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Solving problems on generalized convex graphs via mim-width ☆

Flavia Bonomo-Braberman [a,b], Nick Brettell [c,*], Andrea Munaro [d], Daniël Paulusma [e]

[a] Universidad de Buenos Aires, Facultad de Ciencias Exactas y Naturales, Departamento de Computación, Buenos Aires, Argentina
[b] CONICET-Universidad de Buenos Aires, Instituto de Investigación en Ciencias de la Computación (ICC), Buenos Aires, Argentina
[c] School of Mathematics and Statistics, Victoria University of Wellington, New Zealand
[d] Department of Mathematical, Physical and Computer Sciences, University of Parma, Italy
[e] Department of Computer Science, Durham University, UK

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ABSTRACT

A bipartite graph $G = (A, B, E)$ is $\mathcal{H}$-convex for some family of graphs $\mathcal{H}$ if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the neighbours in A of each $b \in B$ induce a connected subgraph of H. Many NP-complete problems are polynomial-time solvable for $\mathcal{H}$-convex graphs when $\mathcal{H}$ is the set of paths. The underlying reason is that the class has bounded mim-width. We extend this result to families of $\mathcal{H}$-convex graphs where $\mathcal{H}$ is the set of cycles, or $\mathcal{H}$ is the set of trees with bounded maximum degree and a bounded number of vertices of degree at least 3. As a consequence, we strengthen many known results via one general and short proof. We also show that the mim-width of $\mathcal{H}$-convex graphs is unbounded if $\mathcal{H}$ is the set of trees with arbitrarily large maximum degree or an arbitrarily large number of vertices of degree at least 3.

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1. Introduction

Many computationally hard graph problems can be solved efficiently if we place constraints on the input. Instead of solving individual problems in an ad hoc way we may try to decompose the vertex set of the input graph into large sets of "similarly behaving" vertices and to exploit this decomposition for an algorithmic speed up that works for many problems simultaneously. This requires some notion of an "optimal" vertex decomposition, which depends on the type of vertex decomposition used and which may relate to the minimum number of sets or the maximum size of a set in a vertex decomposition. An optimal vertex decomposition gives us the "width" of the graph.

A graph class has *bounded width* if every graph in the class has width at most some constant c. Boundedness of width is often the underlying reason why a graph-class-specific algorithm runs efficiently: in this case, the proof that the algorithm

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* Corresponding author.
E-mail addresses: fbonomo@dc.uba.ar (F. Bonomo-Braberman), nick.brettell@vuw.ac.nz (N. Brettell), a.munaro@qub.ac.uk (A. Munaro), daniel.paulusma@durham.ac.uk (D. Paulusma).

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is efficient for some special graph class reduces to a proof showing that the width of the class is bounded by some constant. We will give examples, but also refer to the surveys [26,34,39,44,66] for further details and examples.

Width parameters differ in strength. A width parameter p *dominates* a width parameter q if there is a function f such that $p(G)$ is at most $f(q(G))$ for every graph G. If p dominates q but q does not dominate p, then we say that p is *more powerful* than q. If both p and q dominate each other, then p and q are *equivalent*. If neither p is more powerful than q nor q is more powerful than p, then p and q are *incomparable*. If p is more powerful than q, then the class of graphs for which p is bounded is larger than the class of graphs for which q is bounded and so efficient algorithms for bounded p have greater applicability with respect to the graphs under consideration. The trade-off is that fewer problems exhibit an efficient algorithm for the parameter p, compared to the parameter q.

This notion of powerfulness leads to a large hierarchy of width parameters, in which new width parameters continue to be defined, for example, graph functionality [1] in 2019 and twin-width [5] in 2020. The well-known parameters boolean-width, clique-width, module-width and rank-width are equivalent to each other [17,55,60]. They are more powerful than the equivalent parameters branch-width and treewidth [24,62,66] but less powerful than mim-width [66], which is less powerful than sim-width [45]. To give another example, thinness is more powerful than path-width [53], but less powerful than (linear) mim-width and incomparable to clique-width or treewidth [6].

For each group of equivalent width parameters, a growing set of NP-complete problems is known to be tractable on graph classes of bounded width. Proving the latter for some graph class often immediately tells us that many problems are tractable for that class without the need for constructing algorithms for each problem. However, there are still large families of graph classes, and many width parameters, for which it is not known whether the class has bounded width.

1.1. Our focus

We consider the relatively new width parameter *mim-width*, which we define below. Recently, we showed in [14,15] that boundedness of mim-width is the underlying reason why some specific hereditary graph classes, characterized by two forbidden induced subgraphs, admit polynomial-time algorithms for a range of problems including k-Colouring and its generalization List k-Colouring (the algorithms are given in [23,25,35]). Here we prove that the same holds for a number of superclasses of convex graphs known in the literature. Essentially all the known polynomial-time algorithms for such classes are obtained by reducing to the class of convex graphs. We show that our new approach via mim-width *simplifies the analysis, unifies the sporadic approaches and explains the reductions to convex graphs.*

1.2. Mim-width

A set of edges M in a graph G is a *matching* if no two edges of M share an endpoint. A matching M is *induced* if there is no edge in G between vertices of different edges of M. Let $(A, \overline{A})$ be a partition of the vertex set of a graph G. Then $G[A, \overline{A}]$ denotes the bipartite subgraph of G induced by the edges with one endpoint in A and the other in $\overline{A}$. Vatshelle [66] introduced the notion of *maximum induced matching width*, also called mim-width. Mim-width measures the extent to which it is possible to decompose a graph G along certain vertex partitions $(A, \overline{A})$ such that the size of a maximum induced matching in $G[A, \overline{A}]$ is small. The kind of vertex partitions permitted stems from classical branch decompositions, as we explain below.

A *branch decomposition* for a graph G is a pair (T, δ), where T is a subcubic tree and δ is a bijection from $V(G)$ to the leaves of T. Every edge $e \in E(T)$ partitions the leaves of T into two classes, L_e and $\overline{L_e}$, depending on which component of $T - e$ they belong to. Hence, e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. Let $\mathrm{cutmim}_G(A_e, \overline{A_e})$ be the size of a maximum induced matching in $G[A_e, \overline{A_e}]$. Then the *mim-width* $\mathrm{mimw}_G(T, \delta)$ of (T, δ) is the maximum value of $\mathrm{cutmim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$. The *mim-width* $\mathrm{mimw}(G)$ of G is the minimum value of $\mathrm{mimw}_G(T, \delta)$ over all branch decompositions (T, δ) for G. See Fig. 1 for an example of a branch decomposition for a graph.

A *caterpillar* is a tree T that contains a path P, the *backbone* of T, such that every vertex not on P has a neighbour on P (the tree in Fig. 1 (b) is an example of a caterpillar). A *linear branch decomposition* for a graph G is a pair (T, δ), where T is a caterpillar and δ is a bijection from $V(G)$ to the leaves of T. The *linear mim-width* of G is the minimum value of $\mathrm{mimw}_G(T, \delta)$ over all linear branch decompositions (T, δ) for G. Note that Fig. 1 is in fact an example of a linear branch decomposition.

The problems of computing the mim-width and linear mim-width of a graph are NP-hard [63]. Moreover, approximating the mim-width in polynomial time within a constant factor of the optimal is not possible unless $\mathsf{NP} = \mathsf{ZPP}$ [63]. It is not known how to compute in polynomial time a branch decomposition for a graph G whose mim-width is bounded by some function in $\mathrm{mimw}(G)$. Nevertheless, if G is from some graph class $\mathcal{G}$ of bounded mim-width, then this is often possible. In that case, the mim-width of $\mathcal{G}$ is said to be *quickly computable*. One can then try to develop a polynomial-time algorithm for the graph problem under consideration via dynamic programming over the computed branch decomposition. We give examples of such problems later.

1.3. Convex graphs and generalizations

A bipartite graph $G = (A, B, E)$ is *convex* if there exists a path P with $V(P) = A$ such that the neighbours in A of each $b \in B$ induce a connected subpath of P. Convex graphs generalize bipartite permutation graphs (see, e.g., [11]) and form a

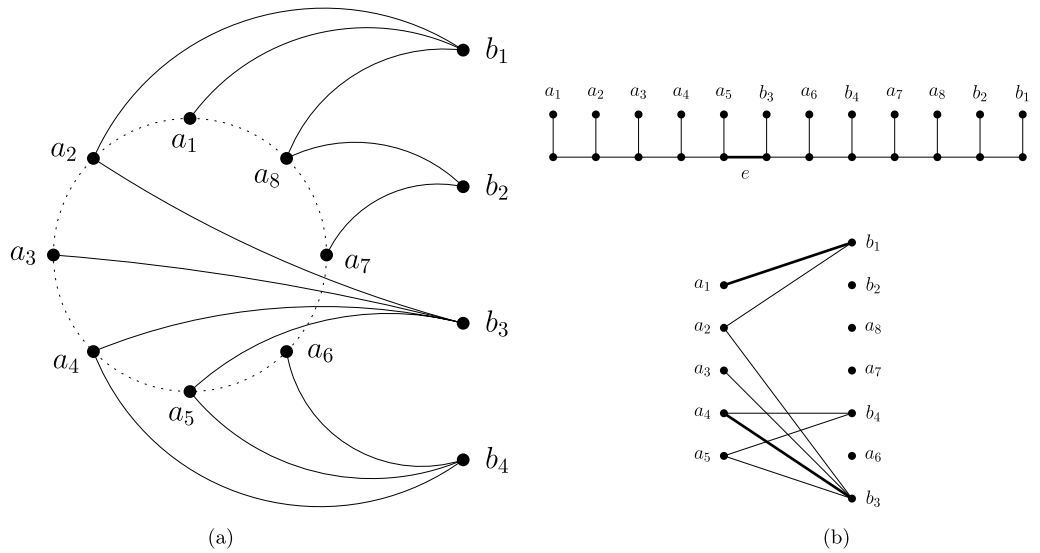

Fig. 1. (a) A circular convex graph $G = (A, B, E)$ with a circular ordering on A. (b) A (linear) branch decomposition (T, δ) for G, where T is a caterpillar with a specified edge e, together with the graph $G[A_e, \overline{A_e}]$. The bold edges in $G[A_e, \overline{A_e}]$ form an induced matching and it is easy to see that $\mathrm{cutmim}_G(A_e, \overline{A_e}) = 2$.

well-studied graph class. They were introduced in the sixties, by Glover [31], to solve a special type of matching problem arising in some industrial application. Another early paper solving matching problems on convex graphs is by Lipski and Preparata [48].

Belmonte and Vatshelle [2] proved that the mim-width of convex graphs is bounded and quickly computable. We consider superclasses of convex graphs and research to what extent mim-width can play a role in obtaining polynomial-time algorithms for problems on these classes.

Let $\mathcal{H}$ be a family of graphs. A bipartite graph $G = (A, B, E)$ is $\mathcal{H}$-convex if there exists a graph $H \in \mathcal{H}$ with $V(H) = A$ such that the set of neighbours in A of each $b \in B$ induces a connected subgraph of H. If $\mathcal{H}$ consists of all paths, we obtain the class of convex graphs. Recall that a caterpillar is a tree T that contains a path P (the backbone), such that every vertex not on P has a neighbour on P. A caterpillar with a backbone consisting of one vertex is a *star*. A *comb* is a caterpillar such that every backbone vertex has exactly one neighbour outside the backbone. The *subdivision* of an edge uv replaces uv by a new vertex w and edges uw and wu. A *triad* is a tree that can be obtained from a 4-vertex star after a sequence of subdivisions. For $t, \Delta \geq 0$, a (t, Δ)-*tree* is a tree with maximum degree at most Δ and containing at most t vertices of degree at least 3; note that, for example, a triad is a $(1, 3)$-tree. We can now define the following classes of $\mathcal{H}$-convex graphs:

- convex graphs if $\mathcal{H}$ consists of all paths
- *circular convex* graphs if $\mathcal{H}$ consists of all cycles
- *tree convex* graphs if $\mathcal{H}$ consists of all trees
- *star convex* graphs if $\mathcal{H}$ consists of all stars
- *triad convex* graphs if $\mathcal{H}$ consists of all triads
- *comb convex* graphs if $\mathcal{H}$ consists of all combs
- (t, Δ)-*tree convex* graphs if $\mathcal{H}$ consists of all (t, Δ)-trees.

See Fig. 1 for an example of a circular convex graph (this class was introduced by Liang and Blum [47] to model certain scheduling problems).

To show the relationships between the above graph classes we need some extra terminology. Let $\mathcal{C}_{t,\Delta}$ be the class of (t, Δ)-tree convex graphs. For fixed t or Δ, we have increasing sequences $\mathcal{C}_{t,0} \subseteq \mathcal{C}_{t,1} \subseteq \cdots$ and $\mathcal{C}_{0,\Delta} \subseteq \mathcal{C}_{1,\Delta} \subseteq \cdots$. For $t \in \mathbb{N}$, the class of (t, ∞)-*tree convex graphs* is $\bigcup_{\Delta \in \mathbb{N}} \mathcal{C}_{t,\Delta}$, denoted by $\mathcal{C}_{t,\infty}$. Similarly, for $\Delta \in \mathbb{N}$, the class of (∞, Δ)-*tree convex graphs* is $\bigcup_{t \in \mathbb{N}} \mathcal{C}_{t,\Delta}$, denoted by $\mathcal{C}_{\infty,\Delta}$. Hence, $\mathcal{C}_{t,\infty}$ and $\mathcal{C}_{\infty,\Delta}$ are the set-theoretic limits of the increasing sequences $\{\mathcal{C}_{t,\Delta}\}_{\Delta \in \mathbb{N}}$ and $\{\mathcal{C}_{t,\Delta}\}_{t \in \mathbb{N}}$, respectively. The class of (∞, ∞)-*tree convex graphs* is $\bigcup_{t,\Delta \in \mathbb{N}} \mathcal{C}_{t,\Delta}$, which coincides with the class of tree convex graphs. Notice that the class of convex graphs coincides with $\mathcal{C}_{t,2}$ for any $t \in \mathbb{N} \cup \{\infty\}$, and with $\mathcal{C}_{0,\Delta}$ for any $\Delta \in (\mathbb{N} \setminus \{0, 1\}) \cup \{\infty\}$. The class of star convex graphs coincides with $\mathcal{C}_{1,\infty}$. Moreover, each triad convex graph belongs to $\mathcal{C}_{1,3}$ and each comb convex graph belongs to $\mathcal{C}_{\infty,3}$.

A bipartite graph is *chordal bipartite* if every induced cycle in it has exactly four vertices. Every convex graph is chordal bipartite (see, e.g., [11]) and every chordal bipartite graph is tree convex (see [42,49]). In Fig. 2 we display the relationships between these classes.

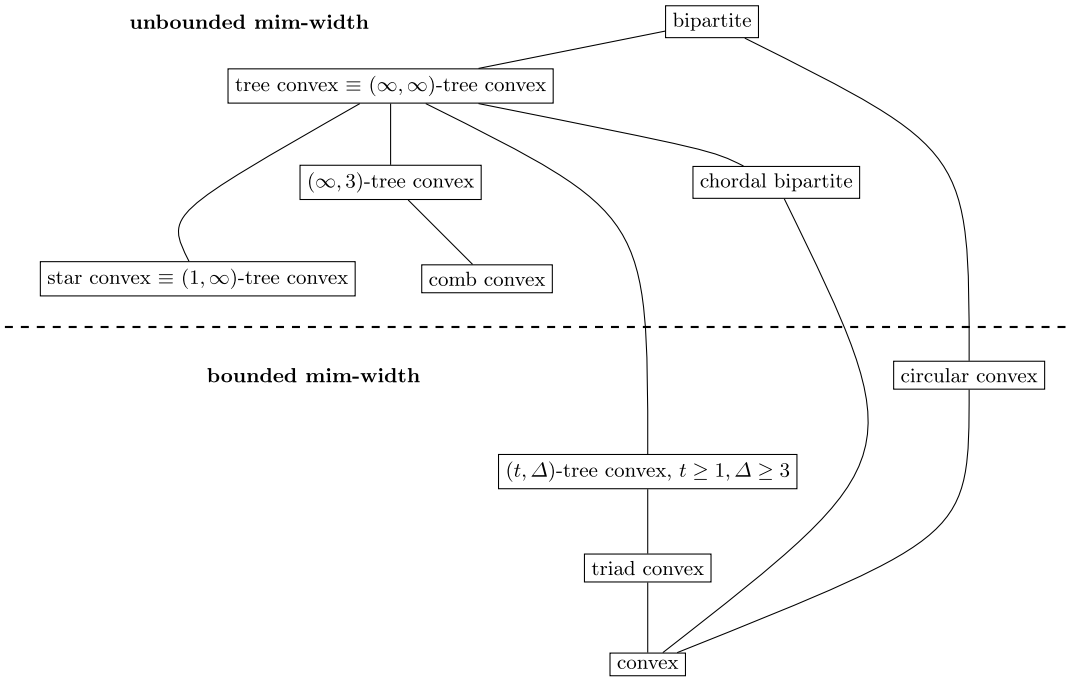

Fig. 2. The inclusion relations between the classes we consider. A line from a lower-level class to a higher one means the first class is contained in the second.

Brault-Baron et al. [13] proved that chordal bipartite graphs have unbounded mim-width. Hence, the result of [2] for convex graphs cannot be generalized to chordal bipartite graphs. We determine the mim-width of the other classes in Fig. 2 but first discuss known algorithmic results for these classes.

1.4. Known results

Belmonte and Vatshelle [2] and Bui-Xuan et al. [18] proved that so-called Locally Checkable Vertex Subset and Vertex Partitioning (LC-VSVP) problems, first defined in [65], are polynomial-time solvable on graph classes whose mim-width is bounded and quickly computable. This result was extended by Bergougnoux and Kanté [3] to variants of such problems with additional constraints on connectivity or acyclicity. Each of the problems mentioned below is a special case of a Locally Checkable Vertex Subset (LCVS) problem possibly with one of the two extra constraints. We refer to the listed papers for the definitions of the problems, as we do not need them here.

Panda et al. [57] proved that Induced Matching is polynomial-time solvable for circular convex and triad convex graphs, but NP-complete for star convex and comb convex graphs. Pandey and Panda [58] proved that Dominating Set is polynomial-time solvable for circular convex, triad convex and $(1, \Delta)$-tree convex graphs for every $\Delta \geq 1$. Liu et al. [51] proved that Connected Dominating Set is polynomial-time solvable for circular convex and triad convex graphs. Chen et al. [20] showed that Dominating Set, Connected Dominating Set and Total Dominating Set are NP-complete for star convex and comb convex graphs.

Lu et al. [52] proved that Independent Dominating Set is polynomial-time solvable for circular convex and triad convex graphs. The latter result was previously shown in [64] using a dynamic programming approach instead of a reduction to convex graphs [52]. Song et al. [64] showed in fact a stronger result, namely that Independent Dominating Set is polynomial-time solvable for (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$. They also showed that Independent Dominating Set is NP-complete for star convex and comb convex graphs [64]. Hence, they obtained a dichotomy: Independent Dominating Set is polynomial-time solvable for (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$ but NP-complete for $(\infty, 3)$-tree convex graphs and $(1, \infty)$-tree convex graphs.

The same dichotomy (explicitly formulated in [68]) holds for Feedback Vertex Set and is obtained similarly. Namely, Jiang et al. [43] proved that this problem is polynomial-time solvable for triad convex graphs and mentioned that their algorithm can be generalized to (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$. Jiang et al. [42] proved that Feedback Vertex Set is NP-complete for star convex and comb convex graphs. In addition, Liu et al. [50] proved that Feedback Vertex Set is polynomial-time solvable for circular convex graphs, whereas Jiang et al. [42] proved that the Weighted Feedback Vertex Set problem is polynomial-time solvable for triad convex graphs.

It turns out that the above problems are polynomial-time solvable on circular convex graphs and subclasses of (t, Δ)-tree convex graphs, but NP-complete for star convex graphs and comb convex graphs. In contrast, Panda and Chaudhary [56]

proved that Dominating Induced Matching is not only polynomial-time solvable on circular convex and triad convex graphs, but also on star convex graphs. Nevertheless, we notice *a common pattern*: many dominating set, induced matching and graph transversal type of problems are polynomial-time solvable for (t, Δ)-tree convex graphs, for every $t \geq 1$ and $\Delta \geq 3$, and NP-complete for comb convex graphs, and thus for $(\infty, 3)$-tree convex graphs, and star convex graphs, or equivalently, $(1, \infty)$-tree convex graphs. Moreover, essentially all the polynomial-time algorithms reduce the input to a convex graph.

1.5. Our results

We simplify the analysis, unify the above approaches and explain the reductions to convex graphs, using mim-width. We prove three results that, together with the fact that chordal bipartite graphs have unbounded mim-width [13], explain the dotted line in Fig. 2. The first two results generalize the result of [2] for convex graphs. The third result gives two new reasons why tree convex graphs (that is, (∞, ∞)-tree convex graphs) have unbounded mim-width. In Sections 2–4, respectively, we show how these results follow from some stronger statements (see also Section 1.7).

Theorem 1. *The (linear) mim-width of the class of circular convex graphs is bounded and quickly computable.*

Theorem 2. *For every $t, \Delta \in \mathbb{N}$ with $t \geq 1$ and $\Delta \geq 3$, the (linear) mim-width of the class of (t, Δ)-tree convex graphs is bounded and quickly computable.*

Theorem 3. *The class of star convex graphs and the class of comb convex graphs have unbounded mim-width.*

Hence, we obtain a structural dichotomy (recall that star convex graphs are the $(1, \infty)$-tree convex graphs and that comb convex graphs are $(\infty, 3)$-tree convex):

Corollary 1. *Let $t, \Delta \in \mathbb{N} \cup \{\infty\}$ with $t \geq 1$ and $\Delta \geq 3$. The class of (t, Δ)-tree convex graphs has bounded mim-width if and only if $\{t, \Delta\} \cap \{\infty\} = \varnothing$.*

1.6. Algorithmic consequences

As discussed, the following six problems were shown to be NP-complete for star convex and comb convex graphs, and thus for $(1, \infty)$-tree convex graphs and $(\infty, 3)$-tree convex graphs:

1. Feedback Vertex Set [3,42]
2. Dominating Set [20]
3. Connected Dominating Set [20]
4. Total Dominating Set [20]
5. Independent Dominating Set [64]
6. Induced Matching [57].

These problems are examples of LCVS problems, possibly with connectivity or acyclicity constraints. Hence, they are polynomial-time solvable for every graph class whose mim-width is bounded and quickly computable [2,3,18]. Recall that the same holds for Weighted Feedback Vertex Set [40] and (Weighted) Subset Feedback Vertex Set [4]; these three problems generalize Feedback Vertex Set and are thus NP-complete for star convex graphs and comb convex graphs. Combining these results with Corollary 1 yields the following complexity dichotomy which extends and unifies the aforementioned algorithmic results from ten papers [20,42,43,50–52,57,58,64,68].

Corollary 2. *Let $t, \Delta \in \mathbb{N} \cup \{\infty\}$ with $t \geq 1$, $\Delta \geq 3$ and Π be one of the problems mentioned above, restricted to (t, Δ)-tree convex graphs. If $\{t, \Delta\} \cap \{\infty\} = \varnothing$, then Π is polynomial-time solvable; otherwise, Π is NP-complete.*

It is worth noting that this complexity dichotomy does not hold for all LCVS problems; recall that Dominating Induced Matching is polynomial-time solvable on star convex graphs [56]. We also extend this result from the literature. Namely, Theorems 1 and 2, combined with the result of [18], imply that Dominating Induced Matching is also polynomial-time solvable on circular convex graphs and (t, Δ)-tree convex graphs for every $t \geq 1$ and $\Delta \geq 3$.

Our results have further algorithmic consequences. For every fixed integer $k \geq 1$, the k-Colouring problem is an example of an LC-VSVP problem. Kwon [46] observed that even its generalization List k-Colouring is polynomial-time solvable on graph classes whose mim-width is bounded and quickly computable (see [15] for details). Hence, Theorems 1 and 2, combined with Kwon's observation [46], also generalize a result of Díaz et al. [27] for List k-Colouring on convex graphs to circular convex and (t, Δ)-tree convex graphs (for any fixed $t \geq 1$, $\Delta \geq 3$). We prove the following result (note that the complexity of List 3-Colouring for comb convex graphs is still unresolved).

Theorem 4. *For $k \geq 4$,* List k-Colouring *is NP-complete for star convex graphs and comb convex graphs, while* List 3-Colouring *is polynomial-time solvable for star convex graphs.*

Thus, in addition to Corollary 2, we obtain the following corollary for yet another problem.

Corollary 3. *For $k \geq 4$ and $t, \Delta \in \mathbb{N} \cup \{\infty\}$ with $t \geq 1$, $\Delta \geq 3$,* List k-Colouring, *restricted to (t, Δ)-tree convex graphs, is polynomial-time solvable if $\{t, \Delta\} \cap \{\infty\} = \varnothing$, and NP-complete otherwise.*

1.7. Outline

As mentioned, we prove Theorems 1–3 in Sections 2–4, respectively, by showing additional structural results. First, in Section 2, we prove Theorem 1 by showing that circular convex graphs have linear mim-width at most 2. Next, in Section 3, we prove a stronger version of Theorem 2; namely we prove the result for thinness, a relatively recent graph width parameter introduced in [53]. This replaces the more direct proof given in [8]. Finally, in Section 4, we prove that Theorem 3 follows from a stronger result as well: we prove that star convex graphs and comb convex graphs have unbounded sim-width. Recall that the latter width parameter is more powerful than mim-width [45].

The above additional results naturally beg the question for a more refined analysis on width parameters for generalized convex classes. We perform this analysis in Section 5. There, we consider a hierarchy of width parameters, which include mim-width and thinness. We determine exactly which of the generalized convex classes considered in the previous sections have bounded width for each of these parameters. We are not yet aware of any new algorithmic results arising from this analysis, but in Section 7 we give some directions for future research, after proving Theorem 4 in Section 6.

1.8. Preliminaries

We consider only finite graphs $G = (V, E)$ with no loops and no multiple edges. For $v \in V$, the *neighbourhood* $N_G(v)$ is the set of vertices adjacent to v. The *degree* $d(v)$ of a vertex $v \in V$ is the size $|N_G(v)|$. A vertex of degree k is a *k-vertex*. A graph is *subcubic* if every vertex has degree at most 3. We let $\Delta(G) = \max\{d(v) : v \in V\}$. For disjoint $S, T \subseteq V$, we say that S is *complete to* T if every vertex of S is adjacent to every vertex of T. For $S \subseteq V$, $G[S] = (S, \{uv : u, v \in S, uv \in E\})$ is the subgraph of G *induced* by S. The *disjoint union* $G + H$ of graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A graph is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. A 2-partite graph is also called *bipartite*. A graph G is a *support* for a hypergraph $H = (V, \mathcal{S})$ if the vertices of G correspond to the vertices of H and, for each hyperedge $S \in \mathcal{S}$, the subgraph of G induced by S is connected. When a bipartite graph $G = (A, B, E)$ is viewed as a hypergraph $H = (A, \{N(b) : b \in B\})$, then a support T for H with $T \in \mathcal{H}$ is a witness that G is $\mathcal{H}$-convex.

2. The proof of Theorem 1

We need the following known lemma on recognizing circular convex graphs.

Lemma 1 *(see, e.g., Buchin et al. [16]). Circular convex graphs can be recognized and a cycle support computed, if it exists, in polynomial time.*

We now define a special type of caterpillar. For an integer $\ell \geq 1$, an *ℓ-caterpillar* is a subcubic tree T on 2ℓ vertices with $V(T) = \{s_1, \ldots, s_\ell, t_1, \ldots, t_\ell\}$, such that $E(T) = \{s_i t_i : 1 \leq i \leq \ell\} \cup \{s_i s_{i+1} : 1 \leq i \leq \ell - 1\}$. Note that we label the leaves of an ℓ-caterpillar $t_1, t_2, \ldots, t_\ell$, in this order. Given a total ordering $\prec$ of length ℓ, we say that (T, δ) is *obtained from* $\prec$ if T is an ℓ-caterpillar and δ is the natural bijection from the ℓ ordered elements to the leaves of T. We can now state our first result.

Theorem 5. *Every circular convex graph G has linear mim-width at most 2. Moreover, we can construct a linear branch decomposition (T, δ) for G with $\mathrm{mimw}_G(T, \delta) \leq 2$ in polynomial time.*

Proof. Let $G = (A, B, E)$ be a circular convex graph with a circular ordering on A. By Lemma 1, we construct in polynomial time such an ordering $a_1, \ldots, a_n$, where $n = |A|$. Let $B_1 = N(a_n)$ and $B_2 = B \setminus B_1$. We obtain a total ordering $\prec$ on $V(G)$ by extending the ordering $a_1, \ldots, a_n$ as follows. Each $b \in B_1$ is inserted after a_n, breaking ties arbitrarily. Each $b \in B_2$ is inserted immediately after the largest element of A it is adjacent to (hence immediately after some a_i with $1 \leq i < n$), breaking ties arbitrarily. We refer to Fig. 1 for an example.

Let T be the $|V(G)|$-caterpillar obtained from $\prec$. We will prove that $\mathrm{mimw}_G(T, \delta) \leq 2$. Let $e \in E(T)$. We may assume without loss of generality that e is not incident to a leaf of T. Let M be a maximum induced matching of $G[A_e, \overline{A_e}]$. As e is

not incident to a leaf, we may assume without loss of generality that each vertex in $\overline{A_e}$ is larger than any vertex in A_e in the ordering $\prec$.

We first observe that at most one edge of M has one endpoint in B_2. Indeed, suppose there exist two edges $xy, x'y' \in M$, each with one endpoint in B_2, say $\{y, y'\} \subseteq B_2$. Since each vertex in B_2 is adjacent only to smaller vertices, $\{y, y'\} \subseteq \overline{A_e}$ and $\{x, x'\} \subseteq A_e$. Without loss of generality, assume that $x \prec x'$. As $x' \prec y$ and y is adjacent to x, we find that y is also adjacent to x', by construction. This contradicts the fact that M is induced.

We now show that at most two edges in M have an endpoint in B_1 and, if exactly two such edges are in M, then no edge with an endpoint in B_2 is. First suppose that three edges of M have one endpoint in B_1 and let u_1, u_2, u_3 be these endpoints. Since $N(u_1)$, $N(u_2)$ and $N(u_3)$ are intervals of the circular ordering on A all containing a_n, one of these neighbourhoods is contained in the union of the other two, contradicting the fact that M is induced.

Finally, suppose that exactly two edges u_1v_1 and $u_2v_2 \in M$ have one endpoint in B_1 and thus their other endpoint in A. Let $\{u_1, u_2\} \subseteq \overline{A_e}$ and $\{v_1, v_2\} \subseteq A_e$. Then, as each vertex in $\overline{A_e}$ is larger than any vertex in A_e in $\prec$, we find that u_1 and u_2 belong to B_1 and thus $\{v_1, v_2\} \subseteq A$. Now if there is some edge $u_3v_3 \in M$ such that $u_3 \in B_2$, then $u_3 \in \overline{A_e}$. Recall that $N(u_1)$ and $N(u_2)$ are intervals of the circular ordering on A both containing a_n. Since M is induced, for each $i, j \in \{1, 2\}$, we have that $v_i \in N(u_j)$, if $i = j$, and $v_i \notin N(u_j)$, if $i \neq j$. This implies that one of v_1 and v_2 is larger than v_3 in $\prec$ and so it is contained in $N(u_3)$, contradicting the fact that M is induced. This concludes the proof. □

The proof of Theorem 1 is now an easy consequence of the above result.

Theorem 1 (restated). *The (linear) mim-width of the class of circular convex graphs is bounded and quickly computable.*

Proof. By definition, the mim-width of a graph G is at most the linear mim-width of G. Hence, the result follows from Theorem 5. □

3. The proof of Theorem 2

We need the following lemma on recognizing (t, Δ)-tree convex graphs.[1]

Lemma 2. *For $t, \Delta \in \mathbb{N}$, (t, Δ)-tree convex graphs can be recognized and a (t, Δ)-tree support computed, if it exists, in $O(n^{t+3})$ time.*

Proof. Given a hypergraph $H = (V, \mathcal{S})$ together with degrees d_i for each $i \in V$, Buchin et al. [16] provided an $O(|V|^3 + |\mathcal{S}||V|^2)$ time algorithm that solves the following decision problem: Is there a tree support for H such that each vertex i of the tree has degree at most d_i? If it exists, the algorithm computes a tree support satisfying this property. Given as input a bipartite graph $G = (A, B, E)$, we consider the hypergraph $H = (A, \mathcal{S})$, where $\mathcal{S} = \{N(b) : b \in B\}$. For each of the $\binom{|A|}{t} = O(|A|^t)$ subsets $A' \subseteq A$ of size t we proceed as follows: we assign a degree Δ to each of its elements and a degree 2 to each element in $A \setminus A'$. We then apply the algorithm in [16] to the $O(|A|^t)$ instances thus constructed. If G is (t, Δ)-tree convex, then the algorithm returns a (t, Δ)-tree support for H. □

As mentioned in Section 1, we will prove Theorem 2 as a consequence of a stronger result for a less powerful width parameter, namely thinness. This notion was introduced by Mannino, Oriolo, Ricci and Chandran [53] as a generalization of interval graphs, in such a way that interval graphs are exactly the graphs that are 1-thin, according to characterizations by Ramalingam and Pandu Rangan [59] and Olariu [54]. A graph $G = (V, E)$ is *k-thin* if there exists an ordering $v_1, \ldots, v_n$ of V and a partition of V into k classes $(V^1, \ldots, V^k)$ such that, for every triple (r, s, t) with $r < s < t$, the following holds: if v_r and v_s belong to the same class V^i for some $i \in \{1, \ldots, k\}$ and moreover $v_rv_t \in E$, then we have $v_sv_t \in E$. In this case, the ordering $v_1, \ldots, v_n$ and the partition $(V^1, \ldots, V^k)$ are said to be *consistent* and form a *k-thin representation* for G. The *thinness* $\text{thin}(G)$ of G is the minimum integer k such that G is k-thin. See Fig. 3 for an example of a 3-thin graph. As shown by, for example, Bonomo and de Estrada [6], it is not difficult to construct graphs of arbitrarily large thinness.

As mentioned, thinness is more powerful than path-width [53], but less powerful than linear mim-width [6]; see also Fig. 4. We refer to [6,7,19,53] for a wide range of NP-complete graph problems that become polynomial-time solvable on graph classes of bounded thinness; in particular these problems include the framework of weighted list matrix partition problems [30], such as List k-Colouring, List H-Homomorphism, Equitable k-Colouring and many other graph partitioning problems into a fixed number of sets. However, polynomial-time algorithms for graph classes of thinness at most k still need, as part of the input, an $f(k)$-thin representation for some function f that only depends on k. Whether such an $f(k)$-thin representation can be computed in polynomial time is currently unknown. Bonomo and de Estrada [6] provided a polynomial-time algorithm for finding a smallest partition of the vertex set of a graph that is consistent with a given vertex ordering.

The following theorem shows that every class of (t, Δ)-tree convex graphs ($t \geq 1$ and $\Delta \geq 3$) has bounded thinness.

[1] Jiang et al. [42] proved that Weighted Feedback Vertex Set is polynomial-time solvable for triad convex graphs if a triad support is given as input. They observed that an associated tree support can be constructed in linear time, but this does not imply that a triad support can be obtained. Lemma 2 shows that indeed a triad support can be obtained in polynomial time and need not be provided on input.

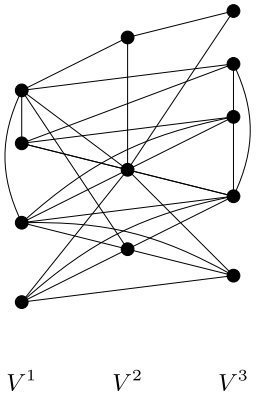

Fig. 3. A 3-thin representation of a graph. The vertices are ordered increasingly by their y-coordinate, and the classes correspond to the vertical lines.

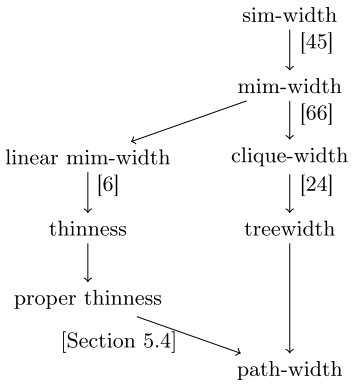

Fig. 4. The relationships between the different width parameters that we consider in Section 5. Parameter p is more powerful than parameter q if and only if there exists a directed path from p to q. To explain the incomparabilities, proper interval graphs have proper thinness 1 [6] and unbounded clique-width [32], whereas trees have tree-width 1 and unbounded linear mim-width [36]. Moreover, trees have unbounded path-width (see [28]) and convex graphs have bounded thinness (Theorem 6) but unbounded proper thinness (Theorem 8). Unreferenced arrows follow from the definitions of the width parameters involved and the above statements, except for the arrow from proper thinness to path-width whose proof we give in Section 5.4.

Theorem 6. *For every $t, \Delta \in \mathbb{N}$ with $t \geq 1$ and $\Delta \geq 3$, the class of (t, Δ)-tree convex graphs has thinness at most $2 + t(\Delta - 2)$, and a corresponding $(2 + t(\Delta - 2))$-thin ordering and partition can be computed in polynomial time.*

Proof. Let $G = (A, B, E)$ be a (t, Δ)-tree convex graph for some $t \geq 1$ and $\Delta \geq 3$. Let $V^1 = B$. We can compute a (t, Δ)-tree support in polynomial time by Lemma 2. We root this tree at an arbitrary leaf and assign the root to the class V^2. If a node has one child, then it is assigned to the same class as its parent. If it has more than one child, then one child is assigned to the same class as its parent, and each of the other children starts a new class. So, we have at most $2 + t(\Delta - 2)$ classes (including V^1).

We order the vertices of A by postorder of the support tree, and insert each vertex b of B into the order in such a way that the greatest vertex of A smaller than it is the greatest neighbour of b in A. Note that each class of vertices of A is a linearly ordered path of the support tree.

We will now show that the order and the partition are consistent. The vertices of B are an independent set, the vertices of A are an independent set, and no vertex of A has a neighbour in B smaller than it. So, suppose that $a < a' < b$, with $a, a' \in V^j$ for some $j \geq 2$, $b \in V^1$, and $ab \in E$.

Let a'' be the greatest neighbour of b in A. By construction of the order, a'' is the greatest vertex of A that is smaller than b. Hence, $a < a' \leq a'' < b$ (note that $a' = a''$ is possible). Moreover, $ab \in E$ and $a''b \in E$. If $a'' \in V^j$, then clearly a' belongs to the unique path joining a and a'' in the support tree, so $a'b \in E$. If $a'' \in V^{j'}$, for $j' \neq j$, then the common ancestor of a and a'' in the support tree is also a neighbour of b. Since we have ordered the tree by postorder, this common ancestor is precisely a''. Also in this case, a' belongs to the unique path joining a and a'' in the support tree, so $a'b \in E$.

From the above we conclude that G has thinness at most $2+t(\Delta-2)$. Moreover, we can construct the $(2+t(\Delta-2))$-thin ordering and partition above in polynomial time. □

We also need the following known result, which shows that linear mim-width (and thus mim-width) is a more powerful width parameter than thinness.

Lemma 3 *([6]). For every graph G, the linear mim-width of G is at most the thinness of G. Moreover, from every k-thin representation for G we can obtain, in linear time, a linear branch decomposition for G of width at most k.*

We are now ready to prove Theorem 2.

Theorem 2 (restated). *For every $t, \Delta \in \mathbb{N}$ with $t \geq 1$ and $\Delta \geq 3$, the (linear) mim-width of the class of (t, Δ)-tree convex graphs is bounded and quickly computable.*

Proof. By definition, the mim-width of a graph G is at most the linear mim-width of G. Combining Theorem 6 with Lemma 3 immediately yields that for every $t, \Delta \in \mathbb{N}$ with $t \geq 1$ and $\Delta \geq 3$, the class of (t, Δ)-tree convex graphs has (linear) mim-width at most $2 + t(\Delta - 2)$, and that a corresponding branch decomposition can be computed in polynomial time. □

4. The proof of Theorem 3

For proving Theorem 3, we need the following lemma.

Lemma 4 *(see Wang et al. [67]). Let $G = (A, B, E)$ be a bipartite graph and G' be the bipartite graph obtained from G by making k new vertices complete to B. If $k = 1$, then G' is star convex. If $k = |A|$, then G' is comb convex.*

As mentioned in Section 1, we prove Theorem 3 as a consequence of as stronger result, which requires some extra terminology given below.

Consider a branch decomposition (T, δ) for a graph G. Recall that every edge $e \in E(T)$ partitions the leaves of T into the classes L_e and $\overline{L_e}$ and that e induces a partition $(A_e, \overline{A_e})$ of $V(G)$, where $\delta(A_e) = L_e$ and $\delta(\overline{A_e}) = \overline{L_e}$. Let $\mathrm{cutsim}_G(A_e, \overline{A_e})$ be the size of a maximum induced matching M in G such that every edge of M has one end-vertex in A_e and the other end-vertex in $\overline{A_e}$. Then the *sim-width* $\mathrm{simw}_G(T, \delta)$ of (T, δ) is the maximum value of $\mathrm{cutsim}_G(A_e, \overline{A_e})$ over all edges $e \in E(T)$. The *sim-width* $\mathrm{simw}(G)$ of G is the minimum value of $\mathrm{simw}_G(T, \delta)$ over all branch decompositions (T, δ) for G.

We now prove the following result.

Theorem 7. *The class of star convex graphs and the class of comb convex graphs have unbounded sim-width.*

Proof. We use the fact that grids are bipartite and have unbounded sim-width [45]. Moreover, just as for mim-width, it is readily seen that adding a vertex to a graph does not decrease the sim-width. Hence, we can apply Lemma 4: for any grid on partition classes A and B, by adding a vertex complete to B we obtain a star convex graph, and by adding $|A|$ new vertices complete to B we obtain a comb convex graph. Thus the class of star convex graphs and the class of comb convex graphs have unbounded sim-width. □

We also need the following observation of Kang, Kwon, Strømme and Telle.

Lemma 5 *([45]). For every graph G, the sim-width of G is at most the mim-width of G.*

We can now prove Theorem 3.

Theorem 3 (restated). *The class of star convex graphs and the class of comb convex graphs have unbounded mim-width.*

Proof. The result immediately follows from Theorem 7 after applying Lemma 5. □

5. A refined parameter analysis

We proved Theorems 1–3 by showing stronger results for other width parameters. In this section, we extend this more refined analysis on width parameters for the graph classes listed in Fig. 2. We will consider the graph width parameters listed in Fig. 4, which are path-width, treewidth, clique-width, (linear) mim-width, sim-width, thinness and proper thinness. The results in this section are summarized in Fig. 5. Note that we provide a *complete* picture with respect to the width parameters and graph classes considered.

We prove the three parts in Fig. 5 that are separated by the dotted lines in Sections 5.1–5.3, respectively; note that some results are shown already in the previous sections. Afterwards, we prove in Section 5.4 the (only) unreferenced arrow in Fig. 4.

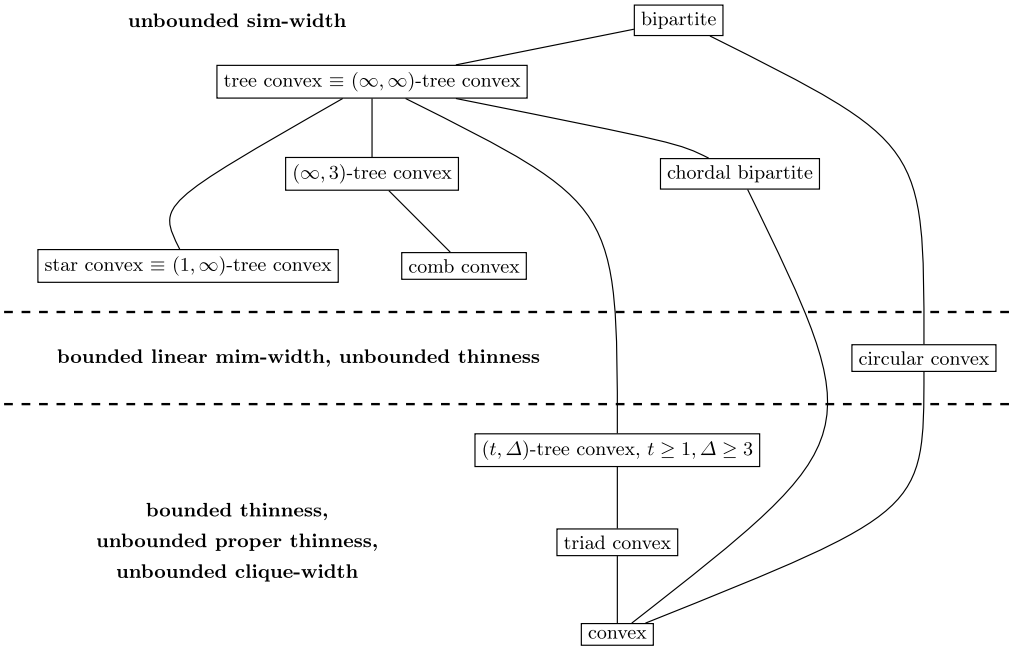

Fig. 5. The inclusion relations between the classes we consider. A line from a lower-level class to a higher one means the first class is contained in the second.

5.1. Bounded thinness but unbounded proper thinness and clique-width

In this section we consider the bottom row in Fig. 5. In Section 3, we already proved that for every $t, \Delta \in \mathbb{N}$ with $t \geq 1$ and $\Delta \geq 3$, the class of (t, Δ)-tree convex graphs has bounded thinness (Theorem 6). Hence, it remains to show that convex graphs have unbounded proper thinness and unbounded clique-width.

The notion of proper thinness was introduced by Bonomo and de Estrada [6], in such a way that proper interval graphs are exactly the graphs that are proper 1-thin, according to a characterization of Roberts [61]. Hence, proper k-thinness generalizes the notion of proper interval graphs, just like k-thinness does for interval graphs. A graph $G = (V, E)$ is *proper k-thin* if there exists an ordering $v_1, \ldots, v_n$ of V and a partition of V into k classes $(V^1, \ldots, V^k)$ such that for each triple (r, s, t) with $r < s < t$ the following holds:

- if the vertices v_r and v_s are in the same class V^i for some $i \in \{1, \ldots, k\}$ and $v_r v_t \in E$, then $v_s v_t \in E$; and
- if the vertices v_s and v_t are in the same class V^i for some $i \in \{1, \ldots, k\}$ and $v_r v_t \in E$, then $v_r v_s \in E$.

In this case, the ordering $v_1, \ldots, v_n$ and the partition $(V^1, \ldots, V^k)$ are *strongly consistent*. The *proper thinness* $\mathrm{pthin}(G)$ of G is the minimum integer k such that G is proper k-thin. We refer to Fig. 4 for an illustration of the modelling power of proper thinness with respect to other graph width parameters. The algorithmic motivation for introducing proper thinness lies in the fact that weighted list matrix-partition problems enhanced with domination constraints are polynomial-time solvable on graphs of bounded proper thinness [6] (the computational complexity of such problems is still open for graphs of bounded thinness). However, just as for thinness, polynomial-time algorithms for graph classes of proper thinness at most k still need, as part of the input, a proper $f(k)$-thin representation for some function f that only depends on k. Whether a proper $f(k)$-thin representation can be computed in polynomial time is still open.

We cannot strengthen Theorem 6 to proper thinness, nor to clique-width, due to the following result for convex graphs.

Theorem 8. *The class of convex graphs has unbounded proper thinness and unbounded clique-width.*

Proof. For the second part of the statement, we note that the class of bipartite permutation graphs, which form a subclass of convex graphs (see, e.g., [11]), has unbounded clique-width [12]. Hence, it remains to show the first part of the statement.

Given a vertex order $<$ and a subset $S \subseteq V(G)$, a set of vertices $X \subseteq S$ is *consecutive in S according to $<$* if there is no z in $S \setminus X$ such that $\min(X) < z < \max(X)$. Notice that for each vertex $u \in S$, there are at most two vertices x in S such that $\{u, x\}$ is consecutive in S according to $<$. Namely, if $S = s_1 < \cdots < s_r$ and $u = s_i$, such vertices are s_{i-1} when $i > 1$ and s_{i+1} when $i < r$.

We prove the following claim.

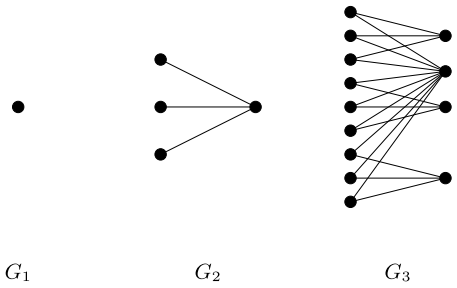

Fig. 6. The bipartite convex graphs G_1, G_2, G_3 from the family of graphs $\{G_k\}_{k\geq 1}$ in the proof of Theorem 8.

Claim 1. An order $<$ and a partition $V^1, \ldots, V^k$ are strongly consistent if and only if for every $v \in V(G)$ and every $1 \leq j \leq k$, the set $N[v] \cap (V^j \cup \{v\})$ is consecutive in $V^j \cup \{v\}$ according to $<$. In particular, $N[v] \cap V^j$ is consecutive in V^j according to $<$.

We prove Claim 1 as follows. $\Rightarrow$) Let $v \in V(G)$ and $1 \leq j \leq k$. Let $X = N[v] \cap (V^j \cup \{v\})$ and suppose there is some $z \in (V^j \cup \{v\}) \setminus X$ (or equivalently, $z \in V^j \setminus X$ as $v \in X$) such that $\min(X) < z < \max(X)$. If $v < z$, then $v < z < \max(X)$; $z, \max(X) \in V^j$; $v \max(X) \in E(G)$; and $vz \notin E(G)$, contradicting that the order and partition are strongly consistent. If $v > z$, then $\min(X) < z < v$; $z, \min(X) \in V^j$; $v \min(X) \in E(G)$; and $vz \notin E(G)$, again contradicting that the order and partition are strongly consistent.

$\Leftarrow$) Let $r < s < t$ such that $rt \in E(G)$. Suppose first $r, s \in V^j$ for some $1 \leq j \leq k$. Since the vertices in $N[t] \cap (V^j \cup \{t\})$ are consecutive in $V^j \cup \{t\}$ according to $<$, $r, t \in N[t] \cap (V^j \cup \{t\})$, and $s \in V^j$, then $s \in N[t]$. The proof for $s, t \in V^j$ for some $1 \leq j \leq k$ is analogous by using the property for $N[r]$.

As an immediate consequence, we have that $N[v] \cap V^j$ is consecutive in $V^j \cup \{v\}$ according to $<$. This completes the proof of Claim 1.

Let $\{G_k\}_{k\geq 1}$ be a family of bipartite convex graphs defined recursively as follows: G_1 is the 1-vertex graph, with the partition (A, B) of $V(G_1)$ such that $|A| = 1$ and $B = \varnothing$. For $k \geq 2$, we define $G_k = (A, B)$ from the disjoint union of three copies $H_i = (A_i, B_i)$, $i = 1, 2, 3$, of G_{k-1} by adding a new vertex u that we make complete to $A = A_1 \cup A_2 \cup A_3$ (thus, $B = B_1 \cup B_2 \cup B_3 \cup \{u\}$). Note that for every $k \geq 2$, both A and B are nonempty. The graphs G_1, G_2 and G_3 are displayed in Fig. 6.

As each G_k is convex, it remains to show the following claim.

Claim 2. For every $k \geq 1$, $\text{pthin}(G_k) = k$.

We first prove by induction that $\text{pthin}(G_k) \geq k$. For $k = 1$, the statement is true, as G_1 is the 1-vertex graph.

Let $k \geq 2$. The graph $G_k = (A, B)$ arises from the disjoint union of three copies $H_i = (A_i, B_i)$, $i = 1, 2, 3$, of G_{k-1}, by adding a vertex u complete to $A = A_1 \cup A_2 \cup A_3$ (thus $B = B_1 \cup B_2 \cup B_3 \cup \{u\}$). Notice that for $v \in A_i$, $w \in A_j$, $i \neq j$, it holds that $N[v] \cap N[w] = \{u\}$. On the other hand, every vertex of H_i belongs to $N[v]$ for some $v \in A_i$.

Now let $<$ be an ordering of $V(G_k)$ and let $(V^1, \ldots, V^r)$, for some integer r, be a partition of $V(G_k)$, such that $<$ and $(V^1, \ldots, V^r)$ are strongly consistent. Let j be such that $u \in V^j$. There are at most two vertices x in V^j such that $\{u, x\}$ is consecutive in V^j according to $<$, so there is at least one value i, $1 \leq i \leq 3$, such that none of them belongs to H_i. We claim that $V^j \cap V(H_i) = \varnothing$. Suppose, on the contrary, that there exists a vertex w in $V(H_i) \cap V^j$, and let $v \in A_i$ such that $w \in N[v]$. Then $N[v] \cap (V^j \cup \{v\})$ is not consecutive in $V^j \cup \{v\}$, which contradicts Claim 1.

Finally, since $<$ and the partition restricted to $V(H_i)$ are also strongly consistent, and, by the inductive hypothesis, $\text{pthin}(H_i) = k - 1$, it follows that $r \geq k$, so $\text{pthin}(G_k) \geq k$.

We now prove that $\text{pthin}(G_k) \leq k$. In order to do this, we will inductively build an ordering and a partition of $V(G_k)$ into k classes that are strongly consistent, and such that $V^1 = A$. For $k = 1$, this is straightforward. For $k \geq 2$, by induction, we can find an ordering and a partition into $k - 1$ classes for the vertex set of each $H_i \cong G_{k-1}$, such that the ordering and partition are strongly consistent. We concatenate the orderings, and take the union of the corresponding classes, to obtain an ordering and partition for $G_k - u$. In particular, $V^1 = A_1 \cup A_2 \cup A_3$. Finally, we create a new class $V^k = \{u\}$, and make u the greatest vertex in the order. It is readily seen that the order and the partition that we created are strongly consistent for G_k. □

5.2. Bounded linear mim-width but unbounded thinness

By Theorem 5, the class of circular convex graphs has bounded linear mim-width. Below we prove that circular convex graphs have unbounded thinness. Hence, for circular convex graphs we cannot obtain the same result as for (t, Δ)-tree convex graphs in Theorem 6.

Theorem 9. *The class of circular convex graphs has unbounded thinness.*

Proof. The *crown* H_n is the graph on $2n$ vertices that is obtained from a complete bipartite graph after removing a perfect matching. The class of crown graphs has unbounded thinness [9] and is readily seen to be circular convex. □

5.3. Unbounded sim-width

From Theorem 7 we know that both the class of star convex graphs and the class of comb convex graphs have unbounded sim-width. Hence, it remains to prove that chordal bipartite graphs have unbounded sim-width, which we do below. For two graphs H_1 and H_2, a graph G is (H_1, H_2)-*free* if G has no induced subgraph isomorphic to H_1 or H_2.

Theorem 10. *The class of chordal bipartite graphs has unbounded sim-width.*

Proof. Let $K_3 \boxminus S_3$ be the graph that consists of a triangle on vertices x, y, z to which we add three new vertices x', y', z' with edges xx', yy' and zz'. Let $K_3 \boxminus K_3$ be the graph obtained from two triangles on vertices a_1, b_1, c_1 and a_2, b_2, c_2, respectively, to which we add the edges a_1a_2, b_1b_2 and c_1c_2. Kang et al. [45] proved that every class of $(K_3 \boxminus S_3, K_3 \boxminus K_3)$-free graphs of unbounded mim-width has unbounded sim-width. Recall that the class of chordal bipartite graphs has unbounded mim-width [13]. It remains to observe that every chordal bipartite graph is $(K_3 \boxminus S_3, K_3 \boxminus K_3)$-free. □

Theorem 10 strengthens the aforementioned result that chordal bipartite graphs have unbounded mim-width [13].

5.4. The missing relationship in Fig. 4

A *path decomposition* of a graph $G = (V, E)$ is a sequence of subsets of vertices whose union is V and such that: (1) for each edge $vw \in E$, there exists a subset containing both v and w; and (2) for each $v \in V$ the subsets containing v are consecutive in the sequence. The width of a path decomposition is one less than the maximum size of a subset. The *path-width* of a graph G, denoted $\mathrm{pw}(G)$, is the minimum possible width over all possible path decompositions of G.

Mannino, Oriolo, Ricci and Chandran [53] proved that $\mathrm{thin}(G) \leq \mathrm{pw}(G) + 1$. We slightly modify their proof to obtain the following result.

Theorem 11. *For a graph G, $\mathrm{pthin}(G) \leq 2^{\mathrm{pw}(G)}(\mathrm{pw}(G) + 1)$. Moreover, given a path decomposition of width q, a vertex ordering and a strongly consistent partition into at most $2^q(q+1)$ independent sets can be found in polynomial time.*

Proof. In [53], it is proved that, given a path decomposition of width q, one can find a vertex ordering and a consistent partition into at most $q+1$ independent sets (a colouring) in polynomial time, with the additional property that each vertex has at most one neighbour smaller than itself of each colour. By consistency, that possible neighbour is the greatest vertex smaller than itself of that colour, if such a vertex exists.

We refine that partition to make it strongly consistent with the order, splitting each colour class into at most 2^q sets according to whether or not it has a neighbour smaller than itself in each of the other colour classes. Notice that refining a partition maintains consistency, so, in order to prove strong consistency, let $u < z < v$, $uv \in E(G)$, with z, v in the same refined set. Since the colour classes are independent sets, u and v are of distinct colours, say a and b, respectively, and u is the greatest vertex smaller than v of colour a. By the way of refining the colour classes, z is of colour b and since v does have a neighbour of colour a smaller than itself, so does z. Since u is the greatest vertex smaller than z of colour a, $uz \in E(G)$. So, we have defined a vertex ordering and a partition into at most $2^q(q+1)$ independent sets that are strongly consistent and can be found in polynomial time. □

It is readily seen that complete graphs have proper thinness 1 but unbounded path-width. Combining this observation with Theorem 11 shows that proper thinness is more powerful than path-width.

6. The proof of Theorem 4

In this section we prove Theorem 4. Let G be a graph. A function $c : V(G) \to \{1, 2, \ldots\}$ is a *colouring* of G if $c(u) \neq c(v)$ for every pair of adjacent vertices u and v. A *list assignment* of a graph $G = (V, E)$ is a function L that prescribes a list of "admissible" colours $L(u) \subseteq \{1, 2, \ldots\}$ to each $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V$. For an integer $k \geq 1$, if $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then L is a *list k-assignment*. The List k-Colouring problem is to decide if a graph G with a list k-assignment L has a colouring that respects L. Note that if every list is $\{1, \ldots, k\}$, we obtain the classical k-Colouring problem.

We need the following well-known result regarding the 2-List Colouring problem. For this problem, a graph G with a list assignment L is given as input, where $|L(u)| \leq 2$ for every $u \in V(G)$.

Theorem 12 *([29]). The* 2-List Colouring *problem is linear-time solvable.*

We are now ready to prove the main result of this section.

Theorem 4 (restated). *For $k \geq 4$, List k-Colouring is NP-complete for star convex graphs and comb convex graphs, while List 3-Colouring is polynomial-time solvable for star convex graphs.*

Proof. We start with proving the hardness results. It is well-known that for $k \geq 3$, List k-Colouring is NP-complete for bipartite graphs, even for various subclasses of bipartite graphs (see, for example, [22,38]). Hence, let $k \geq 3$ and (G, L) be an instance of List k-Colouring, where G is a bipartite graph with partition classes A and B.

We let G^* be the bipartite graph obtained from G by adding $|A|$ new vertices to A that we make complete to B. Then, by Lemma 4, G^* is comb convex. Let L^* be the extension of L where each new vertex has been given the list $\{k+1\}$. We observe that (G, L) is a yes-instance of List k-Colouring if and only if (G^*, L^*) is a yes-instance of List $(k+1)$-Colouring.

We let G' be the bipartite graph obtained from G by adding one new vertex to A that we make complete to B. By Lemma 4, G' is star convex. We give the new vertex the list $\{k+1\}$ and repeat the argument above.

We now prove the polynomial-time result, which only holds for star convex graphs. So consider a star convex graph $G = (A, B, E)$ with a list 3-assignment L. We may assume without loss of generality that for every $u \in B$, it holds that $L(u)$ has size at least 2; else if $L(u) = \{i\}$ for some $u \in B$ and $i \in \{1, 2, 3\}$, then we delete colour i from the list of every neighbour of u in A and delete u from G.

Let $a \in A$ correspond to the centre of the star that is a support for G. Then, every vertex in B is either adjacent to a or has degree 1. We may remove every vertex $u \in B$ of degree 1 from G: as $|L(u)| \geq 2$, we are always able to assign u a colour from $L(u)$ after restoring it. Afterward removing vertices of B that have degree 1 from G, every vertex of B is adjacent to a.

We now consider all of the at most three options of colouring a with a colour from $L(a)$. For each chosen colour $c(a) \in L(a)$, we remove $c(a)$ from the list of every vertex of B and moreover, we assign $c(a)$ to every other vertex $a' \in A$ with $c(a) \in L(a')$. Afterwards we obtain an instance of 2-List Colouring, which means we can apply Theorem 12. □

7. Final remarks

In this paper we generalized and unified a number of algorithmic results for generalized convex graphs by showing boundedness of mim-width. In a recent arXiv paper, Jaffke, Kwon and Telle [41] introduced the notion of bi-mim-width for directed graphs. As a consequence of their study, they considered *H-convex graphs* for a fixed graph H, which in our terminology corresponds to $\mathcal{H}$-convex graphs where $\mathcal{H}$ consists of all subdivisions of H. For example, when H is the cycle on two vertices with two edges between them, we obtain the class of circular convex graphs. They showed that the linear mim-width of an H-convex graph is at most $6|E(H)|$. For circular convex graphs this leads to a bound of 12. If H is a tree of maximum degree at most Δ with at most t vertices of degree at least 3, then the bound of $6|E(H)|$ leads to a bound of $6(t\Delta - t + 1)$, as H has at most $t\Delta - t + 1$ edges. Note that combining Theorem 6 with Lemma 3 yields a bound of $2 + t(\Delta - 2)$.

We are not aware of any new algorithmic implications due to our refined width parameter analysis in Section 5. We do observe that there exist problems that are NP-complete for graphs of bounded (linear) mim-width, but polynomial-time solvable for graphs of bounded thinness. Namely, Vatshelle [66] proved that Clique (the problem of deciding if a graph has a clique of size at least k for some given integer k) is NP-complete for graphs of mim-width at most 6; in fact the branch decomposition in Vatshelle's proof turns out to be even linear. On the other hand, Clique belongs to a large framework of graph problems that are polynomial-time solvable for graphs of bounded thinness [6]. However, Clique is trivial on bipartite graphs, and all the graph classes we consider in this paper are bipartite. It would therefore be interesting to study the following problems.

Open Problem 1. Do there exist (natural) problems that are NP-complete for bipartite graphs of bounded mim-width but polynomial-time solvable for graphs of bounded thinness or bounded linear mim-width, given an associated decomposition?

Open Problem 2. Do there exist (natural) problems that are NP-complete for graphs of bounded thinness, but polynomial-time solvable for graphs of bounded proper thinness?

For answering the above questions, more structural results may be needed. For instance, finding classifications of (H_1, H_2)-free graphs and H-free (circular) convex graphs of bounded thinness might help.

The following problem is also interesting.

Open Problem 3. Is it possible to characterize convex graphs of proper thinness at most k and circular convex graphs of thinness at most k by some obstruction set, for example, a forbidden set of induced subgraphs?

Naturally, it would also be interesting to obtain dichotomies for more graph problems restricted to classes of generalized convex graphs. We ask this question in particular for graph problems known to be solvable in polynomial time for graph classes whose mim-width is bounded and quickly computable. In our paper we gave examples of ten of such problems (see Corollaries 2 and 3).

We may be able to find more of such graph problems by considering other setting where generalized convex graphs play a role. For example, Chen et al. [21] considered the problem Subset Interconnection Design, which is to decide if a bipartite graph belongs to a class of $\mathcal{H}$-convex graphs. This problem and its variants have several applications, for example in the design of scalable overlay networks and vacuum systems [21], combinatorial auctions [33] and fair allocation of indivisible goods [10]. We leave the following problem for future research:

Open Problem 4. Are the problems in the settings of [10,21,33] solvable in polynomial time for graph classes whose mim-width is bounded and quickly computable?

We finish our paper with some open problems on List k-Colouring. The first open problem results from Section 6.

Open Problem 5. Determine the computational complexity of List 3-Colouring for comb convex graphs.

Our second open problem was also asked by Huang et al. [37], who proved that for all $k \geq 4$, List k-Colouring is NP-complete for P_8-free chordal bipartite graphs.

Open Problem 6. Determine the computational complexity of List 3-Colouring for chordal bipartite graphs.

Answering both questions would complete the results for List k-Colouring for the graph classes displayed in Fig. 2.

CRediT authorship contribution statement

Flavia Bonomo-Braberman: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Writing – original draft, Writing – review & editing. **Nick Brettell:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Writing – original draft, Writing – review & editing. **Andrea Munaro:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Writing – original draft, Writing – review & editing. **Daniël Paulusma:** Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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Data availability

No data was used for the research described in the article.

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