are given by

$$\delta v_\tau = \frac{i}{2}\zeta\bar{\lambda} + \frac{i}{2}\bar{\zeta}\lambda\,, \tag{E.32}$$

$$\delta\sigma = -\delta v_\tau\,, \tag{E.33}$$

$$\delta\lambda = \zeta\left(D_t\sigma + iD\right) + 2i\sigma\partial_\tau\zeta\,, \tag{E.34}$$

$$\delta\bar{\lambda} = \bar{\zeta}\left(D_t\sigma - iD\right) + 2i\sigma\partial_\tau\bar{\zeta}\,, \tag{E.35}$$

$$\delta D = \frac{1}{2}\left(D_t^+\left(\zeta\bar{\lambda}\right) + 2i\bar{\lambda}\partial_\tau\zeta\right) - \frac{1}{2}\left(D_t^+\left(\bar{\zeta}\lambda\right) + 2i\lambda\partial_\tau\bar{\zeta}\right)\,. \tag{E.36}$$

We can use these variations to compute the action of δ^2 on the field components:

$$\delta^2\sigma = D_\tau\left(\bar{\zeta}\zeta\sigma\right)\,, \tag{E.37}$$

We see that it reproduces the algebra in Eq 5.38 [6]

$$\{Q_2, Q_3\}\sigma = L_0\sigma\,, \qquad \{Q_5, Q_6\}\sigma = L_0\sigma\,, \tag{E.38a}$$

$$\{Q_2, Q_6\}\sigma = \left(L_1 + i\epsilon^{i\tau}\right)\sigma\,, \qquad \{Q_3, Q_5\}\sigma = \left(L_{-1} - i\epsilon^{-i\tau}\right)\sigma\,. \tag{E.38b}$$

For example, σ is consistently uncharged under the R-symmetry J and that σ has Weyl weight 1. For the gauginos we obtain

$$\delta^2\lambda = D_\tau^+\left(\bar{\zeta}\zeta\lambda\right) + \bar{\zeta}\lambda\partial_\tau\zeta\,, \qquad \delta^2\bar{\lambda} = D_\tau^+\left(\bar{\zeta}\zeta\bar{\lambda}\right) + \zeta\bar{\lambda}\partial_\tau\bar{\zeta}\,, \tag{E.39}$$

which yields

$$\{Q_2, Q_3\}\lambda = \left(L_0 + \frac{i}{2}\right)\lambda\,, \qquad \{Q_5, Q_6\}\lambda = \left(L_0 - \frac{i}{2}\right)\lambda\,, \tag{E.40a}$$

$$\{Q_2, Q_6\}\lambda = \left(L_1 + \frac{3i}{2}\epsilon^{i\tau}\right)\lambda\,, \qquad \{Q_3, Q_5\}\lambda = \left(L_{-1} - \frac{3i}{2}\epsilon^{-i\tau}\right)\lambda\,. \tag{E.40b}$$

Thus, λ has Weyl weight 3/2 and has charge $-1/2$ under J. For $\bar{\lambda}$ we get the same result but with opposite J-charge.

We also provide the embedding for the chiral field

$$\phi_1 = \Phi_{1\mathrm{i}}\,, \tag{E.41}$$

$$\psi_1 = e^{-\frac{i}{2}\tau}\frac{(1+i)}{\sqrt{2}}\lambda_{1,1\mathrm{i}}\,, \tag{E.42}$$

with the supersymmetry transformations

$$\delta\phi_1 = -\zeta\psi_1\,, \tag{E.43}$$

$$\delta\psi_1 = \left(iD_t^+\left(\bar{\zeta}\phi_1\right) - \phi_1\partial_\tau\bar{\zeta}\right)\,. \tag{E.44}$$

[6] We are omitting the gauge transformations, which are not relevant for the discussion.

The action of δ^2 on ϕ_1 is

$$\{Q_2, Q_3\}\phi_1 = (L_0 - i)\,\phi_1\,, \qquad \{Q_5, Q_6\}\phi_1 = (L_0 + i)\,\phi_1\,, \tag{E.45a}$$

$$\{Q_2, Q_6\}\phi_1 = \left(L_1 + i\epsilon^{i\tau}\right)\phi_1\,, \qquad \{Q_3, Q_5\}\phi_1 = \left(L_{-1} - i\epsilon^{-i\tau}\right)\phi_1\,. \tag{E.45b}$$

We see that ϕ_1 is charged under J with charge 1 and that ϕ_1 has Weyl weight 1. For ψ_1, we get

$$\{Q_2, Q_3\}\psi_1 = \left(L_0 - \frac{i}{2}\right)\psi_1\,, \qquad \{Q_5, Q_6\}\psi_1 = \left(L_0 + \frac{i}{2}\right)\psi_1\,, \tag{E.46a}$$

$$\{Q_2, Q_6\}\psi_1 = \left(L_1 + \frac{3}{2}i\epsilon^{i\tau}\right)\psi_1\,, \qquad \{Q_3, Q_5\}\psi_1 = \left(L_{-1} - \frac{3}{2}i\epsilon^{-i\tau}\right)\psi_1\,. \tag{E.46b}$$

Thus, we see that we can assign a Weyl weight $3/2$ to ψ_1 and a J charge $1/2$. In conclusion, we have shown concretely that our δ reproduces the algebra of the latitude Wilson loop at $\nu = 1$.

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