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# Green's functions for the load sharing in multiple insulating glazing units

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**Abstract**

Insulating Glazing Units (IGUs) are glass panes held together by structural edge seals, entrapping a gas for thermal and acoustic insulation. The bending of one pane induces pressure variations in the cavities, which produce the load sharing through the gas with the other panes. Under mild hypotheses, the whole IGU behaves as a linear elastic system. Hence, for IGUs with panes and cavities of any number, shape, size and edge constraints, the Green's functions can be defined for the pressure variations and the pane displacements, so that these quantities can be expressed as convolution integrals with any applied load. The proposed Green's functions are calculated with a tailored used of Betti's reciprocal work theorem, referred to as Betti's Analytical Method (BAM), here generalized to multiple IGUs with $N$ panes. Remarkably, the Green's functions for the pressure variations in the cavities are completely determined by the deformation field of a simply-supported plate, with the same shape of the IGU, under uniform pressure: this represents the universal expression, possibly tabulated, covering all possible cases. The proposed approach is

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validated through comparisons with prescriptions by standards and numerical simulations. Parametric analyses show the effect of the IGU geometry on the load sharing.

Keywords: Insulating Glazing Units (IGUs); composite structure; Green's function; analytical approach; Betti's Analytical Method (BAM).

# 1 Introduction

Insulating Glass Units (IGUs) consist of two or more glass panes, either monolithic or laminated, held together by structural edge seals, separated by one or more air or noble gas-filled spaces, used to reduce the heat transfer in the building envelope. When composed by two or three glass panes they are respectively referred to as Double Glazing Units (DGUs), or Triple Glazing Units (TGUs). When an IGU is subjected to external loads, either concentrated or distributed, there is an interaction among the glass panes and the gas filling the cavities, which redistributes the loads among the glass panes. This effect, referred to as *load sharing*, depends upon the flexural stiffness of the panes and the gas compressibility, since it is due to the pressure variation in the interpane cavities consequent to the applied actions. Also changes in the barometric pressure and in the gas temperature inside the cavity, with respect to the time of sealing, unbalance the external and internal pressures, developing effects in all the panes. Extensive research on the load sharing between the lites has been pursued for DGUs, but the case of TGUs or multiple IGUs appears to be less investigated. Many engineers often design TGUs by simply ignoring the contribution of the middle lite to load sharing [24], but this approach may be too much conservative.

Various approximate methods have been proposed to evaluate the *load sharing* in multiple IGUs. The simpler one is the "thickness cubed" method proposed by the ASTM E1300 [1] for DGUs and TGUs under uniformly-distributed loads (wind, snow, self weight). The method assumes the gas in the cavity to be incompressible, so that the deflection of the glass panes is inversely proportional to the cube of their thickness. Coefficients are proposed to account for the pressure variations in the cavities due to changes in barometric pressure and temperature. A more refined method accounting for the gas compressibility has been proposed by the European Norm EN 16612 [28], based on the work by Siebert and Maniatis [30]. By requiring that the sealed spaces are in equilibrium under the ideal gas law, *insulating-unit factors* for calculation are presented in tables for DGUs and TGUs under external uniform pressure and changes in barometric pressure and temperature. Both the aforementioned Standards do not provide any formula for the calculation of IGUs other than rectangular and simply supported at the borders, and they cover neither the case of more than three glass panes, nor line-distributed or concentrated loads.

To our knowledge, the only method to design IGUs composed by an arbitrary number of glass panes is that proposed by Feldmeier [10, 12], which requires to determine, either numerically or analytically, the volume enclosed between the deformed plate under the considered action. This allows to define coefficients, which have been tabulated in [13] for rectangular DGUs as a function of the aspect ratio of the panes, supposed linear elastic, simply supported at the borders, under either line distributed or concentrated loads. Based on results from [11], formulas and coefficients to calculate circular and triangular shaped geometries are recorded in [30]. Additionally, iterative numerical methods [34, 35, 39], in general implemented with Finite Elements [37], have been proposed. A comparison of the different approaches is presented in [24].

More recently, the "Betti's Analytical Method" (BAM), based on a tailored use of the Reciprocal Work Theorem [32] by Enrico Betti [4], has been proposed in [16] to evaluate the load sharing in DGUs when the glass panes are modelled as Kirchhoff-Love plates [33]. The strength of this approach is that, in order to calculate the internal gas pressure for DGUs of any size, shape, boundary and load conditions, there is no need to solve the elastic problem for each specific external load condition. The method simply requires to evaluate the deformation of a simply supported plate, of the same shape of the DGU, under uniform pressure. This represents a strong advantage with respect to the method by Feldmeier, which requires a table of coefficients for each kind of loading. It has been verified [17] that, when the effects of the geometric nonlinearities plays an important role (large and thin plates), the BAM approach, despite being based on linear elasticity theory, still allows to capture, with very good accuracy, the pressure variation in the cavity. The effective state of stress can be determined *a posteriori*, performing a non-linear geometric numerical analysis of each plate, loaded by the external action and the internal pressure variation.

Here, our major result is that, under mild hypotheses, an IGU of arbitrary geometry and with various types of external restraints at the borders, composed by any number of panes entrapping a perfect gas, responds as a *linear elastic* system, so that one can calculate the corresponding Green's functions for the pressure variations in the gas and the displacement of the glass panes. A Green's function for a linear elastic plate [22, 29], possibly laminated [38], corresponds to its out-of-plane deflection resulting from the action of an unitary concentrated force applied on an arbitrary point in the interior of the plate. Green's function are also used for studying the dynamic response of plates [26, 21], especially under impulsive loadings. Green's functions and impulse responses are formally identical, except that the former are used to solve boundary value problems, whereas the latter are involved in initial value problems [27]. The major advantage is that, also for IGUs, once the Green's function is known, the solution for any arbitrary forcing function can be expressed as the convolution integral [20, 19, 25] of the Green's functions and the load distribution itself.

The presented formulation assumes that the spacer connections are stiff, although it has been proved [2] that their deformation may influence the gross response of the IGU. There is no difficulty in incorporating, within the theoretical framework of BAM, the compliance of the spacer joints, at least when it is determined by a linear elastic constitutive law. However, this is not done here but postponed to future work, because the description of the degree of constraint offered by the spacers, in terms of elongation and bending, would require the discussion of a great number of possible design details for the insulated elements. This would imply the consideration of many possible subcases, adding a noteworthy complication to the equations, but no significant achievement from a theoretical point of view.

To calculate the Green's functions, we show that the BAM approach can be generalized to cover IGUs with by an arbitrary number of glass panes. This is presented in Section 2 for TGUs and extended in Section 3 to IGUs with $N > 3$ glass panes, requires the use of $N - 1$ auxiliary systems to apply Betti's Theorem. The method allows to evaluate the pressure variations in all the chambers, for any type of edge constraint and external actions, as well as for climatic loads. Remarkably, also for multiple IGUs, all the relevant expressions depend only upon the deformation of a simply supported plate, of the same shape of the DGU, under uniform pressure, which thus represents a "universal" function for this type of problems. In Section 4, the accuracy of the Green's function approach for TGUs is demonstrated by comparisons with the formulae proposed in standards, for the simple cases that these methods can handle, and with numerical analyses, performed with MEPLA ISO software [5], for more general cases. Parametric analyses are presented to illustrate the influence on the load-sharing capacity of the size and shape of the IGU, glass and spacer thickness and type of loading.

# 2 The use of Green's functions for triple glazing units

Betti's Analytical Method (BAM), presented in [16], is here generalized and extended to derive the Green's functions for Triple Glazing Units (TGUs).

## 2.1 Green's function for monolithic plates

The Green's function for a linear-elastic Kirchhoff-Love plate [33] provides its out-of-plane displacement field consequent to the action of a concentrated unitary force at a generic point [29]. Once the Green's function is known, the solution for any arbitrary load can be expressed as a convolution integral.

For a plate of thickness $h$, introduce a reference system with the axes $(x, y)$ lying in its middle plane and the axis $z$ in the out-of-plane direction. If the forces *per* unit area $f(\mathbf{x})$ are applied at $\mathbf{x} = (x, y)$, the out-of-plane displacement $w(\mathbf{x})$ solves the Germain-Lagrange plate equation [33]

$$\Delta\Delta w(\mathbf{x}) = \frac{f(\mathbf{x})}{D}\,, \tag{2.1}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the plate bending stiffness, with $E$ the Young's modulus and $\nu$ the Poisson's ratio, and $\Delta\Delta(\cdot) = \partial^4(\cdot)/\partial x^4 + 2\partial^4(\cdot)/(\partial x^2 \partial y^2) + \partial^4(\cdot)/\partial y^4$ is the double Laplacian operator. The Green's function $\mathcal{G}(\mathbf{x}, \mathbf{x}^*)$ for the out-of-plane displacement is the solution of

$$\Delta\Delta \mathcal{G}(\mathbf{x}, \mathbf{x}^*) = \frac{\delta(\mathbf{x} - \mathbf{x}^*)}{D}\,, \tag{2.2}$$

where $\delta(\mathbf{x}-\mathbf{x}^*)$ is the two-dimension Dirac Delta function [9], representing a concentrated unitary out-of-plane force acting at $\mathbf{x}^*$. The non-dimensional counterpart $\zeta(\mathbf{x},\mathbf{x}^*)$ of $\mathcal{G}(\mathbf{x},\mathbf{x}^*)$ is obtained by writing

$$\mathcal{G}(\mathbf{x},\mathbf{x}^*)=\frac{A}{D}\,\zeta(\mathbf{x},\mathbf{x}^*)\,, \tag{2.3}$$

where $A$ denotes the plate area.

Thanks to the linearity of the problem, the solution of (2.1) can be written in the form

$$w(\mathbf{x})=\mathcal{G}(\mathbf{x},\mathbf{x}^*)*f(\mathbf{x})=\int_\Omega \mathcal{G}(\mathbf{x},\mathbf{x}^*)f(\mathbf{x}^*)\;d\mathbf{x}^*\,, \tag{2.4}$$

where "$*$" denotes convolution, and $\Omega$ is the reference domain of the plate in the $(x,y)$ plane.

When the plate is subjected to a uniform pressure $q$, the out-of-plane displacement can be written as

$$w_q(\mathbf{x})=\mathcal{G}(\mathbf{x},\mathbf{x}^*)*q=q\;\frac{A}{D}\int_\Omega \zeta(\mathbf{x},\mathbf{x}^*)\;d\mathbf{x}^*=q\;\frac{A^2}{D}\varphi(\mathbf{x})\,, \tag{2.5}$$

with

$$\varphi(\mathbf{x})=\frac{1}{A}\;\int_\Omega \zeta(\mathbf{x},\mathbf{x}^*)\;d\mathbf{x}^*\,. \tag{2.6}$$

This is the non-dimensional shape function for the deflection of the plate under a uniform pressure, which depends only upon the shape of the plate and its boundary conditions. For a great number of cases, its form is recorded in the technical literature [33].

## 2.2 Green's functions for triple glazing units via Betti's analytical method

Consider a Triple Glazing Unit (TGU) of arbitrary shape formed by three glass panes labelled as "1","2" and "3", of thickness $h_i$, $i = 1, 2, 3$, confining two hermetic gas-filled cavities of thickness $s_j$, $j = 1, 2$, for which the out-of-plane displacement is variously constrained[1], as schematically indicated in Figure 1. Consider, again, a reference system with the $(x, y)$ axes parallel to the plane of the panes and $z$ at right angle to that.

The reference state *at the time of sealing* is defined by $V_{0j}$, the reference volume of the $j$-th cavity for $j = 1, 2$, and by $p_0$, the internal glass pressure that equals the external atmospheric pressure. When external loads are applied the gas volume varies, producing a pressure variation of the gas in the $j$-th cavity $\Delta p_j$, $j = 1, 2$, which sums up with the applied loads and contributes to the panels deformation. This effect is in general beneficial, because it allows to *share* the applied loads on all the panes. The *load sharing*, associated with the pressure variation in the two cavities, is now evaluated by extending to this case the "BAM" approach [16].

In order to account for loads of any type, the key point is to evaluate the Green's function for the whole TGU. Consider then, as shown in Figure 2a, the action of a concentrated unitary load, identified by the two-dimensional Dirac Delta function $\delta_1(\mathbf{x} - \mathbf{x}^*)$, applied on glass pane "1" at point $\mathbf{x}^* = (x^*, y^*)$. No particular hypotheses are made about the type of boundary conditions for the out-of-plane displacement at the borders of the TGU, which can be simply or elastically supported, either continuously or point-wise[2].

Betti's *Reciprocal Work Theorem* [4, 32] states that, for a *linear elastic* structure under two equilibrated sets of force, the work done by the first load set through the displacements produced by the second set is equal to the work done by the second

[1] The spacer joints are assumed to provide an internal constraint to the panes equivalent to a reciprocal simple support, i.e., rotations are not refrained, while the out-of-plane displacement at the borders is constrained to be the same in all the panels. Hence, external restraints that prevent the out-of-plane displacement can be applied indifferently on one pane or another.

[2] As discussed in [16], only the cases of clamped edges and of IGUs supported in an interior point cannot be handled with this simple method.

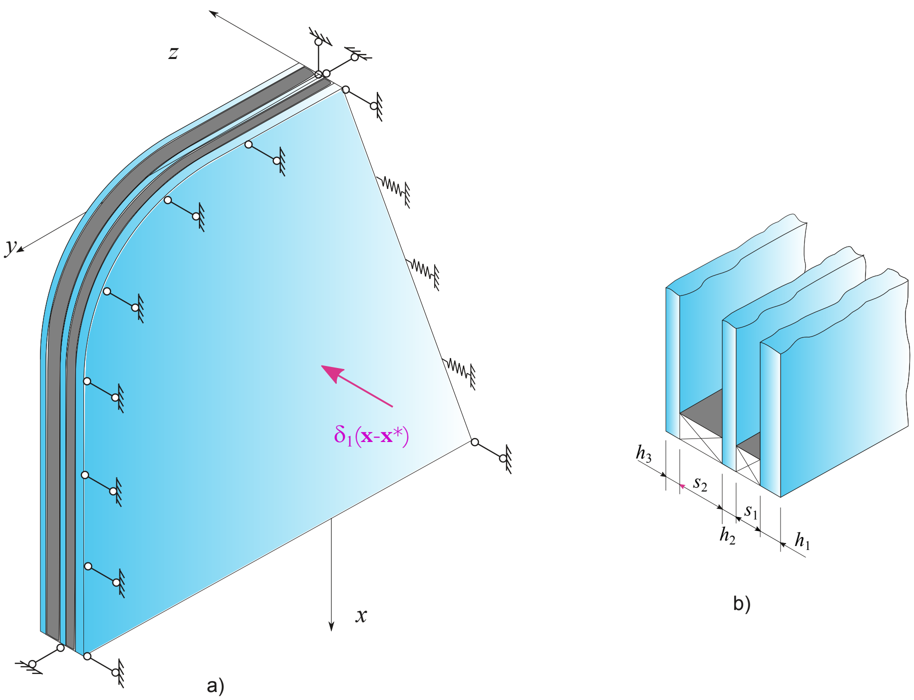



Figure 1: Triple Glazing Unit of arbitrary shape, arbitrarily supported at the borders: a) axonometric view and b) section of the unit (not in the same scale).

load set through the displacements produced by the first set. We consider the TGU subjected to three load sets, referred to as $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, as represented in Figure 2.

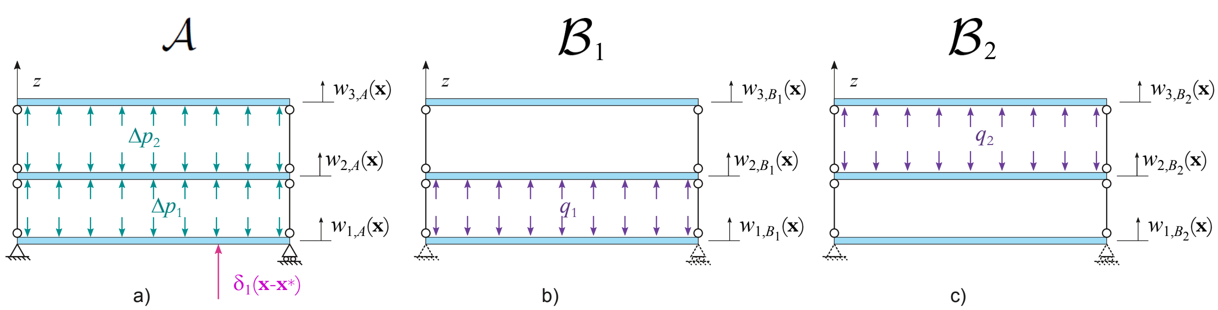



Figure 2: Schematic representation of the load systems $\mathcal{A}$, $\mathcal{B}_1$ and $\mathcal{B}_1$ for the application of Betti's theorem.

The load set $\mathcal{A}$, shown in Figure 2a, consists of the unitary concentrated load $\delta_1(\mathbf{x}-\mathbf{x}^*)$ and the consequent (unknown) pressure variations $\Delta p_1$ and $\Delta p_2$. Denote by $w_{i,\mathcal{A}}(\mathbf{x})$ the out-of-plane displacement in the positive $z$ direction of the point $(\mathbf{x})$ on the $i$-th plate, $i = 1, 2, 3$. The *auxiliary* systems $\mathcal{B}_1$ and $\mathcal{B}_2$, shown in Figures 2b and

2c, respectively, correspond to the application of the *fictitious* internal pressures $q_1$ and $q_2$, acting in the gas cavities "1" and "2". Indicate with $w_{i,\mathcal{B}_j}(\mathbf{x})$ the corresponding out-of-plane displacement of the $i$-th plate, $i = 1, 2, 3$, in the configuration $\mathcal{B}_j$, $j = 1, 2$.

As demonstrated in [16], in the auxiliary systems the external constraint reactions are null since the effect of the internal pressure on two facing panes is counterbalanced by mutual internal constrain represented by the edge spacer. We neglect the extensional deformation of the edge seal[3], so that the constraint offered by the edge spacers can be schematized by a continuous line distribution of simple pendulums along the edges, as indicated in Figure 2.

The mutual works of systems $\mathcal{A}$ and $\mathcal{B}_1$, considering that $w_{3,\mathcal{B}_1}(\mathbf{x}) = 0$, read

$$
\begin{aligned}
\mathcal{L}_{\mathcal{A}\mathcal{B}_1} &= w_{1,\mathcal{B}_1}(\mathbf{x}^*) + \Delta p_1 \int_\Omega [w_{2,\mathcal{B}_1}(\mathbf{x}) - w_{1,\mathcal{B}_1}(\mathbf{x})]\ d\mathbf{x} - \Delta p_2 \int_\Omega w_{2,\mathcal{B}_1}(\mathbf{x})\ d\mathbf{x}\,,\\
\mathcal{L}_{\mathcal{B}_1\mathcal{A}} &= q_1 \int_\Omega [w_{2,\mathcal{A}}(\mathbf{x}) - w_{1,\mathcal{A}}(\mathbf{x})]\ d\mathbf{x}\,,
\end{aligned} \tag{2.7}
$$

where, again, $\Omega$ denotes the reference domain in the $(x, y)$ plane. Analogously, the mutual works of systems $\mathcal{A}$ and $\mathcal{B}_2$ (considering that $w_{1,\mathcal{B}_2}(\mathbf{x}) = 0$) take the form

$$
\begin{aligned}
\mathcal{L}_{\mathcal{A}\mathcal{B}_2} &= \Delta p_1 \int_\Omega w_{2,\mathcal{B}_2}(\mathbf{x})\ d\mathbf{x} + \Delta p_2 \int_\Omega [w_{3,\mathcal{B}_2}(\mathbf{x}) - w_{2,\mathcal{B}_2}(\mathbf{x})]\ d\mathbf{x}\,,\\
\mathcal{L}_{\mathcal{B}_2\mathcal{A}} &= q_2 \int_\Omega [w_{3,\mathcal{A}}(\mathbf{x}) - w_{2,\mathcal{A}}(\mathbf{x})]\ d\mathbf{x}\,.
\end{aligned} \tag{2.8}
$$

Observe that [16] the integrals appearing in $\mathcal{L}_{\mathcal{B}_1\mathcal{A}}$ and $\mathcal{L}_{\mathcal{B}_2\mathcal{A}}$ correspond to the volume variations, in system $\mathcal{A}$, of cavities "1" and "2", respectively. These are related to the pressure variation in the same cavities, supposed to respect the Ideal Gas Law. By considering isothermal transformations, one has

[3]According to [7], this assumption usually leads to conservative results in the evaluation of the stress state in the glass plies.

$$
\begin{aligned}
\int_\Omega [w_{2,\mathcal{A}}(\mathbf{x}) - w_{1,\mathcal{A}}(\mathbf{x})] \; d\mathbf{x} &= -\frac{\Delta p_1}{p_0 + \Delta p_1} V_{01} \simeq -\frac{\Delta p_1}{p_0} \; V_{01} \, , \\
\int_\Omega [w_{3,\mathcal{A}}(\mathbf{x}) - w_{2,\mathcal{A}}(\mathbf{x})] \; d\mathbf{x} &= -\frac{\Delta p_2}{p_0 + \Delta p_2} V_{02} \simeq -\frac{\Delta p_2}{p_0} \; V_{02} \, .
\end{aligned} \tag{2.9}
$$

The approximation that has been done here comes from the observation that $\Delta p_1$ and $\Delta p_2$ are much smaller than $p_0$. In fact, the pressure variation is usually of the order of 1 kPa, whereas the atmospheric pressure $p_0$ is of the order of 100 kPa, so that the error is the order of 1%. The simplification in (2.9) is of paramount importance because it states that the volume variation in the cavity is a linear function of the pressure variation. This implies that the system composed by the glass panes and the gas trapped in a cavity is "approximately" a linear elastic system, for which a Green's function can be defined.

Furthermore, it is important to observe that the displacement fields $w_{1,\mathcal{B}_1}(\mathbf{x})$ and $w_{2,\mathcal{B}_1}(\mathbf{x})$, for system $\mathcal{B}_1$, and $w_{2,\mathcal{B}_2}(\mathbf{x})$ and $w_{3,\mathcal{B}_2}(\mathbf{x})$, for system $\mathcal{B}_2$, all correspond to the deformation of simply supported plates under the uniform pressures $q_1$ and $q_2$, respectively, whatever the edge conditions of the IGU. In fact, it is possible to demonstrate that the internal pressure acting in the cavity of each auxiliary system is supported by the internal constraint offered by the edge seal only. This is because both panes confining the pressurized cavity are in equilibrium under the same bending moments *per* unit length and edge reactions, even if the deformation and the state of stress in each pane scale according to third and second power of the pane thickness, respectively. Due to the similarity in the deformations in pressurized plates of different thickness, there is no warping of the edges, so that the external constraints of the IGU remain inactive. Thus, the displacements can be written as

$$
\begin{aligned}
w_{1,\mathcal{B}_1}(\mathbf{x}) &= -\frac{q_1}{D_1} A^2 \varphi(\mathbf{x}) \, , \qquad w_{2,\mathcal{B}_1}(\mathbf{x}) = \frac{q_1}{D_2} A^2 \varphi(\mathbf{x}) \, , \\
w_{2,\mathcal{B}_2}(\mathbf{x}) &= -\frac{q_2}{D_2} A^2 \varphi(\mathbf{x}) \, , \qquad w_{3,\mathcal{B}_2}(\mathbf{x}) = \frac{q_2}{D_3} A^2 \varphi(\mathbf{x}) \, ,
\end{aligned} \tag{2.10}
$$

where $D_i = \frac{Eh_i^3}{12(1-\nu^2)}$ is the flexural stiffness of the $i$-th plate, $i = 1, 2, 3$, and $\varphi(\mathbf{x})$ is the non-dimensional shape function, defined by (2.6).

Betti's theorem states that $\mathcal{L}_{\mathcal{A}\mathcal{B}_1} = \mathcal{L}_{\mathcal{B}_1\mathcal{A}}$ and $\mathcal{L}_{\mathcal{A}\mathcal{B}_2} = \mathcal{L}_{\mathcal{B}_2\mathcal{A}}$. Denoting with $\overline{\varphi}_A$ the mean value of the shape function on the plate area $A$, i.e., $\overline{\varphi}_A := \frac{1}{A}\int_\Omega \varphi(\mathbf{x})d\mathbf{x}$, by using (2.9) and (2.10) the aforementioned equalities read

$$
\begin{aligned}
&-\frac{A^2}{D_1}\,\varphi(\mathbf{x}^*) + \Delta p_1\left(\frac{1}{D_1}+\frac{1}{D_2}\right)\,A^3\bar{\varphi}_A - \Delta p_2\,\frac{A^3}{D_2}\,\bar{\varphi}_A = -\frac{\Delta p_1}{p_0}\,V_{01}\,,\\
&-\Delta p_1\,\frac{A^3}{D_2}\,\bar{\varphi}_A + \Delta p_2\left(\frac{1}{D_2}+\frac{1}{D_3}\right)A^3\,\bar{\varphi}_A = -\frac{\Delta p_2}{p_0}\,V_{02}\,.
\end{aligned}
\tag{2.11}
$$

If the gas was incompressible, $\Delta V_{0i} = 0$, and the second terms of (2.11) would be null. This leads to a strong simplification of (2.11), in the form

$$
\Delta p_{1,\delta 1} = \frac{D_2 + D_3}{\bar{\varphi}_A\,(D_1 + D_2 + D_3)}\frac{\varphi(\mathbf{x}^*)}{A}\,, \qquad \Delta p_{2,\delta 1} = \frac{D_3}{\bar{\varphi}_A\,(D_1 + D_2 + D_3)}\frac{\varphi(\mathbf{x}^*)}{A}\,. \tag{2.12}
$$

In words, the load sharing depends only on the plates stiffness if the gas is incompressible.

However, the gas compressibility usually plays a relevant role. To solve the system (2.11), it is useful to define the non-dimensional coefficients[4]

$$
\begin{aligned}
\mu_1^- &= \frac{A^3}{D_1}\frac{p_0}{V_{01}}\,, \quad \mu_1^+ = \frac{A^3}{D_2}\frac{p_0}{V_{01}}\,,\\
\mu_2^- &= \frac{A^3}{D_2}\frac{p_0}{V_{02}}\,, \quad \mu_2^+ = \frac{A^3}{D_3}\frac{p_0}{V_{02}}.
\end{aligned}
\tag{2.13}
$$

By solving the system (2.11), one finds

[4]These are qualitatively similar to the coefficients $\alpha_k^-$ e $\alpha_k^+$ defined in [28].

$$\Delta p_{1,\delta 1} = \frac{K_{1,\delta 1}}{A}\,\varphi(\mathbf{x}^*)\,, \quad \Delta p_{2,\delta 1} = \frac{K_{2,\delta 1}}{A}\,\varphi(\mathbf{x}^*)\,, \tag{2.14}$$

where $K_{1,\delta 1}$ and $K_{2,\delta 1}$ are non dimensional coefficients[5], which read

$$\begin{aligned} K_{1,\delta 1} = & \frac{\mu_1^-[\bar{\varphi}_A(\mu_2^- + \mu_2^+) + 1]}{\bar{\varphi}_A^2(\mu_1^-\mu_2^- + \mu_1^+\mu_2^+ + \mu_1^-\mu_2^+) + \bar{\varphi}_A(\mu_1^- + \mu_1^+ + \mu_2^- + \mu_2^+) + 1}\,, \\ K_{2,\delta 1} = & \frac{\bar{\varphi}_A\mu_1^-\mu_2^-}{\bar{\varphi}_A^2(\mu_1^-\mu_2^- + \mu_1^+\mu_2^+ + \mu_1^-\mu_2^+) + \bar{\varphi}_A(\mu_1^- + \mu_1^+ + \mu_2^- + \mu_2^+) + 1}\,. \end{aligned} \tag{2.15}$$

It is thus evident that the pressure variations depend upon the external loading, the IGU shape and size, the bending stiffness of the glass panes, but not upon the fictitious *arbitrary* loads $q_1$ and $q_2$ used in Betti's theorem.

When $D_3 \to 0$, so that $\mu_2^+ \to \infty$, one obtains

$$\Delta p_{1,\delta 1} = \frac{\mu_1^-}{\bar{\varphi}_A(\mu_1^+ + \mu_1^-) + 1}\frac{\varphi(\mathbf{x}^*)}{A}\,, \quad \Delta p_{2,\delta 1} = 0\,, \tag{2.16}$$

which coincides with the expressions presented in [16] for the case of DGUs under concentrated unitary force.

The expressions (2.14), which provide the pressure variations due to an unitary concentrated action, represent the *Green's functions for the pressure variations* in the TGU. Consequently, when a generic load per unit area $f_1(\mathbf{x})$ is applied on plate "1", the consequent pressure variations may be evaluated as

$$\Delta p_{j,f1} = \Delta p_{j,\delta 1} *\ f_1(\mathbf{x}) = \frac{K_{j,\delta 1}}{A}\int_\Omega \varphi(\mathbf{x}^*) f_1(\mathbf{x}^*)\, d\mathbf{x}^*\,,\ \ j = 1, 2\,. \tag{2.17}$$

Once $\Delta p_{j,\delta 1}$, $j = 1, 2$, are known from (2.14), it is possible to evaluate the out-of-plane displacement of all panels. Panel "1" is subjected to the concentrated force

[5] The first index denotes the cavity where the pressure variation arises, while the second indicates that the unitary concentrated load acts on plate 1.

and to $-\Delta p_{1,\delta 1}$; panel "2" is bent by the pressure resultant $\Delta p_{1,\delta 1} - \Delta p_{2,\delta 1}$; panel "3" is under $\Delta p_{2,\delta 1}$. These displacement fields are consequent to a concentrated unitary force $\delta_1(\mathbf{x} - \mathbf{x}^*)$ applied on pane "1". Therefore, they correspond to the *Green's functions for displacements* and they will be denoted with $\mathcal{G}_{i,\delta 1}(\mathbf{x}, \mathbf{x}^*)$, $i = 1, 2, 3$.

Remarkably, the expressions (2.14) are valid whatever the edge condition of the IGU. In the particular case in which *the IGU is simply supported along the whole border*, from (2.3) and (2.14) one obtains

$$
\begin{aligned}
\mathcal{G}_{1,\delta 1}(\mathbf{x}, \mathbf{x}^*) &= \frac{A}{D_1}\,\zeta(\mathbf{x}, \mathbf{x}^*) - \frac{\Delta p_{1,\delta 1}}{D_1} A^2\,\varphi(\mathbf{x}) = \frac{A}{D_1}\left[\zeta(\mathbf{x}, \mathbf{x}^*) - K_{1,\delta 1}\,\varphi(\mathbf{x})\,\varphi(\mathbf{x}^*)\right], \\
\mathcal{G}_{2,\delta 1}(\mathbf{x}, \mathbf{x}^*) &= \frac{A}{D_2}\left(K_{1,\delta 1} - K_{2,\delta 1}\right)\,\varphi(\mathbf{x})\,\varphi(\mathbf{x}^*), \\
\mathcal{G}_{3,\delta 1}(\mathbf{x}, \mathbf{x}^*) &= \frac{A}{D_3} K_{2,\delta 1}\,\varphi(\mathbf{x})\,\varphi(\mathbf{x}^*). 
\end{aligned} \tag{2.18}
$$

When an arbitrary load per unit area $f_1(\mathbf{x})$ is applied on glass pane "1", the corresponding out-of-plane displacements of the glass panes can be evaluated as the convolution integral of the Green's functions and the external load, *as per* eq. (2.4). Under diverse boundary conditions for the IGU, the Green's functions for displacements shall take into account the actual effect of the constraints at the edges. They could be directly calculated by considering the package of panes loaded by pressure variations in the cavities and the external concentrated force.

If the TGU is subjected to a unitary concentrated load $\delta_3(\mathbf{x} - \mathbf{x}^*)$, applied on pane "3" and acting in the direction of positive $z$, the same procedure presented above provides the gas pressure variations

$$
\Delta p_{j,\delta 3} = \frac{K_{j,\delta 3}}{A}\,\varphi(\mathbf{x}^*), \quad j = 1, 2, \tag{2.19}
$$

with

$$
\begin{aligned}
K_{1,\delta 3} &= -\frac{\mu_1^+ \mu_2^+ \bar{\varphi}_A}{\bar{\varphi}_A^2(\mu_1^- \mu_2^- + \mu_1^+ \mu_2^+ + \mu_1^- \mu_2^+) + \bar{\varphi}_A(\mu_1^- + \mu_1^+ + \mu_2^- + \mu_2^+) + 1}\,, \\
K_{2,\delta 3} &= -\frac{\mu_2^+[\bar{\varphi}_A(\mu_1^- + \mu_1^+) + 1]}{\bar{\varphi}_A^2(\mu_1^- \mu_2^- + \mu_1^+ \mu_2^+ + \mu_1^- \mu_2^+) + \bar{\varphi}_A(\mu_1^- + \mu_1^+ + \mu_2^- + \mu_2^+) + 1}\,.
\end{aligned}
\tag{2.20}
$$

where the minus sign is due to the fact that the concentrated force is directed outwards. Correspondingly, the Green's functions for displacements, when the IGU is simply supported at the border, may be written as

$$
\begin{aligned}
\mathcal{G}_{1,\delta 3}(\mathbf{x}, \mathbf{x}^*) &= -\frac{A}{D_1} K_{1,\delta 3}\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\,, \\
\mathcal{G}_{2,\delta 3}(\mathbf{x}, \mathbf{x}^*) &= \frac{A}{D_2}\,(K_{1,\delta 3} - K_{2,\delta 3})\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\,, \\
\mathcal{G}_{3,\delta 3}(\mathbf{x}, \mathbf{x}^*) &= \frac{A}{D_3}\,[\zeta(\mathbf{x}, \mathbf{x}^*) + K_{2,\delta 3}\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)]\,.
\end{aligned}
\tag{2.21}
$$

If a generic distributed load *per* unit area $f_3(\mathbf{x})$ is applied on glass pane "3", the pressure variations and out-of-plane displacement fields can be evaluated through expressions analogous to (2.17) and (2.4), respectively.

## 2.3 The effect of arbitrary external loads

The most common case in the design practice is that of a uniformly distributed external load, i.e. $f_1(\mathbf{x}) = \overline{p}$. In this case, from (2.17), one has

$$
\Delta p_1 = K_{1,\delta 1}\ \bar{\varphi}_A\ \overline{p}\,, \quad \Delta p_2 = K_{2,\delta 1}\ \bar{\varphi}_A\ \overline{p}\,, \tag{2.22}
$$

where, again, $\bar{\varphi}_A$ is the mean value of $\varphi(\mathbf{x})$ on the plate area. Graphs and tables for the evaluation of $\bar{\varphi}_A$ for plates with different shapes are recorded in [14].

The displacement fields are calculated as the convolution integral of the Green's functions and the external load. For a simply supported IGU, recalling (2.6) and (2.18), one obtains

$$
\begin{aligned}
w_1(\mathbf{x}) &= \overline{p}\ \frac{A}{D_1}\left[\int_\Omega \zeta(\mathbf{x},\mathbf{x}^*)d\mathbf{x}^* - K_{1,\delta 1}\varphi(\mathbf{x})\int_\Omega \varphi(\mathbf{x}^*)d\mathbf{x}^*\right] = \overline{p}\ \frac{A^2}{D_1}\left[1 - K_{1,\delta 1}\bar{\varphi}_A\right]\varphi(\mathbf{x})\,,\\
w_2(\mathbf{x}) &= \overline{p}\ \frac{A^2}{D_2}\ (K_{1,\delta 1} - K_{2,\delta 1})\ \bar{\varphi}_A\ \varphi(\mathbf{x})\,,\\
w_3(\mathbf{x}) &= \overline{p}\ \frac{A^2}{D_3}\ K_{2,\delta 1}\ \bar{\varphi}_A\ \varphi(\mathbf{x})\,.
\end{aligned}
\tag{2.23}
$$

In the ideal case in which the gas filling the cavities is incompressible, $\Delta p_{1,\delta 1}$ and $\Delta p_{2,\delta 1}$, by using (2.12), are given by

$$
\Delta p_{1,\delta 1} = \frac{D_2 + D_3}{(D_1 + D_2 + D_3)}\ \overline{p}\,, \qquad \Delta p_{2,\delta 1} = \frac{D_3}{(D_1 + D_2 + D_3)}\ \overline{p}\,. \tag{2.24}
$$

Therefore, the pressure resultant of the uniformly-distributed loads on panes “1”, “2” and “3” reads

$$
\begin{aligned}
r_1 &= \overline{p} - \Delta p_1 = \frac{D_1}{(D_1 + D_2 + D_3)}\ \overline{p}\,,\\
r_2 &= \Delta p_1 - \Delta p_2 = \frac{D_2}{(D_1 + D_2 + D_3)}\ \overline{p}\,,\\
r_3 &= \Delta p_2 = \frac{D_3}{(D_1 + D_2 + D_3)}\ \overline{p}\,.
\end{aligned}
\tag{2.25}
$$

These expressions coincide with the so-called “thickness cubed” method [1], according to which each pane carries a part of the total load proportional to its flexural stiffness.

Consider now the case of a uniformly line-distributed load $\overline{H}$, acting on a smooth curve $\Sigma$ of length $L$. If $l$ denotes a curvilinear abscissa on $\Sigma$, with $0 \leq l \leq L$, and $\mathbf{x} = \tilde{\mathbf{x}}(l)$ the point of the curve at $l$, following [31, 36, 42] it is possible to define a “line Delta function” $\delta_\Sigma$, associated with the curve $\Sigma$, such that for any test function

$g(\mathbf{x})$

$$\int_\Omega \delta_\Sigma(\mathbf{x}^* - \tilde{\mathbf{x}}(l))g(\mathbf{x}^*)\ d\mathbf{x}^* = \int_\Sigma g(\tilde{\mathbf{x}}(l))\ dl\,. \tag{2.26}$$

By writing the uniformly curve-distributed load on pane "1" as $f_1(\mathbf{x}) = \overline{H}\delta_\Sigma(\mathbf{x} - \tilde{\mathbf{x}})$, the pressure variations, evaluated *as per* (2.17), are

$$\Delta p_1 = K_{1,\delta 1}\ \bar{\varphi}_\Sigma\ \frac{\overline{H}L}{A}\,, \quad \Delta p_2 = K_{2,\delta 1}\ \bar{\varphi}_\Sigma\ \frac{\overline{H}L}{A}\,, \tag{2.27}$$

where $\bar{\varphi}_\Sigma := \frac{1}{L}\int_\Sigma \varphi(\tilde{\mathbf{x}}(l))\ dl$ is the mean value of the non-dimensional shape function $\varphi(\mathbf{x})$ on the curve $\Sigma$ where the load is applied. Values of $\bar{\varphi}_\Sigma$ for triangular and rectangular plates with different aspect ratio are recorded in [14] when $\Sigma$ is a generic straight line.

The out-of-plane displacements are evaluated with (2.4). Acconting for (2.18), for a simply supported IGU they read

$$\begin{aligned}
w_1(\mathbf{x}) &= \overline{H}\ \frac{A}{D_1}\left[\int_\Omega \zeta(\mathbf{x},\mathbf{x}^*)\delta_\Sigma(\mathbf{x}-\tilde{\mathbf{x}})d\mathbf{x}^* - K_{1,\delta 1}\varphi(\mathbf{x})\int_\Omega \varphi(\mathbf{x}^*)\delta_\Sigma(\mathbf{x}-\tilde{\mathbf{x}})d\mathbf{x}^*\right]\\
&= \overline{H}\ \frac{A}{D_1}\ \left[\int_\Sigma \zeta(\mathbf{x},\tilde{\mathbf{x}}(l))dl - K_{1,\delta 1}\varphi(\mathbf{x})\ L\bar{\varphi}_\Sigma\right]\,,\\
w_2(\mathbf{x}) &= \overline{H}\ \frac{A}{D_2}\ (K_{1,\delta 1} - K_{2,\delta 1})\ L\bar{\varphi}_\Sigma\ \varphi(\mathbf{x})\,,\\
w_3(\mathbf{x}) &= \overline{H}\ \frac{A}{D_3}\ K_{2,\delta 1}\ L\bar{\varphi}_\Sigma\ \varphi(\mathbf{x})\,,
\end{aligned} \tag{2.28}$$

Observe that the integral appearing in the expression for $w_1(\mathbf{x})$ represents the shape function for the out-of-plane displacement due to the line-distributed load.

When panel "1" of the TGU is subjected to the concentrated force $F_1$ at $\mathbf{x}^*$, the pressure variations are given by

$$\Delta p_j = K_{j,\delta 1}\varphi(\mathbf{x}^*)\frac{F_1}{A}\,, \ \ j = 1, 2\,. \tag{2.29}$$

The out-of-plane displacement fields obviously correspond to the Green's functions multiplied by $F_1$.

It can be directly verified that, by setting $D_3 \to 0$ (equivalently, $\mu_2^+ \to \infty$) in (2.22), (2.27) and (2.29), the equations obtained in [16] for IGUs composed by only two glass panes are recovered.

It should be remarked that, to evaluate the pressure variation in the cavities of an IGU of arbitrary geometry and edge conditions, due to the application of external loads of any kind, what is necessary is only to evaluate the deformation $\varphi(\mathbf{x})$ of a simply supported plate under uniform pressure. This represents, in our opinion, a noteworthy advancement with respect to the approach by Feldmeier [10, 13], where the deformation of the pane under the effective external loads needs to be calculated.

## 2.4 The effect of climatic actions

Variations of the barometric pressure $\overline{\Delta p}$ with respect to the sealing site, as well as of the environmental temperature $\overline{\Delta T}$ with respect to the absolute temperature $T_0$ at the time of sealing, may also produce the deflection of the glass panes.

The former case may be treated by considering an external uniform pressure $\overline{\Delta p}$ acting on both the outer glass panes, considered positive if the actual pressure is higher than $p_0$. This case may be studied by considering the uniformly distributed load $f_1(\mathbf{x}) = \overline{\Delta p}$ on pane "1" and $f_3(\mathbf{x}) = -\overline{\Delta p}$ on pane "3", and superposing the effects. Hence, the pressure variation in the $j$-th cavity, $j = 1, 2$, can be evaluated as

$$\Delta p_j = \Delta p_{j,\delta 1} * f_1(\mathbf{x}) + \Delta p_{j,\delta 3} * f_3(\mathbf{x}) = (K_{j,\delta 1} - K_{j,\delta 3})\ \bar{\varphi}_A\ \overline{\Delta p}\,. \tag{2.30}$$

Under balanced external pressures, for the same argument used while discussing (2.10), the edge constraints of the IGU are inactive, and each pane behaves as if it was simply supported on the edge seal. Therefore, the displacement fields may be evaluated as $w_i(\mathbf{x}) = \mathcal{G}_{i,\delta 1} * f_1(\mathbf{x}) + \mathcal{G}_{i,\delta 3} * f_3(\mathbf{x}) = (\mathcal{G}_{i,\delta 1} - \mathcal{G}_{i,\delta 3}) * \overline{\Delta p}$, $i = 1, 2, 3$. Using

(2.6), (2.18) and (2.21), one obtains

$$
\begin{aligned}
w_1(\mathbf{x}) &= \overline{\Delta p}\, \frac{A^2}{D_1} \left[1 - (K_{1,\delta 1} - K_{1,\delta 3})\, \bar{\varphi}_A\right] \varphi(\mathbf{x})\,, \\
w_2(\mathbf{x}) &= \overline{\Delta p}\, \frac{A^2}{D_2}\, \left(K_{1,\delta 1} - K_{2,\delta 1} - K_{1,\delta 3} + K_{2,\delta 3}\right) \bar{\varphi}_A\, \varphi(\mathbf{x})\,, \\
w_3(\mathbf{x}) &= \overline{\Delta p}\, \frac{A^2}{D_3} \left[-1 + (K_{2,\delta 1} - K_{2,\delta 3})\, \bar{\varphi}_A\right] \varphi(\mathbf{x})\,.
\end{aligned}
\tag{2.31}
$$

It may be verified that, for an incompressible gas, expressions analogue to (2.25) are recovered.

Likewise, changes in the temperature of the gas in the cavities, with respect to the absolute temperature $T_0$ at the time of sealing, can produce an increase of internal pressure which deforms the glass panes. According to [28], the temperature in the cavities in the steady state can be calculated as a function of the temperatures of the inner and outer glass panes, as well as of the heat transfer coefficient of the glass plies and of the cavity.

Here, for completeness, we record the relevant expressions. Consider the TGU of Figure 1, subjected to temperature variations $\overline{\Delta T}_1$ and $\overline{\Delta T}_2$ in cavities "1" and "2", respectively. For a compressible ideal gas, the volume, absolute pressure and absolute temperature variations are related by the Ideal Gas Law, so that

$$
\begin{aligned}
\frac{p_0\, V_{01}}{(p_0 + \Delta p_1)(V_{01} + \Delta V_1)} &= \frac{T_0}{T_0 + \overline{\Delta T}_1}\,, \\
\frac{p_0\, V_{02}}{(p_0 + \Delta p_2)(V_{02} + \Delta V_2)} &= \frac{T_0}{T_0 + \overline{\Delta T}_2}\,.
\end{aligned}
\tag{2.32}
$$

Here, $\Delta V_1$ and $\Delta V_2$ are the volume variations due to the deflection of the glass panes under the action of the pressure variations $\Delta p_1$ and $\Delta p_2$. Again, under balanced pressures the edge constraints of the IGU are inactive, as indicated while discussing (2.10). Thus, recalling that

$$w_1(\mathbf{x}) = -\Delta p_1 \,\frac{A^2}{D_1}\varphi(\mathbf{x})\,, \quad w_2(\mathbf{x}) = (\Delta p_1 - \Delta p_2)\,\frac{A^2}{D_2}\varphi(\mathbf{x})\,, \quad w_3(\mathbf{x}) = \Delta p_2\,\frac{A^2}{D_3}\varphi(\mathbf{x})\,, \tag{2.33}$$

using the quantities defined in (2.13), one obtains

$$\begin{aligned} \Delta V_1 &= \int_\Omega [w_2(\mathbf{x}) - w_1(\mathbf{x})]\; d\mathbf{x} = \frac{V_{01}}{p_0}\left[(\mu_1^- + \mu_1^+)\Delta p_1 - \mu_1^+\Delta p_2\right]\bar{\varphi}_A\,, \\ \Delta V_2 &= \int_\Omega [w_3(\mathbf{x}) - w_2(\mathbf{x})]\; d\mathbf{x} = \frac{V_{02}}{p_0}\left[(\mu_2^- + \mu_2^+)\Delta p_2 - \ \mu_2^-\Delta p_1\right]\bar{\varphi}_A\,. \end{aligned} \tag{2.34}$$

The non-linear system of equations obtained by substituting (2.34) into (2.32) is difficult to solve. However, it can be observed that the terms of the type $\Delta p_j \Delta p_k$, $j, k = 1, 2$, are much smaller than those of types $\Delta p_j p_0$ and $p_0^2$, and hence they can be neglected. Therefore, an approximated solution can be found in the form

$$\Delta p_j = \frac{p_0}{T_0}\left[\frac{K_{j,\delta 1}}{\mu_1^-}\;\overline{\Delta T}_1 - \frac{K_{j,\delta 3}}{\mu_2^+}\;\overline{\Delta T}_2\right]\,, \quad j = 1, 2\,. \tag{2.35}$$

Once the pressure variations are known, the out-of-plane displacement may be found through equations (2.33).

It may be verified that, for $D_3 \to 0$, eq.s (2.30) and (2.35) allow to recover the expression obtained in [14] for DGUs.

# 3 Extension to insulating units with arbitrary number of glass panes

We now show how the method can be extended to Insulating Glazing Units (IGUs) composed by an arbitrary number of panes. The relevant Green's functions are explicitly calculated for symmetric IGUs with four panes and for IGUs with five identical glass panes, with edge spacers all of the same thickness.

## 3.1 Generalization of Betti's analytical method

Consider an IGU with arbitrary shape and edge constraints, composed by $N$ glass panes, of thickness $h_i$, $i = 1 \dots N$, connected by $N-1$ spacers of thickness $s_j$, $j = 1 \dots N-1$. Introduce a reference system similar to that indicated in Figure 1. As in Section 2.2, in the configuration $\mathcal{A}$ panel "1" of the IGU is subjected to a concentrated unitary force $\delta_1(\mathbf{x} - \mathbf{x}^*)$ at the generic point $\mathbf{x}^*$ and undergoes the pressure variation $\Delta p_j$ in the $j$-th cavity, $j = 1 \dots N-1$. In the framework of the BAM approach, the load sharing is determined by considering the $N-1$ auxiliary load sets shown in Figure 3, referred to as $\mathcal{B}_j$, $j = 1 \dots N-1$. In the $j$-th system, the IGU is subjected to a self-equilibrated load distribution given by a fictitious uniform pressure $q_j$, acting in the $j$-th cavity.

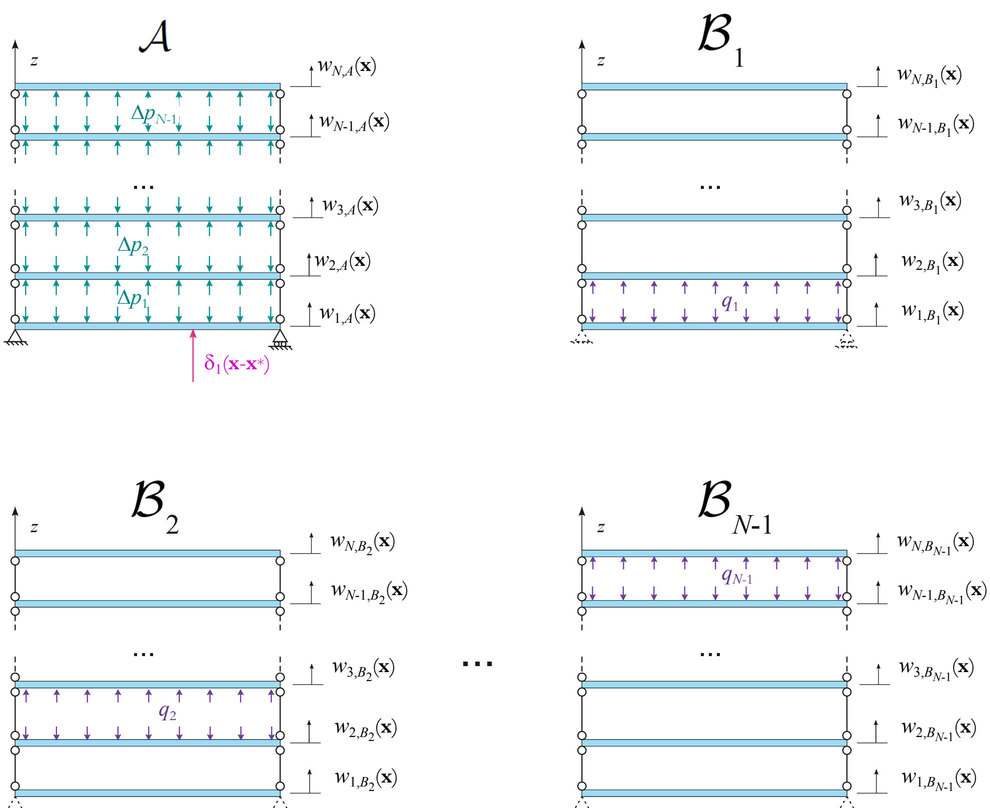



Figure 3: Schematic representation of the load systems for the application of Betti's theorem.

Let $w_{i,\mathcal{B}_j}(\mathbf{x})$ represent the displacement field in the $i$-th glass pane, $i = 1 \dots N$, for the $\mathcal{B}_j$ load set, $j = 1 \ldots N-1$. The mutual works $\mathcal{L}_{\mathcal{A}\mathcal{B}_j}$ done by the load set $\mathcal{A}$ through the displacements produced by the load set $\mathcal{B}_j$, $j = 1 \ldots N-1$, are given by

$$\begin{aligned}
\mathcal{L}_{\mathcal{A}\mathcal{B}_1} = &\quad w_{1,\mathcal{B}_1}(\mathbf{x}^*) + \Delta p_1 \int_\Omega [w_{2,\mathcal{B}_1}(\mathbf{x}) - w_{1,\mathcal{B}_1}(\mathbf{x})] \; d\mathbf{x} - \Delta p_2 \int_\Omega w_{2,\mathcal{B}_2}(\mathbf{x}) \; d\mathbf{x} \,, \\
\mathcal{L}_{\mathcal{A}\mathcal{B}_j} = &\quad \Delta p_{j-1} \int_\Omega w_{j,\mathcal{B}_j}(\mathbf{x}) + \Delta p_j \int_\Omega [w_{j+1,\mathcal{B}_j}(\mathbf{x}) - w_{j,\mathcal{B}_j}(\mathbf{x})] \; d\mathbf{x} \\
&\quad - \Delta p_{j+1} \int_\Omega w_{j+1,\mathcal{B}_j}(\mathbf{x}) \; d\mathbf{x} \,, \qquad j = 2 \ldots N-2 \,, \\
\mathcal{L}_{\mathcal{A}\mathcal{B}_{N-1}} = &\Delta p_{N-2} \int_\Omega w_{N-1,\mathcal{B}_{N-1}}(\mathbf{x}) + \Delta p_{N-1} \int_\Omega [w_{N,\mathcal{B}_{N-1}}(\mathbf{x}) - w_{N-1,\mathcal{B}_{N-1}}(\mathbf{x})] \; d\mathbf{x} \,.
\end{aligned} \tag{3.1}$$

On the other hand, by recalling (2.9), the mutual work $\mathcal{L}_{\mathcal{B}_j\mathcal{A}}$, done by the $j$-th load set $\mathcal{B}_j$ through the displacements in system $\mathcal{A}$, can be written as

$$\mathcal{L}_{\mathcal{B}_j\mathcal{A}} = q_j \int_\Omega [w_{j+1,\mathcal{A}}(\mathbf{x}) - w_{j,\mathcal{A}}(\mathbf{x})] \; d\mathbf{x} \simeq -\frac{\Delta p_j}{p_0} \; V_{0j} \,, \quad j = 1 \ldots N-1 \,, \tag{3.2}$$

where the same simplification of (2.9) has been made.

The $N-1$ equalities $\mathcal{L}_{\mathcal{A}\mathcal{B}_j} = \mathcal{L}_{\mathcal{B}_j\mathcal{A}}$ of the mutual works, can be written in compact form by defining, in analogy with (2.13), the following coefficients for the $j$-th cavity

$$\mu_j^- = \frac{A^3}{D_j} \frac{p_0}{V_{0j}} \,, \quad \mu_j^+ = \frac{A^3}{D_{j+1}} \frac{p_0}{V_{0j}} \,, \;\; j = 1 \ldots N-1 \,, \tag{3.3}$$

and read

$$
\begin{aligned}
&\mu_1^- \ \frac{\varphi(\mathbf{x}^*)}{A} + \Delta p_1(\mu_1^- + \mu_1^+)\bar{\varphi}_A - \Delta p_2 \ \mu_1^+ \ \bar{\varphi}_A = -\Delta p_1\,, \\
&\left[\Delta p_{j-1}\mu_j^- + \Delta p_j(\mu_j^- + \mu_j^+) - \Delta p_{j+1}\mu_j^+\right]\bar{\varphi}_A = -\Delta p_j\,, \quad j = 2 \ldots N-2\,, \\
&\left[\Delta p_{N-2} \ \mu_{N-1}^- + \Delta p_{N-1}(\mu_{N-1}^- + \mu_{N-1}^+)\right]\bar{\varphi}_A = -\Delta p_{N-1}\,.
\end{aligned}
\tag{3.4}
$$

The solution of this system provides the pressure variations $\Delta p_j$, $j = 1 \ldots N-1$, for IGUs composed by an arbitrary number $N$ of glass plies. When the pressure variations $\Delta p_j$, $j = 1 \ldots N-1$ are known, the Green's function for the out-of-plate displacements could be found, following the same procedure illustrated in Section 2.2.

The solution of (3.4) can be found in closed forms, but it is quite lengthy because, analogously to (2.15), it entails terms of the $N-1$ order in $\bar{\varphi}_A$ and $\mu_j^-$, $\mu_j^+$. However, great simplifications can be obtained for $N = 4$ when the IGU is symmetrically composed, as well as for $N = 5$ when the thickness is the same for all glass panes.

## 3.2 Green's Function for symmetric insulating units with four glass panes

Consider a symmetric IGU composed by four glass panes, the two external ones of thickness $h_1 = h_4$, and the inner ones of thickness $h_2 = h_3$. In order to obtain compact formulae, denote with $D_2 = D_3 = D$ the flexural stiffness of the inner panes, and by $D_1 = D_4 = D/\alpha$ that of the outer panes. Moreover, indicate the initial volume of the cavity comprised between panes "2" and "3" with $V_0$, and that between panes "1" and "2", equal to the cavity between panes "3" and "4", with $V_0/\beta$.

By defining $\mu = \frac{A^3}{D}\frac{p_0}{V_0}$, in agreement with (3.3) the non dimensional coefficients may be written as

$$
\mu_1^- = \alpha\beta\mu\,, \quad \mu_1^+ = \beta\mu\,, \quad \mu_2^- = \mu_2^+ = \mu\,, \quad \mu_3^- = \beta\mu\,, \quad \mu_3^+ = \alpha\beta\mu\,. \tag{3.5}
$$

Under these assumptions, the system of equations (3.4) may be solved to obtain

$$\Delta p_{j,\delta 1} = \frac{K_{j,\delta 1}}{A}\ \varphi(\mathbf{x}^*)\,, \qquad j = 1\ldots 3\,, \tag{3.6}$$

where

$$\begin{aligned}
K_{1,\delta 1} &= \frac{\left[\beta\mu^2\bar{\varphi}_A^2\,(2\alpha+1)+\mu\,\bar{\varphi}_A\,(\alpha\,\beta+\beta+2)+1\right]\mu\,\alpha\,\beta}{2\,\alpha\,\beta^2\mu^3\bar{\varphi}_A^3\,(\alpha+1)+\beta\,\mu^2\bar{\varphi}_A^2\,(\alpha^2\beta+2\,\alpha\,\beta+4\,\alpha+\beta+2)+2\mu\,\bar{\varphi}_A\,(\alpha\,\beta+\beta+1)+1}\\
K_{2,\delta 1} &= \frac{\alpha\,\bar{\varphi}_A\,\mu^2\beta}{2\,\alpha\,\beta\,\mu^2\bar{\varphi}_A^2+\mu\,\bar{\varphi}_A\,(\alpha\,\beta+\beta+2)+1}\,,\\
K_{3,\delta 1} &= \frac{\beta^2\mu^3\bar{\varphi}_A^2\alpha}{2\,\alpha\,\beta^2\mu^3\bar{\varphi}_A^3\,(\alpha+1)+\beta\,\mu^2\bar{\varphi}_A^2\,(\alpha^2\beta+2\,\alpha\,\beta+4\,\alpha+\beta+2)+2\,\mu\,\bar{\varphi}_A\,(\alpha\,\beta+\beta+1)+1}\,.
\end{aligned} \tag{3.7}$$

In the particular case of a simply supported IGU, the corresponding Green's functions for the out-of-plane displacement of panes are

$$\begin{aligned}
\mathcal{G}_{1,\delta 1}(\mathbf{x},\mathbf{x}^*) &= \frac{A}{D}\left[\zeta(\mathbf{x},\mathbf{x}^*)-K_{1,\delta 1}\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\right]\,,\\
\mathcal{G}_{2,\delta 1}(\mathbf{x},\mathbf{x}^*) &= \frac{\alpha A}{D}\left(K_{1,\delta 1}-K_{2,\delta 1}\right)\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\,,\\
\mathcal{G}_{3,\delta 1}(\mathbf{x},\mathbf{x}^*) &= \frac{\alpha A}{D}\left(K_{2,\delta 1}-K_{3,\delta_1}\right)\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\,,\\
\mathcal{G}_{4,\delta 1}(\mathbf{x},\mathbf{x}^*) &= \frac{A}{D}\ K_{3,\delta 1}\ \varphi(\mathbf{x})\ \varphi(\mathbf{x}^*)\,.
\end{aligned} \tag{3.8}$$

Once the Green's functions are known, the out-of-plane deflection of the glass panes under arbitrary external loading and under climatic[6] actions may be evaluated by following a procedure analogue to that presented in Sections 2.3 and 2.4, respectively.

[6] To our knowledge, current standards do not provide explicit formulae to evaluate the temperature changes in the different cavities for IGUs composed by more than 3 glass panes subjected to temperature gradients. As observed in [40], in this case software tools are needed.

## 3.3 Green's Function for insulating units with five identical glass panes

Consider an IGU composed by five glass panes with the same thickness $h$ and equal cavities, with identical edge spacers. Denote by $D$ the flexural stiffness of each glass pane, and by $V_0$ the reference volume of the cavities. In this case, one has $\mu_j^- = \mu_j^+ = \mu = \frac{A^3}{D}\frac{p_0}{V_0}$, $j = 1\ldots4$, and the equation system (3.4) can be strongly simplified. The resulting formulae for the pressure variation are identical to (3.6) but with $j = 1\ldots4$, where the non dimensional coefficients $K_{j,\delta1}$ take the form

$$
\begin{aligned}
K_{1,\delta1} &= \frac{\mu\,(2\,\mu\,\bar{\varphi}_A+1)\,(2\,\mu^2\bar{\varphi}_A^2+4\,\mu\,\bar{\varphi}_A+1)}{(\mu^2\bar{\varphi}_A^2+3\,\mu\,\bar{\varphi}_A+1)\,(5\,\mu^2\bar{\varphi}_A^2+5\,\mu\,\bar{\varphi}_A+1)}\\
K_{2,\delta1} &= \frac{\mu^2\bar{\varphi}_A\,(\mu\,\bar{\varphi}_A+1)\,(3\,\mu\,\bar{\varphi}_A+1)}{(\mu^2\bar{\varphi}_A^2+3\,\mu\,\bar{\varphi}_A+1)\,(5\,\mu^2\bar{\varphi}_A^2+5\,\mu\,\bar{\varphi}_A+1)}\,,\\
K_{3,\delta1} &= \frac{\mu^3\bar{\varphi}_A^2\,(2\,\mu\,\bar{\varphi}_A+1)}{(\mu^2\bar{\varphi}_A^2+3\,\mu\,\bar{\varphi}_A+1)\,(5\,\mu^2\bar{\varphi}_A^2+5\,\mu\,\bar{\varphi}_A+1)}\,,\\
K_{4,\delta1} &= \frac{\mu^4\bar{\varphi}_A^3}{(\mu^2\bar{\varphi}_A^2+3\,\mu\,\bar{\varphi}_A+1)\,(5\,\mu^2\bar{\varphi}_A^2+5\,\mu\,\bar{\varphi}_A+1)}\,.
\end{aligned}
\tag{3.9}
$$

For simply supported IGUs, the Green's functions are analogous to (3.8) and read

$$
\begin{aligned}
\mathcal{G}_{1,\delta1}(\mathbf{x},\mathbf{x}^*) &= \frac{A}{D}\left[\zeta(\mathbf{x},\mathbf{x}^*) - K_{1,\delta1}\;\varphi(\mathbf{x})\;\varphi(\mathbf{x}^*)\right]\,,\\
\mathcal{G}_{i,\delta1}(\mathbf{x},\mathbf{x}^*) &= \frac{A}{D}\left(K_{i-1,\delta1} - K_{i,\delta1}\right)\;\varphi(\mathbf{x})\;\varphi(\mathbf{x}^*)\,,\qquad i = 2,3,4\,,\\
\mathcal{G}_{5,\delta1}(\mathbf{x},\mathbf{x}^*) &= \frac{A}{D}\;K_{4,\delta1}\;\varphi(\mathbf{x})\;\varphi(\mathbf{x}^*)\,.
\end{aligned}
\tag{3.10}
$$

Again, the out-of-plane deflection of the glass panes may be evaluated with the convolutions of the Green's functions with the external loads or climatic actions, by following the same procedure illustrated in Sections 2.3 and 2.4, respectively.

# 4 Examples and comparisons

The analytical results obtained in the previous Sections are now compared with those obtainable with either the recommendations by current Standards, in particular the European Norm EN 16612 [28], and the freeware software MEPLA ISO [5]. Reference is made to a rectangular TGUs because, whereas the BAM approach can be used for IGUs of any shape, composition and type of constraint, neither the EN 16612, nor MEPLA ISO, can consider shapes other than rectangular and packages with more than three glass panes. All the aforementioned approaches consider rigid spacer joints that constrain the relative out-of-plane displacement of the glass panes. This is somehow a limitation, since the spacer may have relevant effects on the gross bending response of glass panels in IGUs, as indicated in [2], where original experimental tests were proposed for IGUs with different types of spacer connections. However, a detailed discussion of this aspect goes beyond our scope here.

For an $a \times b$ rectangular TGU, simply supported on the four sides, consider two different packages[7]: a *symmetric* TGU, with $h_1 = h_3 = 10$ mm, $h_2 = 6$ mm, $s_1 = s_2 = 12$ mm, hereafter referred to as the "*S-TGU*", and an *uneven* TGU, with $h_1 = 6$ mm, $h_2 = 8$ mm, $h_3 = 10$ mm, $s_1 = 12$ mm, $s_2 = 16$ mm, denoted in the sequel as the "*U-TGU*". Gas pressure at the time of sealing is considered equal to the standard atmospheric pressure $p_0 = 101325$ Pa, and the initial volumes of the cavities are $V_{0j} = s_j\, A$, $j = 1, 2$.

## 4.1 Comparisons with consolidated methods proposed by standards

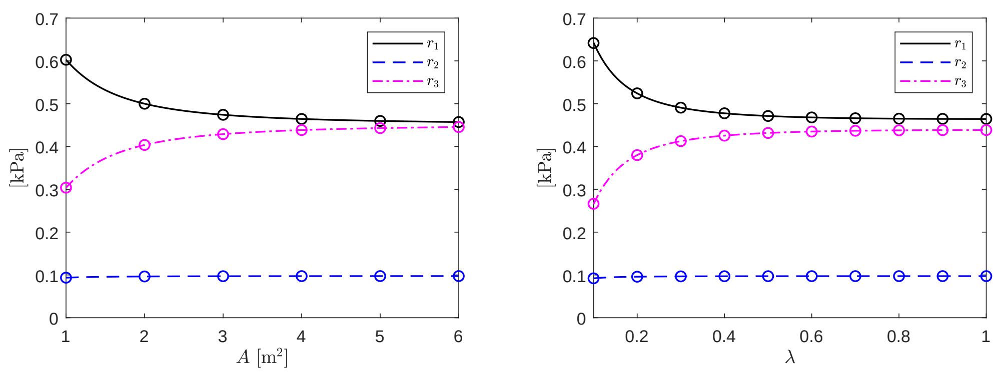
The European Norm EN 16612 [28] covers only simply supported IGUs under uniformly-distributed loads, also associated with barometric pressure variations and temperature changes. Since the standard does not provide formulae for deflections, the compar-

[7] Usually, the inner glass ply of a TGU is thinner than the other ones, since its predominant task is to seperate the cavities to improve the thermal properties, while the outer panes are subject to wind (outside) and balustrade (inner) loads. The selected geometries are of interest from a theoretical point of view.

isons are made in terms of the load resultant on the different glass panes.

### 4.1.1 TGUs under external uniform pressure

Consider the action of a uniform pressure $\overline{p} = 1$ kPa, for which the pressure variations $\Delta p_1$ and $\Delta p_2$ are obtained from (2.22). For the symmetric *S-TGU* simply supported on the four sides, Figure 4a shows the resulting uniformly distributed loads[8] $r_1 = \overline{p} - \Delta p_1$ on pane "1", $r_2 = \Delta p_1 - \Delta p_2$ on pane "2" and $r_3 = \Delta p_2$ on pane "3". These are plotted for *square* panels of different size as a function of $A = ab$, varying between 1 m$^2$ and 6 m$^2$. Figure 4b records the same quantities for a fixed area $A = 4$ m$^2$ as a function of the TGU aspect ratio $\lambda = a/b$, with $0.1 \leq \lambda \leq 1$. The comparison is between the results from the BAM method and the formula suggested by [28], evidenced with circular dots. The agreement is excellent.




(a) (b)

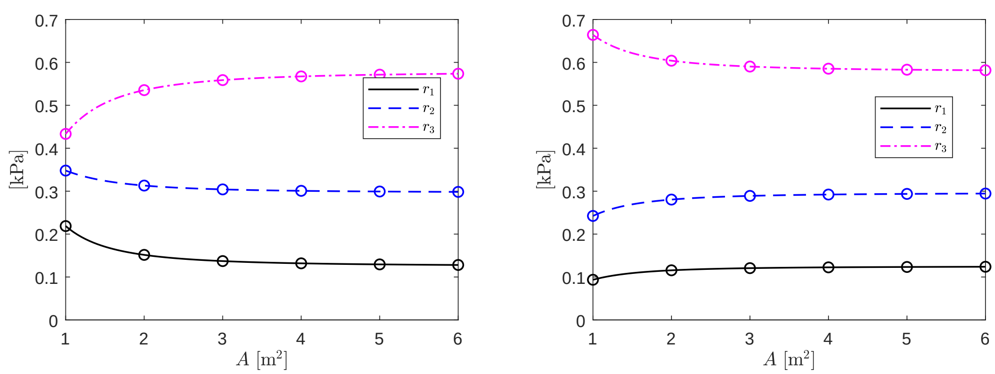
Figure 4: *S-TGU*s under uniform pressure $\overline{p} = 1$ kPa. Load resultant on the different panes for a) $\lambda = a/b = 1$, as a function of $A = ab$ and b) $A = 4$ m$^2$, as a function of $\lambda$. Comparison between the proposed approach (lines) and EN 16612 (circles).

Observe that, in agreement with [16, 14], the higher the compliance of the constituent panes is, the lower is the influence of the gas deformation. In fact, for high values of $A$ and $\lambda$ (large panels, approximately square), the load resultants tend to those for isochoric gas transformations ($r_1 = r_3 = 0.4513\,\overline{p}$ and $r_2 = 0.0975\,\overline{p}$), evaluated with the "thickness cubed method" [1] and re-obtained in (2.25) as a particular

[8]Here, the uniform loads are assumed to be positive if in the direction of positive $z$, *as per* Figure 1.

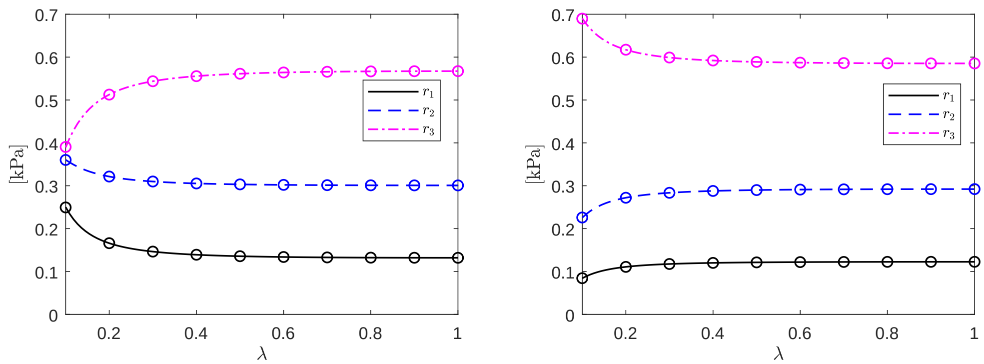
case. Notice that this limit is attained for values of $A$ higher than 6 m$^2$. For low values of $A$ and/or $\lambda$, the gas compressibility does not allow for the optimal sharing of the loads among the different panes: as $A$ and/or $\lambda$ are diminished, $r_1$ increases (directly loaded pane), while $r_3$ decreases. It may be verified that also $r_2$ (central pane) decreases but, because of symmetry, the variation is very small.

The same comparison is made for the uneven *U-TGU*, for which it is of interest to consider loads applied on pane "1", as before, or on pane "3", for which the pressure variations come from (2.19). These two cases are illustrated in Figures 5a and 5b, respectively, for square *U-TGU*s ($\lambda = 1$) as a function of the area $A$. As already discussed, for large $A$ the load resultants tend to those for incompressible gas ($r_1 = 0.125\ \overline{p}$, $r_2 = 0.2963\ \overline{p}$ and $r_3 = 0.5787\ \overline{p}$), evaluated with (2.25). In small TGUs, on the contrary, the directly-loaded pane carries the higher percentage of the external load. In general, if the directly loaded panel is the thick one, the load sharing reduces because its deformation is small, and viceversa. The uneven deformability of the panes also affects the resultant $r_2$ on the central pane. A qualitatively similar trend may be recognized in graphs in Figure 6, referring to U-TGUs with area $A = 4$ m$^2$ and varying aspect ratio $\lambda$. In any case, there is an excellent agreement between the results from the proposed approach and the method of EN 16612.




(a) (b)

Figure 5: Load resultant on the different panes of the *U-TGU* from uniformly distributed load $\overline{p} = 1$ kPa applied on a) pane "1" or b) pane "3", for $\lambda = 1$ and varying $A$. Comparison between the proposed approach (lines) and EN 16612 (circles).




(a) (b)

Figure 6: Load resultant on the different panes of the $U$-$TGU$ from uniformly distributed load $\overline{p} = 1$ kPa applied on a) pane "1" or b) pane "3", for $A = 4$ m$^2$ and varying $\lambda$. Comparison between the proposed approach (lines) and EN 16612 (circles).

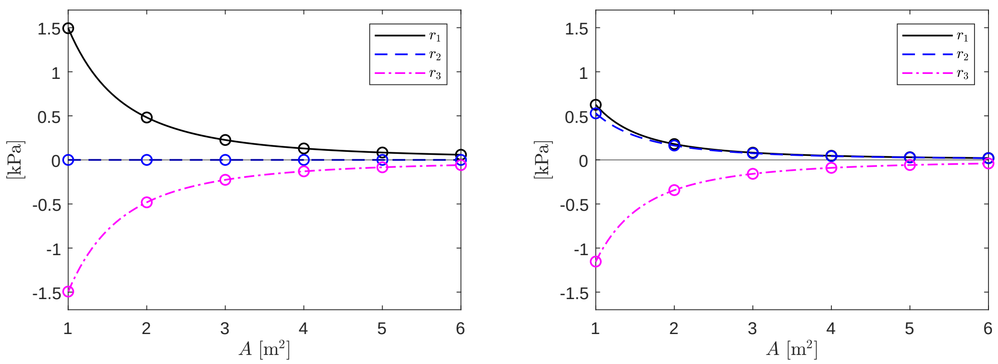
### 4.1.2 TGUs under barometric pressure variations

A barometric pressure variation $\overline{\Delta p} = 5 \cdot 10^{-3}$ MPa produces the pressure variations inside cavities $\Delta p_1$ and $\Delta p_2$ as *per* (2.30). We again compare the resultants of the load per unit area $r_1 = \overline{\Delta p} - \Delta p_1$, $r_2 = \Delta p_1 - \Delta p_2$ and $r_3 = \Delta p_3 - \overline{\Delta p}$.

Figure 7a is the counterpart of Figure 4a for the symmetric package $S$-$TGU$ with $\lambda = 1$ and varying $A$. Because of symmetry, the pressure variations in the two cavities coincide and their resultant on the inner pane is null. Figure 7b, which refers to the asymmetric case $U$-$TGU$, shows that the pressure variations are not equal because of the different stiffness of the glass plates, so that the central pane is stressed. Again, the more compliant the glass panes (large $A$), the higher the pressure variations in the cavities, which equilibrate the barometric pressure and release the glass panes.

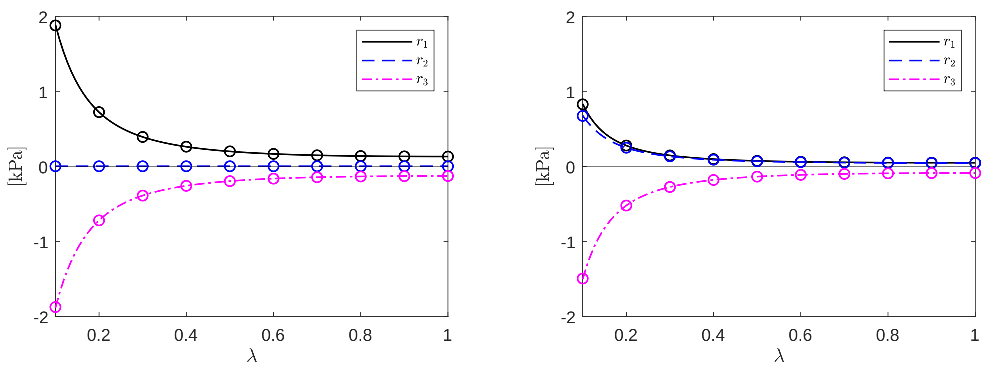
To investigate the influence of the aspect ratio, Figure 8a and Figure 8b refer to the case $A = 4$ m$^2$ and varying $\lambda$ for $S$-$TGU$ and $U$-$TGU$, respectively. Also in this case, symmetry annihilates the load on the inner lite. The stress in the panes diminishes by increasing the aspect ratio of the TGU, being minimum for square elements.

Again, there is an excellent agreement between the results from the proposed method and the EN 16612 approach.




(a) (b)

Figure 7: Load acting on the different panes for a barometric pressure variation $\overline{\Delta p} = 5 \cdot 10^{-3}$ MPa in TGUs with $\lambda = 1$ and varying $A$: a) *S-TGU* and b) *U-TGU*. Comparison between the proposed approach (lines) and EN 16612 methods (circles).



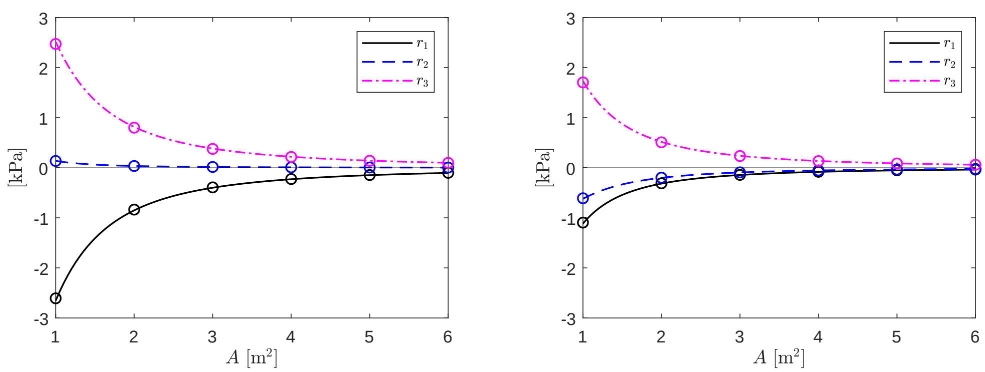

(a) (b)

Figure 8: Load acting on the different panels for a barometric pressure variation $\overline{\Delta p} = 5 \cdot 10^{-3}$ MPa in TGUs with $A = 4$ m$^2$ and varying $\lambda$: a) *S-TGU* and b) *U-TGU*. Comparison between the proposed approach (lines) and EN 16612 methods (circles).

### 4.1.3 TGUs under temperature variations

Suppose that the gas trapped in the cavities "1" and "2" undergoes two different temperature variations $\overline{\Delta T}_1 = 30°\text{C}$ and $\overline{\Delta T}_2 = 20°\text{C}$, respectively. In this case, the pressure variations $\Delta p_1$ and $\Delta p_2$ are evaluated through (2.35).

In Figures 9a and 9b, $r_1 = -\Delta p_1$, $r_2 = \Delta p_1 - \Delta p_2$, and $r_3 = \Delta p_3$ are plotted for *S-TGU*s and *U-TGU*s, respectively, for $\lambda = 1$ and varying $A$. The stiffer the panes (small $A$), the higher they are stressed. In *U-TGU*s the stiffer panes carry the most of the temperature-induced pressure change.




(a) (b)

Figure 9: Load acting on the different panes for $\overline{\Delta T}_1 = 30°\text{C}$ and $\overline{\Delta T}_2 = 20°\text{C}$ in TGUs with $\lambda = 1$ and varying $A$: a) *S-TGU* and b) *U-TGU*. Comparison between the proposed approach (lines) and EN 16612 methods (circles).

A similar behavior can be observed in Figure 10, which shows results for *S-TGU*s and *U-TGU*s with the same area $A = 4$ m$^2$ with different aspect ratio $\lambda$. The effect of the climatic actions is confirmed to be higher for small IGUs and for low aspect ratios.

One should observe that, even if the variation of internal pressure due to climatic actions is much higher than the effects of wind and snow, typically of the order of 1 kPa, the assumption $\Delta p_1, \Delta p_2 \ll p_0$, which is at the base of the simplification (2.9) and used to evaluate (2.35), is fully satisfied. This is confirmed by the excellent agreement between the considered approaches.

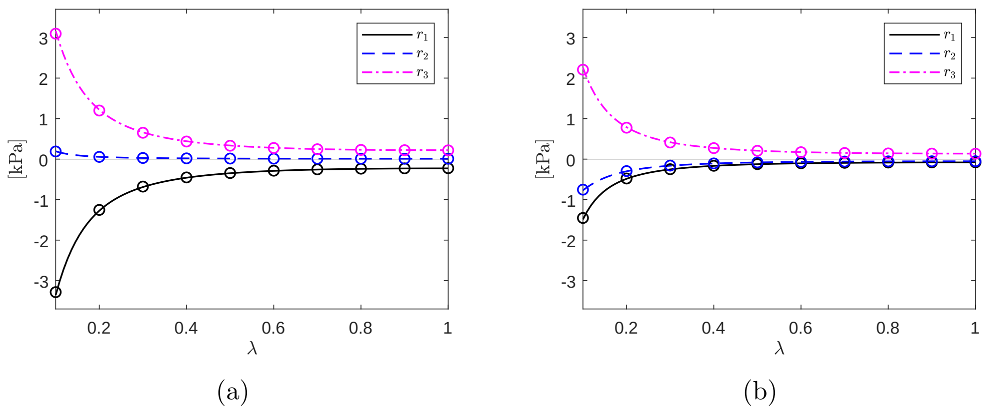



Figure 10: Load acting on the different panes for $\overline{\Delta T}_1 = 30°$C and $\overline{\Delta T}_2 = 20°$C in TGUs with $A = 4$ m$^2$ and varying $\lambda$: a) *S-TGU* and b) *U-TGU*. Comparison between the proposed approach (lines) and EN 16612 methods (circles).

## 4.2 Comparisons with numerical results

Since current standards do not provide compact formulae for the load sharing in IGUs under loads either than uniformly distributed, for such cases the comparison is now made with the results from MEPLA ISO [5], in terms of out-of-plane displacement of the panes, consistently with the output of the software. The external load is applied on pane "1", and the maximum deflection of panes "2" and "3", subjected to $(\Delta p_1 - \Delta p_2)$ and $\Delta p_2$, respectibely is calculated. We consider the same *S-TGU*s and *U-TGU*s, under a uniformly-distributed line-load in the $x$ direction (Figure 11a), a concentrated force (Figure 11b) and, for the sake of comparison with the FEM code, under uniform pressure. The numerical analyses have been performed by meshing the panes with 30 mm × 30 mm elements.

### 4.2.1 TGUs under external uniform pressure

Consider the action of a uniform pressure $\overline{p} = 1$ kPa. In the proposed approach, the out-of-plane displacement is evaluated *via* the Green's function as *per* (2.23). For *S-DGU*s and *U-TGU*s with area of $A = 4$ m$^2$ and variable $\lambda$, Figure 12 shows the comparison in terms of $w_{max;2}$ and $w_{max;3}$, i.e., the maximum deflection of panes "2" and "3", respectively.

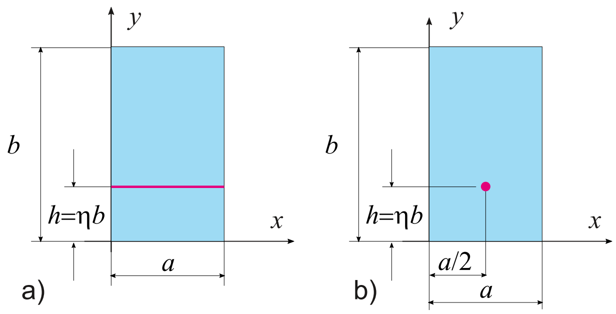



Figure 11: Rectangular TGU under a) uniformly-distributed line-load and b) point load used in the numerical analysis.

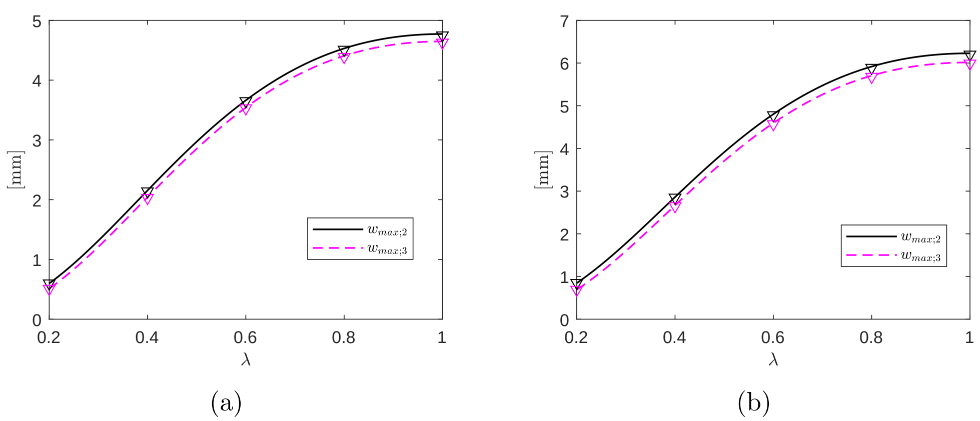


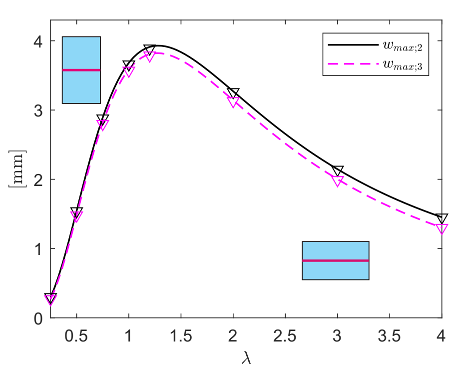

Figure 12: Out-of-plane displacements of panes "2" and "3" in a) $S$-$TGU$s and b) $U$-$TGU$s, under uniform pressure $\overline{p} = 1$ kPa acting on pane "1", for $A = 4$ m$^2$ and varying $\lambda$. Proposed approach (lines) and MEPLA ISO (triangles).

It is evident that, due to the load sharing, the maximum deflections of panes "2" and "3" are very similar, for all the considered geometries. In the ideal situation of incompressible gas response, each of the three glass ply carries a part of the distributed load proportional to the flexural stiffness *as per* (2.25) and, hence, their deflection are identical. The little difference here recorded is due to the gas compressibility. The difference between numerical and analytical results is less than 0.7%.

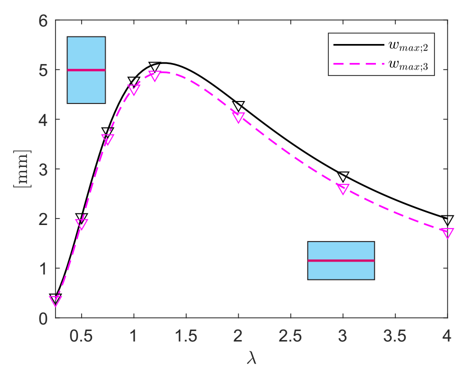
### 4.2.2 TGUs under external line-distributed load

With reference to Figure 11a, consider a uniformly-distributed line-load $\overline{H} = 1$ kN/m, applied on the straight line $y = h = \eta\, b$, with $\eta \in [0, 1]$. The pressure variations are given by (2.27) and the out-of-plane displacements of the glass panes are evaluated with (2.28).

Considered first the case $\eta = 1/2$ for TGUs with $A = 4$ m$^2$ and variable $\lambda$: when $\lambda < 1$ ($\lambda > 1$) the line load is applied on the short (long) side. Figure 13a and 13b show the maximum deflections of panes "2" and "3", for *S-TGU* and *U-TGU*, respectively. In order to investigate the effect of the load position, in Figure 14 the maximum deflection of pane "3" is plotted as a function of the parameter $\eta$, for IGU with $A = 4$ m$^2$ and different aspect ratio ($\lambda = 0.5$, $\lambda = 1$ and $\lambda = 2$).




(a)




(b)

Figure 13: Out-of-plane displacements of panes "2" and "3" in a) $S$-$TGU$s and b) $U$-$TGU$s under the uniformly-distributed line-load $\overline{H} = 1$ kN/m acting on pane "1" at mid-height, for $A = 4$ m$^2$ and varying $\lambda$. Proposed approach (lines) and MEPLA ISO (triangles).

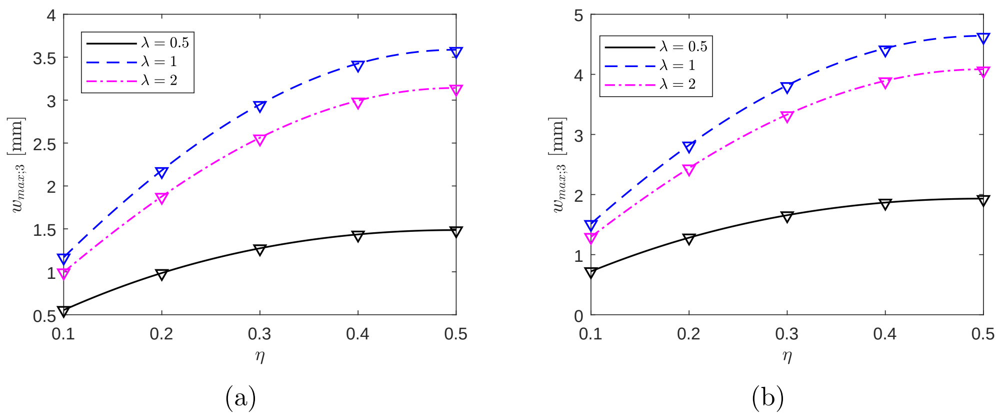



Figure 14: Out-of-plane displacement of pane "3" in a) *S-TGU*s and b) *U-TGU*s under the uniformly-distributed line-load $\overline{H} = 1$ kN/m, for $A = 4$ m$^2$, $\lambda = 0.5, 1, 2$ and varying $\eta$. Proposed approach (lines) and MEPLA ISO (triangles).

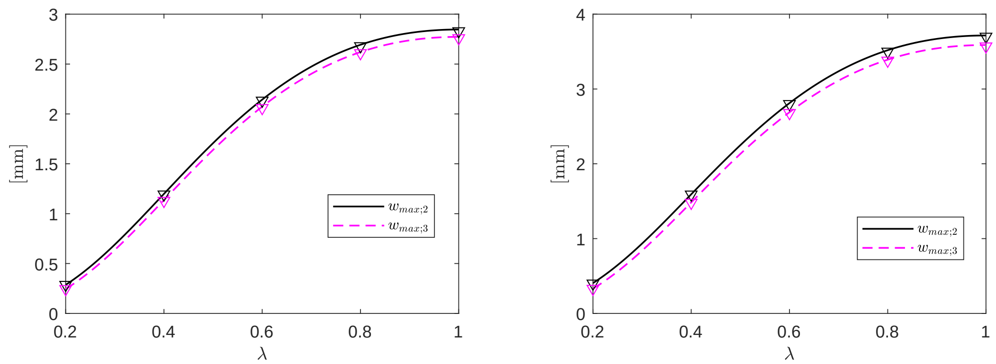
Also in this case, there is an excellent agreement between analytical and numerical results, with maximum error lower than 0.7%.

### 4.2.3 TGUs under concentrated load

Consider now the case of Figure 11b, where a concentrated load $\overline{F} = 1$ kN is applied at $(x^* = a/2,\ y^* = h = \eta\, b)$, with $\eta \in [0, 1]$. The pressure variations $\Delta p_1$ and $\Delta p_2$ are evaluated with (2.29), while the out-of-plane displacements correspond to the Green's functions (2.18), multiplied by the value of $\overline{F}$. In the numerical analyses, the load has been modelled as smeared on a 20 mm $\times$ 20 mm square, centered at $(x^*,\ y^*)$.

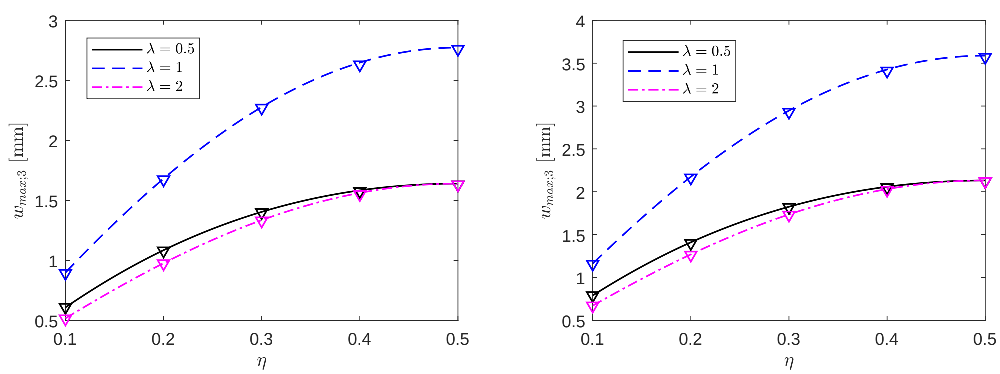
For a concentrated load applied at the plate center ($\eta = 1/2$), Figure 15 shows the maximum deflections of panes "2" and "3" in *S-TGU*s and *U-TGU*s, evaluated with the BAM approach and through MEPLA ISO, plotted as a function of $\lambda$ when $A = 4$ m$^2$. Then, to investigate the influence of the load position, $\eta$ is varied between 0.1 and 0.5. Figures 16a and 16b are the counterparts of Figure 14. Obviously, the cases $\lambda = 0.5$ and $\lambda = 2$ coincides for $\eta = 0.5$, since the load is applied at the plate center.

Also under concentrated load, the maximum error is very low, of the order of 0.6%, confirming the high accuracy of the proposed approach.




(a) (b)

Figure 15: Out-of-plane displacements of panes "2" and "3" in a) $S$-$TGU$s and b) $U$-$TGU$s under a concentrated load $\overline{F} = 1$ kN at the center of pane "1", for $A = 4$ m$^2$ and varying $\lambda$. Proposed approach (lines) and MEPLA ISO (triangles).




(a) (b)

Figure 16: Out-of-plane displacements of pane "3" in a) $S$-$TGU$s and b) $U$-$TGU$s under ua concentrated load $\overline{F} = 1$ kN at the center of pane "1", for $A = 4$ m$^2$, $\lambda = 0.5, 1, 2$ and varying $\eta$. Proposed approach (lines) and MEPLA ISO (triangles).

# 5 Conclusions

Insulating Glazing Units (IGUs) are schematized as the assembly of edge-sealed Kirchhoff-Love glass plates entrapping an ideal gas. Under mild simplifying hypotheses, i.e., the pressure variations inside the cavities consequent to applied external loads are small with respect to the atmospheric pressure, the whole IGU behaves as a linear elastic system. Therefore, the *Green's functions* for the pressure variations and the out-of-plane displacements of the glass panes, associated with the effects of a concentrated force applied on one external pane, can be defined. Once the Green's functions are known, the effect of any arbitrary load field can be evaluated as the convolution integral of the Green's functions and the load itself. To calculate the Green's functions, Betti's Analytical Method (BAM), recently proposed in [16] for Double Glazing Units (DGUs) only, has been extended to multiple IGUs, composed by an arbitrary number of glass panes. Remarkably, for any shape, type of loading and edge-constraint of the IGU, the Green's function for the pressure variation in the cavities can be expressed as a function of the deformation under uniform pressure of a simply-supported plate, with the same shape of the IGU. Because of this, we repute that this approach represents an improvement with respect to other methods proposed in the technical literature [13], which require to determine the pane deflections under the specific external load and an iterative procedure to calculate the volume change in the cavities.

For rectangular Triple Gazing Units (TGUs) under uniformly distributed load, or subjected to a variation in barometric pressure or environmental temperature with respect to time of sealing, the results from the Green's function method are in excellent agreement with the formulae proposed by current standards [28]. The efficiency of the proposed approach is also confirmed by comparisons with the numerical results performed with a dedicated software [5], for which the maximum differences are less then 0.7% for TGUs of various size and aspect ratio, under either uniform, line distributed or concentrated external actions.

The proposed approach, which strongly exploits the properties of the Green's functions, could be of particular interest also in the study of the dynamic response of IGUs subjected to the impact of objects or people, including the modelling of the pendulum test [6, 8, 23], as well as to blast loading [3, 18, 41]. The potentiality of the method in the dynamical case are yet to be fully appreciated.

Of course the results here presented only refer to monolithic glass panes, considered as linear-elastic Kirchhoff-Love plates. However, recent numerical experiments [17] on DGUs have shown, albeit tentatively, that the variation of the internal glass pressure due to the application of external loads is only marginally effected by the geometric non-linearity of the glass plates. Hence, one could consider the proposed method, based on Green's functions, to determine the internal pressure variations, using such values to calculate, *a posteriori*, the deflection and stress in the glass panes taking into account membrane effects. Furthermore, laminated-glass panes could be covered in this approach by considering them as monolithic plates characterized by the corresponding effective thickness in terms of deflection [15]. The design details of the insulated elements, in particular the various possible shapes of the spacer joints, have not been considered here although it has been demonstrated [2] that they can have a great influence for the state of stress and deflection of the IGU. With a more in-depth analysis of the technological aspects, it will be possible to proceed with a validation of the BAM analytical approach with a dedicated experimental campaign. All these aspects will be the subject of further work.

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