Decisions in Economics and Finance (2025) 48:775–793
https://doi.org/10.1007/s10203-024-00471-z

On *Specimen Theoriae Novae de Mensura Sortis* of Daniel Bernoulli

Paola Modesti[1]

Received: 11 October 2023 / Accepted: 22 July 2024 / Published online: 7 August 2024

Abstract
This piece in the *Milestones* series is dedicated to the paper "*Specimen Theoriae Novae de Mensura Sortis*" by Daniel Bernoulli, published in 1738 on the *Commentarii Academiae Scientiarum Imperialis Petropolitanae.*

Keywords Expected utility theory · Logarithmic utility · St. Petersburg paradox

Mathematics Subject Classification 91B16

JEL Classification D81

1 Introduction

We owe the little jewel *Specimen Theoriae Novae de Mensura Sortis* (1738) to Daniel Bernoulli (1700–1782), a 'complete scientist' with a talent for mathematics which, although the technical and historical context were not ripe, meant he had strong and enlightening intuitions.

His treatise "has profoundly influenced economic theory, portfolio theory, and operations research and has growing influence in evolutionary biology and behavioural ecology" (Stearns 2000, p. 221). Its English translation (Bernoulli 1738b) was cited 1341 times [1985 in 2023] according to the WoS Core Collection "by people writing on decision theory, risk management, mathematical probability, expected utility, cognition and choice, ecology, evolutionary ecology, marketing, preference structures, and engineering design - not bad for a paper written by a mathematical physicist 262 [285 in 2023] years ago" (p. 221).

But, above all, the memoir was one of the earliest applications of differential calculus in general, and certainly the first with significant economic content. It was therefore

I wish to thank Marzia De Donno, the Associate Editor and two referees for their comments.

✉ Paola Modesti
paola.modesti@unipr.it

1 Department of Economics and Management, Università degli Studi di Parma, via Kennedy, 6, Parma, Italy

the first published paper in which mathematics was applied to economics. Considering that it predates Adam Smith's (1723–1790) *An Inquiry into the Nature and Causes of the Wealth of Nations* (1776), which is often taken as the birth of economics, we might argue that mathematics for economics existed before economics itself officially existed.

Bernoulli published the memoir in 1738 in *Commentarii Academiae Scientiarum Imperialis Petropolitanae* (*Tomus* V, 1730–1731), a Russian academic journal appearing from 1726 to 1746, which featured six of his seven papers on probability. The work was written in Latin, the usual language for mathematical papers in the 18th century. But because few modern scientists knew Latin, it remained almost unknown until it was translated, first into German by A. Pringsheim (Father-in-law of Thomas Mann) in 1896 (Bernoulli 1738a), then into English by L. Sommer and published in *Econometrica* in 1954 (Bernoulli 1738b). R. Mille translated Sommer's version into French (Bernoulli 1738c) and in 1985 R. Charreton made a new translation from Latin into French (Bernoulli 1738d). Finally, in 2008, P. Agnoli and P. Piccolo translated the original essay into Italian (Bernoulli 1738e).

The German version contains an interesting introduction by L. Fick, but it was the English translation, commissioned by W. Simpson, editor of *Econometrica*, that made the paper widely accessible. In a footnote Simpson wrote: "In view of the frequency with which Bernoulli's famous paper has been referred to in recent economic discussion, it has been thought appropriate to make it more generally available by publishing this English version" (Bernoulli 1738b, p. 23). As Heukelom (2011) explains, "Early [in] 1952, Simpson commissioned Latin professor Louise Sommers of American University [...] to translate the text, with the purpose of publishing it later that year. But as Sommers was not an economist, Simpson first asked William Baumol to check a few economic words and concepts in the translation to see if they fitted contemporary economic jargon" (p. 2). Translating the work proved to be harder than expected, and mathematician Karl Menger (1902–1985) was called in to write four technical footnotes. The translation was published two years later.

Bernoulli was more than 200 years ahead of economists and his intuitions were not fully developed until the mid-19th century. They inspired many generations of researchers. In 2000, Daniel Bernoulli would have been 300 years old and his St. Petersburg paradox,[1] one of the *gems* of the paper, also celebrated three centuries in 2013. Numerous studies were published to celebrate these anniversaries (see, among others, Polasek 2000; Stearns 2000 and Seidl 2013), and interest has not waned in more recent years.

The present tribute opens in Sect. 2 by presenting the remarkable Bernoulli family. Section 3 describes the scientific background to Bernoulli's paper. Section 4 then illustrates the contents of the memoir and the debate which ensued. In Sect. 5, the *focus* shifts to the paper's influence on later scientific work.

[1] The paradox was proposed by Nicolaus Bernoulli in 1713. See Sect. 4.3.

2 The Bernoulli family

"There is a famous statement of Newton's [...]: 'If I have seen further, it is by standing on the shoulders of giants'. [...] If we can see further - and I believe we can... - it is by standing on the shoulders of a remarkable collection of scientists named Bernoulli, who walked the streets of Basel over two centuries ago" (Stigler 1996, p. 12).

From 1654 to 1789, three generations of Bernoullis produced eight mathematicians[2] who fundamentally changed the face of science (see, among others, Todhunter 1865; Bell 1937; Boyer 1968; Samuelson 1977; Stigler 1996; Polasek 2000; Stearns 2000 and Salov 2014). "Of these eight, seven were born in Basel and five died there [...] . In addition, five held chairs at the University of Basel" (Stigler 1996, p. 8). The Bernoulli family in fact lends credibility to the theory that genius can be hereditary (see, among others, Bell 1937 and Stigler 1996).

The Bernoullis settled in Basel in 1620, arriving from Antwerp which they fled in 1583 to escape persecution of the Huguenots. The Basel progenitor was Nicolaus the Elder (1623–1708). He and his family were merchants. His three sons, Jacob I (1654–1705), Nicolaus I (1662–1716) and Johann I (1667–1748), made significant contributions to mathematics and science. Jacob I was the first to give a theoretical basis to probability theory and to apply it in contexts other than games of chance. His masterpiece was *Ars Conjectandi*, published posthumously in 1713, in which he presented a radically new approach to probability theory. We owe to him an estimate of the number e, binomial distribution, polar coordinates and the term 'integral'.

Johann I became first a physician, against the wishes of his father who wanted him to go into business, and then, under the guidance of Jacob I, a mathematician. He achieved significant results in mathematics, physics, chemistry and astronomy. It was he, for instance, who formulated what is called the Rule of De l'Hôpital, after one of his students to whom he probably sold it. "After Newton's death in 1727, Johann I Bernoulli was unchallenged as the leading mathematical preceptor to all Europe" (Polasek 2000, p. 36). He had a quarrelsome disposition, as did other family members, and he himself tried to steal results from his brother Jacob I and his son Daniel. Both Jacob I and Johann I had minds of great power and innovation.

Nicolaus I also began not as a mathematician: he first earned the titles of Doctor of Philosophy and Doctor of Law. He later taught at St. Petersburg Academy and, when he died, the Empress Catherine I of Russia held a grand official funeral in his honour. Nicolaus III (1695–1726), Johann II (1710–1790) and Daniel, sons of Johann I, were all eminent mathematicians, as were Johann III (1746–1807) and Jacob II (1759–1789), sons of Johann II. The members of the second and third generations studied two new disciplines in depth, mathematical analysis and probability theory, and Jacob I, Johann I and Daniel were the most brilliant from a mathematical point of view. This was odd, as mathematics was not the first chosen field of study for any of them.

Stigler (1996) writes: "there are other instances where a single family has maintained supremacy over several generations, but only with the use of force of arms or

[2] To date there have been at least 120 members of the Bernoulli family distinguished in sciences, law, literature, arts and various professions (Bell 1937).

the power of great wealth. Only in the case of the Bernoullis [...] has such preeminence been maintained for such a period by so many individuals through intellectual power alone" (p. 12).

The Bernoulli S-type asteroid 2034, discovered in 1973, and the Bernoulli lunar crater are both named after the family.

Focusing on the author of *Specimen Theoriae Novae de Mensura Sortis*, Daniel Bernoulli was born in Groningen in the Netherlands. His father Johann I wanted him to go into business, but Daniel chose first medicine, and mathematics and physics only later. "He studied medicine in Basel, Heidelberg and Strasbourg and wrote a dissertation *De respiratione* on the mechanics of breathing in 1721" (Polasek 2000, pp. 36–37). A member of the Academies of Paris, Berlin, St. Petersburg and of the Royal Academy in London, he won the award of the Paris *Académie des sciences* ten times, and was a close friend of L. Euler (1707–1783) with whom he had fruitful scientific collaboration. In 1725, Daniel became a professor of Mathematics in St. Petersburg, and stayed there 8 years, before returning to Basel where he taught physics and philosophy. His studies focused on differential and integral calculus, probability theory, the kinetic theory of gases, the theory of vibrating strings, actuarial sciences, analysis of inoculation, anatomy and botany. His best known findings were in the field of hydrodynamics: Bernoulli's Principle is a key concept without which we would not know how to fly airplanes (Bernoulli 2005a1). As well as *Specimen Theoriae Novae de Mensura Sortis*, the most significant of his numerous important contributions was probably *Dijudicatio maxime probabilis plurium observationum discrepantium atque verisimillima inductio inde formanda* (Bernoulli 1778), a precursor to the estimation of maximum likelihood.

An anecdote (Bell 1937) which reflects the high regard in which Daniel Bernoulli was held, was that once, while traveling, he introduced himself to a stranger, saying, "My name is Daniel Bernoulli". Thinking he was being laughed at, the stranger replied, "And my name is Isaac Newton". The delighted Bernoulli said this was the best compliment he had ever received.

3 Before Bernoulli

To fully appreciate the value of Bernoulli's contribution in *Specimen Theoriae Novae de Mensura Sortis*, we need to take into account its economic and mathematical cultural background.

3.1 Utility

The concept of utility, the *essence* of the essay, comes from the distant past. From the start, it had been closely related to value theory.

For early Greek philosophers, accustomed to looking for a single cause for every phenomenon, the value of a good lay in the good itself. Plato (428/427 B.C. - 348/347 B.C.) observed that "if you had no foresight, you wouldn't be able to look forward to enjoying it in the future. Your life wouldn't be that of a man. It would be the life of an oyster" (*Philebus*, 21). Plato's words sound quite modern, but Aristotle's (384 B.C. - 322 B.C.) are even more surprising. In the treatise *Politics*, he introduced a distinction between use value and exchange value, and in the *Nicomachean Ethics*, explored how a measure of the value in use could be applied to the value in exchange.

Hellenists differed in their approach. For Epicurus (341 B.C. - 270 B.C.), at the basis of society there is only the search for what is useful. There are no universal laws and there is no good or evil: there exist only conventional laws inspired by subjective utility criteria. For the Stoics, what is useful coincides with the power of *logos*, which, in turn, coincides with the essence of Man. In the period preceding the mid-18th century, the concept of 'useful' appeared in the works of many philosophers. In particular, Thomas Hobbes (1588–1679) highlighted the relation between good and evil and pleasure and pain, Baruch Spinoza (1632–1677) held that the leading principle for humanity is utility, which can be maximized through knowledge, and David Hume (1711–1776) saw public utility as the sole origin of justice.

3.2 Probability

In 1865, Isaac Todhunter published in Cambridge, *A History of the Mathematical Theory of Probability*: *from the Time of Pascal to that of Laplace*. In the opening lines, he writes "The practice of games of chance must at all times have directed attention to some of the elementary considerations of the Theory of Probability" (p. 1). He notes that one of the first known references,[3] to gambling occurs in the *Divine Comedy* by Dante Alighieri (1265–1321), who mentions '*zara*', a very popular game played with three dice in the Middle Ages[4].

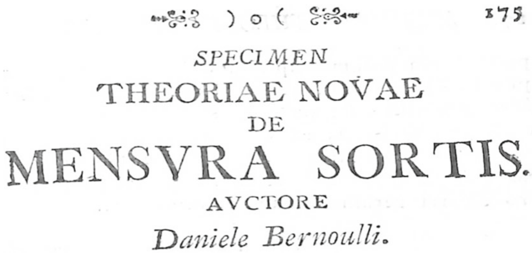
The theory of probability appeared during the 17th century,[5] mainly thanks to Fermat (1601–1665), Pascal (1623–1662) and Huygens (1629–1695), when insurance companies, along with mainly aristocratic card and dice players, acted as a stimulus to mathematical thinking. In 1654, the year of birth of Jacob I Bernoulli, the nobleman gambler Antoine Gombaud, *alias* Chevalier de Méré (1607–1684), brought the problem of the fair division of the stakes in an interrupted game of chance to the attention of Pascal. Pascal discussed this, the *problem of points*, with Fermat, and the correspondence between the two mathematicians solved the problem in two different ways, laying the foundations of probability theory.

In 1657, Christiaan Huygens published his first essay on probability; in 1694, Edmond Halley (1656–1742) dealt with insurance issues and life tables; and Isaac Newton's (1642–1727) methods and results, which were often used by probabilists as well, soon became central to European culture. In 1713, Jacob I Bernoulli's *Ars*

[3] Indeed, gambling was mentioned much earlier in the 4th century B.C. Sanskrit epic *Mahabharata* where, in the 32nd century B.C., Yudhishtira gambled away his kingdom, his wife, his wealth, the freedom of his brothers, and even himself.

[4] "*Quando si parte il gioco de la zara* ", *Divine Comedy*, *Purgatory*, *Canto* VI, 1.

[5] In previous centuries studies by Girolamo Cardano (1501–1576), Johannes Kepler (1571–1630), and Galileo Galilei (1564–1642) appeared.


175

SPECIMEN
THEORIAE NOVAE
DE
MENSVRA SORTIS.
AVCTORE
Daniele Bernoulli.

Fig. 1 The frontispiece of the work (Bernoulli 1738)

Conjectandi was published posthumously. Other prominent probabilists of the Enlightenment include Pierre Rémond de Montmort (1676–1719) and Abraham de Moivre (1667–1754).

4 *Specimen Theoriae Novae de Mensura Sortis*

In the first half of the 18th century, probabilists focused their attention on the determination of the fair price of a game which gives the monetary amount x_i $(i = 1, \ldots, n)$ with probability p_i $(0 \le p_i \le 1, \sum_{i=1}^{n} p_i = 1)$. The first price proposed was the expected value of the random variable which takes the values of the winnings of the game: $\sum_{i=1}^{n} p_i x_i$, that is, the amount of money which makes null the expected win.

But this *Principle of expected value* has several shortcomings,[6]. “The first to doubt the general validity of this principle was Daniel Bernoulli, and to this day he is considered the originator of modern utility theory” (Jensen 1967, p. 165).

In 1738, Bernoulli published the *Specimen Theoriae Novae de Mensura Sortis* (Fig. 1).

The memoir contains 19 sections in which three closely related subjects are discussed: the concept of utility (1–9), logarithmic utility (10–16) and the St. Petersburg paradox (17–19).

4.1 Bernoulli’s concept of utility

Bernoulli rejected the principle of expected value, pointing out that generally, two individuals who face the same risk do not evaluate it in the same way. He gave the following example: “Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. [...] all men cannot use the same rule to evaluate the gamble. [...] the determination of the *value*

[6] Presumably, the principle of expected value derived from the law of large numbers (in the long run, the average gain tends to its expected value), even though an individual may play only a limited number of times. The law of large numbers was introduced first by Jacob Bernoulli in 1713 in his *Ars Conjectandi* but, strangely, was not mentioned by his nephew Daniel Bernoulli.

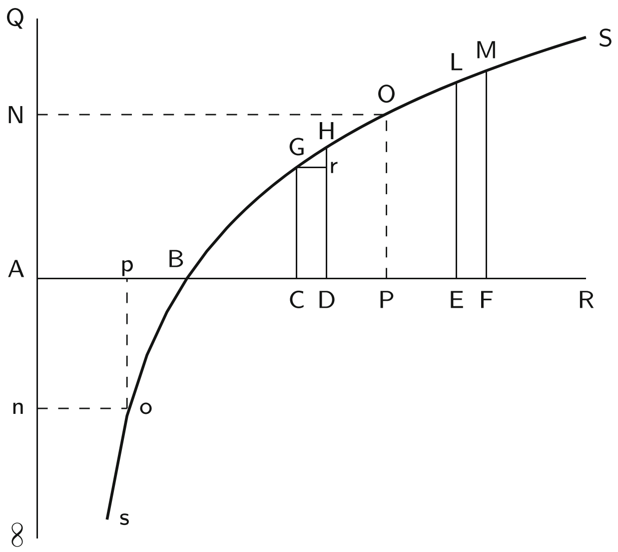


Fig. 2 Bernoulli's utility function graph

of an item must not be based on its *price*, but rather on the *utility* it yields. [...] Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount" (Bernoulli 1738b, p. 24).

Utility[7], as it is used in decision theory, was born.

Once established that utility is the correct criterion of evaluation, the value of a game is the profit (not the utility) corresponding to what is called the expected value of utility (mean utility). This is the sum of all *addenda* obtained by multiplying the utility of every win by its relative probability. Note that shifting from expected utility to the equivalent expected gain was to appear later as de Finetti's concept of Certainty Equivalent (1931).

Although utility is an 'individual concept' which depends on circumstances, Bernoulli found that "it is highly probable that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed*" (p. 25). This is the notion of decreasing marginal utility, although the assumption of inverse proportionality appears too restrictive.

Bernoulli then gives a geometrical representation of a utility function with some comments (Fig. 2).

Let AB be the initial wealth. "Then after extending AB, a curve $BGLS$ must be constructed, whose ordinates CG, DH, EL, FM, etc., designate *utilities* corresponding to the abscissas BC, BD, BE, BF, etc., designating gains in wealth" (p. 26). The

7 Bernoulli writes '*emolumentum*' and '*emolumentum medium*' (1738, p. 176 and following). Heukelom (2011) writes: "Particularly difficult was the word 'emolumentum', used by Bernoulli to denote the purchasing power of various monetary outcomes. [...] Thus, Baumol reasoned, emolumentum is an objective measure of the value of different outcomes to the individual [...] and inserted 'utility' as its translation" (p. 3).

expected utility is:

$$PO = \frac{mCG + nDH + pEL + qFM + \ \dots}{m + n + p + q + \ \dots} \qquad (1)$$

where m is the number of the cases in which the gain BC can occur and $n, p, q, \dots$ are defined analogously.

Thus "the straight line NO - AB represents the gain which may properly be expected, or the value of the risky proposition in question" (p. 26) and it is the maximum price that can be accepted. Bernoulli next observes that, if every gain is directly proportional to its utility (as in the principle of expected value), the curve that represents the utility function becomes a straight line and the corresponding price is the expected value of the random variable that describes the risk.

This geometric construction has a *general* value: the utility curve is any monotonically increasing, concave function, and it doesn't necessarily have to be logarithmic.

4.2 Logarithmic utility

In a subsequent section, Bernoulli goes back to his more restrictive initial hypothesis. He considers the infinitesimally small difference between the gains BD and BC. The gain in utility due to the transition from BC to BD is assumed to be proportional to $CD = \mathrm{d}x$ and inversely proportional to $AC = x$. If $CG = y$, $rH = \mathrm{d}y$, $AB = \alpha$ and b is a convenient constant, we[8] have:

$$\mathrm{d}y = b\frac{\mathrm{d}x}{x}$$

that, with the initial condition[9] $y(\alpha) = 0$, gives:

$$y = b \ln \frac{x}{\alpha}$$

Here, we find the very first utility function for money in the literature.

Now, applying this result to (1), "it follows that

$$b \ln \frac{AP}{AB} = \frac{mb \ln \frac{AC}{AB} + nb \ln \frac{AD}{AB} + pb \ln \frac{AE}{AB} + qb \ln \frac{AF}{AB} + \ \dots}{m + n + p + q + \ \dots}$$

and therefore

$$AP = \left(AC^m \cdot AD^n \cdot AE^p \cdot AF^q \cdot \ \dots\right)^{1/m+n+p+q+\dots}$$

and if we subtract AB from this, the remaining magnitude, BP, will represent the value of the risky proposition in question" (Bernoulli 1738b, p. 28).

[8] Bernoulli implicitly assumes the differentiability of y.

[9] From $y = b(\ln x + C)$ and the initial condition, it follows that $C = -\ln \alpha$ which gives $y = b(\ln x - \ln \alpha)$. Moreover, note that $y = b \ln \frac{x}{\alpha}$ is equivalent to $y = \ln x$, up to changes of scale.

Some applications, illustrated by simple numerical examples, follow.

- "First, it appears that in many games, even those that are absolutely fair, both of the players may expect to suffer a loss; indeed this is Nature's admonition to avoid the dice altogether..." (p. 29): the concavity of the utility function implies that even a fair game is not profitable. The (Jensen's) inequality that characterizes concave functions implies that everyone should prefer the *status quo* to any fair game. Bernoulli thus anticipates a concept that would be officially introduced in the 1960s: his decision makers, with their concave utilities, are risk adverse, according to the well-known definition of Arrow-Pratt;
- "to inquire how great an advantage the gambler must enjoy over his opponent in order to avoid any expected loss" (p. 29), Bernoulli presents an example of a bet from which he deduces that, however great the possible gain may be, the risked loss has to be smaller than the initial capital.

The following two applications concern insurance problems.

- In the first, a merchant would like to insure his goods that are to be shipped from Amsterdam to St. Petersburg. Bernoulli's method makes it possible to determine the initial wealth the merchant should possess to make the insurance worthwhile: "A man less wealthy than this would be foolish to provide the surety, but it makes sense for a wealthier man to do so" (p. 30). Note that Bernoulli explicitly refers to the fact that the merchant might not buy the insurance at "an amount which he considers outrageously high" (p. 30), a consideration still reported in intermediate microeconomics textbooks;
- in the second application, Bernoulli appears to anticipate the diversification principle underlying the modern theory of portfolio selection by more than 200 years. He writes: "Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together" (p. 30). The idea is illustrated with a charming example. "Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. [...] of ten ships one perishes" (p. 30). Bernoulli's utility theory shows clearly that it is safer and more economical to entrust the goods to two or more ships instead of one, even if Sempronius' expectation will never increase above a specific threshold value, because the risk can never be completely eliminated. The conclusion is surprisingly modern: "This counsel will be equally serviceable for those who invest their fortunes in foreign bills of exchange and other hazardous enterprises" (p. 30). This sentence distills the advice of spreading risk. Thus Bernoulli appears to have been the first scientist to point out the importance of diversification, which strengthens the claim that he might have written the first paper in "mathematical methods for economics" as noted in the Introduction.

4.3 The St. Petersburg paradox

The second part of Bernoulli's memoir describes the St. Petersburg paradox, probably the oldest and best known paradox in decision theory.

In the years 1713–1738, the problem, initially proposed by Nicolaus II Bernoulli (1687–1759), son of Nicolaus I and cousin of Daniel, generated a wide discussion (see, among others, Salov 2014 and Seidl 2012) among numerous eminent mathematicians across half of Europe, from P.R. de Montmort, G. Cramer (1704–1752), G.L.-L. Buffon (1707–1788), J.B. D'Alembert (1717–1783), L. Euler up to Daniel Bernoulli himself. It was probably the St. Petersburg paradox that inspired him to develop his utility theory.

4.3.1 The genesis

In September 1713, Nicolaus II Bernoulli, writing to French mathematician de Montmort[10], submitted the following[11] problem: A gives to B one *scudo* if, rolling a die, the face six comes up, two *scudo* if the face six comes at the second roll, four *scudo* if the lucky roll is the third, *etc.*. The question is: what is the expectation of B?

In May 1728, Gabriel Cramer, eminent professor of Mathematics at the University of Geneva, wrote to N. Bernoulli from London and proposed two possible solutions to the problem, in which he substituted the die with a fair coin (see Bernoulli 2013).

In October 1728, N. Bernoulli presented the problem to Daniel Bernoulli, in a letter to him in St. Petersburg.

In July 1731, Daniel wrote his solution to Nicolaus II and in the same year presented his conclusions to the Imperial Academy of Sciences of St. Petersburg. Only in 1732 did Nicolaus II send Cramer's suggestions to Daniel, who then included them in the final version of his paper.

4.3.2 Daniel Bernoulli's version of the paradox

Daniel Bernoulli mentions the correspondence with Nicolaus II and presents the problem in these terms: "Peter tosses a coin and continues to do so until it should land 'heads' when it comes to the ground. He agrees to give Paul one ducat if he gets 'heads' on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation" (Bernoulli 1738b, p. 31). This is the classical St. Petersburg paradox, that is probably named after *Commentarii* of the Academy of St. Petersburg in which it first appeared.

With the principle of expected value, the price of this game is infinitely large[12], whereas any "reasonable man would sell his chance, with great pleasure, for twenty[13]

[10] The correspondence was added in the second edition of 1713 of *Essay D'analyse Sur Les Jeux de Hazard* (de Montmort 1708, 1713).

[11] It is the fifth of five problems (de Montmort 1713, p. 402).

[12] If X is the random variable that describes the game, $\mathrm{E}[X] = \sum_{n=1}^{+\infty} \frac{1}{2^n} 2^{n-1} = +\infty$.

[13] A simulation of the game is available at https://mathematik.com/Petersburg/Petersburg.html. Simulations show that the game is always disadvantageous if the partecipation fee is 20.

ducats" (p. 31). Bernoulli applies his theory and finds that, if the initial capital of Paul is α, the value of the game[14] is:

$$\left[(\alpha+1)^{1/2}(\alpha+2)^{1/4}\cdot\ \ldots\ \cdot\left(\alpha+2^{n-1}\right)^{1/2^n}\cdot\ \ldots\right]-\alpha$$

This quantity increases with α, but it stays finite if α is finite.

4.3.3 The contribution of Gabriel Cramer

Bernoulli's memoir ends with a letter in French from Cramer. Bernoulli observes: "Indeed I have found his theory so similar to mine that it seems miraculous that we independently reached such close agreement on this sort of subject" (p. 33).

Cramer deals with the St. Petersburg paradox in two ways. His first suggestion is to treat all the amounts of money greater than 2^{24} ducats as equivalent to one other, probably because, if n is a large sum of money, n and $n+1$ give the same utility[15]. In this way, with the principle of expected value, the fair price of the game is 13 ducats.

Cramer's second suggestion introduces a real utility function, according to which a gain should be evaluated through its square root. It was Cramer who made the fascinating observation that "we suppose the moral value of goods to be directly proportionate to the square root of their mathematical quantities" (p. 34). The expected value of this utility function is $(2+\sqrt{2})/2$. But this is not the fair price which has to be such that "the pain caused by its loss is equal to the moral expectation of the pleasure I hope to derive from my gain" (p. 34). The fair price, according to this rule, turns out to be 2.9 ducats. This approach is indeed surprisingly similar to that of Bernoulli, regardless of the utility function chosen. In the history of science, this is not the only example of two researchers who achieve the same result separately: when the time is right, ideas align and science advances.

Bernoulli developed his theory in 1730–1731 and became aware of Cramer's letter in 1732. It was both generous and honest of him to highlight Cramer's contribution. Savage comments: "Daniel Bernoulli's paper reproduces portions of a letter from Gabriel Cramer to Nicolaus Bernoulli, which establishes Cramer's chronological priority to the idea of utility and most of the other main ideas of Bernoulli's paper. But it is Bernoulli's formulation together with some of the ideas that were specifically his that became popular and have had widespread influence to the present day" (1954, p. 92).

14 See Bernoulli 1738b, pp. 31–32, and, in particular, Menger's Footnotes 9 and 10. Note that Menger corrects an imprecision by Bernoulli who, in the discussion of the paradox, uses expressions like "in one half of the cases", "in one quarter of the cases", *etc.* which are inappropriate, as the number of cases is infinite. See also Todhunter (1865), pp. 220–221.

15 In the original letter reported by Bernoulli (1738), Cramer refers to the utility concept as '*l'usage qu'ils peuvent faire*' (p. 191), ' *valeur morale*', '*plaisir*' and '*esperance morale*' (p. 192).

4.4 The debate

Specimen Theoriae Novae de Mensura Sortis is unanimously recognized as a small masterpiece because of the modernity of the concepts, the depth of its insights, and the elegance and simplicity of the writing. In just a few pages, Bernoulli laid the groundwork for modern utility theory, complete with Certainty Equivalent, for marginal utility and, although in embryonic form, for portfolio selection.

The paper has three cornerstones: the notion of utility, the principle of decreasing marginal utility, and the reference to the individual's initial wealth. Its main intuition is undoubtedly that what is important is the individual appreciation of a result: the value of a bet, or in modern terms, a lottery, cannot be reduced to the expected value of the random variable which describes its stakes (objective measure); we need a *subjective evaluation*. Bernoulli suggests associating every lottery with prizes $c_1, c_2, \ldots, c_n$ and probabilities $p_1, p_2, \ldots, p_n$ to a utility function equal, by definition, to the expected value[16] of the utilities u of $c_1, c_2, \ldots, c_n$: $\sum_{i=1}^{n} p_i u(c_i)$.

As often happens, revolutionary findings are *simple*. The key is the change of a word ("the mere paraphrasing" (Bernoulli 1738b, p. 24)), from the expected value of a random variable to the expected value of its utility:

$$\begin{bmatrix} \text{money} & \rightarrow & \text{material happiness} \\ \\ \text{utility} & \rightarrow & \text{moral happiness} \end{bmatrix}$$

For the first time, a preference functional is introduced and the risk evaluation considers the *satisfaction* derived from the winnings as well as the probabilities. In 1921, Keynes writes: "Bernoulli's formula crystallises the undoubted truth that the value of a sum of money to a man varies according to the amount he already possesses" (1957, p. 320). And Daboni (1982) remarks that Bernoulli is famous for logarithmic utility and the St. Petersburg paradox, but his greatest *discovery* is utility: his main innovation is that, to compare random amounts, we make use of a modified scale of monetary values through the introduction of a utility function (p. 100).

In just a few lines, Bernoulli captures the essence of how humans behave when facing risk. It was a courageous choice, because, until that time, the principle of expected value seemed unquestionable. But "neither of the relevant arguments justifies categorial acceptance of the principle. None the less, the principle was at first so categorically accepted that it seemed paradoxical to mathematicians of the early eighteenth century that presumably prudent individuals reject the principle in certain real and hypothetical decision situations" (Savage 1954, p. 92). The St. Petersburg paradox requires a special consideration. It appears that Bernoulli's aim was to find an example that would be sufficiently indisputable to necessitate the replacement of the principle of expected value. As a result, the *challenging* St. Petersburg paradox has a methodological significance that Bernoulli feared could not be found in the other examples provided in the paper. He wrote in fact that they could be neglected as "abstractions resting upon precarious hypotheses" (Bernoulli 1738b, p. 31).

16 Note that this approach also makes it possible to associate a utility with lotteries with no monetary prize.

In his splendid paper of 1977, "another milestone publication" (Salov 2014, p. 8), Paul A. Samuelson writes: "The St. Petersburg paradox constitutes a fascinating chapter in the history of ideas" (p. 24). " [...] I was seduced by the antiquarian charms of the problem" (p. 25). So it is not surprising that there is a great deal of literature about it: "the list of writers connected with the St. Petersburg paradox reads like a veritable who's who in probability and the social sciences" (p. 24) and includes, among many others, Keynes (1921), Coolidge (1949), Stigler (1950), Arrow (1951, 1971), Brito (1975), Aumann (1977) and Shapley (1977a, b).

The paper had a fresh and innovative structure, and it accurately illustrated theory with examples and applications. The introduction of initial wealth to decision processes added to the modern 'tone' of the theory, and although it applied only in a certain context, Bernoulli's insight about the decreasing marginal utility sparked passionate debate in the 19th century.

The *explosive power* of Bernoulli's work is well described by Stigler (1996): "It is impossible to pick up a current issue of a journal in economics or econometrics without seeing some of the fruits of the seed Daniel Bernoulli planted 255 [285 in 2023] years ago" (pp. 9–10). The overall view of Bernoulli is so modern that it took more than two hundred years before John von Neumann (1903–1957) and Oskar Morgenstern (1902–1977) formalized it. It turned out to be so powerful that it became the *pivot* of utility theory.

Of course, the memoir is not entirely free of criticism, but this in no way diminishes the value of this *avant-garde* work, written in 1731, when Bernoulli was only 31 years old.

First, Bernoulli continuously distinguishes among the rich and the poor, and probably because of his Calvinist origins, often indicates behaviour as virtuous. His presentation of utility as a moral concept may irritate the modern reader, and appears to base the theory on a mystic principle. Von Neumann and Morgenstern (1947) write: "We are entitled to use the unmodified 'mathematical expectation' since we are satisfied with the simplified concept of utility [...] . This excludes in particular all those more elaborate concepts of 'expectation', which are really attempts at improving that naive concept of utility. (E.g. D. Bernoulli's 'moral expectation' in the 'St. Petersburg Paradox')" (p. 83, Footnote 2).

But the main problem with Bernoulli's utility function is its claim to represent preferences of individuals: axioms are missing. It was not until the 20th century that possible axiomatizations were presented and the theory was completed. Schoemaker (1982) writes: "Bernoulli's theory is mostly a descriptive model, even though the expectation principle at the time may have enjoyed much face validity normatively" (p. 531). Note however that Bernoulli points out that there can be many exceptions to the behaviour he regards as usual or desirable, some of which he illustrates through examples. In this he almost anticipates the distinction between descriptive and normative validity.

Moreover, although Bernoulli became famous mainly for logarithmic utility, paradoxically that utility function turns out to be a potential weak point, as it relies on hypotheses that are too restrictive. Stigler (1950) comments: "Bernoulli was right in seeking the explanation [of the paradox] in utility [...], and he was wrong only in making a special assumption with respect to the shape of the utility curve for which there was no evidence and which he submitted to no tests" (p. 375). On the other hand, Giocoli (1998) suggests a possible explanation, based again on the success of the expected value principle: "Bernoulli is aware of the fact that the numerous useful applications of the expected value rule force him to make his own criterion operational". He thus added the hypothesis which gives rise to logarithmic utility in order to provide "an easy operational rule to calculate the value of a game" (p. 14).

Much has been written about *Specimen Theoriae Novae de Mensura Sortis*. Early approval from prestigious mathematicians came from P.S. Laplace (1749–1827), who, in his *Théorie Analitique des Probabilités* of 1812, presented the main points of the paper, adding some original contributions based on mathematical notions not available to Bernoulli.

From a technical point of view, two issues, in particular, attracted attention: small probabilities and possible boundedness of the utility function.

Buffon met Cramer in Geneva in 1730 and agreed with the proposals of Cramer and Bernoulli. He observed that, in the St. Petersburg paradox, if heads did not fall until after the twenty-ninth throw, all the fortune of the Kingdom of France would not be enough to pay the player. On this point, Shapley (1977a), coauthor of the first *Milestones* paper (LiCalzi 2022), points out that there is a missing link: "One assumes that the subject believes the offer to be genuine, i.e., that he will actually be paid, no matter how much he may win" (p. 441)). Buffon suggested ignoring any event with probability less than 1/10,000 (2010, p. 39). Indeed, numerous experiments show that individuals usually neglect events with small probabilities and, in general, "The treatment of very small probabilities remains a very controversial issue today" (Hey, Neugebauer, Pasca, in Buffon 2010, p. 7, Footnote IV). Other mathematicians who fix a threshold for probabilities and/or for winnings, as, in a sense, Cramer also did, include Fontaine (1704–1771), D'Alembert (1717–1783), Condorcet (1743–1794), Poisson (1781–1840) and Cournot (1801–1877).

The discussion on the boundedness of the utility is more recent. "After 1738, nothing earthshaking was added to the findings of Daniel Bernoulli [...] until the quantum jump in analysis provided by Karl Menger (1934)" (Samuelson 1977, p. 25). And in Footnote 2, he adds: "By contrast to the 'ordinary' paradox of 1713–1738, there is the Super - Petersburg paradox,[17] of Menger (1934), in which the gains at the i^{th} toss grow enough faster than 2^i to make expected utility infinite when utility is unbounded - even though it is strictly concave" (p. 24). Menger shows that the value of the St. Petersburg game is finite only if the utility function of the gambler is bounded and, subsequently, Arrow (1971) proves that, to avoid the paradox, the utility function has to be bounded from above and from below (see also Aumann 1977).

[17] It is easy to verify that one small change in the text of the St. Petersburg paradox is enough to obtain an infinite expected utility. Menger (1934) suggested using $u(x) = \ln x$ and $e, e^2 \ldots, e^{2^{n-1}}, \ldots$ as the stakes of the different rolls of the dice. It was Samuelson (1977) who called the new game 'Super - Petersburg paradox of Menger'.

5 After Bernoulli

"In spite of its content, Bernoulli's essay remained almost unknown to economists for over 140 years" (Giocoli 1998, p. 8).

In those years, however, from a strictly economic point of view, utility theory was coming into being. At the end of the 18th century, Jeremy Bentham (1748–1832), the founder of Utilitarianism, discussed pain, pleasure and utility and, subsequently J. Dupuit (1804–1866) and H.H. Gossen (1810–1858), among others, also took up the theme. But at the beginning of the 19th century, Utilitarianism suffered a setback, in part due to the lack of mathematical support, for which the economic environment was not yet ready.

In the 1870s, however, a real revolution caught on. Great Britain, 1871: William Stanley Jevons (1835–1882) published the *Theory of Political Economy*; Germany 1871: Carl Menger (1840–1921) published *Principles of Economics*; France 1874: Léon Walras published the *Éléments d'Économie Politique Pure*. Almost simultaneously in their substantially equivalent works, these three authors laid the foundations of *Marginalism* that would legitimize utility theory in a certain context. But it was only in 1906, with the second edition of the *Manuale di Economia politica* by Vilfredo Pareto, that utility theory under certainty, complete with ordinal utility and preferences, was formalized and began to take the form we know today.

"With the development of the utility theory of value in 1870s, Bernoulli's proposal was found to fit in very well, especially in view of the common assumption of a diminishing marginal utility of income. Marshall ascribes the risk-aversion he observes in business to this cause" (Arrow 1951,p. 421). As Savage (1954) writes: "Economists were for a time enthusiastic about the principle of diminishing marginal utility and they saw what they believed to be reflections of it in many aspects of everyday life" (p. 95). Partly thanks to a quotation appearing in Jevons' *Theory of Political Economy*, "the name of Daniel Bernoulli finally became familiar in post-1870 Economics" (Giocoli 1998, p. 8).

In fact, for a long time, it had been precisely the concavity of Bernoulli's utility function that led to the principle of maximization of expected utility being rejected. Friedman and Savage (1948), in the first important review on utility theory under risk, comment: "If the marginal utility diminishes, an individual seeking to maximize utility will never participate in a 'fair game of chance' [...]. But this implication is clearly contradicted by actual behavior. [...] Even since the widespread recognition that the assumption of diminishing marginal utility is unnecessary to explain riskless choices, writers have continued to reject maximization of expected utility as 'unrealistic'. This rejection of maximization of expected utility has been challenged by John von Neumann and Oskar Morgenstern" (pp. 280–281), who proposed utility functions for risk choices that are not necessarily concave.

"In the 1860s, Bernoulli's hypothesis [about logarithmic utility] received some corroboration from the newly emerging field of psychophysics. The so-called Weber-Fechner Law held that a just noticeable difference in sensation is directly proportional to the intensity of the stimulus received: sensation is a logarithmic function of stimulus" (Blaug 1997, p. 318). For decades such a function, initially confirmed by numerous experiments "on weight, temperature, tonal, and other type of discriminations which the formula fitted very well [...], was a lively topic of discussion" (Stigler 1950, pp. 375–376).

Throughout the 19th century, utility theory under uncertainty was essentially ignored. At the beginning of the 20th century, despite various misapprehensions which weakened the clarifying power of Bernoulli's mathematical tool, a new concept of utility, based on what was termed neo-Bernoullian[18] thought, made its appearance. It summarised some of the numerous intuitions of the two previous centuries and some new issues. The originator of this new approach was F.P. Ramsey (1903–1930), who, in 1926, wrote the article *Truth and Probability* (Ramsey 1931), which was relatively unknown until 1954, when its main themes were advanced by L.J. Savage (1917–1971). Ramsey proposed a joint axiomatic approach to the study of utility under uncertainty, which although it was incomplete and somewhat cryptic, paved the way for utility theory and probability theory. "Ramsey improves on Bernoulli in that he defines utility operationally in terms of the behavior of a person constrained by certain postulates. Ramsey's essays, though now much appreciated, seem to have had relatively little influence" (Savage 1954, p. 96).

Concerning the axiomatic approach, we note that in 1899, David Hilbert (1862–1943) published *The Foundations of Geometry*, which revolutionized mathematics. In 1895, Hilbert became professor of Mathematics at the University of Gottingen and it was in Gottingen that his assistant John von Neumann learned the axiomatic method. At the same time, in Vienna, the economist Oskar Morgenstern was conceiving the project of introducing axiomatization to economics. Von Neumann and Morgenstern met in Princeton in 1939, and in 1944 published *The Theory of Games* (1944, 1947), one of the highest points of economic theory, where the axiomatic theory of expected utility under risk appeared for the first time. Von Neumann and Morgenstern, together with, a few years later, Savage, definitively vindicated Bernoulli's work and confirmed his intuition by providing his theory with a set of axioms which guarantee the existence of a utility function. They wrote: "We have practically defined numerical utility as being that thing for which the calculus of mathematical expectations is legitimate" (1947, p. 28). And, in Footnote 2: "Thus Daniel Bernoulli's well known suggestion to 'solve' the 'St. Petersburg Paradox' by the use of the so-called 'moral expectation' (instead of the mathematical expectation) means defining the utility numerically as the logarithm of one's monetary possessions".

As Savage (1954) writes: "The von Neumann-Morgenstern theory of utility [...] gives strong intuitive grounds for accepting the Bernoullian utility hypothesis, as a consequence of well-accepted maxims of behavior" (p. 97). He continues: "the main function of the von Neumann and Morgenstern postulates themselves is to put the essential content of Daniel Bernoulli's 'postulate' into a form that is less gratuitous in appearance" (p. 99).

But that is another story.

The importance of *Specimen Theoriae Novae de Mensura Sortis* for utility theory under uncertainty is well established. Over the years, its influence has extended in all

[18] More precisely, the neo-Bernoullians are the scholars who provided Bernoulli's utility with an axiomatic structure. The term was mainly used by Maurice Allais and Daniel Ellsberg to characterize a decision theory to which they objected.

directions, from neuroeconomics to communication networks, and from agriculture to natural sciences and beyond. The scientific output inspired by the memoir is immense. And in recent literature, the appeal of the St. Petersburg paradox has stimulated theoretical and experimental research (see, among others, Blavatskyy 2005; Teira 2006; Hayden and Platt 2009; Marcondes et al. 2017; Cox et al. 2017; Nobandegani and Shultz 2020; Gasparian et al. 2021; Yukalov 2021; Peterson 2022). It has inspired many different *puzzles*, including the Pasadena game (Nover and Hájek 2004) as well as artistic enterprise: in 2014, a sculpture series, *Sankt Petersburg Paradox* by the German artist Sarah Ortmeyer, was shown at the Swiss Institute in New York.

The moving words of airplane pilot Scheer (2018) can provide a fitting conclusion: "I find it fascinating that Daniel Bernoulli, who gave us the principle known to every student of the wing, also gave us some of the earliest work on the calculation of risk. There you have the two challenges given to we aviators—how to fly the contraption and how to keep it from killing us".

Funding Open access funding provided by Università degli Studi di Parma within the CRUI-CARE Agreement.

Declarations

Conflict of interest The author did not receive support from any organization for the submitted work. She has no financial or proprietary interest to disclose.

References

Arrow, K.J.: Alternative approaches to the theory of choice in risk-taking situations. Econometrica **19**, 404–437 (1951)

Arrow, K.J.: Essays in the Theory of Risk-Bearing. North-Holland, Amsterdam (1971)

Aumann, R.J.: The St. Petersburg paradox: a discussion of some recent comments. J. Econ. Theory **14**, 443–445 (1977)

Bell, E.T.: Men of Mathematics. Simon and Schuster, New York (1937)

Bernoulli, J.: Ars Conjectandi. Impensis Thurnisiorum, fratrum, Basel (1713)

Bernoulli, D.: Hydrodynamica. J.R. Dulsecker, Strasbourg (1738a1). **Translated by Mikhailov, G.K. In Grattan-Guinness, I. (ed.). Landmark Writings in Western Mathematics 1640–1940, pp. 131-142. Elsevier, Amsterdam (2005)**

Bernoulli, D.: *Specimen Theoriae Novae de Mensura Sortis*. Commentari Academiae Scientiarum Imperialis Petropolitanae **5**, 175–192 (1738). (**Translated by: (a) Pringsheim, A.: Die Grundlage der modernen Wertlehre. Duncker & Humblot, Leipzig (1896). (b) Sommer, L.: Exposition of a New Theory on the Measurement of Risk. Econometrica 22, 23-36 (1954). (c) Mille, R.: Expos'e d'une théorie nouvelle sur l'évaluation du risque. Revue de Statistique Appliquée 19, 5-18 (1971). (d) Charreton, R.: Esquisse d'une théorie nouvelle de mesure du sort, Cahiers du sé minaire**

d'histoire des Mathématiques 6, 61-77 (1985). (e) Agnoli, P., Piccolo, F.: Presentazione di una nuova teoria sulla valutazione del rischio e concetto di utilità. Bollettino telematico di filosofia politica. https://archiviomarini.sp.unipi.it/592 (2008). Accessed 14 March 2023)

Bernoulli, D.: Dijudicatio maxime probabilis plurium observationum discrepantium atque verisimillima inductio inde formanda. Acta Academiae Scientiarum Imperialis Petropolitanae for 1777, pars prior: 3–23 (1778). **Translated by Allen, G.D.: The most probable choice between several discrepant observations and the formation therefrom of the most likely induction. Biometrika 48, 3-18 (1961)**

Bernoulli, N.: 1713–1732. Correspondence of Nicolas Bernoulli concerning the St. Petersburg Game. **Translated by Pulskamp, R.J. (2013). https://www.google.com/search?client=firefox-b-d&q=Correspondence+of +Nicolas+Bernoulli+concerning+the+St.+ Petersburg+game Accessed 14 May 2023**

Blaug, M.: Economic theory in retrospect. Richard D. Irwin Press, Homewood (1962). 5th Edition Cambridge University Press (1997)

Blavatskyy, P.R.: Back to the St. Petersburg paradox? Manag. Sci. **51**, 677–678 (2005)

Boyer, C.B.: A History of Mathematics. Wiley, New York (1968)

Brito, D.L.: Becker's theory of the allocation of time and the St. Petersburg paradox. J. Econ. Theory **10**, 123–26 (1975)

Buffon, G.L.L: Essai d'Arithmetique Morale. In Supplément à l'Histoire Naturelle, vol. 4, Supplement (1777). **Translated by Hey, J.D, Neugebauer, T.M., Pasca, C.M.: Essays on Moral Arithmetic. Research Paper Series No. 10-06, University of Luxembourg, Luxembourg (2010)**

Coolidge, J.W.: The mathematics of great amateurs. Dover Publications (1949)

Cox, J.C., Kroll, E.B., Lichters, M., Sadiraj, V., Vogt, B.: The St Petersburg paradox despite risk-seeking preferences: an experimental study. Bus. Res. **12**, 27–44 (2017)

Daboni, L.: Sulla nozione di utilità bernoulliana. In: Franco, G. (ed.) Saggi in onore di Bruno Menegoni, pp. 97–114. Alceo, Padua (1982)

de Finetti, B.: Sul concetto di media. Giornale degli Economisti **2**, 369–396 (1931)

de Montmort, P.R.: Essay D'analyse Sur Les Jeux de Hazard. Quillau, Paris (1708). 2nd Edition Quillau (1713)

Friedman, M., Savage, L.J.: The utility analysis of choices involving risk. J. Polit. Econ. **56**, 279–304 (1948)

Gasparian, M.S., Kiseleva, I.A., Titov, V.A., Sadovnikova, N.A.: St Petersburg paradox: adoption of decisions on the basis of data mining and development of software in the sphere of business analytics. Nexo **34**, 1370–1380 (2021)

Giocoli, N.: The 'true' hypothesis of Daniel Bernoulli: What did the marginalists really know? Hist. Econ. Ideas **6**, 7–43 (1998)

Hayden, B.Y., Platt, M.L.: The mean, the median, and the St. Petersburg paradox. Judgm. Decis. Mak. **4**, 256–272 (2009)

Heukelom, F.: What's in a name?. https://www.ineteconomics.org/perspectives/blog/whats-in-a-name (2011). Accessed 6 June 2023

Jensen, N.E.: An introduction to Bernoullian utility theory. Swed. J. Econ. **69**, 163–183 (1967)

Keynes, J.M.: A treatise on probability. Macmillan, London (1921, reprinted 1957)

Laplace, P.S.: Théorie Analytique des Probabilités. Courcier, Paris (1812)

LiCalzi, M.: Bipartite choices. Decis. Econ. Finan. **45**, 551–568 (2022)

Marcondes, D., Peixoto, C., Souza, K., Wechsler, S.: Entrance fees and a Bayesian approach to the St. Petersburg paradox. Philosophies **2**, 11 (2017)

Menger, K.: Das Unsicherheits-moment in der Wertlehre. Z. Nationaldkon, **51**, 459–485 (1934). **Translated by Schoellkopf, W.: The role of uncertainty in economics. In: Shubik, M. (ed.) Essays in Honor of Oskar Morgenstern, pp. 211-232. Princeton University Press, Princeton (1967)**

Nobandegani, A.S., Shultz, T.R.: A resource-rational, process-level account of the St. Petersburg paradox. Top. Cogn. Sci. **12**, 417–432 (2020)

Nover, H., Hájek, A.: Vexing expectations. Mind **113**, 237–249 (2004)

Peterson, M.: The St. Petersburg paradox. In: Zalta, E.N. (ed.) The Stanford Encyclopedia of Philosophy. https://plato.stanford.edu/archives/sum2022/entries/paradox-stpetersburg/. Accessed 11 July 2023

Polasek, W.: The Bernoullis and the origin of probability theory: looking back after 300 years. Resonance **5**, 26–42 (2000)

Ramsey, F.P.: Truth and probability. In: The Foundations of Mathematics and other Logical Essays, pp. 156–198. Routledge and Kegan Paul, London (1931)

Salov, V.: "The Gibbon of Math History". Who invented the St. Petersburg paradox? Khinchin's resolution. arXiv:1403.3001 (2014). Accessed 14 May 2023
Samuelson, P.A.: St. Petersburg paradoxes: defanged, dissected, and historically described. J. Econ. Lit. **15**, 24–55 (1977)
Savage, L.J.: The Foundations of Statistics. Wiley, New York (1954)
Scheer, G.: Rolling the dice with the Bernoulli brothers. https://www.thisaviationlife.com/rolling-dice-bernoulli-brothers/ (2018). Accessed 27 July 2023
Schoemaker, P.J.H.: The expected utility model: its variants, purposes, evidence and limitations. J. Econ. Lit. **2**, 529–563 (1982)
Seidl, C.: The St. Petersburg paradox at 300. J. Risk Uncertain. **46**, 247–264 (2013)
Shapley, L.S.: The Petersburg paradox-A con game? J. Econ. Theory **14**, 439–444 (1977a)
Shapley, L.S.: Lotteries and Menus: a comment on unbounded utilities. J. Econ. Theory **14**, 446–453 (1977b)
Stearns, S.C.: Daniel Bernoulli (1738): evolution and economics under risk. J. Biosci. **25**, 221–228 (2000)
Stigler, G.J.: The development of utility theory, parts I and II. J. Polit. Econ. **58**(307–327), 373–396 (1950)
Stigler, S.M.: The Bernoullis of Basel. J. Econom. **75**, 7–13 (1996)
Teira, D.: On the normative dimension of the St Petersburg paradox. Stud. Hist. Philos. Sci. Part A **37**, 210–223 (2006)
Todhunter, I.: A History of the Mathematical Theory of Probability: From the Time of Pascal to that of Laplace. Cambridge University Press, Cambridge (1865)
von Neumann, J., Morgenstern, O.: The Theory of Games and Economic Behavior, 2nd edn. Princeton University Press, Princeton (1944)
Yukalov, V.I.: A resolution of St. Petersburg paradox. J. Math. Econ. **97**, 102537 (2021)