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Variational Methods in Material Science

Coordinatore:
Prof. Leonardo Biliotti

Tutore:
Prof.ssa Michela Eleuteri

Cotutore:
Prof. Luca Lussardi

Dottorando: **Andrea Torricelli**

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Andrea Torricelli

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Abstract

English: In this thesis we will study the properties of a number of functionals related to Material Science. In particular we are interested in the energy of models such as Peridynamic, Micromagnetism, Elasticity and many others. It is well known that Calculus of Variations is a key tool in order to understand these models. Our aim is to show some examples of how classical results of Calculus of Variations and Functional Analysis such as the Direct Method of Calculus of Variations and Γ-convergence, can be used to study these energies.

Italiano: In questa tesi studiamo le proprietà di alcuni funzionali relativi alla Scienza dei Materiali. In particolare siamo interessati dall'energia di modelli come la Peridinamica, il Micromagnetismo, l'Elasticità e molti altri. E' risaputo che il Calcolo delle Variazioni è uno strumento fondamentale per lo studio di questi modelli. Il nostro scopo è quello di mostrare alcuni esempi di come risultati classici del Calcolo delle Variazioni e dell'Analisi Funzionale come il Metodo Diretto del Calzolo delle Variazioni e la Gamma-convergenza possono essere utilizzati per studiare queste energie.

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Part I

Introduction and Preliminaries

Introduction

This thesis is structure as follows:

In **Chapter 1** we recall the majority of the tools we will use in further chapters. First we introduce the problem from which the Direct Method of Calculus of Variations originated, then we consider the cases where the method fails and we briefly explain how to treat them. Finally we introduce further tools such as the BV functions and Γ-convergence.

In **Chapter 2** we study the interaction between the linearized elastic energy and the Micromagnetic model. First we introduce the Micromagnetic model and then we use Γ-convergence in order to study the dimensional reduction of the linearized elastic energy, obtaining an explicit expression for the Γ-limit.

In **Chapter 3** we consider the problem of the relaxation of a particular class of non-local supremal functionals. In this Chapter we also state some of the more direct consequences of our result for differential inclusions and non-local indicator functionals. Finally, we use our main theorem in order to prove a dimensional reduction result for our non-local supremal functionals. The results stated in this Chapter can be found in the paper

Relaxation of one-dimensional non-local supremal functionals in the Sobolev setting, A. Torricelli, E. Zappale, https://doi.org/10.48550/arXiv.2307.13597, (2023)

In **Chapter 4** we study a family of functionals defined on sets. Such functionals have many applications and our aim is to investigate their asymptotic behavior. In particular, in the direction of studying the Γ-limit. The results presented in this Chapter are contained in the paper

Asymptotic Analysis of a Family of Non-local Functionals on Sets, M. Eleuteri, L. Lussardi, A. Torricelli, ESAIM: COCV 29 (2023) 1.

In **Chapter 5** and **Chapter 6** we consider the obstacle problem. Due to its very nature, the Obstacle Problem is related to a variety of different topics, even outside of the field of Material Science. In this Chapters we give a brief overview of such topics and we discuss two problems related to the Obstacle Problem: the extremality of its solutions and their higher differentiability with respect to the differentiability of the Obstacle. The results contained in this Chapter can be found in

Regularity Results for Bounded Solutions to Obstacle Problems with Non-standard Growth Conditions, A. Gentile, R. Giova, A. Torricelli, Mediterr. J. Math. 19:270, (2022), and

A necessary condition for extremality of solutions to autonomous obstacle problems with general growth, S. Riccò, A. Torricelli, Nonlinear Analysis: Real World Applications 76, (2024),

and

Monotonic approximation of differentiable convex functions with applications to general minimization problems, P. Harjulehto, P.Hästö, A. Torricelli, https://doi.org/10.48550/arXiv.2310.11920, (2023).

Chapter 1

Mathematical preliminaries

In this chapter we recall some classical notions of Calculus of Variations, mainly the Direct Method of Calculus of Variations, Ekeland Variational Principle, Γ-convergence and the space of BV functions. The aim is to motivate the existence of the solution to the classical problem of Calculus of Variation, since such solutions play a central role in this thesis. The results stated in this chapter are widely known, hence we will not give proof of every statement, providing just corresponding references. Moreover, we assume the reader knowledge of some of the most basic definitions such as compactness, measurable function, Borelian functions, etc...

This chapter is structured as follow. First we state the Direct Method of Calculus of Variations, then we cover the cases where the Method fails. After that, we introduce the definition of BV functions and some of their properties. Finally we introduce the notion of Γ-convergence and its properties.

1.1 Direct Method of Calculus of Variations

Throughout this chapter we will denote with X a topological space and with $\overline{\mathbb{R}}$ the extended set of real numbers.

Definition 1.1.1 (Compactness)**.** Given a subset K of X we say that K is

i) *sequentially compact* if every sequence of points of K has a subsequence converging to a point of K,

ii) *relatively compact* if its closure is a compact set,

iii) *relatively sequentially compact* if its closure is sequentially compact.

Definition 1.1.2 (Lower semicontinuity)**.** Given a function $F : X \to \overline{\mathbb{R}}$ we say that F is

i) *lower semicontinuous* if for all $t \in \mathbb{R}$ the set $\{x \in X : F(x) > t\}$ is open,

ii) *sequentially lower semicontinuous* if for every point $x \in X$ and for every sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converging to x it holds

$$F(x) \leqslant \liminf_{n\to\infty} F(x_n).$$

Definition 1.1.3 (Coercivity)**.** Given a function $F : X \to \overline{\mathbb{R}}$ we say that F is *coercive* if for all $t \in \mathbb{R}$ the set $\{x \in X : F(x) \geqslant t\}$ is compact, while we say that F is *sequentially coercive* if for all $t \in \mathbb{R}$ the set $\{x \in X : F(x) \geqslant t\}$ is sequentially compact.

Remark 1.1.4. It is well known that lower semicontinuity and coercivity are weaker than their own sequential counterpart, i.e. that lower semicontinuity and coercivity imply respectively sequential lower semicontinuity, and sequential coercivity. Moreover, it is also well known that the previous notions coincide with their sequential counterpart if the topological space X is a metric space. Finally, if X is a metric space, then the notions of sequential compactness and compactness coincide.

Theorem 1.1.5 (Direct Method of Calculus of Variations)**.** *Given $F : X \to \overline{\mathbb{R}}$ sequentially coercive and sequentially lower semicontinuous, then F attains its minimum on X. Also, if F is not identically to $+\infty$, every minimizing sequence has a converging subsequence.*

Proof. We can assume without loss of generality that F is not identically $+\infty$ and so that $\inf_{x\in X} F(x) < +\infty$. Fixed a minimizing sequence $(x_n)_n$, it holds by definition that

$$\lim_{n\to+\infty} F(x_n) = \inf_{x\in X} F(x).$$

Since F is sequentially coercive, we can assume that $(x_n)_n \subset K$, where K is some proper sequentially compact subset of X. Since K is sequentially compact, it follows that we can extract a non relabeled converging subsequence $(x_n)_n$. Let x be the limit of x_n. Since F is sequentially lower semicontinuous we get

$$\inf_{x\in X} F(x) \leqslant F(x) \leqslant \liminf_{n\to\infty} F(x_n) = \inf_{x\in X} F(x)$$

□

Remark 1.1.6. It is interesting to observe that the requirements to use the Direct Method of Calculus of Variations are paradoxical. Indeed the weaker the topology chosen on X, the easier is for F to be sequentially coercive and the harder is for F to be sequentially lower semicontinuous.

In this thesis we will work with integral functionals, i.e. functionals $F : W^{1,p}(\Omega, \mathbb{R}^N) \to \overline{\mathbb{R}}$ such as

$$F(u) := \int_\Omega f(x, u, \nabla u)\, dx, \tag{1.1.7}$$

where $1 \leqslant p \leqslant +\infty$ and Ω is an open and bounded subset of $\mathbb{R}^n$. Thus the Direct Method gives the solution to the problem

$$\min_{u\in W^{1,p}(\Omega,\mathbb{R}^N)} \int_\Omega f(x, u, \nabla u)\, dx, \tag{1.1.8}$$

when $1 < p < +\infty$. In order to make this thesis self contained, we will now state a well known result on the necessary and sufficient condition for the lower semicontinuity and coercivity of such functionals. In order to state such results, we need to introduce some definitions.

Definition 1.1.9 (Carathéodory function)**.** Given a function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^{N\times n}$ we say that f is a *Carathéodory function* if for every $x \in \Omega$ the function $f(x, \cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^{N\times n}$ and for every $(\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}^{N\times n}$ the function $f(\cdot, \eta, \xi)$ is Lebesgue measurable.

Remark 1.1.10. It should be clear that if f is a Carathéodory function and $u : \Omega \to \mathbb{R}^n$ and $v : \Omega \to \mathbb{R}^{N\times n}$ are Lebesgue measurable functions then the function $f(x, u(x), v(x))$ is again a Lebesgue measurable function.

Definition 1.1.11 (Quasiconvexity)**.** Given a Borel measurable and locally bounded function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ we will say that f is *quasiconvex* if for every open and bounded set $D \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N\times n}$ and every $\phi \in W_0^{1,\infty}(D, \mathbb{R}^N)$ it holds

$$f(\xi) \leqslant \frac{1}{|D|}\int_D f(\xi + \nabla\phi(x))dx$$

Remark 1.1.12. It is not difficult to prove that any convex function is also quasiconvex.

Definition 1.1.13 (p-growth)**.** Given a function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ and $1 \leqslant p < +\infty$ we say that f has *p-growth* (or *Standard growth*) if there exist a positive constant ν such that for a.e. $x \in \Omega$ it holds

$$\frac{1}{\nu}(|u|^p + |\xi|^p) \leqslant f(x, u, \xi) \leqslant \nu(1 + |u|^p + |\xi|^p). \tag{1.1.14}$$

If $p = 1$ we say that f has *Linear growth.* We say that f has *(p,q)-growth* (or *Non-standard growth*) if there exists $1 \leqslant p < q < +\infty$ and $\nu > 0$ such that

$$\frac{1}{\nu}(|u|^p + |\xi|^p) \leqslant f(x, u, \xi) \leqslant \nu(1 + |u|^q + |\xi|^q), \tag{1.1.15}$$

for a.e. $x \in \Omega$

The proof of the next theorems can be found for instance in [45].

Theorem 1.1.16 (Necessary and Sufficient Condition)**.** *Given $\Omega \subset \mathbb{R}^n$ open and bounded and a Caratheodory function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ with p-growth, for some $1 \leqslant p < +\infty$, then the functional F defined as*

$$F(u) := \int_\Omega f(x, u, \nabla u)dx,$$

is weakly lower semicontinuous in $W^{1,p}(\Omega, \mathbb{R}^N)$ if and only if $f(x, u, \cdot)$ is quasiconvex for every $x \in \Omega$ and $u \in \mathbb{R}^N$.

Remark 1.1.17. If f has p-growth for some $p > 1$, then fixed $t \in \mathbb{R}$ and $u \in \{u \in W^{1,p}(\Omega, \mathbb{R}^N) : F(u) \leqslant t\}$ it follows that

$$||u||_{W^{1,p}(\Omega)} \leqslant \nu t,$$

i.e. the sublevel set $\{u \in W^{1,p}(\Omega, \mathbb{R}^N) : F(u) \leqslant t\}$ is bounded with respect to the norm $W^{1,p}(\Omega, \mathbb{R}^N)$. Moreover, since $p > 1$ then $W^{1,p}(\Omega, \mathbb{R}^N)$ is reflexive and thus the sublevel set $\{u \in W^{1,p}(\Omega, \mathbb{R}^N) : F(u) \leqslant t\}$ is compact with respect to the weak topology of $W^{1,p}(\Omega, \mathbb{R}^N)$. Consequently F is weakly coercive.

From the previous remark, the next Corollary follows.

Corollary 1.1.18. *Given $\Omega \subset \mathbb{R}^n$ open, $p > 1$, and a Caratheodory function $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}$ quasiconvex in the third variable and with p-growth, then the integral functional*

$$F(u) := \int_\Omega f(x, u, \nabla u)dx,$$

has at least one minimizer in $W^{1,p}(\Omega, \mathbb{R}^N)$ and every minimizing sequence in $W^{1,p}(\Omega, \mathbb{R}^N)$ converges weakly to a minimizer of F.

Remark 1.1.19. It is not difficult to prove that if both $f(x, \cdot, \xi)$ and $f(x, u, \cdot)$ are strictly convex functions then the solution to the minimum problem is unique. See [45], Theorem 3.30 for more details.

When $N = 1$ and $p > 1$, it is possible to generalize the previous method for a functional defined on a subset S of $W^{1,p}(\Omega)$, assuming that S has the property that the seminorm

$$u \mapsto ||\nabla u||_{L^p(\Omega)}$$

induces the same topology on S as the usual norm of $W^{1,p}(\Omega)$. Some examples of sets satisfying this property are

$$\begin{aligned}
S_1 &:= \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u\, dx = m \right\}, \text{ for some positive } m,\\
S_2 &:= \left\{ u \in W^{1,p}(\Omega) : ||u||_{L^p(\Omega)} = m \right\}, \text{ for some positive } m,\\
S_3 &:= \left\{ u \in W^{1,p}(\Omega) : u - u_0 \in W_0^{1,p}(\Omega) \right\}, \text{ for some } g \in W^{1,p}(\Omega),\\
S_4 &:= \{ u \in S_3 : u \geqslant \psi \text{ a.e. in } \Omega \}, \text{ for some function } \psi \in W^{1,p}(\Omega).
\end{aligned}$$

It can be shown that the property is satisfied on the previous sets by means of Poincaré and Poincaré-Wirtinger inequality (for more details we refer to [133]). The sets S_3 will be of particular relevance for the results presented in Chapter 6. The set will be intuitively denoted by $W^{1,p}_{u_0}(\Omega)$, i.e.

$$W^{1,p}_{u_0}(\Omega) := \left\{ u \in W^{1,p}(\Omega) : u - u_0 \in W_0^{1,p}(\Omega) \right\}.$$

The case when $p = 1$, i.e. if we search for minimizers of F in $W^{1,1}(\Omega)$ or in one of its subsets, is more complicate. Indeed since $W^{1,1}(\Omega)$ is not reflexive, then the assumption of linear growth is not enough to grant any coercivity for f. One way to approach the problem is Ekeland Variational Principle. For the proofs of the following theorems see [143].

Theorem 1.1.20 (Ekeland Variational Principle)**.** *Let (X, d) be a complete metric space and $F : X \longrightarrow \overline{\mathbb{R}}$ be lower semicontinuous and bounded from below. Given $\varepsilon > 0$ and $x \in X$ such that*

$$F(x) \leqslant \inf_X F + \varepsilon,$$

then for every $\lambda > 0$ there exists $x_\lambda \in V$ such that

i) $d(x, x_\lambda) \leqslant \lambda$,

ii) $F(x_\lambda) \leqslant F(x)$,

iii) x_λ *is the unique minimizer of the functional* $x \longmapsto F(x) + \varepsilon\lambda^{-1}d(x, x_\lambda)$.

By taking ε and λ as (proper) vanishing functions of $k \in \mathbb{N}$, Ekeland Variational Principle generates a minimizing sequence with the property that each point of the sequence is the unique minimizer of a perturbed problem. It is then clear that this does not ensure the existence of a minimizer for F, since we have no control on the asymptotic behavior of the minimizing sequence. The next theorem will give us an idea of what are the conditions under which we can expect the weak convergence for the minimizing sequence. For more details see [67].

Theorem 1.1.21 (Compactness Criterion)**.** *Let* Ω *be of finite Lebesgue measure. Given a sequence of functions* $(u_k)_k \subset L^1(\Omega, \mathbb{R}^N)$*, then the following statements are equivalent:*

i) *We can extract a (non relabeled) subsequence* $(u_k)_k$ *that converges weakly in* $L^1(\Omega, \mathbb{R}^N)$.

ii) *For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* k

$$\int_{\{|u_k|>\delta\}} |u_k| dx < \varepsilon.$$

iii) *For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every* k

$$\int_B |u_k| dx < \varepsilon,$$

for every measurable subset $B \subset \Omega$ *of measure* $|B| < \delta$.

iv) *There exists a positive, Borel and superlinear function* $\theta : [0, +\infty) \to \overline{\mathbb{R}}$ *such that*

$$\sup_k \int_\Omega \theta(|u_k|) dx < +\infty.$$

The condition $ii)$ is called *Equi-integrability* or *Uniform integrability*. The chain of implications $i) \Leftrightarrow ii) \Leftrightarrow iii)$ is known as Dunford-Pettis Compactness Criterion, while the implication $ii) \Leftrightarrow iv)$ is known as de la Vallée-Poussin Criterion (or Theorem). Names aside, the previous theorem tells us that the superlinearity of the integrand f is, together with the lower semicontinuity of F, a sufficient condition for the existence of the minimizers. For the sake of completeness, we recall the definition of superlinearity.

Definition 1.1.22. Given a function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ we say that f is *superlinear* if

$$\lim_{|\xi| \to \infty} \frac{f(\xi)}{|\xi|} = +\infty.$$

The lower semicontinuity of F is ensured by the superlinearity of f and by Theorem 1.1.16. Indeed it can be shown that the theorem holds true if f is superlinear.

Clearly, the only case that has not yet been considered is the case when the integrand f has linear growth. In this case the idea is to consider the minimum problem on a wider space than $W^{1,1}(\Omega)$ (the space of BV functions) and then consider an extension $\overline{F}$ of the functional F and then to formulate the minimum problem on the new space with respect to the new functional

$\overline{F}$. In particular it is important to construct $\overline{F}$ in such a way that the minimum of $\overline{F}$ on BV coincides with the infimum of F on $W^{1,1}$. This case presents many more complications then the previous ones and thus we refer to [3], [133], or [70] for and deeper discussion of on the topic. Functions of bounded variation and their properties will be useful in Chapter 4 so we will proceed with a brief overview on the subject.

Definition 1.1.23 (BV Function)**.** Given $u \in L^1(\Omega)$, u is said to be a function of *Bounded Variation* if its distributional derivatives are representable by finite Radon measures in Ω, i.e. if

$$\int_\Omega u\frac{\partial \phi}{\partial x_i}dx = -\int_\Omega \phi d(D_i u), \quad \phi \in C_0^\infty(\Omega), i = 1, ..., n.$$

for some vector valued Radon measure $Du = (Du_1, ..., Du_n)$. The space of function of bounded variation on Ω is denoted with $BV(\Omega)$.

An alternative definition is the following.

Proposition 1.1.24. *Given $u \in L^1(\Omega)$ then u is a BV function if and only if*

$$\sup\left\{\int_\Omega u\mathrm{div}\phi dx : \phi \in C_0^\infty(\Omega), ||\phi||_{L^\infty(\Omega)} \leqslant 1\right\} < +\infty.$$

Definition 1.1.25 (Total Variation)**.** Given $u \in L^1(\Omega)$ the quantity

$$\sup\left\{\int_\Omega u\mathrm{div}\phi dx : \phi \in C_0^\infty(\Omega), ||\phi||_{L^\infty(\Omega)} \leqslant 1\right\},$$

is called *Total Variation* of u on Ω and it is denoted with $|Du|(\Omega)$.

The previous proposition tells us that BV functions are those functions that have finite total variation on Ω. On another note, it is easy to see that $W^{1,1}(\Omega) \subset BV(\Omega)$ and for every $u \in W^{1,1}(\Omega)$ it holds $Du = \nabla u\mathcal{L}^n$, where $\mathcal{L}^n$ denote the $n-$dimensional Lebesgue measure. In this thesis we will denote with χ_E the characteristic function of a set E. If E has finite measure then $\chi_E \in L^1(\Omega)$. Moreover, if we suppose that $\mathcal{L}^n(E \cap \Omega) > 0$, by Proposition 1.1.24, it is clear that $\chi_E \in BV(\Omega)$ and thus that its total variation

$$|D\chi_E|(\Omega) = \sup\left\{\int_E \mathrm{div}\phi dx : \phi \in C_0^\infty(\Omega), ||\phi||_{L^\infty(\Omega)} \leqslant 1\right\}$$

is finite. Furthermore, since $\chi_E \in BV(\Omega)$ it should be evident that $BV(\Omega)$ is not a subset of $W^{1,1}(\Omega)$ since Sobolev functions do not admit jump discontinuities.

Definition 1.1.26 (Perimeter)**.** Given $E \subset \mathbb{R}^n$ Lebesgue measurable, the *Perimeter* of E in Ω is defined as

$$\mathcal{P}(E, \Omega) := |D\chi_E|(\Omega).$$

We will denote with $\mathcal{P}_n(\Omega)$ the set of sets of finite perimeter in Ω. Moreover if $\Omega = \mathbb{R}^n$ then we will denote the perimeter as $\mathcal{P}(E)$.

It is only natural to wonder if there exists a relation between this definition of perimeter and the more common $n-1$ dimensional Hausdorff measure. In order to answer this question it is necessary the definition of Reduced Boundary. In the following we will denote the n-dimensional Hausdorff measure as H^n.

Definition 1.1.27 (Reduced Boundary)**.** Given $E \subset \mathbb{R}^n$ Lebesgue measurable, its *Reduced Boundary* is defined as the set $\partial^* E$ of points $x \in supp\,|D\chi_E|$ such that

- $\exists \lim_{h\to 0} \frac{D\chi_E(B(x,h))}{|D\chi_E|(B(x,h))} =: \nu_E(x)$,
- $||\nu_E(x)|| = 1$.

Moreover, the unit vector $\nu_E(x)$ is called *Inner Unit Normal* of E in x.

Theorem 1.1.28 (De Giorgi, [3])**.** *Given $E \subset \mathbb{R}^n$ Lebesgue measurable, then*

- *$\partial^* E$ is countably $n-1$ rectifiable and $|D\chi_E| = H^{n-1} \llcorner \partial^* E$,*
- *for any $x \in \partial^* E$ the set $\frac{E-x}{h}$ converges locally in measure in $\mathbb{R}^n$ to the halfspace $H_{E,x}$ orthogonal to $\nu_E(x)$ and containing $\nu_E(x)$, i.e.*

$$H_{E,x} := \{\xi \in \mathbb{R}^n : \xi \cdot \nu_E(x) \geqslant 0\}.$$

- *(Gauss-Green formula)*

$$\int_E \mathrm{div}\psi dx = -\int_{\partial^* E} \psi \cdot \nu_E \, dH^{n-1}$$

The halfspace $H_{E,x}$ is called *Tangent Halfspace* of E in x. Clearly, the previous theorem shows the relation between the perimeter and the Hausdorff measure. Moreover, it gives a more understandable meaning to the reduced boundary of a set: it is the "effective" boundary of the set, i.e. the set of points that are "seen" by the Hausdorff measure. For more details on the theory of sets of finite perimeter see [3] and [70]

In order to make a topological space out of BV we need to consider a topology. There are three kinds of convergences (and thus topologies) associated to BV. The strong BV convergence, i.e. the usual norm convergence, the weak*-BV convergence and the so called intermediate (or strict) convergence. We will define only the first two, since we will not use the third one in this thesis.

Definition 1.1.29 (BV norm)**.** Given $u \in BV(\Omega)$ the BV norm of u is defined as

$$\begin{aligned} ||u||_{BV(\Omega)} &:= \int_\Omega |u| dx + |Du|(\Omega) \\ &= ||u||_{L^1(\Omega)} + |Du|(\Omega). \end{aligned}$$

Clearly, given a sequence $(u_h)_h \subset BV(\Omega)$, then $u_h \overset{BV}{\to} u$ as $h \to 0$ if $||u_h - u||_{BV(\Omega)} \to 0$ as $h \to 0$.

Definition 1.1.30 (Weak*-BV convergence)**.** Given a sequence $(u_h)_h \subset BV(\Omega)$ and $u \in BV(\Omega)$ then u_h converges weakly* to u in BV if u_h converges to u in $L^1(\Omega)$ and Du_h converges to Du weakly*, i.e. if

$$\int_\Omega \phi d(Du_h) \to \int_\Omega \phi d(Du), \quad \phi \in C_0(\Omega)$$

as $h \to 0$.

Obviously the convergence in norm is stronger than the weak* convergence. We will now introduce some properties of the weak* convergence. The proof of the following theorem can be found in [70].

Theorem 1.1.31 (Compactness for BV functions)**.** *Given $\Omega \subset \mathbb{R}^n$ open, bounded and with Lipschitz boundary and a sequence $(u_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ satisfying*

$$\sup_k ||u_k||_{BV(\Omega)} < +\infty,$$

then there exists a (non relabeled) subsequence $(u_k)_k$ and a function $u \in BV(\Omega)$ such that $u_k \to u$ in $L^1(\Omega)$ as $k \to +\infty$.

The proof of the following theorem can be found in [3].

Theorem 1.1.32 (Characterization of weakly*-BV convergence)**.** *Given a sequence $(u_k)_{k\in\mathbb{N}} \subset BV(\Omega)$ and a function $u \in BV(\Omega)$, then $u_k \rightharpoonup^* u$ in $BV(\Omega)$ if and only if $(u_k)_{k\in\mathbb{N}}$ is bounded in $BV(\Omega)$ and $u_k \to u$ in $L^1(\Omega)$.*

The last notion we would like to introduce in this section is the notion of Relaxed Functional or Lower Semicontinuous Envelope.

Definition 1.1.33 (Lower semicontinuous envelope)**.** Let X be a metric space. Given a functional $F : X \to \overline{\mathbb{R}}$, its *Lower semicontinuous envelope* $\overline{F}$ is defined as

$$\overline{F}(x) := \inf \left\{ \liminf_{n\to\infty} F(x_n) : x_n \to x \text{ in } X \right\}$$

It is not difficult to prove that $\overline{F}$ is lower semicontinuous with respect to the topology of X. Indeed, it actually possible to prove that $\overline{F}$ is the greater lower semicontinuous functional smaller than F, i.e.

$$\overline{F}(x) = \sup \left\{ \phi(x) | \phi : X \to \overline{\mathbb{R}}, \phi(x) \leqslant F(x), \phi \text{ lower semicontinuous in } X \right\}.$$

From now on, we will respectively write l.s.c and l.s.c envelope instead of lower semicontinuous and lower semicontinuos envelope.

1.2 $\Gamma-$convergence

In this subsection we will recall the basic notion of Γ-convergence introduced by De Giorgi in [52] in the 1970's. Γ-convergence is a variational notion of convergence that was constructed in order to study the asymptotic behavior of families of functionals parametrized by a small

parameter. This is because it is charaterized by the property that, if a family of functionals $(F_h)_h$ Γ-converges to a functional F and if the minimizer u_h of F_h are converging to a point u, then u is a minimizer of F.
As before we will denote with X a topological space. Furthermore, given a point $x \in X$ we will denote with $N(x)$ the set of all the open neighborhood of x in X. Finally we will denote with h an infinitesimal sequence of real numbers. For more details on $\Gamma-$convergence, see for instance [3], [46].

Definition 1.2.1 (Γ-convergence)**.** Given a sequence of functionals $(F_h)_h$ on X, its *Upper* $\Gamma-$*limit* and *Lower* $\Gamma-$*limit* are functionals respectively defined as

$$(\Gamma - \limsup_{h\to 0} F_h)(x) := \sup_{U\in N(x)} \limsup_{h\to 0} \inf_{y\in U} F_h(y)$$
$$(\Gamma - \liminf_{h\to 0} F_h)(x) := \sup_{U\in N(x)} \liminf_{h\to 0} \inf_{y\in U} F_h(y).$$

If there exists a functional F on X such that

$$F(x) = (\Gamma - \limsup_{h\to 0} F_h)(x) = (\Gamma - \liminf_{h\to 0} F_h)(x),$$

then F is called Γ-*limit of* F_h with respect to the topology of X. Oftentimes we will write $\Gamma - \lim_{h\to 0} F_h = F$ or $F_h \xrightarrow{\Gamma} F$ when F_h is $\Gamma-$converging to F. Also, the Lower and Upper $\Gamma-$limit are often respectively called $\Gamma - \liminf$ and $\Gamma - \limsup$ functionals.

It is clear that the previous definition is too general to be used as it is. In order to make the definition more convenient, we can assume further structure on the topological space X. If we suppose X to be a metric space then it is possible to prove (see [46]) that the previous definition is equivalent to the following,

Definition 1.2.2 (Sequential $\Gamma-$convergence)**.** Given a sequence of functionals $(F_h)_h$ on X and a functional F, it is said that F is the *Sequential* $\Gamma-$*limit* of F_h if it holds true that

i) for every $(x_h)_h \subset X$ converging in X to some point x then

$$F(x) \leqslant \liminf_{h\to 0} F_h(x_h),$$

ii) for every point $x \in X$ there exists a sequence $(x_h)_h \subset X$ converging to x in X and such that

$$F(x) \geqslant \limsup_{h\to 0} F_h(x_h).$$

The sequence $(x_h)_h$ in *ii*) is usually called *Recovery sequence.* Moreover, the *Sequential* $\Gamma - \liminf$ and *Sequential* $\Gamma - \limsup$ are respectively defined as

$$(\Gamma - \liminf_{h\to 0} F_h)(x) := \inf \left\{ \liminf_{h\to 0} F_h(x_h) : x_h \to x \text{ in } X \right\},$$
$$(\Gamma - \limsup_{h\to 0} F_h)(x) := \inf \left\{ \limsup_{h\to 0} F_h(x_h) : x_h \to x \text{ in } X \right\},$$

As it should be clear from the previous definitions, the Γ-limit of a sequence of functionals depends heavily on the topology of X and its other topological properties. In this thesis we will work mainly in $W^{1,p}$ (and some of its subsets) endowed with either the weak or strong topology. Since for $1 < p < \infty$ the space $W^{1,p}$ is a Banach and reflexive space then, any norm bounded subset of $W^{1,p}$ is metrizable. Because of this observation we will suppose from now on that X is a metric space. We will now state some known properties of the Γ-convergence.

Lemma 1.2.3. *The upper and lower Γ-limit are lower semicontinuous with respect to the topology of X.*

Lemma 1.2.4. *Given a functional $F : X \to \overline{\mathbb{R}}$ and the sequence $(F_h)_h$ definded as $F_h = F$, then the Γ-limit of F_h is the l.s.c envelope of F.*

Lemma 1.2.5. *Given a sequence of functionals $(F_h)_h$ on X and Γ-converging to F and given $G : X \to \overline{\mathbb{R}}$ continuous on X, then*

$$F_h + G \xrightarrow{\Gamma} F + G$$

Theorem 1.2.6 (Convergence of minimizers, [3])**.** *Given a sequence of functionals $(F_h)_h$ on X such that $F_h \xrightarrow{\Gamma} F$ and that there exists a compact set K such that for every h*

$$\inf_K F_h = \inf_X F_h. \tag{1.2.7}$$

Then,

$$\inf_X F_h \to \min_X F,$$

and any limit point of any sequence $(x_h)_h \subset X$ such that

$$\lim_{h\to 0}[F_h(x_h) - \inf_{x\in X} F(x)] = 0,$$

is a minimizer of F.

Proof. Given a sequence $(x_h)_h \subset X$ satisfying

$$\lim_{h\to 0}[F_h(x_h) - \inf_{x\in X} F(x)] = 0,$$

then it follows from (1.2.7) that $(x_h)_h$ has at least one limit point. Fix one of such limit points x and a subsequence $(x_{h_k})_k$ converging to x. It follows from the definition of Γ-limit that

$$F(x) \leqslant \liminf_{k\to\infty} F_{h_k}(x_{h_k}) = \liminf_{k\to\infty} \inf_X F_{h_k}.$$

Fixed a point $x' \in X$ and a recovery sequence x'_h converging to x', then, again from the definition of Γ-limit we get

$$F(x') \geqslant \limsup_{h\to 0} F_h(x'_h) \geqslant \limsup_{k\to\infty} \inf_X F_{h_k}.$$

Combining both inequalities we get

$$F(x) \leqslant \liminf_{k\to\infty} \inf_X F_{h_k} \leqslant \limsup_{k\to\infty} \inf_X F_{h_k} \leqslant F(x'),$$

which proves that x is a minimum point for F. Moreover, taking $x = x'$ gives

$$\min_X F = \lim_{k\to\infty} \inf_X F_{h_k}.$$

Since x is arbitrary the theorem follows. $\square$

Remark 1.2.8. It is easy to see that the previous theorem still holds true for a sequence of almost minimizers such as the one generated by Ekeland Variational Principle.

As it is clear from the proof, a condition equivalent to (1.2.7) is that any minimum point x_h of F_h is contained in a compact subset of X such that that any sequence of minimizer has at least a converging subsequence. This is exactly the same requirement that we have when we want to apply the Direct Method of Calculus of Variations and thus it will be no surprise that the previous theorem holds true if we suppose that the family $(F_h)_h$ is equi-coercive instead than (1.2.7), where equi-coercivity is defined as follows.

Definition 1.2.9 (Equi-coercivity)**.** The sequence $(F_h)_h$ is *Equi-coercive* on X if there exists a lower semicontinuous functional $\Phi : X \to \overline{\mathbb{R}}$ such that for every $h > 0$ it holds $\Phi \leqslant F_h$ for every $x \in X$.

It is important to emphasize the importance of the topology in the definition of equi-coercivity. Indeed the topology that makes $(F_h)_h$ equi-corcive will be the most suitable topology to use in order to pass to the Γ-limit. Since the L^p (or $W^{1,p}$) norm is lower semicontinuous with respect to the weak topology (and thus with respect to the strong topology) of L^p (or $W^{1,p}$), then it will be no surprise that a family of integral functionals, having densities satisfying p-growth conditions, will be equi-coercive with respect the the weak L^p (or $W^{1,p}$) topology.

Part II

Micromagnetism and Elasticity

Based on the model of the Landau-Lifshitz micromagnetic theory of ferromagnetic materials [12, 22, 23, 103, 110], the states of a rigid single-crystal ferromagnet, with a reference configuration $\Omega \subset \mathbb{R}^3$, subject to a given external magnetic field ξ_e, are described by a vector field, the magnetization M, verifying the so-called *fundamental constraint of micromagnetic theory*: a ferromagnetic body is always locally saturated, i.e. there exists a positive constant M_s such that

$$|M| = M_s(T) \quad \text{a.e. in } \Omega.$$

The saturation magnetization M_s depends on the chosen material and on the temperature T, and vanishes above a certain temperature, fixed for every crystal type, known as Curie point. In what follows, we will assume that the specimen is at a fixed temperature below the Curie point of the material, thus the value M_s will be regarded as a material-dependent function, i.e. a constant function in our framework. In this Chapter we will express the magnetization M under the form $M = M_s(T)m$, where $m : \Omega \to \mathbb{S}^2$ is a vector field which takes values on the unit sphere $\mathbb{S}$ of $\mathbb{R}^3$. On one hand the magnitude of the magnetization vector is constant in space, on the other hand, in general, it is not the case for its direction. The observable states of Ω can be mathematically characterized as local minimizers of the *Gibbs–Landau free energy functional* associated with the single-crystal ferromagnetic particle

$$\begin{aligned} \mathscr{G}_{\mathscr{L}}(m) &:= \int_\Omega a_{ex}|\nabla m|^2 dx + \int_\Omega \varphi_{an}(m)dx + \frac{\mu_0}{2}\int_{\mathbb{R}^3} |\xi[M_s m]|^2\, dx - \mu_0 \int_\Omega \xi_e \cdot M_s m\, dx \\ &:= \mathscr{E}(m) + \mathscr{A}(m) + \mathscr{W}(m) + \mathscr{Z}(m), \end{aligned}$$

where μ_0 denotes the vacuum pemeability. The first term, $\mathscr{E}(m)$, is called *exchange energy* and penalizes spatial variations of m. The factor a_{ex} in the first term is a positive material constant that summarizes the effect of short-range exchange interactions. The second term, $\mathscr{A}(m)$, is called *anisotropy energy* and it models the existence of preferred directions for the magnetization, the so-called *easy axes*, which usually depend on the crystallographic structure of the material. The anisotropy energy density $\phi_{an} : \mathbb{S}^2 \to [0, +\infty)$ is assumed to be even and globally Lipschitz continuous function that vanishes only on a finite set of unit vectors (the easy axes). The third term, $\mathscr{W}(m)$, called the *magnetostatic self-energy*, is the energy due to the (dipolar) magnetic field, also known in literature as the *stray field*, $\xi[m]$ generated by m. From the mathematical point of view, assuming Ω open, bounded and with a Lipschitz boundary, a given magnetization $m \in L^2(\Omega, \mathbb{R}^3)$ generates the stray field $\xi[m] = \nabla u_m$ where the potential u_m solves

$$\Delta u_m = -\operatorname{div} m\chi_\Omega \quad \text{in } \mathscr{D}'(\mathbb{R}^3),$$

where we denote with $m\chi_\Omega$ the extension of m to $\mathbb{R}^3$ that vanishes outside of Ω. By Lax–Milgram theorem, the existence of a solution to the previous equation is granted. Moreover, the solution is unique in the Beppo–Levi space

$$BL^1(\mathbb{R}^3) := \left\{ u \in \mathscr{D}'(\mathbb{R}^3) : \frac{u(\cdot)}{\sqrt{1+|\cdot|^2}} \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\}.$$

Finally, the stray field is a positive semidefinite operator of norm one. The fourth term, $\mathscr{Z}(m)$, is called the *interaction energy* (or *Zeeman energy*), and models the tendency of the crystal to have its magnetization aligned with the (constant in space) external applied field $\xi_e \in \mathbb{R}^3$, which is assumed to be unaffected by variations of m.

The competition of those four terms explain most of the striking pictures of the magnetization that one can see in most ferromagnetic material [93], in particular the so-called domain structure, that is large regions of uniform or slowly varying magnetization (the magnetic domains) separated by very thin transition layers (the domain walls).

In the last fifteen years, magnetic systems with the shape of thin film have been subject to extensive research (nanotubes [91, 105], 3d helices [115, 140], thin spherical shells [54, 116, 138, 139]). In the case of curved thin films, the embedding of two-dimensional structures in the three-dimensional space permits to alter the magnetic properties of the system by tailoring its local curvature. Under technical hypotheses on the geometry of the thin-film, it is well-known that the demagnetizing field behaves like the projection of the magnetization on the normal to the thin film. This result was proved by Gioia and James [85]. In particular, they showed that in planar thin films the effect of the demagnetizing field operator is represented by an easy-surface anisotropy term. A extension of this result to the case of curved thin films is given in [28], although under the technical assumption that the thin geometry is generated by a surface diffeomorphic to the closed unit disk of $\mathbb{R}^2$. Moreover, in [135], a Γ-convergence analysis is performed on pillow-like shells, i.e., on shells of small thickness $\varepsilon > 0$ having the form $\Omega_\varepsilon := \{(x, z) \in \omega \times \mathbb{R} : \varepsilon y_1(x) \leqslant z \leqslant \varepsilon y_2(x)\}$ with $\omega \subset \mathbb{R}^2$ and y_1, y_2 functions vanishing on the boundary of ω. All these studies, by their own a local nature, do not cover significant scenarios like the one of a spherical thin film [54, 138, 139]. In [53], Di Fratta addressed the question but under a convexity assumption on the surface generating the thin film. While, in [55] the authors established three distinct variational principles for the magnetostatic self-energy that, through the explicit construction of suitable families of scalar and vector potential, allow to circumvent the technical difficulties in of managing the demagnetizing field operator, at least in the stationary case.

Chapter 2

Dimensional Reduction of a Magnetoelastic Single-crystal Particle

In this Chapter we investigate the dimension reduction of the family of functionals $\mathbb{E}$

$$\mathbb{E}(u,m) = \int_\Omega \varepsilon^2|\nabla m|^2 + \varphi(m) - \xi_e \cdot m\, dx + \int_{\mathbb{R}^3} |\xi|^2\, dx + \int_\Omega \mathcal{C}(e(u) - e_0(m)) : (e(u) - e_0(m))\, dx,$$

i.e. we suppose that Ω is a cylinder of height $h > 0$ and we find its Γ-limit as the height h goes to zero. In addition to the usual free energy associated with a single-crystal ferromagnetic particle we consider also the term

$$\int_\Omega \mathcal{C}(e(u) - e_0(m)) : (e(u) - e_0(m))\, dx,$$

which represent the elastic energy of a (very) small deformation u of the cylinder Ω. The asymptotic behavior of such an energy when the height of the domain is going to 0 is well known (see [107] and [24]), but it is also well known that for a general elastic energy it is not always possible to deduce an explicit expression for the density of such Γ-limit. In this chapter we will show that due to the simplicity of our energy, that is not the case in linearized elasticity.

In what follows we assume that $\Omega \subset \mathbb{R}^3$ is the *reference configuration* of our material, $m : \mathbb{R}^3 \to \mathbb{R}^3$ is the *normalized magnetization*, and it is such that $|m| = 1$ a.e. in Ω, $m = 0$ on $\mathbb{R}^3 \setminus \Omega$. ξ_e is the *external magnetic field*, $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ is the *induced magnetic field* and it is related to m through Maxwell's equations

$$div(\nabla\xi + m) = 0.$$

Moreover, $\varphi : \mathbb{S}^2 \to (0, +\infty)$ is the *anysotropic energy density*, it is even and continuous, $u : \Omega \to \mathbb{R}^3$ is the *material displacement*, $e(u)$ is the *symmetrized gradient operator*, defined as

$$e(u) := \frac{\nabla u + \nabla u^T}{2}.$$

Finally, e_0 is the *lattice misfit function* and it is a continuous function depending only on m. $\mathcal{C}$ is a fourth order, symmetric and positive definite tensor such that for every $F \in \mathbb{R}^{3\times 3}$ it holds

$$\frac{1}{c}|e(F)|^2 \leqslant \mathcal{C}(F) : F \leqslant c|F|^2, \tag{H1}$$

for some $c > 0$.

Our main result is the following.

Theorem 2.0.1. *The family of functionals*

$$J_h(u,m) := \begin{cases} \frac{1}{h}\mathbb{E}^h(u,m) & \textit{if } (u,m) \in L^2(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathbb{S}^2) \\ +\infty & \textit{elsewhere}, \end{cases}$$

Γ-converges to the functional

$$J(w,m) = \int_\omega \varepsilon^2 |\nabla' m|^2 + \varphi(m) - \xi_e \cdot m + \frac{1}{2} m_3^2 + F_0(\nabla w^*) - 2\mathcal{C}(e_0(m)) : e_w \\ + \mathcal{C}(e_0(m)) : e_0(m)\, dx'$$

if $(w,m) \in V \times W^{1,2}(\omega,\mathbb{R}^3)$ and $J(w,m) = +\infty$ elsewhere, where

$$V := \{w \in W^{1,2}(\Omega,\mathbb{R}^3) | \exists \eta \in W^{1,2}(\omega,\mathbb{R}^2), \eta_3 \in W^{2,2}(\omega) : \\ w = \begin{pmatrix} \eta \\ \eta_3 \end{pmatrix} - x_3 \begin{pmatrix} \nabla' \eta_3 \\ 0 \end{pmatrix} \}.$$

and

$$F_0(\nabla w) := \int_{-\frac{1}{2}}^{\frac{1}{2}} f_0(\nabla w^*)\, dx_3.$$

Here $*$ denotes the 2x2 principal submatrix and $f_0(A) := \min_{b \in \mathbb{R}^3} \{\mathcal{C}(M_b^A) : M_b^A\}$ for

$$M_b^A = \begin{pmatrix} & A & b_1 \\ & & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

2.1 Preliminaries

In this section we will state some useful definition and results.

Definition 2.1.1. Given a matrix $A \in \mathbb{R}^{3\times 3}$, we define the matrix A^* as

$$A^* = \sum_{i,j=1}^{2} A_{ij} e_i \otimes e_j,$$

where e_1, e_2 and e_3 are the vectors of the canonical base of $\mathbb{R}^3$.

Remark 2.1.2. Given a matrix $A \in \mathbb{R}^{3\times 3}$ the matrix A^* is the matrix 2×2 obtained by "cutting off" the third row and column from A.

An important theorem that will be fundamental for our result is Korn's inequality (see [24]).

Theorem 2.1.3. *Let $p > 1$, $\Omega \in \mathbb{R}^3$ be open, connected, bounded, and with Lipschitz boundary and $\psi \in W^{1,p}(\Omega, \mathbb{R}^3)$, then*

$$||\psi||^2_{W^{1,p}(\Omega,\mathbb{R}^3)} \leqslant C_K \left(||\psi||^2_{L^p(\Omega,\mathbb{R}^3)} + ||e(\nabla\psi)||^2_{L^p(\Omega,\mathbb{R}^{3\times3})} \right),$$

where C_K is a positive constant that depends only on p and Ω.

In particular we will use Korn's inequality for thin domains (see [24, Appendix]).

Theorem 2.1.4. *Let $\omega \in \mathbb{R}^2$ be open, connected, bounded, and with Lipschitz boundary and let $I := \left(-\frac{1}{2}, +\frac{1}{2}\right)$ and $\psi \in W^{1,2}(\omega \times I, \mathbb{R}^3)$, then*

$$||(\psi_1, \psi_2, h\psi_3)||^2_{W^{1,2}(\omega\times I,\mathbb{R}^3)} \leqslant C_K \left(||(\psi_1, \psi_2, h\psi_3)||^2_{L^2(\omega\times I,\mathbb{R}^3)} + ||e(\nabla_h\psi)||^2_{L^2(\omega\times I,\mathbb{R}^{3\times3})} \right),$$

where $\nabla_h := [\partial_{y_1}|\partial_{y_2}|\frac{1}{h}\partial_{y_3}]$ and C_K is a positive constant that depends only on ω.

Definition 2.1.5. Given a sequence of functions $(u_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ we say that it *Converges in displacement* if there exists a couple of functions $(w, v) \in W^{1,2}(\omega, \mathbb{R}^2) \times W^{2,2}(\omega)$, and there exists a sequence of functions $(\psi_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ such that $(\psi_1, \psi_2, h\psi_3) \to 0$ in L^2 and

$$u_h = \begin{pmatrix} w \\ \frac{v}{h} \end{pmatrix} - x_3 \begin{pmatrix} \nabla' v \\ 0 \end{pmatrix} + \psi_h$$

for every $h > 0$.

Remark 2.1.6. Basically a sequence of functions $(u_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ converges in displacement to the couple $(w, v) \in W^{1,2}(\omega, \mathbb{R}^2) \times W^{2,2}(\omega)$ if $(u^1_h, u^2_h) \to w - x_3\nabla' v$ in L^2 and $hu^3_h \to v$ in L^2.

Remark 2.1.7. In [24], Bukal and Velčić proved the Γ-convergence with respect to the convergence in displacement of the functional

$$\mathbf{J}_h(u) := \int_{\Omega_h} Q^h(x, Sym(\nabla u))dx,$$

where $\Omega_h = \omega \times (-\frac{h}{2}, +\frac{h}{2})$, $\omega \subset \mathbb{R}^2$ and Q^h is assumed to be a L^∞ function in x and a coercive, uniformly bounded quadratic form in the second variable and $Sym(\nabla u)$ is the symmetric part of the gradient of u. In particular, under appropriate hypothesis on Q^h, they proved that fixed a subsequence $(h_n)_{n\in\mathbb{N}}$ and if u_{h_n} converges in displacements to (w, v), then $\mathbf{J}_{h_n}$ Γ-converges to

$$\mathbf{J}(w, v) := \int_{\omega\times I} Q_0(x, w, v)dx,$$

where the function Q_0 depends on the sequence $(h_n)_{n\in\mathbb{N}}$. Moreover, they cannot prove an explicit representation. In this Chapter we exploit the independence from h and x of our function

$$Q^h(x, Sym(\nabla u))) = \mathcal{C}(\nabla u) : \nabla u$$

in order to find an explicit expression for our Q_0 and also with the aim to prove that in our case Q_0 doesn't depends on $(h_n)_{n\in\mathbb{N}}$

2.1.1 Notation

In contrast to the notation used elsewhere in this thesis, we will denote with ∂_{x_i}, with $i = 1, 2, 3$, the distributional derivative of a function with respect to x_i.

Suppose that $\Omega_h := \omega \times (-\frac{h}{2}, +\frac{h}{2})$ for $h > 0$, $\omega \subset \mathbb{R}^2$ open, bounded, connected and with Lipschitz boundary, and $\Omega := \Omega_1 = \omega \times (-\frac{1}{2}, +\frac{1}{2})$. We want to study the asymptotic behavior as $h \to 0$ of the family of functionals

$$\mathbb{E}(u, m) := \mathbb{E}_{mag}(m) + \mathbb{E}_i(m) + \mathbb{E}_{ela}(u, m), \quad (u, m) \in W^{1,2}(\Omega_h, \mathbb{R}^3) \times W^{1,2}(\Omega_h, \mathbb{S}^2),$$

where the *Magnetization energy* is given by

$$\mathbb{E}_{mag}(m) := \int_{\Omega_h} \varepsilon^2 |\nabla m|^2 + \varphi(m) - \xi_e \cdot m \, dx,$$

the *Induced magnetization energy* is defined as

$$\mathbb{E}_i(m) := \int_{\mathbb{R}^3} |\xi_h|^2 \, dx,$$

and the *Elastic energy* is

$$\mathbb{E}_{ela}(u, m) := \int_{\Omega_h} \mathcal{C}(e(u) - e_0(m)) : (e(u) - e_0(m)) \, dx.$$

Rescaling the energy with the change of variables

$$y_1 := x_1, \quad y_2 := x_2, \quad y_3 := \frac{1}{h} x_3,$$

we can rewrite $\mathbb{E}$ as $h\mathbb{E}^h$ with

$$\mathbb{E}^h(u, m) := \mathbb{E}^h_{mag}(m) + \mathbb{E}^h_i(m) + \mathbb{E}^h_{ela}(u, m), \quad (u, m) \in W^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,2}(\Omega, \mathbb{S}^2),$$

and

$$\begin{aligned}
\mathbb{E}^h_{mag}(m) &:= \int_{\Omega} \varepsilon^2 |\nabla_h m|^2 + \varphi(m) - \xi_e \cdot m \, dx, \\
\mathbb{E}^h_i(m) &:= \int_{\mathbb{R}^3} |\xi_h|^2 \, dx \\
\mathbb{E}^h_{ela}(u, m) &:= \int_{\Omega} \mathcal{C}(e^h(u) - e_0(m)) : (e^h(u) - e_0(m)) \, dx.
\end{aligned}$$

with

$$e^h(u) := \frac{\nabla_h u + \nabla_h u^T}{2} = e(\nabla_h u), \quad \nabla_h = \left[\partial_{y_1} | \partial_{y_2} | \frac{1}{h} \partial_{y_3} \right].$$

In order to prove Theorem 2.0.1, we study separately the Γ-limit of of the functionals $\mathbb{E}^h_{mag}, \mathbb{E}^h_i$ and $\mathbb{E}^h_{ela}$.

2.1.2 Induced Magnetization Energy

In [85] Gioia and James proved the following Proposition.

Proposition 2.1.8. *Given* $(m_h)_{h>0} \subset L^2(\Omega, \mathbb{S}^2)$ *such that* $m_h \to m$ *in* $L^2(\Omega, \mathbb{S}^2)$ *as* $h \to 0$, *and given the sequence* $(\xi_h)_{h>0}$ *such that for every* $h > 0$

$$div(\nabla \xi_h + m_h \chi_\Omega) = 0 \text{ in } \mathbb{R}^3,$$

where $m_h \chi_\Omega$ *denotes the extension to* 0 *of* m_h *outside of* Ω, *then* $\nabla \xi_h \to 0$ *in* $L^2(\Omega, \mathbb{R}^3)$ *and* $\frac{1}{h}\partial_{y_3}\xi_h \to m_3$ *in* $L^2(\Omega, \mathbb{R}^3)$. *Moreover*

$$\mathbb{E}_i^h(m_h) \to E_i^0(m) = \int_\Omega \frac{1}{2} m_3^2 \, dx.$$

It is thus clear that the Γ-limit of E_i^h with respect to the $L^2(\Omega, \mathbb{R}^3)$-norm is E_i^0.

2.1.3 Magnetization Energy

About E_{mag}^h, we can notice that the term

$$\int_\Omega \xi_e \cdot m \, dx$$

is linear, while the term

$$\int_\Omega \varphi(m) \, dx$$

is continuous, and so the Γ-limit of E_{mag}^h with respect to the weak-$W^{1,2}(\Omega, \mathbb{R}^3)$ topology can be reduced to the study of the asymptotic behavior of

$$E_h(m) = \int_\Omega \left(|\nabla' m|^2 + \frac{1}{h^2} |\partial_{x_3} m|^2 \right) \, dx,$$

with $\nabla' := [\partial_{x_1}, \partial_{x_2}]$.

Lemma 2.1.9. *Given* $(m_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ *such that* $\sup_{h>0} E_h(m_h) < +\infty$, *then there exists a subsequence (non relabeled)* $(m_h)_{h>0}$ *such that* $m_h \rightharpoonup m$ *in* $W^{1,2}(\Omega, \mathbb{R}^3)$ *and* $\partial_{x_3} m = 0$. *Moreover, for such a subsequence, it holds that*

$$||\nabla' m||_{L^2(\omega, \mathbb{R}^{3\times 2})} \leqslant \liminf_{h \to 0} E_h(m_h).$$

Proof. If $\sup_{h>0} E_h(m_h) < +\infty$ then

$$||\nabla m_h||_{L^2(\Omega, \mathbb{R}^{3\times 3})} \leqslant C, \tag{2.1.10}$$

for some $C > 0$, since for $h < 1$

$$||\nabla' m_h||_{L^2(\Omega, \mathbb{R}^{3\times 2})} \leqslant C, \quad ||\partial_{x_3} m_h||_{L^2(\Omega, \mathbb{R}^3)} \leqslant \frac{1}{h^2} ||\partial_{x_3} m_h||_{L^2(\Omega, \mathbb{R}^3)} \leqslant C. \tag{2.1.11}$$

From (2.1.10) we deduce that there exists a subsequence (non relabeled) $(m_h)_{h>0}$ such that $m_h \rightharpoonup m$ in $W^{1,2}(\Omega, \mathbb{R}^3)$, while from the second inequality of (2.1.11) we deduce that $||\partial_{x_3} m_h||_{L^2(\Omega,\mathbb{R}^3)}$ has to converge L^2-strongly to 0, and so $\partial_{x_3} m = 0$. Since the L^2-norm is lower semicontinuous with respect to the weak convergence then it holds

$$||\nabla m||_{L^2(\Omega,\mathbb{R}^{3\times 3})} \leqslant \liminf_{h\to 0} ||\nabla m_h||_{L^2(\Omega,\mathbb{R}^{3\times 3})} \leqslant \liminf_{h\to 0} E_h(m_h).$$

It follow from $\partial_{x_3} m = 0$ that $||\nabla m||_{L^2(\Omega,\mathbb{R}^{3\times 3})} = ||\nabla' m||_{L^2(\omega,\mathbb{R}^{3\times 2})}$, and this conclude the proof. $\square$

To conclude, note that if for every $m \in W^{1,2}(\Omega, \mathbb{R}^3)$ such that $||\partial_{y_3} m|| = 0$ we define $m_h = m$, then it holds that

$$||\nabla' m||_{L^2(\omega,\mathbb{R}^{3\times 2})} = E_h(m_h) = \lim_{h\to 0} E_h(m_h),$$

so $(m_h)_{h>0}$ is a proper recovery sequence for the functional E_h and the Γ-limit of $\mathbb{E}^h_{mag}$ with respect to the weak-$W^{1,2}(\Omega, \mathbb{R}^3)$ topology is

$$\mathbb{E}^0_{mag}(m) = \int_\omega \left(|\nabla' m|^2 + \varphi(m) - \xi_e \cdot m\right) dx'.$$

2.2 The Elastic Energy

At this point, we can focus on the functional $\mathbb{E}^h_{ela}$. Since for every $(u_h, m_h)_h \subset W^{1,2}(\Omega, \mathbb{R}^3) \times W^{1,2}(\Omega, \mathbb{S}^2)$ it holds

$$\begin{aligned}\mathbb{E}^h_{ela}(u_h, m_h) &= \int_\Omega (e^h(u_h) - e_0(m_h)) : \mathcal{C}(e^h(u_h) - e_0(m_h))\, dy \\ &= \int_\Omega e^h(u_h) : \mathcal{C}(e^h(u_h))\, dy - 2\int_\Omega e^h(u_h) : \mathcal{C}(e_o(m_h))\, dy + \int_\Omega e_0(m_h) : \mathcal{C}(e_0(m_h))\, dy,\end{aligned}$$

then, as before, we notice that the second term of the sum is continuous in $L^2(\Omega)$ with respect to the m-variable and it is linear with respect to the u-variable; on the other hand the third term of the sum is also continuous in $L^2(\Omega)$ so we can focus on the study of the Γ-convergence of the first term.

We define

$$I_h(u) = \int_\Omega f(e^h(u))\, dy$$

with $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ defined as $f(e) = \mathcal{C}(e) : e$.
It is clear from hypothesis (H1) that if $\sup_{h>0} I_h(u_h) < +\infty$ for some $(u_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$, then

$$||e^h(u_h)||_{L^2(\Omega,\mathbb{R}^{3\times 3})} < C$$

for every $h > 0$, so there exists a subsequence (non relabeled) $(e^h(u_h)))_h$ such that $e^h(u_h) \rightharpoonup e_0$ in $L^2(\Omega, \mathbb{R}^{3\times 3})$, for some $e_0 \in L^2(\Omega, \mathbb{R}^{3\times 3})$. From this we cannot deduce any kind of compactness for $(u_h)_{h>0}$, indeed we cannot apply Korn's inequality since Korn's constant would depend on h. In order to recover some kind of compactness, we have to rescale the function u, i.e. we have to use Korn's inequality for thin domains. Fixed u we define

$$v^u = (v_1.v_2, v_3) := (u_1, u_2, hu_3),$$

such that

$$e^h(u) = \begin{bmatrix} \partial_{y_1} u_1 & \frac{\partial_{y_1} u_2 + \partial_{y_2} u_1}{2} & \frac{1}{2}(\partial_{y_1} u_3 + \frac{\partial_{y_3} u_1}{h}) \\ \frac{\partial_{y_1} u_2 + \partial_{y_2} u_1}{2} & \partial_{y_2} u_2 & \frac{1}{2}(\partial_{y_2} u_3 + \frac{\partial_{y_3} u_2}{h}) \\ \frac{1}{2}(\partial_{y_1} u_3 + \frac{\partial_{y_3} u_1}{h}) & \frac{1}{2}(\partial_{y_2} u_3 + \frac{\partial_{y_3} u_2}{h}) & \frac{1}{h}\partial_{y_3} u_3 \end{bmatrix}$$
$$= \begin{bmatrix} \partial_{y_1} v_1 & \frac{\partial_{y_1} v_2 + \partial_{y_2} v_1}{2} & \frac{1}{2h}(\partial_{y_1} v_3 + \partial_{y_3} v_1) \\ \frac{\partial_{y_1} v_2 + \partial_{y_2} v_1}{2} & \partial_{y_2} v_2 & \frac{1}{2h}(\partial_{y_2} v_3 + \partial_{y_3} v_2) \\ \frac{1}{2h}(\partial_{y_1} v_3 + \partial_{y_3} v_1) & \frac{1}{2h}(\partial_{y_2} v_3 + \partial_{y_3} v_2) & \frac{1}{h^2}\partial_{y_3} v_3 \end{bmatrix} =: \bar{e}^h(v^u).$$

Lemma 2.2.1. *Given $(u_h)_{h>0} \subset W^{1,2}(\Omega, \mathbb{R}^3)$ such that $\sup_{h>0} I_h(u_h) < C$ for some $C > 0$, then*

$$(u_h^1, u_h^2, hu_h^3) \xrightarrow{L^2} \nu \in V,$$

with

$$V := \{\nu \in W^{1,2}(\Omega, \mathbb{R}^3) | \exists \eta \in W^{1,2}(\omega, \mathbb{R}^2), \eta_3 \in W^{2,2}(\omega) :$$
$$\nu = \begin{pmatrix} \eta \\ \eta_3 \end{pmatrix} - x_3 \begin{pmatrix} \nabla \eta_3 \\ 0 \end{pmatrix} \}.$$

Proof. If $\sup_{h>0} I_h(u_h) < C$, if we define $v_h^{u_h} = (v_h^1, v_h^2, v_h^3) = (u_h^1, u_h^2, hu_h^3)$ then

$$||\bar{e}^h(v_h^{u_h})||_{L^2(\Omega, \mathbb{R}^{3\times 3})} < +\infty$$

for every $h > 0$, thus there exists a subsequence (not relabeled) such that

$$\bar{e}^h(v_h^{u_h}) \rightharpoonup e_\nu \text{ in } L^2(\Omega, \mathbb{R}^{3\times 3}), \quad \text{for some } e_\nu \in L^2(\Omega, \mathbb{R}^{3\times 3}).$$

By the definition of $\bar{e}^h$ it is clear that

$$(\bar{e}^h)^*((v_h^{u_h})) = e^*((v_h^{u_h})) \rightharpoonup e_v^* \text{ in } L^2(\Omega, \mathbb{R}^{2\times 2}),$$
$$e_{i3}(v_h) \to 0 \text{ in } L^2(\Omega), \text{ for } i = 1, 2, 3.$$

From Korn's inequality, we get that $||v_h^{u_h}||_{W^{1,2}(\Omega, \mathbb{R}^3)} < C$, for every $h > 0$, so a suitable non relabeled subsequence

$$v_h^{u_h} \rightharpoonup \nu \text{ in } W^{1,2}(\Omega, \mathbb{R}^3),$$

and also

$$e(v_h^{u_h}) \rightharpoonup e(\nu) \text{ in } L^2(\Omega, \mathbb{R}^{3\times 3}),$$
$$e(\nu) = \begin{pmatrix} e_\nu^* & 0_{2\times 1} \\ 0_{1\times 2} & 0 \end{pmatrix}. \tag{2.2.2}$$

If $\nu \in V$ then clearly $e(\nu)$ has the form (2.2.2). On the other hand, if $e(\nu)$ has the form (2.2.2) and $\nu \in W^{1,2}(\Omega, \mathbb{R}^3)$, then it holds that $\partial_{x_3}\nu_3 = 0$, so there exists $\eta_3 \in W^{1,2}(\omega)$ such that $\nu_3 = \eta_3$. Since for $i = 1, 2$

$$e(\nu)_{i3} = e(\nu)_{3i} = \partial_{x_i}\nu_3 + \partial_{x_3}\nu_i = 0,$$

then

$$\partial^2_{x_3}\nu_i = -\partial_{x_1}(\partial_{x_3}\nu_3) = 0 \text{ in the sense of distributions.}$$

Therefore it has to be

$$\nu_i = \eta_i + x_3\psi_i$$

for some $\eta_i, \psi_i \in W^{1,2}(\omega)$, but then

$$0 = \partial_{x_i}\nu_3 + \partial_{x_3}\nu_i = \partial_{x_i}\nu_3 + \psi_i,$$

and so $\psi_i = -\partial_{x_i}\nu_3$. From this, we deduce that $\nu_3 \in W^{2,2}(\omega)$ and that $\nu \in V$. $\square$

Remark 2.2.3. Given $(u_h)_{h>0} \subset W^{1,2}(\Omega,\mathbb{R}^3)$ such that (u^1_h, u^2_h, hu^3_h) converges to $\nu \in V$ in L^2, then u_h is converging in displacements to ν. To see this, it is enough to consider

$$\psi_h = u_h - \begin{pmatrix} \eta \\ \frac{\eta_3}{h} \end{pmatrix} + x_3 \begin{pmatrix} \nabla\eta_3 \\ 0 \end{pmatrix},$$

and then apply Definition 2.1.5

Remark 2.2.4. Since the function $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ defined as $f(e) = \mathcal{C}(e) : e$ is continuous and convex then the functional

$$u \in W^{1,2}(\Omega,\mathbb{R}^3) \mapsto \int_\Omega f(\nabla u)dy$$

is weak lower semicontinuous in $W^{1,2}(\Omega,\mathbb{R}^3)$ and so it holds that given $(u_h)_{h>0} \subset W^{1,2}(\Omega,\mathbb{R}^3)$ such that (u^1_h, u^2_h, hu^3_h) converges weakly to $w \in V$ in $W^{1,2}$ and such that $e^h((u^1_h, u^2_h, hu^3_h))$ converges weakly to e_0 in L^2 then

$$\liminf_{h\to 0} \int_\Omega f(\nabla u_h)dy \geqslant \int_\Omega f(e_0)dy \geqslant \int_\Omega f_0(e_0)dy = \int_\Omega f_0(e^*(w))dy,$$

where $*$ denotes the 2x2 principal submatrix and $f_0(A) = \min_{b\in\mathbb{R}^3}\{M^A_b : \mathcal{C}(M^A_b)\}$ for

$$M^A_b = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & b_1 \\ & & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix}.$$

Theorem 2.2.5. *The family of functionals $(I_h)_{h>0}$ Γ-converges to the functional I with respect to the topology induced by the convergence in displacements, with*

$$I(u) := \begin{cases} \int_\omega F_0(\nabla u)\, dx_1 dx_2 \text{ if } u \in V \\ +\infty \qquad \text{elsewhere} \end{cases}$$

where

$$F_0(\nabla u) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} f_0(\nabla u^*)\, dx_3.$$

Proof. First we notice that the $\liminf$ inequality follows from Remark 2.2.4. In order to construct a recovery sequence, we search for an explicit expression for f_0. Chosen $A \in \mathbb{R}^{2\times 2}$, we define

$$f_A(b_1, b_2, b_3) = f\begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \begin{matrix} b_1 & b_2 \end{matrix} & b_3 \end{pmatrix}.$$

Now, using the symmetry of the tensor $\mathcal{C}$, we have that

$$\begin{cases} \frac{df_A(b_1,b_2,b_3)}{db_1} = 8b_1 C_{1133} + 8b_2 C_{1233} + 4b_3 C_{1333} + \sum_{jl=1}^{2} C_{1j3l}A_{jl} \\ \frac{df_A(b_1,b_2,b_3)}{db_2} = 8b_1 C_{1233} + 8b_2 C_{2233} + 4b_3 C_{2333} + \sum_{jl=1}^{2} C_{2j3l}A_{jl} \\ \frac{df_A(b_1,b_2,b_3)}{db_3} = 4b_1 C_{1333} + 4b_2 C_{2333} + 2b_3 C_{3333} + \sum_{jl=1}^{2} C_{3j3l}A_{jl}; \end{cases} \tag{2.2.6}$$

indeed if we define

$$e = \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} b_1 \\ b_2 \end{matrix} \\ \begin{matrix} b_1 & b_2 \end{matrix} & b_3 \end{pmatrix}$$

then

$$\begin{aligned} f(e) = f_A(b_1, b_2, b_3) &= \sum_{ijkl=1}^{3} C_{ijkl} e_{ik} e_{jl} \\ &= \sum_{ijkl=1}^{2} C_{ijkl} A_{ik} A_{jl} + 2\sum_{jl=1}^{3} C_{13jl} b_1 e_{jl} + 2\sum_{jl=1}^{3} C_{23jl} b_2 e_{jl} + \sum_{jl=1}^{3} C_{33jl} b_3 e_{jl}. \end{aligned}$$

The second term of the previous sum is

$$\sum_{jl=1}^{3} C_{13jl} b_1 e_{jl} = b_1 \left(\sum_{jl=1}^{2} C_{13jl} A_{jl} + 2C_{1133} b_1 + 2C_{1233} b_2 + C_{1333} b_3 \right),$$

and since a similar form holds also for the third and fourth term, then deriving $f_A(b_1, b_2, b_3)$ with respect to b_1, b_2 and b_3 we obtain (2.2.6). The matrix associated to the system (2.2.6) is

$$P = \begin{pmatrix} 8b_1 C_{1133} & 8b_2 C_{1233} & 4b_3 C_{1333} \\ 8b_1 C_{1233} & 8b_2 C_{2233} & 4b_3 C_{2333} \\ 4b_1 C_{1333} & 4b_2 C_{2333} & 2b_3 C_{3333} \end{pmatrix}$$

and so by Cramer's rule the solution to (2.2.6) is $b_i = G_i(A)$ with $i = 1, 2, 3$ and

$$G_1(A) = \frac{\begin{vmatrix} -\sum_{jl=1}^{2} C_{1j3l}A_{jl} & 8b_2 C_{1233} & 4b_3 C_{1333} \\ -\sum_{jl=1}^{2} C_{2j3l}A_{jl} & 8b_2 C_{2233} & 4b_3 C_{2333} \\ -\sum_{jl=1}^{2} C_{3j3l}A_{jl} & 4b_2 C_{2333} & 2b_3 C_{3333} \end{vmatrix}}{det\, P}$$

$$G_2(A) = \frac{\begin{vmatrix} 8b_1 C_{1133} & -\sum_{jl=1}^{2} C_{1j3l}A_{jl} & 4b_3 C_{1333} \\ 8b_1 C_{1233} & -\sum_{jl=1}^{2} C_{2j3l}A_{jl} & 4b_3 C_{2333} \\ 4b_1 C_{1333} & -\sum_{jl=1}^{2} C_{3j3l}A_{jl} & 2b_3 C_{3333} \end{vmatrix}}{det\, P}$$

$$G_3(A) = \frac{\begin{vmatrix} 8b_1C_{1133} & 8b_2C_{1233} & -\sum_{jl=1}^2 C_{1j3l}A_{jl} \\ 8b_1C_{1233} & 8b_2C_{2233} & -\sum_{jl=1}^2 C_{2j3l}A_{jl} \\ 4b_1C_{1333} & 4b_2C_{2333} & -\sum_{jl=1}^2 C_{3j3l}A_{jl} \end{vmatrix}}{det\, P}.$$

From this we get that

$$f_0(A) = f\begin{pmatrix} & A & & G_1(A) \\ & & & G_2(A) \\ G_1(A) & G_2(A) & & G_3(A) \end{pmatrix}.$$

If $A = e^*(w)$ with $w \in V$ then G_1, G_2 and G_3 will be affine functions of the third variable since

$$\begin{aligned} A_{jj} &= \partial_{x_j}\eta_j - x_3\partial^2_{x_j}\eta_3 \\ A_{jl} &= \frac{1}{2}(\partial_{x_1}\eta_2 + \partial_{x_2}\eta_1) - x_3\partial^2_{x_1x_2}\eta_3. \end{aligned}$$

Since each $G_i(e(w)^*)$, with $i = 1, 2, 3$, is an affine function of the third variable then for each i there exists $g_i^1, g_i^2 \in W^{1,2}(\omega)$ such that

$$G_i(e(w)^*) = g_i^1 + x_3 g_i^2, \quad i = 1, 2, 3.$$

Fixed $\varepsilon > 0$ we can find $\phi_i, \psi_i \in C_c^\infty(\omega)$ such that

$$||g_i^1 - \phi_i||_{L^2(\omega)} < \varepsilon, \quad ||g_i^2 - \psi_i||_{L^2(\omega)} < \varepsilon.$$

Fixed

$$w = \begin{pmatrix} \eta \\ \eta_3 \end{pmatrix} - x_3 \begin{pmatrix} \nabla\eta_3 \\ 0 \end{pmatrix} \in V$$

we define $(v_h)_{h>0}$ such that

$$\begin{cases} v_h^1 = \eta_1 + \partial_{x_1}\eta_3 + h(x_3\phi_1 + \frac{x_3^2}{2}\psi_1) \\ v_h^2 = \eta_2 + \partial_{x_2}\eta_3 + h(x_3\phi_2 + \frac{x_3^2}{2}\psi_2) \\ v_h^3 = \eta_3 + h^2(x_3\phi_3 + \frac{x_3^2}{2}\psi_3). \end{cases}$$

Clearly $v_h \to w$ pointwise as $h \to 0$. If we define

$$u_h = (u_h^1, u_h^2, u_h^3) = (v_h^1, v_h^2, \frac{v_h^3}{h}),$$

then

$$
\begin{aligned}
&e^h(u_h)\\
&=\begin{pmatrix} \partial_{x_1} v_h^1 & \frac{1}{2}(\partial_{x_2} v_h^1 + \partial_{x_1} v_h^2) & \frac{1}{2h}(\partial_{x_3} v_h^1 + \partial_{x_1} v_h^3) \\ \frac{1}{2}(\partial_{x_2} v_h^1 + \partial_{x_1} v_h^2) & \partial_{x_2} v_h^2 & \frac{1}{2h}(\partial_{x_2} v_h^3 + \partial_{x_3} v_h^2) \\ \frac{1}{2h}(\partial_{x_3} v_h^1 + \partial_{x_1} v_h^3) & \frac{1}{2h}(\partial_{x_2} v_h^3 + \partial_{x_3} v_h^2) & \frac{1}{h^2}\partial_{x_3} v_h^3 \end{pmatrix}\\
&=\begin{pmatrix} \begin{matrix} e(w)^* + hC \end{matrix} & \begin{matrix} \phi_1 + x_3\psi_1 \\ \phi_2 + x_3\psi_2 \end{matrix} \\ \begin{matrix} \phi_1 + x_3\psi_1 & \phi_2 + x_3\psi_2 \end{matrix} & \phi_3 + x_3\psi_3 \end{pmatrix}\\
&=\begin{pmatrix} \begin{matrix} hC \end{matrix} & \begin{matrix} \phi_1 + x_3\psi_1 - (g_1^1 + x_3 g_2^1) \\ \phi_2 + x_3\psi_2 - (g_2^1 + x_3 g_2^1) \end{matrix} \\ \begin{matrix} \phi_1 + x_3\psi_1 - (g_1^1 + x_3 g_2^1) & \phi_2 + x_3\psi_2 - (g_2^1 + x_3 g_2^1) \end{matrix} & \phi_3 + x_3\psi_3 - (g_3^1 + x_3 g_3^2) \end{pmatrix}\\
&+\begin{pmatrix} \begin{matrix} e(w)^* \end{matrix} & \begin{matrix} g_1^1 + x_3 g_2^1 \\ g_2^1 + x_3 g_2^1 \end{matrix} \\ \begin{matrix} g_1^1 + x_3 g_2^1 & g_2^1 + x_3 g_2^1 \end{matrix} & g_3^1 + x_3 g_3^2 \end{pmatrix}
\end{aligned}
$$

where C is a 2×2 matrix depending on $\eta_1, \eta_2, \nabla'\eta_3, \phi_1, \phi_2, \psi_1, \psi_2$ and their derivatives. Let us denote the first matrix as $B_{h,\varepsilon}$ and the second one as A. Plugging $e^h(u_h)$ in f we get

$$f(e^h(u_h)) = f(A) + f(B_{h,\varepsilon}) + 2\mathcal{C}(A) : B_{h,\varepsilon}.$$

Passing to the $\limsup_{h\to 0}$ and remembering that ε is arbitrary then we get

$$\limsup_{h\to 0} f(e^h(u_h)) = f(A).$$

$\square$

Part III

Non-local Functionals

A growing interest is developing towards non-local functionals in recent years, due to their many applications, both from the application view point (peridynamics, image processing, artificial intelligence, etc), and the theoretical one, e.g. [11, 117, 47, 64, 97, 41] among a plethora of scientific contributions. In particular a big attention is devoted to non-local integrals, we refer to [128] and the bibliography contained therein for an overview and to [20] for the interactions with local energies, among the wide literature. On the other hand, L^∞ variational problems arise when studying optimal design problems, in the control theory, optimal transport, etc, and in these past decades a wide theory has been built, we refer to [9, 1] for the pioneering theoretical papers and to [62, 131, 39] and the bibliography contained therein just to mention some more recent theoretical contribution among a vast literature.

In **Chapter 3** we study the lower semicontinuity of the non-local supremal functional

$$\operatorname*{ess\,sup}_{x,y\in I} W(\nabla u(x), \nabla u(y)), \quad u \in W^{1,\infty}(I, \mathbb{R}^d),$$

with $I \subset \mathbb{R}$ open and bounded. In particular, we characterize its lower semicontinuous envelope and we prove a necessary and sufficient condition on the supremand for the lower semicontinuity of the functional. The study of the lower semicontinuity of a non-local functional is a well known field. In [98] Kreisbeck and Zappale investigated the necessary and sufficient condition that makes the non-local supremal functionals

$$L^\infty(\Omega) \ni u \mapsto \operatorname*{ess\,sup}_{x,y\in\Omega} f(u(x), u(y)), \quad \Omega \subset \mathbb{R}^n \text{ open and bounded},$$

lower semicontinuous when tested along sequences of L^∞ functions converging weakly*. In their paper, they prove that such condition is the separate level convexity of the supremand function. Moreover, they also prove that the relaxed functional is structure preserving, i.e. that it is still a supremal functional, and that its supremand is the separately convex envelope of W. This seems to be only natural since in [9] the authors proved that level convexity is a necessary and sufficient condition for the lower semicontinuity in the local case, i.e. for the functional

$$L^\infty(\Omega) \ni u \mapsto \operatorname*{ess\,sup}_{x\in\Omega} f(u(x)), \quad \Omega \subset \mathbb{R}^n \text{ open and bounded},$$

while in [1, 29] and [130] the authors proved that its relaxation is, again, structure preserving. Furthermore, in [96] Kreisbeck, Ritorno and Zappale study the non-local case when the functional is

$$L^\infty(\Omega, \mathbb{R}^d) \ni u \mapsto \operatorname*{ess\,sup}_{x,y\in\Omega} f(u(x), u(y)), \quad \Omega \subset \mathbb{R}^n \text{ open and bounded},$$

with $d \in \mathbb{N}$. In this case the necessary and sufficient condition on the supremand f is called Cartesian convexity. The notion of Cartesian convexity coincides with the separate level convexity when $d = 1$, but it is strictly weaker for $d > 1$. As extensively explained in both [96] and [98], non-local supremal functionals are strongly connected to non-local differential inclusions and non-local integral functionals and in particular to non-local indicator functionals. On one hand, in [10], the authors prove a characterization of the weak lower semicontinuity in L^p of the non-local integral functional

$$\int\int_{\Omega\times\Omega} f(x, y, u(x), u(y))dxdy, \quad u \in W^{1,p}(\Omega, \mathbb{R}^d),\ \Omega \subset \mathbb{R}^n \text{ open and bounded},$$

On the other hand, in [13] it has been proved that the relaxation of a non-local integral functional defined in open subsets of $\mathbb{R}$, does not admit, in general, a representation of the same type (see also [10, 99, 21]). To ease the parallel with the non-local integral setting, it is worth to mention that in [120], Muñoz proved that separate convexity of the integrand is a necessary and sufficient condition for the weak lower semicontinuity of the functional

$$\iint_{\Omega\times\Omega} f(\nabla u(x), \nabla u(y))dxdy, \quad u \in W^{1,p}(\Omega),\, \Omega \subset \mathbb{R}^n \text{ open and bounded.}$$

The same question in the supremal setting is currently still open and it is not clear if the Cartesian level convexity of the supremand W is also necessary for the lower semicontinuity of

$$\tilde{J}(u) := \operatorname*{ess\,sup}_{\Omega\times\Omega} W(\nabla u(x), \nabla u(y)), \quad u \in W^{1,\infty}(\Omega, \mathbb{R}^d),\, \Omega \subset \mathbb{R}^n \text{ open and bounded,}$$

with $d \geqslant 1$. In the case where ∇u is replaced by u, the question has been completely answered in [96], while, in the case of gradients, a sufficient condition on W ensuring lower semicontinuity has been provided in [77]. The study of the lower semicontinuity of $\tilde{J}$ with respect to the $W^{1,\infty}$-weak* topology is equivalent to characterize, for every $c \in \mathbb{R}$, the $W^{1,\infty}$-weak* closure of $B_{L_c(W)}$, with $L_c(W)$ being c-(sub)level set of W, i.e.

$$L_c(W) := \left\{(\xi, \eta) \in \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} : W(\xi, \eta) \leqslant c\right\},$$

and, for every $K \subset \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$

$$B_K := \left\{u \in W^{1,\infty}(\Omega; \mathbb{R}^d) : (\nabla u(x), \nabla u(y)) \in K \text{ for a.e. } x \in \Omega \times \Omega\right\}.$$

In this Chapter, stemming from the results in [96, 98], we provide an answer when $n = 1$ in terms of the notion of Cartesian convexity introduced in [96] for the set K and an equivalent characterization in term of separate convexity when also $d = 1$. Moreover, we also provide the supremal counterpart of the results for non-local integral functionals depending on derivatives of Sobolev functions contained in [13].

In **Chapter 4** we focus on a different type of non-local functional. In particular we study a non-local perimeter-like functional. As anticipated in Chapter 2, the perimeter of a set in the Euclidean space $\mathbb{R}^d$ as a measure of the locus that circumscribes the set itself. This concept is captured by the analytical theory of finite perimeters sets. This theory is grounded on Caccioppoli's seminal works [25] and on the ideas developed by De Giorgi [50, 51]. The foundamental idea behind this theory is the following: identify a set with its characteristic function, consider the distributional gradient of the latter and then define its total variation as perimeter of the given set. Indeed, fundamental results by De Giorgi and Federer [71, 51] show that, if we retain this definition, the perimeter coincides with the $d-1$-dimensional Hausdorff measure of a certain subset of the topological boundary. The class of sets with finite perimeter has good compactness properties, hence it is possible to tackle various problems that are formulated in geometric terms using the Direct Method of Calculus of Variations. As an example of this we can quote the Plateau's problem, which has been solved with this method in [13]. Beyond this well-established theory, several authors have grown interested in some set functionals that are globally referred to as non-local perimeters. A well known example of non-local perimeter is the fractional perimeters that were introduced by Caffarelli et al. in [26] and that were later

generalized and largely investigated (see for instance [4, 27, 38, 108, 129]). The study of non-local perimeters is motivated by both a theoretical viewpoint and refering to applications, as explained in the brief account given by Cinti et al. in [38]. The defnition of these functionals might seem distant from the original De Giorgi's one, but non-local perimeters resemble the classical one from various perspectives. In particular, it can be proves that they are indeed perimeters in the sense proposed by [35], where Chambolle, Morini and Ponsiglione collect some properties that a set functional should have in order to deserve such a label. The authors also show examples of functionals that fit in this framework and naturally, De Giorgi's perimeter is one of them.
In this Chapter we study the asymptotic behavior, as $\varepsilon \to 0$, of the family of functionals

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon}\int_{E^c\cap\Omega} f(G_\varepsilon * \chi_{E\cap\Omega})\,dx.$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $N > 1$, E is a set of finite perimeter in Ω, f is given and $G_\varepsilon(z) = \frac{1}{\varepsilon^N}G(\frac{z}{\varepsilon})$ where G is a suitable kernel. Our analysis has been inspired by a paper by Miranda et al. in [119] where the case $f(t) = t$ is considered and G is the Gauss-Weierstrass kernel, namely $G(z) = \frac{1}{(4\pi)^{N/2}}e^{-|z|^2/4}$ (see also [84] and [106] for smoother sets and [2] for similar convolution approximation). More precisely, in [119] it is proven that the pointwise limit is, up to a constant, the perimeter of E in Ω. A more general kernel G has been investigated, in the context of optimal partition problems, by Esedoḡlu and Otto [66] where G is smooth and non-negative, radially symmetric and satisfying the following conditions:

$$\int_{\mathbb{R}^N} G(z)\,dz = 1, \quad \int_{\mathbb{R}^N} |z|G(z)\,dz < +\infty, \quad |\nabla G(z)| \lesssim G\left(\frac{z}{2}\right), \quad \nabla G(z)\cdot z \leqslant 0.$$

On the other hand, as in [66], Esedoḡlu and Otto consider only the case $f(t) = t$, but they prove a complete Γ-convergence result for the family $\{\mathscr{F}_\varepsilon\}_{\varepsilon>0}$ on finite perimeter sets with respect to the strong L^1-convergence. We point out that for the simpler class of functionals with $f(t) = t$ there is a much simpler proof of Gamma Convergence in L^1 subsequently given by Esedoḡlu and Jacobs in [65]. In particular, they deal with more general convolution kernels, which in particular can be non-radially symmetric, thus including anisotropy, and even change sign to a certain extent, which turns out to be necessary for the approximation of certain anisotropies. A very similar result has been obtained more recently by Berendsen and Pagliari [13]. As far as we know, the last result in this field is due to Pagliari [130] where he essentially removes the radial symmetry of G and he obtains, as limit, an anisotropic perimeter.

Chapter 3

Relaxation of non-local supremal functionals and dimensional reduction

3.1 Setting and Main Results

When Ω is a bounded open subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$, it is well known that in general there is no explicit representation for the relaxation of nonlocal supremal functionals of the type

$$\tilde{J}(u) := \operatorname*{ess\,sup}_{\Omega\times\Omega} W(\nabla u(x), \nabla u(y)),\ u \in W^{1,\infty}(\Omega, \mathbb{R}^d),\ \Omega \subset \mathbb{R}^n \text{ open and bounded}, \quad (3.1.1)$$

with respect to the $W^{1,\infty}$-weak* topology. We will characterize the weak*-$W^{1,\infty}$ lower semicontinuity of the functional $\tilde{J}$ in (3.1.1) in terms of properties on the supremand W when the open set Ω is an open interval on the real line. For the sake of exposition we will mainly focus on the case when $d = 1$, i.e. the field u is scalar valued. On the other hand the results concerning fields in $W^{1,\infty}(I, \mathbb{R}^d)$ can be deduced exploiting analogous arguments, see Theorem 3.1.5. Actually, Theorem 3.1.8 could be deduced as a corollary of Theorem 3.1.5, but we opted for presenting the proof in the scalar valued setting for the sake of exposition.

Furthermore, we will show that, under suitable assumptions on W, in the case $n = 1$, the relaxed functional, can be represented in supremal form with a suitable density. Furthermore if also $d = 1$, without any restriction on W, the relaxed energy is of the same type with a density coinciding with the separately level convex envelope of the original one. In the vector valued case the relaxation will be deduced in view of [96, Theorem 1.2], see Theorem 3.1.7.

In order to state our results we need to introduce some definitions.

Definition 3.1.2 (Maximal Cartesian square, [98])**.** Given $K \subset \mathbb{R}^n \times \mathbb{R}^n$ we say that $P \subset K$ is a *Maximal Cartesian set* if $P = A \times A$ for some $A \subset \mathbb{R}^n$ and if it holds true that given $B \subset \mathbb{R}^n$ such $A \subset B$ and $B \times B \subset K$ then $A = B$

Definition 3.1.3 (Cartesian convex set, [96])**.** Consider a subset $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and a function $f : \mathbb{R}^n \times \mathbb{R}^n \to \overline{\mathbb{R}}$,

1. E is said *Cartesian convex* if $A \times A \subset E$ implies $A^{co} \times A^{co} \subset E$, where A^{co} denotes the convex envelope of the set A.

2. Assuming E not Cartesian convex, we define its Cartesian convex hull as the smallest Cartesian convex set containing E. The Cartesian convex hull of E will be denoted by $E^{\times sc}$. Moreover, we denote with $\mathcal{P}_E$ the set of all the maximal Cartesian squares

3. We say E admits a *basic Cartesian convexification* if $E^{\times xc} = \bigcup_{A\times A\in\mathcal{P}_E} A^{co}\times A^{co}$.

4. f is said to be *Cartesian level convex* if for every $c\in\mathbb{R}$ the set $L_c(f)$ is Cartesian convex.

5. Assuming f is not Cartesian level convex, the corresponding envelope is defined as follow

$$f^{\times lc}(\xi,\eta) := \sup\{g(\xi,\eta)|\, f:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R} \text{ is Cartesian level convex and } g\leqslant f\},$$

Remark 3.1.4. It was proven in [96] that the Cartesian level convex envelope of f can be described as

$$f^{\times lc}(\xi,\eta) = \inf\{c\in\mathbb{R} : (\xi,\eta)\in L_c(f)^{\times c}\}$$

for $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d$. Moreover, it is important to notice that in $\mathbb{R}\times\mathbb{R}$ the notion of Cartesian level convexity coincides with the notion of separate level convexity.

The main results presented in this Chapter can be stated as follows.

Theorem 3.1.5. *Let $J : W^{1,\infty}(I;\mathbb{R}^d)\to\mathbb{R}$ be defined as*

$$J(u) := \operatorname*{ess\,sup}_{I\times I} W(u'(x),u'(y)), \tag{3.1.6}$$

with $W:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ coercive, lower semicontinuous, diagonal and symmetric, then J is lower semicontinuous with respect to the $W^{1,\infty}-weak^*$ topology if and only if W is Cartesian level convex.

Moreover,

Theorem 3.1.7. *Let W and J be as in Theorem 3.1.5. Let J^{rlx} be the sequentially $W^{1,\infty}$–weak* lower semicontinuous envelope of J, i.e.*

$$J^{rlx} := \inf\{\liminf_{j\to+\infty} J(u_j) : u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(I;\mathbb{R}^d)\}.$$

If every sublevel set of W has a basic Cartesian convexification, then

$$J^{rlx}(u) = \operatorname*{ess\,sup}_{I\times I} W^{\times lc}(u'(x),u'(y)),$$

for every $u\in W^{1,\infty}(I;\mathbb{R}^d)$, where $W^{\times lc}$ stands for the Cartesian level convex envelope of W.

From the previous theorem, it follows that

Theorem 3.1.8. *Let $J_{1d} : W^{1,\infty}(I)\to\mathbb{R}$ be defined as*

$$J_{1d}(u) := \operatorname*{ess\,sup}_{I\times I} V(u'(x),u'(y)), \tag{3.1.9}$$

with $V:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ coercive, lower semicontinuous, diagonal and symmetric, then J_{1d} is lower semicontinuous with respect to the $W^{1,\infty}-weak^*$ topology if and only if V is separately level convex.

Furthermore,

Theorem 3.1.10. *Let V and J_{1d} be as in Theorem 3.1.8. Let J_{1d}^{rlx} be the sequentially $W^{1,\infty}-$weak* lower semicontinuous envelope of J_{1d}, i.e.*

$$J_{1d}^{rlx} := \inf\{\liminf_{j\to+\infty} J_{1d}(u_j) : u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(I)\}. \tag{3.1.11}$$

Then

$$J_{1d}^{rlx}(u) = \operatorname*{ess\,sup}_{I\times I} V^{slc}(u'(x), u'(y)),$$

for every $u \in W^{1,\infty}(I)$, where V^{slc} is the separately level convex envelope of V.

Before concluding this section we would like to point out that the weak* topology is metrizable on bounded sets (see [73, A.1.5]) and thus from now on we will omit the word "sequential".

3.2 Preliminaries

In what follows n ad d will denote elements of $\mathbb{N}$ and the set $\overline{\mathbb{R}}$ will denote the closure of the set of real numbers, i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. In this subsection we will give some definition and recall some results that will be necessary for the proofs of our theorems.

First we recall the notions of *level convexity, separate convexity* and *separate level convexity.*

Definition 3.2.1. A set $A \subset \mathbb{R}^d \times \mathbb{R}^d$ is said to be *separately convex* if for every $(\xi_1, \xi_2), (\eta_1, \eta_2) \in \mathbb{R}^d \times \mathbb{R}^d$ with $\xi_1 = \eta_1$ or $\xi_2 = \eta_2$ it holds

$$t(\xi_1, \xi_2) + (1-t)(\eta_1, \eta_2) \in A,$$

for every $t \in (0, 1)$. We denote with A^{sc} the separate convex hull of A, i.e. the smallest separately convex set containing A.

Given a function $f : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$ and any real number c, we denote with $L_c(f)$ the c-sublevel set of f, i.e.

$$L_c(f) := \{(\xi, \eta) \in \mathbb{R}^d \times \mathbb{R}^d : f(\xi, \eta) \leqslant c\},$$

while for every $K \subset \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$

$$B_K := \left\{u \in W^{1,\infty}(\Omega; \mathbb{R}^d) : (\nabla u(x), \nabla u(y)) \in K \text{ for a.e. } x \in \Omega \times \Omega\right\}. \tag{3.2.2}$$

Definition 3.2.3. A function $f : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is said to be

1. *level convex* if for every $c \in \mathbb{R}$ the set $L_c(f)$ is convex;
2. *separately convex* if for every $\xi \in \mathbb{R}^d$ both $f(\cdot, \xi)$ and $f(\xi, \cdot)$ are convex functions.
3. *separately level convex* if $L_c(f)$ is separately convex for every $c \in \mathbb{R}$.

We denote the separately convex envelope of f, i.e. the greatest convex function less than or equal to f by f^{sc} and we denote with f^{slc} the separately level convex envelope of f.

Definition 3.2.4. Given $W : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$ we say that W is symmetric if $W(\xi,\eta) = W(\eta,\xi)$ for every $(\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d$. Also, we say that W is diagonal if

$$(\xi,\eta) \in L_c(W) \Rightarrow (\xi,\xi),(\eta,\eta) \in L_c(W).$$

If f is symmetric it holds (see [96, eq. (2.8)]),

$$W(\xi,\eta) = \max\{W(\xi,\xi), W(\xi,\eta), W(\eta,\eta)\}.$$

Definition 3.2.5. We say that a function $W : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$ is coercive if there exists $C' > 0$ such that

$$C'|(\xi,\eta)| \leqslant W(\xi,\eta),$$

for every $\xi,\eta \in \mathbb{R}^d$.

Remark 3.2.6. It important to notice that if $A \subset \mathbb{R}^d \times \mathbb{R}^d$ is open, then so is A^{sc}. This is in general not true for compactness, see [45, Remark 7.18 (ii)]. On the other hand, as proved in [95, Proposition 2.3], the compactness is preserved if $A \subset \mathbb{R} \times \mathbb{R}$. Moreover, from Definitions of seperate convexity 3.2.1 and separately level convexity 3.2.3, for any $W : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$,

$$L_c(W)^{sc} \subset L_c(W^{slc}),$$

While the inverse inclusion is in general not true, it has been proved in [98, Lemma 7.4] that if $d = 1$, then

$$L_c(W)^{sc} = L_c(W^{slc}),$$

if W is symmetric and diagonal. As we will see in the next paragraph, assuming W symmetric and diagonal is not restrictive.

We will now prove that, as anticipated, there is no loss of generality in assuming that the supremands V and W in (3.1.9) and (3.1.6) are diagonal and symmetric.

Definition 3.2.7. Given $K \subset \mathbb{R}^n \times \mathbb{R}^n$ we define the set $\widehat{K}$ as follows

$$\widehat{K} := \{(\xi,\eta) \in K : (\eta,\xi),(\eta,\eta),(\xi,\xi) \in K\} \tag{3.2.8}$$

Consider now $W : \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} \to \mathbb{R}$ and define the function $\widehat{W} : \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} \to \mathbb{R}$ by

$$\widehat{W}(\xi,\eta) := \inf\left\{c \in \mathbb{R} : (\xi,\eta) \in \widehat{L_c(W)}\right\}. \tag{3.2.9}$$

By definition $\widehat{L_c(W)}$ it follows that for every $c \in \mathbb{R}$

$$L_c(\widehat{W}) = \widehat{L_c(W)}. \tag{3.2.10}$$

Moreover, $\widehat{W}$ is symmetric and diagonal and $\widehat{W} \geqslant W$. From the previous inequality it follows that the coercivity of W is inherited by $\widehat{W}$. Furthermore, define for $\Omega \subset \mathbb{R}^n$ and every $K \subset \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$

$$A_K := \left\{v \in L^\infty(\Omega;\mathbb{R}^{d\times n}) : (v(x),v(y)) \in K \text{ for a.e. } x \in \Omega\times\Omega\right\}. \tag{3.2.11}$$

Consider now the following Proposition.

Proposition 3.2.12 ([98], Proposition 5.1)**.** *Given $E, F \subset \mathbb{R}^n \times \mathbb{R}^n$ closed, then $A_E = A_F$ if and only if $\widehat{E} = \widehat{F}$. In particular, $A_E = A_{\widehat{E}}$.*

From the previous Proposition and identity (3.2.10) follows that for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^d)$,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\Omega\times\Omega} \widehat{W}(\nabla u(x), \nabla u(y)) = \inf\left\{c \in \mathbb{R} : \nabla u \in A_{L_c(\widehat{W})}\right\} \\
&= \inf\left\{c \in \mathbb{R} : \nabla u \in A_{\widehat{L_c(W)}}\right\} = \inf\left\{c \in \mathbb{R} : \nabla u \in A_{L_c(W)}\right\} \\
&= \operatorname*{ess\,sup}_{\Omega\times\Omega} W(\nabla u(x), \nabla u(y)).
\end{aligned}$$

Clearly, the identity holds in the case $W = V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\Omega = I$.

Before proceeding with the proofs of Theorem 3.1.5 and 3.1.7 we would like to highlight a couple of facts. On one hand, our results can be seen as results on non-local differential inclusions. Indeed, let K be a compact subset of $\mathbb{R} \times \mathbb{R}$. Corollary 3.1.10 says that, given a sequence $(u_j)_j \subset W^{1,\infty}(I)$ such that

$$(u_j'(t), u_j'(s)) \in K,$$

for a.e. $(s,t) \in I \times I$, and such that $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(I)$ then,

$$(u'(t), u'(s)) \in \widehat{K}^{sc},$$

where $\widehat{K}$ is defined as in Definition 3.2.8 and the superscript sc stands for its separately level convex hull. It is then possible to show that Corollary 3.1.8 states that $(u'(t), u'(s)) \in K$ for a.e. $(s,t) \in I \times I$ if and only if $\widehat{K}$ is separately convex. This means that, if we consider the set B_K defined as

$$B_K := \left\{u \in W^{1,\infty}(I) : (u'(x), u'(y)) \in K \text{ for a.e. } (x,y) \in I \times I\right\}, \tag{3.2.13}$$

then B_K is closed if and only if K is symmetric, diagonal and separately convex.

On the other hand, our results can be understood as results on the sequential weak* lower semicontinuity and relaxation of indicator functionals. Given a $K \subset \mathbb{R} \times \mathbb{R}$ we define its indicator function as

$$\chi_K(\xi, \eta) := \begin{cases} 0 \text{ if } (\xi,\eta) \in K, \\ \infty \text{ otherwise,} \end{cases}$$

and the related indicator functional $I : W^{1,\infty}(I) \to \overline{\mathbb{R}}$ as

$$I(u) := \iint_{I\times I} \chi_K(u'(x), u'(y))dxdy.$$

By definition of χ_K and B_K

$$\iint_{I\times I} \chi_K(u'(x), u'(y))dxdy = \begin{cases} 0 \text{ if } u \in B_K, \\ \infty \text{ otherwise} \end{cases},$$

so I is sequentially weak* lower semicontinuous if and only if for every sequence $(u_j)_j \subset B_K$ converging weakly* to u it holds that $u \in B_K$ and this is true if and only if K is symmetric, diagonal and separately convex. It is not difficult to prove that all this is still true if we suppose $K \subset \mathbb{R}^n \times \mathbb{R}^n$ and switching the level convexity with the Cartesian level convexity.

3.3 Necessary and sufficient condition

In this section we will prove Theorem 3.1.8 and 3.1.5. In particular we will prove first Theorem 3.1.8 and then, from there, we will show how to obtain Theorem 3.1.5 when the dimension is higher than one.

The set $B_{L_c(V)}$ (see (3.2.13)) is the c-sublevel set of J_{1d}. Moreover, we recall that a functional is weakly* lower semicontinuous if and only if for every $c \in \mathbb{R}$ its sublevel set is weakly* closed. It follows that J_{1d} is weakly* lower semicontinuous if and only if $B_{L_c(V)}$ is weakly* closed for every $c \in \mathbb{R}$. In order to prove Corollary 3.1.8 we have to prove that the separate level convexity of the supremand V appearing in (3.1.9) is a sufficient and necessary condition for the weak* lower semicontinuity of fhe functional J_{1d} defined in (3.1.9). To prove this, we will prove that $B_{L_c(V)}$ is weakly* closed if and only if $B_{L_c(V)} = B_{L_c(V)^{sc}}$ and by Remark 3.2.6 we have that this is true if and only if V is separately level convex. The first step in order to prove Corollary 3.1.8 consists in proving that there exists a dense subset $D^\infty(I)$ in $B_{L_c(V)}$. In what follows I will be an bounded interval (a, b) of $\mathbb{R}$.

Before proving Theorem 3.1.8 we must recall some notions. Given $E \subset \mathbb{R} \times \mathbb{R}$, let $S_E^\infty(I)$ be defined as

$$S_E^\infty(I) := \{s : I \to \mathbb{R}|\ \text{s is a simple function, } (s(x), s(y)) \in E \text{ for a.e. } x, y \in I\}.$$

Moreover, for a measurable function $f : I \to \mathbb{R}$ we consider the integral function u_f defnied as

$$u_f(x, c) := c + \int_a^x f(t)dt, \quad x \in I \tag{3.3.1}$$

Finally, we define the set $D^\infty(I)$ as the set of functions u_s with $s \in S_E^\infty(I)$.

Remark 3.3.2. We recall that, given $v \in L^\infty(I)$, then for every $c \in \mathbb{R}$ function $u_v(\cdot, c) \in L^\infty(I)$. Indeed,

$$u_v(x, c) = c + \int_a^x v(t)dt \leqslant c + ||v||_{L^\infty(I)}(b - a) < +\infty.$$

Moreover, the distributional derivative exists also in the classical sense and

$$\frac{du_v(x, c)}{dx} = v(x) \quad \text{for a.e. } x \in I,$$

and $u_v \in W^{1,\infty}(I)$. In what follows, the role of c can be neglected. We will omit it from the notation and denote the integral function just by u_v. To conclude, given a function u_v with $v \in L^\infty(I)$ and a sequence of functions $(v_j)_{j\in\mathbb{N}} \subset L^\infty(I)$ such that $v_j \overset{*}{\rightharpoonup} v$ in $L^\infty(I)$, then $u_{v_j} \overset{*}{\rightharpoonup} u_v$ in $W^{1,\infty}(I)$ (see [74], [5]).

Lemma 3.3.3. *Given $E \subset \mathbb{R} \times \mathbb{R}$ symmetric and diagonal, let B_E be as in* (3.2.13)*, then for every $u \in B_E$ there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset B_E \cap D^\infty(I)$ such that $u_j \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(I)$.*

Proof. Since $B_E \subset W^{1,\infty}(I)$ we can define $B'_E := \{u' : u \in B_E\}$. $B'_E \subset L^\infty(I)$ and it is a subset of $A_E := \{v \in L^\infty(I) : (v(x), v(y)) \in E\}$ in (3.2.11). By [98, Lemma 5.4 and

Corollary 5.5] every element v of A_E can be approximated with respect to the $L^\infty(I)$ strong convergence by a sequence $\{s_j\}_j \subset A_E \cap S^\infty(I)$. In particular this happens for $v = u' \in B'_E$. It follows by Remark 3.3.2 that, given $u \in B_E$, there exists $\{u_{s_j}\}_j \subset D_I^\infty \cap B_E$ weakly* converging to u in $W^{1,\infty}(I)$. □

In the previous lemma we supposed that $E \subset \mathbb{R} \times \mathbb{R}$ is symmetric and diagonal, but from Proposition 3.2.10 it is clear that the lemma holds true also for every closed $E \subset \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n}$.

We can now prove Theorem 3.1.8.

Proof of Theorem 3.1.8. The sufficiency of the separate level convexity of V follows from one of the implications of [98, Theorem 1.3(i)], taking into account the weak* convergence of $(u'_k)_k$ for every sequence $(u_k)_k$ weakly* converging to u in $W^{1,\infty}(I)$.

For what concerns the reverse implication, we claim that the $W^{1,\infty}-$weak* lower semicontinuity of the functional J_{1d} in (3.1.9) ensures the $L^\infty-$weak* lower semicontinuity of the functional in $L^\infty(I)$ defined as $\operatorname{ess\,sup}_{I\times I} V(v(x), v(y))$ along sequences $(v_k)_k \subset L^\infty(I)$. Indeed, if $v_k \overset{*}{\rightharpoonup} v$, in $L^\infty(I)$, then, as observed in Remark 3.3.2, it is possible to define

$$u_k(x) := \int_a^x v_k(t)dt \in L^\infty(I)$$

and u_k is in $L^\infty(I)$. Indeed the fact that v_k is bounded in $L^\infty(I)$, entails also that

$$\operatorname*{ess\,sup}_{x\in I} |u_k(x)| \leqslant \operatorname*{ess\,sup}_{x\in I} \int_a^x |v_k|(x)dx \leqslant \|v_k\|_{L^\infty(I)}(b-a).$$

Moreover $\frac{du_k}{dx}$ exists for a.e. $x \in I$ and coincides with v_k. Hence $u_k \in W^{1,\infty}(I)$. Finally u_k is uniformly bounded in the $W^{1,\infty}$ norm, thus, up to a not relabeled subsequence, $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(I)$ with $\frac{du}{dx} = v$ a.e. in I. This proves our claim. Consequently the proof is concluded by the reverse implication of [98, (i)Theorem 1.3]. □

Remark 3.3.4. It is important to notice that the level convexity of $W : \mathbb{R}^{d\times n} \times \mathbb{R}^{d\times n} \to \mathbb{R}$ is still a sufficient condition to ensure the weak* lower semicontinuity of the functional $\operatorname{ess\,sup}_{\Omega\times\Omega} W(\nabla u(x), \nabla u(y))$. As anticipated, the reverse implication does not hold true (see [96, Theorem 1.1]), even if one assumes $n = 1$.

Now we show how to derive the proof of Theorem 3.1.5 from the previous proof.

Proof of Theorem 3.1.5. The proof follows along the lines of Theorem 3.1.8, considering vector valued functions in $W^{1,\infty}(I;\mathbb{R}^d)$, replacing [98, Theorem 1.3(ii)] by [96, Theorem 1.1], i.e. exploiting the necessary and sufficient condition given by the Cartesian level convexity of W for non-local supremal functionals depending on $L^\infty(I;\mathbb{R}^d)$ fields, instead of the separate level convexity. □

3.4 Relaxation

In this subsection we will prove Theorem 3.1.10 and Theorem 3.1.7. To do so, we will rely on the properties of the Γ-convergence.

Proof of Theorem 3.1.10. The proof follows by [98, Proposition 7.5], Theorem 3.1.8 and Lemma 1.2.4. First we prove the liminf inequality. Consider

$$J^{slc}(u) := \operatorname*{ess\,sup}_{I\times I} V^{slc}(u'(x), u'(y)).$$

By definition of separately level convex envelope (see Definition 3.2.3) we get

$$V^{slc} \leqslant V,$$

By [98, Proposition 7.5] for every $(u_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(I)$ such that $u_k \overset{*}{\rightharpoonup} u$ in $W^{1,\infty}(I)$ we have that $u_k' \overset{*}{\rightharpoonup} u'$ in $L^\infty(I)$, hence

$$J^{slc}(u) \leqslant \liminf_{k\to\infty} J^{slc}(u_k) \leqslant \liminf_{k\to\infty} J_{1d}(u_k).$$

And this proves the liminf inequality. Now we have to prove the existence of a recovery sequence and the limsup inequality. Fix $u \in W^{1,\infty}(I)$. By [98, Proposition 7.5] there exists a sequence $(v_k)_{k\in\mathbb{N}} \subset L^\infty(I)$ such that $v_k \overset{*}{\rightharpoonup} u'$ in $L^\infty(I)$ as $k \to +\infty$ and

$$\lim_{k\to+\infty} \operatorname*{ess\,sup}_{I\times I} V(v_k(x), v_k(y)) = \operatorname*{ess\,sup}_{I\times I} V^{slc}(u'(x), u'(y)). \tag{3.4.1}$$

Defining $u_k(x) := \int_a^x v_k(x)dx$ (see Remark 3.3.2) allows us to say that $(u_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(I)$, and $u_k \overset{*}{\rightharpoonup} w$ in $W^{1,\infty}(I)$ with $\frac{dw}{dx} = u'$, a.e. in I, hence $w(x) = u(x) + C$. This fact, combined with (3.4.1) guarantees that

$$\lim_{k\to+\infty} \operatorname*{ess\,sup}_{I\times I} V(u_k'(x), u_k'(y)) = \operatorname*{ess\,sup}_{I\times I} V^{slc}(u'(x), u'(y)),$$

which in turn ensures the existence of a recovery sequence, i.e. $(u_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(I)$, weakly* converging to u and that concludes the proof. □

The proof of Theorem 3.1.7 can be obtained from the previous one.

Proof of Theorem 3.1.7. The proof relies on the same arguments of Theorem 3.1.10, exploiting [96, Theorem 1.2] instead of [98, Proposition 7.5]. □

To conclude this section we show an application of our results. In particular we will use Theorem 3.1.10 to prove the dimensional reduction of a non-local supremal functional.

3.5 Dimensional reduction

In this section we study the dimensional reduction of the functional

$$J_{2d}^h(u) := \operatorname*{ess\,sup}_{(x,y)\in I_h^2\times I_h^2} \max\{W(\nabla u(x), \nabla u(y)), h^\alpha ||\nabla u||_{L^\infty(I_h^2)}\},$$

with $I_h^2 := (0,1) \times (0,h)$, $u \in W^{1,\infty}(I_h^2)$, and $0 < \alpha < 1$. Moreover, we also assume that for every $\xi, \eta \in I^2$ it holds $W(\xi,\eta) = W_1(\xi_1,\eta_1)$ for some function $W_1 : I \times I \to [0,+\infty]$.

Finally, we assume W coercive, continuous, symmetric and diagonal. The study is done by means of the Γ-convergence of the rescaled functional

$$J^h(u) := \operatorname*{ess\,sup}_{(x,y)\in I^2\times I^2} \max\{W(\nabla_h u(x), \nabla_h u(y)), h^\alpha ||\nabla_h u||_{L^\infty(I^2)}\},$$

with $\nabla_h u := [\frac{\partial u}{\partial x_1}, \frac{1}{h}\frac{\partial u}{\partial x_2}]$, $u \in W^{1,\infty}(I^2)$. In particular, we prove that under the previous hypotheses the reduced functional is

$$J_{1d}(u) := \operatorname*{ess\,sup}_{(x,y)\in I\times I} W_1^{slc}(u'(x), u'(y)),$$

with $u \in W^{1,\infty}(I)$. First we consider the matter of the compactness.

Lemma 3.5.1. *Given a sequence $(u_h) \subset W^{1,\infty}(I^2)$ such that one of the following holds*

- $\sup_h u_h(0,0) \leqslant C$, $C \in \mathbb{R}$,
- $u_h(0,0) = (l_1, l_2)$, *with* $(l_1, l_2) \in \mathbb{R}^2$, *for all* $h > 0$,
- $(u_h)_{h>0}$ *is bounded in* $L^\infty(I^2)$,

and such that $\sup_{h\to+\infty} J^h(u_h) < +\infty$, *then there exist* $u \in W^{1,\infty}(I)$ *and a not relabelled subsequence such that* $u_h \rightharpoonup^* u$ *in* $W^{1,\infty}(I^2)$. *Moreover, it holds*

$$\frac{\partial u(x)}{\partial x_2} = 0 \text{ for every } x \in I^2.$$

Proof. The coerciveness of W, Sobolev's embedding Theorem which guarantees that $(u_h) \subset C(\overline{I^2})$, Rademacher Theorem which provides a.e. differentiability, guarantee that (u_h) is equibounded and equicontinuous, hence, by Ascoli-Arzelà theorem, there exists a not relabeled subsequence $u_h \to u$ in $L^\infty(I^2)$. Moreover, by

$$||u_h||_{W^{1,\infty}(I^2)} \leqslant c,$$

we can deduce that (u_h) is also $W^{1,\infty}$-weakly* converging to u. Since $1 - \alpha > 1$, then by

$$\frac{1}{h^{1-\alpha}}\frac{\partial u_h}{\partial x_2} \leqslant c,$$

$\frac{\partial u_h}{\partial x_2}$ is converging strongly in $L^\infty(I^2)$ to 0, which means that $\frac{\partial u}{\partial x_2} = 0$ for every $x \in I^2$. □

We recall that the space $W^{1,\infty}(I)$ is continuously embedded in $W^{1,\infty}(I^2)$ and that it is isomorphic the the space $V := \{u \in W^{1,\infty}(I^2) : \frac{\partial u}{\partial x_2} = 0 \text{ for a.e. } x_2 \in I\}$. From here onwards, given $u \in W^{1,\infty}(I)$ we will denote with $\overline{u}$ its representative in $W^{1,\infty}(I^2)$, i.e. its correspondig element in V. Notice that the previous lemma tells us that u_h is weakly* converging in $W^{1,\infty}(I^2)$ to some $\overline{u} \in V$ for some $u \in W^{1,\infty}(I)$.

We now prove the Γ-liminf and the Γ-limsup inequalities. The lower bound follows from the fact that for every $(u_h)_h \subset W^{1,\infty}(I^2)$ such that $u_h \rightharpoonup^* \overline{u}$, with $\overline{u} \in V$, it holds

$$\begin{aligned} &W_1^{slc}\left(\frac{\partial u_h(x)}{\partial x_1}, \frac{\partial u_h(y)}{\partial y_1}\right) = W^{slc}(\nabla_h \overline{u}_h(x), \nabla_h \overline{u}_h(y)) \\ &\leqslant W(\nabla_h \overline{u}_h(x), \nabla_h \overline{u}_h(y)) \leqslant \max\{W(\nabla_h \overline{u}_h(x), \nabla_h \overline{u}_h(y)), h^\alpha ||\nabla_h \overline{u}_h||_{L^\infty(I^2)}\}. \end{aligned}$$

Passing to the $\operatorname{ess\,sup}$ in the previous inequality we have that by Theorem 3.1.10 the left hand side is weak* lower semicontinuous and that $\frac{\partial u_h}{\partial x_1} \rightharpoonup^* \frac{\partial \overline{u}}{\partial x_1}$, with $\frac{\partial \overline{u}(x)}{\partial x_1} = u'$ for every $x_1 \in I$ and for a.e. $x_2 \in I$, leading to

$$\begin{aligned} J_{1d}(u) &= \operatorname*{ess\,sup}_{(x_1,y_1)\in I\times I} W_1^{slc}(u'(x_1), u'(y_1)) \\ &\leqslant \liminf_{h\to 0} \operatorname*{ess\,sup}_{(x,y)\in I^2\times I^2} \max\{W(\nabla_h \overline{u}_h(x), \nabla_h \overline{u}_h(y)), h^\alpha ||\nabla_h \overline{u}_h||_{L^\infty(I^2)}\}. \end{aligned}$$

The recovery sequence is constructed using a classical argument by Le Dret and Raoult (see [107]). Fix $u, v \in W^{1,\infty}(I)$ and consider the sequence

$$u_h(x) := \overline{u}(x) + hx_2 v(x_1).$$

By construction $u_h \rightharpoonup^* \overline{u}$ in $W^{1,\infty}(I^2)$ Moreover,

$$\nabla_h u_h(x) = \left(\frac{\partial \overline{u}(x)}{\partial x_1} + hx_2 v'(x_1), v(x_1)\right),$$

so by continuity of W,

$$W(\nabla_h u_h(x), \nabla_h u_h(y)) \to W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right) \text{ in } L^\infty(I^2) \text{ as } h \to 0,$$

and

$$\begin{aligned} &\lim_{h\to 0} \max\{W(\nabla_h u_h(x), \nabla_h u_h(y)), h^\alpha ||\nabla_h u_h||_{L^\infty(I^2)}\} \\ &= \max\{W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right), 0\}, \end{aligned}$$

in $L^\infty(I^2)$ as $h \to 0$. Since W is positive it follows that

$$\begin{aligned} &\max\{W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right), 0\} \\ &= W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right). \end{aligned}$$

Moreover, for every $x, y \in I^2$,

$$W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right) = W_1(u'(x_1), u'(y_1)).$$

By definition of Γ-convergence and the previous identities,

$$\begin{aligned} &\left(\Gamma - \lim_{h\to 0} J^h\right)(\overline{u}) \leqslant \lim_{h\to 0} \operatorname*{ess\,sup}_{(x,y)\in I^2\times I^2} \max\{W(\nabla_h u_h(x), \nabla_h u_h(y)), h^\alpha ||\nabla_h u_h||_{L^\infty(I^2)}\} \\ &= \operatorname*{ess\,sup}_{(x,y)\in I^2\times I^2} \max\{W\left(\left(\frac{\partial \overline{u}(x)}{\partial x_1}, v(x_1)\right), \left(\frac{\partial \overline{u}(y)}{\partial y_1}, v(y_1)\right)\right), 0\} \\ &= \operatorname*{ess\,sup}_{(x_1,y_1)\in I\times I} W_1(u'(x_1), u'(y_1)). \end{aligned}$$

The lef-thand side is weak* lower semicontinous in $W^{1,\infty}(I^2)$, so in particular it is lower semicontinuous along every sequence of elements of V generated by a sequence of elements of $W^{1,\infty}(I)$.Thus, the left-hand side must by lower (or equal) than the lower semicontinuous envelope of the right-hand side. By Theorem 3.1.8 we get

$$\left(\Gamma - \lim_{h\to 0} J^h\right)(\overline{u}) \leqslant \operatorname*{ess\,sup}_{(x_1,y_1)\in I\times I} W_1^{slc}(u'(x_1), u'(y_1)),$$

and this concludes the proof.

Chapter 4

Study of the asymptotic behavior of a family of non-local functional

4.1 Setting and Main Results

In this Chapter we consider the family of functionals $(\mathscr{F}_\varepsilon)_\varepsilon$, with $\varepsilon > 0$, defined as

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E^c \cap \Omega} f(G_\varepsilon * \chi_{E\cap\Omega})\, dx,$$

with $\Omega \subset \mathbb{R}^N$ is open and bounded, $N > 1$, E is a set of finite perimeter in Ω, f is given and $G_\varepsilon(z) = \frac{1}{\varepsilon^N} G(\frac{z}{\varepsilon})$ where G is a suitable kernel. In particular we assume that G is even, non-negative, supported on the unit closed ball and with $\int_{\mathbb{R}^N} G(z)\, dz = 1$. First of all we are able to compute the pointwise limit, as $\varepsilon \to 0$, of $\mathscr{F}_\varepsilon(E)$.

Theorem 4.1.1 (Pointwise limit)**.** *Assume f is of class C^1, non-decreasing and $f(0) = 0$. Let $E \in \mathcal{P}_N(\Omega)$. Then*

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(E) = \mathscr{F}(E)$$

with

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(E) = \int_{\partial^* E\cap\Omega} \int_0^1 f\left(\int_{\{z\cdot\nu_E(x) \geqslant t\}} G(z)\, dz \right) dt\, d\mathcal{H}^{N-1}(x),$$

where $\partial^* E$ is the reduced boundary of E and $\nu_E(x)$ is the outer unit normal at E.

In view to have a Γ-convergence result we investigate also the lower estimate. Unfortunately, the technique of Esedoḡlu and Otto [66] does not work in our situation: it is crucial for them to switch the order of integration, that is impossible for us since we have f between the exterior integral and the convolution one. It seems that this difficulty cannot be easily overcome in the general situation. We are able to show a Γ-liminf inequality only on graphs of C^1 functions with respect to the C^1-uniform convergence.

Theorem 4.1.2 (Lower bound)**.** *Let $D \subset \mathbb{R}^{N-1}$ be open and bounded with Lipschitz boundary, let $u_h, u \in C^{1,1}(D)$, with $u_h, u > 0$ on D such that $u_h \to u$ uniformly in $C^1(D)$. Let E_h, E be given by*

$$E_h = \{(x,y) \in \mathbb{R}^{N-1}\times\mathbb{R} : x \in D,\, 0 \leqslant y \leqslant u_h(x)\},$$

$$E = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D,\, 0 \leqslant y \leqslant u(x)\}.$$

Then, for any positive infinitesimal sequence (ε_h) *it holds*

$$\liminf_{h \to +\infty} \mathscr{F}_{\varepsilon_h}(E_h) \geqslant \mathscr{F}(E).$$

Actually, it is easy to generalize such a inequality in the case of sets which are locally graphs of C^1 functions with respect to a suitable convergence. Finally, we also prove that if f is also convex, then the pointwise limit is lower semicontinuous with respect to the strong L^1-convergence.

Theorem 4.1.3. *If f is convex then $\mathscr{F}$ is lower semicontinuous with respect to the strong $L^1(\Omega)$ topology.*

The previous Theorem suggests that for f convex the Γ-limit in the strong L^1-convergence should be the pointwise limit. At the end of the Chapter we will also discuss some examples.

4.2 Notation, Preliminaries and Setting

In what follows, for any $r > 0$ and $x \in \mathbb{R}^N$ the notation $B_r^N(x)$ will denote for the open ball in $\mathbb{R}^N$ centered at x with radius r, while $\mathbb{S}^{N-1} = \partial B_1^N(0)$. If $A \subseteq \mathbb{R}^N$ we also denote by $\mathcal{H}^k(A)$ the Hausdorff measure of A of dimension $k \in \{0, 1, \ldots, N\}$ (where $\mathcal{H}^0$ is the counting measure). If for $h > 0$ A_h, A are measurable subsets of $\mathbb{R}^N$, then $A_h \to A$ in $L^1(\mathbb{R}^N)$ (or $L^1_{\text{loc}}(\mathbb{R}^N)$) means that $\chi_{A_h} \to \chi_A$ in $L^1(\mathbb{R}^N)$ (respectively $L^1_{\text{loc}}(\mathbb{R}^N)$). Finally, for any $A \subseteq \mathbb{R}^N$ we let $A^c = \mathbb{R}^N \setminus A$.

4.2.1 Sets of finite perimeter

We recall some notions on finite perimeter sets in euclidean space, for details see [3]. Let Ω be an open subset of $\mathbb{R}^N$. A measurable set $E \subseteq \mathbb{R}^N$ is said to be a *set of finite perimeter in* Ω if

$$\mathcal{P}(E, \Omega) = \sup\left\{\int_E \operatorname{div} X(x)\, dx : X \in C^1_c(\Omega; \mathbb{R}^N),\ \|X\|_\infty \leqslant 1\right\} < +\infty.$$

The quantity $\mathcal{P}(E, \Omega)$ is called *perimeter of E in Ω*, see Definition 1.1.26. Finite perimeter sets have "nice" boundary in the sense that one can define a subset of E as the set of points x where there exists a unit vector $\nu_E(x)$ such that

$$\frac{x - E}{r} \to \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \geqslant 0\}, \text{ in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ as } r \to 0. \tag{4.2.1}$$

$\nu_E(x)$ is called the *outer normal to E at x*, see Definition 1.1.27. The set where $\nu_E(x)$ exists is called the *reduced boundary of E* and is denoted by $\partial^* E$. It is possible to show that for any E set of finite perimeter in Ω, we have $\mathcal{P}(E, \Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega)$. Moreover, the reduced boundary of E plays the role of the topological boundary also in the sense of the integration by parts, see Theorem 1.1.28. Indeed, one can show that, if E is a set of finite perimeter in Ω, then the following Gauss-Green formula holds true

$$\int_E \operatorname{div} X(x)\, dx = \int_{\partial^* E} X(x) \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x), \quad \forall X \in C^1_c(\Omega; \mathbb{R}^N). \tag{4.2.2}$$

Finally, Finite perimeter sets have also good properties from a variational point of view. Indeed, the perimeter is l.s.c. with respect to the L^1 convergence, i.e. if E_h, E have finite perimeter in Ω and $E_h \xrightarrow{L^1} E$, then

$$\mathcal{P}(E,\Omega) \leqslant \liminf_{h\to+\infty} \mathcal{P}(E_h,\Omega).$$

4.2.2 Setting

In what follows, we suppose that $G\colon \mathbb{R}^N \to [0,+\infty)$ be of class C^∞ and such that

$$\operatorname{supp} G = \overline{B_1^N(0)}, \quad G(-x) = G(x), \quad \int_{\mathbb{R}^N} G(x)\, dx = 1.$$

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^N$,

$$G_\varepsilon(x) := \frac{1}{\varepsilon^N} G\left(\frac{x}{\varepsilon}\right).$$

Moreover, we consider a continuous and non-decreasing function $f\colon [0,+\infty) \to \mathbb{R}$ with $f(0) = 0$. For $\varepsilon > 0$, we define the functional $\mathscr{F}_\varepsilon\colon \mathcal{P}_N(\Omega) \to [0,+\infty)$ as

$$\mathscr{F}_\varepsilon(E) := \frac{1}{\varepsilon} \int_{E^c\cap\Omega} f(G_\varepsilon * \chi_{E\cap\Omega})\, dx. \tag{4.2.3}$$

In order to state our main results, we consider the function $\theta\colon \mathbb{S}^{N-1} \to [0,+\infty)$ defined as

$$\theta(\nu) := \int_0^1 f\left(\int_{\{x\cdot\nu\geqslant t\}} G(x)\, dx\right) dt, \tag{4.2.4}$$

and the functional $\mathscr{F}\colon \mathcal{P}_N(\Omega) \to [0,+\infty)$

$$\mathscr{F}(E) := \int_{\partial^* E\cap\Omega} \theta(\nu_E(x))\, d\mathcal{H}^{N-1}(x).$$

4.3 Pointwise limit and recovery sequence

In this subsection we prove the pointwise limit of $\mathscr{F}_\varepsilon$, i.e. Theorem 4.1.1, and we make some consideration on what its existence entails. First we will prove the following theorem. The main idea comes from the technique used in [119]. For the reader's convenience the proof is divided in four steps.

STEP 1. We prove that for any $E \in \mathcal{P}_N$ it holds

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x)\, d\eta\, d\mathcal{H}^{N-1}(x), \tag{4.3.1}$$

where fixed $\eta > 0$ and $x \in \partial^* E$ then $X(\eta, x)$ is defined as

$$X(\eta, x) := \frac{1}{\eta^N} \int_{E^c} f'(G_\eta * \chi_E(y)) G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta}\, dy.$$

For any $\eta > 0$ and any $y \in \mathbb{R}^N$ we have by the Gauss-Green formula (4.2.2),

$$
\begin{aligned}
&\frac{d}{d\eta} f(G_\eta * \chi_E(y)) \\
&\quad = -f'(G_\eta * \chi_E(y)) \frac{1}{\eta^{N+1}} \int_{\mathbb{R}^N} \left(NG\left(\frac{y-x}{\eta}\right) + \nabla G\left(\frac{y-x}{\eta}\right) \cdot \frac{y-x}{\eta} \right) \chi_E(x)\, dx \\
&\quad = f'(G_\eta * \chi_E(y)) \frac{1}{\eta^N} \int_{\mathbb{R}^N} \mathrm{div}_x \left(G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \right) \chi_E(x)\, dx \\
&\quad = f'(G_\eta * \chi_E(y)) \frac{1}{\eta^N} \int_{\partial^* E} G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x).
\end{aligned}
$$

Since $G_\varepsilon * \chi_E \to \chi_E$ in $L^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, then for any $y \in \mathbb{R}^N$ holds

$$
f(G_\varepsilon * \chi_E(y)) - f(\chi_E(y)) = \int_0^\varepsilon \frac{d}{d\eta} f(G_\eta * \chi_E(y))\, d\eta.
$$

From the previous identity we get, using the fact that $f(0) = 0$, that

$$
\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon} \int_{E^c} f(G_\varepsilon * \chi_E(y)) - f(\chi_E(y))\, dy \\
&= \frac{1}{\varepsilon} \int_{E^c} \int_0^\varepsilon \frac{d}{d\eta} f(G_\eta * \chi_E(y))\, d\eta\, dy \\
&= \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon \frac{1}{\eta^N} \int_{E^c} f'(G_\eta * \chi_E(y)) G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta}\, dy\, d\eta \cdot \nu_E(x)\, d\mathcal{H}^{N-1}(x) \\
&= \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x)\, d\eta\, d\mathcal{H}^{N-1}(x)
\end{aligned}
$$

and hence (4.3.1).

STEP 2. Now we claim that for any $x \in \partial^* E$ holds

$$
\lim_{\varepsilon \to 0} X(\varepsilon, x) = \int_{\{z \cdot \nu_E(x) \geqslant 0\}} f'\left(\int_{\{(v-z) \cdot \nu_E(x) \geqslant 0\}} G(v)\, dv \right) G(z) z\, dz. \tag{4.3.2}
$$

First we notice that

$$
\begin{aligned}
X(\varepsilon, x) &= \frac{1}{\varepsilon^N} \int_{E^c} f'(G_\varepsilon * \chi_E(y)) G\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon}\, dy \\
&= \frac{1}{\varepsilon^N} \int_{E^c} f'\left(\frac{1}{\varepsilon^N} \int_E G\left(\frac{y-w}{\varepsilon}\right) dw \right) G\left(\frac{y-x}{\varepsilon}\right) \frac{y-x}{\varepsilon}\, dy.
\end{aligned}
$$

Performing first the change of variable $y = x + \varepsilon z$ and then $w = x + \varepsilon z - \varepsilon v$, we obtain

$$
\begin{aligned}
X(\varepsilon, x) &= \int_{\frac{E^c - x}{\varepsilon}} f'\left(\frac{1}{\varepsilon^N} \int_E G\left(\frac{x + \varepsilon z - w}{\varepsilon}\right) dw \right) G(z) z\, dz \\
&= \int_{\frac{E^c - x}{\varepsilon}} f'\left(\int_{\frac{x-E}{\varepsilon} + z} G(v)\, dv \right) G(z) z\, dz.
\end{aligned}
$$

Passing to the limit for $\varepsilon \to 0$, using (4.2.1) and applying the Dominated Convergence Theorem obtain (4.3.2).

STEP 3. Now we prove that for any $x \in \partial^* E$ we have

$$\int_{\{z\cdot\nu_E(x)\geqslant 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geqslant 0\}} G(v)\,dv\right) G(z)z\,dz\cdot\nu_E(x) = \theta(\nu_E(x)). \tag{4.3.3}$$

First we recall that any $z \in \mathbb{R}^N$ with $z\cdot\nu_E(x) \geqslant 0$ can be written (in a unique way) as $z = \bar{z} + t\nu_E(x)$ with $\bar{z}\cdot\nu_E(x) = 0$ and $t \geqslant 0$. Moreover, $z\cdot\nu_E(x) = (\bar{z}+t\nu_E(x))\cdot\nu_E(x) = t$. Furthermore, since G is supported on $\overline{B_1^N(0)}$, we can consider $t \in [0,1]$ obtaining

$$\begin{aligned}
&\int_{\{z\cdot\nu_E(x)\geqslant 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geqslant 0\}} G(v)\,dv\right) G(z)z\,dz\cdot\nu_E(x)\\
&\qquad = \int_{\{z\cdot\nu_E(x)\geqslant 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geqslant 0\}} G(v)\,dv\right) G(z)z\cdot\nu_E(x)\,dz\\
&\qquad = \int_0^1\int_{\{\bar{z}\cdot\nu_E(x)=0\}} f'\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right) G(\bar{z}+t\nu_E(x))\,t\,dt\,d\bar{z}\\
&\qquad = \int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\int_{\{\bar{z}\cdot\nu_E(x)=0\}} G(\bar{z}+t\nu_E(x))\,t\,d\bar{z}\,dt\\
&\qquad = \int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\int_{\{z\cdot\nu_E(x)=t\}} G(z)\,d\mathcal{H}^{N-1}(z)\,t\,dt.
\end{aligned}$$

Finally, we notice that

$$\begin{aligned}
&\frac{d}{dt}\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\\
&\qquad = \lim_{h\to 0}\frac{1}{h}\left(\int_{\{v\cdot\nu_E(x)\geqslant t+h\}} G(v)\,dv - \int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\\
&\qquad = -\lim_{h\to 0}\frac{1}{h}\int_{\{t\leqslant v\cdot\nu_E(x)\leqslant t+h\}} G(v)\,dv\\
&\qquad = -\int_{\{v\cdot\nu_E(x)=t\}} G(v)\,dv,
\end{aligned}$$

and integrating by parts we get

$$\begin{aligned}
&\int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\int_{\{z\cdot\nu_E(x)=t\}} G(z)\,d\mathcal{H}^{N-1}(z)\,t\,dt\\
&\quad = -\int_0^1 \frac{d}{dt} f\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\,t\,dt\\
&\quad = -f\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\,t\Big|_0^1 + \int_0^1 f\left(\int_{\{v\cdot\nu_E(x)\geqslant t\}} G(v)\,dv\right)\,dt\\
&\quad = \theta(\nu_E(x))
\end{aligned}$$

where θ has been introduced in (4.2.4). This concludes the proof of (4.3.3).

STEP 4. We conclude using (4.3.1), (4.3.2), (4.3.3), De l'Hôpital rule and the Dominated Convergence Theorem and we deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\partial^* E}\int_0^\varepsilon X(\eta,x)\cdot\nu_E(x)\,d\eta\,d\mathcal{H}^{N-1}(x)\\ &= \int_{\partial^* E}\lim_{\varepsilon\to 0}X(\eta,x)\cdot\nu_E(x)\,d\eta\,d\mathcal{H}^{N-1}(x)\\ &= \int_{\partial^* E}\theta(\nu_E(x))\,d\mathcal{H}^{N-1}(x).\end{aligned}$$

This ends the proof of the theorem.

The Theorem 4.1.1 shows that $\mathscr{F}$ is a suitable candidate for the Γ-limit of $\mathscr{F}_\varepsilon$ with respect to the L^1 convergence. Indeed, it proves that fixed $E \in \mathcal{P}_N(\Omega)$, then the constant sequence $E_\varepsilon = E$ is a recovery sequence for E with respect to the L^1 topology and that

$$\limsup_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E) \leqslant \mathscr{F}(E).$$

4.4 Lower bound (and the lack of it)

Now we prove Theorem 4.1.2. In order to prove that $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ with respect to the L^1 convergence, we should now prove that fixed any sequence $(E_\varepsilon)_\varepsilon$ converging in L^1 to E it holds

$$\mathscr{F}(E) \leqslant \liminf_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E_\varepsilon).$$

Unfortunately, we are not able to prove this lower bound estimate with respect to the L^1 convergence. Indeed, the classical methods to prove this kind of estimates (such as the slicing method or the blow up method) fail due to the non-local nature of the functional $\mathscr{F}$. We are, instead, able to prove the Γ-liminf with respect to a weaker type of convergence, namely the $C^{1,1}$ convergence.

Definition 4.4.1 (Uniformly $C^{1,1}$-regular set)**.** Given $E \subset \mathbb{R}^N$ we say that E is a *uniformly $C^{1,1}$-regular* set in Ω if there exist $L, \delta > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subseteq \mathbb{R}^{N-1}$ open and a function $u^x \in C^{1,1}(D^x)$ such that:

- $\partial E \cap \Omega \cap B^N_\delta(x)$ is the graph of u^x;
- $\|\nabla u^x\|_\infty \leqslant L$.

In other words a uniformly $C^{1,1}$-regular set is a set whose boundary can be locally parametrized by a Lipschitz continuous function. The set of uniformly $C^{1,1}$-regular sets in Ω is contained in $\mathcal{P}_N(\Omega)$, i.e. in the set of sets of finite perimeter in Ω. On the set of uniformly $C^{1,1}$-regular sets in Ω we consider the following convergence.

Definition 4.4.2. Given a sequence $(E_\varepsilon)_\varepsilon$ of sets of $\mathbb{R}^N$ we say that E_h converges to $E \subset \mathbb{R}^N$ if there exist $\delta, L > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subseteq \mathbb{R}^{N-1}$ open and functions $u^x_h, u^x \in C^{1,1}(D^x)$ such that:

- $\partial E_h \cap \Omega \cap B^N_\delta(x), \partial E \cap \Omega \cap B^N_\delta(x)$ are the graphs of u^x_h, u^x respectively;
- $\|\nabla u^x_h\|_\infty \leqslant L$ and $\|\nabla u^x\|_\infty \leqslant L$;
- $u^x_h \to u^x$ uniformly in $C^1(D^x)$.

With these notions in mind we are able to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. We recall that for any $x \in D$, the unit vector

$$\nu_h(x) = \frac{(-\nabla u_h(x), 1)}{\sqrt{1 + |\nabla u_h(x)|^2}}$$

coincides with the exterior unit normal to $\partial^* E_h$ at $(x, u_h(x))$ (see Definition 1.1.27). Moreover, for any $\eta > 0$ small enough, we define

$$D^\eta := \{x \in D : d(x, \partial D) > \eta\}.$$

Clearly enough, $D^\eta \nearrow D$ in L^1 as $\eta \to 0^+$. Finally, for $z \in \mathbb{R}^N$, we use the notation $z := (\bar{z}, z^N)$. We divide the proof in five steps.

STEP 1: First we prove that for any $\sigma > 0$, $x \in D^{3\sigma}$ and $h \in \mathbb{N}$ with $\varepsilon_h < \sigma$ it holds

$$\overline{B^{N-1}_2(0)} \subset \frac{x - D^\sigma}{\varepsilon_h}. \tag{4.4.3}$$

Indeed, if $x \in D^{3\sigma}$ then $\overline{B^{N-1}_{2\sigma}(x)} \subset D^\sigma$. If we assume $z \in \mathbb{R}^{N-1}$ and $|z| \leqslant 2$, then it follows that $|x - \varepsilon_h z - x| \leqslant 2\varepsilon_h < 2\sigma$, which in turns implies that $x - \varepsilon_h z \in D^\sigma$ and so (4.4.3) follows.

STEP 2: Fixed $x \in D^{3\sigma}$, $s \in [0,1]$ and $\xi \in \mathbb{R}^{N-1}$ we define

$$a_h(x, s, \xi) := \frac{u_h(x) - u_h(x + \varepsilon_h s\overline{\nu_h(x)} - \varepsilon_h \xi)}{\varepsilon_h} + s\nu_h(x)^N.$$

We claim that

$$\lim_{h \to +\infty} a_h(x, s, \xi) = \nabla u(x) \cdot (\xi - s\overline{\nu(x)}) + s\nu(x)^N. \tag{4.4.4}$$

Indeed,

$$\begin{aligned}
&\frac{u_h(x + \varepsilon_h s\overline{\nu_h(x)} - \varepsilon_h \xi) - u_h(x)}{\varepsilon_h} \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \frac{d}{dt} u_h(x + t(s\overline{\nu_h(x)} - \xi))\, dt \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u_h(x + t(s\overline{\nu_h(x)} - \xi)) \cdot (s\overline{\nu_h(x)} - \xi)\, dt \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u_h(x + t(s\overline{\nu_h(x)} - \xi)) - \nabla u(x + t(s\overline{\nu_h(x)} - \xi))) \cdot (s\overline{\nu_h(x)} - \xi)\, dt \\
&\qquad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u(x + t(s\overline{\nu_h(x)} - \xi)) - \nabla u(x + t(s\overline{\nu(x)} - \xi))) \cdot (s\overline{\nu_h(x)} - \xi)\, dt \\
&\qquad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u(x + t(s\overline{\nu(x)} - \xi)) \cdot (s\overline{\nu_h(x)} - \xi)\, dt =: I_1 + I_2 + I_3.
\end{aligned}$$

Concerning the first integral, we have

$$|I_1| \leqslant (s + |\xi|)\|\nabla u_h - \nabla u\|_\infty \to 0 \;\text{ as } h \to +\infty.$$

For the second integral we have that if L is the Lipschitz constant of ∇u, then

$$I_2 \leqslant L(s + |\xi|)\|\overline{\nu_h} - \overline{\nu}\|_\infty \to 0 \;\text{ as } h \to +\infty.$$

Finally, for the third integral, let $g(t) := \nabla u(x + t(s\overline{\nu(x)} - \xi))$. We observe that g is continuous, so

$$\lim_{h\to+\infty} \frac{1}{\varepsilon_h}\int_0^{\varepsilon_h} g(t)\, dt = g(0)$$

and from this follows that

$$\lim_{h\to+\infty} I_3 = \nabla u(x) \cdot (s\overline{\nu(x)} - \xi),$$

as claimed.

STEP 3: We define $M = \sup_h \|u_h\|_\infty$ and fix $\sigma \in (0, M/2)$. We prove now that, for any $h \in \mathbb{N}$ with $\varepsilon_h < \sigma$, it holds

$$\mathscr{F}_{\varepsilon_h}(E_h) \geqslant \int_{D^{3\sigma}} \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{a_h(x,s,\xi)}^1 G(\xi,\eta)\, d\eta\, d\xi\right) ds \sqrt{1 + |\nabla u_h(x)|^2}\, dx. \quad (4.4.5)$$

First we notice that

$$\{z \in E_h^c : B_{\varepsilon_h}(z) \cap E_h \neq \emptyset\} \supset \{(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N) : x \in D^\sigma,\, r \in (0, \varepsilon_h)\},$$

It follows that,

$$\mathscr{F}_{\varepsilon_h}(E_h) \geqslant \frac{1}{\varepsilon_h} \int_{D^{3\sigma}} \int_0^{\varepsilon_h} f\left(G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N)\right)\, dr \sqrt{1 + |\nabla u_h(x)|^2}\, dx.$$

We concentrate now on the term $G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N)$ and we rewrite it in a suitable way by performing some changes of variables. First, by noticing that $E_h = \{(z,w) \in D \times \mathbb{R} : 0 \leqslant w \leqslant u_h(z)\}$, we can get

$$\begin{aligned}
&G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N) \\
&\geqslant \int_{D^\sigma} \frac{1}{\varepsilon_h^N} \int_0^{u_h(z)} G\left(\frac{x - r\overline{\nu_h(x)} - z}{\varepsilon_h}, \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}\right) dw\, dz.
\end{aligned}$$

We now perform the change of variables in the following order

$$\eta = \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}, \quad \xi = \frac{x + r\overline{\nu_h(x)} - z}{\varepsilon_h},$$

obtaining

$$G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N) \geqslant \int_{\frac{x + r\overline{\nu_h(x)} - D^\sigma}{\varepsilon_h}} \int_{a_h(x, r/\varepsilon_h, \xi)}^{\frac{u_h(x) + r\nu_h(x)^N}{\varepsilon_h}} G(\xi,\eta)\, d\eta\, d\xi.$$

Since f is non-decreasing then by the change of variable $r = \varepsilon_h s$ we arrive to

$$\mathscr{F}_{\varepsilon_h}(E_h) \geqslant \int_{D^{3\sigma}} \int_0^1 f\left(\int_{\frac{x-D^\sigma}{\varepsilon_h} + s\overline{\nu_h(x)}} \int_{a_h(x,s,\xi)}^{\frac{u_h(x)}{\varepsilon_h} + s\nu_h(x)^N} G(\xi,\eta)\, d\eta\, d\xi \right) ds \sqrt{1 + |\nabla u_h(x)|^2}\, dx.$$

By (4.4.3) we deduce that for any $x \in D^{3\sigma}$ and for any $s \in [0,1]$

$$\frac{x - D^\sigma}{\varepsilon_h} + s\overline{\nu_h(x)} \supset \overline{B_2^{N-1}(0)} + s\overline{\nu_h(x)} \supset \overline{B_1^{N-1}(0)}.$$

Furthermore, using $\sigma < M/2$, we also get

$$\frac{u_h(x)}{\varepsilon_h} + s\nu_h(x)^N > 1.$$

As a consequence, recalling that G is supported on $\overline{B_1^N(0)}$, we obtain (4.4.5).

STEP 4: Passing to the limit as $h \to +\infty$ in (4.4.5), using Fatou's Lemma (4.4.4) and the Dominated Convergence Theorem we obtain

$$\begin{aligned}
&\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \\
&\qquad \geqslant \int_{D^{3\sigma}} \int_0^1 f\left(\liminf_{h\to+\infty} \int_{\overline{B_1^{N-1}(0)}} \int_{a_h(x,s,\xi)}^1 G(\xi,\eta)\, d\eta\, d\xi \right) ds \sqrt{1 + |\nabla u(x)|^2}\, dx \\
&\qquad = \int_{D^{3\sigma}} \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)}) + s\nu(x)^N}^1 G(\xi,\eta)\, d\eta\, d\xi \right) ds \sqrt{1 + |\nabla u(x)|^2}\, dx.
\end{aligned}$$

Since σ is arbitrary, for σ small enough we get

$$\begin{aligned}
&\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \\
&\qquad \geqslant \int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)}) + s\nu(x)^N}^1 G(\xi,\eta)\, d\eta\, d\xi \right) ds \sqrt{1 + |\nabla u(x)|^2}\, dx.
\end{aligned}$$

STEP 5: We conclude the proof by showing that

$$\int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)}) + s\nu(x)^N}^1 G(\xi,\eta)\, d\eta\, d\xi \right) ds \sqrt{1 + |\nabla u(x)|^2}\, dx = \mathscr{F}(E).$$

First, we observe that

$$\eta = \nabla u(x) \cdot (\xi - s\overline{\nu(x)}) + s\nu(x)^N = \nabla u(x) \cdot \xi + s\sqrt{1 + |\nabla u(x)|^2}$$

is the equation of an affine hyperplane in $\mathbb{R}^N$ orthogonal to $\nu(x)$ whose distance from the origin is

$$\frac{s\sqrt{1 + |\nabla u(x)|^2}}{\sqrt{1 + |\nabla u(x)|^2}} = s.$$

It follows that,

$$\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)})+s\nu(x)^N}^{1} G(\xi,\eta)\,d\eta\,d\xi = \int_{\{z\cdot\nu_E(x,u(x))\geqslant s\}} G(z)\,dz$$

from which

$$\begin{aligned}
&\int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)})+s\nu(x)^N}^{1} G(\xi,\eta)\,d\eta\,d\xi\right) ds\ \sqrt{1+|\nabla u(x)|^2}\,dx \\
&\quad = \int_D \int_0^1 f\left(\int_{\{z\cdot\nu_E(x,u(x))\geqslant s\}} G(z)\,dz\right) ds\ \sqrt{1+|\nabla u(x)|^2}\,dx \\
&\quad = \int_{\partial E} \int_0^1 f\left(\int_{\{z\cdot\nu_E(y)\geqslant s\}} G(z)\,dz\right) ds\,d\mathcal{H}^{N-1}(y) = \mathscr{F}(E)
\end{aligned}$$

and this concludes the proof. □

It is clear that we can extend this theorem to the uniformly $C^{1,1}$-regular sets and thus prove that for any sequence of $(E_h)_h$ of uniformly $C^{1,1}$-regular sets converging to a set E with respect to the corresponding notion of convergence, then it holds

$$\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \geqslant \mathscr{F}(E).$$

Observing that the uniformly $C^{1,1}$ convergence is stronger than the L^1 convergence then we have proved that $\mathscr{F}_\varepsilon \xrightarrow{\Gamma} \mathscr{F}$ with respect to the uniformly $C^{1,1}$ convergence.

To end this section, we would like to point out that the proof of Theorem 4.1.1 becomes much more straightforward if we suppose that E is a $C^{1,1}$-regular set in Ω. We report the proof for the readers convenience. For such sets, the following geometric property holds true (for details see [142, Section I.2]): there exists $r > 0$ such that the map

$$\Psi_r \colon \partial E \times [0,r] \to \{y \in E^c : d(y,\partial E) \leqslant r\}, \quad \Psi_r(x) = x + t\nu_E(x)$$

is a $C^{1,1}$-diffeomorphism. Thus, performing the change of variable $x = y - \varepsilon z$, we have

$$\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon}\int_{\{y\in E^c : d(y,\partial E)\leqslant\varepsilon\}} f\left(\frac{1}{\varepsilon^N}\int_E G\left(\frac{y-x}{\varepsilon}\right)dx\right)\,dy \\
&= \frac{1}{\varepsilon}\int_{\{y\in E^c : d(y,\partial E)\leqslant\varepsilon\}} f\left(\int_{\frac{y-E}{\varepsilon}} G(z)dz\right)\,dy \\
&= \frac{1}{\varepsilon}\int_{\Psi_\varepsilon(\partial E\times[0,\varepsilon])} f\left(\int_{\frac{y-E}{\varepsilon}} G(z)dz\right)\,dy.
\end{aligned}$$

For any $(x,t) \in \partial E \times [0,\varepsilon]$ let $J_\varepsilon(x,t) = |\det D\Psi_\varepsilon(x)|$. Then, using $t = \varepsilon s$, we get

$$\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon}\int_{\partial E}\int_0^\varepsilon f\left(\int_{\frac{x-E}{\varepsilon}+\frac{t}{\varepsilon}\nu_E(x)} G(z)dz\right)\,J_\varepsilon(x,t)\,d\mathcal{H}^{N-1}(x)\,dt \\
&= \int_{\partial E}\int_0^1 f\left(\int_{\frac{x-E}{\varepsilon}+s\nu_E(x)} G(z)dz\right)\,J_\varepsilon(x,\varepsilon s)\,d\mathcal{H}^{N-1}(x)\,ds.
\end{aligned}$$

By the regularity of E we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(x,\varepsilon s) = 1,$$

from which, applying again (4.2.1),

$$\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E) = \int_{\partial E}\int_0^1 f\left(\int_{\{v\cdot\nu_E(x)\geqslant t\})} G(z)\,dz\right)\,dt\,d\mathcal{H}^{N-1}(x) = \mathscr{F}(E).$$

4.5 Compactness

In this subsection we will address the issue of the compactness of the sequences along which we test the functionals. Our interest in such a problem is twofold. On one hand, in order to apply Theorem 1.2.6, we have to ensure that any sequence of minimizers has a limit point, while on the other hand we need to ensure that such a sequence of minimizers actually exists. As anticipated in the first section, the existence of minimizers for integral functionals is strictly tied to the coercivity of that functional and thus, in our case, to the equicoercivity of the whole sequence of functionals.

In order to investigate this issues we consider a sequence $(E_\varepsilon)_\varepsilon \subset \mathcal{P}_N(\Omega)$ such that

$$\limsup_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E_\varepsilon) \leqslant c < +\infty, \quad c\in\mathbb{R}.$$

On one hand, if we assume that the integrand f has linear growth, i.e. that there exists $\mu\in\mathbb{R}$ such that

$$\mu|\xi| \leqslant f(\xi)$$

then we can prove that $(\mathscr{F}_\varepsilon)_\varepsilon$ is equicoercive with respect to the strong topology of L^1. Indeed, we have

$$\frac{\mu}{\varepsilon}\int_{E_\varepsilon^c\cap\Omega} G_\varepsilon * \chi_{E_\varepsilon\cap\Omega}\,dx \leqslant \mathscr{F}_\varepsilon(E_\varepsilon) \leqslant c.$$

From [66, Lemma A.2] we have that the functional

$$\mathscr{E}_\varepsilon(E) := \frac{\mu}{\varepsilon}\int_{E^c\cap\Omega} G_\varepsilon * \chi_{E\cap\Omega}\,dx,$$

is lower semicontinuous on $\mathcal{P}_N(\Omega)$ with respect to the strong $L^1(\Omega)$ topology. Thus, $\mathscr{F}_\varepsilon$ is equicoercive. Moreover, from [66, Lemma A.2] (see also [2, Thm. 3.1]), it follows that $(E_\varepsilon)_\varepsilon$ must be precompact with respect to the strong L^1 topology and that its limit points must be sets of finite perimeter in Ω.

On the other hand, we cannot expect compactness with respect to the topology we endowed on the set of uniformly $C^{1,1}$-regular sets. This is because, in order to have such property, we need for every sequence of functions parametrizing the boundary of the sets E_ε to be compact, but Ascoli-Arzelà Theorem states as follows.

Theorem 4.5.1 (Ascoli-Arzelà, [134]). *Let X be a compact topologiacal space. Given a sequence of functions $(u_\varepsilon)_\varepsilon \subset C(X)$ pointwise bounded and equicontinuous, then $(u_\varepsilon)_\varepsilon$ is uniformly bounded and there exists a subsequence that converges uniformly.*

We recall that a sequence of functions $(u_k)_k$ on $\mathbb{R}$ is said to be equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every k it holds

$$|x - y| < \varepsilon \implies |u(x)_k - u(y)_k| < \delta.$$

As far as we know, there is no assumption on f such that it ensures the equicontinuiuty of the functions parametrizing the boundary of each E_ε or their pointwise boundedness.

4.6 Lower semicontinuity and Some Examples

In this section we prove Theorem 4.1.3. Moreover, we also show some example of $\mathscr{F}$.

Proof. Proof of Theorem 4.1.3 It is well known (see for instance [3, Theorem. 5.14]) that is sufficient to check that the positively one-homogeneous extension of θ given by

$$\tilde{\theta}(v) = \begin{cases} |v|\theta\left(\dfrac{v}{|v|}\right)\,dt & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

is convex. First, we notice that for each $v \in \mathbb{R}^N$ with $v \neq 0$ we have

$$\theta\left(\frac{v}{|v|}\right) = \int_0^1 f\left(\int_{\{z\cdot v \geqslant |v|t\}} G(z)\,dz\right)\,dt \overset{|v|t=s}{=} \frac{1}{|v|}\int_0^{|v|} f\left(\int_{\{z\cdot v\geqslant s\}} G(z)\,dz\right)\,ds,$$

from which we obtain

$$\tilde{\theta}(v) = \begin{cases} \displaystyle\int_0^{|v|} f\left(\int_{\{z\cdot v\geqslant s\}} G(z)\,dz\right)\,ds & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

It is easy to see that $\tilde{\theta}$ is convex. Indeed, since f is convex, there exist $(\alpha_h), (\beta_h)$ such that

$$f = \lim_{h\to+\infty} f_h \text{ uniformly on compact sets, where } f_h(t) = \alpha_h t + \beta_h.$$

For any $h \in \mathbb{N}$ let

$$\tilde{\theta}_h(v) = \begin{cases} \displaystyle\int_0^{|v|} f_h\left(\int_{\{z\cdot v\geqslant s\}} G(z)\,dz\right)\,ds & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Since $f_h \to f$ uniformly on $[0,1]$ then $\tilde{\theta}_h \to \tilde{\theta}$ pointwise. In order to conclude, it is sufficient to show that $\tilde{\theta}_h$ is convex. For any $v \neq 0$ we let $\hat{v} = \frac{v}{|v|}$. Then

$$
\begin{aligned}
\tilde{\theta}_h(v) &= \alpha_h \int_0^{|v|} \int_{\{z\cdot v \geqslant s\}} G(z)\,dz\,ds + \beta_h |v| \\
&= \alpha_h \int_0^{|v|} \int_{\{\bar{z}\cdot v = 0\}} \int_{s/|v|}^{+\infty} G(\bar{z} + t\hat{v})\,d\bar{z}\,dt\,ds + \beta_h |v| \\
&= \alpha_h \int_0^{+\infty} \int_{\{\bar{z}\cdot v = 0\}} \int_0^{t|v|} G(\bar{z} + t\hat{v})\,ds\,dt\,d\bar{z} + \beta_h |v| \\
&= \alpha_h |v| \int_0^{+\infty} \int_{\{\bar{z}\cdot v = 0\}} tG(\bar{z} + t\hat{v})\,dt\,d\bar{z} + \beta_h |v| \\
&= \alpha_h \int_{\{\bar{z}\cdot v \geqslant 0\}} G(z) z\cdot v\,dz + \beta_h |v| \\
&= \frac{\alpha_h}{2} \int_{\mathbb{R}^N} G(z)|z\cdot v|\,dz + \beta_h |v|
\end{aligned}
$$

where the last equality follows since G is even. Notice that the last expression is convex in v and this ends the proof. $\square$

Now we will present some examples of $\mathscr{F}$. This will show that our Γ-limit is consistent with the existent literature on the topic and that, for a particular choice of f, then $\mathscr{F}$ is indeed the Γ-limit with respect to the strong L^1 convergence.

G radially symmetric

Assume $G(z) = g(|z|)$ for some $g\colon [0,+\infty) \to \mathbb{R}$. Take $\nu \in \mathbb{S}^{N-1}$ and $t \geqslant 0$. Notice that the quantity

$$
\int_{\{z\cdot\nu \geqslant t\}} G(z)\,dz
$$

does not depend on ν. Take now $E \in \mathcal{P}_N$ and $x \in \partial^* E$. We have

$$
\int_0^1 f\left(\int_{\{z\cdot\nu_E(x) \geqslant t\}} G(z)\,dz\right)dt = c
$$

where c is a constant that depends only on N, f and G. Then

$$
\mathscr{F}(E) = c\,\mathcal{H}^{N-1}(\partial^* E).
$$

The case $f(t) = t$

When f is the identity function for any $E \in \mathcal{P}_N$ and for any $x \in \partial^* E$ we have

$$
\begin{aligned}
\theta(\nu_E(x)) &= \int_0^1 \int_{H_{\nu_E(x)}+t\nu_E(x)} G(z)\,dz\,dt \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} \int_t^1 G(\bar{z}+s\nu_E(x))\,ds\,d\bar{z}\,dt \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} \int_0^s G(\bar{z}+s\nu_E(x))\,dt\,d\bar{z}\,ds \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} s\,G(\bar{z}+s\nu_E(x))\,d\bar{z}\,ds \\
&= \int_{H_{\nu_E(x)}} G(z) z\cdot\nu_E(x)\,dz \\
&= \frac{1}{2}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,dz.
\end{aligned}
$$

Then the limit $\mathscr{F}$ is given by

$$
\mathscr{F}(E) = \frac{1}{2}\int_{\partial^* E}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,dz\,d\mathcal{H}^{N-1}(x).
$$

This is in accordance to [130]. Moreover, if $N > 1$ and G is radially symmetric we have, if $g\colon [0,+\infty) \to \mathbb{R}$ is such that $G(z) = g(|z|)$,

$$
\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,dz &= \frac{1}{2}\int_{\mathbb{R}^N} g(|z|)|z\cdot\nu_E(x)|\,dz \\
&= \frac{1}{2}\int_0^{+\infty}\int_{\mathbb{S}^{N-1}} g(r)r|\xi\cdot\nu_E(x)|\,dr\,d\mathcal{H}^{N-1}(\xi) \\
&= |B_1^{N-1}(0)|\int_0^{+\infty} g(r)r\,dr \\
&= \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}\int_{\mathbb{R}^N} G(z)|z|\,dz
\end{aligned}
$$

since it is well known that for any $\nu \in \mathbb{S}^{N-1}$ it holds

$$
\frac{1}{2}\int_{\mathbb{S}^{N-1}} |\xi\cdot\nu|\,d\mathcal{H}^{N-1}(\xi) = |B^{N-1}(0)|.
$$

We thus deduce that

$$
\mathscr{F}(E) = c_{N,G}\,\mathcal{H}^{N-1}(E), \quad c_{N,G} = \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}\int_{\mathbb{R}^N} G(z)|z|\,dz,
$$

and this is in accordance to [66].

Part IV

The Obstacle Problem

The Obstacle Problem appeared in the mathematical literature in the work by Stampacchia [141] in the special case $\psi = \chi_E$; in an earlier independent work, Fichera [72] solved the first unilateral problem, the so-called Signorini problem in elastostatics. The regularity of the solutions to the obstacle problem are strongly related to the regularity of the obstacle. To quote a few results in this direction we recall [56] in the setting of Morrey and Campanato spaces and [57], [59], [14], [123] in the setting of nonstandard growth conditions (see also [58], [15], [124] for Calderón-Zygmund case). In this thesis we are concerned with the following formulation of the Obstacle Problem:

$$\min_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega F(x, Dv(x))\, dx,$$

where $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary, $F : \mathbb{R}^n \longrightarrow [0, +\infty)$ is a C^1 function and $u_0 \in W^{1,p}(\Omega)$ is a boundary data such that $F(\cdot, Du_0) \in L^p(\Omega)$. Moreover, the function $\psi \in W^{1,p}_{u_0}(\Omega)$ is called *obstacle* and it is such that $F(\cdot, D\psi) \in L^p(\Omega)$, with $p \geqslant 1$. Finally, the class of the admissible functions $\mathbb{K}_\psi(\Omega)$ is defined as

$$\mathbb{K}_\psi(\Omega) := \left\{ v \in W^{1,p}_{u_0}(\Omega) \,:\, v \geqslant \psi \text{ a.e. on } \Omega,\, F(\cdot, Dv) \in L^p(\Omega) \right\}.$$

If F is superlinear and lower semicontinuous, the existence of a solution to the Obstacle Problem follows from the Direct Method of Calculus of Variation (see Chapter 2). Moreover, if $F(x, \xi)$ is strictly convex with respect to ξ the solution to the problem is unique (see [45, Theorem 3.30]). In the uncostrained case, i.e. for the classical problem of Calculus of Variations, it well known that under suitable hypotheses the minimizers are also solutions to the Euler-Lagrange equation. In particular the solution to the Euler-Lagrange equation are called *Extremals*.

Definition 4.6.1. Fixed a boundary data $u_0 \in W^{1,p}(\Omega)$ consider the problem

$$\min_{v \in W^{1,p}_{u_0}(\Omega)} \int_\Omega F(x, Dv(x))\, dx,$$

for a C^1 function F. A mapping $u \in W^{1,p}_{u_0}(\Omega)$ is called *Extremal* if $F(\cdot, Du) \in L^p(\Omega)$ and

$$\int_\Omega F'(x, Du) \cdot D\phi\, dx = 0, \quad \text{for any } \phi \in C_0^\infty(\Omega),$$

where with F' we denote the derivative of F with respect to the second variable.

In the case of the obstacle problem the definition must be modified to adapt to the additional constraint imposed by the presence of the obstacle.

Definition 4.6.2. Consider the problem Obstacle Problem defined above. A mapping $u \in \mathbb{K}_\psi(\Omega)$ is called *Extremal* if $F(\cdot, Du) \in L^p(\Omega)$ and

$$\int_\Omega F'(x, Du) \cdot D\phi\, dx \geqslant 0, \quad \text{for any } \phi \in C_0^\infty(\Omega), \phi ge 0$$

There are many works about regularity theory in variational problems and elliptic systems with non-standard growth (see Definition 1.1.14), but the papers which paved the way were the famous [111] and [112] by Marcellini; since they were published, a lot of new ideas have been applied to this research branch and many results have been proved in several directions (see for example [114] by Marcellini, or [118] by Mingione for a general exposition and further references). However, regarding the obstacle problems there are still some issues which have not been studied in an exhaustive way yet. In this thesis we are concerned with two aspects of the regularity of solutions: the extremality of the solutions and their higher differentiability.

In **Chapter 5** we deal with the relation between minima and extremals: it is common knowledge that, for both the constrained and unconstrained problems, the regularity of the solutions often comes from the fact that the minimizers are extremals too, i.e. they solve a corresponding variational equality or inequality. Note though that there are examples of variational problems whose minimizers do not satisfy the Euler-Lagrange equation in the weak sense, as proved by Ball and Mizel in [8]. While in the case of standard growth conditions the situation is well established (see for instance the book [45] by Dacorogna), in the case of non-standard growth conditions, the relation between extremals and minima is an issue that requires a careful investigation. In 2014 Carozza, Kristensen and Passarelli di Napoli investigated exactly this topic in [31] in the case of convex integral functionals, with the aim of showing that their minimizers are characterized to be the energy solutions to the Euler-Lagrange systems for the functionals under non-standard growth conditions. The main tool they use is a particular regularization procedure: the integrand F is approximated by a sequence of strictly convex and uniformly elliptic integrands F_k which satisfy standard $p-$growth conditions and whose minimizers u_k strongly converge to the minimizer u in $W^{1,p}$. With that said, according to the standard duality theory for convex problems, every such minimizer u_k is associated to a row-wise solenoidal matrix field denoted by σ_k. Finally, for the pairing (Du_k, σ_k), suitable pointwise estimates that are preserved while passing to the limit are proved. Such estimates then provide conditions which allow the Euler-Lagrange system to hold for an F-minimizer. In a subsequent paper, i.e. [32], the same achievement has been carried on under more general growth assumptions, covering a wide class of functionals, from those with almost-linear growth to the ones with exponential growth and beyond, by the use of Ekeland variational principle (see Theorem 1.1.20) and Young measures, to obtain the necessary estimates for the pairing (Du_k, σ_k) to be able to pass to the limit. In [31], [32], by Carrozza, Kristensen and Passarelli di Napoli, and [63], by Eleuteri and Passarelli di Napoli, the concept of convex duality is exploited. While this seems to be very natural, its use is not so common in the context of convex variational integrals with non-standard growth conditions. We should note though that, before these papers, also in [18] and [19] the authors (respectively Bonfanti and Cellina for the first one and Bonfanti, Cellina and Mazzola for the second) make use of it and, in particular, in [19] is also addressed the question of energy-extremality of minimizers. They work in the context of more general variational integrals $v \longmapsto \int_\Omega F(x, v, Dv)\, dx$ in the multi-dimensional scalar case with $n \geqslant 2$, $N = 1$ under convexity and regularity hypotheses on the gradient. Talking about more general functionals, we remark that the results obtained in [31] can be generalized to minimizers of the general autonomous convex variational integral $\int_\Omega F(v, Dv)\, dx$, just under the hypothesis that the integrand $F = F(\eta, \xi)$ is jointly convex. Similar remarks, together with the precise statements and sketches of the proofs, were given in [30] by the same authors. Eventually, in the paper [63] Eleuteri and Passarelli di Napoli address the analogue issue of [31] in the case of constrained minimizers with a very general obstacle quasi-continuous up to a subset of zero capacity. Let us mention that relying on techniques of convex analysis, Scheven and Schmidt in [136] and [137] investigated the Dirichlet minimization problem for the total variation and the area functional with one-sided obstacle. The main point is that they were able to identify certain dual maximization problems for bounded divergence-measure fields and to establish duality formulas and point-wise relations between (generalized) BV minimizers and dual maximizers. Their results are very general and apply to very general obstacles, such as BV obstacles and the obstacle considered in [63]; the proofs of their results crucially depend on a new version of the Anzellotti-type pairing (see [6]) which involves general divergence measure fields and specific representatives of BV functions, by employing several fine results on capacities and one-sided approximation. This framework is proved to be the right one in order to extend the results in [31] to very general obstacle problems, as long as, by means of the Anzellotti-type pairing, they are able to express the natural counterpart of the variational inequality in this very general setting, which will reduce to the usual one once they have the right summability for the functions involved.

In **Chapter 6** we are interested in higher differentiability results for solutions to obstacle problems. Such a topic has been object of intense interest in the last years and as we already remarked, it has been usually observed that the regularity of the obstacle influences the regularity of the solutions to the problem: for linear problems the solutions are as regular as the obstacle; this is no longer the case in the nonlinear setting for general integrands without any specific structure. Hence along the years, there has been an intense research activity in which extra regularity has been imposed on the obstacle to balance the nonlinearity (see [17, 16, 75, 76]). In [34, 36, 62, 78, 79, 82, 90, 109, 144] the authors analyzed how an extra differentiability of integer or fractional order of the gradient of the obstacle provides an extra differentiability to the gradient of the solutions, also in case of standard growth. However, since no extra differentiability properties for the solutions can be expected even if the obstacle ψ is smooth, unless some assumption is given on the x-dependence of the operator $D_\xi F(x, \xi)$, the higher differentiability results for the solutions of systems or for the minimizers of functionals in the case of unconstrained problems (see [7, 40, 42, 60, 80, 81, 86, 87, 88, 126, 127]) have been useful and source of inspiration also for the constrained case. Differentiability results for solutions defined by duality when the coefficients are in $W^{1,n}$ can be found in [104]. For higher regularity results for solutions to non-autonomous elliptic problems, we also refer to [102], and to the recent paper [49], where both unconstrained and constrained problems are treated and the optimal assumption for the obstacle is given in order to get the Lipschitz regularity for the solutions. It is well known that, for unconstrained problems with (p, q)-growth, the boundedness of the minimizers can play a crucial role in order to get regularity for the gradient, under weaker assumptions on the gap between p and q and on the data of the problem (see [30]). In this Chapter we prove that the same phenomenon happens for the bounded solutions to obstacle problems with (p, q)-growth.

Chapter 5

A Necessary Condition for the Extremality of Solution to the Obstacle Problem with General Growth

5.1 Setting and Main Results

This Chapter is divided in two parts: in the first part we will be concerned with the autonomous problem, while in the second part we will study the non-autonomous case. In this Chapter we aim to extend the results of [32] in the constrained optimization problem, but we make use of different hypotheses on the obstacle and on the Lagrangian, in particular a superlinear growth condition at infinity and convexity guaranteed by the hypotheses on the function that bounds the Lagrangian from below in the autonomous case and supercorecivity and local boundedness in the non-autonomous one. As in [31], [32] and [63], we use the concept of convex duality in various steps of the proof. The most challenging steps of the proof under our hypotheses are the passage to the limit, where we have to pay attention to the presence of the obstacle, and the proof of the variational inequality.

The aim of this Chapter is therefore to investigate the relation between minimizers of the obstacle problem and its extremals. In particular we find a necessary condition for the extremality of solutions of the obstacle problem. We consider the autonomous obstacle problems of the form

$$\min_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega F(Dv(x))\, dx, \tag{5.1.1}$$

where Ω is an open and bounded subset of $\mathbb{R}^n$ and $\mathbb{K}_\psi(\Omega)$ is the class of admissible functions, defined as

$$\mathbb{K}_\psi(\Omega) = \left\{ v \in W^{1,1}_{u_0}(\Omega) \,:\, v \geqslant \psi \text{ a.e. on } \Omega,\ F(Dv) \in L^1(\Omega) \right\}, \tag{5.1.2}$$

where $u_0 \in W^{1,1}(\Omega)$ is a boundary data such that $F(Du_0) \in L^1(\Omega)$ and where $\psi \in W^{1,1}(\Omega)$ is a function called *obstacle* such that $F(D\psi) \in L^1(\Omega)$. The focus of the first part of this Chapter is how minimizers of the above-defined constrained problem could be characterized, exploiting a primal-dual formulation of the optimization problem. We assume the Lagrangian satisfies a superlinear growth condition at infinity, although it is not subjected to any growth condition from above. Moreover, we assume that the Lagrangian is bounded from below by a given convex and superlinear function, that brings the Lagrangian to inherit the same degree of convexity that it has. We state now our main results. In what follows we will denote with F^* the conjugate function of a function F, in the sense of convex analysis (see (5.2.1)), and with F' the gradient of F.

Theorem 5.1.3. *Let F be a non-negative function in $C^1(\mathbb{R}^n)$, satisfying* (H1) *and* (H2) *with ϕ defined as in* (5.1.7)*, and let $u_0 \in W^{1,1}(\Omega)$ be such that $F(Du_0), F(t\,Du_0) \in L^1(\Omega)$ for some $t > 1$. Then, for the unique minimizer $u \in W^{1,1}_{u_0}(\Omega)$ of the minimum problem* (5.1.1) *it holds*

$$F^*(F'(Du)) \in L^1(\Omega), \qquad F'(Du) \cdot Du \in L^1(\Omega) \tag{5.1.4}$$

and

$$\operatorname{div} F'(Du) \leqslant 0 \quad \textit{in distributional sense.} \tag{5.1.5}$$

Moreover it holds the following identity

$$\int_\Omega F(Du)\,dx = [\![F'(Du), \psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega F^*(F'(Du))\,dx. \tag{5.1.6}$$

In order to understand this result, we need to introduce some notation. Consider a C^1 and strictly convex function $\phi : \mathbb{R}^n \longrightarrow [0, +\infty)$ such that

$$\phi(\xi) := \theta(|\xi|) \qquad \text{for all} \quad \xi \in \mathbb{R}^n \tag{5.1.7}$$

for a function $\theta : [0, +\infty) \longrightarrow [0, +\infty)$ superlinear. We will suppose that

$$F - \phi \qquad \text{is a convex function,} \tag{H1}$$

which implies that F is also strictly convex. Furthermore, we will assune that there exist $c, r \in \mathbb{R}$ such that

$$F(\xi) \geqslant \frac{1}{2}\phi(\xi) + c, \tag{H2}$$

for all $\xi \in \mathbb{R}^n$ such that $|\xi| > r$. The previous hypothesis clearly implies that F is superlinear. In order to prove our claim, the superlinearity of F would be enough. We decided to use hypothesis (H2) in order to stress the fact that F inherits the same properties of ϕ.

Consider now the following set of functions

$$S_-(\Omega) := \{\sigma \in L^1_{loc}(\Omega, \mathbb{R}^n) \,:\, \operatorname{div}\sigma \leqslant 0 \ \text{ in distributional sense}\}.$$

With $\operatorname{div}\sigma$ we denote the *distributional divergence* of σ which is defined as the measure that realize the following identity

$$\int_\Omega \varphi\, d(-\operatorname{div}\sigma) = \int_\Omega \sigma \cdot D\varphi\, dx \qquad \forall\, \varphi \in C_0^\infty(\Omega). \tag{5.1.8}$$

Moreover, we say that the distributional divergence is non positive is distributional sense if it holds true that

$$\int_\Omega \varphi\, d(-\operatorname{div}\sigma) \geqslant 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega),\ \varphi \geqslant 0.$$

Fixed $\sigma \in S_-(\Omega)$ and $u \in W^{1,1}_{u_0}(\Omega)$ we define the measure

$$[\![\sigma, u]\!]_{u_0}(\Omega) \;:=\; \int_\Omega (u - u_0)\, d(-\mathrm{div}\,\sigma) + \int_\Omega \sigma \cdot Du_0\, dx, \tag{5.1.9}$$

where, we recall, $u_0 \in W^{1,1}(\Omega)$ is the boundary data of the obstacle problem. It is important to notice that the measure $[\![\sigma, u]\!]_{u_0}(\Omega)$ may eventally take the value $+\infty$ for certain choices of $u \in W^{1,1}(\Omega)$ and $\sigma \in S_-(\Omega)$.

Remark 5.1.10. By the definition of distributional divergence follows that if $u \in W^{1,1}_{u_0}(\Omega)$ then

$$\int_\Omega (u - u_0)\, d(-\mathrm{div}\,\sigma) \;=\; \int_\Omega \sigma \cdot (Du - Du_0)\, dx.$$

thus we get that

$$[\![\sigma, u]\!]_{u_0}(\Omega) = \sigma \cdot Du \in L^1(\Omega).$$

Moreover we are also able to prove the following result.

Theorem 5.1.11. *Let $F \in C^1(\mathbb{R}^n)$ be strictly convex and Lipschitz continuous and such that there exists a solution u of the problem*

$$\min_{v \,\in \mathbb{K}_\psi(\Omega)} \int_\Omega F(Dv(x))\, dx, \tag{5.1.12}$$

where the class $\mathbb{K}_\psi(\Omega)$ is defined as in (5.1.2) *and $u_0 \in W^{1,1}(\Omega)$ is such that $F(Du_0) \in L^1(\Omega)$. Then the following holds true*

$$\min_{v \in \mathbb{K}_\psi(\Omega)} \int_\Omega F(Dv)\, dx = \max_{\sigma \in S^\infty_-(\Omega)} \left([\![\sigma, \psi]\!]_{u_0}(\Omega) - \int_\Omega F^*(\sigma)\, dx \right), \tag{5.1.13}$$

with

$$S^\infty_-(\Omega) \;:=\; \{\sigma \in L^\infty(\Omega, \mathbb{R}^n) \;:\; \mathrm{div}\,\sigma \leqslant 0 \;\; \textit{in distributional sense}\}.$$

In particular,

$$\int_\Omega F(Du)\, dx = [\![F'(Du), \psi]\!]_{u_0}(\Omega) - \int_\Omega F^*(F'(Du))\, dx. \tag{5.1.14}$$

It is important to notice that in this case, since F is Lipschitz, then F cannot be superlinear and thus the Direct Method cannot grant us the existence of a minimizer. It follows that the existence of such minimizer has to be assumed. Still, if a minimizer u exists, it will be unique since we assume F strictly convex. Moreover, we want to stress the fact that if F is Lipschitz continuous, then there is no need to prove the equivalence of (5.1.4) and (5.1.5). Indeed, if F is Lipschitz continuous, then $F' \in L^\infty(\Omega)$ and thus

$$F'(Du) \cdot Du \in L^1(\Omega).$$

Since F is convex and lower semicontinuous, then considering Fenchel's identity it follows that $F^*(F'(Du)) \in L^1(\Omega)$. The inequality (5.1.5) instead follows from a classical argument of Calculus of Variation. We write it for the reader convenience. Assume that u is the minimizer of (5.1.1). Since $\mathbb{K}_\psi(\Omega)$ is convex, then fixed $0 < h < 1$ and $v \in \mathbb{K}_\psi(\Omega)$ it follows that the convex combination

$$w = u + h(v - u)$$

is an element of $\mathbb{K}_\psi(\Omega)$. Since u is the unique minimizer of it follows that

$$\begin{aligned}
0 &\leqslant \frac{1}{h}\int_\Omega F(Dw) - F(Du)dx \\
&= \frac{1}{h}\int_\Omega F(Du + hD(v-u)) - F(Du)dx \\
&= \frac{1}{h}\int_\Omega \int_0^1 (F'(Du + hsD(v-u))) \cdot (hD(v-u))dsdx \\
&= \int_\Omega \int_0^1 (F'(Du + hsD(v-u))) \cdot (D(v-u))dsdx.
\end{aligned}$$

Since F is Lipschitz continuous then $F' \in L^\infty(\mathbb{R}^n)$ so we can use the Dominated Convergence Theorem in the last line to pass to the limit for $h \to 0$. From this we get

$$\int_\Omega F'(Du) \cdot (D(v-u))dx \geqslant 0.$$

If we choose $v = u + \eta$, with $\eta \in W^{1,1}(\Omega)$ and $\eta \geqslant 0$ we get the inequality (5.1.5).

Now we extend Theorem 5.1.3 to the non-autonomous case. The next result will further develop the study of the relation between solution to the obstacle problem and its extremals. In particular we prove a more general result and then we apply it at the case of the obstacle problem. As an added value to our main result, we refine techniques used in the proof of Theorem 5.1.3 in order to make the proof of our next result clearer in its key point. We consider the problem

$$\min_{v \in K} \int_\Omega F(x, Dv(x))\, dx, \tag{5.1.15}$$

where $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is a Carathéodory function and K is a closed and convex subset of $W^{1,1}_{u_0}(\Omega)$. For simplicity, and with minor loss of generality, we assume that $F(x, 0) = 0 = F'(x, 0)$ for every $x \in \Omega$. In order to state our main result we need to introduce a couple of definitions. We say that $G : \Omega \times \mathbb{R}^n \to [0, \infty]$ is is *uniformly superlinear at infinity* (also known as supercoercive) if

$$\lim_{\xi \to \infty} \inf_{x \in \Omega} \frac{G(x, \xi)}{|\xi|} = \infty,$$

and we denote $M_G(A) := \sup_{x \in \Omega, \xi \in A} G(x, \xi)$. We say that G is *locally bounded* if $M_G(A) < \infty$ for every bounded $A \subset \mathbb{R}^n$.

Our main result is the following

Theorem 5.1.16. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Assume that $F : \Omega \times \mathbb{R}^n \to [0, \infty)$ is strictly convex and C^1 in the second variable with continuity modulus independent of x, uniformly superlinear at infinity and locally bounded. If there exist $t > 1$*

and $w_0 \in K$ such that $F(\cdot, tDw_0) \in L^1(\Omega)$, then there exists a unique minimizer $u \in K$ of (5.1.15)*, and it satisfies*

$$F^*(\cdot, F'(\cdot, Du)) \in L^1(\Omega) \quad \textit{and} \quad F'(\cdot, Du) \cdot Du \in L^1(\Omega).$$

If $\eta + K \subset K$ for every $\eta \in C_0^\infty(\Omega)$ with $\eta \geqslant 0$ or without restriction, then

$$\mathrm{div} F'(x, Du) \leqslant 0 \quad \textit{or} \quad \mathrm{div} F'(x, Du) = 0,$$

respectively, in a distributional sense.

A couple examples of sets K which are closed and convex in $W^{1,1}(\Omega)$ and satisfy the condition $\eta + K \subset K$ of the Theorem 5.1.16 are $K = W^{1,1}_{u_0}(\Omega)$ or

$$K = \left\{ v \in W^{1,1}_{u_0}(\Omega) : v \geqslant \psi \text{ a.e. in } \Omega \right\},$$

where $\psi : \Omega \to [-\infty, \infty)$ is the *obstacle*. Moreover, the condition $F(\cdot, tDv) \in L^1(\Omega)$ for some $t > 1$ implies that $F'(\cdot, Dv) \cdot Dv \in L^1(\Omega)$ which in turn implies that $F(\cdot, Dv) \in L^1(\Omega)$; this follows from the point-wise inequality

$$F(\xi) \leqslant \xi \cdot F'(\xi) \leqslant \frac{F(t\xi) - F(\xi)}{t-1} \leqslant \tfrac{1}{t-1} F(t\xi).$$

Before proving our results, we would like to recall some notions about Functional Analysis and Generalized Young Measures.

5.2 Preliminaries

In this section we introduce some useful notations and some basic definitions of Convex and Functional Analysis and Generalized Young Measures. We denote by $C(\mathbb{R}^n)$ the space of the continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$ and by $C_0(\mathbb{R}^n)$ the completion of continuous functions with compact support with respect to the supremum norm. We denote by $M(\mathbb{R}^n)$ the space of finite Radon measures on $\mathbb{R}^n$ and, given a measure μ, we will denote its support with $\operatorname{supp} \mu$. We denote by $Pr(\mathbb{R}^n)$ the space of the probability measures defined on the Borel sets of $\mathbb{R}^n$. Finally, we denote by $L^\infty_w(\Omega, M(\mathbb{R}^n))$ the set of the essentially bounded and weakly* measurable functions (see [133]) from Ω to $M(\mathbb{R}^n)$. Throughout this Chapter, we always assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with Lipschitz boundary and that $K \subset W^{1,1}_{u_0}$ is closed and convex. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0)$ is the open ball with center x_0 and radius r. If its center is clear or irrelevant, we write $B_r = B_r(x_0)$. The *characteristic function* χ_E of $E \subset \mathbb{R}^n$ is defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. Moreover, let $f, g : E \to \mathbb{R}$ be measurable in $E \subset \mathbb{R}^n$. We denote the integral average of f over E with $0 < |E| < \infty$ by $(f)_E := \fint_E f\,dx := \frac{1}{|E|}\int_E f\,dx$. The gradient of f is denoted Df. If $E \subset \mathbb{R}$, then f is said to be *almost increasing* with constant $L \geqslant 1$ if $f(s) \leqslant Lf(t)$ whenever $s \leqslant t$. If $L = 1$, f is *increasing*. Similarly, we define an *almost decreasing* or *decreasing* function. We write $f \lesssim g$, $f \approx g$ and $f \simeq g$ if there exists $C \geqslant 1$ such that $f(y) \leqslant Cg(y)$, $C^{-1}f(y) \leqslant g(y) \leqslant Cf(y)$ and $f(C^{-1}y) \leqslant g(y) \leqslant f(Cy)$ for all $y \in E$, respectively. We use c and C as a generic constants whose value may change between appearances.

5.2.1 Functional analysis

We recall that if a convex function G is differentiable, then

$$G(\xi) \geqslant G(\zeta) + G'(\zeta) \cdot (\xi - \zeta)$$

for all $\xi, \zeta \in \mathbb{R}^n$ and G' is *monotone*, i.e.

$$(G'(\xi) - G'(\zeta)) \cdot (\xi - \zeta) \geqslant 0$$

by [121, Theorems 3.8.1 and 3.8.3]. Furthermore, the *conjugate* $G^* : \mathbb{R}^n \to [0, \infty]$ of $G : \mathbb{R}^n \to [0, \infty]$ is defined as

$$G^*(z) := \sup_{\xi \in \mathbb{R}^n} (\xi \cdot z - G(\xi)). \tag{5.2.1}$$

Fenchel's inequality (also known as Young's inequality)

$$\xi \cdot z \leqslant G(z) + G^*(\xi)$$

holds by definition. Moreover, if G is differentiable, then *Fenchel's identity*

$$G(G'(\xi)) + G^*(\xi) = \xi \cdot G'(\xi)$$

holds for all $\xi \in \mathbb{R}^n$ [121, Proposition 6.1.1]. If G is convex and lower semicontinuous, then $G^{**} = G$ [121, Theorem 6.1.2]. We will apply these and other results pointwise to functions of type $G : \Omega \times \mathbb{R}^n \to [0, \infty]$, convex in the second variable.

The many properties of the conjugate function are well known. In particular we will need the following result. This property is obtained as a corollary of [83, Lemma 3.1].

Lemma 5.2.2. *If $F : \mathbb{R}^n \longrightarrow \overline{\mathbb{R}}$ is convex, non-negative, superlinear and $F(0) = 0$ and has as effective domain the whole $\mathbb{R}^n$, then its polar F^* verifies the same properties of F.*

Remark 5.2.3. It is worth noticing that the hypothesis $F(0) = 0$ is not necessary for the proof of the Lemma, in the sense that it is only a property the polar function inherits from the original function F.

The following result connects the concepts of supercoercivity and local boundedness and the conjugate function.

Lemma 5.2.4. *If $G : \Omega \times \mathbb{R}^n \to [0, \infty]$ is locally bounded or uniformly superlinear at infinity, then the conjugate $G^* : \Omega \times \mathbb{R}^n \to [0, \infty]$ is uniformly superlinear at infinity or locally bounded, respectively.*

Proof. With the choice $\xi := r\frac{z}{|z|}$, we see that $G^*(x, z) \geqslant r\,|z| - M_G(B_r)$. If G is locally bounded this implies that $\liminf_{z\to\infty} \inf_{x\in\Omega} \frac{G^*(x,z)}{|z|} \geqslant r$. Since this holds for any $r > 0$, G^* is uniformly superlinear at infinity. If G is uniformly superlinear at infinity, then for any $k > 0$ we can choose $r > 0$ such that $\inf_{x\in\Omega} \frac{G(x,\xi)}{|\xi|} \geqslant k$ for every $|\xi| \geqslant r$. When $z \in B_k$, we obtain that

$$\begin{aligned} G^*(x, z) &= \max\left\{ \sup_{\xi\in B_r} (\xi \cdot z - G(x, \xi)), \sup_{\xi\in\mathbb{R}^n\setminus B_r} (\xi \cdot z - G(x, \xi)) \right\} \\ &\leqslant \max\left\{ r\,|z|, \sup_{\xi\in\mathbb{R}^n\setminus B_r} \big(k\,|\xi| - \inf_{x\in\Omega} G(x, \xi)\big) \right\} \leqslant kr. \end{aligned}$$

Thus G^* is locally bounded. □

5.2.2 Generalized Young measure

The theory of generalized Young measures is wide and well known. In order to prove our claim we will need only few of their property. In order to make this thesis self contained, we will now state some well known results about Generalized Young measures. For more details on Young measures and Generalized Young measures see [133].

Definition 5.2.5. Given a Carathéodory function $f : \mathbb{R}^n \to \mathbb{R}$ with linear growth we call Recession function (or Recession integrand) the function f^∞ defined as

$$f^\infty(\xi) := \lim_{t\to+\infty} \frac{f(t\,\xi)}{t},$$

if this limit exists.

Definition 5.2.6. A generalized Young measure on the open and bounded set $\Omega \subset \mathbb{R}^n$ is a triple $\nu = ((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\Omega})$ where $\nu_x \in Pr(\mathbb{R}^n)$ for a.e. $x \in \Omega$, λ is a positive finite measure on $\overline{\Omega}$ and $\nu_x^\infty \in Pr(\mathbb{S}^{n-1})$ such that

i) $x \mapsto \nu_x$ is weakly* measurable with respect to Lebesgue measure,

ii) $x \mapsto \nu_x^\infty$ is weakly* measurable with respect to λ,

iii) $\int_\Omega \int_{\mathbb{R}^n} |\xi|\, d\nu_x(\xi)\, dx < \infty$.

We refer to [133] for the definition of the weak* measuability conditions in the previous definition, since are not needed here. The next proposition contains many known results, mainly [133, Theorem 12.5] and [133, Lemma 12.14].

Proposition 5.2.7. *Let $(v_k)_k \subset W^{1,1}(\Omega)$ with $\sup_k \|\nabla v_k\|_{L^1(\Omega)} < \infty$. Then there exist a generalized Young measure $((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\overline{\Omega}})$ such that*

$$\lim_{k\to\infty} \int_\Omega F(x, \nabla v_k)\, dx = \int_\Omega \int_{\mathbb{R}^n} F(x,\xi)\, d\nu_x(\xi)\, dx + \int_{\overline{\Omega}} \int_{\partial B_1} F^\infty(x,\xi)\, d\nu_x^\infty(\xi)\, d\lambda(x)$$

for any $F \in C(\overline{\Omega} \times \mathbb{R}^n)$ with $F^\infty \in C(\overline{\Omega} \times \mathbb{R}^n)$.

Moreover, $(\nabla v_k)_k$ is equi-integrable if and only if $\lambda = 0$. Furthermore, if $\lambda = 0$ and $v_k \to v$ in $L^1(\Omega)$ then $v \in W^{1,1}(\Omega)$ and for a.e. $x \in \Omega$,

$$\nabla v(x) = \int_{\mathbb{R}^n} \xi\, d\nu_x(\xi).$$

5.3 Proof of theorem 5.1.3

The proof is divided in six steps. First we will construct a sequence of approximating functions $(F_k)_k$. Next, we will consider a family of perturbed obstacle problems generated by the approximating function $(F_k)_k$. Then we will study the asymptotic behavior of the solution of the perturbed problems. Finally, we will show that (5.1.4), (5.1.5) and (5.1.6) hold true.

STEP 1 Consider the conjugate function F^* of F. Thanks to Lemma 5.2.2, since F is superlinear, convex and $\mathcal{C}^1$, then its polar

$$F^*(z) := \sup_{\xi \in \mathbb{R}^n} (z \cdot \xi - F(\xi)) \qquad \forall\, z \in \mathbb{R}^n$$

is a real-valued, strictly convex and superlinear at infinity function. Fixed $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$, we can define

$$\overline{F}_k^{**}(\xi) := \sup_{|z| \leqslant k} (\xi \cdot z - F^*(z))\,.$$

We can observe that $\overline{F}_k^{**}$ is a real-valued, convex and k-Lipschitz function and, since F^{**} is lower semicontinuous, then

$$\overline{F}_k^{**} \overset{k\to\infty}{\longrightarrow} F^{**} = F \quad \text{pointwise,}$$

sinec F is lower semicontinuous and convex. Now we define

$$\overline{G}_k^{**}(\xi) := \max\left\{\overline{F}_k^{**}(\xi), \theta(|\xi|)\right\}. \tag{5.3.1}$$

Again, $\overline{G}_k^{**} \overset{k\to\infty}{\longrightarrow} F^{**} = F$ pointwise, but for each $k \in \mathbb{N}$ there also must exists $r_k > 0$ such that $r_k \overset{k\to\infty}{\longrightarrow} +\infty$ and such that

$$\overline{G}_k^{**}(\xi) = \theta(|\xi|)$$

when $|\xi| \geqslant r_k$. Now we define

$$H_k(\xi) := \begin{cases} \overline{G}_k^{**}(\xi) & \text{if } |\xi| < r_k, \\ \dfrac{\theta(r_k)}{r_k}\,|\xi| & \text{if } |\xi| \geqslant r_k. \end{cases}$$

Again, it is possible to prove that H_k is a convex and m_k-Lipschitz function, with

$$m_k := \frac{\theta(r_k)}{r_k}$$

for all $k \in \mathbb{N}$. Now we regularize H_k by means of the convolution kernels

$$\Phi_\varepsilon(\xi) := \varepsilon^{-n}\,\Phi\left(\frac{\xi}{\varepsilon}\right),$$

where

$$\Phi(\xi) := \begin{cases} c\,\exp\left(\dfrac{1}{|\xi|^2 - 1}\right) & \text{if } |\xi| < 1, \\ 0 & \text{if } |\xi| \geqslant 1 \end{cases}$$

and where c is chosen such that $\int_{\mathbb{R}^n} \Phi(\xi)\, d\xi = 1$. In particular, we consider the function $\Phi_\varepsilon * H_k(\xi)$ and we remark that this is a convex, $\mathcal{C}^\infty$ and m_k-Lipschitz function for which it holds

$$H_k(\xi) \leqslant \Phi_\varepsilon * H_k(\xi) \leqslant H_k(\xi) + \varepsilon\, m_k, \tag{5.3.2}$$

for each $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$. If we define

$$\delta_k := \frac{1}{k^2 m_k},$$
$$\mu_k := \frac{1}{k-1}$$

and

$$F_k(\xi) := \Phi_{\delta_k} * H_k(\xi) - \mu_k, \tag{5.3.3}$$

then it is possible to prove that it holds true that $F_k(\xi) \leqslant F_{k+1}(\xi)$ for all $k \in \mathbb{N}$ and $\xi \in \mathbb{R}^n$ (see [31]) and that $F_k \nearrow F$ pointwise as $k \to \infty$.

STEP 2 Since, by construction, the functions F_k are not superlinear, we cannot guarantee the existence of solutions to the obstacle problems

$$\min_{w \in \mathbb{K}_\psi(\Omega)} \int_\Omega F_k(Dw)\, dx.$$

In order to bypass this issue, we will use a version of Ekeland variational principal for metric spaces, namely Theorem 1.1.20. We define for every $k \in \mathbb{N}$ the integral functional

$$I_k(w) := \int_\Omega F_k(Dw)\, dx, \qquad \text{for all } w \in \mathbb{K}_\psi(\Omega).$$

For every $k \in \mathbb{N}$ we have that $F_k \leqslant F$, so

$$\inf_{w\in\mathbb{K}_\psi(\Omega)} I_k(w) \leqslant \inf_{w\in\mathbb{K}_\psi(\Omega)} I(w) = \int_\Omega F(Du)\, dx. \tag{5.3.4}$$

By definition of infimum, we can find for every $k \in \mathbb{N}$ a function $v_k \in \mathbb{K}_\psi(\Omega)$ such that

$$I_k(v_k) \leqslant \inf_{w\in\mathbb{K}_\psi(\Omega)} I_k(w) + \frac{1}{k}.$$

Noticing that for every $k \in \mathbb{N}$ holds

$$F_k(\xi) \geqslant \min\left\{\frac{\theta(r_k)}{r_k}\,|\xi|, \theta(|\xi|)\right\} \quad \text{for all } \ \xi \in \mathbb{R}^n,$$

then by (5.3.4) we get that $(v_k)_k \subset \mathbb{K}_\psi(\Omega)$ must be bounded in $W^{1,1}(\Omega)$, which implies that there exists a not relabeled subsequence $(v_k)_k$ converging weakly* in BV to some v. Moreover, by Proposition 5.2.7 we get that there exists a generalized Young measure $\nu = ((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\Omega})$ such that, for every $k \in \mathbb{N}$,

$$\lim_{j\to+\infty} \int_\Omega F_j(Dv_j)\, dx \geqslant \lim_{j\to+\infty} \int_\Omega F_k(Dv_j)\, dx \tag{5.3.5}$$
$$= \int_\Omega \int_{\mathbb{R}^n} F(x,\xi)\, d\nu_x(\xi)\, dx\, dx + \int_{\overline{\Omega}} \int_{S^{n-1}} F_k^\infty(\xi)\, d\nu_x^\infty(\xi)\, d\lambda. \tag{5.3.6}$$

We observe now that, by construction,

$$F_k^\infty(\xi) = \lim_{t\to+\infty} \frac{F_k(t\,\xi)}{t} = \frac{\theta(r_k)}{r_k}|\xi|,$$
$$\int_{S^{n-1}} F_k^\infty(\xi)\, d\nu_x^\infty(\xi) = \frac{\theta(r_k)}{r_k}.$$

Moreover,

$$\lim_{j\to+\infty}\int_\Omega F_j(Dv_j)\,dx \leqslant \lim_{j\to+\infty}\left(\inf_{w\in\mathbb{K}_\psi(\Omega)} I_j(w) + \frac{1}{j}\right) \leqslant \int_\Omega F(Du)\,dx, \tag{5.3.7}$$

so passing to the limit as $k\to\infty$ in (5.5.5), by the Monotone Convergence Theorem and the fact that $\frac{\theta(r_k)}{r_k}\to+\infty$ as $k\to+\infty$, we get that $\lambda=0$ and

$$\int_\Omega \langle F,\nu_x\rangle\,dx \leqslant \lim_{j\to+\infty}\left(\inf_{w\in\mathbb{K}_\psi(\Omega)}\int_\Omega F_j(Dw)\,dx\right). \tag{5.3.8}$$

Since $\lambda=0$, then $(Dv_k)_k$ is equi-integrable by Proposition 5.2.7, implying that $(v_k)_k$ is converging weakly in $W^{1,1}(\Omega)$ to v. Since $\mathbb{K}_\psi(\Omega)$ is weakly closed then also $v\in\mathbb{K}_\psi(\Omega)$. Furthermore, by Jensen's inequality and Proposition 5.2.7 we get

$$\int_\Omega F(Dv)\,dx \leqslant \int_\Omega \langle F,\nu_x\rangle\ dx. \tag{5.3.9}$$

This proves that

i) $v=u$,

ii) $\inf_{w\in\mathbb{K}_\psi(\Omega)}\int_\Omega F_j(Dw)\,dx \longrightarrow \int_\Omega F(Du)\,dx$ as $j\to+\infty$.

Indeed by (5.3.7), (5.3.8), and (5.3.9) we have that

$$\int_\Omega F(Dv)\,dx \leqslant \int_\Omega\langle F,\nu_x\rangle\,dx \leqslant \lim_{j\to+\infty}\left(\inf_{w\in\mathbb{K}_\psi(\Omega)}\int_\Omega F_j(Dw)\,dx\right) \leqslant \int_\Omega F(Du)\,dx,$$

and since u is a minimizer and it is unique we get i) and ii). On the other hand, (ii) allows us to say that there exists $(\varepsilon_k)_k\subset\mathbb{R}$ such that $\varepsilon_k\to0$ as $k\to\infty$ and

$$\inf_{w\in\mathbb{K}_\psi(\Omega)}\int_\Omega F_k(Dw)\,dx \leqslant \int_\Omega F_k(Du)\,dx \leqslant \int_\Omega F(Du)\,dx = \varepsilon_k^2 + \inf_{w\in\mathbb{K}_\psi(\Omega)}\int_\Omega F_k(Dw)\,dx.$$

By applying Theorem 1.1.20 with $\lambda=\varepsilon_k$, we get that there exists a sequence $(u_k)_k\subset\mathbb{K}_\psi(\Omega)$ such that $Du_k\to Du$ in $L^1(\Omega)$ and such that

$$\int_\Omega F_k(Du_k)\,dx \leqslant \int_\Omega F_k(Du)\,dx \leqslant \int_\Omega F(Du)\,dx.$$

Finally, such that for every $k\in\mathbb{N}$, u_k is the unique minimizer of

$$w\longmapsto \int_\Omega \left[F_k(Dw) + \varepsilon_k|Dw-Du_k|\right]\,dx,$$

which is an integral functional with a Lipschitz continuous integrand. If we define

$$\sigma_k := F_k'(Du_k),$$

then we can derive a related important variational inequality. Indeed, fixed $\eta \in \mathbb{K}_\psi(\Omega)$ and $0 < \varepsilon < 1$, we have that, defining

$$v_k := u_k + \varepsilon(\eta - u_k) \in \mathbb{K}_\psi(\Omega),$$

we obtain that

$$\frac{1}{\varepsilon}\int_\Omega [F_k(Dv_k) + \varepsilon_k|Dv_k - Du_k| - F_k(Du_k)]\, dx \geqslant 0.$$

On the other hand,

$$\begin{aligned}&\frac{1}{\varepsilon}\int_\Omega [F_k(Dv_k) + \varepsilon_k|Dv_k - Du_k| - F_k(Du_k)]\, dx\\ =&\frac{1}{\varepsilon}\int_\Omega [F_k(Du_k + \varepsilon D(\eta - u_k)) + \varepsilon\varepsilon_k|D\eta - Du_k| - F_k(Du_k)]\, dx,\end{aligned}$$

so

$$\frac{1}{\varepsilon}\int_\Omega [F_k(Du_k + \varepsilon D(\eta - u_k)) - F_k(Du_k)]\, dx \geqslant -\varepsilon_k \int_\Omega |D\eta - Du_k|\, dx.$$

Since

$$\begin{aligned}&\frac{1}{\varepsilon}\int_\Omega [F_k(Du_k + \varepsilon D(\eta - u_k)) - F_k(Du_k)]\, dx\\ &= \int_\Omega \int_0^1 F_k'(Du_k + s\varepsilon D(\eta - u_k)) \cdot D(\eta - u_k)\, ds dx,\end{aligned}$$

then, by the Dominated Convergence Theorem and the uniform boundedness of $(Du_k)_k$ in $L^1(\Omega)$, we get

$$\int_\Omega F_k'(Du_k) \cdot D(\eta - u_k)\, dx \geqslant -\varepsilon_k \int_\Omega |D\eta - Du_k|\, dx \geqslant -\varepsilon_k \left(C + \int_\Omega |D\eta|\, dx \right). \quad (5.3.10)$$

STEP 3 Now we want to have informations on the asymptotic behavior of σ_k as $k \to \infty$. Since $F_k \nearrow F$ pointwise and $F_{k+1} \geqslant F_k$ for all $k \in \mathbb{N}$, it follows in particular from Dini's Lemma that the convergence is locally uniform in ξ, so we can now prove that $\sigma_k = F_k'(Du_k) \longrightarrow F'(Du)$ locally uniformly. To that end, we consider $\xi \in \mathbb{R}^n$ and $(F_k'(\xi_k))_k$ where $\xi_k \to \xi$ as $k \to \infty$. Because difference-quotients of convex functions are increasing in the increment, we have for all $\eta \in \mathbb{R}^n$ and $0 < |t| \leqslant 1$ that

$$\begin{aligned}|(F_k'(\xi_k) - F'(\xi)) \cdot \eta| &\leqslant \left| \frac{F_k(\xi_k + t\,\eta) - F_k(\xi_k) - F'(\xi) \cdot t\eta}{t} \right|\\ &\leqslant |F_k(\xi_k + \eta) - F_k(\xi_k) - F'(\xi) \cdot \eta|.\end{aligned}$$

Consequently, we get

$$\limsup_{k \to +\infty} |(F_k'(\xi_k) - F'(\xi)) \cdot \eta| \leqslant |F(\xi + \eta) - F(\xi) - F'(\xi) \cdot \eta|,$$

for all $\eta \in \mathbb{R}^n$. Hence, for all $0 < s \leqslant 1$, we obtain that

$$\limsup_{k\to+\infty} \frac{|(F_k'(\xi_k) - F'(\xi)) \cdot s\eta|}{s} \leqslant \frac{|F(\xi + s\,\eta) - F(\xi) - F'(\xi)\cdot s\eta|}{s},$$

which is equivalent to

$$\limsup_{k\to+\infty} |(F_k'(\xi_k) - F'(\xi)) \cdot \eta| \leqslant \left| \frac{F(\xi + s\,\eta) - F(\xi)}{s} - F'(\xi)\cdot \eta \right|.$$

If we let $s \to 0$ and recall that F is differentiable in ξ, we conclude that the left-hand side must vanish. This proves the local uniform convergence of derivatives, so it follows that $\sigma_k = F_k'(Du_k) \longrightarrow F'(Du)$ locally uniformly and, in particular, in measure on Ω.
We now point out that the following identity holds, namely

$$\sigma_k \cdot Du_k = F_k^*(\sigma_k) + F_k(Du_k), \tag{5.3.11}$$

and it has been deduced by Fenchel's indentity and using the definition of σ_k, recalling that $F_k^{**} = F_k$ and choosing $\xi = Du_k$. Moreover, the same identity holds for the function F, namely

$$\sigma^* \cdot Du = F^*(\sigma^*) + F(Du), \tag{5.3.12}$$

with $\sigma^* := F'(Du)$.

STEP 4 Since $u_0 \in \mathbb{K}_\psi(\Omega)$, we can use (5.5.6) choosing $\eta = u_0$, thus getting

$$\int_\Omega \sigma_k \cdot (Du_0 - Du_k)\, dx \geqslant -\varepsilon_k \left(C + \int_\Omega |Du_0|\, dx \right) \qquad \forall\, k \in \mathbb{N},$$

that is equivalent to

$$\int_\Omega \sigma_k \cdot Du_0\, dx + \varepsilon_k \left(C + \int_\Omega |Du_0|\, dx \right) \geqslant \int_\Omega \sigma_k \cdot Du_k\, dx \qquad \forall\, k \in \mathbb{N}.$$

Now, integrating (5.3.11) over Ω and using the previous inequality, we have that, chosen $t > 1$,

$$\begin{aligned}
&\int_\Omega F_k^*(\sigma_k)\, dx \\
= \;& \int_\Omega \sigma_k \cdot Du_k\, dx - \int_\Omega F_k(Du_k)\, dx \\
\leqslant \;& \int_\Omega \sigma_k \cdot Du_0\, dx + \varepsilon_k \left(C + \int_\Omega |Du_0|\, dx \right) - \int_\Omega F_k(Du_k)\, dx \\
= \;& \frac{1}{t} \int_\Omega \sigma_k \cdot tDu_0\, dx + \varepsilon_k \left(C + \int_\Omega |Du_0|\, dx \right) - \int_\Omega F_k(Du_k)\, dx \\
\leqslant \;& \frac{1}{t} \int_\Omega F_k^*(\sigma_k)\, dx + \frac{1}{t} \int_\Omega F_k(t\, Du_0)\, dx + \varepsilon_k \left(C + \int_\Omega |Du_0|\, dx \right) - \int_\Omega F_k(Du_k)\, dx,
\end{aligned}$$

where we also used Fenchel's inequality, exploiting the convexity and lower semicontinuity of F_k. Reabsorbing the first term in the right-hand side by the left-hand side and using the fact that $F_k(\xi) \leqslant F(\xi)$ for all $\xi \in \mathbb{R}^n$ we obtain that

$$\begin{aligned}
&\int_\Omega F_k^*(\sigma_k)\,dx \\
\leqslant\;& \frac{1}{t-1}\int_\Omega F_k(t\,Du_0)\,dx - \frac{t}{t-1}\int_\Omega F_k(Du_k)\,dx + \frac{t\,\varepsilon_k}{t-1}\left(C + \int_\Omega |Du_0|\,dx\right) \\
\leqslant\;& \frac{1}{t-1}\int_\Omega F(t\,Du_0)\,dx - \frac{t}{t-1}\int_\Omega F_k(Du_k)\,dx + \frac{t\,\varepsilon_k}{t-1}\left(C + \int_\Omega |Du_0|\,dx\right) \\
\leqslant\;& C_1\int_\Omega F(t\,Du_0)\,dx - C_2\int_\Omega F(Du_0)\,dx + C_3\left(C + \int_\Omega |Du_0|\,dx\right), \qquad (5.3.13)
\end{aligned}$$

where in the last line we used the fact that

$$\int_\Omega F_k(Du_k)\,dx \leqslant \int_\Omega F(Du)\,dx \leqslant \int_\Omega F(Du_0)\,dx,$$

and that since $(\varepsilon_k)_k$ is converging then it is bounded. Recalling that $F_k^* \searrow F^*$ and, by the hypotheses, that $F(t\,Du_0) \in L^1(\Omega)$ with $t > 1$, we also obtain that

$$\int_\Omega F^*(\sigma_k)\,dx \leqslant C_1\int_\Omega |F(t\,Du_0)|\,dx + C_2\int_\Omega F(Du_0)\,dx + C_3\left(C + \int_\Omega |Du_0|\,dx\right) < +\infty.$$

Since we already observed that $\sigma_k \longrightarrow F'(Du)$ locally uniformly, by the previous estimate and Fatou's Lemma we have that

$$\int_\Omega F^*(F'(Du))\,dx \leqslant \liminf_{k\to+\infty}\int_\Omega F^*(\sigma_k)\,dx < +\infty.$$

Thus

$$F^*(F'(Du)) \in L^1(\Omega).$$

Whence, by (5.3.12), we also have

$$F'(Du)\cdot Du \in L^1(\Omega),$$

since $F(Du) \in L^1(\Omega)$ by the definition of minimizer.

STEP 5 Now we want to prove the validity of the variational inequality (5.1.5). Since F^* is superlinear at infinity by Lemma 5.2.2 and since $F_k^* \searrow F^*$, then there exists $\overline{\theta} : [0,+\infty) \longrightarrow [0,+\infty)$ increasing and superlinear at infinity such that

$$\overline{\theta}(|\xi|) \leqslant F^*(\xi) \leqslant F_k^*(\xi) \qquad \forall\,\xi \in \mathbb{R}^n. \qquad (5.3.14)$$

Moreover, using (5.5.7) we have that

$$\sup_{k\in\mathbb{N}}\int_\Omega \overline{\theta}(|\sigma_k|)\,dx \leqslant \int_\Omega F_k^*(\sigma_k)\,dx < +\infty,$$

thus we can use De La Vallè-Poussin's Theorem in order to obtain the equi-integrability for $(\sigma_k)_k$. Since $(\sigma_k)_k$ converges in measure to σ^*, then we can apply Vitali's Convergence Theorem which proves that $(\sigma_k)_k$ converges to σ^* in $L^1(\Omega)$. Now we observe that (5.5.6) implies that

$$\int_\Omega \sigma_k\cdot D\varphi\,dx \geqslant -\varepsilon_k\int_\Omega |D\varphi|\,dx, \quad \forall\,\varphi \in C_0^\infty(\Omega), \varphi \geqslant 0. \qquad (5.3.15)$$

Indeed this is implied by taking $\eta = u_k + \varphi$ in (5.5.6). It follows that if we pass to the limit as $k \longrightarrow +\infty$ in (5.3.15) and, by the $L^1(\Omega)$ convergence of $(\sigma_k)_k$ to σ^*, it yields

$$\int_\Omega \sigma^* \cdot D\varphi \, dx \geqslant 0 \qquad \forall\, \varphi \in C_0^\infty(\Omega), \;\; \varphi \geqslant 0.$$

This implies that $\operatorname{div} \sigma^* \leqslant 0$ in the distributional sense, i.e. (5.1.5).

STEP 6 We should notice that, up to subsequences, it holds true that

$$\int_\Omega \sigma^* \cdot Du \, dx \leqslant \liminf_{k \to +\infty} \int_\Omega \sigma_k \cdot Du_k \, dx. \tag{5.3.16}$$

To prove this, we observe that, by (5.3.1), (5.3.2) and (5.3.3), we have

$$F_k(\xi) \geqslant -\mu_k \qquad \forall\, \xi \in \mathbb{R}^n.$$

Now, by (5.3.11), (5.3.14) and the fact that $\overline{\theta}(|\xi|) \geqslant 0$ for all $\xi \in \mathbb{R}^n$, we get that

$$\sigma_k \cdot Du_k = F_k^*(\sigma_k) + F_k(Du_k) \geqslant \overline{\theta}(|\sigma_k|) - \mu_k \geqslant -\mu_k.$$

Therefore we can apply Fatou's Lemma to the sequence of functions $(\sigma_k \cdot Du_k + \mu_k)_k$ that, thanks to the definition of $(\mu_k)_k$, converges a.e. to $\sigma^* \cdot Du$ up to subsequences, letting us deduce that

$$\int_\Omega \sigma^* \cdot Du \, dx \leqslant \liminf_{k \to +\infty} \int_\Omega \left(\sigma_k \cdot Du_k + \mu_k\right) \, dx \leqslant \liminf_{k \to +\infty} \int_\Omega \sigma_k \cdot Du_k \, dx,$$

as we wanted to prove. Using (5.5.6) with ψ in place of η, since $\psi \in \mathbb{K}_\psi(\Omega)$, we get that

$$\int_\Omega \sigma_k \cdot Du_k \, dx \leqslant \int_\Omega \sigma_k \cdot D\psi \, dx + \varepsilon_k \left(C + \int_\Omega |D\psi| \, dx \right)$$

and the $L^1(\Omega)$ convergence of $(\sigma_k)_k$ to σ^* implies that

$$\begin{aligned}
&\liminf_{k \to +\infty} \int_\Omega \sigma_k \cdot Du_k \, dx \\
&\leqslant \liminf_{k \to +\infty} \left[\int_\Omega \sigma_k \cdot D\psi \, dx + \varepsilon_k \left(C + \int_\Omega |D\psi| \, dx \right) \right] \\
&= \int_\Omega \sigma^* \cdot D\psi \, dx.
\end{aligned}$$

Combining this with (5.3.12) and (5.3.16), we obtain that

$$\int_\Omega F(Du) \, dx + \int_\Omega F^*(\sigma^*) \, dx = \int_\Omega \sigma^* \cdot Du \, dx \leqslant \int_\Omega \sigma^* \cdot D\psi \, dx = [\![\sigma^*, \psi]\!]_{u_0}(\overline{\Omega}),$$

where we use the measure defined in (5.1.9). Now we have to prove the reverse inequality, i.e.

$$\int_\Omega F(Du) \, dx + \int_\Omega F^*(\sigma^*) \, dx \geqslant [\![\sigma^*, \psi]\!]_{u_0}(\overline{\Omega}). \tag{5.3.17}$$

First we observe that, thanks to (5.1.8),

$$\begin{aligned}
[\![\sigma^*, \psi]\!]_{u_0}(\overline{\Omega}) &= \int_\Omega (\psi - u_0)\, d(-\mathrm{div}\,\sigma^*) + \int_\Omega \sigma^* \cdot Du_0\, dx \\
&= \int_\Omega (\psi - u + u - u_0)\, d(-\mathrm{div}\,\sigma^*) + \int_\Omega \sigma^* \cdot Du_0\, dx \\
&\leqslant \int_\Omega (u - u_0)\, d(-\mathrm{div}\,\sigma^*) + \int_\Omega \sigma^* \cdot Du_0\, dx,
\end{aligned}$$

where we used the fact that

$$\int_\Omega (u - \psi)\, d(-\mathrm{div}\,\sigma^*) \geqslant 0$$

because $-\mathrm{div}\,\sigma^*$ is a non-negative Radon measure and $u \geqslant \psi$ a.e. on Ω. By (5.1.8) and by (5.3.12), we get that

$$\begin{aligned}
[\![\sigma^*,\, \psi]\!]_{u_0}(\overline{\Omega}) &\leqslant \int_\Omega (u - u_0)\, d(-\mathrm{div}\,\sigma^*) + \int_\Omega \sigma^* \cdot Du_0\, dx \\
&= \int_\Omega \sigma^* \cdot Du\, dx \\
&= \int_\Omega F(Du)\, dx + \int_\Omega F^*(\sigma^*)\, dx. \qquad (5.3.18)
\end{aligned}$$

Combining (5.3.17) with the previous inequality, we get that it is in fact an equality, i.e.

$$[\![\sigma^*,\, \psi]\!]_{u_0}(\overline{\Omega}) = \int_\Omega F(Du)\, dx + \int_\Omega F^*(\sigma^*)\, dx.$$

Notice that this also implies that the measure $[\![\sigma^*,\, \psi]\!]_{u_0}(\overline{\Omega})$ is finite for σ^* and ψ.

5.4 Proof of theorem 5.1.11

The proof of Theorem 5.1.11 leans on a similar method to the one used in STEP 6, in particular exploiting the definition (5.1.9) and the techniques of Convex Analysis. We write the details for the reader's convenience.
First of all, let us recall that since u is a solution to (5.1.12) and G is Lipschitz continuous, then it holds that

$$\int_\Omega G'(Du) \cdot (D\phi - Du) dx \geqslant 0 \qquad \forall \phi \in \mathbb{K}_\psi(\Omega). \qquad (5.4.1)$$

Consider $\sigma \in S_-(\Omega)$ and $v \in \mathbb{K}_\psi(\Omega)$. Since $-\mathrm{div}\sigma$ is a non-negative Radon measure and $v \geqslant \psi$ a.e. on Ω, it holds that

$$\int_\Omega (v - \psi)\, d(-\mathrm{div}\sigma) \geqslant 0.$$

By definition (5.1.9) and by the previous inequality we get that

$$
\begin{aligned}
[\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) &= \int_\Omega (\psi - u_0)\, d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\, dx \\
&= \int_\Omega (\psi - v + v - u_0)\, d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\, dx \\
&\leqslant \int_\Omega (v - u_0)\, d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\, dx.
\end{aligned}
$$

Since $v, u_0 \in W^{1,1}_{u_0}(\Omega)$, then $v = u_0$ on $\partial\Omega$ and using (5.1.8) on the first integral of the last expression, we get that

$$
\begin{aligned}
[\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) &\leqslant \int_\Omega \sigma\cdot(Dv - Du_0)\, dx + \int_\Omega \sigma\cdot Du_0\, dx \\
&= \int_\Omega \sigma\cdot Dv\, dx \\
&\leqslant \int_\Omega G(Dv)\, dx + \int_\Omega G^*(\sigma)\, dx,
\end{aligned}
$$

by means of Fenchel's inequality and the fact that $G = G^{**}$. The previous inequality implies that

$$
\int_\Omega G(Dv)\, dx \geqslant [\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\, dx,
$$

so passing to the minimum on $v \in \mathbb{K}_\psi(\Omega)$ and to the maximum on $\sigma \in S^\infty_-(\Omega)$ we get

$$
\min_{v\in\mathbb{K}_\psi(\Omega)} \int_\Omega G(Dv)\, dx \geqslant \max_{\sigma\in S_-(\Omega)} \left([\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\, dx \right).
$$

We have now to prove the reverse inequality. We consider Fenchel's identity with $F = G$ and $\xi = Du$ and, since $u \in \mathbb{K}_\psi(\Omega)$ is the unique solution of (5.1.12), then

$$
\begin{aligned}
\int_\Omega G(Du)\, dx &= \int_\Omega G'(Du)\cdot Du\, dx - \int_\Omega G^*(G'(Du))\, dx \\
&= \int_\Omega G'(Du)\cdot Du - Du_0\, dx + \int_\Omega G'(Du)\cdot Du_0 - \int_\Omega G^*(G'(Du))\, dx.
\end{aligned}
$$

Since G satisfies the variational inequality (5.4.1) then it holds true also that

$$
\int_\Omega G'(Du)\cdot D\phi\, dx \geqslant 0 \qquad \forall\phi\in C^\infty_0(\Omega), \phi \geqslant 0,
$$

and so if we set $\sigma = G'(Du)$ then we have that $\sigma \in S^{\infty}_{-}(\Omega)$, thus we have

$$\begin{aligned}
\int_\Omega G(Du)\,dx &= \int_\Omega \sigma\cdot(Du - Du_0)\,dx + \int_\Omega \sigma\cdot Du_0\,dx - \int_\Omega G^*(\sigma)\,dx \\
&= \int_\Omega (u-u_0)\,d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\,dx - \int_\Omega G^*(\sigma)\,dx \\
&= \int_\Omega (\psi-\psi+u-u_0)\,d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\,dx - \int_\Omega G^*(\sigma)\,dx \\
&= \int_\Omega (\psi-u_0)\,d(-\mathrm{div}\sigma) - \int_\Omega (\psi-u)\,d(-\mathrm{div}\sigma) + \int_\Omega \sigma\cdot Du_0\,dx - \int_\Omega G^*(\sigma)\,dx \\
&= [\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega \sigma\cdot(D\psi - Du)\,dx - \int_\Omega G^*(\sigma)\,dx \\
&\leqslant [\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\,dx,
\end{aligned}$$

where we used once again (5.1.8) and the fact that it holds (5.4.1), where we chose $\phi = \psi \in \mathbb{K}_\psi(\Omega)$. Finally, we have that

$$\begin{aligned}
\min_{v\in\mathbb{K}_\psi(\Omega)} \int_\Omega G(Dv)\,dx &= \int_\Omega G(Du)\,dx \\
&\leqslant [\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\,dx \\
&\leqslant \max_{\sigma\in S_-(\Omega)} \left([\![\sigma,\psi]\!]_{u_0}(\overline{\Omega}) - \int_\Omega G^*(\sigma)\,dx \right),
\end{aligned}$$

Since this also implies that the measure $[\![\sigma,\,\psi]\!]_{u_0}(\overline{\Omega})$ is finite for σ and ψ, this concludes the proof.

5.5 Examples

In this section we considered a very general growth on the convex Lagrangian F. Assumption (H2) allows us to cover a wide class of functionals from those with almost-linear growth to the ones with exponential growth and beyond. For specific examples of functionals with general growth, see Section 3 of [61]. We now state some examples of Lagrangians to which Theorem 5.1.3 applies to. Such functions will be always called F_i for a certain index $i \in \mathbb{N}$, while auxiliary functions will be denoted with different letters.

Example 5.5.1. For every $\xi \in \mathbb{R}^n$, we can consider

$$F_1(\xi) := |\xi|^\alpha \qquad \text{with } \alpha > 1.$$

Example 5.5.2. For every $\xi \in \mathbb{R}^n$, we can consider

$$F_2(\xi) := \cosh|\xi|.$$

Example 5.5.3. For every $\xi \in \mathbb{R}^n$ we consider the function

$$G(\xi) := |\xi|\ln|\xi|$$

and we refer to one of its global minimum points as ξ_0, namely such that $|\xi_0| = \frac{1}{e}$ and $G(\xi_0) = -\frac{1}{e}$. If we define

$$F_3(\xi) := (|\xi| + |\xi_0|)\ln(|\xi| + |\xi_0|) - G(\xi_0) = \left(|\xi| + \frac{1}{e}\right)\ln\left(|\xi| + \frac{1}{e}\right) + \frac{1}{e},$$

then it satisfies the hypotheses of Theorem 5.1.3.

5.5.1 Proof of Theorem 5.1.16

We divide the proof in several steps. First, we construct a sequence of approximating energy functionals. Then, we generate a sequence of almost-minimizers through Theorem 1.1.20 and we study the asymptotic behavior of that sequence. Finally, we prove the claims of our Theorem.

Construnction of approximating functionals

We consider the a variant of the second conjugate function F^{**}, where the set in the supremum is restricted to a ball. For $k \in \mathbb{N}$ (or indeed $k > 0$) and $\xi \in \mathbb{R}^n$, we define

$$F_k(x,\xi) := \sup_{|z|\leqslant k} (\xi \cdot z - F^*(x,z)).$$

Previous studies [31, 32] have applied a sequence of additional regularizations to F_k. However, we show that this is not necessary by proving appropriate regularity of F_k, which is of independent interest. In particular, the additional regularizations do not preserve the normalization $F(0) = 0 = F'(0)$, which is of relevance in the non-autonomous case.

The choices $z = \xi/|\xi|$ and $z = k\xi/|\xi|$ give the lower bounds

$$F_k(x,\xi) \geqslant \max\left\{|\xi| - M_{F^*}(1), k\,|\xi| - M_{F^*}(k)\right\}.$$

If F has superlinear growth at infinity, then Lemma 5.2.4 implies that F^* is locally bounded, i.e. $M_{F^*}(t) < \infty$ for every $t > 0$. The sequence F_k is increasing and

$$F_k(x,\cdot) \nearrow F^{**}(x,\cdot) = F(x,\cdot)$$

for every $x \in \Omega$. For simplicity we write the next result without x-dependence; it is later applied pointwise.

Theorem 5.5.4. *If $F \in C^1(\mathbb{R}^n)$ with $F(0) = 0 = F'(0)$ is strictly convex and superlinear at infinity, then $F_k \in C^1(\mathbb{R}^n)$ is k-Lipschitz and coincides with F in the set $(F')^{-1}(B_k)$.*

Proof. Since F is differentiable, strictly convex and superlinear at infinity, F^* is strictly convex and C^1 [121, Theorem 6.2.4]. Unfortunately, this proof of C^1-smoothness does not work when the set in the supremum is restricted, as is the case for F_k. Therefore, we employ a more direct approach.

The supremum of $z \mapsto \xi \cdot z - F(z)$ in the definition of F_k is obtained at an interior point where its gradient vanishes or at a boundary point where its gradient is perpendicular to the boundary:

$$\xi - (F^*)'(z) = 0 \quad \text{for } z \in B_k$$

or

$$\xi - (F^*)'(z) = sz \quad \text{for } z \in \partial B_k, s \geqslant 0.$$

Here $s \geqslant 0$ because this gives the exterior normal and thus the maximum. Note that $(F^*)'(z) = (F')^{-1}(z)$ by [121]. In the case $z \in B_k$ the gradient vanishes when $z = F'(\xi)$ so Fenchel's identity gives $F_k(\xi) = \xi \cdot F'(\xi) - F^*(F'(\xi)) = F(\xi)$ and thus F_k coincides with F in $(F')^{-1}(B_k)$.

Let us define $H : \mathbb{R}^n \to \overline{B_k}$ by $H(\xi) := \arg\max_{|z| \leqslant k}(\xi \cdot z - F^*(z))$; this is well-defined since the function is strictly concave and the set is convex. Then $F_k(\zeta) = \zeta \cdot H(\zeta) - F^*(H(\zeta))$ and $F_k(\xi) \geqslant \xi \cdot H(\zeta) - F^*(H(\zeta))$ and vice versa for $H(\xi)$ so that

$$(\xi - \zeta) \cdot H(\zeta) \leqslant F_k(\xi) - F_k(\zeta) \leqslant (\xi - \zeta) \cdot H(\xi).$$

Since $|H| \leqslant k$, this gives that F_k is k-Lipschitz. Once we prove that H is continuous, this implies that $F_k'(\xi) = H(\xi)$ and so $F_k \in C^1$.

Define a continuous mapping $G : \partial B_k \times (0, \infty) \to \mathbb{R}^n$ by

$$G(z, r) := \begin{cases} (F^*)'(\frac{r}{k} z) & \text{if } r \in (0, k), \\ (r - k)z + (F^*)'(z) & \text{if } r \in [k, \infty). \end{cases}$$

This amounts in solving the zeros of the derivative in ξ, so this is some kind of inverse to H. Strict monotonicity of $(F^*)'$ implies that $G|_{\partial B_k \times (0,k]}$ is an injection. For $\partial B_k \times [k, \infty)$, we consider the equation

$$sz + (F^*)'(z) = tw + (F^*)'(w).$$

Since $(F^*)'$ is monotone, we obtain that

$$0 \geqslant \big((F^*)'(w) - (F^*)'(z)\big) \cdot (z - w) = (sz - tw) \cdot (z - w) = s|z|^2 + t|w|^2 - (s + t) z \cdot w.$$

If $z, w \in \partial B_k$, then this is only possible when $z = w$. From the original equation, we then see that also $s = t$. Thus $G|_{\partial B_k \times [k,\infty)}$ is an injection. Suppose next that $z \in \partial B_k$, $w \in B_k$, $s > 0$ and $t = 0$. The inequality becomes $z \cdot w \geqslant |z|^2$, which is impossible. We have thus checked all combinations, so we conclude that G is an injection. It follows from Brouwer's Theorem on the invariance of domain [94, Theorem VI 9] that G^{-1} is continuous. Let $P : \partial B_k \times (0, \infty) \to \mathbb{R}^n$ be the projection (in polar coordinates) onto the ball $\overline{B_k}$:

$$P(z, s) = \begin{cases} \frac{s}{k} z & \text{if } s \in (0, k), \\ z & \text{if } s \in [k, \infty). \end{cases}$$

Since P and G^{-1} are continuous, the same is true for $H = P \circ (G^{-1})$ in $\mathbb{R}^n \setminus \{0\}$. In the neighborhood $(F')^{-1}(B_k)$ of 0 we showed that $H = F'$, so it is continuous. Thus H is continuous in all of $\mathbb{R}^n$. □

Let $R > 0$. Since F is locally bounded, there exists $k_0 \in \mathbb{N}$ such that $F'(x, \xi) \in B_{k_0}$ for all $x \in \Omega$ and $\xi \in B_R$. Thus $F_k = F$ and $F_k' = F'$ in B_R when $k \geqslant k_0$. In particular, $F_k \to F$ and $\partial F_k \to F'$ locally uniformly.

Suppose that $Du_k \to Du$ in $L^1(\Omega; \mathbb{R}^n)$ and define $\sigma, \sigma_k \in L^0(\Omega; \mathbb{R}^n)$ as $\sigma_k := F_k'(\cdot, Du_k)$ and $\sigma := F'(\cdot, Du)$. Let us show that $\sigma_k \to \sigma$ in measure on Ω. Let $R > 0$ and $k_0 > 0$ be from the previous paragraph. For $\epsilon > 0$, since the continuity modulus of F' is independent of x, we can choose $\delta > 0$ such that $|F'(x, \xi') - F'(x, \xi)| < \epsilon$ for all $\xi' \in \overline{B}_R$ and $|\xi' - \xi| < \delta$.

Since $Du_k \to Du$ in $L^1(\Omega;\mathbb{R}^n)$ we can choose $k_1 > 0$ such that $|\{|Du_k - Du| \geqslant \delta\}| < \epsilon$ for all $k > k_1$. Finally, since $\|Du_k\|_1$ is a bounded sequence, we can choose $k_2 > 0$ such that $|\{|Du_k| \geqslant R\}| < \epsilon$ for all $k > k_2$. From the inequality

$$|\sigma_k(x) - \sigma(x)| \leqslant |F_k'(x, Du_k) - F'(x, Du_k)| + |F'(x, Du_k) - F'(x, Du)|,$$

we conclude when $k \geqslant \max\{k_0, k_1, k_2\}$ that

$$\begin{aligned}\big\{|\sigma_k - \sigma| > \epsilon\big\} &\subset \big\{|F_k'(\cdot, Du_k) - F'(\cdot, Du_k)| > 0\big\} \cup \big\{|F'(\cdot, Du_k) - F'(\cdot, Du)| > \epsilon\big\}\\ &\subset \big\{|Du_k| \geqslant R\big\} \cup \big\{|Du_k - Du| \geqslant \delta\big\}.\end{aligned}$$

Therefore $\big|\big\{|\sigma_k - \sigma| > \epsilon\big\}\big| < 2\epsilon$ when $k \geqslant \max\{k_0, k_1, k_2\}$ and so we have proved convergence in measure.

Almost-minimizers and their limit

Once more, by construction, the functions F_k have linear growth at infinity, so we cannot guarantee the existence of solutions to the approximating obstacle problems

$$\min_{w \in K} \int_\Omega F_k(x, Dw)\, dx.$$

Instead, we use a version of the Ekeland variational principal for metric spaces (Theorem 1.1.20) to generate a sequence of functions $(u_k)_k$ that plays the same role. We define

$$I(w) := \int_\Omega F(x, Dw)\, dx \quad \text{and} \quad I_k(w) := \int_\Omega F_k(x, Dw)\, dx \qquad \text{for} \quad w \in K.$$

Since $F_k \leqslant F$ for $k \in \mathbb{N}$, $\inf_{w\in K} I_k(w) \leqslant \inf_{w\in K} I(w)$. By the definition of infimum, we can find $v_k \in K$ for every $k \in \mathbb{N}$ such that

$$I_k(v_k) \leqslant \inf_{w\in K} I_k(w) + \frac{1}{k} \leqslant \inf_{w\in K} I(w) + 1 < \infty.$$

Using $F_k(x, \xi) \geqslant |\xi| - M_{F^*}(1)$, we get that $(Dv_k)_k$ is bounded in $L^1(\Omega)$. Since $v_k \in K \subset W^{1,1}_{u_0}(\Omega)$, the Sobolev embedding implies that $(v_k)_k$ is bounded in $W^{1,1}(\Omega)$. Since Ω has Lipschitz boundary, there exists a non-relabeled subsequence $(v_k)_k$ converging to some u in $L^1(\Omega)$ [3, Theorem 3.23]. By Proposition 5.2.7, this sequence generates a generalized Young measure $((\nu_x)_{x\in\Omega}, \lambda, (\nu_x^\infty)_{x\in\Omega})$ such that

$$\begin{aligned}\lim_{k\to\infty} I_k(v_k) &\geqslant \lim_{k\to\infty} \int_\Omega F_j(x, Dv_k)\, dx\\ &= \int_\Omega \int_{\mathbb{R}^n} F_j(x, \xi)\, d\nu_x(\xi)\, dx + \int_{\overline{\Omega}} \int_{\partial B_1} F_j^\infty(x, \xi)\, d\nu_x^\infty(\xi)\, d\lambda(x)\end{aligned} \tag{5.5.5}$$

for every $j \in \mathbb{N}$. From the estimates $j\,|\xi| - M_{F^*}(j) \leqslant F_j(x, \xi) \leqslant j\,|\xi|$, we conclude that

$$F_j^\infty(x, \xi) = \lim_{x'\to x, \xi'\to\xi, t\to\infty} \frac{F_j(x', t\,\xi')}{t} = j$$

for $\xi \in \partial B_1$. The left-hand side of (5.5.5) is finite, so passing to the limit as $j \to \infty$ we obtain by monotone convergence that $\lambda = 0$ and

$$\int_\Omega \int_{\mathbb{R}^n} F(x,\xi)\, d\nu_x(\xi)\, dx \leqslant \lim_{k\to\infty} I_k(v_k).$$

Since $\lambda = 0$, $(v_k)_k$ is equi-integrable by Proposition 5.2.7 and so it converges weakly in $W^{1,1}(\Omega)$ to u. Since K is convex and closed, it is weakly closed [67, p. 6], and so $v \in K$. Furthermore, by Jensen's inequality for $\xi \mapsto F(x,\xi)$ with fixed x and Proposition 5.2.7 we get

$$\int_\Omega \int_{\mathbb{R}^n} F(x,\xi)\, d\nu_x(\xi)\, dx \geqslant \int_\Omega F\left(x, \int_{\mathbb{R}^n} \xi\, d\nu_x(\xi)\right) dx = \int_\Omega F(x, Dv)\, dx.$$

We have shown that

$$I(u) \leqslant \int_\Omega \int_{\mathbb{R}^n} F(x,\xi)\, d\nu_x(\xi)\, dx \leqslant \lim_{k\to\infty} I_k(v_k) \leqslant \inf_{w\in K} I(w).$$

Thus u is a minimizer of (5.1.15). Since F is strictly convex, the minimizer is unique (see, e.g., [133, Proposition 2.10].)

Furthermore, we have shown that $\inf_{w\in K} I_k(w) \to I(v)$ as $k \to \infty$. Define

$$\epsilon_k := \sqrt{I_k(u) - \inf_{w\in K} I_k(w)}.$$

Since $I_k(u) \nearrow I(u)$ by monotone convergence, we obtain that $\epsilon_k \to 0^+$. From Theorem 1.1.20 with $\mathcal{F}(w) := \int F_k(x, Dw)\, dx$ and parameters $\epsilon = \epsilon_k^2$ and $\lambda = \epsilon_k$, we obtain $u_k \in K$ with $\|u_k - u\|_{W^{1,1}(\Omega)} \leqslant \epsilon_k$ and

$$\int_\Omega F_k(x, Du_k)\, dx \leqslant \int_\Omega F_k(x, Du)\, dx,$$

which is the unique minimizer of

$$w \longmapsto \int_\Omega \big[F_k(x, Dw) + \epsilon_k |Dw - Du_k|\big]\, dx.$$

Fix $w \in K$ and $0 < h < 1$, and define

$$w_k := u_k + h(w - u_k).$$

Since $w_k = (1-h)u_k + hw$, it belongs to K. The minimizing property implies that

$$\frac{1}{h}\int_\Omega \left[F_k(x, Dw_k) + \epsilon_k |Dw_k - Du_k| - F_k(x, Du_k)\right]\, dx \geqslant 0.$$

By the definition of w_k, $\epsilon_k |Dw_k - Du_k| = h\epsilon_k |Dw - Du_k|$, so that

$$\int_\Omega \frac{F_k(x, Du_k + hD(w - u_k)) - F_k(x, Du_k)}{h}\, dx \geqslant -\epsilon_k \int_\Omega |Dw - Du_k|\, dx.$$

Since F_k is k-Lipschitz, $k|D(w - u_k)|$ is a majorant for the integrand on the left-hand side and since F_k is C^1 we have convergence to the derivative:

$$\frac{F_k(x, Du_k + hD(w - u_k)) - F_k(x, Du_k)}{h} \to F_k'(x, Du_k) \cdot D(w - u_k)$$

as $h \to 0^+$ for a.e. $x \in \Omega$. Henec we obtain for any $w \in K$ that

$$\int_\Omega F_k'(x, Du_k) \cdot D(w - u_k)\, dx \geqslant -\epsilon_k \int_\Omega |Dw - Du_k|\, dx \geqslant -c\epsilon_k, \tag{5.5.6}$$

where $c := \|Dw\|_1 + \sup \|Du_k\|_1 < \infty$.

The validity of the integrability claims

Recall that $\sigma_k = F_k'(x, Du_k)$. We choose $w := w_0 \in K$ in (5.5.6) and obtain that

$$\int_\Omega \sigma_k \cdot Dw_0 \, dx + \epsilon_k c \geqslant \int_\Omega \sigma_k \cdot Du_k \, dx.$$

Now, integrating the inequality $F_k^*(x, \sigma_k) = \sigma_k \cdot Du_k - F_k(x, Du_k) \leqslant \sigma_k \cdot Du_k$ over Ω and using the previous inequality, we have for $t > 1$ from the assumption of the theorem that

$$\begin{aligned}\int_\Omega F_k^*(x, \sigma_k) \, dx \leqslant \int_\Omega \sigma_k \cdot Du_k \, dx &\leqslant \frac{1}{t} \int_\Omega \sigma_k \cdot tDw_0 \, dx + c\epsilon_k \\ &\leqslant \frac{1}{t} \int_\Omega F_k^*(x, \sigma_k) \, dx + \frac{1}{t} \int_\Omega F_k(x, t\, Dw_0) \, dx + c\epsilon_k,\end{aligned}$$

where we also used Fenchel's inequality. Reabsorbing the first term on the right-hand side into the left-hand side and using $F_k \leqslant F$ and $c\,|\epsilon_k| \leqslant C_2$ for some constant, we obtain that

$$\int_\Omega F_k^*(x, \sigma_k) \, dx \leqslant \frac{1}{t-1} \int_\Omega F_k(x, t\, Dw_0) \, dx + \frac{t\, \epsilon_k}{t-1} c \leqslant C_1 \int_\Omega F(x, t\, Dw_0) \, dx + C_2.$$

Recalling that $F_k^*(x, \cdot) \searrow F^*(x, \cdot)$ for every $x \in \Omega$ and that $F(\cdot, t\, Du_0) \in L^1(\Omega)$ by assumption, we conclude that

$$\int_\Omega F^*(x, \sigma_k) \, dx \leqslant C_1 \int_\Omega F(x, t\, Dw_0) \, dx + C_2' < \infty. \tag{5.5.7}$$

Since we already observed that $\sigma_k \to \sigma = F'(x, Du)$ in measure, we have by the previous estimate and Fatou's Lemma that

$$\int_\Omega F^*(x, F'(x, Du)) \, dx \leqslant \liminf_{k \to \infty} \int_\Omega F^*(x, \sigma_k) \, dx < \infty.$$

Thus $F^*(\cdot, F'(\cdot, Du)) \in L^1(\Omega)$. Since $F'(x, Du) \cdot Du \leqslant F^*(x, F'(x, Du)) + F(x, Du)$ and $F(\cdot, Du) \in L^1(\Omega)$ by the definition of minimizer, we also have $F'(\cdot, Du) \cdot Du \in L^1(\Omega)$.

The validity of the variational inequality

Let $\theta(t) := \sup_{x \in \Omega, \xi \in B(0,t)} F(x, \xi)$. Since F is locally bounded, θ is, as well. By Lemma 5.2.4, θ^* is superlinear at infinity. Since $F_k(x, \xi) \leqslant F(x, \xi) \leqslant \theta(|\xi|)$, the opposite inequality holds for the conjugates. By (5.5.7),

$$\sup_{k \in \mathbb{N}} \int_\Omega \theta^*(|\sigma_k|) \, dx \leqslant \sup_{k \in \mathbb{N}} \int_\Omega F^*(x, \sigma_k) \, dx < \infty,$$

and so the De La Vallè-Poussin Theorem 1.1.21 implies the equi-integrability of $(\sigma_k)_k$. Since $(\sigma_k)_k$ converges in measure to σ, it converges almost everywhere (up to a subsequence) and Vitali's Convergence Theorem [134, p. 133] yields $(\sigma_k)_k \to \sigma$ in $L^1(\Omega)$. With this we choose $w := u_k + \eta$ in (5.5.6) and obtain that

$$\int_\Omega \sigma \cdot D\eta \, dx = \lim_{k \to \infty} \int_\Omega \sigma_k \cdot D\eta \, dx \geqslant 0$$

when $w \in K$. If $\eta + K \subset K$ for every $\eta \in C_0^\infty(\Omega)$ this gives $\mathrm{div}\sigma = 0$ in the distributional sense; if $\eta + K \subset K$ only when $\eta \geqslant 0$ we get $\mathrm{div}\sigma \leqslant 0$, instead.

Chapter 6

Higher differentiability

6.1 Setting and Main Result

In this Chapter we will see an example of how the regularity of the obstacle affects the regularity of the solutions to the obstacle problem. Although we are working in the field of the obstacle problem, our aim is different compared to the previous Chapter. We will thus work in a slightly different setting. The main difference is the set over which we search for the solution, meaning that the problem will be now formulated as follow:

$$\min_{v \in \mathcal{K}_\psi(\Omega)} \int_\Omega F(x, Dv)\,\mathrm{d}x, \tag{6.1.1}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n > 2$, $\psi : \Omega \mapsto [-\infty, +\infty)$ belonging to the Sobolev class $W^{1,p}_{\mathrm{loc}}(\Omega)$ is the *obstacle* and

$$\mathcal{K}_\psi(\Omega) = \{v \in u_0 + W^{1,p}_0(\Omega, \mathbb{R}) : v \geqslant \psi \text{ a.e. in } \Omega\}$$

is the class of the admissible functions, with $u_0 \in W^{1,p}(\Omega)$ a fixed boundary data and the integrand $F(x, Dv)$ satisfies non standard (p, q)-growth conditions (see [111, 112]).

In contrast with the rest of this thesis, in the following we will denote with $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^n$ and with $D_x F$ and $D_\xi F$ the derivative of F with respect to the first and second variable respectively. We consider an integrand F such that $\xi \mapsto F(x, \xi)$ is C^2 and there exists $\tilde{F} : \Omega \times [0, \infty) \to [0, \infty)$ such that

$$F(x, \xi) = \tilde{F}(x, |\xi|).$$

Furthermore, we assume that there exist positive constants $\tilde{\nu}, \tilde{L}$, and exponents p, q with $2 \leqslant p < q < p + 1 < +\infty$ and a parameter $0 \leqslant \mu \leqslant 1$ such that the following assumptions are satisfied

$$\langle D_{\xi\xi} F(x, \xi)\lambda, \lambda\rangle \geqslant \tilde{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\lambda|^2 \tag{F1}$$

$$|D_{\xi\xi} F(x, \xi)| \leqslant \tilde{L}\left[(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-2}{2}}\right] \tag{F2}$$

for almost every $x \in \Omega$ and every $\xi, \lambda \in \mathbb{R}^n$.

Arguing as in [44], the assumptions (F1) and (F2) combined with the radiality F, in the second variable, imply that there exists a positive constant $\tilde{\ell}$ such that

$$\frac{1}{\tilde{\ell}}(|\xi|^2 - \mu^2)^{\frac{p}{2}} \leqslant F(x,\xi) \leqslant \tilde{\ell}\left[(\mu^2 + |\xi|^2)^{\frac{p}{2}} + (\mu^2 + |\xi|^2)^{\frac{q}{2}}\right] \tag{F3}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$, i.e. the function F has non-standard growth conditions of (p,q)-type as defined and introduced by Marcellini in [111, 112] Regarding, the dependence of F on the $x-$variable, we will work under the assumption that there exists a non negative function $k(x) \in L^{\frac{p+2}{p-q+1}}$ such that

$$|D_{x\xi}F(x,\xi)| \leqslant k(x)\left[(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-1}{2}}\right] \tag{F4}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^n$.

As anticipated before, in case of standard growth conditions, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a solution to the obstacle problem (6.1.1) in $\mathcal{K}_\psi(\Omega)$ if and only if $u \in \mathcal{K}_\psi(\Omega)$ and u is a solution to the variational inequality

$$\int_\Omega \langle A(x, Du(x)), D(\varphi(x) - u(x))\rangle \, \mathrm{d}x \geqslant 0 \qquad \forall \varphi \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \,\text{and}\, \varphi \geqslant \psi, \tag{6.1.2}$$

where the operator $A(x,\xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$A(x,\xi) := D_\xi F(x,\xi).$$

In case of standard growth, by a density argument it possible to show the validity of (6.1.2) for every $\varphi \in \mathcal{K}_\psi(\Omega)$. Since we are dealing with non standard growth, it is worth noticing that (6.1.2) holds true also for solutions to (6.1.1) due to our assumptions $q - p < 1$ on the gap between the ellipticity exponent p and the growth exponent q. The validity of (6.1.2) can be easily checked as done at the beginning of the proof of the Theorem 6.1.3 below.
We stress that this is not obvious in the general case of non standard growth conditions. Even for unconstrained problems, the relation between minima and extremals, i.e. solutions of the corresponding Euler Lagrange system, is an issue that requires a careful investigation, as alredy discussed in the previous Chapter. From assumptions (F1) $-$ (F4), we derive the existence of positive constants ν, L, ℓ such that the following p-ellipticity and q-growth conditions are satisfied by the map A:

$$\langle A(x,\xi) - A(x,\eta), \xi - \eta\rangle \geqslant \nu|\xi - \eta|^2\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \tag{A1}$$

$$|A(x,\xi) - A(x,\eta)| \leqslant L|\xi - \eta|\left[\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} + \left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{q-2}{2}}\right] \tag{A2}$$

$$|A(x,\xi)| \leqslant \ell\left[\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} + \left(\mu^2 + |\xi|^2\right)^{\frac{q-1}{2}}\right], \tag{A3}$$

$$|D_x A(x,\xi)| \leqslant k(x)\left[(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-1}{2}}\right] \tag{Ã4}$$

for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^n$.
Thanks to a characterization of the Sobolev spaces due to Hajlasz [92], we deduce from $(\tilde{\mathrm{A}}4)$ that there exists a non-negative function $\kappa \in L_{\mathrm{loc}}^{\frac{p+2}{p-q+1}}(\Omega)$ such that

$$|A(x,\xi) - A(y,\xi)| \leqslant (\kappa(x) + \kappa(y)) |x-y| \left[\left(\mu^2 + |\xi|^2\right)^{\frac{p-1}{2}} + \left(\mu^2 + |\xi|^2\right)^{\frac{q-1}{2}} \right] \tag{A4}$$

for almost every $x, y \in \Omega$ and for all $\xi \in \mathbb{R}^n$.

Under the previous assumption we will prove the following result.

Theorem 6.1.3. *Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem* (6.1.1) *and let $A(x,\xi)$ satisfy the assumptions (A1)–(A4) with $2 \leqslant p < q < \min\{p+1, p^* = \frac{np}{n-p}\}$. Then, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ the following implication holds*

$$D\psi \in W_{\mathrm{loc}}^{1,\frac{p+2}{p+2-q}}(\Omega) \implies \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

with the following estimate

$$\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2 \,\mathrm{d}x \leqslant \frac{c(\|\psi\|^2_{L^\infty} + \|u\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{2}}} \cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}} + |Du(x)|^p \right] \mathrm{d}x. \tag{6.1.4}$$

It is worth noticing that the assumption of boundedness of the obstacle ψ is needed in order to get the boundedness of the solutions (see Theorem 6.2.7). If we want to remove the hypothesis $\psi \in L^\infty$, it is sufficient sufficient to assume the minimizers a priori bounded. Moreover, in this case we can remove also the hypothesis $q < p^*$.

The proof of Theorem 6.1.3 is structured as follow: first we verify the validity of the variational inequality (6.1.2) also in the case of non standard growth, then we combine an a priori estimate for the second derivatives of the local solutions, obtained using the well known difference quotient method, with a suitable approximation argument. The local boundedness of the obstacle (and thus of the solutions) allows us to use two interpolation inequalities that give the higher local integrability $L^{\frac{2(p+2)}{p+2-q}}$ for the gradient of the obstacle and the higher local integrability L^{p+2} of the gradient of the solutions. The higher integrability is what will let us weaken the assumption on κ, that is the function that control the dependence on x-variable of the operator A. We conclude noticing that, if the minimizer u is assumed a priori in L^r with $r > \frac{np}{n-p-2}$, instead of $u \in L^\infty$, then the interpolation inequality of Lemma 6.2.1 still gives a higher integrability result for Du, precisely $Du \in L^{\frac{r}{r+2}(p+2)}$. Such higher integrability allows us to obtain the same higher differentiability result of Theorem 6.1.3 assuming $\kappa \in L^{\frac{r}{(r-p)}\frac{(p+2)}{p-q+1}}$. We'd like to point out that for $p < n-2$ and $q < \frac{1}{n}(n - \frac{r}{r-p})p + \frac{1}{n}(n - 2\frac{r}{r-p})$ we get $\frac{r}{(r-p)}\frac{(p+2)}{p-q+1} < n$. This means that we obtain the regularity result again, but under a Sobolev assumption on the dependence on the x-variable below the critical one $W^{1,n}$.

6.2 Preliminaries and Notation

We will denote by C or c a general constant. Sunch constants may vary on occasions, even within the same line of estimates. Dependencies on parameters and other constants will be suitably emphasized using either parentheses or subscripts. Moreover, with $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ we will denote the ball centered at x of radius r. We will omit the dependence on the center when no confusion arises. Now we will state some Gagliardo-Nirenberg-type inequalities. Such inequalities will be stated as lemmas. The proofs of inequalities (6.2.2) and (6.2.3) can be found in [30, Appendix A], while the proof of (6.2.5) see for example [122].

Lemma 6.2.1. *For any $\phi \in C_0^1(\Omega)$ with $\phi \geqslant 0$, and any C^2 map $v : \Omega \to \mathbb{R}^N$, we have*

$$\begin{aligned}
&\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)|Dv(x)|^{\frac{m}{m+1}(p+2)}\mathrm{d}x \\
\leqslant&(p+2)^2\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)|v(x)|^{2m}\mathrm{d}x\right)^{\frac{1}{m+1}} \cdot \left[\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\left|D\phi(x)\right|^2\left|Dv(x)\right|^p \mathrm{d}x\right)^{\frac{m}{m+1}}\right. \\
&\left.+n\left(\int_\Omega \phi^{\frac{m}{m+1}(p+2)}(x)\left|Dv(x)\right|^{p-2}\left|D^2v(x)\right|^2\mathrm{d}x\right)^{\frac{m}{m+1}}\right],
\end{aligned} \tag{6.2.2}$$

for any $p \in (1,\infty)$ and $m > 1$. Moreover, for any $\mu \in [0,1]$

$$\begin{aligned}
&\int_\Omega \phi^2(x)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}|Dv(x)|^2\mathrm{d}x \\
\leqslant& c\|v\|^2_{L^\infty(\mathrm{supp}(\phi))}\int_\Omega \phi^2(x)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p-2}{2}}\left|D^2v(x)\right|^2\mathrm{d}x \\
&+c\|v\|^2_{L^\infty(\mathrm{supp}(\phi))}\int_\Omega\left(\phi^2(x)+|D\phi(x)|^2\right)\left(\mu^2+|Dv(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x,
\end{aligned} \tag{6.2.3}$$

for a constant $c = c(p)$.

Lemma 6.2.4. *Let $u \in L^p(\Omega) \cap W^{2,r}(\Omega)$ with $1 \leqslant p \leqslant \infty$ and $1 \leqslant r \leqslant \infty$. Then $u \in W^{1,q}(\Omega)$ where q is such that $\frac{1}{q} = \frac{1}{2}\left(\frac{1}{p}+\frac{1}{r}\right)$ and*

$$\|Du\|_{L^q} \leqslant C\|u\|^{\frac{1}{2}}_{W^{2,r}}\|u\|^{\frac{1}{2}}_{L^p} \tag{6.2.5}$$

The following theorem is an higher differentiability result for the solutions to (6.1.1) when f satisfies standard growth conditions. The proof can be found in [34].

Theorem 6.2.6. *Let $A(x,\xi)$ satisfy the conditions (A1)–(A4) with $p = q \geqslant 2$ and let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to the obstacle problem* (6.1.2). *Then, if $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$ the following implication*

$$D\psi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega) \implies \left(\mu^2+|Du|^2\right)^{\frac{p-2}{4}}Du \in W^{1,2}_{\mathrm{loc}}(\Omega),$$

holds true.

The proof of the next result can be found in [33, Theorem 1.1].

Theorem 6.2.7. *Let u in $K_\psi(\Omega)$ be a solution of* (6.1.1) *under the assumptions* (A1) and (A2) with $2 \leqslant p \leqslant q$ such that

$$p \leqslant q < p^* = \tfrac{np}{n-p} \quad \text{if} \quad p < n$$

$$p \leqslant q < \infty \qquad\qquad \text{if} \quad p \geqslant n$$

If the obstacle $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and the following estimate

$$\sup_{B_{R/2}} |u| \leqslant \left[\sup_{B_R} |\psi| + \left(\int_{B_R} |u(x)|^{p^*} \mathrm{d}x\right)\right]^\gamma \tag{6.2.8}$$

holds for every ball $B_R \Subset \Omega$, for $\gamma(n,p,q) > 0$ and $c = c(\ell, \nu, p, q, n)$.

In the proof of Theorem 6.1.3, we will make use of the auxiliary function $V_p : \mathbb{R}^n \to \mathbb{R}^n$, defined as

$$V_p(\xi) := \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi. \tag{6.2.9}$$

For such a function the following estimates hold (see [89]).

Lemma 6.2.10. *Let $1 < p < \infty$. There is a constant $c = c(n,p) > 0$ such that*

$$c^{-1}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \leqslant \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi - \eta|^2} \leqslant c\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}, \tag{6.2.11}$$

for any $\xi, \eta \in \mathbb{R}^n$ and $\xi \neq \eta$. Moreover, for a C^2 function g, there is a constant $C(p)$ such that

$$C^{-1}\left|D^2 g\right|^2 \left(\mu^2 + |Dg|^2\right)^{\frac{p-2}{2}} \leqslant |D\left(V_p(Dg)\right)|^2 \leqslant C\left|D^2 g\right|^2 \left(\mu^2 + |Dg|^2\right)^{\frac{p-2}{2}}. \tag{6.2.12}$$

Now we state a well known iteration lemma. The proof can be found in [89, Lemma 6.1]).

Lemma 6.2.13 (Iteration Lemma)**.** *Let $h : [\rho, R] \to \mathbb{R}$ be a nonnegative bounded function, $0 < \theta < 1$, $A, B \geqslant 0$ and $\gamma > 0$. Assume that*

$$h(r) \leqslant \theta h(s) + \frac{A}{(s-r)^\gamma} + B$$

for all $\rho \leqslant r < s \leqslant R_0 < R$. Then

$$h(\rho) \leqslant \frac{cA}{(R_0 - \rho)^\gamma} + cB,$$

where $c = c(\theta, \gamma) > 0$.

As anticipated, an important tool in proving Theorem 6.1.3 will be the different quotients method. We will now recall some related definitions and properties. The proof of the following results can be found [89].

Definition 6.2.14. Given $h \in \mathbb{R}^n$, for every function $F : \mathbb{R}^n \to \mathbb{R}$ the *Finite difference operator* is defined by

$$\tau_h F(x) = F(x+h) - F(x).$$

Proposition 6.2.15. *Let F and G be two functions such that $F, G \in W^{1,p}(\Omega)$, with $p \geqslant 1$, and let us consider the set*

$$\Omega_{|h|} := \{x \in \Omega : dist(x, \partial\Omega) > |h|\}.$$

Then

$(d1)$ $\tau_h F \in W^{1,p}(\Omega_{|h|})$ *and*

$$D_i(\tau_h F) = \tau_h(D_i F).$$

$(d2)$ *If at least one of the functions F or G has support contained in $\Omega_{|h|}$ then*

$$\int_\Omega F(x)\, \tau_h G(x)\, \mathrm{d}x = \int_\Omega G(x)\, \tau_{-h} F(x)\, \mathrm{d}x.$$

$(d3)$ *We have*

$$\tau_h(FG)(x) = F(x+h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result is the equivalent integral version of Lagrange Theorem.

Lemma 6.2.16. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$, and $F, DF \in L^p(B_R)$ then*

$$\int_{B_\rho} |\tau_h F(x)|^p \, dx \leqslant c(n,p)|h|^p \int_{B_R} |DF(x)|^p \, \mathrm{d}x.$$

Moreover

$$\int_{B_\rho} |F(x+h)|^p \, dx \leqslant \int_{B_R} |F(x)|^p \, \mathrm{d}x.$$

Lemma 6.2.17. *Let $F : \mathbb{R}^n \to \mathbb{R}^N$, $F \in L^p(B_R)$ with $1 < p < +\infty$. Suppose that there exist $\rho \in (0, R)$ and $M > 0$ such that*

$$\sum_{s=1}^{n} \int_{B_\rho} |\tau_{s,h} F(x)|^p \mathrm{d}x \leqslant M^p |h|^p$$

for every $h < \frac{R-\rho}{2}$. Then $F \in W^{1,p}(B_R, \mathbb{R}^N)$. Moreover

$$\|DF\|_{L^p(B_\rho)} \leqslant M.$$

To conclude this section we state a Lemma that will be the main tool in the second part of the proof of Theorem 6.1.3. The proof of the following lemmma can be found in [43].

Lemma 6.2.18. *Let $f : \Omega \times \mathbb{R}^n \to [0, \infty)$ be a Carathéodory function such that $\xi \mapsto f(x, \xi)$ is $\mathcal{C}^2$ and there exists $\tilde{f} : \Omega \times [0, \infty) \to [0, \infty)$ such that $f(x, \xi) = \tilde{f}(x, |\xi|)$. Moreover, let us assume that f satisfies assumptions (F1)–(F4). Then there exists a sequence $(f_\varepsilon)_\varepsilon$ of Carathéodory functions $f_\varepsilon : \Omega \times \mathbb{R}^n \to [0, \infty)$, monotonically convergent to f, such that*

(i) *for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$ and for every $\varepsilon_1 < \varepsilon_2$, we have*

$$f_{\varepsilon_2}(x, \xi) \leqslant f_{\varepsilon_1}(x, \xi) \leqslant f(x, \xi)$$

(ii) *there exists $\bar{\nu} > 0$ depending only on p and $\tilde{\nu}$ such that*

$$\langle D_{\xi\xi} f_\varepsilon(x,\xi)\lambda, \lambda\rangle \geqslant \bar{\nu}(\mu^2 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$,

(iii) *there exist K_0, K_1 independent of ε and $\bar{K}_1$ depending on ε such that*

$$K_0(|\xi|^p - \mu^2) \leqslant f_\varepsilon(x,\xi) \leqslant K_1\left[(\mu^2 + |\xi|^2)^{\frac{p}{2}} + (\mu^2 + |\xi|^2)^{\frac{q}{2}}\right],$$
$$f_\varepsilon(x,\xi) \leqslant \bar{K}_1(\varepsilon)(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$,

(iv) *there exists a constant $C(\varepsilon) > 0$ such that*

$$|D_{x\xi} f_\varepsilon(x,\xi)| \leqslant k(x)\left[(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} + (\mu^2 + |\xi|^2)^{\frac{q-1}{2}}\right]$$

$$|D_{x\xi} f(x,\xi)| \leqslant C(\varepsilon)k(x)(\mu^2 + |\xi|^2)^{\frac{p-1}{2}}$$

for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^n$.

We can now turn to the proof of our main result.

6.3 Proof of Theorem 6.1.3

The proof of the theorem is obtained in two steps: first we establish the a priori estimate and then we conclude through an approximation argument.

Step 1: The a priori estimate.
In order to get the a priori estimate we first need to prove the validity of the variational inequality (6.1.2) also in the case of non-standard growth conditions.
Suppose that u is a local solution to the obstacle problem in $\mathcal{K}_\psi(\Omega)$ such that

$$Du \in W^{1,2}_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad \left(\mu^2 + |Du|^2\right)^{\frac{p-2}{4}} Du \in W^{1,2}_{\mathrm{loc}}(\Omega). \tag{6.3.1}$$

Thanks to our assumptions on the exponents p and q we can deduce from Theorem 6.2.7 that the solution u to (6.1.1) is bounded. Such boundedness, with the a priori assumption (6.3.1) on the second derivatives of u, allows us to apply Lemma 6.2.1 to get the higher integrability $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$.
Concerning the obstacle ψ, by the assumptions $\psi \in L^\infty(\Omega)$ and $D^2\psi \in L^{\frac{p+2}{p+2-q}}(\Omega)$, applying Lemma 6.2.4, we have $D\psi \in L^{\frac{2(p+2)}{p+2-q}}(\Omega) \hookrightarrow L^{p+2}(\Omega)$.

Note that $Du \in L^{p+2}_{\mathrm{loc}}(\Omega)$ (and then, obviously, $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$) implies that the variational inequality (6.1.2), by a simple density argument, holds true for every $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$.

Indeed, since $u \in \mathcal{K}_\psi(\Omega)$, for every $v \geqslant 0$ and every $\varepsilon > 0$ it results $u + \varepsilon v \geqslant \psi$, therefore if $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$ by minimality of u

$$\int_\Omega F(x, Du(x))\, dx \leqslant \int_\Omega F(x, Du + \varepsilon Dv(x))\, \mathrm{d}x$$

or equivalently

$$\int_\Omega \Big[F(x, Du(x) + \varepsilon Dv(x)) - F(x, Du(x)) \Big]\, \mathrm{d}x \geqslant 0.$$

Hence, we have

$$\varepsilon \int_\Omega \int_0^1 \langle D_\xi F(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x) \rangle\, \mathrm{d}\theta\, \mathrm{d}x \geqslant 0$$

and also

$$\int_\Omega \int_0^1 \langle D_\xi F(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x) \rangle\, \mathrm{d}\theta\, \mathrm{d}x \geqslant 0$$

where we divided both side of previous inequality by ε. We observe that

$$\begin{aligned}
0 &\leqslant \int_\Omega \int_0^1 \langle D_\xi F(x, Du(x) + \theta\varepsilon Dv), Dv \rangle\, \mathrm{d}\theta\, \mathrm{d}x \\
&\leqslant \int_\Omega \int_0^1 |D_\xi F(x, Du + \theta\varepsilon Dv(x))||Dv(x)|\, \mathrm{d}\theta\, \mathrm{d}x \\
&\leqslant \int_\Omega \int_0^1 (\mu^2 + |Du + \theta\varepsilon Dv(x)|^2)^{\frac{q-1}{2}} |Dv(x)|\, \mathrm{d}\theta\, \mathrm{d}x \\
&\leqslant c \int_\Omega (\mu^2 + |Du(x)|^2 + \varepsilon^2 |Dv(x)|^2)^{\frac{q-1}{2}} |Dv(x)|\, \mathrm{d}x, \qquad (6.3.2)
\end{aligned}$$

where in the last inequality we used Lemma 8.3 in [89].
Therefore, since $v \in W^{1,q}_{\mathrm{loc}}(\Omega)$, by the growth assumption (A3), assuming without loss of generality $\varepsilon < 1$, we get

$$\int_0^1 \langle D_\xi F(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x) \rangle\, \mathrm{d}\theta \leqslant \mu^q + |Du|^q + |Dv|^q \in L^1(\Omega).$$

Then, applying dominated convergence theorem in (6.3.2), we have

$$\lim_{\varepsilon \to 0} \int_\Omega \int_0^1 \langle D_\xi F(x, Du(x) + \theta\varepsilon Dv(x)), Dv(x) \rangle\, d\theta\, \mathrm{d}x = \int_\Omega \langle D_\xi F(x, Du(x)), Dv(x) \rangle\, \mathrm{d}x \geqslant 0$$

for every $v \in W^{1,q}_0(\Omega)$, $v \geqslant 0$. At this point it is standard to verify the inequality (6.1.2)

$$\int_\Omega \langle D_\xi F(x, Du(x)), D\varphi(x) - Du(x) \rangle\, \mathrm{d}x \geqslant 0.$$

Now we have to choose suitable test functions φ in (6.1.2) that involve the different quotient of the solution and at the same time satisfy the conditions $\varphi \in W^{1,q}_{\mathrm{loc}}(\Omega)$ and $\varphi \geqslant \psi$ in Ω. In order to do this, we proceed similarly to what has been done in [34, 62].

Let us fix a ball $B_R \Subset \Omega$ and arbitrary radii $\frac{R}{2} < r < s < t < \lambda r < R$, with $1 < \lambda < 2$. Let us consider a cut off function $\eta \in C_0^\infty(B_t)$ such that $\eta \equiv 1$ on B_s and $|D\eta| \leqslant \frac{c}{t-s}$. From now on, with no loss of generality, we suppose $R < 1$.

Let $v \in W_0^{1,q}(\Omega)$ be such that

$$u - \psi + \tau v \geqslant 0 \qquad \forall \tau \in [0,1], \tag{6.3.3}$$

and observe that $\varphi = u + \tau v \geqslant \psi$ for all $\tau \in [0,1]$. For $|h| < \frac{R}{4}$, we consider

$$v_1(x) = \eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right],$$

so we have $v_1 \in W_0^{1,p+2}(\Omega)$, and, for any $\tau \in [0,1]$, v_1 satisfies (6.3.3). Indeed, for a.e. $x \in \Omega$ and for any $\tau \in [0,1]$

$$\begin{aligned} u(x) - \psi(x) + \tau v_1(x) &= u(x) - \psi(x) + \tau\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right] \\ &= \tau\eta^2(x)(u-\psi)(x+h) + (1-\tau\eta^2(x))(u-\psi)(x) \geqslant 0, \end{aligned}$$

since $u \in \mathcal{K}_\psi(\Omega)$ and $0 \leqslant \eta \leqslant 1$. Therefore, from $q - p < 1$ we have $L^{p+2}(\Omega) \hookrightarrow L^q(\Omega)$ and so we can use $\varphi = u + \tau v_1$ as a test function in inequality (6.1.2), thus getting

$$0 \leqslant \int_\Omega \left\langle A(x, Du(x)), D\left[\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right]\right]\right\rangle \mathrm{d}x. \tag{6.3.4}$$

Similarly, we define

$$v_2(x) = \eta^2(x-h)\left[(u-\psi)(x-h) - (u-\psi)(x)\right],$$

and we have $v_2 \in W_0^{1,p+2}(\Omega)$, the inequality (6.3.3) still is satisfied for any $\tau \in [0,1]$, and we can use $\varphi = u + \tau v_2$ as test function in (6.1.2), obtaining

$$0 \leqslant \int_\Omega \left\langle A(x, Du(x)), D\left[\eta^2(x-h)\left[(u-\psi)(x-h) - (u-\psi)(x)\right]\right]\right\rangle \mathrm{d}x,$$

and by means of a change of variable, we have

$$0 \leqslant \int_\Omega \left\langle A(x+h, Du(x+h)), D\left[\eta^2(x)\left[(u-\psi)(x) - (u-\psi)(x+h)\right]\right]\right\rangle \mathrm{d}x. \tag{6.3.5}$$

Now we can add (6.3.4) and (6.3.5), thus getting

$$\begin{aligned} 0 \leqslant &\int_\Omega \left\langle A(x, Du(x)), D\left[\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right]\right]\right\rangle \mathrm{d}x \\ &+ \int_\Omega \left\langle A(x+h, Du(x+h)), D\left[\eta^2(x)\left[(u-\psi)(x) - (u-\psi)(x+h)\right]\right]\right\rangle \mathrm{d}x, \end{aligned}$$

that is

$$0 \leqslant \int_\Omega \left\langle A(x, Du(x)) - A(x+h, Du(x+h)), D\left[\eta^2(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right]\right]\right\rangle \mathrm{d}x,$$

which implies

$$0 \geqslant \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), \eta^2(x) D\left[(u-\psi)(x+h) - (u-\psi)(x)\right]\right\rangle \mathrm{d}x$$
$$+ \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x, Du(x)), 2\eta(x) D\eta(x)\left[(u-\psi)(x+h) - (u-\psi)(x)\right]\right\rangle \mathrm{d}x.$$

Previous inequality can be rewritten as follows

$$\begin{aligned}
0 \geqslant & \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x+h, Du(x)), \eta^2(x)(Du(x+h) - Du(x))\right\rangle \mathrm{d}x \\
& - \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x+h, Du(x)), \eta^2(x)(D\psi(x+h) - D\psi(x))\right\rangle \mathrm{d}x \\
& + \int_\Omega \left\langle A(x+h, Du(x+h)) - A(x+h, Du(x)), 2\eta(x) D\eta(x) \tau_h (u-\psi)(x)\right\rangle \mathrm{d}x \\
& + \int_\Omega \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2(x)(Du(x+h) - Du(x))\right\rangle \mathrm{d}x \\
& - \int_\Omega \left\langle A(x+h, Du(x)) - A(x, Du(x)), \eta^2(x)(D\psi(x+h) - D\psi(x))\right\rangle \mathrm{d}x \\
& + \int_\Omega \left\langle A(x+h, Du(x)) - A(x, Du(x)), 2\eta(x) D\eta(x) \tau_h (u-\psi)(x)\right\rangle \mathrm{d}x \\
=: & \, I + II + III + IV + V + VI,
\end{aligned} \tag{6.3.6}$$

so we have

$$I \leqslant |II| + |III| + |IV| + |V| + |VI|. \tag{6.3.7}$$

The ellipticity assumption (A1) implies

$$I \geqslant \nu \int_\Omega \eta^2(x) |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} \mathrm{d}x. \tag{6.3.8}$$

By virtue of assumption (A2), we have

$$\begin{aligned}
|II| \leqslant & L \int_\Omega \eta^2(x) |\tau_h Du(x)| \Big[(\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} \\
& \qquad + (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{q-2}{2}} \Big] |\tau_h D\psi(x)| \mathrm{d}x \\
= & L \int_\Omega \eta^2(x) |\tau_h Du(x)| (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{p-2}{2}} |\tau_h D\psi(x)| \mathrm{d}x \\
& + L \int_\Omega \eta^2(x) |\tau_h Du(x)| (\mu^2 + |Du(x)|^2 + |Du(x+h)|^2)^{\frac{q-2}{2}} |\tau_h D\psi(x)| \mathrm{d}x \\
=: & II_1 + II_2.
\end{aligned} \tag{6.3.9}$$

Let us consider the term II_1. If we apply Young's inequality with exponents $(2,2)$ and Hölder's inequality with exponents $\left(\frac{p+2}{4}, \frac{p+2}{p-2}\right)$, we get

$$\begin{aligned}
II_1 \leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon \int_\Omega \eta^2(x)|\tau_h D\psi(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
\leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon \left(\int_{B_t} |\tau_h D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_t} (\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}},
\end{aligned}$$

where we also used the properties of η. Since $D\psi \in W^{1,\frac{p+2}{p+2-q}}_{\mathrm{loc}}(\Omega)$ and $2 \leqslant p < q < p+1$, we also have $D\psi \in W^{1,\frac{p+2}{2}}_{\mathrm{loc}}(\Omega)$, and using both estimates of Lemma 6.2.16, we get

$$\begin{aligned}
II_1 \leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}.
\end{aligned} \tag{6.3.10}$$

In order to estimate the term II_2, applying Young's inequality with exponents $(2,2)$ and Hölder's inequality with exponents $\left(\frac{p+2}{2(p+2-q)}, \frac{p+2}{2q-p-2}\right)$, we get

$$\begin{aligned}
II_2 \leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon \int_\Omega \eta^2(x)|\tau_h D\psi(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{2q-p-2}{2}}\mathrm{d}x \\
\leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon \left(\int_{B_t} |\tau_h D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}} \\
&\cdot \left(\int_{B_t} (\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}
\end{aligned}$$

where we used also the properties of η. Since $D\psi \in W^{1,\frac{p+2}{p+2-q}}_{\mathrm{loc}}(\Omega)$, we may use the first and the second estimate of Lemma 6.2.16, thus obtaining

$$\begin{aligned}
II_2 \leqslant& \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}} \cdot \left(\int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.
\end{aligned} \tag{6.3.11}$$

Plugging (6.3.10) and (6.3.11) into (6.3.9), we get

$$
\begin{aligned}
|II| \leqslant & 2\varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
& + c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}} \\
& + c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.
\end{aligned}
\tag{6.3.12}
$$

Arguing analogously, by virtue of assumption (A2) we have

$$
\begin{aligned}
|III| \leqslant & 2L\int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|\left[\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\right. \\
& \left. +\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{q-2}{2}}\right]|\tau_h\left(u-\psi\right)(x)|\mathrm{d}x \\
= & c\int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}|\tau_h\left(u-\psi\right)(x)|\mathrm{d}x \\
& + \int_\Omega \eta(x)|D\eta(x)||\tau_h Du(x)|\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{q-2}{2}}|\tau_h\left(u-\psi\right)(x)|\mathrm{d}x \\
& := III_1+III_2.
\end{aligned}
\tag{6.3.13}
$$

Using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{p+2}{4},\frac{p+2}{p-2}\right)$, and the properties of η, we have

$$
\begin{aligned}
III_1 \leqslant & \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
& + c_\varepsilon\int_\Omega|\tau_h(u-\psi)(x)|^2|D\eta(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
\leqslant & \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
& + \frac{c_\varepsilon}{(t-s)^2}\left(\int_{B_t}|\tau_h(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \\
& \quad \cdot\left(\int_{B_t}(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}},
\end{aligned}
$$

and Lemma 6.2.16 implies

$$
\begin{aligned}
III_1 \leqslant & \varepsilon\int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
& + \frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}.
\end{aligned}
\tag{6.3.14}
$$

Similarly, using Young's inequality with exponents $(2,2)$, Hölder's inequality with exponents $\left(\frac{p+2}{2(p+2-q)}, \frac{p+2}{2q-p-2}\right)$, the properties of η and Lemma 6.2.16, we get

$$
\begin{aligned}
III_2 \leqslant\, & \varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}} |D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}} \cdot \left(\int_{B_{\lambda r}} \left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.
\end{aligned} \tag{6.3.15}
$$

Plugging (6.3.14) and (6.3.15) into (6.3.13), we get

$$
\begin{aligned}
|III| \leqslant\, & 2\varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}} |D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}} \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}} |D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}} \cdot \left(\int_{B_{\lambda r}} \left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}.
\end{aligned} \tag{6.3.16}
$$

For what concerns the term IV, assumption (A4) implies

$$
\begin{aligned}
|IV| \leqslant & |h| \int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right) \qquad (6.3.17)\\
& \left[\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}+\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}\right]|\tau_h Du(x)|\mathrm{d}x \\
= & |h| \int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h Du(x)|\mathrm{d}x \\
& + |h| \int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}|\tau_h Du(x)|\mathrm{d}x \\
& =: IV_1 + IV_2. \qquad (6.3.18)
\end{aligned}
$$

If we use Young's inequality with exponents $(2,2)$ and the properties of η, we obtain

$$
\begin{aligned}
IV_2 \leqslant & |h| \int_\Omega \eta^2(x)\left(\kappa(x+h)+\kappa(x)\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}|\tau_h Du(x)|\mathrm{d}x \\
\leqslant & \varepsilon \int_\Omega \eta^2(x)\left|\tau_h Du(x)\right|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x \\
& + c_\varepsilon |h|^2 \int_{B_t} \left(\kappa(x+h)+\kappa(x)\right)^2\left(\mu^2+|Du(x)|^2\right)^{\frac{2q-p}{2}}\mathrm{d}x.
\end{aligned}
$$

Using Hölder's inequality with exponents $\left(\frac{p+2}{2(p-q+1)}, \frac{p+2}{2q-p}\right)$ and Lemma 6.2.16 we have

$$
\begin{aligned}
IV_2 \leqslant & \varepsilon \int_\Omega \eta^2(x) \left|\tau_h Du(x)\right|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} \mathrm{d}x \\
& + c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2} + |Du(x)|^{p+2}\right) \mathrm{d}x\right)^{\frac{2q-p}{p+2}} .
\end{aligned} \tag{6.3.19}
$$

Analogously, since $\kappa \in L^{\frac{p+2}{p-q+1}}_{\mathrm{loc}}(\Omega) \hookrightarrow L^{p+2}_{\mathrm{loc}}(\Omega)$, using Young's inequality with exponents $(2,2)$ Hölder's inequality with exponents $\left(\frac{p+2}{2}, \frac{p+2}{p}\right)$, the properties of η and Lemma 6.2.16, we get

$$
\begin{aligned}
IV_1 \leqslant & \varepsilon \int_\Omega \eta^2(x) \left|\tau_h Du(x)\right|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} \mathrm{d}x \\
& + c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} \kappa^{p+2}(x) \mathrm{d}x\right)^{\frac{2}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2} + |Du(x)|^{p+2}\right) \mathrm{d}x\right)^{\frac{p}{p+2}} .
\end{aligned} \tag{6.3.20}
$$

Plugging (6.3.19) and (6.3.20) into (6.3.17), we obtain

$$
\begin{aligned}
|IV| \leqslant & 2\varepsilon \int_\Omega \eta^2(x) \left|\tau_h Du(x)\right|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}} \mathrm{d}x \\
& + c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2} + |Du(x)|^{p+2}\right) \mathrm{d}x\right)^{\frac{2q-p}{p+2}} \\
& + c_\varepsilon |h|^2 \left(\int_{B_{\lambda r}} \kappa^{p+2}(x) \mathrm{d}x\right)^{\frac{2}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2} + |Du(x)|^{p+2}\right) \mathrm{d}x\right)^{\frac{p}{p+2}} .
\end{aligned} \tag{6.3.21}
$$

The condition (A4) also entails

$$
\begin{aligned}
|V| \leqslant & |h| \int_{\Omega} \eta^2(x) \left(\kappa(x+h)+\kappa(x)\right) \left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}} |\tau_h D\psi(x)| \,\mathrm{d}x \\
& + |h| \int_{\Omega} \eta^2(x) \left(\kappa(x+h)+\kappa(x)\right) \left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}} |\tau_h D\psi(x)| \,\mathrm{d}x \\
\leqslant & |h| \left(\int_{B_t} \left(\kappa(x+h)+\kappa(x)\right)^{p+2} \mathrm{d}x \right)^{\frac{1}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2}+|Du(x)|^{p+2}\right) dx \right)^{\frac{p-1}{p+2}} \\
& \cdot \left(\int_{B_t} |\tau_h D\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x \right)^{\frac{2}{p+2}} \\
& + |h| \left(\int_{B_t} \left(\kappa(x+h)+\kappa(x)\right)^{\frac{p+2}{p-q+1}} \mathrm{d}x \right)^{\frac{p-q+1}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2}+|Du(x)|^{p+2}\right) dx \right)^{\frac{q-1}{p+2}} \\
& \cdot \left(\int_{B_t} |\tau_h D\psi(x)|^{\frac{p+2}{2}} \mathrm{d}x \right)^{\frac{2}{p+2}} \\
\leqslant & c|h|^2 \left(\int_{B_{\lambda r}} \kappa^{p+2}(x) \mathrm{d}x \right)^{\frac{1}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2}+|Du(x)|^{p+2}\right) \mathrm{d}x \right)^{\frac{p-1}{p+2}} \\
& \cdot \left(\int_{B_{\lambda r}} \left|D^2\psi(x)\right|^{\frac{p+2}{2}} \mathrm{d}x \right)^{\frac{2}{p+2}} \\
& + c|h|^2 \left(\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x \right)^{\frac{p-q+1}{p+2}} \cdot \left(\int_{B_t} \left(\mu^{p+2}+|Du(x)|^{p+2}\right) \mathrm{d}x \right)^{\frac{q-1}{p+2}} \\
& \cdot \left(\int_{B_{\lambda r}} \left|D^2\psi(x)\right|^{\frac{p+2}{2}} \mathrm{d}x \right)^{\frac{2}{p+2}} \qquad (6.3.22)
\end{aligned}
$$

where we used Hölder's inequality with exponents $\left(p+2, \frac{p+2}{p-1}, \frac{p+2}{2}\right)$ and $\left(\frac{p+2}{p-q+1}, \frac{p+2}{q-1}, \frac{p+2}{2}\right)$, the properties of η and Lemma 6.2.16.

Finally, using again assumption (A4), the properties of η, Hölder's inequality and Lemma 6.2.16, we have

$$
\begin{aligned}
|VI| \leqslant & 2|h| \int_\Omega \eta(x)\,|D\eta(x)|\,(\kappa(x+h)+\kappa(x))\left(\mu^2+|Du(x)|^2\right)^{\frac{p-1}{2}}|\tau_h(u-\psi)(x)|\,\mathrm{d}x \\
& + 2|h| \int_\Omega \eta(x)\,|D\eta(x)|\,(\kappa(x+h)+\kappa(x))\left(\mu^2+|Du(x)|^2\right)^{\frac{q-1}{2}}|\tau_h(u-\psi)(x)|\,\mathrm{d}x \\
\leqslant & \frac{c|h|}{t-s}\left(\int_{B_t}(\kappa(x+h)+\kappa(x))^{p+2}\,\mathrm{d}x\right)^{\frac{1}{p+2}} \cdot \left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}} \\
& \cdot \left(\int_{B_t}|\tau_h(u-\psi)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}} \\
& + \frac{c|h|}{t-s}\left(\int_{B_t}(\kappa(x+h)+\kappa(x))^{\frac{p+2}{p-q+1}}\,\mathrm{d}x\right)^{\frac{p-q+1}{p+2}} \cdot \left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}} \\
& \cdot \left(\int_{B_t}|\tau_h(u-\psi)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}} \\
\leqslant & \frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}} \cdot \left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}} \\
& \cdot \left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}} \\
& + \frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}} \cdot \left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}} \\
& \cdot \left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\,\mathrm{d}x\right)^{\frac{2}{p+2}}. \qquad (6.3.23)
\end{aligned}
$$

Inserting (6.3.8), (6.3.12), (6.3.16), (6.3.21), (6.3.22) and (6.3.23) into (6.3.7) we infer

$$
\begin{aligned}
& \nu \int_\Omega \eta^2(x)|\tau_h Du(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\,\mathrm{d}x \\
\leqslant \;& 6\varepsilon \int_\Omega \eta^2(x)|\tau_h Du(x)|^2(\mu^2+|Du(x)|^2+|Du(x+h)|^2)^{\frac{p-2}{2}}\mathrm{d}x \\
& + c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}} \\
& + c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}} \cdot \left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}} \\
& + \frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}} \cdot \left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}} \\
& + \frac{c_\varepsilon|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}} \cdot \left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}} \\
& + c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}} \cdot \left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}
\end{aligned}
$$

$$
\begin{aligned}
&+c_\varepsilon|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}\\
&\qquad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\
&\qquad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\qquad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\
&\qquad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}
\end{aligned}
$$

Choosing $\varepsilon=\frac{\nu}{12}$, we can reabsorb the first term from the right-hand side to the left-hand one, thus getting

$$
\begin{aligned}
&\nu\int_\Omega\eta^2(x)|\tau_hDu(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
\leqslant\ & c|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2-q)}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&+\frac{c|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{4}{p+2}}\cdot\left(\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{p-2}{p+2}}\\
&+\frac{c|h|^2}{(t-s)^2}\left(\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{2(p+2)-2q}{p+2}}\cdot\left(\int_{B_{\lambda r}}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p-2}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{2p-2q+2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{2q-p}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{2}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{p-1}{p+2}}
\end{aligned}
$$

$$
\begin{aligned}
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+c|h|^2\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)\mathrm{d}x\right)^{\frac{q-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D^2\psi(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{p+2}\mathrm{d}x\right)^{\frac{1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{p-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}\\
&+\frac{c|h|^2}{t-s}\left(\int_{B_{\lambda r}}\kappa(x)^{\frac{p+2}{p-q+1}}\mathrm{d}x\right)^{\frac{p-q+1}{p+2}}\cdot\left(\int_{B_t}\left(\mu^{p+2}+|Du(x)|^{p+2}\right)dx\right)^{\frac{q-1}{p+2}}\\
&\quad\cdot\left(\int_{B_{\lambda r}}\left|D\left(u-\psi\right)(x)\right|^{\frac{p+2}{2}}\mathrm{d}x\right)^{\frac{2}{p+2}}.
\end{aligned}
$$

Now we apply Young's inequalities and since $u\in\mathcal{K}_\psi(\Omega)$, we have

$$
\begin{aligned}
&\nu\int_\Omega\eta^2(x)|\tau_hDu(x)|^2\left(\mu^2+|Du(x+h)|^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\mathrm{d}x\\
\leqslant\;&10\varepsilon|h|^2\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\\
&+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon|h|^2\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D(u-\psi)(x)|^{\frac{p+2}{2}}\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p+2-q}}}\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon|h|^2\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x. \qquad (6.3.24)
\end{aligned}
$$

By Young's inequalities of exponents $\left(p+2-q,\frac{p+2-q}{p+1-q}\right)$ we can estimate the thirdlast integral appearing in the right hand side of the previous inequality as

$$
\begin{aligned}
&\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D(u-\psi)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x\\
\leqslant&\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|Du(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x\\
\leqslant& c_\varepsilon|h|^2\int_{B_{\lambda r}}|Du(x)|^{p+2}\,\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x,\\
\leqslant&\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\varepsilon|h|^2\int_{B_{\lambda r}}(\mu^{p+2}+|Du(x)|^{p+2})\,\mathrm{d}x+\frac{c_\varepsilon|h|^2}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x,
\end{aligned}
$$

and similarly, using Young's inequality with exponents $(2,2)$, we get

$$\begin{aligned}
&\frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D(u-\psi)|^{\frac{p+2}{2}}\,\mathrm{d}x \\
\leqslant & \frac{c_\varepsilon |h|^2}{(t-s)^{p+2}}|B_R| + \varepsilon |h|^2 \int_{B_{\lambda r}} (\mu^{p+2} + |Du(x)|^{p+2})\,\mathrm{d}x + \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}}\,\mathrm{d}x.
\end{aligned}$$

So, from (6.3.24), we get

$$\begin{aligned}
&\nu \int_\Omega \eta^2(x) |\tau_h Du(x)|^2 \left(\mu^2 + |Du(x+h)|^2 + |Du(x)|^2\right)^{\frac{p-2}{2}}\,\mathrm{d}x \\
\leqslant\; & 12\varepsilon |h|^2 \int_{B_{\lambda r}} (\mu^2 + |Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{p-q+2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x + c_\varepsilon |h|^2 \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + c_\varepsilon |h|^2 \int_{B_{\lambda r}} \kappa^{p+2}(x)\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R| + \frac{c_\varepsilon |h|^2}{(t-s)^{p+2}}|B_R|. \qquad (6.3.25)
\end{aligned}$$

Using, in the left hand side of the previous estimate, the right-hand side of the inequality (6.2.11) in Lemma 6.2.10 , we get

$$\begin{aligned}
&\nu \int_\Omega \eta^2(x) \left|\tau_h V_p\left(Du(x)\right)\right|^2\mathrm{d}x \\
\leqslant\; & 12\varepsilon |h|^2 \int_{B_{\lambda r}} (\mu^2 + |Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{p-q+2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{2}}} \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x + c_\varepsilon |h|^2 \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x \\
&+ c_\varepsilon |h|^2 \int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + c_\varepsilon |h|^2 \int_{B_{\lambda r}} \kappa^{p+2}(x)\mathrm{d}x \\
&+ \frac{c_\varepsilon |h|^2}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R| + \frac{c_\varepsilon |h|^2}{(t-s)^{p+2}}|B_R|.
\end{aligned}$$

Dividing both sides by $|h|^2$ and using Lemma 6.2.17 and the properties of η, we have

$$\begin{aligned}
&\nu \int_{B_s} \left|DV_p\left(Du(x)\right)\right|^2\mathrm{d}x \\
\leqslant\; & 12\varepsilon \int_{B_{\lambda r}} (\mu^2 + |Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\\
&+c_\varepsilon\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+c_\varepsilon\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\\
&+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\frac{c_\varepsilon}{(t-s)^{p+2}}|B_R|.
\end{aligned}
\tag{6.3.26}
$$

Now, by virtue of left-hand side of inequality (6.2.12) of Lemma 6.2.10

$$
\begin{aligned}
&\int_{B_s}\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\left|D^2u(x)\right|^2\mathrm{d}x\leqslant\int_{B_s}|DV_p(Du(x))|^2\mathrm{d}x\\
\leqslant\;&12\varepsilon\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\\
&+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x\\
&+c_\varepsilon\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+c_\varepsilon\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\\
&+\frac{c_\varepsilon}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\frac{c_\varepsilon}{(t-s)^{p+2}}|B_R|.
\end{aligned}
\tag{6.3.27}
$$

By virtue of the local boundedness of u, the second interpolation inequality of Lemma 6.2.1 yields

$$
\begin{aligned}
&\int_\Omega\eta^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\mathrm{d}x\\
&\leqslant c\|u\|^2_{L^\infty(\mathrm{supp}(\eta))}\int_\Omega\eta^2(x)\left(\mu^2+|Du(x)|^2\right)^{\frac{p-2}{2}}\left|D^2u(x)\right|^2\mathrm{d}x\\
&\quad+c\|u\|^2_{L^\infty(\mathrm{supp}(\eta))}\int_\Omega\left(|\eta(x)|^2+|D\eta(x)|^2\right)\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x.
\end{aligned}
$$

and so, combining this last estimate with (6.3.27), and using the properties of η, we get

$$
\begin{aligned}
&\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\mathrm{d}x\\
\leqslant\;&12\varepsilon c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}(\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\mathrm{d}x\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{2}}}\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\\
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}|B_R|+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{p+2}}|B_R|\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^2}\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x. \qquad (6.3.28)
\end{aligned}
$$

Now let us notice that, since $2\leqslant p<q<p+1$, we have $\frac{p+2}{2}<\frac{p+2}{p+2-q}$ and $p+2<\frac{p+2}{p-q+1}$. So using Young's inequality with exponents $\left(\frac{2}{q-p},\frac{2}{p+2-q}\right)$, we get

$$
\begin{aligned}
\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x &\leqslant c|B_R|^{\frac{q-p}{2}}\left(\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right)^{\frac{p+2-q}{2}}\\
&\leqslant c|B_R|+c\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x,
\end{aligned}
$$

and similarly

$$
\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{2}}\mathrm{d}x \leqslant c|B_R|+c\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x.
$$

Moreover, since $p+2<\frac{p+2}{p-q+1}$, by Young's inequality with exponents $\left(\frac{1}{q-p},\frac{1}{p-q+1}\right)$, we have

$$
\begin{aligned}
\int_{B_{\lambda r}}\kappa^{p+2}(x)\mathrm{d}x &\leqslant c|B_R|^{q-p}\left(\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right)^{p-q+1}\\
&\leqslant c|B_R|+c\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x.
\end{aligned}
$$

So, since $t-s<1$, (6.3.28)becomes

$$
\begin{aligned}
&\int_{B_r}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}|Du(x)|^2\,\mathrm{d}x\leqslant 12\varepsilon c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p+2}{2}}\mathrm{d}x\\
&+c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}\left[\int_{B_{\lambda r}}\kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x+\int_{B_{\lambda r}}|D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+\frac{c_\varepsilon\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}}\left[\int_{B_{\lambda r}}\left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x+\int_{B_{\lambda r}}|D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x+|B_R|\right],
\end{aligned}
$$

and since $0 \leqslant \mu \leqslant 1$, we get

$$\begin{aligned}
&\int_{B_r} |Du(x)|^{p+2}\,\mathrm{d}x \qquad (6.3.29)\\
&\leqslant \int_{B_r} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}} |Du(x)|^2\,\mathrm{d}x\\
&\leqslant 12\varepsilon c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})} \int_{B_{\lambda r}} (\mu^2+|Du(x)|^2)^{\frac{p+2}{2}}\,\mathrm{d}x\\
&+ c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})} \left[\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+ \frac{c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}} \left[\int_{B_{\lambda r}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x + \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + |B_R|\right]\\
&\leqslant 12\varepsilon c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})} \int_{B_{\lambda r}} |Du(x)|^{p+2}\mathrm{d}x\\
&+ c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})} \left[\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+ \frac{c_\varepsilon \|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}} \left[\int_{B_{\lambda r}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x + \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + |B_R|\right],
\end{aligned}$$

Choosing ε such that $12\varepsilon\,\|u\|^2_{L^\infty(B_R)} \leqslant \frac{1}{2}$, previous estimate becomes

$$\begin{aligned}
&\int_{B_r} |Du(x)|^{p+2}\,\mathrm{d}x \leqslant \frac{1}{2}\int_{B_{\lambda r}} |Du(x)|^{p+2}\mathrm{d}x\\
&+c\|u\|^2_{L^\infty(B_{\lambda r})} \left[\int_{B_{\lambda r}} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x + \int_{B_{\lambda r}} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x\right]\\
&+\frac{c\|u\|^2_{L^\infty(B_{\lambda r})}}{(t-s)^{\frac{p+2}{p-q+1}}} \left[\int_{B_{\lambda r}} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\mathrm{d}x + \int_{B_{\lambda r}} |D\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + |B_R|\right] \qquad (6.3.30)
\end{aligned}$$

where $c = c(p, q, L, \nu, \mu)$ is independent of t and s.

Since (6.3.30) is valid for any $\frac{R}{2} < r < s < t < \lambda r < R < 1$, taking the limit as $s \to r$ and $t \to \lambda r$, we get

$$\begin{aligned}
&\int_{B_r} |Du(x)|^{p+2}\,\mathrm{d}x \leqslant \frac{1}{2}\int_{B_{\lambda r}} |Du(x)|^{p+2}\,\mathrm{d}x\\
&+c\|u\|^2_{L^\infty(B_R)} \left[\int_{B_R} |D^2\psi(x)|^{\frac{p+2}{p+2-q}}\mathrm{d}x + \int_{B_R} \kappa^{\frac{p+2}{p-q+1}}(x)\mathrm{d}x\right]\\
&+\frac{c\|u\|^2_{L^\infty(B_R)}}{r^{\frac{p+2}{p-q+1}}(\lambda-1)^{\frac{p+2}{p-q+1}}} \left[|B_R| + \int_{B_R} |D\psi(x)|^{\frac{p+2}{p+2-q}}\,\mathrm{d}x + \int_{B_R} \left(\mu^2+|Du(x)|^2\right)^{\frac{p}{2}}\,\mathrm{d}x\right] \qquad (6.3.31)
\end{aligned}$$

Now, setting

$$h(r) = \int_{B_r} |Du(x)|^{p+2}\,\mathrm{d}x,$$

$$A = c\|u\|^2_{L^\infty(B_R)} \left[|B_R| + \int_{B_R} |D\psi(x)|^{\frac{p+2}{p+2-q}} \, \mathrm{d}x + \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} \, \mathrm{d}x \right],$$

and

$$B = c\|u\|^2_{L^\infty(B_R)} \left[\int_{B_R} |D^2\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \int_{B_R} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x \right],$$

we obtain

$$h(r) \leqslant \frac{1}{2} h(\lambda r) + \frac{A}{r^{\frac{p+2}{p-q+1}} (\lambda - 1)^{\frac{p+2}{p-q+1}}} + B$$

Thus, we can apply Lemma 6.2.13, with

$$\theta = \frac{1}{2} \qquad \text{and} \qquad \gamma = \frac{p+2}{p-q+1},$$

obtaining

$$\begin{aligned}
&\int_{B_r} |Du(x)|^{p+2} \, \mathrm{d}x \\
\leqslant & c\|u\|^2_{L^\infty(B_R)} \left[\int_{B_R} |D^2\psi(x)|^{\frac{p+2}{p+2-q}} \mathrm{d}x + \int_{B_R} \kappa^{\frac{p+2}{p-q+1}}(x) \mathrm{d}x + \right] \\
&+ \frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p-q+1}}} \left[|B_R| + \int_{B_R} |D\psi(x)|^{\frac{p+2}{p+2-q}} \, \mathrm{d}x + \int_{B_R} \left(\mu^2 + |Du(x)|^2\right)^{\frac{p}{2}} \, \mathrm{d}x \right]
\end{aligned}$$

Since $R < 1$, the previous estimate can be written as follows

$$\begin{aligned}
&\int_{B_{\frac{R}{2}}} |Du(x)|^{p+2} \mathrm{d}x \\
\leqslant & \frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p+2-q}}} \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p \right] \mathrm{d}x.
\end{aligned} \tag{6.3.32}$$

Plugging the last inequality in (6.3.26) and choosing $\eta \in C_0^\infty(B_{\frac{R}{2}})$ such that $\eta \equiv 1$ on $B_{\frac{R}{4}}$ we get

$$\begin{aligned}
&\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2 \, \mathrm{d}x \leqslant \\
&\frac{c\|u\|^2_{L^\infty(B_R)}}{R^{\frac{p+2}{p+2-q}}} \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p \right] \mathrm{d}x.
\end{aligned}$$

that by virtue of estimate (6.2.8), gives us the a priori estimate with

$$\int_{B_{\frac{R}{4}}} |DV_p(Du(x))|^2 \,\mathrm{d}x \tag{6.3.33}$$

$$\leqslant \frac{c(\|\psi\|^2_{L^\infty} + \|u\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{p+2-q}}}$$

$$\cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}}(x) + |Du(x)|^p\right] \mathrm{d}x. \tag{6.3.34}$$

with $c = c(p, q, L, \nu, \mu)$.

Step 2: The approximation. Now we conclude the proof by passing to the limit in the approximating problem. The limit procedure is standard see, e.g., ([42]).

Let $u \in \mathcal{K}_\psi(\Omega)$ be a solution to (6.1.1) and let F_ε be the sequence obtained applying Lemma 6.2.18 to the integrand F. Let us fix a ball $B_R \Subset \Omega$ and let $u_\varepsilon \in u + W^{1,p}_0(B_R)$ be the solution to the minimization problem

$$\min_{v \in \mathcal{K}_\psi(B_R)} \int_{B_R} F_\varepsilon(x, Dv(x))\mathrm{d}x$$

By Theorem 6.2.6, the minimizers u_ε satisfy the a priori assumptions at (6.3.1), i.e. $\left(\mu^2 + |Du_\varepsilon|^2\right)^{\frac{p-2}{4}} Du_\varepsilon \in W^{1,2}_{\mathrm{loc}}(\Omega)$, and therefore we are legitimated to use estimate (6.3.33) thus obtaing

$$\int_{B_{\frac{R}{4}}} |DV_p(Du_\varepsilon(x))|^2 \,\mathrm{d}x \tag{6.3.35}$$

$$\leqslant \frac{c(\|\psi\|^2_{L^\infty} + \|u_\varepsilon\|^2_{L^{p^*}(B_R)})}{R^{\frac{p+2}{p+2-q}}}$$

$$\cdot \int_{B_R} \left[1 + \left|D^2\psi(x)\right|^{\frac{p+2}{p+2-q}} + |D\psi(x)|^{\frac{p+2}{p+2-q}} + \kappa^{\frac{p+2}{p-q+1}} + |Du_\varepsilon(x)|^p\right] \mathrm{d}x.$$

By the first inequality of growth conditions at (iii) of Lemma 6.2.18 and the minimality of u_ε we get

$$\begin{aligned}
\int_{B_R} |Du_\varepsilon(x)|^p \,\mathrm{d}x &\leqslant C(K_0) \int_{B_R} F_\varepsilon(x, Du_\varepsilon(x)) \,\mathrm{d}x \\
&\leqslant C(K_0) \int_{B_R} F_\varepsilon(x, Du(x)) \,\mathrm{d}x \\
&\leqslant C(K_0) \int_{B_R} F(x, Du(x)) \,\mathrm{d}x \,,
\end{aligned}$$

where in the last estimate we used the second inequality at (i) of Lemma 6.2.18.

Since $F(x, Du) \in L^1_{\mathrm{loc}}(\Omega)$ by assumption, we deduce, up to subsequences, that there exists $\bar{u} \in W^{1,p}_0(B_R) + u$ such that

$$u_\varepsilon \rightharpoonup \bar{u} \qquad \text{weakly in } W^{1,p}_0(B_R) + u\,.$$

Note that, since $u_\varepsilon \in \mathcal{K}_\psi$ for every ε and $\mathcal{K}_\psi$ is a closed set, we have $\bar{u} \in \mathcal{K}_\psi$. Our next aim is to show that $\bar{u}$ is a solution to our obstacle problem over the ball B_R.

To this aim fix $\varepsilon_0 > 0$ and observe that the lower semicontinuity of the functional $w \mapsto \int_{B_R} F_{\varepsilon_0}(x, Dw)\,\mathrm{d}x$, the minimality of u_ε and the monotonicity of the sequence of F_ε yield

$$\begin{aligned}
\int_{B_R} F_{\varepsilon_0}(x, D\bar{u}(x))\,\mathrm{d}x &\leqslant \lim_{\varepsilon\to 0} \int_{B_R} F_{\varepsilon_0}(x, Du_\varepsilon(x))\,\mathrm{d}x \\
&\leqslant \int_{B_R} F_{\varepsilon_0}(x, Du(x))\,\mathrm{d}x \leqslant \int_{B_R} F(x, Du(x))\,\mathrm{d}x
\end{aligned}$$

We now use monotone convergence Theorem in the left hand side of previous estimate to deduce that

$$\int_{B_R} F(x, D\bar{u}(x))\,\mathrm{d}x = \lim_{\varepsilon_0\to 0} \int_{B_R} F_{\varepsilon_0}(x, D\bar{u}(x))\,\mathrm{d}x \leqslant \int_{B_R} F(x, Du(x))\,\mathrm{d}x$$

Therefore, we have proved that the limit function $\bar{u} \in W^{1,p}(B_R) + u$ is a solution to the minimization problem

$$\min\left\{ \int_\Omega F(x, Dw(x))\mathrm{d}x : w \in W_0^{1,p}(B_R) + u,\ w \in \mathcal{K}_\psi \right\}.$$

Since by the strict convexity of the functional the solution is unique, we conclude that $u = \bar{u}$. It is quite routine to show that the convergence of u_ε to u is strong in $W^{1,p}_{\mathrm{loc}}(B_R)$.
The strong convergence of u_ε to u in $W^{1,p}(B_R)$ implies also that u_ε converges strongly to u in $L^{p^*}(B_R)$ and hence the conclusion follows passing to the limit as $\varepsilon \to 0$ in estimate (6.3.35).

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