THÈSE POUR OBTENIR LE GRADE DE DOCTEUR DE L'UNIVERSITÉ DE MONTPELLIER

En Mathématiques

École doctorale: I2S - Information Structures Systèmes

Unité de recherche: IMAG - Institut Montpelliérain Alexander Grothendieck

Anneaux de Cox des éclatements des espaces multiprojectifs

Présentée par Elena Poma
le 19 Janvier 2024

Sous la direction de Michele Bolognesi
et de Alex Massarenti

Devant le jury composé de

Michele Bolognesi, Professeur, Université de Montpellier	Directeur de thèse
Alex Massarenti, Maître de conference, Université de Ferrara	Codirecteur de thèse
Carolina Araujo, Professeur, Instituto Nacional de Matemática Pura e Aplicada	Examinateur
Boris Pasquier, Professeur, Université de Poitiers	Examinateur
Thomas Dedieu, Maître de conference, Université de Toulouse	Examinateur

UNIVERSITÉ FRANCO ITALIENNE
UNIVERSITÀ ITALO FRANCESE

Cox rings of blow-ups of multiprojective spaces

Elena Poma

January 11, 2024

Agli amici,
alla mia famiglia
e agli altri animali.

Abstract

This thesis deals with birational geometry, which is a subfield of algebraic geometry. In particular, we study Mori dream spaces, which are varieties that are strictly related to the theory of Mori's minimal model program. Mori dream spaces were introduced by Y. Hu and S. Keel in the beginning of the 21th century. Roughly speaking, a Mori dream space is a projective variety, whose cone of effective divisors admits a well-behaved decomposition into convex sets, called Mori chambers. These chambers are the nef cones of the birational models of X. Several geometric objects are of fundamental importance if we want to proceed in the study of the birational geometry of a normal projective variety. In particular its cones of curves and of divisors. Another notion of great importance to establish whether a variety is a Mori dream space or not is the property of being weak or log Fano. Weak Fano varieties are log Fano and log Fano varieties are Mori dream spaces. In 2021, T. Grange, E. Postinghel and A. Prendergast-Smith focussed on blow-ups of $\mathbb{P}^1 \times \mathbb{P}^2$ and of $\mathbb{P}^1 \times \mathbb{P}^3$ in sets of up to six points in very general position. Their main result is the explicit descriptions of the cones of effective divisors on these varieties and the description of the geometry of the generating classes. More explicitly, they proved that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ is weak Fano if and only if the number of blown up points is less or equal than six and that if the number of blown up points is less or equal than six, the variety $\mathbb{P}^1 \times \mathbb{P}^3$ blown-up in those points is log Fano. Hence, these varieties are also Mori dream spaces.

In chapter 2 of this thesis we give an overview on the theory of Cox rings, Mori dream spaces and log Fano varieties. In the first sections we give the definitions of the various cones inside $\mathrm{N}_1(X)$ and inside $\mathrm{N}^1(X)$ and their inclusion relations. We then give a description of the Cox ring of a variety equipped with an algebraic torus action. We conclude the chapter with the result that permits to find generators for the moving cone of a variety from the generators of its Cox ring. It will follow an explanation on the main results concerning Mori dream spaces and log Fano varieties, and many examples. Finally, we introduce the main object of study of this thesis: the variety $X_r^{1,n}$, which is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^n$ in r points in general position. In particular, we focus on $X_{n+1}^{1,n}$, $X_{n+2}^{1,n}$ and on $X_{n+3}^{1,n}$ when $n \leq 4$. In chapter 3 we compute the cone of effective curves of $X_r^{1,n}$ for $r = n+1, n+2$

and $r = n + 3$ when $n \leq 4$. We then prove that $X_r^{1,n}$ is log Fano for $r \leq n + 1$. In chapter 4, we compute generators and relations of the Cox ring of $X_{n+1}^{1,n}$. We then use these generators to compute generators of the moving cone of $X_{n+1}^{1,n}$. In order to do the computation, we wrote some scripts on Maple and Magma, some of which are provided in chapter 6. At the end if chapter 4 we compute the nef cones of $X_r^{1,n}$ for $r = n + 1, n + 2$ and for $r = n + 3$ when $n \leq 4$. Then, in chapter 5 we also give a Mori chamber decomposition of $X_{n+1}^{1,n}$ in Magma and we display the case $n = 2$.

Résumé

Cette thèse traite de la géométrie birationnelle, qui est un sous-domaine de la géométrie algébrique. En particulier, nous étudions les Mori dream spaces, qui sont des variétés strictement liées à la théorie du programme de modèle minimal de Mori. Les Mori dream spaces ont été introduits par Y. Hu et S. Keel au début du 21 ème siècle. En gros, un Mori dream space est une variété projective dont le cône des diviseurs effectifs admet une décomposition bien conçue en ensembles convexes, appelés chambres de Mori. Ces chambres sont les cônes de nef des modèles birationnels de X. Plusieurs objets géométriques sont d'une importance fondamentale si l'on veut poursuivre l'étude de la géométrie birationnelle d'une variété projective normale. En particulier ses cônes de courbes et de diviseurs. Une autre notion de grande importance pour établir si une variété est un Mori dream space ou non est la propriété d'être weak ou log Fano. Les variétés weak Fano sont log Fano et les variétés log Fano sont Mori dream spaces. En 2021, T. Grange, E. Postinghel et A. Prendergast-Smith se sont concentrés sur les éclatements de $\mathbb{P}^1 \times \mathbb{P}^2$ et de $\mathbb{P}^1 \times \mathbb{P}^3$ dans des ensembles allant jusqu'à six points dans une position très générale. Leur résultat principal est la description explicite des cônes de diviseurs effectifs sur ces variétés et la description de la géométrie des classes génératrices. Plus explicitement, ils ont prouvé que l'éclatement de $\mathbb{P}^1 \times \mathbb{P}^2$ est weak Fano si et seulement si le nombre de points éclatés est inférieur ou égal à six et que si le nombre de points éclatés est inférieur ou égal à six, l'éclatement de la variété $\mathbb{P}^1 \times \mathbb{P}^3$ en ces points est log Fano. Par conséquent, ces variétés sont aussi des Mori dream spaces.

Dans le chapitre 2 de cette thèse, nous donnons un aperçu de la théorie des anneaux de Cox, des Mori dream spaces et des variétés log Fano. Dans les premières sections, nous donnons les définitions des différents cônes à l'intérieur de $\mathrm{N}_1(X)$ et à l'intérieur de $\mathrm{N}^1(X)$ et leurs relations d'inclusion. Nous donnons ensuite une description de l'anneau de Cox d'une variété équipée d'une action de tore algébrique. Nous concluons le chapitre avec le résultat qui permet de trouver des générateurs pour le cône des diviseurs mouvables d'une variété à partir des générateurs de son anneau de Cox. Il s'ensuivra une explication des principaux résultats concernant les Mori dream spaces et les variétés log Fano, ainsi que de nombreux exemples.

Enfin, nous introduisons l'objet d'étude principal de cette thèse : la variété $X_r^{1,n}$, qui est l'éclatement de $\mathbb{P}^1 \times \mathbb{P}^n$ en r points en position générale. En particulier, nous nous concentrons sur $X_{n+1}^{1,n}$, $X_{n+2}^{1,n}$ et sur $X_{n+3}^{1,n}$ lorsque $n \leq 4$. Dans le chapitre 3, nous calculons le cône des courbes effectives de $X_r^{1,n}$ pour $r = n+1, n+2$ et $r = n+3$ lorsque $n \leq 4$. Nous prouvons ensuite que $X_r^{1,n}$ est log Fano pour $r \leq n+1$. Dans le chapitre 4, nous calculons les générateurs et les relations de l'anneau de Cox de $X_{n+1}^{1,n}$. Nous utilisons ensuite ces générateurs pour calculer les générateurs du cône des diviseurs mouvables de $X_{n+1}^{1,n}$. Pour effectuer ces calculs, nous avons écrit des scripts sur Maple et Magma, dont certains sont fournis dans le chapitre 6. A la fin du chapitre 4, nous calculons les cônes nef de $X_r^{1,n}$ pour $r = n+1, n+2$ et pour $r = n+3$ lorsque $n \leq 4$. Ensuite, dans le chapitre 5, nous donnons aussi une décomposition en chambre de Mori de $X_{n+1}^{1,n}$ dans Magma et nous montrons le cas $n = 2$.

Riassunto

Questa tesi si occupa di geometria birazionale, che è un sottocampo della geometria algebrica. In particolare, studia i Mori dream spaces, che sono varietà strettamente correlate alla teoria del programma del modello minimale di Mori. I Mori dream spaces sono stati introdotti da Y. Hu e S. Keel all'inizio del XXI secolo. In parole povere, un Mori dream space è una varietà proiettiva il cui cono di divisori effettivi ammette una decomposizione in insiemi convessi, chiamati camere di Mori. Queste camere sono i coni nef dei modelli birazionali di X. Diversi oggetti geometrici sono di fondamentale importanza se si vuole procedere nello studio della geometria birazionale di una varietà proiettiva normale. In particolare i coni delle curve e dei divisori. Un'altra nozione di grande importanza per stabilire se una varietà è un Mori dream space o meno è la proprietà di essere weak o log Fano. Le varietà weak Fano sono log Fano e le varietà log Fano sono Mori dream spaces. Nel 2021, T. Grange, E. Postinghel e A. Prendergast-Smith si sono concentrati sui blow-up di $\mathbb{P}^1 \times \mathbb{P}^2$ e di $\mathbb{P}^1 \times \mathbb{P}^3$ in insiemi di massimo sei punti in posizione molto generale. Il loro risultato principale è la descrizione esplicita dei coni di divisori effettivi su queste varietà e la descrizione della geometria delle classi generatrici. Più esplicitamente, hanno dimostrato che il blow-up di $\mathbb{P}^1 \times \mathbb{P}^2$ è weak Fano se e solo se il numero di punti scoppiati è minore o uguale a sei e che se il numero di punti scoppiati è minore o uguale a sei, la varietà ottenuta scoppiando $\mathbb{P}^1 \times \mathbb{P}^3$ in quei punti è log Fano. Quindi, queste varietà sono anche Mori dream spaces.

Nel capitolo 2 di questa tesi forniamo una panoramica sulla teoria degli anelli di Cox, dei Mori dream spaces e delle varietà log Fano. Nelle prime sezioni diamo le definizioni dei vari coni interni a $\mathrm{N}_1(X)$ e a $\mathrm{N}^1(X)$ e le loro relazioni di inclusione. Diamo poi una descrizione dell'anello di Cox di una varietà dotata di un'azione torica algebrica. Concludiamo il capitolo con il risultato che permette di trovare i generatori del cono dei divisori mobili di una varietà a partire dai generatori del suo anello di Cox. Seguirà una spiegazione dei principali risultati relativi ai Mori dream spaces e alle varietà log Fano, e molti esempi. Infine, introduciamo il principale oggetto di studio di questa tesi: la varietà $X_r^{1,n}$, che è il blow-up di $\mathbb{P}^1 \times \mathbb{P}^n$ in r punti in posizione generale. In particolare, ci concentriamo su $X_{n+1}^{1,n}$, $X_{n+2}^{1,n}$ e su $X_{n+3}^{1,n}$ quando $n \leq 4$. Nel capitolo 3 calcoliamo il cono delle curve

effettive di $X_r^{1,n}$ per $r = n+1, n+2$ e $r = n+3$ quando $n \leq 4$. Dimostriamo poi che $X_r^{1,n}$ è log Fano per $r \leq n+1$. Nel capitolo 4, calcoliamo i generatori e le relazioni dell'anello di Cox di $X_{n+1}^{1,n}$. Utilizziamo poi questi generatori per calcolare i generatori del cono dei divisori mobili di $X_{n+1}^{1,n}$. Per eseguire i calcoli, abbiamo scritto alcuni script su Maple e Magma, alcuni dei quali sono riportati nel capitolo 6. Alla fine del capitolo 4 calcoliamo i coni nef di $X_r^{1,n}$ per $r = n+1, n+2$ e per $r = n+3$ quando $n \leq 4$. Poi, nel capitolo 5 diamo anche una decomposizione della camera di Mori di $X_{n+1}^{1,n}$ in Magma e mostriamo il caso $n = 2$.

Acknowledgment

I would like to express my sincere gratitude to all those who have contributed to the completion of this project.

First and foremost, I extend my deepest appreciation to my advisors, Michele Bolognesi and Alex Massarenti, for their invaluable guidance, unwavering support, and insightful feedback throughout the entire process. Their expertise and encouragement have been instrumental in shaping the direction and quality of this work.

I am also thankful to the Università Italo Francese for their financial support given in the project VINCI, which has enabled me to carry out this research.

I extend my thanks to the members of my thesis referees, Michela Artebani and Elisa Postinghel, for their time and valuable input. Their constructive feedback has greatly enriched the content and presentation of this thesis.

I am grateful to my colleagues and friends I met in Ferrara, who provided assistance and encouragement along the way. Also for making me experience a year full of fun and joy in this beautiful city.

Last but not least, I want to express my deepest appreciation to Francesco and to my family for their unwavering support and understanding during the ups and downs of this journey.

This thesis would not have been possible without the collective contributions of all those mentioned above. Thank you for being an integral part of this endeavor.

Contents

Chapter 1

Introduction

This thesis deals with birational geometry, which is a subfield of algebraic geometry. The goal of birational geometry is to determine when two algebraic varieties are isomorphic outside lower-dimensional subsets. To be more precise, let us give the following definition:

Definition 1.0.1. Let X and Y be two algebraic varieties. A birational map from X to Y is a rational map $f : X \dashrightarrow Y$ such that there is a rational map $Y \dashrightarrow X$ inverse to f.

The existence of a birational map between two varieties X and Y is equivalent to the existence of an isomorphism between two open subsets of X and Y.

A natural question that arises after this definition is to classify algebraic varieties up to birational equivalence. This process takes origin at the end of the 19th century, and still entails open questions. Huge advances were done in the second half of the 20th century, thanks to the works of W.L. Chow (1911-1995), that essentially state that, in order to accomplish this classification, one can focus on particular types of algebraic viariety:

Lemma 1.0.2. *(Chow's lemma). If X is a complete, irreducible variety, then there exists a projective variety X' that is birational to X.*

For a proof we refer to [Sha13].

In 1964 Hironaka, proved that it was possible to resolve singularities of varieties over fields of characteristic zero by repeatedly blowing up along non-singular subvarieties, using a very complicated argument by induction on the dimension. As a consequence, he proved that every variety over a field of characteristic zero is birational to a smooth projective variety. For further details, we refer to [Hir64]. Hence, to study algebraic varieties up to birational equivalence, it is enough to study smooth projective varieties.

Let us give an insight into the birational classification of algebraic varieties depending on their dimension.

The classification of curves is classical and was done in the 19th century:

Theorem 1.0.3. *Every curve is birationally equivalent to a unique nonsingular projective curve. Algebraic curves are classified by their genus* g *and there are three classes: rational curves* $(g = 0)$*, elliptic curves* $(g = 1)$ *and curves of general type* $(g \geq 2)$*.*

Moerover, two smooth curves are birational if and only if they are isomorphic, hence the two types of equivalence, for this low dimension, coincide.

The situation with surfaces is already more complicated. Obviously, there is no more equivalence between birationality and isomorphism, and a key example for this fact is given by the blow-up construction: if we blow-up the projective plane $\mathbb{P}^2$ infinitely many times, we obtain infinitely many non-isomorphic but birational smooth varieties. In this case, every blow-up leads to the formation of a rigid (-1) curve.

The study of the birational geometry of surfaces started around Rome with the Italian school of algebraic geometry, a group of leading mathematicians who made major contributions in birational geometry, especially towards the classification of algebraic surface, roughly from 1885 to 1935. The leadership mainly belonged to G. Castelnuovo (1865-1952) , F. Enriques (1871-1946) and to F. Severi (1879-1961), who gave great contributions to the theory of algebraic surfaces.

The existence of infinitely many birational surfaces led to the need to look for a *special surface* in each equivalence class. So it was natural to ask if, in each birational class, we could identify a *simplest model*, called the *minimal model*, and if we could provide an algorithm or a procedure to find it.

The Italian school came up with the following classical definition and proved the theorems below, which are, for example, stated in [Bea83] and [Mat02].

Definition 1.0.4. A smooth surface S is minimal if any birational morphism $\psi : S \to T$ to another smooth surface T is an isomorphism.

Theorem 1.0.5. *Any smooth surface* S *has a minimal model.*

We denote by $\kappa(X)$ the Kodaira dimension of a variety X. The following theorem gives a description of the minimal models for surfaces.

Theorem 1.0.6. *Let* S *be a smooth surface:*

1. *If* $\kappa(S) \geq 0$*, then the minimal model of* S *is unique.*

2. *If* $\kappa(S) < 0$*, the minimal model is not unique. In this case* S *is birationally equivalent to* $\mathbb{P}^1 \times C$*, for some smooth curve* C*, and we say that* S *is birationally ruled.*

The building block of classical birational geometry of surfaces is the following criterion, due to Castelnuovo which essentially describes the process of constructing a minimal model of any surface.

Theorem 1.0.7. *Let S_1 be a smooth surface and E a smooth rational curve on S_1 with $E^2 = -1$. Then there exists a birational morphism $\phi : S_1 \to S_2$, where S_2 is a smooth surface and ϕ is the blow up of S_2 with exceptional divisor E.*

It is worth pointing out some useful remarks:

Remark 1. *Let us denote by S_{min} the minimal model of S. The canonical bundle of S_{min} has a very interesting property: its intersection with curves is positive. In geometry, this is equivalent to the following property: the canonical divisor of the minimal surface, $K_{S_{min}}$, is nef.*

One of the major obstacles to the generalization of Castelnuovo's criterion to higher dimensions was the lack of a "good" analogue to the notion of minimal model; the classical definition 1.0.4 does not make sense even in dimension three (and, moreover, there is no notion of (-1) curve on a three-fold)!

So, it was not clear which curves one needed to contract in order to follow Castelnuovo's procedure.

It took many decades to understand how to attach the problem for varieties of dimension three or higher. It was by the middle 1980's, that a group of famous mathematicians, such as Kawamata, Kollar, Miyaoka, Mori, Reid, Shokurov, Viehweg and many others proposed a generalization of the minimal model program to higher dimensions. See for details [KM98]. The minimal model program was successfully completed by Mori, in [Mor88], for three-folds.

One of the ingenious insights of Mori was to find the right definition for a variety X to be a minimal model.

Definition 1.0.8. A variety X is minimal if its canonical divisor K_X is nef.

The key idea of Mori's algorithm in all dimensions is to contract all curves C that have negative intersection with K_X. These curves span an extremal ray of the Mori cone, which will be introduced soon. For the purpose of generalizing the minimal model program, Mori introduced new definitions concerning the birational geometry of X.

One of his fundamental results is the *Mori cone theorem*, which describes the cone of effective curves on a smooth projective variety. The cone of effective curves is denoted by $\overline{\mathrm{NE}}(X)$ and it is also called *Mori cone*.

Theorem 1.0.9. *Let X be a smooth projective variety. Then there exist countably many extremal rays R_i of the cone $\mathrm{NE}(X)$ such that $K_X \cdot R_i < 0$ and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum R_i.$$

In particular, for every ample $\mathbb{Q}$-divisor H on X, there exist finitely many such rays R'_i with

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+H\geq 0} + \sum R'_i.$$

Moreover, the rays R_i are discrete in the half-space $\mathrm{NE}(X)_{K_X} < 0$.

For further details, see [Laz04a]. For a complete proof of Theorem 1.0.9 see [KM98] and [Mor82].

Hence, Mori proved that in the closed half-space where K_X is nonnegative, we know nothing, but in the complementary half-space, the cone is spanned by some countable collection of curves which are quite special: they are rational, and their 'degree' is bounded very tightly by the dimension of X.

One of the main problem that appears when we contract extremal rays is that we may run into non smooth varieties. In order to fix this problem, Mori introduced the concept of terminal singularity and, in his minimal model program, allowed also varieties with this mild type of singularities.

The process of the minimal model program encounters difficulties. These issues motivates the introduction of the notion of flip, which can be described as a special codimension-2 surgery, when the dimension is three.

In 1988, Mori's result on existence of flips explained in [Mor88], completed the algorithm for constructing minimal models for threefolds.

In 2010 there was a great breakthrough in the minimal model theory when C. Birkar, P. Cascini, C. Hacon and J. McKernan. In [BCHM08] they proved the existence of minimal models for a large class of varieties, called of general type.

There still remains tons of open problems in the theory of minimal model program: for example, extending the minimal models for many varieties of special type, or proving the termination of flips in dimension higher than four.

Strictly related to the theory of Mori's minimal model program is the notion of *Mori dream space*.

Mori dream spaces were introduced in [HK00] by Y. Hu and S. Keel. They are so called, since they behave very well from the point of view of Mori's minimal model program.

Roughly speaking, a Mori dream space is a projective variety X, whose cone of effective divisors $\mathrm{Eff}(X)$ is rational and polyhedral and admits a well-behaved decomposition into convex sets, called Mori chambers. These chambers are the nef cones of the birational models of X.

From an algebraically point of view, Mori dream spaces can be characterized as varieties whose total coordinate ring, called *Cox ring*, is finitely generated. For this reason, they can be also seen as a natural Mori theoretic generalization of toric varieties. The motivation for introducing the Cox ring of a variety is the following.

The coordinate ring of a variety is a fundamental object in algebraic geometry. But, given a projective variety X, there is no canonical candidate for its coordinate ring, since it depends on the embedding of the variety into a projective space. More precisely, it depends on a choice of a very ample line bundle and a generating set of sections. An attempt to fix this issue was done by Hu and Keel in [HK00], who introduced the *Cox ring*, or *total coordinate ring* of a variety, which is a generalization of Cox's previous construction of a coordinate ring for toric varieties, explained in [Cox95].

We recall the definition of the Cox ring.

Definition 1.0.10. Let X be a normal $\mathbb{Q}$-factorial variety, with free and finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup G of the group of Weil divisors on X such that the canonical map $G \to \mathrm{Cl}(X)$, mapping a divisor $D \in G$ to its class $[D]$, is an isomorphism. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where $D \in G$ represents $[D] \in \mathrm{Cl}(X)$, and the multiplication in $\mathrm{Cox}(X)$ is defined by the standard multiplication of homogeneous sections in the field of rational functions on X.

We remark that this ring need not be finitely generated in general. However, for a variety X, having finite generated Cox ring has important consequences on its birational geometry.

As an example of varieties with non-finitely generated Cox ring, or, equivalently, which are not Mori dream spaces, we mention the following result of Mukai [Muk04]:

Theorem 1.0.11. *Let X be the blow-up of the projective space $\mathbb{P}^{r-1}$ in n points in general position. Then $\mathrm{Cox}(X)$ is not finitely generated if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \leq 1.$$

Examples of Mori dream spaces include flag varieties, projective toric varieties, spherical varieties and smooth Fano varieties.

In addition to this algebraic characterization there are several algebraic varieties characterized by some positivity property of the anti-canonical divisor, such as *weak Fano* and *log Fano* varieties. We recall briefly their definitions.

Definition 1.0.12. Let X be a normal variety and $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-divisor. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X, D) and write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i - \tilde{D}$$

The pair (X, D) is *Kawamata log terminal* (klt) if $a_i > -1$ and $d_j < 1$ for any i, j.

Definition 1.0.13. Let X be a smooth projective $\mathbb{Q}$-factorial variety. We say that X is *weak Fano* if $-K_X$ is nef and big and X is *log Fano* if there exists an effective divisor D such that $-(K_X + D)$ is ample and the pair (X, D) is klt.

The following proposition explain how these varieties are related to Mori dream spaces.

Proposition 1.0.14. *[BCHM08, Corollary* 1.3.2*] Let X be a smooth projective variety. If X is log Fano then X is a Mori dream space .*

If we want to proceed in the study of the birational geometry of a normal projective variety X, it is very useful to learn something about its cones of curves and of divisors. These cones are defined for Cartier divisors and for $1-$ cycles up to an equivalence relation. See [Laz04a] for details.

Definition 1.0.15. We denote by $\mathrm{N}^1(X)$ the vector space of Cartier divisors, with real coefficients, up to numerical equivalence. Dually, we denote by $\mathrm{N}_1(X)$ the vector space of $1-$cycles, again with real coefficients and up to numerical equivalence.

The vector space $\mathrm{N}^1(X)$ contains three important cones. The effective cone is the convex cone spanned by effective divisors, denoted $\mathrm{Eff}(X)$. It is not in general closed. The nef cone $\mathrm{Nef}(X)$ is the cone of classes of divisors in $\mathrm{N}^1(X)$ having nonnegative intersection with all curves in X. This is closed by definition, but it is not in general rational or polyhedral. Finally, the movable cone is the convex cone in $\mathrm{N}^1(X)$, spanned by the classes of movable divisors. The Mori cone $\mathrm{NE}(X)$, already introduced for Theorem 1.0.9, lies inside $\mathrm{N}_1(X)$. Understanding the structure of these cones is therefore a basic problem in algebraic geometry. Especially, one of the goal of birational geometry is to determine when a variety has closed or polyhedral cones.

For example, Fano varieties are algebraic varieties with anticanonical divisor $-K_X$ is ample. Then, by Theorem 1.0.9 they have rational polyhedral nef cones. They also have rational polyhedral effective cones by [BCHM08].

For Mori dream space, the situation is ideal: $\mathrm{Nef}(X)$ and $\mathrm{NE}(X)$ are rational polyhedral cones. In particular, $\mathrm{NE}(X)$ is closed. Also $\mathrm{Mov}(X)$ and $\mathrm{Eff}(X)$ are rational polyhedral cones.

In general, to determine whether a variety is a Mori dream space or not, and in case to study in detail its Mori chamber decomposition is a hard problem. This has been done for instance when X is obtained by blowing-up points in a projective space [Muk04], [CT06], [AM16], [AC17], [BM21], [LP17].

In [CT06, Theorem 1.3], A. M. Castravet and J. Tevelev proved that the blow-up of $(\mathbb{P}^n)^s$ in r general points is a Mori dream space if and only if

$$\frac{1}{r+1} + \frac{1}{s-n-1} + \frac{1}{n+1} > 1.$$

Indeed, when the above inequality is not satisfied the effective cones of these blow-ups are not finitely generated and the proof of this last fact relies on the symmetries of their Picard groups which carry a natural Weyl group action. For products of projective spaces with unbalanced dimensions this is not the case. In this thesis we push a little further the investigation of the birational geometry of these varieties initiated by T. Grange, E. Postinghel and A. Prendergast-Smith in [GPPS22].

In particular, T. Grange, E. Postinghel and A. Prendergast-Smith focussed on blow-ups of $\mathbb{P}^1 \times \mathbb{P}^2$ and of $\mathbb{P}^1 \times \mathbb{P}^3$ in sets of up to six points in very general position. Their main result in [GPPS22, Sections $3-5$] is the explicit descriptions of the cones of effective divisors on these varieties and the description of the geometry of the generating classes. Since these varieties lack of simmetry, they introduced techniques such as induction, restriction and base loci lemmas.

Since being log Fano implies being a Mori dream space, it was natural for the authors to ask whether varieties of the above kind are log Fano. We recall that for blowups of $\mathbb{P}^2$, being log Fano is equivalent to being a Mori dream space and for these cases the Cox rings were described by Batyrev–Popov in [BP04]. More recently, Araujo–Massarenti [AM16] and Lesieutre–Park [LP17] proved that the same holds in higher dimension, namely that blow-ups of $\mathbb{P}^n$ or of products of the form $(\mathbb{P}^n)^m$ in points in very general position are log Fano if and only if they are Mori dream spaces.

It is therefore natural to ask which of the mixed products

$$X_s^{m,n} =: \mathrm{Bl}_s(\mathbb{P}^m \times \mathbb{P}^n)$$

are weak Fano or log Fano.

T. Grange, E. Postinghel and A. Prendergast-Smith show that

Theorem 1.0.16. *[GPPS22, Theorem 6.3] The variety $X_s^{1,2}$ is weak Fano if and only if $s \leq 6$.*

and

Theorem 1.0.17. *[GPPS22, Corollary 6.7] For $s \leq 6$, the variety $X_s^{1,3}$ is log Fano.*

Since weak Fano varieties are log Fano, by Corollary 1.0.14, the varieties $X_s^{1,2}$ and $X_s^{1,3}$ are Mori dream spaces for $s \leq 6$.

In chapter 2 of this thesis we give an overview on the theory of Cox rings, Mori dream spaces and log Fano varieties. In the first sections we give the definitions of the various cones inside $N_1(X)$ and $N^1(X)$ and their inclusion relations. We then give a description of the Cox ring of a variety equipped with an algebraic torus action, following the treatment in [HS10]. We conclude the chapter with the result that permits to find generators for the moving cone of a variety from the generators of its Cox ring, following [ADHL15, Proposition 3.3.2.3]. It will follow an explanation on the main results concerning Mori dream spaces and log Fano varieties, and many examples. Finally, we introduce the main object of study of this thesis: the variety $X_r^{1,n}$, for $r = n+1, n+2$ and for $n+3$ if $n \leq 4$.

In chapter 3 we compute the cone of effective curves of $X_r^{1,n}$, for $r = n+1, n+2$ and for $r = n+3$ if $n \leq 4$. We then prove that $X_r^{1,n}$ is log Fano for $r \leq n+1$.

In chapter 4, we compute generators and relations of the Cox ring of $X_{n+1}^{1,n}$. We then use these generators to compute generators of the moving cone of $X_{n+1}^{1,n}$. In order to do the computation, we wrote some scripts on Maple and Magma, some of which are provided in chapter 6. At the end if chapter 4 we compute the nef cones of $X_r^{1,n}$ for $r = n+1, n+2$ and for $r = n+3$ when $n \leq 4$. Then, in chapter 5 we also give a Mori chamber decomposition of $X_{n+1}^{1,n}$ in Magma and we display the case $n = 2$.

Chapter 2

Cox rings, Mori dream spaces and log Fano varieties

In this chapter we introduce the necessary preliminaries for the rest of the treatment. In the first section we give definitions of the various cones lying inside $\mathrm{N}_1(X)$ and $\mathrm{N}^1(X)$. In the second section we introduce the notion of Cox ring of a variety and its characterization when X is equipped with a torus action. Then we give an important result about the relation between generators of the Cox ring and generators of the moving cone of X.

In the third and in the fourth section we introduce Mori dream spaces, weak Fano varieties and log Fano varieties, as well as the theorems that relates the three properties.

Finally, in the last section, we introduce the varieties which are the main object of this thesis.

2.1 Cones of divisors and of curves

Let X be a normal projective $\mathbb{Q}$-factorial variety. Denote by $\mathrm{Div}(X)$ the group of Cartier divisors on X. We refer to [Laz04a] for a comprehensive treatment of the topics that will be introduced hereafter. Of the various natural equivalence relations defined on $\mathrm{Div}(X)$, we will deal with the weakest:

Definition 2.1.1. Two Cartier divisors

$$D_1, D_2 \in \mathrm{Div}(X)$$

are *numerically equivalent*, written

$$D_1 \equiv_{num} D_2$$

if

$$(D_1 \cdot C) = (D_2 \cdot C),$$

for every irreducible curve $C \subset X$ or, equivalently, if $(D_1 \cdot \gamma) = (D_2 \cdot \gamma)$ for all one-cycles γ on X.

The multiplication is the intersection number between the curve C and the divisor D.

A divisor is *numerically trivial* if it is numerically equivalent to zero, and

$$\mathrm{Num}(X) \subset \mathrm{Div}(X)$$

is the subgroup consisting of all numerically trivial divisors.

Definition 2.1.2. The Néron-Severi group of X is the quotient group

$$\mathrm{N}^1(X) = \mathrm{Div}(X)/\,\mathrm{Num}(X),$$

of numerical equivalence classes of divisors on X.

This group is finitely generated by the Néron–Severi theorem, which was proved by Severi over the complex numbers and by Néron over more general fields:

Proposition 2.1.3. *The Néron-Severi group* $\mathrm{N}^1(X)$ *is a free abelian group of finite rank. The rank of* $\mathrm{N}^1(X)$ *is called the Picard number of* X *and is written* $\rho(X)$.

We want to give an overview of the classical theory of ample divisors. We begin with the following definition:

Definition 2.1.4. Let X be a projective variety and L a line bundle on X.

1. L is *very ample* if there exists a closed embedding $X \subset \mathbb{P}$ of X into some projective space $\mathbb{P} := \mathbb{P}^N$ such that

$$L = \mathcal{O}_X(1) =: \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$

2. L is *ample* if $L^{\otimes m}$ is very ample for some $m > 0$.

A Cartier divisor D on X is ample or very ample if the corresponding line bundle $\mathcal{O}_X(D)$ is so, and a numerical equivalence class in $\mathrm{N}^1(X)$ is *ample* if it is the class of an ample divisor.

Definition 2.1.5. A divisor $D \subset X$ is *numerically effective (nef)* if

$$(D \cdot C) \geq 0,$$

for all irreducible curves $C \subset X$.

This definition only depends on the numerical equivalence class of D, so one has a notion of nef classes in $\mathrm{N}^1(X)$.

As a matter of terminology, if V is a finite dimensional real vector space, a *cone* in V is a set $K \subseteq V$ stable under multiplication by positive scalars. We denote with $\mathrm{Eff}(X)$ the convex cone in $\mathrm{N}^1(X)$ spanned by classes of effective divisors; Its closure $\overline{\mathrm{Eff}(X)}$ is the convex cone of pseudoeffective divisors.

In general, $\mathrm{Eff}(X)$ needs not to be open or closed, but for the varieties treated in chapter 3 it will be rational polyhedral, in particular closed.

We start by defining cones and their relations in $\mathrm{N}^1(X)$.

Definition 2.1.6. The *ample cone*

$$\mathrm{Amp}(X) \subset \mathrm{N}^1(X)$$

of X is the convex cone of classes of ample divisors on X. The *nef cone*

$$\mathrm{Nef}(X) \subset \mathrm{N}^1(X),$$

is the convex cone of classes of nef divisors on X.

Remark 2. *As soon as* $\dim(\mathrm{N}^1(X)) \geq 3$, *the structure of these cones can become quite complicated. For example, they may or may not be polyhedral.*

We view $\mathrm{N}^1(X)$ as a finite dimensional vector space with its standard Euclidean topology. This allows us in particular to discuss closures and interiors of sets of numerical equivalence classes of divisors.

Theorem 2.1.7. *([Laz04a, Theorem* 1.4.23*]) Let X be any projective variety. Then:*

1. *The nef cone is the closure of the ample cone:*

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}.$$

2. *The ample cone is the interior of the nef cone:*

$$\mathrm{Amp}(X) = \mathrm{Int}(\mathrm{Nef}(X)).$$

´

Now we introduce the notion on movable divisor on X and of moving cone of X.

Definition 2.1.8. Let $D \subseteq X$ be a divisor. The *stable base locus* $\mathbb{B}(D)$ of the divisor D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$\mathbb{B}(D) = \bigcap_{s>0} B(sD).$$

Definition 2.1.9. A divisor $D \subset X$ is *movable* if its stable base locus has codimension at least two in X.

The *moving cone* of X is the convex cone $\mathrm{Mov}(X) \subset \mathrm{N}^1(X)$ generated by classes of movable divisors.

We have inclusions

$$\mathrm{Nef}(X) \subset \overline{\mathrm{Mov}(X)} \subset \overline{\mathrm{Eff}(X)}.$$

Dually to the vector space $\mathrm{N}^1(X)$, we introduce $\mathrm{N}_1(X)$, the group of 1-cycles on X modulo numerical equivalence, tensored with $\mathbb{R}$. First, we define the numerical equivalence classes of curves.

Definition 2.1.10. We denote by $\mathrm{Z}_1(X)$ the $\mathbb{R}$-vector space of real one-cycles on X, consisting of all finite $\mathbb{R}$-linear combinations of irreducible curves on X. An element $\gamma \in \mathrm{Z}_1(X)$ is a formal sum

$$\gamma = \sum_i a_i \cdot C_i,$$

where $a_i \in \mathbb{R}$ and $C_i \subseteq X$ is an irreducible curve. Two one-cycles

$$\gamma_1, \gamma_2 \in \mathrm{Z}_1(X)$$

are *numerically equivalent* if

$$(D \cdot \gamma_1) = (D \cdot \gamma_2),$$

for every $D \in \mathrm{Div}(X)$. The corresponding vector space of numerical equivalence classes of one-cycles is written $\mathrm{N}_1(X)$. Thus by construction one has a perfect pairing

$$\mathrm{N}^1(X) \times \mathrm{N}_1(X) \to \mathbb{R}$$
$$(\delta, \gamma) \mapsto (\delta \cdot \gamma) \in \mathbb{R}.$$

In particular, $\mathrm{N}_1(X)$ is a finite dimensional real-vector space on which we put the standard Euclidean topology.

The relevant cones in $\mathrm{N}_1(X)$ are those spanned by effective curves:

Definition 2.1.11. The *cone of curves* (or *Mori cone*) of X,

$$\mathrm{NE}(X) \subseteq \mathrm{N}_1(X)$$

is the cone spanned by the classes of all effective one-cycles on X. Concretely,

$$\mathrm{NE}(X) = \{\sum_i a_i[C_i] \mid C_i \subset X \text{ an irreducible curve, } a_i \geq 0\}.$$

Its closure $\overline{\mathrm{NE}(X)} \subset \mathrm{N}_1(X)$ is the closed cone of *pseudo-effective* curves on X.

We recall the definition of dual cone.

Definition 2.1.12. Suppose that $K \subset V$ is a closed convex cone in a finite dimensional real vector space. The *dual* of K is defined to be the cone in V^* given by

$$K^* = \{\phi \in V^* \mid \phi(x) \geq 0 \ \forall x \in K\}.$$

An important fact is that the cone of pseudoeffective curves $\overline{\mathrm{NE}}(X)$ and the nef cone $\mathrm{Nef}(X)$ are dual. This is the content of the following proposition:

Proposition 2.1.13. *[Laz04a, Proposition* 1.4.28*] The Mori cone* $\overline{\mathrm{NE}}(X)$ *is the closed cone dual to* $\mathrm{Nef}(X)$*, i.e.*

$$\overline{\mathrm{NE}}(X) = \{\gamma \in \mathrm{N}_1(X) \mid (\gamma \cdot \delta) \geq 0, \quad \forall \delta \in \mathrm{Nef}(X)\}.$$

Proof. The duality theorem for cones states that under the natural identification of V^{**} with V, one has $K^{**} = K$. In the situation at hand, take $V = \mathrm{N}_1(X)$ and $K = \overline{\mathrm{NE}}(X)$. Then $\mathrm{Nef}(X) = \overline{\mathrm{NE}}(X)^*$ by definition. Consequently $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)^*$, which concludes the proof. ☐

The notion of extremality will often appear throughout the thesis:

Definition 2.1.14. Let $K \subset V$ be a closed convex cone in a finite dimensional real vector space. An *extremal ray* $r \subseteq K$ is a one-dimensional subcone having the property that if

$$v + w \in r,$$

for some vectors $v, w \in K$, then necessarily $v, w \in r$.

An extremal ray is contained in the boundary of K.

Amplitude of divisors on X can be also characterized by the cone of effective curves on X, in a similar way as the notion of nef divisor.

Consider the following set:

$$D_{>0} = \{\gamma \in \mathrm{N}_1(X) \ \mid (D \cdot \gamma) > 0\},$$

which determines an open halfspace in $\mathrm{N}_1(X)$.

Theorem 2.1.15. *[Laz04a, Theorem 1.4.29] Let X be a projective variety, and let D be an divisor on X. Then D is ample if and only if*

$$\overline{\mathrm{NE}(X)} \setminus \{0\} \subseteq D_{>0}.$$

For a proof we refer to [Laz04a].

Now we give some examples, taken from the lecture notes [Deb11], of varieties and of their cones of effective curves.

Example 2.1.16. There is an isomorphism

$$\mathrm{N}_1(\mathbb{P}^n) \to \mathbb{R}$$

given by the assignement

$$\sum_i \lambda_i [C_i] \longmapsto \sum_i \lambda_i \deg(C_i).$$

So

$$\mathrm{NE}(\mathbb{P}^n) \cong \mathbb{R}^+.$$

Example 2.1.17. If X is a smooth quadric in $\mathbb{P}^3$, and C_1 and C_2 are lines in X which meet, the relations $(C_1 \cdot C_2) = 1$ and $(C_1 \cdot C_1) = (C_2 \cdot C_2) = 0$ imply that the classes $[C_1]$ and $[C_2]$ are independent in $\mathrm{N}_1(X)$. In fact,

$$\mathrm{N}_1(X) = \mathbb{R}[C_1] \oplus \mathbb{R}[C_2]$$

and

$$\mathrm{NE}(X) = \mathbb{R}^+[C_1] \oplus \mathbb{R}^+[C_2].$$

For further details we also refer to [Deb01].

Although the cone $\mathrm{NE}(X)$ is closed in each of the examples above, this is not always the case as we will see in example 2.2.7.

2.2 Cox rings

Cox rings are significant global invariants of algebraic varieties, naturally generalizing homogeneous coordinate rings of projective spaces.

They were first introduced by D. A. Cox for toric varieties [Cox95], and then his construction was generalized to projective varieties in [HK00].

We recall the definition of the Cox ring of an algebraic variety X.

Definition 2.2.1. Let X be a normal $\mathbb{Q}$-factorial variety, with free and finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup G of the group of Weil divisors on X such that the canonical map $G \to \mathrm{Cl}(X)$, mapping a divisor $D \in G$ to its class $[D]$, is an isomorphism. The *Cox ring* of X is defined as

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} H^0(X, \mathcal{O}_X(D))$$

where $D \in G$ represents $[D] \in \mathrm{Cl}(X)$, and the multiplication in $\mathrm{Cox}(X)$ is defined by the standard multiplication of homogeneous sections in the field of rational functions on X.

We remark that this ring need not be finitely generated in general. However, we will see later that for a variety X, having finitely generated Cox ring has important consequences on its birational geometry.

For example, the Cox ring being finitely generated means that the effective and the nef cone are both polyhedral (and hence finitely generated), see [HK00]. In general the contrary is not true.

As an example of varieties with non-finitely generated Cox ring, we show the following result of Mukai [Muk04]:

Theorem 2.2.2. *Let X be the blow-up of the projective space $\mathbb{P}^{r-1}$ in n points in general position. Then $\mathrm{Cox}(X)$ is not finitely generated if and only if*

$$\frac{1}{2} + \frac{1}{r} + \frac{1}{n-r} \leq 1.$$

From this theorem we can deduce that we need to blow-up $n \geq 9$ general points in $\mathbb{P}^2$ and $n \geq 8$ general points in $\mathbb{P}^3$ to find an infinite Cox ring. We will return later on blow-ups of $\mathbb{P}^2$ in general points.

One of the main concerns about the theory of Cox rings is to determine, in the case of finitely generated $\mathrm{Cox}(X)$, explicit generators and relations. More precisely, in this setting we would like to display $\mathrm{Cox}(X)$ as a quotient

$$\mathrm{Cox}(X) = k[x_1, \ldots, x_n]/I_X.$$

Here we consider the natural $\mathrm{Pic}(X)$-grading on $k[x_1, \ldots, x_n]$ and I_X given by letting $\deg(x_i) = D_i$, so that $\mathrm{Cox}(X)$ is in fact a multigraded ring.

An important example is the calculation of the Cox ring of Del Pezzo surfaces, which is the content of the Batyrev-Popov conjecture [BP04]. It is shown that describing the behaviour of the Cox ring under blow-ups is a highly nontrivial problem. The ideals of relations quickly become very complicated, and computer calculations are infeasible.

Examples of varieties for which the Cox ring is simple are toric varieties, since their Cox ring is a polynomial ring.

A stronger claim is true:

Proposition 2.2.3. *[HK00, Corollary* 2.10*] Let X be a smooth projective variety with* $\mathrm{Pic}(X)_{\mathbb{Q}} = \mathrm{N}_1(X)$*. Then X is a toric variety if and only if it has a Cox ring that is a polynomial ring.*

After toric varieties, the next step is to study varieties whose Cox rings have a unique defining relation. Some examples of such spaces are given in [BH04] and [Der13]. Other than this, few actual computations of Cox rings have been carried out.

Another example of varieties with non finitely generated Cox ring is given by the following lemma:

Lemma 2.2.4. *Let X be a surface containing an infinite number of curves of negative self-intersection. Then* $\mathrm{Cox}(X)$ *is not finitely generated.*

Proof. Since for surfaces, the effective cone $\mathrm{Eff}(X)$ coincides with $\mathrm{NE}(X)$ and since having finitely generated effective cone is a necessary condition for the Cox ring to be finitely generated, it suffices show that $\mathrm{NE}(X)$ is not finitely generated.

Suppose that the classes of the curves $C_1, \dots, C_N$ generate $\mathrm{NE}(X)$. Let E be a curve on X with negative self-intersection. Then

$$E \sim \sum_i m_i C_i,$$

for $m_i \geq 0$, since E is effective. Note that

$$E^2 = \sum_i m_i (C_i \cdot E).$$

The right-hand side can only be negative if some $C_i \cdot E < 0$, so E is a component of C_i.

Since each of the C_i can only have finitely many fixed components, and since there is a finite number of C_i, this contradicts the assumption that X had infinitely such E. □

A particular type of curve with negative self-intersection is given by:

Definition 2.2.5. A curve E on X is called an exceptional curve (of the first kind) if it is smooth and rational and $E^2 = -1$.

The following is a result concerning the position of exceptional curves in the closure of the Mori cone of a smooth projective surface.

Proposition 2.2.6. *Let X be a smooth projective surface. We have that:*

1. *The class of an irreducible curve C with* $(C^2) \leq 0$ *is in* $\partial \overline{\mathrm{NE}}(X)$*.*

2. *The class of an irreducible curve C with $(C^2) < 0$ spans an extremal ray of $\overline{\mathrm{NE}}(X)$.*

Proof. Assume $(C^2) = 0$; then $[C]$ has non-negative intersection with the class of any effective divisor, hence with any element of $\overline{\mathrm{NE}}(X)$. Let H be an ample divisor on X. If $[C]$ is in the interior of $\overline{\mathrm{NE}}(X)$, so is $[C] + t[H]$ for all t small enough; this implies

$$0 \leq (C \cdot (C + tH)) = t(C \cdot H),$$

for all t small enough, which is absurd since $(C \cdot H) > 0$. Assume now $(C^2) < 0$ and $[C] = z_1 + z_2$, where z_i is the limit of a sequence of classes of effective $\mathbb{Q}$-divisors $D_{i,m}$. Write

$$D_{i,m} = a_{i,m}C + D^{'}_{i,m}$$

with $a_{i,m} \geq 0$ and $D'_{i,m}$ effective with $(C \cdot D'_{i,m}) \geq 0$. Taking intersections with H, we see that the upper limit of the sequence $(a_{i,m})_m$ is at most 1, so we may assume that it has a limit a_i. In that case, $([D'_{i,m}])_m$ also has a limit $z'_i = z_i - a_i[C]$ in $\overline{\mathrm{NE}}(X)$ which satisfies $C \cdot z'_i \geq 0$. We have then $[C] = (a_1 + a_2)[C] + z'_1 + z'_2$, and by taking intersections with C, we get $a_1 + a_2 \geq 1$. But

$$0 = (a_1 + a_2 - 1)[C] + z'_1 + z'_2$$

and since X is projective, this implies $z'_1 = z'_2 = 0$ and proves 1. and 2.. □

We present a classical example due to Nagata [Nag60] of varieties with infinitely many exceptional curves.

Example 2.2.7. Let $p_1, \dots, p_9$ be points in $\mathbb{P}^2$ which are the nine base-points of a general pencil of cubics.

Let

$$\pi : X \to \mathbb{P}^2$$

be the blow-up of the plane in these points, and let $E_1, \dots, E_9$ be the exceptional divisors.

The anti-canonical system

$$|-K_X| = |3L - E_1 - \cdots - E_9|,$$

is base-point free, hence it defines a morphism

$$\phi : X \to \mathbb{P}^1.$$

The morphism ϕ is an elliptic fibration, hence its smooth fibers are elliptic curves. Among this sections, there are the nine exceptional curves

$$E_1, \dots, E_9.$$

If we choose E_1 as the origin, the smooth fibers of ϕ become abelian groups. Translations by elements E_i then generate a subgroup G of $\mathrm{Aut}(X)$ which can be shown to be isomorphic to $\mathbb{Z}^8$.

For each $\sigma \in G$, the curve $E_\sigma = \sigma(E_0)$ is rational with self-intersection -1 and $(K_X \cdot E_\sigma) = -1$. It follows from Proposition 2.2.6 that $\mathrm{NE}(X)$ has infinitely many extremal rays contained in the open half-space $N_1(X)_{K_X<0}$, which are not locally finite in a neighborhood of $K_X^\perp$, because $(K_X \cdot E_\sigma) = -1$, but $(E_\sigma)_{\sigma\in G}$ is unbounded since the set of classes of irreducible curves is discrete in $\mathrm{N}_1(X)$. So there are infinitely many exceptional curves and by 2.2.4, $\mathrm{Cox}(X)$ cannot be finitely generated. Suppose now that the points are in general position. Also here we get infinitely many exceptional curves. We give a proof of this based on the Cremona transformation, following an exercise in Hartshorne [Har77, V.4.15]. Suppose there are only finitely many exceptional curves. In particular there exists a divisor D with divisor class

$$aL - b_1E_1 - \cdots - b_9E_9,$$

with $b_1 \le b_2 \le \cdots \le b_9$ and maximal $a > 0$. With maximal we mean the curve of maximal degree among the finitely many exceptional ones. Consider the divisor class

$$\begin{aligned}\tilde{D} = (2a - b_1 - b_2 - b_3)L - (a - b_2 - b_3)E_1 - (a - b_1 - b_3)E_2 \\ -(a - b_1 - b_2)E_3 - b_4E_4 - \cdots - b_9E_9.\end{aligned}$$

This divisor class corresponds to the image of D after performing a Cremona transformation based at p_1, p_2, p_3 and in particular, $\tilde{D}$ is the class of an exceptional curve. We claim that $2a - b_1 - b_2 - b_3 > a$, so that $\tilde{D}$ has higher coefficient of L than D, contradicting the maximality of a. Suppose to the contrary that $a - b_1 - b_2 - b_3 \le 0$. Then

$$-K \cdot \tilde{D} = 3a - b_1 - \cdots - b_9$$

$$\le (a - b_1 - b_2 - b_3) + (a - b_1 - b_2 - b_3) + (a - b_1 - b_2 - b_3) \le 0.$$

This contradicts the genus formula since $-K \cdot \tilde{D} = 1$. Hence, $2a - b_1 - b_2 - b_3 > a$ and we are done.

In conclusion, the blow-up of $\mathbb{P}^2$ in nine point in general position $X = \mathrm{Bl}_9\,\mathbb{P}^2$ has non finitely generated Cox ring by Lemma 2.2.4 and the infinitely many exceptional curves spans infinitely many extremal rays of the effective cone by Proposition2.2.6.

2.2.1 Cox rings and torus actions

As we have seen in the previous section, a basic problem is to present $\mathrm{Cox}(X)$ in terms of generators and relations. The knowledge of generators and relations opens a combinatorial approach to geometric properties of X. We follow the treatment explained in [HS10], where J. Hausen and H. Süß investigate the Cox ring of a normal complete variety X, equipped with an algebraic torus action.

Let T be an algebraic torus and consider an effective algebraic torus action

$$T \times X \to X.$$

For a point $x \in X$, denote by $T_x \subset T$ its isotropy group

$$T_x := \{t \in T \mid t \cdot x = x\},$$

and consider the non-empty T-invariant open subset

$$X_0 := \{x \in X \mid \dim(T_x) = 0\} \subset X$$

of points of X with zero-dimensional isotropy group.

There is a geometric quotient

$$q : X_0 \to X_0/T$$

with an irreducible normal but possibly non-separated orbit space X_0/T, and also for X_0/T one can define a Cox ring.

Denote by

$$E_1, \ldots, E_m \subseteq X$$

the T-invariant prime divisors supported in $X \setminus X_0$ and by $D_1, \ldots, D_n \subseteq X$ those T-invariant prime divisors who have a finite generic isotropy group of order $l_j > 1$. Moreover, let 1_{E_k} and 1_{D_j} denote the canonical sections of the divisors E_k and D_j respectively, and let $1_{q(D_j)} \in \mathrm{Cox}(X_0 \setminus T)$ be the canonical section of $q(D_j)$.

Theorem 2.2.8. *[HS10, Theorem 1.1] There is a graded injection*

$$q^* : \mathrm{Cox}(X_0/T) \to \mathrm{Cox}(X)$$

of Cox rings and the assignments

$$S_k \mapsto 1_{E_k}$$

and

$$T_j \mapsto 1_{D_j}$$

induce an isomorphism of Cl(X)-graded rings

$$\mathrm{Cox}(X) \cong \mathrm{Cox}(X_0/T)[S_1, \ldots, S_m, T_1, \ldots, T_n]/\langle T_j^{l_j} - 1_{q(D_j)}; 1 \leq j \leq n\rangle,$$

where the Cl(X)-grading on the right hand side is defined by associating to S_k the class of E_k and to T_j the class of D_j. In particular, $\mathrm{Cox}(X)$ *is finitely generated if and only if* $\mathrm{Cox}(X_0/T)$ *is so.*

The complexity of a T-action on X is the codimension of a generic T-orbit in X. Suppose that the T-action on X is of complexity one. Then, the orbit space X_0/T is of dimension one and has a separation to $\mathbb{P}^1$, that is a rational map:

$$\pi : X_0/T \dashrightarrow \mathbb{P}^1,$$

which is a local isomorphism in codimension one.

Choose $r \geq 1$ and $a_0, \dots, a_r \in \mathbb{P}^1$, such that π is an isomorphism over $\mathbb{P}^1 \setminus \{a_0, \dots, a_r\}$ and all the divisors D_j occur among the

$$D_{i,j} := q^{-1}(y_{i,j}),$$

where $\pi^{-1}(a_i) = \{y_{i,1}, \dots, y_{i,n_i}\}$. Let $l_{ij} \in \mathbb{Z}_{\geq 1}$ denote the order of the generic isotropy group of $D_{i,j}$.
For every $0 \leq i \leq r$, define a monomial

$$f_i := T_{i,1}^{l_{i,1}} \cdots T_{i,n_i}^{l_{i,n_i}} \in \mathbb{C}[T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i].$$

Moreover, write $a_i = [b_i, c_i]$ with $b_i, c_i \in \mathbb{C}$ and for every $0 \leq i \leq r-2$, set $k = j + 1 = i + 2$ and define a trinomial

$$g_i := (c_k b_j - c_j b_k) f_i + (c_i b_k - c_k b_i) f_j + (c_j b_i - c_i b_j) f_k.$$

In this case the Cox ring is finitely generated and the following theorem gives the generators and the relations:

Theorem 2.2.9. *[HS10, Theorem 1.3] Let $T \times X \to X$ be an algebraic torus action of complexity one. Then, in terms of the data defined above, the Cox ring of X is given as*

$$\mathrm{Cox}(X) = \mathbb{C}[S_1, \dots, S_m, T_{i,j}; 0 \leq i \leq r, 1 \leq j \leq n_i]/\langle g_i; 0 \leq i \leq r-2 \rangle,$$

where 1_{E_k} corresponds to S_k, and $1_{D_{i,j}}$ to $T_{i,j}$, and the $\mathrm{Cl}(X)$-grading on the right hand side is defined by associating to S_k the class of E_k and to $T_{i,j}$ the class of $D_{i,j}$. In particular, $\mathrm{Cox}(X)$ is finitely generated.

Theorem 2.2.9 will be crucial in the study of the Cox ring of $X^{1,n}_{n+1}$.

2.2.2 Cox rings and moving cones

In this section we present the following result, which permits to find generators for the moving cone of X from generators of its Cox ring. We refer to [ADHL15] for further details regarding the Cox ring of a variety and their cones of effective, semiample and ample divisors.

Proposition 2.2.10. *[ADHL15, Proposition* 3.3.2.3*] Let X be an irreducible, normal, complete variety with finitely generated divisor class* $\mathrm{Cl}(X)$*. Let $\{f_i\}_{i\in I}$ be any system of pairwise non-associated* $\mathrm{Cl}(X)$*-prime generators for* $\mathrm{Cox}(X)$*. Then, the moving cone is given as:*

$$\mathrm{Mov}(X) = \bigcap_{i\in I} cone(\deg(f_j), j \in I \setminus \{i\}).$$

Note that for non toric varieties, whose Cox ring is not polynomial, this proposition tells us that relations among the generators for $\mathrm{Cox}(X)$ are not necessary to compute $\mathrm{Mov}(X)$.

In order to prove Proposition 2.2.10, we need the following lemma. We denote with $\mathrm{Cox}(X)_w$ the localization of the Cox ring at one of its elements.

Lemma 2.2.11. *[ADHL15, Lemma* 3.3.2.4*] Let X be an irreducible, normal complete variety with* $\mathrm{Cl}(X)$ *finitely generated and let $w \in \mathrm{Cl}(X)$ be effective. Then the following two statements are equivalent.*

1. *The stable base locus of the class $w \in \mathrm{Cl}(X)$ contains a divisor.*

2. *There exist an $w_0 \in \mathrm{Cl}(X)$ with $\dim \Gamma(X, \mathrm{Cox}(X)_{nw_0}) = 1$ for any $n \in \mathbb{Z}_{\geq 0}$ and an $f_0 \in \Gamma(X, \mathrm{Cox}(X)_{w_0})$ such that for any $m \in \mathbb{Z}_{\geq 1}$ and $f \in \Gamma(X, \mathrm{Cox}(X)_{mw})$ one has $f = f'f_0$ with some $f' \in \Gamma(X, \mathrm{Cox}(X)_{mw-w_0})$.*

Proof. The implication (2) $\implies$ (1) is obvious.

So, assume that 1 holds. The class $w \in \mathrm{Cl}(X)$ is represented by some nonnegative divisor D. Let D_0 be a prime component of D that occurs in the base locus of any positive multiple of D, and let $w_0 \in \mathrm{Cl}(X)$ be the class of D_0. Then the canonical section of D_0 defines an element $f_0 \in \Gamma(X, \mathrm{Cox}(X)_{w_0})$ that by [ADHL15, Proposition 1.5.3.5] divides any $f \in \Gamma(X, \mathrm{Cox}(X)_{mw})$, where $m \in \mathbb{Z}_{\geq 1}$. Note that $\Gamma(X, \mathrm{Cox}(X)_{nw_0})$ is of dimension one for every $n \in \mathbb{Z}_{\geq 1}$, because otherwise $\Gamma(X, \mathrm{Cox}(X)_{na_0w_0})$ where $a_0 > 0$ is the multiplicity of D_0 in D, would provide enough sections in $\Gamma(X, \mathrm{Cox}(X)_{nw})$ to move na_0D_0. □

Now we can give the proof of Proposition 2.2.10.

Proof. Set $w_i := \deg(f_i)$.

Denote by $I_0 \subseteq I$ the set of indices with $\dim \Gamma(X, \mathrm{Cox}(X)_{nw_0}) \leq 1$ for all $n \in \mathbb{N}$. Let $w \in \mathrm{Mov}(X)$. Then, lemma 2.2.11 tells us that for any $i \in I_0$, there must be a monomial of the form $\prod_{j\neq i} f_j^{n_j}$ in some $\Gamma(X, \mathrm{Cox}(X)_{nw})$. Consequently, w lies in the cone of the right-hand side. Conversely, consider an elementw of the cone of the right-hand side. Then, for every $i \in I_0$, a product $\prod_{j\neq i} f_j^{n_j}$ belongs to some $\Gamma(X, \mathrm{Cox}(X)_{nw})$. Hence none of the $f_i, i \in I_0$ divides all elements of $\Gamma(X, \mathrm{Cox}(X)_{nw})$. Again by lemma 2.2.11, we conclude $w \in \mathrm{Mov}(X)$. □

We are now able to apply Proposition 2.2.10 to some easy varieties.

First of all, a very naive example, which does not require any help from Maple:

Example 2.2.12. Let us consider the n-dimensional projective space $\mathbb{P}^n$. Then,

$$\mathrm{Cox}(\mathbb{P}^n) = \mathbb{C}[x_1, \dots, x_{n+1}].$$

Moreover, we know that

$$\mathrm{Mov}(\mathbb{P}^n) = \langle H \rangle,$$

where H is the class of an hyperplane. We see that in the setting of Proposition 2.2.10, a system $\{f_i\}i \in I$ of pairwise non-associated $\mathrm{Cl}(X)$-prime generators for $\mathrm{Cox}(X)$ is given by $\{x_i\}$ of $\deg(x_i) = H$ for each $i = 1, \dots, n+1$. Hence, we obtain the right generating set for $\mathrm{Mov}(X)$.

In the following examples, we will denote by H the pull-back of a line in $\mathbb{P}^2$ and by E_i is the class of the exceptional curve over the point p_i.

Example 2.2.13. In this example we want to compute the moving cone of the blow-up of the plane $\mathbb{P}^2$ at a point. It is known that $\mathrm{Bl}_p \mathbb{P}^2$ is a toric variety and hence its Cox ring coincides with the usual homogeneous coordinate ring

$$\mathrm{Cox}(\mathrm{Bl}_p(\mathbb{P}^2)) = \mathbb{C}[x, s_1, s_2, e],$$

where $\deg(x) = H$, $\deg(s_i) = H - E$ and $\deg(e) = E$. We will use a script written in Maple in order to use formula of Proposition 2.2.10. First, we construct the following matrix:

$$A := \begin{matrix} H & E \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 1 & -1 \end{bmatrix},$$

where coefficients of the first column correspond to H and those of the second column to E. Now, we just need to put the transpose of A into the script shown at page 74.

It is found that:

$$\mathrm{Mov}(\mathrm{Bl}_p(\mathbb{P}^2)) = \langle H, H - E \rangle.$$

In fact, we can notice that since $H - E$ is base-point free, it is a nef divisor and hence movable.

In the following examples, we want to compute the moving cone of $X = \mathrm{Bl}_2(\mathbb{P}^n)$, for low numbers of n, starting from the generators of $\mathrm{Cox}(\mathrm{Bl}_2(\mathbb{P}^n))$. Even if the generators for $\mathrm{Mov}(X)$, when $n > 2$, will be shown in example 2.3.7, here we give a different point of view, using the Cox ring and adding also the case $n = 2$.

Example 2.2.14. Let $X = \mathrm{Bl}_2(\mathbb{P}^2)$. It is known that X is a toric variety and hence its Cox ring coincides with the usual homogeneous coordinate ring

$$\mathrm{Cox}(\mathrm{Bl}_2(\mathbb{P}^2)) = \mathbb{C}[x, s_1, s_2, e_1, e_2],$$

where $\deg(x) = H - E_1 - E_2$, $\deg(s_i) = H - E_i$ and $\deg(e_i) = E_i$. So we construct the following matrix:

$$\begin{array}{ccc} H & E_1 & E_2 \end{array}$$

$$A := \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where coefficients of the first column correspond to H, those of the second column to E_1 and those of the third column to E_2. Now, we just need to put the transpose of A into the script shown at page 75, and we find:

$$\mathrm{Mov}(\mathrm{Bl}_2(\mathbb{P}^2)) = \langle H - E_1, H - E_2, H \rangle.$$

Example 2.2.15. Let $X = \mathrm{Bl}_2(\mathbb{P}^3)$. It is known that X is a toric variety and hence its Cox ring coincides with the usual homogeneous coordinate ring

$$\mathrm{Cox}(\mathrm{Bl}_2(\mathbb{P}^3)) = \mathbb{C}[x, y, s_1, s_2, e_1, e_2],$$

where $\deg(x) = \deg(y) = H - E_1 - E_2$, $\deg(s_i) = H - E_i$ and $\deg(e_i) = E_i$. So we construct the following matrix:

$$\begin{array}{ccc} H & E_1 & E_2 \end{array}$$

$$A := \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where coefficients of the first column correspond to H, those of the second column to E_1 and those of the third column to E_2. Now, we just need to put the transpose of A into the script shown at page 76, and we find:

$$\mathrm{Mov}(\mathrm{Bl}_2(\mathbb{P}^2)) = \langle H - E_1 - E_2, H - E_1, H - E_2, H \rangle.$$

Example 2.2.16. Let $X = \mathrm{Bl}_2(\mathbb{P}^4)$. It is known that X is a toric variety and hence its Cox ring coincides with the usual homogeneous coordinate ring

$$\mathrm{Cox}(\mathrm{Bl}_2(\mathbb{P}^4)) = \mathbb{C}[x, y, z, s_1, s_2, e_1, e_2],$$

where $\deg(x) = \deg(y) = \deg(z) = H - E_1 - E_2$, $\deg(s_i) = H - E_i$ and $\deg(e_i) = E_i$. So we construct the following matrix:

$$\begin{matrix} H & E_1 & E_2 \end{matrix}$$

$$A := \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where coefficients of the first column correspond to H, those of the second column to E_1 and those of the third column to E_2. Now, we just need to put the transpose of A into the script shown at page 77, and we find:

$$\mathrm{Mov}(\mathrm{Bl}_2(\mathbb{P}^4)) = \langle H - E_1 - E_2, H - E_1, H - E_2, H \rangle.$$

Blow-ups of $\mathbb{P}^3$ in $r \leq 4$ points in general position are toric varieties. The following example concerns the non toric variety $X = \mathrm{Bl}_5(\mathbb{P}^3)$. The generators and relations have been computed by J. C. Ottem in his master thesis [Ott09]. We present some of his results and we then compute the moving cone using Maple.

Example 2.2.17. Let $X = X_5^3$ be the blow-up of $\mathbb{P}^3$ in points $p_1, \ldots, p_5$ in general position. Let $x_1, \ldots, x_5$ be the generators for the cohomology groups

$$H^0(X, E_1), \ldots, H^0(X, E_5)$$

and let h_{ijk} denote a generator for $H^0(H - E_i - E_j - E_j)$. Geometrically the zero-sections of $x_1, \ldots, x_5$ correspond to the exceptional planes and h_{ijk} corresponds to pullbacks of planes through p_i, p_j, p_k in $\mathbb{P}^3$. Then we have the following:

Theorem 2.2.18. *[Ott09, Theorem 4.5]* $\mathrm{Cox}(X_5^3)$ *is generated by the sections* x_i, h_{ijk} *from the respective divisor classes* $E_1, \ldots, E_5$ *and* $H - E_i - E_j - E_k$, *for* $1 \leq i, j, k \leq 5$ *distinct.*

For a proof we refer to [Ott09]. Relations among these generators are also found in this reference, but it is not necessary to write them down in order to apply Proposition 2.2.10.

Now, with the script shown at page 78, we find 51 generators for the moving cone of X^3_5.

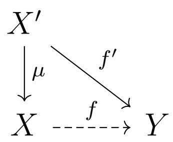
$$\begin{aligned}\operatorname{Mov}(X^3_5) = \langle & 2H - E_{i_1} - E_{i_2} - E_{i_3} - 2E_{i_4} - E_{i_5}, 3H - E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4} - 2E_{i_5},\\ & H - E_i - E_j, 2H - E_{i_1} - E_{i_2} - E_{i_3} - 2E_{i_4}, H - E_i,\\ & 3H - 2E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4}, H\rangle,\end{aligned}$$

for $i_1, \dots, i_5 \in \{1, \dots, 5\}$.

2.3 Mori dream spaces

The notion of Mori dream space was introduced by Y. Hu and S. Keel in [HK00]. This denomination is motivated by the fact that these spaces behave in the best possible way from the point of view of Mori's minimal model program. In this section, we recall the definition of Mori dream space, and their main properties in relation to Fano and log Fano varieties.

Definition 2.3.1. [HK00, Definition 1.1] Let $f : X \dashrightarrow Y$ be a dominant rational map, where Y is normal and projective. We say that f is contracting, or a rational contraction, if there exists a resolution of f



where X' is smooth and projective, μ is birational, and for every μ-exceptional effective divisor E on X' we have

$$f'_*(\mathcal{O}_{X'}(E)) = \mathcal{O}_Y.$$

Remark 3. *Let $f : X \dashrightarrow Y$ be a birational map, with Y normal and projective. Then, f is contracting if and only if there are open subsets $U \subseteq X$ and $V \subseteq Y$ such that f is an isomorphism between U and V, and* $\operatorname{codim}(Y \setminus V) \geq 2$.

Example 2.3.2. The inverse of a blow-up is not a contracting rational map.

Definition 2.3.3. [HK00, Definition 1.8] A small $\mathbb{Q}$-factorial modification (SQM) of X is a birational map

$$g : X \dashrightarrow X',$$

where X' is again normal, projective, and $\mathbb{Q}$-factorial, and g is an isomorphism in codimension 1.

Remark 4. *By remark 3, we see that both g and g^{-1} are contracting. The basic example of a SQM is a flip. Furthermore, it is easy to check that g induces an isomorphism*

$$g^* : \mathrm{N}^1(X') \to \mathrm{N}^1(X)$$

(in particular X and X' have the same Picard number), and that g^* preserves the effective and the movable cones:

$$g^*(\mathrm{Eff}(X')) \subseteq \mathrm{Eff}(X),$$

and

$$g^*(\mathrm{Mov}(X')) \subseteq \mathrm{Mov}(X).$$

In particular, we have

$$g^*(\mathrm{Nef}(X')) \subset \overline{\mathrm{Mov}(X)}.$$

The definition of Mori dream space is the following:

Definition 2.3.4. [HK00, Definition 1.10] Let X be a normal, projective, $\mathbb{Q}$-factorial variety X. We say that X is a Mori dream space (MDS) if the following conditions hold:

1. $\mathrm{Pic}(X)$ is finitely generated, or equivalently $h^1(X, \mathcal{O}_X) = 0$ (so $\mathrm{Pic}(X) \cong \mathrm{N}^1(X)$),
2. $\mathrm{Nef}(X)$ is generated by the classes of finitely many semiample divisors,
3. there is a finite collection of small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$, such that each X_i satisfies the second condition above and

$$\mathrm{Mov}(X) = \bigcup_i f_i^*(\mathrm{Nef}(X_i))$$

.

The collection of all faces of all cones $f_i^*(\mathrm{Nef}(X_i))$ above forms a fan which is supported on $\mathrm{Mov}(X)$. If two maximal cones of this fan, say $f_i^*(\mathrm{Nef}(X_i))$ and $f_j^*(\mathrm{Nef}(X_j))$, meet along a facet, then there exists a normal projective variety Y, a small modification $\phi : X_i \dashrightarrow X_j$, and $h_i : X_i \to Y$, $h_j : X_j \to Y$ small birational morphisms of relative Picard number one such that $h_j \circ \phi = h_i$.

The fan structure on $\mathrm{Mov}(X)$ can be extended to a fan supported on $\mathrm{Eff}(X)$ as follows.

Definition 2.3.5. Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. There are finitely many birational contractions from X to Mori dream spaces, denoted by

$g_i : X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form

$$\mathcal{C}_i = \left\langle g_i^*\Big(\mathrm{Nef}(Y_i)\Big), \mathrm{Exc}(g_i)\right\rangle .$$

We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^\circ$ a *maximal chamber* of $\mathrm{Eff}(X)$. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details.

Proposition 2.3.6. *[HK00, Proposition 2.9] Let X be a normal projective $\mathbb{Q}$-factorial variety such that* $\mathrm{Pic}(X) = \mathrm{N}^1(X)$. *$X$ is a Mori dream space if and only if* $\mathrm{Cox}(X)$ *is finitely generated as an algebra over the base field.*

Example 2.3.7. Let us work out explicitly the cone of effective divisors and the Mori cone of curves of $X := X_2^n$, the blow-up of $\mathbb{P}^n$ at two points $p, q \in \mathbb{P}^n$, with $n > 1$.

Let $H, H_p, H_q, H_{p,q}$ be the strict transforms respectively of a hyperplane, a hyperplane passing through p, through q, and through both p and q. Moreover, let E_p, E_q be the exceptional divisors over p and q respectively. Note that $H_p = H - E_p, H_q = H - E_q$ and $H_{p,q} = H - E_p - E_q$. Then $N^1(X) \cong \mathbb{Z}[H, E_p, E_q]$.

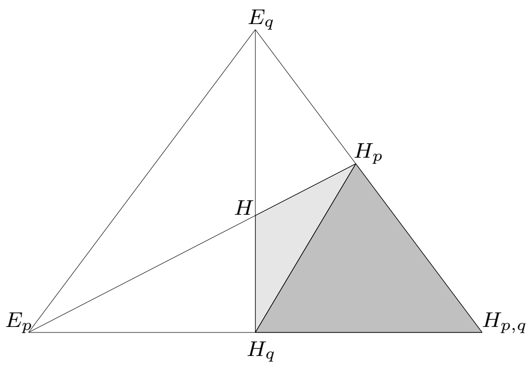
We will denote by h the strict transform of a general line in $\mathbb{P}^n$, and by e_p, e_q classes of lines in E_p and E_q respectively. The intersection pairing is given by $H \cdot h = 1, H \cdot e_p = H \cdot e_q = 0, E_p \cdot e_q = E_q \cdot e_p = 0, E_p \cdot e_p = E_q \cdot e_q = -1$. The last two intersections numbers might be not obvious from a geometrical point of view. To compute them one may reason as follows: the divisor $H - E_p$ represents the strict transform of a general hyperplane through p, and $h - e_p$ represents the strict transform of a general line through p. In the blow-up X these strict transforms do not intersect anymore, so $0 = (H - E_p) \cdot (h - e_p) = H \cdot h - H \cdot e_p - E_p \cdot h + E_p \cdot e_p$ and hence $E_p \cdot e_p = -H \cdot h = -1$.

Now, let $C \subset X$ be an irreducible curve. Then either C is contained in an exceptional divisor and then it is numerically equivalent to a positive multiple of e_p or e_q, or it is mapped by the blow-down map to an irreducible curve $\Gamma \subset \mathbb{P}^n$. Suppose that C it is not contained in an exceptional divisor. Let d, m_p, m_q be respectively the degree of Γ and the multiplicities of Γ at p and q. Then $C \equiv dh - m_p e_p - m_q e_q$. We may write $C \equiv d(h - e_p - e_q) + (d - m_p)e_p + (d - m_q)e_q$. Furthermore, $d - m_p > 0$ otherwise by Bézout's theorem Γ would contain a line through p as a component, and similarly $d - m_q > 0$. Hence $\mathrm{NE}(X)$ is closed and generated by the classes e_p, e_q and $h - e_p - e_q$. Note that the latter is the strict transform of the line in $\mathbb{P}^n$ through p, q. Similarly, it can be shown that $\mathrm{Eff}(X)$ is closed and generated by the classes of E_p, E_q and $H_{p,q}$.

Let us work out the nef cone of X when $n > 2$. This is the cone of divisors intersecting non negatively all the irreducible curves in X. Since any curve in X can be written as a linear combination with non-negative coefficients of the generators of $\mathrm{NE}(X)$, it is enough to check when a divisor intersects non-negatively these generators. Let us write $D \equiv aH + bE_p + cE_q$. Then $D \cdot (h - e_p - e_q) = a + b + c$, $D \cdot e_p = -b$ and $D \cdot e_p = -c$, and $\mathrm{Nef}(X)$ is defined in $N^1(X)_{\mathbb{R}} \cong \mathbb{R}^3$ by the inequalities $a+b+c \geq 0, b \leq 0, c \leq 0$. Hence $\mathrm{Nef}(X)$ is generated by $\langle H, H_p, H_q \rangle$.

Finally, we determine the movable cone of X. The divisor $H_{p,q}$ represents the hyperplanes of $\mathbb{P}^n$ passing through p, q. Hence the stable base locus of $H_{p,q}$ consists of the strict transform of the line through p, q. The stable base locus of all divisors in the cone generated by $\langle H_p, H_q, H_{p,q} \rangle$ is contained in such a strict transform as H_p, H_q have no base loci. Hence all the divisors in this cone are movable when $n > 2$. On the other hand, all divisors in the interior of the cone $\langle H, H_p, E_q \rangle$ contain E_q, all divisors in the interior of the cone $\langle H, H_q, E_p \rangle$ contain E_p, and all divisors in the interior of the cone $\langle H, E_p, E_q \rangle$ contain $E_p \cup E_q$. Therefore, $\mathrm{Mov}(X)$ is the cone generated by $\langle H, H_p, H_q, H_{p,q} \rangle$.

The following picture is a two dimensional cross-section of $\mathrm{Eff}(X)$ displaying its Mori chamber decomposition:



The divisors $H, H_p, H_q, H_{p,q}$ generate $\mathrm{Mov}(X)$, and H, H_p, H_q generate $\mathrm{Nef}(X)$. The chamber delimited by H, H_q, E_p corresponds to the contraction of E_p, similarly the chamber delimited by H, H_p, E_q corresponds to the contraction of E_q, and chamber delimited by H, E_p, E_q corresponds to the contraction of both E_p and E_q.

In the case $n \geq 3$, X admits only one small $\mathbb{Q}$-factorial modification X' corresponding to the chamber delimited by $H_p, H_q, H_{p,q}$. In what follows in this example, we will investigate the geometry of X'.

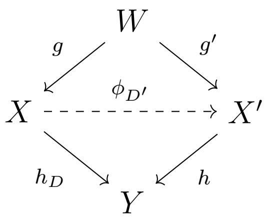
Consider a divisor lying on the wall delimited by H_p and H_q, for instance $D = H_p + H_q = 2H - E_p - E_q$, and let L be the strict transform of the line through p and q. Then $D \cdot L = 0$ and the linear system of quadrics in $\mathbb{P}^n$ through p and q induces a morphism $h_D \colon X \to Y$ contracting L to a point.

On the other hand, a divisor in the maximal chamber delimited by $H_p, H_q, H_{p,q}$ must be ample on X'. We can write such a divisor as $aH_p + bH_q + cH_{p,q}$ with $a, b, c > 0$ and observe that $(aH_p + bH_q + cH_{p,q}) \cdot L = -c < 0$.

Note that the curve L prevents divisors on X in the chamber $\langle H_p, H_q, H_{p,q}\rangle$ from being ample.

Let $g\colon W \to X$ be the blow-up of X along L with exceptional divisor $E_L \subset W$. Observe that E_L is a $\mathbb{P}^{n-2}$-bundle over L.

There is a morphism $g'\colon W \to X'$ contracting E_L, in the direction of L, onto a subvariety $Z \subset X'$ such that $Z \cong \mathbb{P}^{n-2}$. Consider the divisor $D' \equiv H_p+H_q+H_{p,q} \equiv 3H - 2E_p - 2E_q$. The linear system of D' induces a rational map $\phi_{D'}\colon X \dashrightarrow X'$, and we have the following commutative diagram



where $h\colon X' \to Y$ is a small modification contracting $Z \subset X'$ to $h_D(L)$. The rational map $\phi_{D'}\colon X \dashrightarrow X'$ is an isomorphism between $X \setminus L$ and $X' \setminus Z$ and replaces L with the variety Z which is covered by curves having non-negative intersection with all divisors in the chamber $\langle H_p, H_q, H_{p,q}\rangle$. Concretely, in the case $n = 3$ for instance, we can fix homogeneous coordinates $[x : y : z : w]$ on $\mathbb{P}^3$, assume that $p = [1 : 0 : 0 : 0]$, $q = [0 : 0 : 0 : 1]$, and consider the rational maps

$$\alpha\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^7$$

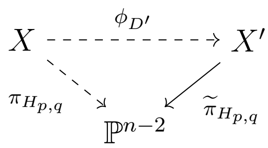
defined by $\alpha([x : y : z : w]) = [xy : xz : xw : y^2 : yz : yw : z^2 : zw]$, that is induced by the quadrics of $\mathbb{P}^3$ passing through p and q, and

$$\beta\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^{11}$$

defined by $\beta([x : y : z : w]) = [xy^2 : xz^2 : xyz : xyw : xzw : y^3 : y^2z : y^2w : yz^2 : yzw : z^3 : z^2w]$, that is induced by the cubics of $\mathbb{P}^3$ having at least double points at p and q. Then Y is the closure of the image of α and X' is the closure of the image of β.

Let us give a geometric description of X'. Let $\Pi \subset X$ be the strict transform of a 2-plane through the line $\overline{pq}$. The plane Π is contracted to a point by the map $\pi_{H_{p,q}}\colon X \dashrightarrow \mathbb{P}^{n-2}$ induced by $H_{p,q}$. Indeed, $\pi_{H_{p,q}}$ is induced by the linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-2}$ with center $\overline{pq}$. Observe that a divisor in the linear system of D' has a base component when restricted to Π, namely the curve L.

Therefore, $\phi_{D'|\Pi}$ is the rational map induced by the linear system of conics through p and q, hence its image is a smooth quadric surface $Q_\Pi \cong \mathbb{P}^1 \times \mathbb{P}^1$. The quadric Q_Π intersects Z at a point. The morphism $\widetilde{\pi}_{H_{p,q}}\colon X' \dashrightarrow \mathbb{P}^{n-2}$, induced by the strict transform of $H_{p,q}$ on X', contracts Q_Π to the point $\pi_{H_{p,q}}(\Pi)$ and maps Z isomorphically onto $\mathbb{P}^{n-2}$. We have the following commutative diagram



and X' has a structure of $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle over $\mathbb{P}^{n-2}$. Summing up, the birational model of X corresponding to the chamber $\langle H_p, H_q, H_{p,q}\rangle$ is a quadric bundle over $\mathbb{P}^{n-2}$ and, as we already noticed, the other chambers $\langle H, H_p, E_q\rangle$, $\langle H, H_q, E_p\rangle$ and $\langle H, E_p, E_q\rangle$ corresponds respectively to $\mathbb{P}^n$ blown-up at q, $\mathbb{P}^n$ blown-up at p and $\mathbb{P}^n$. The chamber $\langle H, H_p, H_q\rangle$ corresponds to X itself.

2.4 Weak and log Fano varieties

In this section we introduce log Fano and weak Fano varieties. First, we define singularities of pairs and log resolutions. We denote by K_X (resp. $(-K_X)$) the canonical (resp. anticanonical) divisor of the variety X.

Let us consider a $\mathbb{Q}$-divisor $D = \sum_i d_i D_i$ on a normal variety X.

We assume that the D_i's are distinct. We want to give a reasonable notion of singularities of the pair (X, D). We require that $K_X + D$ is $\mathbb{Q}$-Cartier.

Then, for a resolution $f : Y \to X$ we have the formula

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i - \tilde{D},$$

where $\tilde{D}$ is the strict transform.

Even when X is smooth, D could be very singular.

Definition 2.4.1. A divisor $D = \sum_i d_i D_i$ on a smooth variety X is simple normal crossing if D is reduced, any component D_i of D is smooth, and D is locally defined in a neighborhood of any point by an equation in local analytic coordinates of the type

$$z_1 \cdot \dots \cdot z_k = 0,$$

with $k \leq \dim(X)$.

Roughly speaking the singularities of D should locally look no worse that those of a union of coordinate hyperplanes.

Example 2.4.2. Let $D = \sum_i D_i$ where the D_i's are hyperplanes in $\mathbb{P}^n$ and let $p_i \in \mathbb{P}^{n*}$ be the point corresponding to D_i. Then D is simple normal crossing if and only if the p_i's are in linear general position.

The following is a consequence of Hironaka's theorem on resolution of singularities, see [Hir64].

Theorem 2.4.3. *Let X be an irreducible algebraic variety over $\mathbb{C}$, and let $D \subset X$ be an effective Cartier divisor on X. Then, there exists a projective birational morphism*

$$f : Y \to X,$$

where X is smooth and $f^{-1}(D) \cup Exc(f)$ is simple normal crossing.

Moreover, the smooth variety Y can be constructed as a sequence of blow-ups along smooth centers supported in the singular loci of D and X. In particular, f is an isomorphism over $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(D))$.

The morphism mentioned in Theorem 2.4.3 has a special name:

Definition 2.4.4. The morphism f is called a *log resolution* of the pair (X, D).

By [Hir64], a log resolution always exists.

Now we can give the following definition, which will be useful in particular for the notion of *klt* pair.

Definition 2.4.5. Let X be a normal variety and $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-divisor. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X, D) and write

$$K_Y = f^*(K_X + \Delta) + \sum_i a_i E_i - \tilde{D}.$$

The pair (X, D) is:

1. terminal if $a_i > 0$ for any i,
2. canonical if $a_i \geq 0$ for any i,
3. klt if $a_i > -1$ and $d_j < 1$ for any i, j,
4. plt if $a_i > -1$ for any i,
5. lc if $a_i \geq -1$ for any i.

Here klt, plt, lc stands for Kawamata log terminal, purely log terminal, and log canonical respectively.

Example 2.4.6. Assume that D is a simple normal crossing divisor, and that X is smooth. Then Id_X is a log resolution.

Example 2.4.7. If $0 < \epsilon < 1$ is a rational number then we have

$$K_X = Id_X^*(K_X + \epsilon D) - \epsilon D.$$

The pair $(X, \epsilon D)$ is Kawamata log terminal. Let $D \subset \mathbb{P}^2$ be an irreducible curve with one node, and let $f : Y \to \mathbb{P}^2$ be the blow-up of the node. Then $f^{-1}D \cup E$ is simple normal crossing. Furthermore $K_Y = f^*K_{\mathbb{P}^2} + E$ and $f^*D = \tilde{D} + 2E$ where $\tilde{D}$ is the strict transform of D, yield

$$K_Y = f^*(K_{\mathbb{P}^2} + D) - \tilde{D} - E.$$

Therefore, the pair $(\mathbb{P}^2, D)$ is log canonical.

Now, we can introduce the notion of log Fano and of weak Fano varieties:

Definition 2.4.8. Let X be a smooth projective $\mathbb{Q}$-factorial variety. We say that X is:

1. weak Fano if $-K_X$ is nef and big,
2. log Fano if there exists an effective divisor D such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal.
3. weak log Fano if there exists an effective divisor D such that $-(K_X + D)$ is nef and big, and the pair (X, D) is Kawamata log terminal.

It follows trivially from Definition 2.4.8 that a smooth weak Fano variety is log Fano. The following Lemma extends the properties to varieties with klt singularities.

Lemma 2.4.9. *[AM16, Lemma* 2.5*] Let X be a normal $\mathbb{Q}$-factorial projective variety with at worst klt singularities. Suppose that $-K_X$ is nef and big (X is weak Fano). Then X is log Fano.*

The bridge between Mori dream spaces and log Fano varieties is the content of the following proposition:

Proposition 2.4.10. *[BCHM08, Corollary* 1.3.2*] Let X be a smooth projective variety. If X is log Fano then X is a Mori dream space .*

Example 2.4.11. In [AM16] and in [LP17] it is proven that blowups of $\mathbb{P}^n$ or of products of the form $(\mathbb{P}^n)^m$ in points in very general position are log Fano if and only if they are Mori dream spaces.

Example 2.4.12. Let write X_k^n for the blow up of $\mathbb{P}^n$ at k points in general position. Results of [Muk04] and [CT06] show that:

1. For $n = 4$, X_k^n is a Mori dream space if and only if $k \leq 8$.
2. For $n > 4$, X_k^n is a Mori dream space if and only if $k \leq n + 3$.

Moreover, for $n = 2$ and $k \leq 8$, $-K_{X_k^n}$ is ample and and for $k \geq 9$, the situation changes drastically. The anti-canonical class of X_k^n is no longer big, and it contains infinitely many (-1)-curves. For $n \geq 3$, X_1^n is a Fano manifold (that is, $-K_{X_1^n}$ is ample), but as soon as $k \geq 2$, X_k^n is no longer Fano. However For $n = 3$ and $k \leq 7$, X_k^n is log Fano, then X_k^n is a Mori dream space also in these cases.

2.5 The varieties $X_r^{1,n}$

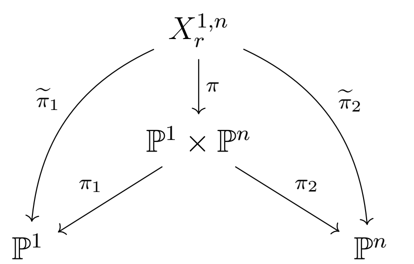
In this section we introduce the main objects of interest in this thesis: blow-ups of the product $\mathbb{P}^1 \times \mathbb{P}^n$ in a collection of points in very general position.

Recall that a collection of points is in very general position in $\mathbb{P}^1 \times \mathbb{P}^n$ if the corresponding element in the Hilbert scheme of s points of $\mathbb{P}^1 \times \mathbb{P}^n$ lies in the complement of a countable union of Zariski closed subsets.

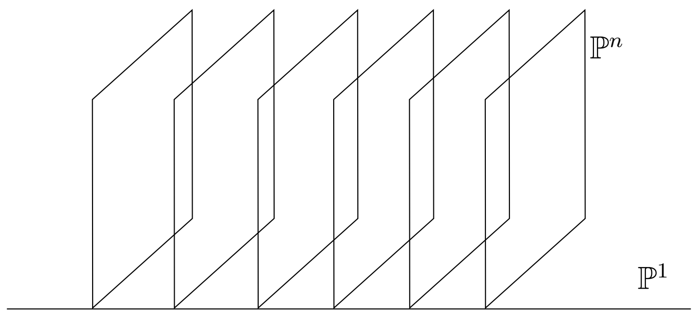
We denote by $X_r^{1,n}$ the blow-up of $\mathbb{P}^1 \times \mathbb{P}^n$ in r points $p_1, \ldots, p_r$ in very general position:

$$X_r^{1,n} := \mathrm{Bl}_r(\mathbb{P}^1 \times \mathbb{P}^n).$$

We will denote by $\pi : X_r^{1,n} \to \mathbb{P}^1 \times \mathbb{P}^n$ the blow-down morphism and by $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^1$, $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^n$ the projections onto the factors. Moreover, let us denote by $\widetilde{\pi}_1, \widetilde{\pi}_2$ the morphisms from $X_r^{1,n}$ to $\mathbb{P}^1$ and $\mathbb{P}^n$ induced by the projections. We summarize the situation in the following diagram:



The product $\mathbb{P}^1 \times \mathbb{P}^n$ is depicted in the following picture:



Let $\mathrm{Pic}(X_r^{1,n})$ denote the Picard group of $X_r^{1,n}$. Throughout the thesis we will denote by:

- H_1 the pull-back of a point of $\mathbb{P}^1$ via $\widetilde{\pi}_1$;
- H_2 the pull-back of a hyperplane in $\mathbb{P}^n$ via $\widetilde{\pi}_2$;
- E_i the exceptional divisor over p_i for $i = 1, \dots, r$;

and by

- h_1 the class of a line contained in a general fiber of $\widetilde{\pi}_2$;
- h_2 the class of a line contained in a general fiber of $\widetilde{\pi}_1$;
- e_i the class of a line in the exceptional divisor E_i for $i = 1, \dots, r$.

We have that

$$\mathrm{Pic}(X_r^{1,n}) = \mathbb{Z}[H_1, H_2, E_1, \dots, E_r].$$

Remark 5. *In the notation previously introduced, we have that $h_1 - e_i$ is the class of a general fiber of $\widetilde{\pi}_2$ passing through the point p_i. Analogously, the class $h_2 - e_i$ the class of a general fiber of $\widetilde{\pi}_1$ passing through the point p_i.*

The following proposition provides informations about generators for $\mathrm{N}^1(X_r^{1,n})$ and $\mathrm{N}_1(X_r^{1,n})$ and the rules of intersection between these generators.

All statements are straightforward consequences of general results about intersection theory of blow-ups; a reference is [EH16, Proposition 13.12]. We follow [GPPS22].

Proposition 2.5.1. *Let $X_r^{1,n}$ as above. Then*

1. *The vector spaces $\mathrm{N}^1(X_r^{1,n})$ and $\mathrm{N}_1(X_r^{1,n})$ have the following bases:*

$$\mathrm{N}^1(X_r^{1,n}) = \langle H_1, H_2, E_1, \dots, E_r \rangle,$$

$$\mathrm{N}_1(X_r^{1,n}) = \langle h_1, h_2, e_1, \dots, e_r \rangle.$$

2. *We have the following intersection numbers among divisors:*

$$\begin{aligned}
&H_1 \cdot H_2^n = 1,\\
&H_1^p \cdot H_2^{1+n-p} = 0 \text{ for } p \neq 1,\\
&H_1^p \cdot E_i^{1+n-p} = H_2^p \cdot E_i^{1+n-p} = 0 \text{ for all } p > 0, i = 1, \dots, s,\\
&E_i^{1+n} = (-1)^n.
\end{aligned}$$

3. *We have the following intersection numbers between divisors and curves:*

$$H_i \cdot h_j = \delta_{ij},$$
$$H_i \cdot e_j = 0 \text{ for } i = 1, 2 \text{ and } j = 1, \ldots, s,$$
$$E_i \cdot e_j = -\delta_{ij}.$$

In particular, Proposition 2.5.1 allows us to determine the numerical classes of curves on $X_r^{1,n}$ as follows.

Corollary 2.5.2. *Let C be the proper transform on $X_r^{1,n}$ of a curve of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^n$ with multiplicity m_i at the point p_i. Then, the class of C in $\mathrm{N}_1(X_r^{1,n})$ is*

$$C \sim d_1 h_1 + d_2 h_2 - \sum_{i=1}^{r} m_i e_i.$$

Proof. We have the intersection numbers

$$C \cdot H_i = d_i, \text{ for} i = 1, 2.$$
$$C \cdot E_j = m_j \text{ for } j = 1, \ldots, r.$$

Since the intersection pairing on $X_r^{1,n}$ is perfect, the given formula then follows from Proposition 2.5.1. □

In particular, we deduce the following formula:

For any divisor D on $X_r^{1,n}$, the class of D can be written in the form

$$d_1 H_1 + d_2 H_2 - \sum_{i=1}^{r} m_i E_i,$$

for some integers $d_1, d_2, m_1, \ldots, m_r$.

In the rest of the thesis we will focus on $X_r^{1,n}$ for $r \leq n+2$ and for $r = n+3$ if $n \leq 4$. However, it is interesting to mention some results on certain asymmetric varieties of blow-up type, which do not fall in the previous cases.

Chapter 3

Mori cones

This chapter consists of five sections. In the first Section, we introduce Del Pezzo surfaces and we write down the generators of their Mori cones. In Sections 2, 3 and 4 we compute the cone of effective curves of $X_r^{1,n}$, for $r \leq n+2$, and for $r = n+3$ when $n \leq 4$. Finally, in section 5 we prove that the varieties $X_r^{1,n}$ are log Fano for $r \leq n+1$.

3.1 Mori cones of Del Pezzo surfaces

In this section we record the effective cones of various Del Pezzo surfaces. These cones are described in standard references such as [Man74].

We briefly recall the notion of Fano varieties and of Del Pezzo surfaces.

Definition 3.1.1. A smooth projective variety X is Fano if the anticanonical divisor $-K_X$ is ample.

Example 3.1.2. Let $X \subset \mathbb{P}^r$ be a smooth hypersurface of degree d. By adjunction

$$K_X = (K_{\mathbb{P}^r} + X)|_X = (d - r - 1)H,$$

where H is the class of a hyperplane. Thus X is Fano if and only if $d \leq r$.

Claim 3.1.3. The product of Fano varieties is Fano. If C is a smooth projective curve then C is Fano if and only if $C \cong \mathbb{P}^1$.

Proof. For a proof see [BP04]. □

Definition 3.1.4. A Del Pezzo surface S is a Fano variety of dimension two. The degree of S is the self intersection number (K_S, K_S) of its canonical class K_S.

The following theorem gives a complete characterization of Del Pezzo surfaces.

Theorem 3.1.5. *Let S be a Del Pezzo surface. Then,*

1. *S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, or*
2. *S is isomorphic to $\mathbb{P}^2$ blown up in $r \leq 8$ points in general position.*

Del Pezzo proved that a Del Pezzo surface has degree at most 9.

Let S_r be the blow-up of $\mathbb{P}^2$ at $p_1, \ldots, p_r \in \mathbb{P}^2$ general points. By Theorem 3.1.5, the surface S_r is Del Pezzo if and only if $0 \leq r \leq 8$. The degree of S_r is $9-r$. In the next subsections, we recall the structure of the Mori cone $\mathrm{NE}(S_r)$ for $r \leq 8$.

We will denote by $\overline{h}$ the pull-back of a line in $\mathbb{P}^2$ and by $\overline{e}_i$ the exceptional divisor over the point p_i, for $i = 1, \ldots, r$. The vector space of one-cycles is given by:

$$\mathrm{N}_1(S_r) = \langle \overline{h}, \overline{e}_1, \ldots, \overline{e}_r \rangle.$$

The only Del Pezzo surface of degree 9 is $\mathbb{P}^2$ and the computation of its Mori cone is straightforward.

3.1.1 Del Pezzo of degree 8

There are two isomorphism types. One is a Hirzebruch surface, given by the blow up of the projective plane at one point, which will be denoted S_1. The other is the product of two projective lines

$$\mathbb{P}^1 \times \mathbb{P}^1,$$

which is the only Del Pezzo surface that cannot be obtained as a blow up of the projective plane.

The Mori cone of S_1 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	1
$\overline{h} - \overline{e}_i$	1

for a total of 2 extremal rays.

3.1.2 Del Pezzo of degree 7

The Mori cone of S_2 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	2
$\overline{h} - \overline{e}_i - \overline{e}_j$	1

for a total of 3 extremal rays.

3.1.3 Del Pezzo of degree 6

The Mori cone of S_3 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	3
$\overline{h} - \overline{e}_i - \overline{e}_j$	3

for a total of 6 extremal rays.

3.1.4 Del Pezzo of degree 5

The Mori cone of S_4 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	4
$\overline{h} - \overline{e}_i - \overline{e}_j$	6

for a total of 10 extremal rays.

3.1.5 Del Pezzo of degree 4

The Mori cone of S_5 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	5
$\overline{h} - \overline{e}_i - \overline{e}_j$	10
$2\overline{h} - \overline{e}_1 - \cdots - \overline{e}_5$	1

for a total of 16 extremal rays.

3.1.6 Del Pezzo of degree 3

The Mori cone of S_6 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	6
$\overline{h} - \overline{e}_i - \overline{e}_j$	15
$2\overline{h} - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5}$	6

for a total of 27 extremal rays. These surfaces are cubic surfaces in $\mathbb{P}^3$.

3.1.7 Del Pezzo of degree 2

The Mori cone of S_7 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	7
$\overline{h} - \overline{e}_i - \overline{e}_j$	21
$2\overline{h} - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5}$	21
$3\overline{h} - 2\overline{e}_i - \overline{e}_{j_1} - \cdots - \overline{e}_{j_6}$	7

for a total of 56 extremal rays.

3.1.8 Del Pezzo of degree 1

The Mori cone of S_8 is generated by the following classes

Divisor class	*Number of extremal rays*
$\overline{e}_i$	8
$\overline{h} - \overline{e}_i - \overline{e}_j$	28
$2\overline{h} - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5}$	56
$3\overline{h} - 2\overline{e}_i - \overline{e}_{j_1} - \cdots - \overline{e}_{j_6}$	56
$4\overline{h} - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{i_3} - \overline{e}_{j_1} - \cdots - \overline{e}_{j_5}$	56
$5\overline{h} - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{i_6} - \overline{e}_{j_1} - \overline{e}_{j_2}$	28
$6\overline{h} - 3\overline{e}_i - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{j_7}$	8

for a total of 240 extremal rays.

3.2 The Mori cone of $X^{1,n}_{n+1}$

In this section we compute de Mori cone of $X^{1,n}_{n+1}$.

Proposition 3.2.1. *The Mori cone of $X^{1,n}_r$ is given by:*

$$\mathrm{NE}(X^{1,n}_r) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle$$

for all $r \leq n+1$.

Proof. It is enough to prove the claim for $r = n+1$. Indeed, if the claim is true for $n+1$, it remains true for $r \leq n+1$, after setting some of the coefficients of the e_i's to zero.

The proof is done by induction on n.

If $n = 1$, then

$$X^{1,1}_2 = \mathrm{Bl}_2(\mathbb{P}^1 \times \mathbb{P}^1).$$

From Proposition 2.5.1 we have

$$\mathrm{N}_1(X_2^{1,1}) = \mathbb{R}[e_1, e_2, h_1, h_2].$$

The well-known isomorphism

$$S_3 \cong X_2^{1,1}$$

implies isomorphisms at the level of one-cycles

$$\mathrm{N}_1(S_3) \cong \mathrm{N}_1(X_2^{1,1})$$

and

$$\mathrm{NE}(S_3) \cong \mathrm{NE}(X_2^{1,1})$$

between Mori cones. The first isomorpshim

$$\mathrm{N}_1(X_2^{1,1}) \to \mathrm{N}_1(S_3)$$

is defined on generators in the following way:

$$\begin{cases} h_1 \longmapsto \overline{h} - \overline{e}_2 \\ h_2 \longmapsto \overline{h} - \overline{e}_1 \\ e_1 \longmapsto \overline{h} - \overline{e}_1 - \overline{e}_2 \\ e_2 \longmapsto \overline{e}_3 \end{cases}$$

Whereas the inverse map is defined as:

$$\begin{cases} \overline{e}_1 \longmapsto h_1 - e_1 \\ \overline{e}_2 \longmapsto h_2 - e_1 \\ \overline{e}_3 \longmapsto e_2 \\ \overline{h} \longmapsto h_1 + h_2 - e_1 \end{cases}$$

The table in 3.1.3 and the isomorphisms above give the following generators for $\mathrm{NE}(X_2^{1,1})$:

$$\mathrm{NE}(X_2^{1,1}) = \langle h_2 - e_i, h_1 - e_i, e_i \rangle,$$

which conclude this case. We prove the induction step.

Suppose that the claim is proved for $X_n^{1,n-1}$ and let $C \subset X_{n+1}^{1,n}$ be an irreducible curve.

If C is contracted by π, then $C \subset E_i$ for some $i = 1, \dots, n+1$, hence C is a multiple of e_i. Otherwise, $\pi(C)$ is a curve in $\mathbb{P}^1 \times \mathbb{P}^n$ of bidegree (d_1, d_2) and multiplicity m_i at the point p_i. By corollary 2.5.2, it can be written as

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \cdots - m_{n+1} e_{n+1}.$$

Relations in 2.5.1 give:

$$C \cdot (H_1 - E_i) = d_1 - m_i.$$

If $d_1 - m_i < 0$, for some i, the curve C is contained in $H_1 - E_i$. The divisor class $H_1 - E_i$ is the class of the strict transform of the fiber of π_1 passing through p_i. Such fiber is isomorphic to $\mathrm{Bl}_{p_i} \mathbb{P}^n$. Hence, since

$$\mathrm{NE}(\mathrm{Bl}_{p_i} \mathbb{P}^n) = \langle e_i, h_2 - e_i \rangle,$$

C can be written as a linear combination with non negative coefficients of e_i and $h_2 - e_i$.

Assume that $d_1 - m_i \geq 0$ for $i = 1, \dots, n+1$.

Consider the projection $\tilde{\pi}_2(C)$ to the second factor $\mathbb{P}^n$. If $\tilde{\pi}_2(C)$ is a point, then C s a linear combination with non negative coefficients of $h_1 - e_i$ and e_i, for some $i = 1, \dots n+1$. Otherwise, $\tilde{\pi}_2(C)$ is a curve in $\mathbb{P}^n$ of degree d_2, passing through the projected points, with multiplicity

$$\mathrm{mult}_{\pi_2(p_i)} \tilde{\pi}_2(C) = m_i,$$

for $i = 1, \dots n+1$.

Let $\Pi_{1,\dots,n} \subset \mathbb{P}^n$ be the hyperplane passing through n of the blown-up points.

By Bézout's theorem we have that either

$$m_1 + \cdots + m_n \leq \deg(\Pi_{1,\dots,n} \cdot \tilde{\pi}_2(C)) = \deg(\Pi_{1,\dots,n}) \deg(\tilde{\pi}_2(C)) = 1 \cdot d_2 = d_2$$

or

$$m_1 + \cdots + m_n > d_2$$

and $\tilde{\pi}_2(C)$ is contained in $\Pi_{1,\dots,n}$:

$$\tilde{\pi}_2(C) \subset \Pi_{1,\dots,n}.$$

Then $\pi(C) \subset \mathbb{P}^1 \times \Pi_{1,\dots,n}$ and hence C is contained in the strict transform of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$, which is isomorphic to $X_n^{1,n-1}$.

By induction, we can conclude that C is a linear combination with nonnegative coefficients of $h_2 - e_i, h_1 - e_i$ and e_i.

On the other hand, if

$$m_1 + \cdots + m_n \leq d_2,$$

C can be written as:

$$C \sim \sum_{i=1}^{n} m_i(h_2 - e_i) + m_{n+1}(h_1 - e_{n+1}) + (d_1 - m_{n+1})h_1 + (d_2 - m_1 - \cdots - m_n)h_2,$$

with non negative coefficients. Since

$$h_1 = (h_1 - e_i) + e_i,$$

and

$$h_2 = (h_2 - e_i) + e_i,$$

the claim is proved. □

Proposition 3.2.2. *If $r > n+1$, the Mori cone of $X^{1,n}_r$ is not generated by $e_i, h_1 - e_i, h_2 - e_i$:*

$$\langle e_i, h_1 - e_i, h_2 - e_i \rangle \subsetneqq \mathrm{NE}(X^{1,n}_r).$$

Proof. It is enough to prove it for $r = n+2$. The curve

$$C \sim h_1 + nh_2 - e_1 - \cdots - e_{n+2}$$

is of bidegree $(1, n)$ and passes through the $n+2$ blown-up points. It is effective, since $\tilde{\pi}_2(C) \subset \mathbb{P}^n$ is the class of a rational normal curve of degree n passing through the $n+2$ projected points. If we try to write it as a linear combination of the generators written above, we would have

$$C \sim \sum_i a_i(h_1 - e_i) + \sum_i b_i(h_2 - e_i) + \sum_i c_i e_i,$$

for some coefficients $a_i, b_i, c_i \geq 0$. Then

$$C \sim (h_1 - e_1) + (nh_2 - e_2 - \cdots e_{n+2}) \sim$$

$$\sim (h_1 - e_1) + \underbrace{(h_2 - e_2) + \cdots + (h_2 - e_{n+1})}_{n} - e_{n+2}.$$

So $a_1 = 1$, $a_j = 0$ for $j \neq 1$, $b_i = 1$ for $i = 2, \dots n+1$ and $c_{n+2} = -1$. In no way we can make c_{n+2} a positive coefficient or tie e_{n+2} to one of the h_i. The claim is proved. □

3.3 The Mori cone of $X^{1,n}_{n+2}$

In this section we compute the Mori cone of $X^{1,n}_{n+2}$.

Consider the Segre embedding

$$\begin{array}{cccc} \sigma_{1,n}: & \mathbb{P}^1 \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^{2n+1} \\ & ([u_0, u_1], [v_0, \dots, v_n]) & \mapsto & [u_0v_0, u_0v_1, \dots, u_1v_n] \end{array}$$

and set $\Sigma^{1,n} := \sigma_{1,n}(\mathbb{P}^1 \times \mathbb{P}^n)$. $\Sigma^{1,n}$ is called the Segre variety and it is a smooth $n+1$-fold in $\mathbb{P}^{2n+1}$. Recall that $\deg(\Sigma^{1,n}) = n+1$.

Proposition 3.3.1. *The Mori cone of $X^{1,n}_{n+2}$ is given by:*

$$\mathrm{NE}(X^{1,n}_{n+2}) = \langle h_1 + nh_2 - e_1 - \cdots - e_{n+2}, h_1 - e_i, h_2 - e_i, e_i \rangle$$

Proof. The proof is done by induction on n.

If $n = 1$, then
$$X^{1,1}_3 = \mathrm{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^1).$$

From Proposition 2.5.1, we have

$$\mathrm{N}_1(X^{1,1}_3) = \mathbb{R}[e_1, e_2, e_3, h_1, h_2].$$

The well-known isomorphism

$$S_4 \cong X^{1,1}_3,$$

implies isomorphisms at the level of one-cycles

$$\mathrm{N}_1(S_4) \cong \mathrm{N}_1(X^{1,1}_3)$$

and
$$\mathrm{NE}(S_4) \cong \mathrm{NE}(X^{1,1}_3)$$

between Mori cones. The first isomorphism

$$\mathrm{N}_1(X^{1,1}_3) \to \mathrm{N}_1(S_4)$$

is defined on generators in the following way:

$$\begin{cases} h_1 \longmapsto \overline{h} - \overline{e}_2 \\ h_2 \longmapsto \overline{h} - \overline{e}_1 \\ e_1 \longmapsto \overline{h} - \overline{e}_1 - \overline{e}_2 \\ e_2 \longmapsto \overline{e}_3 \\ e_3 \longmapsto \overline{e}_4 \end{cases}$$

whose inverse is
$$\begin{cases} \overline{e}_1 \longmapsto h_1 - e_1 \\ \overline{e}_2 \longmapsto h_2 - e_1 \\ \overline{e}_3 \longmapsto e_2 \\ \overline{e}_4 \longmapsto e_3 \\ \overline{h} \longmapsto h_1 + h_2 - e_1 \end{cases}$$

The table in 3.1.4 and the isomorphisms above give the following generators for $\mathrm{NE}(X^{1,1}_3)$:

$$\mathrm{NE}(X^{1,1}_3) = \langle h_1 + h_2 - e_1 - e_2 - e_3, h_1 - e_i, h_2 - e_i, e_i \rangle,$$

which conclude this case.

The proof is by induction.

Suppose that the claim is proved for $X^{1,n-1}_{n+1}$ and let $C \subset X^{1,n}_{n+2}$ be an irreducible curve. If C is contracted by π, then $C \subset E_i$ for some $i = 1, \dots, n+2$. Otherwise, $\pi(C)$ is a curve in $\mathbb{P}^1 \times \mathbb{P}^n$ of bidegree (d_1, d_2) and multiplicity m_i at the point p_i. By corollary 2.5.2, it can be written as

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \dots - m_{n+2} e_{n+2}.$$

If

$$d_1 + d_2 \geq m_1 + \dots + m_{n+2},$$

then there are more terms of the form h_i than terms of the form e_i. So, we can couple each of the e_i with one class between h_1 and h_2, to form classes of the form $h_i - e_j$. If there are some h_i to spare, the curve C can be written as a linear combination of

$$h_i - e_j, h_i.$$

Since

$$h_i = (h_i - e_j) + e_j,$$

C is a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_i, e_i$.

Otherwise, we use the Segre embedding. The curve $\sigma_{1,n}(\pi(C))$ is a curve of degree $d_1 + d_2$ in $\mathbb{P}^{2n+1}$, passing through the $n+2$ points $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$, with multiplicities m_i.

Let us denote the $(n+1)$-dimensional linear space $\Pi_{1,\dots,n+2}$
spanned by $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$. If

$$d_1 + d_2 < m_1 + \dots + m_{n+2},$$

the curve $\sigma_{1,n}(\pi(C))$ is contained in this linear space:

$$\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\dots,n+2}.$$

Since the Segre variety $\Sigma^{1,n}$ has degree $n+1$ in $\mathbb{P}^{2n+1}$, and since the points $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$ are in general position, the intersection

$$\Pi_{1,\dots,n+2} \cap \Sigma^{1,n} = C',$$

is a rational normal curve C' of degree $n+1$, passing through $\sigma_{1,n}(p_1), \dots, \sigma_{1,n}(p_{n+2})$.

In $X^{1,n}_{n+2}$, C' has class

$$h_1 + n h_2 - e_1 - \dots - e_{n+2}.$$

Then, C' is a component of $\sigma_{1,n}(\pi(C))$. By irreducibility of C, they must coincide.

Then

$$C \sim \alpha(h_1 + nh_2 - e_1 - \cdots - e_{n+2}),$$

for some positive coefficient α. This concludes the proof. □

3.4 The Mori cone of $X^{1,n}_{n+3}$ for $n \leq 4$.

In this section we compute the generators for the Mori cone of $X^{1,n}_{n+3}$, when $n \leq 4$. In order to do this, we need to introduce rational normal scrolls. A reference for the theory of scrolls is [GH14].

Let a, b be positive integers with $a \leq b$, and $n = a+b+1$, and let $\Lambda^a, \Lambda^b \subset \mathbb{P}^{a+b+1}$ be complementary linear subspaces of dimension a and b in $\mathbb{P}^n$ (that is, Λ^a and Λ^b are disjoint and span $\mathbb{P}^n$). Choose rational normal curves $C_a \subset \Lambda^a$ and $C_b \subset \Lambda^b$, and an isomorphism $\phi : C_a \to C_b$.

Definition 3.4.1. The surface

$$S_{(a,b)} = \bigcup_{x \in C_a} \langle x, \phi(x) \rangle \subset \mathbb{P}^n,$$

where $\langle x, \phi(x) \rangle$ denotes the line through $x, \phi(x)$, is a rational normal scroll of type (a, b). This is a smooth rational surface of degree $\deg(S_{(a,b)}) = a + b$.

Lemma 3.4.2. *Let $H \subset \mathbb{P}^{2n+1}$ be a general $(n+2)$-plane. Then the intersection $H \cap \Sigma^{1,n}$ is a rational normal scroll $S_{(a,b)}$ with*

$$(a,b) = \begin{cases} (\frac{n+1}{2}, \frac{n+1}{2}) & \text{if } n \text{ is odd;} \\ (\frac{n}{2}, \frac{n+2}{2}) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The Segre variety $\Sigma^{1,n}$ is the projectivization over $\mathbb{P}^1$ of the rank $n+1$ vector bundle $\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1}$. A codimension $n-1$ linear section corresponds to the projectivization of the kernel of a morphism

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^1}^{n-1}$$

which is a rank two vector bundle $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$. To conclude it is enough to note that for H general the splitting type is $(-a, -b)$ is the one given in the statement. □

Lemma 3.4.3. *Let $f : X \to Y$ be a morphism of projective varieties, and* $\mathrm{NE}(f)$ *the cone of curves contracted by f. Then* $\mathrm{NE}(f)$ *is extremal in* $\mathrm{NE}(X)$.

Proof. Let Γ be the class of an irreducible curve in $\mathrm{NE}(f)$, and assume that $\Gamma = \Gamma_1 + \Gamma_2$ for $\Gamma_1, \Gamma_2 \in \mathrm{NE}(X)$. Applying f_* we get $f_*\Gamma_1 + f_*\Gamma_2 = f_*\Gamma = 0$ since Γ is contracted by f. Therefore, $f_*\Gamma_1 = f_*\Gamma_2 = 0$ and hence $\Gamma_1, \Gamma_2 \in \mathrm{NE}(f)$. □

Proposition 3.4.4. *Fix $p_1, \dots, p_{n+3} \in \Sigma^{1,n}$ general points. Set $H = \langle p_1, \dots, p_{n+3} \rangle$. Let $\widetilde{S}_{a,b}$ be the blow-ups of $S_{a,b} = H \cap \Sigma^{1,n}$ at the p_i. Then*

$$\mathrm{NE}(X^{1,n}_{n+3}) = \left\langle \mathrm{NE}(\widetilde{S}_{a,b}), h_1 - e_1, \dots, h_1 - e_{n+3} \right\rangle.$$

Proof. Let $C \sim ah_1 + bh_2 - \sum_{i=1}^{n+3} m_i e_i$ be an irreducible curve in $X^{1,n}_{n+3}$, and $\Gamma \subset \Sigma^{1,n}$ its image in $\mathbb{P}^{2n+1}$. Then $\deg(\Gamma) = a + b$ and $\mathrm{mult}_{p_i} \Gamma = m_i$ for $i = 1, \dots, n+3$.

If $a + b < \sum_{i=1}^{n+3} m_i$ then Γ is contained in all the hyperplanes containing H and hence $\Gamma \subset H \cap \Sigma^{1,n}$. By Lemma 3.4.2 $S_{a,b} = H \cap \Sigma^{1,n}$ is a scroll.

If $a + b \geq \sum_{i=1}^{n+3} m_i$ we can pair each e_i with one among h_1 and h_2, and write C as a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_j, e_k$.

The curves of class $h_2 - e_i$ are numerically equivalent to the strict transform of the line through p_i in the ruling of $S_{a,b}$. The curves of class $h_1 - e_i$ are contracted by $\widetilde{\pi}_2$ and the curves of class $h_2 - e_j$ are contracted by $\widetilde{\pi}_1$. Hence, by Lemma 3.4.3 $h_1 - e_i$ and $h_2 - e_j$ generate extremal rays of $\mathrm{NE}(X^{1,n}_{n+3})$.

Summing-up, we have showed that an irreducible curve $C \subset X^{1,n}_{n+3}$ can be written as a linear combination with non negative coefficients of a curve $\Gamma \subset \widetilde{S}_{a,b}$ and of the $h_1 - e_i$. □

With these tools we are now able to prove that

Proposition 3.4.5. *If $n \leq 4$, the Mori cone of $X^{1,n}_{n+3}$ is given by:*

$$\mathrm{NE}(X^{1,n}_{n+3}) = \langle h_1 + nh_2 - e_{i_1} - \dots - e_{i_{n+2}}, h_1 - e_i, h_2 - e_i, e_i \rangle,$$

for $i, i_1, \dots, i_{n+2} \in \{1, \dots, n+3\}$.

Proof. If $n = 1$, then

$$X^{1,1}_4 = \mathrm{Bl}_4(\mathbb{P}^1 \times \mathbb{P}^1).$$

Proposition 2.5.1 tells that:

$$\mathrm{N}_1(X^{1,1}_4) = \mathbb{R}[e_1, \dots, e_4, h_1, h_2].$$

The well-known isomorphism

$$S_5 \cong X^{1,1}_4,$$

implies that

$$\mathrm{N}_1(S_5) \cong \mathrm{N}_1(X^{1,1}_4)$$

and

$$\mathrm{NE}(S_5) \cong \mathrm{NE}(X_4^{1,1}).$$

The first isomorphism

$$\mathrm{N}_1(X_4^{1,1}) \to \mathrm{N}_1(S_5)$$

is defined on generators in the following way:

$$\begin{cases} h_1 \longmapsto \overline{h} - \overline{e}_2 \\ h_2 \longmapsto \overline{h} - \overline{e}_1 \\ e_1 \longmapsto \overline{h} - \overline{e}_1 - \overline{e}_2 \\ e_2 \longmapsto \overline{e}_3 \\ e_3 \longmapsto \overline{e}_4 \\ e_4 \longmapsto \overline{e}_5 \end{cases}$$

whose inverse is

$$\begin{cases} \overline{e}_1 \longmapsto h_1 - e_1 \\ \overline{e}_2 \longmapsto h_2 - e_1 \\ \overline{e}_3 \longmapsto e_2 \\ \overline{e}_4 \longmapsto e_3 \\ \overline{e}_5 \longmapsto e_4 \\ \overline{h} \longmapsto h_1 + h_2 - e_1 \end{cases}$$

The table in 3.1.5 and the isomorphisms above give the following generators for $\mathrm{NE}(X_4^{1,1})$:

$$\mathrm{NE}(X_4^{1,1}) \cong \langle h_1 + h_2 - e_{i_1} - e_{i_2} - e_{i_3}, h_1 - e_i, h_2 - e_i, e_i \rangle$$

for $i, i_1, \ldots, i_3 \in \{1, \ldots, 4\}$, and this concludes this case.

From now on we consider $n > 1$.

Let $C \subset X_{n+3}^{1,n}$ be an irreducible curve. If C is contracted by π, then $C \subset E_i$ for some $i = 1, \ldots, n+3$, and hence C is a multiple of e_i. Otherwise, $\pi(C)$ is a curve in $\mathbb{P}^1 \times \mathbb{P}^n$ of bidegree (d_1, d_2) and multiplicity m_i at the point p_i. By corollary 2.5.2, it can be written as

$$C \sim d_1 h_1 + d_2 h_2 - m_1 e_1 - \cdots - m_{n+3} e_{n+3}.$$

If

$$d_1 + d_2 \geq m_1 + \cdots + m_{n+3},$$

then there are more terms of the form h_i than terms of the form e_i. So, we can couple each of the e_i with one class between h_1 and h_2, to form classes of the form

$h_i - e_j$. If there are some h_i to spare, the curve C can be written as a linear combination of

$$h_1 - e_j, h_2 - e_i, h_i.$$

Since

$$h_i = (h_i - e_j) + e_j,$$

C is a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_i, e_i$.

Note that the curve $\sigma_{1,n}(\pi(C))$ is a curve of degree d_1+d_2 in $\mathbb{P}^n$, passing through the points $\sigma_{1,n}(p_1), \ldots, \sigma_{1,n}(p_{n+3})$ with multiplicities m_i. Let $\Pi_{1,\ldots,n+3} \subset \mathbb{P}^{2n+1}$ be the $(n+2)$-plane generated by $\sigma_{1,n}(p_1), \ldots, \sigma_{1,n}(p_{n+3})$. If

$$d_1 + d_2 < m_1 + \cdots + m_{n+3},$$

then $\sigma_{1,n}(\pi(C)) \subset \Pi_{1,\ldots,n+3}$. By lemma 3.4.2 the intersection $X = \Pi_{1,\ldots,n+3} \cap \Sigma^{1,n}$ is a degree $n+1$ scroll in $\mathbb{P}^{2n+1}$. By Proposition 3.4.4, the Mori cone of $X^{1,n}_{n+3}$ is generated by the Mori cone of X and by $h_1 - e_i$ for $i = 1, \ldots, n+3$. Since

$$\sigma_{1,n}(\pi(C)) \subset X,$$

in order to find generators for the Mori cone of $X^{1,n}_{n+3}$, we need to compute generators for $\mathrm{NE}(X)$. We will do this by considering separately the cases $n = 2, 3, 4$.

If $n = 2$, X is a cubic scroll in the four dimensional linear space $\Pi_{1,\ldots,5}$. If we look at this surface in the Segre variety of the product $\mathbb{P}^1 \times \mathbb{P}^2$, we see that

$$X = \sigma_{1,2}(\mathrm{Bl}_1(\mathbb{P}^2)),$$

and that the projection $\pi_2 : X \to \mathbb{P}^2$ is the blow-down of the exceptional divisor.

Hence, by blowing up we get that

$$C \subset \mathrm{Bl}_5(X) \overset{(1)}{\cong} S_6,$$

which is a Del Pezzo surface of degree three, that we denoted by S_6. The Mori cone of S_6 is described in table 3.1.6. The isomorphism (1) induces a map

$$\mathrm{N}_1(S_6) \to \mathrm{N}_1(X^{1,2}_5),$$

which is defined on the basis of $\mathrm{N}_1(S_6)$ as follows :

$$\begin{cases} \overline{e}_1 \longmapsto h_1 \\ \overline{h} - \overline{e}_1 \longmapsto h_2 \\ \overline{e}_i \longmapsto e_{i-1} \end{cases},$$

for $i = 2, \ldots, 5$. To find the Mori cone of $\mathrm{Bl}_5(X)$ we write down the images of the generators of $\mathrm{NE}(S_6)$ through the isomorphism above.

First, note that $\overline{e}_i = e_{i-1}$ for $i = 2, \dots, 6$, and $\overline{e}_1 = h_1 = (h_1 - e_i) + e_i$. Furthermore

$$\begin{aligned}
&\overline{h} - \overline{e}_i - \overline{e}_j &&= (h_1 - e_{i-1}) + (h_2 - e_{j-1}) \text{ for } i, j = 2, \dots, 6;\\
&\overline{h} - \overline{e}_1 - \overline{e}_j &&= h_2 - e_{j-1} \text{ for } j = 2, \dots, 6;\\
&2\overline{h} - \overline{e}_2 - \cdots - \overline{e}_6 &&= (h_1 + 2h_2 - e_2 - \cdots - e_6) + (h_1 - e_6)\\
&2\overline{h} - \overline{e}_1 - \overline{e}_{j_1} - \cdots - \overline{e}_{j_4} &&= h_1 + 2h_2 - e_{j_1-1} - \cdots - e_{j_4-1}.
\end{aligned}$$

This computation concludes this case.

If $n = 3$, X is a quartic scroll in $\mathbb{P}^5$. Since the quartic scroll is obtained by intersecting the Segre variety with two general hyperplanes, the corresponding surface in $\mathbb{P}^1 \times \mathbb{P}^3$ is the intersection between two hypersurfaces of bidegree $(1, 1)$:

$$S \subseteq \mathbb{P}^1 \times \mathbb{P}^3 \xrightarrow{\sigma_{1,3}} X \subseteq \Sigma_{1,3} \subseteq \mathbb{P}^7$$

$$\Omega \xmapsto{\sigma_{1,3}} X,$$

with

$$\Omega = \begin{cases} x_0 f_1(y_0, \dots, y_3) + x_1 f_2(y_0, \dots, y_3) = 0 \\ x_0 g_1(y_0, \dots, y_3) + x_1 g_2(y_0, \dots, y_3) = 0 \end{cases},$$

where

$$f_i, g_i \in k[y_0, \dots, y_3]_1.$$

The image of the projection $S \to \mathbb{P}^3$ is the quadric surface $\{f_1 g_2 - f_2 g_1 = 0\} \subset \mathbb{P}^3$. Note that under the generality condition, f_1, f_2, g_1, g_2 do not vanish simultaneously at a point, hence such projection is an isomorphism. Since any quadric surface in $\mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, we can conclude that

$$\Omega \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Hence,

$$C \subset \mathrm{Bl}_6(\Omega) \cong \mathrm{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1) \cong S_7.$$

The isomorphism

$$S_7 \cong \mathrm{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1)$$

gives a map

$$\phi : \mathrm{N}_1(S_7) \longrightarrow \mathrm{N}_1(\mathrm{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1)),$$

given by the following assignement:

$$\begin{cases} \overline{h} \longmapsto \tilde{h}_1 + \tilde{h}_2 - \tilde{e}_1 \\ \overline{e}_1 \longmapsto \tilde{h}_1 - \tilde{e}_1 \\ \overline{e}_2 \longmapsto \tilde{h}_2 - \tilde{e}_1 \\ \overline{e}_i \longmapsto \tilde{e}_{i-1} \end{cases},$$

for $i = 3, \dots, 7$.

We need to understand how Ω is contained in $\mathbb{P}^1 \times \mathbb{P}^3$. We can write

$$\Omega \cong \mathbb{P}^1 \times \mathbb{P}^1 \overset{i}{\hookrightarrow} \mathbb{P}^1 \times \mathbb{P}^3$$

$$((x_0, x_1), (\mu_0, \mu_1)) \longmapsto ((x_0, x_1), (x_1\mu_0, x_0\mu_0, x_1\mu_1, x_0\mu_1)),$$

and notice that for any fixed point $(\bar{x_0}, \bar{x_1}) \in \mathbb{P}^1$, we have

$$i((\bar{x_0}, \bar{x_1}) \times \mathbb{P}^1) = ((\bar{x_0}, \bar{x_1}), (\bar{x_1}\mu_0, \bar{x_0}\mu_0, \bar{x_1}\mu_1, \bar{x_0}\mu_1)),$$

which is a curve in $\mathbb{P}^1 \times \mathbb{P}^3$ with class h_2. Now if we fix a point $(\bar{\mu_0}, \bar{\mu_1})$ in the second factor, we have:

$$i((x_0, x_1), (\bar{\mu_0}, \bar{\mu_1})) = ((x_0, x_1), (x_1\bar{\mu_0}, x_0\bar{\mu_0}, x_1\bar{\mu_1}, x_0\bar{\mu_1})),$$

which is a curve of class $h_1 + h_2$. This inclusion i gives a map

$$i_1 : \mathrm{N}_1(\mathrm{Bl}_6(\mathbb{P}^1 \times \mathbb{P}^1)) \to \mathrm{N}_1(X^{1,3}_6)$$

given by:

$$\Omega = \begin{cases} \tilde{h}_1 \longmapsto h_2 \\ \tilde{h}_2 \longmapsto h_1 + h_2 \\ \tilde{e}_i \longmapsto e_i \end{cases},$$

for $i = 1, \dots, 6$.

Hence, composing the two maps i_1 and ϕ we get a map

$$i_1 \circ \phi : \mathrm{N}_1(S_7) \to \mathrm{N}_1(X^{1,3}_6),$$

which is defined as

$$\Omega = \begin{cases} \overline{h} \longmapsto h_1 + 2h_2 - e_1 \\ \overline{e}_1 \longmapsto h_2 - e_1 \\ \overline{e}_2 \mapsto h_1 + h_2 - e_1 \\ \overline{e}_i \mapsto e_{i-1} \end{cases},$$

for $i = 3, \dots, 7$. Now, consider the table introduced in 3.1.7 with the generators for the Mori cone of S_7. The map $i_1 \circ \phi$ induces an isomorphism

$$\mathrm{NE}(S_7) \xrightarrow{\cong} \mathrm{NE}(X^{1,3}_6).$$

In order to find the generators of $\mathrm{NE}(X^{1,3}_6)$, we need to take the generators of $\mathrm{NE}(S_7)$ and look at their images via the map $i_1 \circ \phi$.

Let us start with the curves $\overline{e}_i$, for $i = 1, \ldots, 7$. We see that $\overline{e}_1$ and $\overline{e}_2$ are sent respectively to $h_2 - e_1$ and to $h_1 + (h_2 - e_1)$. Since $h_1 = (h_1 - e_i) + e_i$, these two curves are linear combination of $h_1 - e_i, h_2 - e_i$ and e_i. The same is true for the image of $\overline{e}_i$ when $i = 3, \ldots 7$: these curves are sent to e_{i-1}. Now we consider the other generators appearing in 3.1.7. The following table must be read in this way: on the left, we have written the generators for $\mathrm{NE}(S_7)$. Since the images of $\overline{e}_i$ depends upon the value of i, we have separated the case for which $i = 1$,$i = 2$ or $i = 3, \ldots 7$. The "$=$" must be intended as "it is sent via $i_1 \circ \phi$ to". On the right, we have the images of the generators written in a form that permits to write them as a linear combination of $h_1 - e_i, h_2 - e_i, e_i$ and $h_1 + 3h_2 - e_{i_1} - \cdots - e_{i_5}$.

$$
\begin{array}{ll}
\overline{h} - \overline{e}_1 - \overline{e}_2 & = e_1; \\
\overline{h} - \overline{e}_1 - \overline{e}_i & = (h_1 - e_1) + (h_2 - e_{i-1}) + e_1; \\
\overline{h} - \overline{e}_2 - \overline{e}_i & = h_2 - e_{i-1}; \\
\overline{h} - \overline{e}_i - \overline{e}_j & = (h_1 - e_1) + (h_2 - e_{i-1}) + (h_2 - e_{j-1}); \\
2\overline{h} - \overline{e}_1 - \overline{e}_2 - \overline{e}_{i_1} - \overline{e}_{i_2} - \overline{e}_{i_3} & = (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + (h_2 - e_{i_3-1}); \\
2\overline{h} - \overline{e}_1 - \overline{e}_{i_1} - \overline{e}_{i_2} - \overline{e}_{i_3} - \overline{e}_{i_4} + & = (h_1 - e_1) + (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) \\
 & (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\
2\overline{h} - \overline{e}_2 - \overline{e}_{i_1} - \overline{e}_{i_2} - \overline{e}_{i_3} - \overline{e}_{i_4} & = h_1 + 3h_2 - e_1 - e_{i_1-1} - e_{i_2-1} - e_{i_3-1} - e_{i_4-1}; \\
3\overline{h} - 2\overline{e}_1 - \overline{e}_2 - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5} & = h_1 + (h_1 + 3h_2 - e_{i_1-1} - \cdots - e_{i_5-1}); \\
3\overline{h} - \overline{e}_1 - 2\overline{e}_2 - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5} & = h_1 + 3h_2 - e_{i_1-1} - \cdots - e_{i_5-1}; \\
3\overline{h} - \overline{e}_1 - \overline{e}_2 - 2\overline{e}_{i_1} - \cdots - \overline{e}_{i_5} & = (h_1 - e_1) + (h_2 - e_{i_1-1}) + \\
 & (h_1 + 3h_2 - e_{i_1-1} - \cdots - e_{i_5-1});
\end{array}
$$

and this concludes the proof, since we see that we can write each class of curve on the right as a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_i, e_i$ and $h_1 + 3h_2 - e_{i_1} - \cdots - e_{i_5}$.

If $n = 4$, X is a quintic scroll in $\mathbb{P}^6$. Since X is the intersection between three general hyperplanes and $\Sigma_{1,4}$, the corresponding surface in $\mathbb{P}^1 \times \mathbb{P}^4$ is a complete intersection of three hypersurfaces of bidegree $(1, 1)$, described by equations:

$$
X = \begin{cases} x_0 f_1(y_0, \ldots, y_4) + x_1 f_2(y_0, \ldots, y_4) = 0 \\ x_0 g_1(y_0, \ldots, y_4) + x_1 g_2(y_0, \ldots, y_4) = 0 \\ x_0 h_1(y_0, \ldots, y_4) + x_1 h_2(y_0, \ldots, y_4) = 0 \end{cases},
$$

where

$$
f_i, g_i \in k[y_0, \ldots, y_4]_1.
$$

Consider the following diagram:

$$\begin{array}{ccc} & S \subset \mathbb{P}^1 \times \mathbb{P}^4 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathbb{P}^1 & & \mathbb{P}^4 \end{array}$$

We have that

$$\pi_{1_{|_S}} : S \to \mathbb{P}^1$$

is a fibration over $\mathbb{P}^1$ whose fibers are lines in $\mathbb{P}^4$ and S of class h_2, and that $\pi_2(S) \subseteq \mathbb{P}^4$ is a cubic 3-fold. The linear system of conics in $\mathbb{P}^2$ through $p = [1 : 0 : 0] \in \mathbb{P}^2$ gives a rational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$$

$$[x, y, z] \mapsto ([xy, xz, y^2, yz, z^2])$$

Hence, blowing up the singularity gives:

$$\begin{array}{ccc} Bl_p(\mathbb{P}^2) & & \\ \downarrow{\scriptstyle\pi} & \searrow^{\tilde{\tau}} & \\ \mathbb{P}^2 & \overset{\tau}{\dashrightarrow} & S \subseteq \mathbb{P}^1 \times \mathbb{P}^4 \end{array}$$

where

$$\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^4$$

is the rational map given by

$$\tau[x, y, z] = ([y, z], [xy, xz, y^2, yz, z^2]),$$

π and $\tilde{\tau}$ its resolution.

Since $\tilde{\tau}$ is an isomorphism onto its image,

$$S \cong \mathrm{Bl}_p \mathbb{P}^2,$$

hence $C \subset \mathrm{Bl}_7(S) \cong \mathrm{Bl}_8(\mathbb{P}^2)$, which is a Del Pezzo surface of degree one, whose generators of the Mori cone are listed in 3.1.8.

The class $\overline{h}$, represented by the line of equation $\{x = 0\}$ in $\mathrm{Bl}_p \mathbb{P}^2$, is sent through the lift of τ to:

$$([y, z], [0, 0, y^2, yz, z^2]) \subseteq \mathbb{P}^1 \times \mathbb{P}^4,$$

with class $h_1 + 2h_2$ in $\mathrm{N}_1(\mathbb{P}^1 \times \mathbb{P}^4)$. Analogously, a line in $\mathrm{Bl}_p(\mathbb{P}^2)$ passing through the point p, of class $\overline{h} - \overline{e}$, represented by the line of equation $\{y = 0\}$, is sent through the lift of τ to:

$$([0, 1], [0, x, 0, 0, z]) \subseteq \mathbb{P}^1 \times \mathbb{P}^4,$$

with class h_2. The calculations above translate into a map $\psi : \mathrm{N}_1(S_8) \to \mathrm{N}_1(X_7^{1,4})$, defined by

$$\begin{cases} \overline{h} \longmapsto h_1 + 2h_2 \\ \overline{e}_1 \longmapsto h_1 + h_2 \\ \overline{e}_i \longmapsto e_{i-1} \end{cases},$$

for $i = 2, \dots, 8$. The map ψ induces an isomorphism

$$\mathrm{NE}(S_8) \xrightarrow{\cong} \mathrm{NE}(X_7^{1,4}).$$

In order to find the generators of $\mathrm{NE}(X_7^{1,4})$, we need to take the generators of $\mathrm{NE}(S_8)$ and look at their images via the map ψ.

Let us start with the curves $\overline{e}_i$, for $i = 1, \dots, 8$. We see that $\overline{e}_1$ is sent to $h_1 + h_2$. Since $h_1 = (h_1 - e_i) + e_i$ and $h_2 = (h_2 - e_i) + e_i$, this curve is a linear combination of $h_1 - e_i, h_2 - e_i$ and e_i. The same it's true for the image of $\overline{e}_i$ when $i = 2, \dots 8$: these curves are sent to e_{i-1}. Now we consider the other generators appearing in 3.1.8. The following table must be read in this way: on the left, we have written the generators for $\mathrm{NE}(S_8)$. Since the images of $\overline{e}_i$ depends upon the value of i, we have separated the case for which $i = 1$ or $i = 2, \dots 8$. The " = " must be intended as "it is sent via ψ to". On the right, we have the images of the generators written in a form that permits to write them as a linear combination of $h_1 - e_i, h_2 - e_i, e_i$ and $h_1 + 4h_2 - e_{i_1} - \dots - e_{i_6}$.

$$
\begin{array}{ll}
\overline{h} - \overline{e}_1 - \overline{e}_j & = h_2 - e_{j-1}; \\
\overline{h} - \overline{e}_i - \overline{e}_j & = (h_1 - e_1) + (h_2 - e_{i-1}) \\
 & +(h_2 - e_{j-1}) + e_1; \\
2\overline{h} - \overline{e}_1 - \overline{e}_{i_1} - \cdots - \overline{e}_{i_4} & = (h_1 - e_{i_1-1}) + (h_2 - e_{i_2-1}) + \\
 & (h_2 - e_{i_3-1}) + (h_2 - e_{i_4-1}); \\
2\overline{h} - \overline{e}_{i_1} - \cdots - \overline{e}_{i_5} & = \sum_{k=1}^{2}(h_1 - e_{i_k-1}) + \\
 & \sum_{t=3}^{5}(h_2 - e_{i_t-1}) + (h_2 - e_1) + e_1; \\
3\overline{h} - 2\overline{e}_i - \overline{e}_{j_1} - \cdots - \overline{e}_{j_6} & = 2(h_1 - e_{i-1}) + (h_1 - e_{j_1-1}) + \\
 & \sum_{k=1}^{6}(h_2 - e_{j_k-1}) + e_{j_1-1}; \\
3\overline{h} - 2\overline{e}_1 - \overline{e}_{j_1} - \cdots - \overline{e}_{j_6} & = h_1 + 4h_2 - e_{j_1-1} - \cdots - e_{j_6-1}; \\
3\overline{h} - 2\overline{e}_i - \overline{e}_1 - \overline{e}_{j_1} - \cdots - \overline{e}_{j_5} & = 2(h_1 - e_{i-1}) + \\
 & (h_2 - e_{j_1-1}) + \cdots + (h_2 - e_{j_5-1}); \\
4\overline{h} - 2\overline{e}_1 - 2\overline{e}_{i_1} - 2\overline{e}_{i_2} - \overline{e}_{j_1} - \cdots - \overline{e}_{j_5} & = 2(h_2 - e_{i_1-1}) + (h_1 - e_{i_2-1}) + \\
 & (h_1 + 4h_2 - e_{i_2-1} - \sum_{k=1}^{5} e_{j_k-1}); \\
4\overline{h} - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{i_3} - \overline{e_1} - \overline{e}_{j_1} - \cdots - \overline{e}_{j_4} & = \sum_{k=1}^{3} 2(h_2 - e_{i_k}) + (h_2 - e_{j_1-1}) + \\
 & \sum_{t=2}^{4}(h_1 - e_{j_t-1}); \\
5\overline{h} - 2\overline{e}_1 - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{i_5} - \overline{e}_{j_1} - \overline{e}_{j_2} & = (h_1 + 4h_2 - 2\sum_{k=1}^{3} e_{i_k-1}) \\
 & +2\sum_{t=1}^{2}(h_t - e_{i_{t+3}-1}) \\
 & +\sum_{k=1}^{2}(h_2 - e_{j_k-1}); \\
5\overline{h} - 2\overline{e}_{i_1} - \cdots - 2\overline{e}_{i_6} - \overline{e}_1 - \overline{e}_j & = \sum_{k=1}^{4} 2(h_2 - e_{i_k-1}) + \\
 & \sum_{t=5}^{6} 2(h_1 - e_{i_t-1}) + (h_2 - e_{j-1}); \\
6\overline{h} - 2\overline{e}_1 - 3\overline{e}_2 - 2\overline{e}_3 - \cdots - 2\overline{e}_8 & = 2(h_1 + 4h_2 - e_3 - \cdots - e_8) + \\
 & (h_1 - e_2) + 2(h_2 - e_2); \\
6\overline{h} - 3\overline{e}_1 - 2\overline{e}_2 - \cdots - 2\overline{e}_8 & = 2(h_1 + 4h_2 - e_2 - \cdots - e_7) + \\
 & (h_1 - e_8) + (h_2 - e_8);
\end{array}
$$

and this concludes the proof, since we see that we can write each class of curve on the right as a linear combination with non negative coefficients of $h_1 - e_i, h_2 - e_i, e_i$ and $h_1 + 4h_2 - e_{i_1} - \cdots - e_{i_6}$.

□

3.5 The variety $X_r^{1,n}$ is log Fano for $r \leq n+1$

As a consequence of Proposition 3.2.1 we want to prove the following proposition:

Proposition 3.5.1. *The variety $X_r^{1,n}$ is log Fano for $r \leq n+1$.*

Consider the case $r = n+1$. The result for $r < n+1$ will then follow from [PS07, Theorem 2.9]. Recall that the anti-canonical divisor on $X_{n+1}^{1,n}$ is

$$-K_{X_{n+1}^{1,n}} = 2H_1 + (n+1)H_2 - nE_1 - \cdots - nE_{n+1}.$$

According to definition 2.4.8, we need to find an effective $\mathbb{Q}$-divisor Δ on $X^{1,n}_{n+1}$ such that $-K_{X^{1,n}_{n+1}} - \Delta$ is ample and the pair $(X^{1,n}_{n+1}, \Delta)$ is Kawamata log terminal (see Definition 1.0.12 for the introduction of the notion of klt). Consider the divisor

$$D = (n+1)H_2 - nE_1 - \cdots - nE_{n+1}.$$

The divisor D is effective by construction, since it is the pull-back of the effective divisor consisting in the union of the hyperplanes in $\mathbb{P}^n$ passing through n among the $n+1$ projections of the blown-up points. Note that there are exactly $n+1$ hyperplanes in this union and that any projected point is contained in exactly n different hyperplanes.

Set $\Delta_\epsilon = \epsilon D$, with $\epsilon \in \mathbb{Q}_{>0}$.

We need to establish for which ϵ the divisor

$$-K_{X^{1,n}_{n+1}} - \Delta_\epsilon = 2H_1 + ((n+1) - \epsilon(n+1))H_2 + (n\epsilon - n)\sum_{i=1}^{n+1} E_i$$

is ample.

By Proposition 3.2.1, the Mori cone of $X^{1,n}_{n+1}$ is

$$NE(X^{1,n}_{n+1}) = \langle h_1 - e_i, h_2 - e_i, e_i \rangle.$$

Since

$$\begin{aligned}
(-K_{X^{1,n}_{n+1}} - \Delta_\epsilon) \cdot e_i = n - \epsilon n > 0 \quad & \text{if and only if} \quad \epsilon < 1;\\
(-K_{X^{1,n}_{n+1}} - \Delta_\epsilon) \cdot (h_1 - e_i) = 2 - n + \epsilon n > 0 \quad & \text{if and only if} \quad \epsilon > (n-2)/n;\\
(-K_{X^{1,n}_{n+1}} - \Delta_\epsilon) \cdot (h_2 - e_i) = n + 1 - \epsilon(n+1) - n + \epsilon n > 0 \quad & \text{if and only if} \quad \epsilon < 1;
\end{aligned}$$

Hence $-K_{X^{1,n}_{n+1}} - \Delta_\epsilon$ is ample if and only if $(n-2)/n < \epsilon < 1$.

Now we show that the pair $(X^{1,n}_{n+1}, \Delta_\epsilon)$ is Kawamata log terminal.

In order to do this we find the singular locus of the divisor. It suffices to show that Δ_ϵ is simple normal crossing, that is a divisor that can be written as a sum $D = \sum D_i$, where D_i are smooth and they intersect transversely. Consider the images $q_i = \pi_2(p_i)$ of the blown-up points via the second projection. There exist $n+1$ hyperplanes in $\mathbb{P}^n$ through each subset of n points among the q_i, and through each q_i there pass n of them. Denote by $\tilde{H}_i \subset X^{1,n}_{n+1}$ the strict transforms of the inverse image via the second projection of these hyperplanes. The inverse image of n of these hyperplanes intersect in $\mathbb{P}^1 \times \mathbb{P}^n$ along the curve $\pi_2^{-1}(q_i)$. Hence, for all $i = 1, \ldots, n+1$, the divisor D, which is the strict transform of the union of these hyperplanes, has multiplicity n along the strict transforms of the curves $\pi_2^{-1}(q_i)$. Since the strict transform of the curve has codimension n in $X^{1,n}_{n+1}$, any n among the divisors $\tilde{H}_i$ intersect transversally along one of the strict transforms of these curves, and this is true for all $i = 1, \ldots, n+1$.

Now, we can generalize this argument for other varieties in the singular locus. Fix a set of m points among the q_i, with $1 \leq m \leq (n-1)$, and denote by Λ_m their linear span. Denote by $\tilde{\Lambda}_m$ the strict transform of $\pi_2^{-1}(\Lambda_m)$. Any $(n-m+1)$ among the $\tilde{H}_i$ intersect along $\tilde{\Lambda}_m$, so this linear span is in the singular locus of D and D has multiplicity $n+1-m$ along $\tilde{\Lambda}_m$. Since $\tilde{\Lambda}$ has codimension $(n-m+1)$, the $\tilde{H}_i$ intersect transverally along the subvarieties $\tilde{\Lambda}_m$.

Then, Δ_ϵ is a simple normal crossing effective divisor for all $\epsilon > 0$.

We conclude that for $(n-2)/n < \epsilon < 1$, the divisor $-K_{X_{n+1}^{1,n}} - \Delta_\epsilon$ is ample and the pair $(X_{n+1}^{1,n}, \Delta_\epsilon)$ is Kawamata log terminal, concluding the proof.

Chapter 4

The Cox ring of $X^{1,n}_{n+1}$ and its moving cone

This chapter consists of two sections. In the first section we give a complete description of the Cox ring of $X^{1,n}_{n+1}$. Since $X^{1,n}_{n+1}$ is equipped with a torus action, we can use the procedure developed in [HS10] and explained in the second section of Chapter 2. In the second section we compute the generators for the moving cone of $X^{1,n}_{n+1}$, following Proposition 2.2.10.

4.1 The Cox ring of $X^{1,n}_{n+1}$

The aim of this section is to compute generators for the Cox ring of $X^{1,n}_{n+1}$. For this purpose, we need to equip $X^{1,n}_{n+1}$ with a torus action.

The n-dimensional complex torus $T = (\mathbb{C}^*)^n$ acts on $(\mathbb{P}^1 \times \mathbb{P}^n)$ as follows:

$$\begin{array}{ccc} T \times (\mathbb{P}^1 \times \mathbb{P}^n) & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^n \\ ((t_1, \ldots, t_n), ([x_0, x_1], [y_0, \ldots, y_n])) & \longmapsto & ([x_0, x_1], [y_0, t_1y_1, t_2y_2, \ldots, t_ny_n]). \end{array} \tag{4.1}$$

We can suppose that the $n+1$ general blown-up points in $\mathbb{P}^1 \times \mathbb{P}^n$ are the following points:

$$\begin{aligned}
&p_1 = ([0,1],[1,0,\ldots,0]);\\
&p_2 = ([1,0],[0,1,0,\ldots,0]);\\
&p_3 = ([1,1],[0,0,1,0,\ldots,0]);\\
&p_4 = ([\alpha_4,\beta_4],[0,0,0,1,0,\ldots,0]);\\
&\vdots\\
&p_{n+1} = ([\alpha_{n+1},\beta_{n+1}],[0,\ldots,0,1]);
\end{aligned}$$

where
$$[\alpha_i, \beta_i] \notin \{[0,1],[1,0],[1,1]\},$$
for $i = 4, \dots, n+1$ and
$$[\alpha_i, \beta_i] \neq [\alpha_j, \beta_j],$$
for $i \neq j$.

Since the blown-up points are fixed by the algebraic torus T, i.e.

$$T(p_i) = p_i,$$

the action 4.1 lifts to an action of T on $X^{1,n}_{n+1}$. Note that the lifted action fixes every exceptional divisor E_i.

Now that $X^{1,n}_{n+1}$ is equipped with an algebraic torus action, we can apply the construction developed in [HS10] and explained in Proposition 2.2.9. First, we need to construct the open subset of points in $X^{1,n}_{n+1}$ having zero-dimensional isotropy group
$$X_0 := \{x \in X^{1,n}_{n+1} \mid \dim(T_x) = 0\} \subseteq X^{1,n}_{n+1}.$$

Let us examine the action of T on a point p which is not a blown up point. We can assume that the point p has coordinate

$$([\alpha, \beta], [y_0, y_1, \dots, y_n]).$$

The lift of the action sends p to

$$([\alpha, \beta], [y_0, t_1 y_1, \dots, t_n y_n]).$$

If $\pi_2(p)$ has some zero coordinate, its isotropy group has dimension greater or equal than one. Indeed, suppose that there exists $i \in \{0, \dots, n\}$ such that $y_i = 0$.

If $i = 0$, we have

$$([\alpha, \beta], [0, y_1, \dots, y_n]) \mapsto ([\alpha, \beta], [0, t_1 y_1, \dots, t_n y_n]),$$

and the isotropy group
$$T_x = \{(t, \dots, t)\} \subseteq T$$
has dimension one.

If $i > 0$, we have

$$([\alpha, \beta], [y_0, y_1, y_2, \dots, 0, \dots, y_n]) \mapsto ([\alpha, \beta], [y_0, t y_1, t_2 y_2, \dots, 0, \dots, t_n y_n]),$$

and the isotropy group

$$T_x = \{(1, \dots, t_i, \dots, 1)\} \subseteq T$$

has dimension one.

Viceversa, if all the coordinates of $\pi_2(p)$ are nonzero, the isotropy group T_p is the trivial subgroup of T.

Hence, consider the hyperplanes in $\mathbb{P}^n$ of equation

$$\{y_i = 0\},$$

for $i = 1, \ldots, n+1$, and denote by $\widetilde{H_i}$ the strict transform of $\pi_2^{-1}(\{y_i = 0\})$.

We have proved that

$$X_{n+1}^{1,n} \setminus X_0 \supseteq \bigcup_{i=0}^{n} \widetilde{H_i}$$

and it follows that

$$X_0 \subseteq X_{n+1}^{1,n} \setminus (\bigcup_{i=0}^{n} \widetilde{H_i}).$$

Since any exceptional divisor E_i is isomorphic to $\mathbb{P}^n$, the lift of the action acts on it in such a way that a general point of E_i has trivial isotropy group. Denote by $R_i \subset E_i$ the union of the T-invariant divisors of the lift of the action restricted to E_i. Then

$$X_0 = X_{n+1}^{1,n} \setminus \left(\bigcup_{i=0}^{n} \widetilde{H_i} \cup \bigcup_{j=1}^{n+1} R_j \right) \subset X_{n+1}^{1,n}.$$

Since the biggest T-orbits of the lift of the action are the fibers $\mathbb{P}^n$, the action of T on $X_{n+1}^{1,n}$ is of complexity one. Hence, the orbit space X_0/T is of dimension one and has a separation to $\mathbb{P}^1$, namely a rational map

$$X_0/T \dashrightarrow \mathbb{P}^1,$$

which is a local isomorphism in codimension one.

Since the general points of each exceptional divisor E_i has trivial isotropy group, the only T-invariant prime divisors supported in $X_{n+1}^{1,n} \setminus X_0$ are the $\widetilde{H_i}$.

We need to find the T-invariant prime divisors with finite generic isotropy group of order $l_j > 1$, denoted by D_i.

Let D be a T-invariant divisor in $X_{n+1}^{1,n}$. Then D is either an exceptional divisor or it is a divisor in $\mathbb{P}^1 \times \mathbb{P}^n$.

Since the general point of E_i has trivial isotropy group, its order l_j is not strictly greater than one.

If D is a T-invariant divisor in $\mathbb{P}^1 \times \mathbb{P}^n$, either D is a fiber $\mathbb{P}^n$, whose general point has trivial isotropy group, or D intersects a fiber $\mathbb{P}^n$ in a divisor of $\mathbb{P}^n$.

In the last case, since the lift of the action 4.1 is transitive on each fiber, the intersection between D and the fiber is fixed. Hence it is one of the $\pi_2^{-1}(\{y_i = 0\})$.

But these divisors have non finite isotropy group. We conclude that there are no divisors of type D_i.

Since the lift of the action is of complexity one, the orbit space X_0/T has a separation to $\mathbb{P}^1$, namely a rational map

$$X_0/T \dashrightarrow \mathbb{P}^1,$$

which gives an isomorphism

$$X_0/T \overset{\cong}{\to} \mathbb{P}^1 \setminus \{\pi_1(p_1), \dots, \pi_1(p_{n+1})\},$$

where $\pi_1(p_1), \dots, \pi_1(p_{n+1})$ are the blown-up points.

Infact, by the definition of the action, we can construct the following diagram which shows the composition of two projections: the map q is the projection to the quotient and π is the morphsim given by the separation to $\mathbb{P}^1$.

$$\begin{array}{c} X_0 \\ \big\downarrow{\scriptstyle q} \\ X_0/T \\ \big\downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 \end{array}$$

We see that the composition $q \circ \pi$ is a separation too. Let us find the preimage of the points $\{\pi_1(p_i)\}$ via the maps π and q: we have that

$$\pi^{-1}(\pi_1(p_i)) = \{y_{i_1}, y_{i_2}\},$$

where y_{i_1}, y_{i_2} are the two classes in the quotient X_0/T representing the two following classes in X_0:

$$D_{i_1} := q^{-1}(y_{i_1}) \quad \text{and} \quad D_{i_2} := q^{-1}(y_{i_2}).$$

D_{i_1} is isomorphic to $\mathbb{P}^n$ blown-up in a point with exceptional divisor $\overline{E}_i \cong \mathbb{P}^{n-1}$, and D_{i_2} is isomorphic to E_i. Note that

$$D_{i_1} \cap D_{i_2} = \overline{E}_i.$$

The isotropy group of a general point of D_{i_1} and D_{i_2} is trivial, hence of order one.

Now, associate to the $\widetilde{H}_i$ variables S_i, to the D_{i_1} variables $T_{i,1}$ and to the D_{i_2} variables $T_{i,2}$. For $1 \leq i \leq n-1$, set $k = j+1 = i+2$ and define the trinomial

$$g_i = (\beta_k\alpha_j - \beta_j\alpha_k)T_{i,1}T_{i,2} + (\beta_i\alpha_k - \beta_k\alpha_i)T_{j,1}T_{j,2} + (\beta_j\alpha_i - \beta_i\alpha_j)T_{k,1}T_{k,2}.$$

For example, the first one of these trinomials is

$$g_1 = -T_{1,1}T_{1,2} - T_{2,1}T_{2,2} + T_{3,1}T_{3,2}.$$

Then we have the following result:

Theorem 4.1.1. *For the Cox ring of $X^{1,n}_{n+1}$ we have the following explicit presentation*

$$\mathrm{Cox}(X^{1,n}_{n+1}) \cong \frac{\mathbb{C}[S_0,\dots,S_n,T_{1,1},T_{1,2},\dots,T_{n+1,1},T_{n+1,2}]}{\langle g_1,\dots,g_{n-1}\rangle}$$

where S_i is the section associated to $H_2 - E_1 - \cdots - E_{i-1} - E_{i+1} - \cdots - E_{n+1}$, $T_{i,1}$ is the section associated to $H_1 - E_i$ and $T_{i,2}$ is the section associated to E_i.

Proof. In the previous set up it is enough to apply Theorem 2.2.9. □

Proposition 4.1.2. *[BP04, Remark* 1.4*] If a variety X has a finitely generated Cox ring, we have the following formula:* $\dim(\mathrm{Cox}(X)) = \dim(X) + \mathrm{rk}(\mathrm{Pic}(X))$.

We verifiy this formula in our setting. Since

$$\mathrm{Pic}(X^{1,n}_{n+1}) = \langle H_1, H_2, E_1, \dots, E_{n+1}\rangle,$$

We have that

$$\dim(X^{1,n}_{n+1}) = n+1,$$

and that

$$\mathrm{rk}(\mathrm{Pic}(X^{1,n}_{n+1})) = n+3.$$

We can compute the dimension of $\mathrm{Cox}(X^{1,n}_{n+1})$ from Theorem 4.1.1. There are $(n+1)$ generators of the form S_i, $(n+1)$ generators of the form $T_{i,1}$ and $(n+1)$ generators of the form $T_{i,2}$. There are $(n-1)$ relations g_i and each relation removes one dimension from the space spanned by the generators. Hence we have:

$$\dim(\mathrm{Cox}(X^{1,n}_{n+1})) = (3n+3) - (n-1) = 2n+4. \tag{4.2}$$

Instead, using formula 4.1.2 we get that:

$$\dim(\mathrm{Cox}(X^{1,n}_{n+1})) = \dim(X^{1,n}_{n+1}) + \mathrm{rk}(\mathrm{Pic}(X^{1,n}_{n+1})) = (n+1) + (n+3) = 2n+4,$$

which agrees with the previous computation.

4.2 The moving cone of $X^{1,n}_{n+1}$

In order to compute the moving cone of $X^{1,n}_{n+1}$ we first compute it for $n \leq 4$ using Maple and then we generalize this construction to all n.

4.2.1 $\mathrm{Mov}(X^{1,n}_{n+1})$ for $n \leq 4$

As a consequence of Theorem 4.1.1, we can apply Theorem 2.2.10 to find the moving cone of $X^{1,n}_{n+1}$. The procedure will be the following:

We compute the generators of $\mathrm{Mov}(X^{1,n}_{n+1})$ for low values of n, with Maple. Later, we will find the generators of $\mathrm{Mov}(X^{1,n}_{n+1})$ for all values of n.

The script written in Maple takes in input a matrix A, where each line corresponds to a generator of $\mathrm{N}_1(X^{1,n}_{n+1})$ and each column correspond to a generator of $\mathrm{Cox}(X^{1,n}_{n+1})$. Hence A is a matrix of size $(n+3)\times(2n+4)$.

Example 4.2.1. Let us set $n = 1$.

Then, Theorem 4.1.1 gives

$$\mathrm{Cox}(X^{1,1}_2) \cong \mathbb{C}[H_2 - E_1, H_2 - E_2, H_1 - E_1, H_1 - E_2, E_1, E_2].$$

Then, the Maple script at page 83 gives:

$$\mathrm{Mov}(X^{1,1}_2) = \langle H_1, H_2, H_1 + H_2 - E_1 - E_2, H_1 + H_2 - E_1, H_1 + H_2 - E_2\rangle.$$

Example 4.2.2. For $n = 2$, Theorem 4.1.1 gives

$$\mathrm{Cox}(X^{1,2}_3) \cong \mathbb{C}[H_2 - E_1 - E_2, H_2 - E_2 - E_3, H_2 - E_1 - E_3,$$

$$E_1, E_2, E_3, H_1 - E_1, H_1 - E_2, H_1 - E_3].$$

The Maple script at page 84 gives the following generators for the movable cone:

$$\mathrm{Mov}(X^{1,2}_3) = \langle H_1, H_2, H_2 - E_1, H_2 - E_2, H_2 - E_3, H_1 + H_2 - E_1 - E_2 - E_3,$$

$$H_1 + H_2 - E_2 - E_3, H_1 + H_2 - E_1 - E_3,$$

$$H_1 + H_2 - E_1 - E_2, 2H_2 - E_1 - E_2 - E_3\rangle.$$

Example 4.2.3. For $n = 3$, Theorem 4.1.1 gives

$$\mathrm{Cox}(X^{1,3}_4) \cong \mathbb{C}[H_2 - E_{i_1} - E_{i_2} - E_{i_3}, E_1, E_2, E_3, E_4, H_1 - E_1, H_1 - E_2, H_1 - E_3, H_1 - E_4],$$

where $i \in \{1, 2, 3, 4\}$. The Maple script at page 86 gives the following generators for the movable cone:

$$\mathrm{Mov}(X^{1,3}_4) = \langle H_1, H_2, H_2 - E_i, H_2 - E_i - E_j, H_1 + H_2 - E_{i_1} - E_{i_2} - E_{i_3},$$

$$H_1 + H_2 - E_1 - E_2 - E_3 - E_4, 2H_2 - 2E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}, 3H_2 - 2E_1 - 2E_2 - 2E_3 - 2E_4\rangle,$$

where $i \in \{1, 2, 3, 4\}$.

Example 4.2.4. For $n = 4$, Theorem 4.1.1 gives

$$\mathrm{Cox}(X^{1,4}_5) \cong \mathbb{C}[H_2 - E_{i_1} - E_{i_2} - E_{i_3} - E_{i_4}, E_1, E_2, E_3, E_4, E_5, H_1 - E_1,$$
$$H_1 - E_2, H_1 - E_3, H_1 - E_4, H_1 - E_5],$$

where $i \in \{1, 2, 3, 4, 5\}$. The Maple script at page 88 gives the following generators for the movable cone:

$$\mathrm{Mov}(X^{1,4}_5) = \langle H_1, H_2, H_2-E_i, H_2-E_i-E_j, H_2-E_{i_1}-E_{i_2}-E_{i_3}, H_1+H_2-E_{i_1}-E_{i_2}-E_{i_3}-E_{i_4},$$
$$H_1 + H_2 - E_1 - E_2 - E_3 - E_4 - E_5, 2H_2 - 2E_{i_1} - 2E_{i_2} - E_{i_3} - E_{i_4} - E_{i_5},$$
$$3H_2 - 3E_{i_1} - 2E_{i_2} - 2E_{i_3} - 2E_{i_4} - 2E_{i_5}, 4H_2 - 3E_1 - 3E_2 - 3E_3 - 3E_4 - 3E_5\rangle,$$

where $i, j, k \in \{1, \ldots, 5\}$.

4.2.2 $\mathrm{Mov}(X^{1,n}_{n+1})$ for all n

Recall that, given a finitely generated cone

$$C = \langle a_i : i = 1, \ldots, |C|\rangle \subset V,$$

where $i = 1, \ldots, |C|$, its dual is the cone defined as

$$C^* = \{u \in V \mid < u, v > \geq 0 \ \forall v \in C\}.$$

Then, we can find the equations for the rays of C^*, by considering the equations

$$\langle u, a_i\rangle = 0,$$

for $i = 1, \ldots, |C|$.

By analyzing the previous cases, we can recognize a pattern among the generators of $\mathrm{Mov}(X^{1,n}_{n+1})$. Consider the following divisor classes on $X^{1,n}_{n+1}$:

$$\begin{array}{lcl}
D_1 & = & H_1; \\
D_h & = & H_2 - E_{i_1} - \cdots - E_{i_h}; \\
D_{i_1,\ldots,i_n} & = & H_1 + H_2 - E_{i_1} - \cdots - E_{i_n}; \\
D_{1,\ldots,n+1} & = & H_1 + H_2 - E_1 - \cdots - E_{n+1}; \\
D_k & = & kH_2 - kE_{i_1} - \cdots - kE_{i_{(n-k)}} - (k-1)E_{i_{(n-k+1)}} - \cdots - (k-1)E_{i_{n+1}}; \\
D_n & = & nH_2 - (n-1)E_1 - \cdots - (n-1)E_{(n+1)}.
\end{array},$$

for $2 \leq k \leq n - 1$, $0 \leq h \leq n - 1$ and $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n + 1\}$.

Set

$$\mathcal{C} := \langle D_1, D_h, D_{i_1,\ldots,i_n}, D_{1,\ldots,n+1}, D_k, D_n\rangle.$$

Furthermore, for an algebraic variety X of dimension n, we recall the notion of $k-$moving curves, for $1 \leq k \leq n - 1$. For further details about these topics see [Pay05].

Definition 4.2.5. An irreducible curve $C \subset X$ is $k-$moving if it belongs to an algebraic family of curves, whose irreducible elements cover a Zariski open subset of an effective cycle of dimension at least $n-k$. We define $\mathrm{Mov}_k(X)$ to be the cone generated by the classes of $k-$moving curves in $\mathrm{N}_1(X)$.

Proposition 4.2.6. *Let X be a $\mathbb{Q}$-factorial projective variety. We have that*

$$\mathrm{Mov}_1(X) \subseteq \mathrm{Mov}(X)^*.$$

In general, the inclusion is strict, but there are examples of varieties for which the equality holds.

Proof. Let C be a $1-$ moving curve in X. Suppose that C is not in the dual of $\mathrm{Mov}(X)$. Then, there exists a divisor $D \in \mathrm{Mov}(X)$ such that

$$D \cdot C < 0.$$

Hence, the divisor covered by the algebraic family of C is a fixed component of the base locus of D. But this is not possible, since the divisor D is movable and its stable base locus has codimension at least two. □

Let $\mathrm{Mov}_1(X^{1,n}_{n+1})$ be the cone of moving curves covering a divisor in $X^{1,n}_{n+1}$. We can prove the following proposition:

Now, given the extremal rays of the cone

$$\mathcal{C} := \langle D_1, D_h, D_{i_1,\dots,i_n}, D_{1,\dots,n+1}, D_k, D_n \rangle,$$

we want to compute the extremal rays of its dual $\mathcal{C}^*$.

Recall that the basis of the ambient space of $\mathcal{C}$ is:

$$\{H_1, H_2, E_1, \dots, E_{n+1}\}.$$

Then, the dual basis is

$$\{h_1, h_2, -e_1, \dots, -e_{n+1}\}.$$

Since for $n \leq 4$ we already proved that $\mathrm{Mov}(X^{1,n}_{n+1}) = \mathcal{C}$, we can compute with Maple the dual cone $\mathrm{Mov}(X^{1,n}_{n+1})^* = \mathcal{C}^*$.

Example 4.2.7. For $n = 1$, we have that $X^{1,1}_2 = \mathrm{Bl}_2(\mathbb{P}^1 \times \mathbb{P}^1)$.

The Maple script at page 83 gives:

$$\mathrm{Mov}(X^{1,1}_2)^* = \langle h_1 - e_1, h_1 - e_2, e_1, e_2, h_2 - e_1, h_2 - e_2 \rangle.$$

Example 4.2.8. For $n = 2$, we have that $X^{1,2}_3 = \mathrm{Bl}_3(\mathbb{P}^1 \times \mathbb{P}^2)$. The Maple script at page 84 gives:

$$\mathrm{Mov}(X^{1,2}_3)^* = \langle h_1, h_2-e_1, h_2-e_2, h_2-e_3, h_1+h_2-e_1-e_2, h_1+h_2-e_1-e_3, h_1+h_2-e_2-e_3, e_1, e_2, e_3 \rangle.$$

Example 4.2.9. For $n = 3$, we have that $X^{1,3}_4 = \mathrm{Bl}_4(\mathbb{P}^1 \times \mathbb{P}^3)$. The Maple script at page 86 gives:

$$\mathrm{Mov}(X^{1,3}_4)^* = \langle h_1, h_2 - e_1, h_2 - e_2, h_2 - e_3, h_2 - e_4,$$

$$h_1 + 2h_2 - e_1 - e_2 - e_3, h_1 + 2h_2 - e_1 - e_2 - e_4, h_1 + 2h_2 - e_2 - e_3 - e_4,$$
$$h_1 + 2h_2 - e_1 - e_3 - e_4, e_1, e_2, e_3, e_4\rangle.$$

Example 4.2.10. For $n = 4$, we have that $X^{1,4}_5 = \mathrm{Bl}_5(\mathbb{P}^1 \times \mathbb{P}^4)$. The Maple script at page 88 gives:

$$\mathrm{Mov}(X^{1,4}_5)^* = \langle h_1, h_2-e_1, h_2-e_2, h_2-e_3, h_2-e_4, h_2-e_5, h_1+3h_2-e_1-e_2-e_3-e_4,$$

$$h_1+3h_2-e_1-e_2-e_4-e_5, h_1+3h_2-e_2-e_3-e_4-e_5, h_1+3h_2-e_1-e_3-e_4-e_5,$$
$$h_1 + 3h_2 - e_1 - e_2 - e_3 - e_5, e_1, e_2, e_3, e_4, e_5\rangle.$$

A straightforward computation shows that

$$\mathcal{C}^* = \langle h_1, h_2 - e_i, h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n}, e_i \rangle$$

for $1 \leq i \leq n+1$, $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+1\}$. The following proposition is crucial for the proof of Proposition 4.2.12. Indeed, it proves that every extremal ray in $\mathcal{C}^*$ corresponds to a curve that covers a divisor in $X_{n+1}^{1,n}$, which was the one of the last assumptions of the proof of Proposition 4.2.12.

Proposition 4.2.11. *Every extremal ray of the dual cone $\mathcal{C}^*$ corresponds to a curve that covers a divisor in $X_{n+1}^{1,n}$, that is:*

$$\mathcal{C}^* \subseteq \mathrm{Mov}_1(X_{n+1}^{1,n}).$$

Proof. Since h_1 is the class of a general fiber of $\tilde{\pi}_2$, it covers the whole variety $X_{n+1}^{1,n}$ and (hence) a divisor. $h_2 - e_i$ is the class of a general fiber of $\tilde{\pi}_1$ and it covers the fiber $\mathbb{P}^n$ over the blown-up point p_i. The class e_i corresponds to the class of a line in the exceptional divisor E_i, so it covers this exceptional divisor. Finally, consider the class

$$h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n}.$$

First, we analyze the case $n = 2$. Consider the class $h_1 + h_2 - e_i - e_j$ in $X_3^{1,2}$. We prove that this curve covers a divisor in $X_3^{1,2}$.

First, we consider the projected curve $\tilde{\pi}_2(h_1 + h_2 - e_i - e_j)$ in $\mathbb{P}^2$. The projected curve is a line $l_{i,j}$ passing through the points $\pi_2(p_i)$ and $\pi_2(p_j)$ with class $h_2 - e_i - e_j$.

Consider the Segre variety $\Sigma^{1,2} \subset \mathbb{P}^5$, and consider the subvariety $\mathbb{P}^1 \times l_{i,j} \subset \mathbb{P}^1 \times \mathbb{P}^2$. Denote by $X' := \sigma_{1,2}(\mathbb{P}^1 \times l_{i,j})$ its image in the Segre variety. Notice that X' is contained in a $\mathbb{P}^3$.

For each point $q \in \mathbb{P}^1 \times l_{i,j}$, consider the plane Π_q passing through q and containing the line $l_{i,j}$. Then

$$X' \cap \Pi_q = \sigma_{1,2}(h_1 + h_2 - e_i - e_j).$$

Hence, $\pi(h_1 + h_2 - e_i - e_j)$ covers the divisor

$$\mathbb{P}^1 \times l_{i,j} \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^2$$

and $h_1 + h_2 - e_i - e_j$ covers the divisor

$$X_2^{1,1} \subset X_3^{1,2},$$

where $X_2^{1,1}$ is the strict transform of $\mathbb{P}^1 \times l_{i,j}$ in $X_3^{1,2}$.

We can generalize this proof to the general case. Consider the class

$$h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n}$$

in $X^{1,n}_{n+1}$ of a degree n curve passing through n among the blown-up points. Consider the projected curve $\tilde{\pi}_2(h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n})$ in $\mathbb{P}^n$. It is a degree $n-1$ curve in $\mathbb{P}^n$, passing through the projection of n points among $p_1, \ldots, p_n$. Its class is:

$$(n-1)h_2 - e_{i_1} - \cdots - e_{i_n}.$$

Denote by $H \subset \mathbb{P}^n$ the hyperplane spanned by the n projected points. Consider the Segre variety $\Sigma^{1,n} \subset \mathbb{P}^{2n+1}$ and consider the subvariety $\mathbb{P}^1 \times H \subset \mathbb{P}^1 \times \mathbb{P}^n$. Denote by

$$X' := \sigma_{1,n}(\mathbb{P}^1 \times H)$$

its image in the Segre variety. Note that X' is contained in a $\mathbb{P}^{2n-1}$.

For each point $q \in \mathbb{P}^1 \times H$, consider the linear n-dimensional subspace Π_q passing through q and through the n projection of points $p_{i_1}, \ldots, p_{i_n}$. Then,

$$X' \cap \Pi_q = \sigma_{1,n}(h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n}).$$

Hence, $\pi(h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n})$ covers the divisor

$$\mathbb{P}^1 \times H \cong \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^1 \times \mathbb{P}^n$$

and $h_1 + (n-1)h_2 - e_{i_1} - \cdots - e_{i_n}$ covers the divisor

$$X^{1,n-1}_n \subset X^{1,n}_{n+1},$$

where $X^{1,n-1}_n$ is the strict transform of $\mathbb{P}^1 \times H$ in $X^{1,n}_{n+1}$. □

To conclude we need the following Proposition:

Proposition 4.2.12. *The movable cone of $X^{1,n}_{n+1}$ is given by*

$$\mathrm{Mov}(X^{1,n}_{n+1}) = \langle D_1, D_h, D_{i_1,\ldots,i_n}, D_{1,\ldots,n+1}, D_k, D_n \rangle$$

for $2 \le k \le n-1$, $0 \le h \le n-1$ and $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+1\}$.

Proof. Consider the cone $\mathcal{C} := \langle D_1, D_h, D_{i_1,\ldots,i_n}, D_{1,\ldots,n+1}, D_k, D_n \rangle$.

Since every divisor appearing in $\mathcal{C}$ is a movable divisor, we have the inclusion

$$\mathcal{C} \subseteq \mathrm{Mov}(X^{1,n}_{n+1}).$$

Then, taking the dual cones, we get

$$\mathrm{Mov}(X^{1,n}_{n+1})^* \subseteq \mathcal{C}^*.$$

Proposition 4.2.6 gives the following chain of inclusions:

$$\mathrm{Mov}_1(X_{n+1}^{1,n}) \subseteq \mathrm{Mov}(X_{n+1}^{1,n})^* \subseteq \mathcal{C}^*.$$

If we prove that every extremal ray in $\mathcal{C}^*$ corresponds to a curve that covers a divisor in $X_{n+1}^{1,n}$, then we would have:

$$\mathrm{Mov}_1(X_{n+1}^{1,n}) = \mathrm{Mov}(X_{n+1}^{1,n})^* = \mathcal{C}^*$$

and hence

$$\mathrm{Mov}(X_{n+1}^{1,n}) = \mathcal{C},$$

concluding the proof. □

Chapter 5

Nef cones and Mori chamber decomposition of $X^{1,n}_{n+1}$

In this chapter we compute the Nef cones of $X^{1,n}_{n+1}, X^{1,n}_{n+2}$ and of $X^{1,n}_{n+3}$ for $n \leq 4$. Moreover, we also give a script that computes the Mori chamber decomposition of $X^{1,n}_{n+1}$ and a concrete execution of the script for $X^{1,2}_3$. We developed in Magma [BCP97] all the scripts.

The Magma library containing the scripts can be downloaded at the following link:

`https://github.com/msslxa/Cox-rings-of-blow-ups-of-multiprojective-spaces`

5.1 The computation of $\mathrm{Nef}(X^{1,n}_{n+1})$

As we have seen in Proposition 2.1.13, the nef cone of a variety X is the dual cone of the Mori cone of curves. In this section, we want to compute the generators of the nef cones of $X^{1,n}_{n+1}$. Proposition 3.2.1 tells us that

$$\mathrm{NE}(X^{1,n}_{n+1}) = \langle e_i, h_1 - e_i, h_2 - e_i \rangle ,$$

for $i = 1, \ldots, n+1$.

Proposition 5.1.1. *The nef cone of $X^{1,n}_{n+1}$ is generated by:*

$$\mathrm{Nef}(X^{1,n}_{n+1}) = \langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r} \rangle,$$

with $r = 1, \ldots n+1$.

It has $2^{n+1} + 1$ rays.

Proof. To compute generators of $\mathrm{Nef}(X)$ it is enough to compute the dual cone of $\mathrm{NE}(X)$. We show a script, reported at page 80, that computes the dual cone of $\mathrm{NE}(X^{1,n}_{n+1})$ for a fixed value of n and gives its generators. This gives us a pattern for the general case. But in order to prove it rigorously one needs to compute the dual cone of the Mori cone given in 3.2.1. To obtain the number of rays, it is enough to notice that there are

$$\sum_{r=1}^{n+1} \binom{n+1}{r}$$

rays of type

$$H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}.$$

Then, the total number of extremal rays is obtained by adding the two rays H_1 and H_2:

$$\sharp\,\mathrm{Rays}(\mathrm{Nef}(X^{1,n}_{n+1})) = 2 + \sum_{r=1}^{n+1} \binom{n+1}{r} = 2 + 2^{n+1} - 1 = 2^{n+1} + 1.$$

The last equality follows from:

$$2^{n+1} = \sum_{r=1}^{n+1} \binom{n+1}{r} + 1.$$

□

5.2 The computation of $\mathrm{Nef}(X^{1,n}_{n+2})$

In this section we compute the nef cone of $X^{1,n}_{n+2}$. Proposition 3.3.1 tells us that

$$\mathrm{NE}(X^{1,n}_{n+2}) = \langle e_i, h_1 - e_i, h_2 - e_i, h_1 + nh_2 - e_1 - \cdots - e_{n+2} \rangle ,$$

for $i = 1, \ldots, n+2$.

Proposition 5.2.1. *The nef cone of $X^{1,n}_{n+2}$ is generated by:*

$$\mathrm{Nef}(X^{1,n}_{n+2}) = \langle H_1, H_2, H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}, 2H_1 + H_2 - E_1 - \cdots - E_{n+2},$$
$$nH_1 + (n+1)H_2 - n\sum_{i=1}^{n+2} E_i \rangle,$$

with $r = 1, \ldots n+1$.

It has $2^{n+2} + 2$ rays.

Proof. The script at page 81 computes the dual cone of $\mathrm{NE}(X^{1,n}_{n+2})$ and gives its generators, for a fixed value of n. This gives us a pattern for the general case. But in order to prove it rigorously one needs to compute the dual cone of the Mori cone given in **??**. We compute the number of extremal rays. There are four rays

$$H_1, H_2, 2H_1 + H_2 - E_1 - \cdots - E_{n+2}, nH_1 + (n+1)H_2 - n\sum_{i=1}^{n+2} E_i.$$

The rays

$$H_1 + H_2 - E_{i_1} - \cdots - E_{i_r},$$

for $r = 1, \ldots n+1$, add up to

$$\sum_{r=1}^{n+1} \binom{n+2}{r}.$$

Since

$$2^{n+2} = \sum_{r=0}^{n+2} \binom{n+2}{r} = 2 + \sum_{r=1}^{n+1} \binom{n+2}{r},$$

we have that

$$\sum_{r=1}^{n+1} \binom{n+2}{r} = 2^{n+2} - 2$$

and the total number of rays is:

$$\sharp\,\mathrm{Rays}(\mathrm{Nef}((X^{1,n}_{n+2}))) = 2^{n+2} + 2 - 2 + 1 + 1 = 2^{n+2} + 2.$$

□

5.3 The computation of $\mathrm{Nef}(X^{1,n}_{n+3})$ for $n \leq 4$.

In this section we compute the nef cone of $X^{1,n}_{n+3}$ for $n \leq 4$. Since we don't know if $\mathrm{NE}(X^{1,n}_{n+1})$ is finitely generated when $n > 4$, we cannot give any result on the nef cone when $n > 4$.

By Proposition 3.4.5,

$$\mathrm{NE}(X^{1,n}_{n+3}) = \langle e_i, h_1 - e_i, h_2 - e_i, h_1 + nh_2 - e_{i_1} - \cdots - e_{i_{n+2}} \rangle,$$

for $i_1, \ldots, i_{n+2}, i \in \{1, \ldots, n+3\}$.

Consider the following divisor classes:

$$\begin{array}{lcl} D_1 & = & H_1; \\ D_2 & = & H_2; \\ D_{i_1,\ldots,i_t} & = & H_1 + H_2 - E_{i_1} - \cdots - E_{i_t}; \\ D_{i_1,\ldots,i_r} & = & 2H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}; \\ D_{i_1,\ldots,i_h} & = & nH_1 + (n+1)H_2 - nE_{i_1} - \cdots - nE_{i_h}; \\ D_{i_1,\ldots,i_s} & = & kH_1 + kH_2 - kE_{i_1} - \cdots - kE_{i_s} - (k-1)E_{i_{(s+1)}} - \cdots - (k-1)E_{i_{n+3}}; \end{array} ,$$

Proposition 5.3.1. *If $n \leq 4$, the nef cone of $X^{1,n}_{n+3}$ is generated by:*

$$\mathrm{Nef}(X^{1,n}_{n+3}) = \langle D_1, D_2, D_{i_1,\ldots,i_t}, D_{i_1,\ldots,i_r}, D_{i_1,\ldots,i_h}, D_{i_1,\ldots,i_s} \rangle,$$

with $t = 1, \ldots n+1$, $r, h = n+2, n+3$, $k = 2, \ldots, n+1$ and $s = n+3-(k+1)$.

Proof. The script at page 82 gives the generators of $\mathrm{Nef}(X^{1,n}_{n+3})$. □

Proposition 5.3.2. *The Nef cone of X has $2^{n+4} - 1/2(n+3)(n+2)$ rays for $n \leq 4$.*

Proof. We count the different rays appearing in the list of Proposition 5.3.1. There are two rays H_1 and H_2. The rays of type

$$H_1 + H_2 - E_{i_1} - \cdots - E_{i_t}$$

for $t = 1, \ldots, n+1$ add up to

$$\sum_{t=1}^{n+1} \binom{n+3}{t}.$$

The rays

$$2H_1 + H_2 - E_{i_1} - \cdots - E_{i_r}$$

and

$$nH_1 + (n+1)H_2 - nE_{i_1} - \cdots - nE_{i_h}$$

for $r, h = n+2, n+3$, are respectively

$$\binom{n+3}{n+2} = (n+3)$$

and

$$\binom{n+3}{n+3} = 1.$$

Then, the last kind of rays,

$$kH_1 + kH_2 - kE_{i_1} - \cdots - kE_{i_s} - (k-1)E_{i_{s+1}} - \cdots - (k-1)E_{i_{n+3}},$$

for $k = 2, \ldots, n+1$ and $s = (n+3) - k - 1$ adds up to

$$\sum_{k=1}^{n} \binom{n+3}{k}.$$

The total number of rays is then

$$2 + 2\sum_{k=1}^{n} \binom{n+3}{k} + \binom{n+3}{n+1} + 2 + 2(n+3).$$

Since
$$2^{n+3}=\sum_{k=0}^{n+3}\binom{n+3}{k},$$
we have that
$$\sum_{k=1}^{n}\binom{n+3}{k}=2^{n+3}-\binom{n+3}{0}-\binom{n+3}{n+1}-\binom{n+3}{n+2}-\binom{n+3}{n+3}=$$
$$=2^{n+3}-2-\binom{n+3}{n+1}-\binom{n+3}{n+2}.$$
Thus
$$2\sum_{k=1}^{n}\binom{n+3}{k}+\binom{n+3}{n+1}=2(2^{n+3}-2-\binom{n+3}{n+1}-\binom{n+3}{n+2})+\binom{n+3}{n+1}=$$
$$=2^{n+4}-4-2\binom{n+3}{n+1}-2\binom{n+3}{n+2}+\binom{n+3}{n+1}=$$
$$=2^{n+4}-4-\binom{n+3}{n+1}-2\binom{n+3}{n+2}=$$
$$=2^{n+4}-4-\frac{1}{2}(n+3)(n+2)-2(n+3).$$
In conclusion we have:
$$\sharp\,\mathrm{Rays}(\mathrm{Nef}(X_{n+3}^{1,n}))=2+2^{n+4}-4-\frac{1}{2}(n+3)(n+2)-2(n+3)+2(n+3)+2=$$
$$=2^{n+4}-\frac{1}{2}(n+3)(n+2).$$
□

5.4 Mori chamber decomposition of $X_{n+1}^{1,n}$

We managed to compute the Mori chamber decomposition of $X_{n+1}^{1,n}$ for $n=2,3,4$, and we got $92, 550$ and 6307 chambers respectively. Finally, we would like to mention that the Mori chamber decomposition of $X_6^{1,2}$ has been fully computed by T. Grange in [Gra22, Chapter 3].

Example 5.4.1. We show the script of Magma written to compute the Mori chamber decomposition of $X_3^{1,2}$. It displays the 92 chambers of the Mori chamber decomposition of $X_3^{1,2}$. Among them, there is one chamber with nine generators which corresponds to the Nef cone of $X_3^{1,2}$:

```
(0,  1,  0,  0,  0),
(1,  0,  0,  0,  0),
(1,  1, -1, -1, -1),
(1,  1, -1, -1,  0),
(1,  1, -1,  0, -1),
(1,  1, -1,  0,  0),
(1,  1,  0, -1, -1),
(1,  1,  0, -1,  0),
(1,  1,  0,  0, -1),
```

The first column corresponds to the coeffcients of H_1, the second to the coefficients of H_2 and the other columns to the coefficients of E_i, for $i = 1, 2, 3$. Hence, the Nef cone of $X_3^{1,2}$ is given by:

$$\mathrm{Nef}(X_3^{1,2}) = \langle H_1, H_2, H_1 + H_2 - E_1 - E_2 - E_3, H_1 + H_2 - E_i, H_1 + H_2 - E_{i_1} - E_{i_2} \rangle,$$

for $i, i_1, i_2 \in \{1, 2, 3\}$.

The following is the entire script that computes the Mori chamber decomposition of $X_{n+1}^{1,n}$:

```
// Input: an ideal I or an integer
// Output: the F-faces indices of I.
//
// If I is an integer it returns
// all the non-empty subsets of {1,..,I}.
// (uses Remark 3.1.1.11)

Ffaces := function(I)
if Type(I) eq RngIntElt then
return Subsets({1..I}) diff {{}};
end if;
B := Basis(I);
R := Parent(I.1);
n := Rank(R);
faces := {};
for S in Subsets({1..n}) diff {{}} do
BS := [Evaluate(g,[(i in S) select R.i else
0 : i in [1..n]]) : g in B];
if &*[R.i : i in S] notin Radical(Ideal(BS)) then
Include(~faces,S);
end if;
end for;
return faces;
end function;


// Input: grading matrix
// Output: effective cone

Eff := function(Q)
n := Ncols(Q);
K := ToricLattice(Nrows(Q));
return Cone([K!Eltseq(r) : r in Rows(Transpose(Q))]);
end function;
```

```
// Input: grading matrix
// Output: moving cone

Mov := function(Q)
n := Ncols(Q);
K := ToricLattice(Nrows(Q));
L := [K!Eltseq(r) : r in Rows(Transpose(Q))];
return &meet([Cone([L[j] : j in Remove([1..#L],i)])
: i in [1..#L]]);
end function;
      // Input: (F-faces,grading matrix)
// Output: orbit cones
OrbitCones := function(F,Q)
n := Ncols(Q);
K := ToricLattice(Nrows(Q));
w := [K!Eltseq(r) : r in Rows(Transpose(Q))];
if #F eq 0 then F := Subsets({1..n}) diff {{}}; end if;
return {Cone([w[i] : i in S]) : S in F};
end function;

// Input: (orbit cones, a class)
// Output: GIT chamber

GitChamber := function(orb,w)
K := Ambient(Random(orb));
w := K!Eltseq(w);
return &meet{C : C in orb | w in C};
end function;

// Input: (orbit cones, a class)
// Output: bunch of cones
BunchCones := function(orb,w)
return {C : C in orb | w in C};
end function;
// Input: (bunch, class, class)
// Output: boolean
//
// It returns true if the two classes
// have the same stable base locus
SameSbl := function(bun,w1,w2)
K := Ambient(Random(bun));
w1 := K!Eltseq(w1);
w2 := K!Eltseq(w2);
return {C : C in bun | w1 in C} eq {C : C in bun | w2 in C};
end function;
```

```
// Input: orbit cones
// Output: GIT fan

GitFan := function(orb)
Eff := Cone(&cat[Rays(C) : C in orb]);
SH := {SupportingHyperplane(Eff,C) : C in Facets(Eff)};
repeat
W := Random(orb);
K := Ambient(W);
w := &+Rays(W);
la := &meet{C : C in orb | w in C};
until Dimension(la) eq Dimension(K);
L := {la};
F := {C : C in Facets(la) | SupportingHyperplane(la,C) notin SH};
repeat
ff := Random(F);
la := Random([C : C in L | IsFace(C,ff)]);
H := K!SupportingHyperplane(la,ff);
w := &+Rays(ff);
if Dimension(Cone([w,w+H]) meet la) eq 0
then e := 1;
else e := -1;
end if;
n := 1;
repeat
u := w + e/10^n*H;
n := n+1;
until u in Eff;
repeat
lb := &meet{C : C in orb | u in C};
u := w + e/10^n*H;
n := n + 1;
until Dimension(lb) eq Dimension(K) and lb meet la eq ff;
L := L join {lb};
Fb :={C : C in Facets(lb) | SupportingHyperplane(lb,C) notin SH};
F := (F join Fb) diff (F meet Fb);
until IsEmpty(F);
return L;
end function;


// Input: (grading matrix, orbit cones, git fan)
// Output: list of triples
// (git chamber, git chamber, git chamber)
//
// In each triple the first is the ample
// chamber and the remaining two are two
// git chambers which lie in the same sbl
```

```
FindTriples := function(Q,orb,gfan)
K := Ambient(Random(gfan));
mov := Cone([K!w : w in RGenerators(Mov(Q))]);
triples := [];
L := Setseq(gfan);
M := [C : C in L | &and[w in mov : w in Rays(C)]];
for C in M do
bun := BunchCones(orb,&+Rays(C));
ll := [[C,L[i],L[j]] : i,j in [1..#L] |
i lt j and L[i] ne C and L[j] ne C
and SameSbl(bun,&+Rays(L[i]),&+Rays(L[j]))];
if #ll ne 0 then
Append(~triples,ll);
end if;
end for;
return triples;
end function;

FF := Rationals();
A<T1,T2,T3,T4,T5,T6,T7,T8,T9> := AffineSpace(Rationals(),9);
I := Ideal(T1*T7-T3*T9+T2*T8);

Q := Matrix(5,9,[0, 0, 0, 1, 1, 1, 0, 0, 0,1, 1, 1,
0, 0, 0, 0, 0, 0,-1,
0, 0, -1, -1, 0, 1, 0, 0,0,
-1, 0, -1, 0, -1, 0, 1, 0,0, 0, -1, 0, -1,
-1, 0, 0, 1]);
F := Ffaces(I);
orb := OrbitCones(F,Q);
gfan := GitFan(orb);
gfan;
```

Chapter 6

Magma and Maple scripts

The following script allows to compute the moving cone of $\mathrm{Bl}_p(\mathbb{P}^2)$ on Maple:

```
// Input: matrix with generators of Cox(Bl_{p}(\mathbb{P}^{2}))
// Output: generators of Mov(Bl_{p}(\mathbb{P}^{2})
A := Matrix([[1, 1, 0, 1], [0, -1, 1, -1]]);

C := ColumnDimension(A);
C := 4

R := RowDimension(A);
R := 2

st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
MovCone := CONE(2, 2, 0, 2, 2)

[[1, -1], [1, 0]]

0.075
```

The following script allows to compute the moving cone of $Bl_2(\mathbb{P}^2)$ on Maple:

```
A := Matrix([[1, 1, 1, 0, 0], [-1, -1, 0, 1, 0],
[-1, 0, -1, 0, 1]]);
C := ColumnDimension(A);
C := 5

R := RowDimension(A);
R := 3

st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
MovCone := CONE(3, 3, 0, 3, 3)

[[1, -1, 0], [1, 0, -1], [1, 0, 0]]

0.024
```

The following script allows to compute the moving cone of $\mathrm{Bl}_2(\mathbb{P}^3)$ on Maple:

```
A := Matrix([[1, 1, 1, 1, 0, 0], [-1, -1, 0, -1, 1, 0],
[-1, -1, -1, 0, 0, 1]]);
C := ColumnDimension(A);
C := 6

R := RowDimension(A);
R := 3

st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
MovCone := CONE(3, 3, 0, 4, 4)

[[1, -1, -1], [1, 0, -1], [1, -1, 0], [1, 0, 0]]
```

The following script allows to compute the moving cone of $Bl_2(\mathbb{P}^4)$ on Maple:

```
A := Matrix([[1, 1, 1, 1, 1, 0, 0], [-1, -1, -1, -1, 0, 1, 0],
[-1, -1, -1, 0, -1, 0, 1]]);
A := [[1, 1, 1, 1, 1, 0, 0],

C := ColumnDimension(A);
C := 7

R := RowDimension(A);
R := 3

st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
MovCone := CONE(3, 3, 0, 4, 4)

[[1, -1, -1], [1, 0, -1], [1, -1, 0], [1, 0, 0]]
```

The following script allows to compute the moving cone of $Bl_5(\mathbb{P}^3)$ on Maple:

```
A := Matrix([[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0],
[-1, -1, -1, 0, 0, 0, 0, -1, -1, -1, 1, 0, 0, 0, 0],
[-1, -1, -1, -1, -1, -1, 0, 0, 0, 0, 0, 1, 0, 0, 0],
[-1, 0, 0, -1, -1, 0, -1, -1, -1, 0, 0, 0, 1, 0, 0],
[0, -1, 0, -1, 0, -1, -1, -1, 0, -1, 0, 0, 0, 1, 0],
[0, 0, -1, 0, -1, -1, -1, 0, -1, -1, 0, 0, 0, 0, 1]]);

C := ColumnDimension(A);
C := 15

R := RowDimension(A);
R := 6

st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
```

```
MovCone := CONE(6, 6, 0, 51, 21)
[[2, -1, -1, -1, -2, -1], [2, -1, -1, -2, -1, -1],
[2, -2, -1, -1, -1, -1], [2, -1, -2, -1, -1, -1],
[2, -1, -1, -1, -1, -2], [3, -1, -2, -2, -2, -2],
[3, -2, -2, -2, -2, -1], [3, -2, -2, -2, -1, -2],
[3, -2, -1, -2, -2, -2], [3, -2, -2, -1, -2, -2],
[1, 0, 0, -1, -1, 0], [1, -1, 0, 0, -1, 0],
[1, 0, -1, 0, -1, 0], [1, 0, 0, 0, -1, -1],
[2, 0, -1, -1, -2, -1], [2, -1, 0, -1, -2, -1],
[2, -1, -1, 0, -2, -1], [1, 0, 0, 0, -1, 0],
[2, -1, -1, -1, -2, 0], [1, -1, 0, -1, 0, 0],
[1, 0, -1, -1, 0, 0], [1, 0, 0, -1, 0, -1],
[2, 0, -1, -2, -1, -1], [2, -1, 0, -2, -1, -1],
[2, -1, -1, -2, 0, -1], [1, 0, 0, -1, 0, 0],
[2, -1, -1, -2, -1, 0], [1, -1, -1, 0, 0, 0],
[1, -1, 0, 0, 0, -1], [2, -2, 0, -1, -1, -1],
[2, -2, -1, 0, -1, -1], [2, -2, -1, -1, 0, -1],
[1, -1, 0, 0, 0, 0], [2, -2, -1, -1, -1, 0],
[1, 0, -1, 0, 0, -1], [2, 0, -2, -1, -1, -1],
[2, -1, -2, 0, -1, -1], [2, -1, -2, -1, 0, -1],
[1, 0, -1, 0, 0, 0], [2, -1, -2, -1, -1, 0],
[2, 0, -1, -1, -1, -2], [2, -1, 0, -1, -1, -2],
[2, -1, -1, 0, -1, -2], [1, 0, 0, 0, 0, -1],
[2, -1, -1, -1, 0, -2], [3, 0, -2, -2, -2, -2],
[3, -2, 0, -2, -2, -2], [3, -2, -2, 0, -2, -2],
[3, -2, -2, -2, 0, -2], [1, 0, 0, 0, 0, 0],
[3, -2, -2, -2, -2, 0]]
```

The following script allows to compute the Mori cone and the Nef cone of $X^{1,n}_{n+1}$ on Magma:

```
// Input: an integer n >= 1;
// Output: the Nef cone of P1xPn blown-up at n+1 general points
in the basis H_1, H_2, -E_1,...,-E_{n+1}.

Nef := function(n)
h1 := [1,0];
h2 := [0,1];
e := [];
for i in [1..n+1] do
e[i] := [0 : j in [1..i+1]] cat [1] cat [0 : j in [i+3..n+3]];
end for;
nege := [];
for i in [1..n+1] do
nege[i] :=[0 : j in [3..i+1]] cat [-1] cat [0 : j in [i+3..n+3]];
end for;
v := [];
for i in [1..n+1] do
v[i] := h1 cat nege[i];
end for;
w := [];
for i in [1..n+1] do
w[i] := h2 cat nege[i];
end for;
NE := Cone(v cat w cat e);
Nef := Dual(NE);
return Nef;
end function;

// Example
Nef(2);
Rays(Nef(2));
```

The following script allows to compute the Mori cone and the Nef cone of $X^{1,n}_{n+2}$ on Magma:

```
// Input: an integer n >= 1;
// Output: the Nef cone of P1xPn blown-up at n+2 general points
in the basis H_1, H_2, -E_1,...,-E_{n+2}.

Nef := function(n)
h1 := [1,0];
h2 := [0,1];
e := [];
for i in [1..n+2] do
e[i] :=[0 : j in [1..i+1]] cat [1] cat [0 : j in [i+3..n+4]];
end for;
nege := [];
for i in [1..n+2] do
nege[i] :=[0 : j in [3..i+1]] cat [-1] cat [0 : j in [i+3..n+4]];
end for;
v := [];
for i in [1..n+2] do
v[i] := h1 cat nege[i];
end for;
w := [];
for i in [1..n+2] do
w[i] := h2 cat nege[i];
end for;
c := [[1,n] cat [-1 : i in [1..n+2]]];
NE := Cone(v cat w cat e cat c);
Nef := Dual(NE);
return Nef;
end function;

// Examples
Rays(Nef(2));
Rays(Nef(3));
```

The following script allows to compute the Mori cone and the Nef cone of $X^{1,n}_{n+3}$, for $n \leq 4$, on Magma:

```
// Input: an integer n >= 1;
// Output: the Nef cone of P1xPn blown-up at n+3 general points
in the basis H_1, H_2, -E_1,...,-E_{n+3}.

Nef := function(n)
h1 := [1,0];
h2 := [0,1];
e := [];
for i in [1..n+3] do
e[i] :=[0 : j in [1..i+1]] cat [1] cat [0 : j in [i+3..n+5]];
end for;
nege := [];
for i in [1..n+3] do
nege[i] :=[0 : j in [3..i+1]] cat [-1] cat [0 : j in [i+3..n+5]];
end for;
v := [];
for i in [1..n+3] do
v[i] := h1 cat nege[i];
end for;
w := [];
for i in [1..n+3] do
w[i] := h2 cat nege[i];
end for;
c := [];
for i in [1..n+3] do
cu := [1,n] cat [-1 : j in [1..i-1]] cat [0]
      cat [-1 : j in[i+1..n+3]];
c := Append(c,cu);
end for;
NE := Cone(v cat w cat e cat c);
Nef := Dual(NE);
return Nef;
end function;

// Examples
Rays(Nef(2));
Rays(Nef(3));
```

This script allows to find generators for the movable cone of $X_2^{1,1}$ and its dual, using Maple.

```
restart;
with(combinat);
with(convex);
with(MDSpackage);
with(ListTools);
with(LinearAlgebra);
A := Matrix([[0, 0, 0, 0, 1, 1], [1, 1, 0, 0, 0, 0],
[-1, 0, 1, 0, -1, 0], [0, -1, 0, 1, 0, -1]]);
C := ColumnDimension(A);
R := RowDimension(A);
st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
MovCone := CONE(4, 4, 0, 5, 6)
[[0, 1, 0, 0], [1, 0, 0, 0], [1, 1, -1, -1], [1, 1, -1, 0],
[1, 1, 0, -1]]
0.030
DM := dual(MovCone);
DM := CONE(4, 4, 0, 6, 5)
rays(DM);
[[1, 0, 1, 0], [1, 0, 0, 1], [0, 0, -1, 0], [0, 0, 0, -1],
[0, 1, 1, 0], [0, 1, 0, 1]]
```

This script allows to find generators for the movable cone of $X_3^{1,2}$ and its dual, using Maple.

```
restart;
with(combinat);
with(convex);
with(MDSpackage);
with(ListTools);
with(LinearAlgebra);
A := Matrix([[0, 0, 0, 0, 0, 0, 1, 1, 1],
[1, 1, 1, 0, 0, 0, 0, 0, 0],
[-1, 0, -1, 1, 0, 0, -1, 0, 0],
[-1, -1, 0, 0, 1, 0, 0, -1, 0], [0, -1, -1, 0, 0, 1, 0, 0, -1]])
C := ColumnDimension(A);
C := 9
R := RowDimension(A);
R := 5
st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
```

```
MovCone := CONE(5, 5, 0, 10, 10)
[[0, 1, 0, -1, 0], [0, 1, -1, 0, 0], [0, 1, 0, 0, -1],
[0, 2, -1, -1, -1], [0, 1, 0, 0, 0], [1, 0, 0, 0, 0],
[1, 1, -1, -1, -1], [1, 1, -1, -1, 0], [1, 1, -1, 0, -1],
[1, 1, 0, -1, -1]]
0.042
DM := dual(MovCone);
DM := CONE(5, 5, 0, 10, 10)
rays(DM);
[[1, 1, 1, 1, 0], [1, 0, 0, 0, 0], [1, 1, 0, 1, 1],
[1, 1, 1, 0, 1], [0, 0, -1, 0, 0], [0, 0, 0, -1, 0],
[0, 0, 0, 0, -1], [0, 1, 1, 0, 0], [0, 1, 0, 1, 0],
[0, 1, 0, 0, 1]]
```

This script allows to find generators for the movable cone of $X_4^{1,3}$ and its dual, using Maple.

```
restart;
with(combinat);
with(convex);
with(MDSpackage);
with(ListTools);
with(LinearAlgebra);
 A := Matrix([[0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1],
 [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0],
 [-1, -1, 0, -1, 1, 0, 0, 0, -1, 0, 0, 0],
 [-1, -1, -1, 0, 0, 1, 0, 0, 0, -1, 0, 0],
 [-1, 0, -1, -1, 0, 0, 1, 0, 0, 0, -1, 0],
 [0, -1, -1, -1, 0, 0, 0, 1, 0, 0, 0, -1]])
 C := ColumnDimension(A);
 C := 12
 R := RowDimension(A);
 R := 6
 st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
```

```
MovCone := CONE(6, 6, 0, 22, 13)
[[0, 1, 0, -1, -1, 0], [0, 1, -1, 0, -1, 0],
[0, 1, 0, 0, -1, -1], [0, 2, -1, -1, -2, -1],
[0, 1, 0, 0, -1, 0], [0, 1, -1, -1, 0, 0],
[0, 1, 0, -1, 0, -1], [0, 2, -1, -2, -1, -1],
[0, 1, 0, -1, 0, 0], [0, 1, -1, 0, 0, -1],
[0, 2, -2, -1, -1, -1], [0, 1, -1, 0, 0, 0],
[0, 2, -1, -1, -1, -2], [0, 1, 0, 0, 0, -1],
[0, 3, -2, -2, -2, -2], [0, 1, 0, 0, 0, 0], [1, 0, 0, 0, 0, 0],
[1, 1, -1, -1, -1, -1], [1, 1, -1, -1, -1, 0],
[1, 1, -1, -1, 0, -1], [1, 1, 0, -1, -1, -1],
[1, 1, -1, 0, -1, -1]]
0.069
DM := dual(MovCone);
DM := CONE(6, 6, 0, 13, 22)
rays(DM);
[[1, 0, 0, 0, 0, 0], [0, 1, 1, 0, 0, 0], [0, 1, 0, 1, 0, 0],
[0, 1, 0, 0, 1, 0], [0, 1, 0, 0, 0, 1], [1, 2, 1, 1, 1, 0],
[1, 2, 1, 1, 0, 1], [1, 2, 0, 1, 1, 1], [1, 2, 1, 0, 1, 1],
[0, 0, -1, 0, 0, 0], [0, 0, 0, -1, 0, 0], [0, 0, 0, 0, -1, 0],
[0, 0, 0, 0, 0, -1]]
```

This script allows to find generators for the movable cone of $X_5^{1,4}$ and its dual, using Maple.

```
restart;
with(combinat);
with(convex);
with(MDSpackage);
with(ListTools);
with(LinearAlgebra);
A := Matrix([[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1],
 [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0],
[-1, -1, -1, -1, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0],
 [-1, -1, -1, 0, -1, 0, 1, 0, 0, 0, 0, -1, 0, 0, 0],
[-1, -1, 0, -1, -1, 0, 0, 1, 0, 0, 0, 0, -1, 0, 0],
 [-1, 0, -1, -1, -1, 0, 0, 0, 1, 0, 0, 0, 0, -1, 0],
 [0, -1, -1, -1, -1, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1]])
C := ColumnDimension(A);
C := 15
R := RowDimension(A);
R := 7
st := time[real]();
for i to C do
Deg[i] := [seq(A[j, i], j = 1 .. R)];
end do;
Degaux := [seq(Deg[j], j = 1 .. C)];
for l to C do
Coneaux[l] := poshull(op(subsop(l = NULL, Degaux)));
end do;
for l to C do
rays(Coneaux[l]);
end do;
MovCone := intersection(seq(Coneaux[i], i = 1 .. C));
rays(MovCone);
time[real]() - st;
```

```
MovCone := CONE(7, 7, 0, 49, 16)
[[0, 1, 0, -1, -1, -1, 0], [0, 1, -1, 0, -1, -1, 0],
[0, 1, 0, 0, -1, -1, -1], [0, 2, -1, -1, -2, -2, -1],
[0, 1, 0, 0, -1, -1, 0], [0, 1, -1, -1, 0, -1, 0],
[0, 1, 0, -1, 0, -1, -1], [0, 2, -1, -2, -1, -2, -1],
[0, 1, 0, -1, 0, -1, 0], [0, 1, -1, 0, 0, -1, -1],
[0, 2, -2, -1, -1, -2, -1], [0, 1, -1, 0, 0, -1, 0],
[0, 2, -1, -1, -1, -2, -2], [0, 1, 0, 0, 0, -1, -1],
[0, 3, -2, -2, -2, -3, -2], [0, 1, 0, 0, 0, -1, 0],
[0, 1, -1, -1, -1, 0, 0], [0, 1, 0, -1, -1, 0, -1],
[0, 2, -1, -2, -2, -1, -1], [0, 1, 0, -1, -1, 0, 0],
[0, 1, -1, 0, -1, 0, -1], [0, 2, -2, -1, -2, -1, -1],
[0, 1, -1, 0, -1, 0, 0], [0, 2, -1, -1, -2, -1, -2],
[0, 1, 0, 0, -1, 0, -1], [0, 3, -2, -2, -3, -2, -2],
[0, 1, 0, 0, -1, 0, 0], [0, 1, -1, -1, 0, 0, -1],
[0, 2, -2, -2, -1, -1, -1], [0, 1, -1, -1, 0, 0, 0],
[0, 2, -1, -2, -1, -1, -2], [0, 1, 0, -1, 0, 0, -1],
[0, 3, -2, -3, -2, -2, -2], [0, 1, 0, -1, 0, 0, 0],
[0, 2, -2, -1, -1, -1, -2], [0, 1, -1, 0, 0, 0, -1],
[0, 3, -3, -2, -2, -2, -2], [0, 1, -1, 0, 0, 0, 0],
[0, 3, -2, -2, -2, -2, -3], [0, 1, 0, 0, 0, 0, -1],
[0, 4, -3, -3, -3, -3, -3], [0, 1, 0, 0, 0, 0, 0],
[1, 0, 0, 0, 0, 0, 0], [1, 1, -1, -1, -1, -1, -1],
[1, 1, -1, -1, -1, -1, 0], [1, 1, -1, -1, -1, 0, -1],
[1, 1, 0, -1, -1, -1, -1], [1, 1, -1, 0, -1, -1, -1],
[1, 1, -1, -1, 0, -1, -1]]
0.153
NULL;
DM := dual(MovCone);
DM := CONE(7, 7, 0, 16, 49)
rays(DM);
[[1, 0, 0, 0, 0, 0, 0], [0, 1, 1, 0, 0, 0, 0],
[0, 1, 0, 1, 0, 0, 0], [0, 1, 0, 0, 1, 0, 0],
[0, 1, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 0, 1],
[1, 3, 1, 1, 1, 1, 0], [1, 3, 1, 1, 1, 0, 1],
[1, 3, 1, 1, 0, 1, 1], [1, 3, 1, 0, 1, 1, 1],
[1, 3, 0, 1, 1, 1, 1], [0, 0, -1, 0, 0, 0, 0],
[0, 0, 0, -1, 0, 0, 0], [0, 0, 0, 0, -1, 0, 0],
[0, 0, 0, 0, 0, -1, 0], [0, 0, 0, 0, 0, 0, -1]]
```

Bibliography

[AC17] C. Araujo and C. Casagrande, *On the Fano variety of linear spaces contained in two odd-dimensional quadrics*, Geom. Topol. **21** (2017), no. 5, 3009–3045. MR 3687113

[ADHL15] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR 3307753

[AM16] C. Araujo and A. Massarenti, *Explicit log fano structures on blow-ups of projective spaces*, Proceedings of the London Mathematical Society **113** (2016), no. 4, 445–473.

[BCHM08] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, 2008.

[BCP97] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478

[BDPS23] M. C. Brambilla, O. Dumitrescu, E. Postinghel, and L. J. Santana Sánchez, *Duality and polyhedrality of cones for mori dream spaces*, 2023.

[Bea83] A. Beauville, *Complex algebraic surfaces*, London Mathematical Society Lecture Note Series, vol. 68, Cambridge University Press, Cambridge, 1983, Translated from the French by R. Barlow, N. I. Shepherd-Barron and M. Reid.

[BH04] F. Berchtold and J. Hausen, *Cox rings and combinatorics*, 2004.

[BM21] M. Bolognesi and A. Massarenti, *Birational geometry of moduli spaces of configurations of points on the line*, Algebra & Number Theory **15** (2021), no. 2, 513–544. MR 4243655

[BP04] V. V. Batyrev and O. N. Popov, *The Cox ring of a del Pezzo surface*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 85–103. MR 2029863

[Cox95] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebraic Geom. **4** (1995), no. 1, 17–50. MR 1299003

[CT06] A-M. Castravet and J. Tevelev, *Hilbert's 14th problem and Cox rings*, Compos. Math. **142** (2006), no. 6, 1479–1498. MR 2278756

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, New York, 2001. MR 1841091

[Deb11] ______, *Introduction to mori theory*, 2011.

[Der13] U. Derenthal, *Singular del pezzo surfaces whose universal torsors are hypersurfaces*, Proceedings of the London Mathematical Society **108** (2013), no. 3, 638–681.

[EH16] D. Eisenbud and J. Harris, *3264 and all that—a second course in algebraic geometry*, Cambridge University Press, Cambridge, 2016. MR 3617981

[GH14] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley Classics Library, Wiley, 2014.

[GOST12] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi, *Characterization of varieties of fano type via singularities of cox rings*, 2012.

[GPPS22] T. Grange, E. Postinghel, and A. Prendergast-Smith, *Log fano blowups of mixed products of projective spaces and their effective cones*, 2022.

[Gra22] T. Grange, *Birational geometry and positivity of blow-ups of products of projective spaces*, 2022.

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Annals of Mathematics **79** (1964), no. 1, 109–203.

[HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. **48** (2000), 331–348, Dedicated to William Fulton on the occasion of his 60th birthday. MR 1786494

[HS10] J. Hausen and H. Süß, *The Cox ring of an algebraic variety with torus action*, Adv. Math. **225** (2010), no. 2, 977–1012. MR 2671185

[KM98] J. Kollar and S. Mori, *Birational geometry of algebraic varieties*, Cambridge University Press, Cambridge, 1998.

[Laz04a] R. Lazarsfeld, *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR 2095471

[Laz04b] ———, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004, Positivity for vector bundles, and multiplier ideals. MR 2095472

[LP17] J. Lesieutre and J. Park, *Log Fano structures and Cox rings of blow-ups of products of projective spaces*, Proc. Amer. Math. Soc. **145** (2017), no. 10, 4201–4209. MR 3690606

[Man74] I.U.I. Manin, *Cubic forms: Algebra, geometry, arithmetic*, North-Holland mathematical library, North-Holland Publishing Company, 1974.

[Mat02] K. Matsuki, *Introduction to the mori program*, Springer New York, NY, New York, NY, 2002.

[Mor82] S. Mori, *Threefolds whose canonical bundles are not numerically effective*, Ann. of Math., 1982.

[Mor88] ———, *Flip theorem and the existence of minimal models for 3-folds*, Jour. Amer. Math. Soc., 1988.

[Muk04] S. Mukai, *Geometric realization of t-shaped root systems and counterexamples to hilbert's fourteenth problem*, 2004.

[Nag60] M. Nagata, *On rational surfaces I. Irreducible curves of arithmetic genus* 0 *or* 1, Memoirs of the College of Science, University of Kyoto. Series A: Mathematics **32** (1960), no. 3, 351 – 370.

[Oka16] S. Okawa, *On images of Mori dream spaces*, Math. Ann. **364** (2016), no. 3-4, 1315–1342. MR 3466868

[Ott09] J. C. Ottem, *Cox rings of projective varieties*, 2009.

[Pay05] S. Payne, *Stable base loci, movable curves, and small modifications, for toric varieties*, 2005.

[PS07] Y. G. Prokhorov and V. V. Shokurov, *Toward the second main theorem on complements: from local to global*, 2007.

[Sha13] I. R. Shafarevich, *Basic algebraic geometry; 3rd ed.*, Springer, Berlin, 2013.