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Developments and perspectives in Nonlinear Potential Theory

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1. Foreword

Nonlinear Potential Theory (NPT) is essentially a part of regularity theory of partial differential equations. Its aim is to study the fine properties of solutions to nonlinear elliptic and parabolic differential equations and derive statements that are as close as possible to those from the classical linear potential theory and from the regularity theory for linear equations. The infancy of NPT dates back to the beginning of the sixities, amid the first developments of the basic work of De Giorgi, Nash and Moser [101, 124, 125]. A turning point is the fundamental paper [75] by Maz'ya & Havin, where a systematic study of various types of potentials is carried out in order to study fine properties of Sobolev Functions. Amongst the other things, in [75] Wolff-Havin-Maz'ya potentials have been introduced. Later fundamental contributions involve the basic work of Hedberg & Wolff [76]; for this and other function theoretic aspects, we refer to the by now classical monograph of Adams & Hedberg [3]. The last years have seen important developments in NPT, with a deeper analysis of the interactions between fine properties of Sobolev functions, regularity theory of nonlinear elliptic equations and nonlinear potentials. A major impulse in this direction has been given by the Finnish school and a comprehensive account of its basic achievements can be found in the by now classical monograph [77] of Heinonen & Kilpeläinen & Martio. On top of this approach there lies the beautiful and foundational work of Kilpeläinen & Malý [82, 83], who proved very fundamental nonlinear potential estimates opening the way to an entire new theory; a different proof has been eventually provided by Trudinger & Wang [129]. For instance, the necessary part of the Wiener criterion of the p-Laplacean equations (whose sufficiency was proved by Maz'ya [107]) has been proved in [83] by means of potential estimates of the type in Theorem 3.3 below (see also [100] for the case $p = n$). Further consequences include for instance important existence theorems for non-homogeneous problems [120, 121]. In all this, De Giorgi's original techniques [52] are pervasive and the reader wishing to have more information in this respect might wish to consult [102]. A further impulse to this line of research has arrived with the possibility of proving pointwise gradient estimates, first observed in [112] for non-degenerate equations. Eventually, a rather comprehensive

theory has been developed in the scalar case [90]. The proof of potential estimates in the vectorial case is instead a more recent fact [93], both for solutions and their gradients. NPT is also linked to another, more recent part of regularity theory, that is, the so-called Nonlinear Calderón-Zygmund Theory (NCZT). This studies the possibility of proving, for nonlinear equations, estimates that are a typical consequence of the classical singular integrals theory [33, 34] in the case of linear equations. Obviously, a common point with the pointwise potential estimates is that the use of fundamental solutions is ruled out. The first papers in this direction containing local estimates are [78, 54, 28, 30], where suitable Harmonic Analysis tools such as various type of maximal operators have been used as a replacement of the missing singular integrals. A different, purely PDE technique has been eventually introduced in [2], where a completely Harmonic Analysis free approach has been developed, allowing to treat cases that were not approachable by other means. The sharp gradient integrability of solutions of degenerate parabolic systems is an instance (see (5.3) below). A different direction has been taken in [110], where for the first time a Calderón-Zygmund theory for elliptic problems with measure data has been formulated in terms of maximal fractional gradient differentiability. This approach has been eventually pushed up the optimal level in [7], where a connection with nonlinear potential estimates has been explained (see Section 7 below for details).

The aim of this note is now to give an account of the latest developments of NPT and NCZT and to establish a framework in which the papers published in this special issue of Nonlinear Analysis can be put. The reader will find several papers contributing different results related to those exposed in the following pages, from a rather wide perspective. The Calculus of Variations, the qualitative analysis of partial differential equations are still themes related to those presented here.

2. The present setting

Here we mainly deal with elliptic equations of the type

$$-\mathrm{div}\, A(Du) = H \qquad \text{in } \Omega \subset \mathbb{R}^n$$

with Ω being an open subset and $n \geq 2$. Notice that, for the choice $A(x, Du) \equiv Du$ and $p = 2$, the last equation reduces to the standard Poisson equation. The right-hand side H will be a distribution that in general will be either a finite mass Borel measure or of the form $H \equiv \mathrm{div}\,(|F|^{p-2}F)$ for some $F \in L^p$, $p > 1$. The assumptions we are going to work with on the vector field $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is that this is assumed to be in $C^0(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$-regular and initially satisfies

$$\begin{cases} |A(z)| + |\partial A(z)||z| \leq L|z|^{p-1} \\ \nu|z|^{p-2}|\xi|^2 \leq \langle \partial A(z)\xi, \xi\rangle \end{cases} \tag{2.1}$$

for every choice of $z \in \mathbb{R}^n \setminus \{0\}, \xi \in \mathbb{R}^n$, and for fixed ellipticity constants $0 < \nu \leq 1 \leq L$. (We preliminary notice that several of the results presented in the following

hold under assumptions that are more general than those in (2.1), but we adopt the ones here for the sake of simplicity). Conditions (2.1) are classical since the work of Ladyzhenskaya & Uraltseva [96] and they are modeled on the p-Laplacean operator, i.e.,

$$A(x,z) = |z|^{p-2}z\ .$$

As said above we shall consider two types of equations. The first one is

$$\operatorname{div} A(Du) = \operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega \subset \mathbb{R}^n\ . \tag{2.2}$$

In this case we consider $F \in L^p_{\mathrm{loc}}(\Omega)$ and distributional solutions (and therefore so-called energy solutions) that is, functions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \langle A(x,Du), D\varphi\rangle\, dx = \int_\Omega \langle |F|^{p-2}F, D\varphi\rangle\, dx \qquad \forall\ \varphi \in C^\infty_0(\Omega)\ .$$

In the second case we shall consider right-hand sides that are not in divergence forms, but that in the most general case are Borel measures μ with finite total mass

$$-\operatorname{div} A(Du) = \mu\ . \tag{2.3}$$

In this case the definition of solution we consider is a notion first used in [20].

Definition 2.1 (Local SOLA)**.** A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an arbitrary open subset and $p > 2 - 1/n$, is a local SOLA to (2.3) under assumptions (2.1), if and only if there exists a sequence of local energy solutions $\{u_k\} \subset W^{1,p}_{\mathrm{loc}}(\Omega)$ to the equations

$$-\operatorname{div} A(x,Du_k) = \mu_k \in L^\infty_{\mathrm{loc}}(\Omega)\,,$$

such that $u_k \rightharpoonup u$ weakly in $W^{1,1}_{\mathrm{loc}}(\Omega)$. Here the sequence $\{\mu_k\}$ converges to μ (locally) weakly* in the sense of measures and satisfies

$$\limsup_k |\mu_k|(B) \le |\mu|(\overline{B})$$

for every ball $B \Subset \Omega$.

SOLA are still distributional solutions, and they are such that

$$Du \in L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{for every } q < \frac{n(p-1)}{n-1}\,, \qquad \text{when } p \le n\ .$$

To be more precise, the limiting integrability of SOLA can be described in terms of weak-Lebesgue spaces, i.e., Marcinkiewicz spaces, that is (for $p \le n$)

$$Du \in \mathcal{M}^{\frac{n(p-1)}{n-1}}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \Longleftrightarrow \sup_{0<\lambda} \lambda^{\frac{n(p-1)}{n-1}} |\{x \in \Omega' \,:\, |Du(x)| > \lambda\}| < \infty\ ,$$

with the last inequality that holds for every open subset $\Omega' \Subset \Omega$. This integrability range is optimal, as decan be seen by looking at the so-called nonlinear fundamental solution

$$G_p(x) \approx \begin{cases} |x|^{\frac{p-n}{p-1}} & \text{if} \quad 1 < p \neq n \\ -\log\ |x| & \text{if} \quad p = n\ , \end{cases} \qquad |x| \neq 0\ , \tag{2.4}$$

which locally solves (as a SOLA) the equation the equation $-\mathrm{div}\,(|Du|^{p-2}Du) = \delta$, and δ is the Dirac measure charging the origin. For an account of such results see for instance [90] and references therein. Notice that the bound $p > 2-1/n$ guarantees that SOLA are Sobolev functions. The range $1 < p \leq 2-1/n$ is more delicate and different notions of solutions must be considered. See for instance [81] and references therein. Finally, some basic notation. We denote by $B_R(x_0) := \{x \in \mathbb{R}^n : |x-x_0| < R\}$. When not important we shall omit denoting the center, i.e., $B_R \equiv B_R(x_0)$. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $f\colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being an integrable map, we shall denote by

$$(f)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} f\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} f(x)\,dx\ ,$$

where $|\mathcal{B}|$ denotes the Lebesgue measure of $\mathcal{B}$. We also identify $L^1_{\mathrm{loc}}(\Omega)$-functions μ with measures, thereby denoting

$$|\mu|(\mathcal{B}) = \int_{\mathcal{B}} |\mu|\,dx \qquad \text{for every measurable subset } \mathcal{B} \Subset \Omega\ .$$

Let $\Omega \subset \mathbb{R}^n$. We denote by $\mathbf{M}_{\mathrm{loc}}(\Omega)$ the space of Borel (signed) measures with locally finite total mass defined on Ω.

3. Estimates via linear and nonlinear potentials

Crucial developments in the last decades are concerned with potential estimates for p-Laplacean type equations (see [131, 102, 103, 90] for basic regularity theory). Two definitions are relevant for this. These have been first introduced in [75] and are described in the following

Definition 3.1. Let $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an open subset; the nonlinear Wolff-Havin-Maz'ya potential $\mathbf{W}^{\mu}_{\beta,p}$ is defined by

$$\mathbf{W}^{\mu}_{\beta,p}(x_0,R) := \int_0^R \left(\frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-\beta p}}\right)^{1/(p-1)} \frac{d\varrho}{\varrho}\ , \qquad \beta > 0$$

whenever $B_R(x_0) \Subset \Omega$.

Wolff-Havin-Maz'ya potentials are a nonlinear version of the more classical (truncated) Riesz potentials, that is

Definition 3.2 (Truncated Riesz potentials)**.** Let $\mu \in \mathbf{M}_{\mathrm{loc}}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ being an open subset; the (truncated) Riesz potential $\mathbf{I}_\beta^\mu$ is defined by

$$\mathbf{I}_\beta^\mu(x_0, R) := \int_0^R \frac{|\mu|(B_\varrho(x_0))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho}, \qquad 0 < \beta < n,$$

whenever $B_R(x_0) \Subset \Omega$.

The relation between the truncated Riesz potentials defined above and the classical ones

$$I_\beta(\mu)(x_0) := \int_{\mathbb{R}^n} \frac{d\mu(x)}{|x - x_0|^{n-\beta}},$$

is trivially given by

$$\mathbf{I}_\beta^\mu(x_0, R) \le c(n) I_\beta(|\mu|)(x_0) \qquad \text{for every } R > 0 . \tag{3.1}$$

When $p = 2$, Wolff and Riesz potentials coincide in the sense of

$$\mathbf{W}_{1,2}^\mu(x_0, R) = \mathbf{I}_2^\mu(x_0, R) \quad \text{and} \quad \mathbf{W}_{1/2,2}^\mu(x_0, R) = \mathbf{I}_1^\mu(x_0, R),$$

In the case of linear problems, Riesz potentials are known to pointwise bound solutions of non-homogeneous equations via convolution with the fundamental solutions. This results in

$$|u(x_0)| \lesssim |I_2(\mu)(x_0)| \qquad \text{and} \qquad |Du(x_0)| \lesssim I_1(|\mu|)(x_0). \tag{3.2}$$

It is a deep result of Kilpeläinen & Malý [82, 83] (with a later proof by Trudinger & Wang [129]) the fact that this principle extends to the case of nonlinear equations provided Wolff-Havin-Maz'ya potentials are used. A statement for general measures is the following, and includes a precise representative criterion in the spirit of the classical linear potential theory:

Theorem 3.3 ([82, 83, 66, 90])**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p > 2-1/n$. Let $B_R(x_0) \Subset \Omega$ be a ball. If $\mathbf{W}_{1,p}^\mu(x_0, R) < \infty$, then x_0 is Lebesgue point of u, in the sense that the limit*

$$\lim_{\varrho \searrow 0} (u)_{B_\varrho(x_0)} =: u(x_0) \tag{3.3}$$

exists and thereby defines the precise representative of u at x_0. Moreover, the pointwise Wolff-Havin-Maz'ya potential estimate

$$|u(x_0)| \le c\mathbf{W}_{1,p}^\mu(x_0, R) + c \fint_{B_R(x_0)} |u| \, dx \tag{3.4}$$

holds with a constant c depending only on n, p, ν, L.

Notice that, when $p = 2$, estimate (3.4) reduces to the classical one valid for the Poisson equation

$$-\triangle u = \mu \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{3.5}$$

that is

$$|u(x_0)| \leq c\mathbf{I}_2^{\mu}(x_0,R) + c \fint_{B_R(x_0)} |u| \, dx \, . \tag{3.6}$$

We also notice that, by (3.3), estimate (3.4) makes sense at every possible point x_0 where the right-hand side is finite and therefore at each point where it does not become trivial. Finally, when $\Omega \equiv \mathbb{R}^n$, and assuming suitable decay at infinity of u, by letting $R \to \infty$ in (3.6) and recalling (3.1), we get the first inequality in (3.2).

The validity of a similar estimate for the gradient has been a debated open problem for a while. The surprising outcome is that, on the contrary of what the orthodoxy of NPT prescribes, Riesz potentials come back into the play. It indeed holds the following:

Theorem 3.4 ([112, 65, 87])**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p > 2 - 1/n$. Let $B_R(x_0) \Subset \Omega$ be a ball. If $\mathbf{I}_1^{\mu}(x_0,R) < \infty$, then x_0 is Lebesgue point of the gradient, in the sense that the limit*

$$\lim_{\varrho \searrow 0} (Du)_{B_\varrho(x_0)} =: Du(x_0)$$

exists and thereby defines the precise representative of Du at x_0. Moreover, the pointwise Riesz potential estimate

$$|Du(x_0)|^{p-1} \leq c\,[\mathbf{I}_1^{\mu}(x_0,R)] + c\left(\fint_{B_R(x_0)} |Du| \, dx \right)^{p-1} \tag{3.7}$$

holds with a constant c depending only on n, p, ν, L.

Again, when $\Omega \equiv \mathbb{R}^n$, and assuming a suitable decay at infinity, letting $R \to \infty$ in (3.7) yields

$$|Du(x_0)|^{p-1} \lesssim \int_{\mathbb{R}^n} \frac{d|\mu|(x)}{|x-x_0|^{n-1}} \, ,$$

that, in turn, gives the second inequality in (3.2) when $p = 2$. Notice that using the monotonicity properties of the vector field $A(\cdot)$, estimate (3.7) can be rephrased as

$$|A(Du(x_0))| \leq c\,[\mathbf{I}_1^{\mu}(x_0,R)]^{1/(p-1)} + c \fint_{B_R(x_0)} |A(Du)| \, dx \, , \tag{3.8}$$

where the presence of p appears only via the dependence of the constants c. Let us observe that potential estimates (3.7)-(3.8) allow to derive a priori gradient bounds in essentially all possible function spaces or, at least in all those where the mapping properties of Riesz potentials are known. An account of this can be found for instance in [39, 90]. Essentially, the final outcome is that estimate (3.7) allows to reduce the problem of finding gradient bounds for solutions to nonlinear, possibly degenerate equations to the one of finding the same bounds for solutions to the Poisson equation. This solution to this last problem is, of course, well-known.

Riesz potentials control the continuity properties of the gradient in the sense of the following:

Theorem 3.5 ([87, 90]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p > 2 - 1/n$. If*

$$\lim_{\varrho \searrow 0} \mathbf{I}_1^{\mu}(x, \varrho) = 0 \quad \textit{holds locally uniformly in } \Omega \textit{ w.r.t. } x\,, \tag{3.9}$$

then Du is continuous in Ω.

The results above hold in the case of scalar equations, while they cannot be true for general elliptic systems (see the counterexamples to regularity [109, 127]). Instead, when restricting to quasi-diagonal structures of the type considered by Uhlenbeck [130], both estimate (3.4) and (3.7) can be recovered. For instance, they hold for SOLA to the p-Laplacean system

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu\,, \tag{3.10}$$

where both u and μ are a vector valued function and a measure, respectively. This result has been achieved in [93] and the proof is quite different from that of the scalar case. It indeed employs regularization arguments that are typical from Geometric Measure Theory and partial regularity theory; these are carefully matched with nonlinear potential theoretic techniques. See also [23] for a related result. We recall that the basic theory for elliptic systems as in (3.10) has been originally established in [61, 62].

The first version of the pointwise gradient estimate (3.7) appears in [112] for the nondegenerate case $p = 2$; the proof in [112] is built on the fractional Caccioppoli type inequalities for measure data problems developed in [110]. An intermediate version for $p > 2$, still using Wolff potentials, has been derived in [66] (where a parabolic version of the gradient estimate has been derived too, but again for the case $p = 2$). The final statement (3.7) has been then derived in [65, 87]. A significant extension to a very general class of uniformly elliptic problems has been proved by Baroni in [10]. A preliminary, level set version of the gradient estimate has been proposed in [111], where the use of fractional maximal operators has been introduced to prove estimates for measure data problems (see [14] for a related parabolic problem); see also [36] for non-necessarily polynomial growth operators. This last approach has been eventually developed by Phuc [118, 119] to get global estimates for the Dirichlet problem. As also shown by Phuc, the approach of [111] can be adapted to get delicate weighted estimates. An interesting global version of a rearrangement baked inequality of the type in (3.7) has been given in [42]. A series of papers [88, 89, 91] proposes a sharp analog of the gradient potential estimates in the case of degenerate parabolic equations, whose prototype is the p-caloric equation with measure data

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu\,.$$

As for the values $p \le 2 - 1/n$ some improvements have been made; in this case one has to consider different, more general versions of solutions than SOLA. For this we refer to the recent papers [113, 128].

Finally, let us also mention that potentials can be used to check whether solutions have a given modulus of continuity. We present a model result (for simplicity

stated in the non-degenerate case $p = 2$) that allows to check Hölder continuity of solutions using suitable potentials (we refer to [90] for a comprehensive presentation of this kind of results).

Theorem 3.6 ([86]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p = 2$. The inequality*

$$\begin{aligned} |u(x_1) - u(x_2)| \quad &\leq \quad c\left[\mathbf{I}^{\mu}_{2-\alpha}(x_1, R) + \mathbf{I}^{\mu}_{2-\alpha}(x_2, R)\right] |x_1 - x_2|^{\alpha} \\ &\quad + c \fint_{B_R} |u| \, dx \cdot \left(\frac{|x_1 - x_2|}{R}\right)^{\alpha} \end{aligned} \tag{3.11}$$

holds uniformly in $\alpha \in [0, 1]$, whenever $B_R \Subset \Omega$ is a ball and $x_1, x_2 \in B_{R/4}$, provided the right-hand side is finite. The constant c depends only on n, p, ν, L.

Notice that, interestingly, estimate (3.11) interpolates between estimate (3.6) (when $\alpha = 0$) and estimate (3.7) (for $\alpha = 1$).

4. Nonlinear Stein theorems

Amongst the other things, estimate (3.7) provides a sharp condition on the right-hand side μ implying the local Lipschitz continuous, and this settles a longstanding problem. Interestingly, the condition on μ is p-independent, as the right-hand side (3.7) clearly implies (it is indeed sufficient to require that $\mathbf{I}^{\mu}_{1}$ is locally bounded). Here we briefly discuss a characterization in terms of a relevant function space. For this let us recall a celebrated result of Stein [126] concerning the limiting Lorentz space $L(n, 1)(\Omega)$. This is given by the measurable functions f such that

$$\int_0^{\infty} |\{x \in \Omega \, : \, |f(x)| > \lambda\}|^{1/n} \, d\lambda < \infty \, .$$

Now, if $Dv \in L(n, 1)$, then Stein's Theorem asserts that v is continuous. Stein's Theorem is, in a sense, the limiting case of the Sobolev-Morrey embedding theorem:

$$Dv \in L^{n+\varepsilon} \Longrightarrow v \in C^{0, \frac{\varepsilon}{n+\varepsilon}} \qquad \text{for } \varepsilon > 0 \, .$$

Letting $\varepsilon \to 0$ would yield $Dv \in L^n$, which anyway only implies $v \in \mathrm{VMO}$ and it is insufficient for continuity. An intermediate limiting space is therefore necessary and $L(n, 1)$ plays this role in the sense that, on finite measure spaces, it holds

$$L^{n+\varepsilon} \subset L(n, 1) \subset L^n \qquad \text{for every } \varepsilon > 0 \, .$$

All the inclusions in the display above are strict. An essentially equivalent way to state Stein's Theorem can be obtained when looking at the gradient regularity of solutions u to the Poisson equation (3.5). Observing that standard CZ theory gives

$$\triangle u \in L(n, 1) \Longrightarrow D^2 u \in L(n, 1)$$

as Lorentz spaces are interpolation spaces, the continuity Du follows again by Stein's Theorem. For a discussion about the optimal role played by the space $L(n,1)$ in the analysis of the Poisson equation we refer to [38].

Now, switching to the general nonlinear case, and observing that $\mu \in L(n,1)$ implies (3.9) (see for instance [87]), we have the following:

Theorem 4.1 ([87]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p > 2 - 1/n$, and such that $\mu \in L(n,1)$ locally in Ω. Then Du is continuous in Ω.*

By different means, Theorem 4.1 continues to hold in the whole range $p > 1$ when considering the p-Laplacean system [92, 93], and therefore vector-valued solutions. A suitable parabolic version of Theorem 4.1 can be also proved as well [88, 89]. We remark that the first result yielding the local boundedness of the gradient when the right-hand side belongs to $L(n,1)$ have been proved, both for equations and systems, in [64]. Global versions have been given, under optimal boundary assumptions and also for systems, in [40, 41]. The space $L(n,1)$ plays a role for more general operators in determining the gradient boundedness, and in this respect recent results dealing with non-uniformly elliptic equations can be found in [18].

5. Nonlinear Calderón-Zygmund theory

This is another part of regularity theory that essentially deals with optimal integrability estimates. In the linear case, the prototype question, rephrased in the way which is mostly close to our setting, is to determine the optimal gradient integrability of solutions to the equation (system)

$$\triangle u = \operatorname{div} Du = \operatorname{div} F \tag{5.1}$$

in terms of the optimal integrability properties of the assigned datum F. The classical Calderón-Zygmund theory [33, 34] then yields

$$F \in L^{\gamma}_{\mathrm{loc}} \Longrightarrow Du \in L^{\gamma}_{\mathrm{loc}} \tag{5.2}$$

for every $\gamma > 1$. The natural nonlinear analog of equation (5.1) is of course given by (2.2), that has been indeed studied at length since the papers [78, 54, 30, 132]. In these papers (5.2) is still proved in the case of nonlinear equations of the type (2.2) but only when $q \geq p$, although a parallel with the linear case would suggest that range of admissible parameters is $q > p - 1$. This remains as an important open problem in the theory. The only progress is contained in the papers [97, 79], where the range is extended to $q > p - \varepsilon$ for some small $\varepsilon \equiv \varepsilon(n, \nu, L) > 0$ which is independent of the solution considered. The original local results in the aforementioned papers have been extended in several directions, including boundary value problems and equations with rough coefficients [24, 25]. On a different direction, very interesting results can be derived in Stochastic Homogenization [4].

All these papers make use of basic Harmonic Analysis tools such as various types of maximal operators. These approaches break down when applied to the degenerate parabolic setting, i.e., they do not allow to get estimates in the case of equations as

$$u_t - \operatorname{div} A(Du) = \operatorname{div}(|F|^{p-2}F) \qquad \text{in } \Omega \subset \mathbb{R}^n \ . \tag{5.3}$$

This problem has been settled by means of a new technique in [2], which is completely free from any Harmonic Analysis tool and which is directly based on direct exit time argument that takes into account the gradient and the datum at different heights. For a comprehensive account on the regularity of equations as in (5.3) we refer to [53]. The techniques introduced in [2] are flexible enough to allow for more general function spaces (see for instance [8, 9]) as well as for several further extensions (including non-uniformly elliptic operators [26, 43] and different function spaces [74]).

6. CZ estimates and non-uniform ellipticity

Both the NPT and the NCZT are rather open fields for non-uniformly elliptic operators. A few particular, yet significant cases have been treated. These relate to non-autonomous energies of the type

$$\begin{aligned} \mathcal{V}(w,\Omega) &:= \int_\Omega |Dw|^{p(x)}\,dx \ , \qquad 1 < \gamma_1 \le p(x) \le \gamma_2 < \infty \\ \mathcal{P}_{p,q}(w,\Omega) &:= \int_\Omega (|Dw|^p + a(x)|Dw|^q)\,dx \ , \qquad 0 \le a(x) \le L \ , \quad 1 < p < q \ . \end{aligned} \tag{6.1}$$

These two functionals, originating in the work of Zhikov [134, 135, 136], also fall in the realm of those with so-called (p,q)-functionals originally considered by Marcellini [104, 105, 106]. This is a topic that has attracted a lot of attention in the recent years. See also the survey paper [109]. The common point of the two functionals in (6.1) is the fact that the growth and/or ellipticity properties of the integrand with respect to the gradient variable depends on the point x. This is at the origin of the non-uniformly elliptic nature of the Euler-Lagrange equations associated to the functionals in (6.1). Indeed, as this is immediately clear for the first functional, in the second case we have that the growth with respect to the gradient variable is p when $a(x) = 0$ and it is q when $a(x) > 0$. By looking at their associated Euler-Lagrange equations, the two equations that are relevant for the corresponding Calderón-Zygmund estimates are

$$\operatorname{div}(Du|^{p(x)-2}Du) = \operatorname{div}(|F|^{p(x)-2}F) \qquad \text{in } \Omega \subset \mathbb{R}^n \ , \tag{6.2}$$

and

$$\operatorname{div} A(x,Du) = \operatorname{div} A(x,F) \qquad \text{in } \Omega \subset \mathbb{R}^n \tag{6.3}$$

where

$$A(z) := (|z|^{p-2} + a(x)|z|^{q-2})z = \frac{H(x,z)}{|z|^2}z$$

and

$$H(x,z) := |z|^p + a(x)|z|^q \qquad \text{for } x \in \Omega \text{ and } z \in \mathbb{R}^n \ .$$

There is by now a large literature on the equation in (6.2). The first paper on the subject is [1], while improvements can be found for instance in [58, 59, 27]. The main common point in all the statements in such papers is the higher integrability formulated in terms of the intrinsic vector field considered in the problem, that is

$$|F|^{p(\cdot)} \in L^\gamma_{\mathrm{loc}}(\Omega) \Longrightarrow |Du|^{p(\cdot)} \in L^\gamma_{\mathrm{loc}}(\Omega) \qquad \text{for every } \gamma > 1 \ .$$

This last assertion holds under the sharp assumption of vanishing log-Hölder continuity on the variable exponent $p(x)$:

$$\lim_{\varrho \to 0} \omega(\varrho) \log\left(\frac{1}{\varrho}\right) = 0 \ , \tag{6.4}$$

where $\omega(\cdot)$ is the modulus of continuity of $p(\cdot)$

$$|p(x) - p(y)| \le \omega(|x-y|) \qquad \forall\, x, y \ .$$

This assumption essentially rules out too large oscillations of the exponent function $p(x)$ resulting in a strong difference between the largest and the smallest eigenvalue of the variable exponent operator over a ball $B \subset \Omega$

$$\frac{\text{highest eigenvalue on } B \text{ of } |Du|^{p(x)-2}}{\text{lowest eigenvalue on } B \text{ of } |Du|^{p(x)-2}} \approx |Du|^{\max_B\, p(x) - \min_B\, p(x)} \ .$$

The last ratio provides a measure of non-uniform ellipticity on B of the operator considered. Similarly, the non-uniform ellipticity of equation in display (6.3), when evaluated on the specific solution u, is measured by the ratio

$$\frac{\text{highest eigenvalue of } \partial_z A(x, Du)}{\text{lowest eigenvalue of } \partial_z A(x, Du)} \approx 1 + a(x)|Du|^{q-p} \ . \tag{6.5}$$

Around phase transition set $\{a(x) = 0\}$, the ratio in (6.5) exhibits blow-up with respect to the gradient with exponent $q-p$; to rebalance, $a(x)$ is required to decay sufficiently fast. This is quantified in the optimal condition

$$a(\cdot) \in C^{0,\alpha}(\Omega) \ , \qquad \frac{q}{p} \le 1 + \frac{\alpha}{n} \tag{6.6}$$

and indeed we have

Theorem 6.1 ([43, 50])**.** *Let $u \in W^{1,1}(\Omega)$ be a distributional solution to equation* (6.3) *such that $H(x, Du), H(x, F) \in L^1(\Omega)$, under the assumptions* (6.6)*. Then*

$$H(\cdot, F) \in L^\gamma_{\mathrm{loc}}(\Omega) \Longrightarrow H(\cdot, Du) \in L^\gamma_{\mathrm{loc}}(\Omega) \qquad \textit{holds for every } \gamma \ge 1 \ .$$

A boundary version of the last result has been obtained in [26]. Let us mention that conditions (6.6) are also relevant for the regularity of minimizers of the

functional $\mathcal{P}_{p,q}$ in (6.1); for this see [11, 12]. The field of non-uniformly elliptic problems, especially in the setting of the Calculus of Variations, has seen a large number of contributions over the last years, starting by the foundational papers of Marcellini [104, 105]. Recent contributions on functionals with non-standard growth conditions and non-uniformly elliptic problems, with special emphasis on double phase and non-autonomous problems, include [11, 12, 18, 21, 19, 35, 46, 47, 45, 48, 49, 67, 68, 114, 115, 116, 133]. In particular, first steps towards a nonlinear potential theory appear in [15, 71, 72, 37]. Moreover, in the papers [70, 73] a very interesting, unified approach to the functionals in (6.1) is proposed, unifying conditions (6.4) and (6.6) in a single one and therefore providing a unified proof for the regularity of minima. We refer to the survey [109] for an overview presentation of functionals with non-standard growth conditions, and to [106] for more recent results. General information on non-autonomous problems, with special emphasis on function spaces, can be found in the monographs [57, 122].

7. CZ theory in the limiting case

Let us recall the basic definition of fractional spaces in their Sobolev-Slobodeckij version. With $\Omega \subset \mathbb{R}^n$ being an open subset, $\alpha \in (0,1)$, $q \in [1,\infty)$, $k \in \mathbb{N}$, the fractional Sobolev space $W^{\alpha,q}(\Omega;\mathbb{R}^k)$ is defined prescribing that $f\colon \Omega \to \mathbb{R}^k$ belongs to $W^{\alpha,q}(\Omega;\mathbb{R}^k)$ if and only if the following Gagliardo-type norm is finite:

$$\begin{aligned} \|f\|_{W^{\alpha,q}(\Omega)} &:= \left(\int_\Omega |f(x)|^q\,dx\right)^{1/q} + \left(\int_\Omega\int_\Omega \frac{|f(x)-f(y)|^q}{|x-y|^{n+\alpha q}}\,dx\,dy\right)^{1/q} \\ &=: \|f\|_{L^q(\Omega)} + [f]_{W^{\alpha,q}(\Omega)} \end{aligned}$$

The local variant $W^{\alpha,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ is defined by requiring that $f \in W^{\alpha,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ if and only if $f \in W^{\alpha,q}_{\mathrm{loc}}(\Omega';\mathbb{R}^k)$ for every open subset $\Omega' \Subset \Omega$. For more we refer to [60].

Fractional Sobolev spaces are very popular these days due to the large interest in studying nonlocal operators. But they have always been used in several different issues from regularity theory. It is for instance worth pointing out their use in establishing Hausdorff dimension estimates for the singular sets of elliptic systems and minimizers of variational problems [109]. Moreover, fractional spaces can be used to establish the existence of boundary regular points [63] for elliptic systems. Here we are interested in a different direction, that is, in the case of equations with measure data, fractional Sobolev spaces can be employed in order to get maximal regularity solutions. This entails a limiting case of CZ estimates. This approach has been initiated in [110] and has been further pursued in [7]. Let us summarize the situation. As it is well-known, solutions to the Poisson equation (3.5) do not possess second order Sobolev derivatives in the case the right-hand side is a measure or just an L^1-function. The same obviously applies to nonlinear equations. Nevertheless, not everything is lost. Indeed, in [110], for the non-degenerate case $p = 2$ is it is proved that SOLA are such that

$$Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \ \sigma \in (0,1)\,. \tag{7.1}$$

The corresponding parabolic version has been derived in [13]. Specifically, (7.1) means that

$$\int_{\Omega'}\int_{\Omega'} \frac{|Du(x)-Du(y)|}{|x-y|^{n+\sigma}}\,dx\,dy < \infty$$

holds for every $\sigma \in (0,1)$ and every bounded open subset $\Omega' \Subset \Omega$. In the case $p>2$, due to the degeneracy of the equation, a substantial amount of differentiability is lost and indeed, again in [110], it is proved that

$$Du \in W^{\frac{\sigma}{p-1},p-1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{for every} \quad \sigma \in (0,1)\,. \tag{7.2}$$

Both the results are optimal in this case and this follows by looking at the fundamental solution (2.4). The loss of derivatives present in (7.2) can be corrected using an intrinsic approach, therefore passing to the vector field $A(Du)$ which incorporates the degeneracy information of the equation. This has been done in [7], where it is indeed proved that

$$A(Du) \in W^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\,. \tag{7.3}$$

Notice that in the case of the purely p-Laplacean operator, when in fact it is $A(z) \equiv |z|^{p-2}z$, the result in the display above amounts to say that

$$|Du|^{p-2}Du \in W^{\sigma,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\,. \tag{7.4}$$

In turn, the result in (7.3) allows to recover the one in (7.2) as the endpoint case of a family of interpolating embeddings as follows:

$$|Du|^{\gamma}Du \in W^{\sigma\frac{\gamma+1}{p-1},\frac{p-1}{\gamma+1}}_{\mathrm{loc}}(\Omega;\mathbb{R}^n) \qquad \text{holds for every } \sigma \in (0,1)\,,$$

and this holds for every γ such that $0 \le \gamma \le p-2$. Notice that, for the choice $\gamma = p-2$, we recover (7.4), while the case $\gamma = 0$ allows to get (7.2) back. We refer to the paper [7] for more on this issue. The qualitative result in (7.3) comes along with an a priori estimate in the sense that, for every $\sigma \in (0,1)$, there exists a constant $c \equiv c(n,p,\nu,L,\sigma)$ such that the following fractional Caccioppoli type inequality

$$\begin{aligned}\fint_{B_{R/2}}\int_{B_{R/2}} &\frac{|A(Du(x))-A(Du(y))|}{|x-y|^{n+\sigma}}\,dx\,dy \\ &\le \frac{c}{R^{\sigma}}\fint_{B_R}|A(Du)|\,dx + \frac{c}{R^{\sigma}}\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]\end{aligned}$$

holds for every ball $B_R \Subset \Omega$. This last estimate can be considered as the singular integral counterpart of the intrinsic potential estimate in (3.8), which, dimensionally speaking, corresponds to the case of fractional integral estimates. Both estimates are formulated involving the direct quantity $A(Du)$, which is consistent with the linearization viewpoint of this approach.

8. Nonlinear potential theory and nonlocal operators

As mentioned above, there has been a large and rapid development of the theory of nonlocal operators. As for regularity, the first contributions towards a systematic development of the theory can be found in the papers [16, 17, 31, 32]. The p-Laplacean fractional theory, which is closely related to our interests here, has instead started with the papers [55, 56], where a complete analog of the standard De Giorgi-Nash-Moser theory has been derived, including Harnack inequalities and local $C^{0,\alpha}$-estimates for solutions. See also [80, 44] for a related approach. Extensions of these results to the case of non-uniformly elliptic operators have been also carried out in [51, 98]. In the papers [55, 56], convolution kernels with measurable coefficients are considered. The related non-linear potential theory has instead started in the paper [94], where a sharp analog of the pointwise potential estimate (3.4) has been proved. Moreover, optimal versions of the Wiener criterion have been proved in [84, 85, 99]. Let us briefly recall the setting of [94] (which is closely related to that of [55, 56]). There, general operators of the type

$$-\mathcal{L}_\Phi u = \mu \qquad \text{in } \Omega \subset \mathbb{R}^n\,, \tag{8.1}$$

are considered, where $-\mathcal{L}_\Phi$ is defined as a distribution via

$$\langle -\mathcal{L}_\Phi u, \varphi\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi(u(x)-u(y))(\varphi(x)-\varphi(y))K(x,y)\,dx\,dy\,,$$

for every smooth function φ with compact support. The function $\Phi : \mathbb{R} \mapsto \mathbb{R}$ is assumed to be continuous, satisfying $\Phi(0) = 0$ together with the monotonicity property

$$\Lambda^{-1}|t|^p \le \Phi(t)t \le \Lambda|t|^p\,, \qquad \forall\, t \in \mathbb{R}\,.$$

Finally, the kernel $K\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is assumed to be measurable, and satisfying the following ellipticity/coercivity properties:

$$\frac{1}{\Lambda|x-y|^{n+sp}} \le K(x,y) \le \frac{\Lambda}{|x-y|^{n+sp}} \qquad \forall\, x,y \in \mathbb{R}^n,\ x \ne y \tag{8.2}$$

where $\Lambda \ge 1$ and

$$s \in (0,1)\,, \qquad p > 2 - \frac{s}{n}\,.$$

When in (8.2) it is $\Lambda = 1$ and $\Phi(t) \equiv |t|^{p-2}t$, i.e., when

$$K(x,y) = \frac{1}{|x-y|^{n+sp}} \tag{8.3}$$

holds, then we have the truly fractional p-Laplacean operator. This naturally emerges by minimizing the Gagliardo norm

$$v \mapsto \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v(x)-v(y))|^p}{|x-y|^{n+sp}}\,dx\,dy$$

in suitable Dirichlet classes. For this we refer to the initial paper [55] and the recent survey [117].

For suitably defined SOLA of such equations (with suitable Dirichlet boundary values prescribed on the complement of Ω) a potential estimate of the type (3.4) holds and involves an additional tail term

$$\mathrm{Tail}(v;x_0,r) := \left[r^{sp} \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{|v(x)|^{p-1}}{|x-x_0|^{n+sp}}\, dx \right]^{1/(p-1)} .$$

in order to encode the nonlocal interactions of the problem. Specifically, it holds that

$$|u(x_0)| \lesssim \mathbf{W}^{\mu}_{s,p}(x_0,r) + \left(\fint_{B_r(x_0)} |u|^{q_*}\, dx \right)^{1/q_*} + \mathrm{Tail}(u;x_0,r)\,,$$

where $q_* := \max\{1, p-1\}$ for every ball $B_r(x_0)$. As in the local case, the precise representative

$$u(x_0) := \lim_{\varrho \to 0} (u)_{B_\varrho(x_0)} = \lim_{\varrho \to 0} \fint_{B_\varrho(x_0)} u\, dx$$

exists as soon as $\mathbf{W}^{\mu}_{s,p}(x_0,r)$ is finite.

As for the higher order theory for operators with measurable coefficients, we mention the results from [95]. By considering the equation (8.1) with $p = 2$ and suitable integrability properties on μ, in [95] it is proved that

$$u \in W^{s,2}_{\mathrm{loc}} \Longrightarrow u \in W^{s+\delta,2+\delta}_{\mathrm{loc}} \qquad \text{for some } \delta > 0\,. \tag{8.4}$$

This surprising gain of differentiability has no local analog for equations with measurable coefficients

$$-\mathrm{div}\,(a(x)Du) = 0\,, \qquad a(x) \approx Id$$

for which it only holds that

$$u \in W^{1,2}_{\mathrm{loc}} \Longrightarrow u \in W^{1,2+\delta}_{\mathrm{loc}} \qquad \text{for some } \delta > 0\,.$$

The result in the last display is known as Meyers estimate. Modern proofs rely on the use of the Gehring lemma and are based on so-called reverse Hölder inequalities. These play an important in several aspects of modern analysis, and especially in the Calculus of Variations [69]. In [95], an approach is presented which is based on the suitable analog of reverse Hölder inequality in the nonlocal case, and this involves a series of delicate off-diagonal estimates. Although the main result in [95] is stated for solutions, the basic analysis only relies on the use of a suitable family of fractional Caccioppoli type inequalities. As such, it allows to several extensions, for instance to those variational problems when the Euler-Lagrange equation does not exist, in the spirit of [69]. The ideas in [95] can be used to deal with new classes of local reverse Hölder inequalities with tails, as shown in [5]. A shorter proof of (8.4), valid only in the linear case (i.e., $\Phi(t) \equiv t$), has been found in [6]. Yet

another very interesting proof valid for solutions and so-called very weak solution (as in the local case defined in [79]) has been found by Schikorra [123] and it is based on a very delicate use of commutator estimates in fractional spaces.

When in (8.2) the dependence on the coefficients is smooth or when we are considering the genuine fractional p-Laplacean operator (that is, we have (8.3)), one expects higher regularity of solutions. This is still a largely open issue. In this direction, a recent interesting result by Brasco & Lindgren & Schikorra [22] claims the higher Hölder continuity of solutions.

9. Another bridge between nonlocal operators and NPT

We finally conclude recalling an approach, introduced in [112], proposing a bridge between nonlocal techniques and nonlinear potential estimates. In particular, we outline the conceptual similarities between the fractional De Giorgi technique first pioneered in [32, 29] (and then widely developed in the following literature) and the one used in [112] to prove estimate (3.7) in the non-degenerate case $p = 2$. The proof in [112] breaks down in two different steps. The first one derives a fractional Caccioppoli estimate and it is in the following:

Theorem 9.1 (Fractional Caccioppoli inequality, [110, 112])**.** *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a local SOLA to* (2.3) *under assumptions* (2.1) *with $p = 2$. If. For every $\sigma < 1/2$ there exists a constant $c \equiv c(n, \nu, L, \sigma)$ such that the estimate*

$$[(|D_m u| - k)_+]_{W^{\sigma,1}(B_{R/2})} \leq \frac{c}{R^\sigma} \int_{B_R} (|D_m u| - k)_+ \, dx + \frac{cR|\mu|(B_R)}{R^\sigma} \tag{9.1}$$

holds whenever $m \in \{1, \dots, n\}$, $k \geq 0$ and $B_R \Subset \Omega$.

The difficult point in the theorem above is that although the problem considered is local, the estimate considered is nonlocal. Inequality (9.1) naturally lead to a suitable definition of fractional De Giorgi's classes which is similar to those used when dealing with nonlocal problems [32, 29, 44]. The main difference is that additional tail terms are involved when considering nonlocal equations. The second step then follows the classical orthodoxy fixed by De Giorgi's approach in that the remaining regularity information is contained in (9.1). Indeed we have

Theorem 9.2 (Potential estimate [112])**.** *Let $\mathbf{w} \in L^1_{\mathrm{loc}}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is an open subset, μ is Borel measure with finite total mass, and $x_0 \in \Omega$. Let $\sigma \in (0, 1)$, and assume that for every radius $r > 0$ such that $B_r(x_0) \Subset \Omega$ the inequality*

$$[(|\mathbf{w}| - k)_+]_{W^{\sigma,1}(B_{r/2}(x_0))} \leq \frac{L}{r^\sigma} \int_{B_r(x_0)} (|\mathbf{w}| - k)_+ \, dx + \frac{Lr|\mu|(B_r)}{r^\sigma}$$

holds for a positive constant L and for every $k \geq 0$; finally, assume that w is continuous at the point x_0. Then the estimate

$$|\mathbf{w}(x_0)| \leq c\mathbf{I}^\mu_1(x_0, R) + c \fint_{B_R(x_0)} (|\mathbf{w}| - k)_+ \, dx + k$$

holds with $c \equiv c(n, L, \sigma)$*, whenever* $B_R(x_0) \Subset \Omega$ *is a ball and* k *is non-negative real number.*

Estimate (3.7) now follows from the last two theorems.

References

[1] Acerbi E. & Mingione G.: Gradient estimates for the $p(x)$-Laplacean system. *J. Reine Angew. Math. (Crelle's J.)* 584 (2005), 117–148.

[2] Acerbi E. & Mingione G.: Gradient estimates for a class of parabolic systems. *Duke Math. J.* 136 (2007), 285–320.

[3] Adams D. R. & Hedberg L. I.: *Function spaces and potential theory.* Grundlehren der Mathematischen Wissenschaften 314. Springer-Verlag, Berlin, 1996.

[4] Armstrong S. & Daniel J.P.: Calderón-Zygmund estimates for stochastic homogenization. *J. Funct. Anal.* 270 (2016), 312–329.

[5] Auscher P. & Bortz S. & Egert M. & Saari O.: On regularity of weak solutions to linear parabolic systems with measurable coefficients. *J. Math. Pures Appl.* 121 (2019), 216–243.

[6] Auscher P. & Bortz S. & Egert M. & Saari O.: Nonlocal self-improving properties: a functional analytic approach. *Tunisian J. Math.* 1 (2019), 151–183.

[7] Avelin B. & Kuusi T. & Mingione G.: Nonlinear Calderón-Zygmund theory in the limiting case *Arch. Ration. Mech. Anal.* 227 (2018), 663–714.

[8] Baroni P.: Lorentz estimates for degenerate and singular evolutionary systems. *J. Differential Equations* 255 (2013), 2927–2951.

[9] Baroni P.: Marcinkiewicz estimates for degenerate parabolic equations with measure data. *J. Funct. Anal.* 267 (2014), 3397–3426.

[10] Baroni P.: Riesz potential estimates for a general class of quasilinear equations. *Calc. Var. Partial Differential Equations* 53 (2015), 803–846.

[11] Baroni P. & Colombo M. & Mingione G.: Harnack inequalities for double phase functionals. *Nonlinear Anal.* 121 (2015), 206–222.

[12] Baroni P. & Colombo M. & Mingione G.: Regularity for general functionals with double phase. *Calc. Var. Partial Differential Equations* 57:62 (2018), 48 pp.

[13] Baroni P. & Habermann J.: Calderón-Zygmund estimates for parabolic measure data problems. *J. Differential Equations* 252 (2012), 412–447.

[14] Baroni P. & Habermann J.: Parabolic Adams theorems. *Houston J. Math* 38 (2012), 855–914.

[15] Baruah D. & Harjulehto P. & Hästö P.: Capacities in generalized Orlicz spaces, *J. Function Spaces* (2018), Article ID 8459874.

[16] Bass R. F. & Kassmann M.: Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.* 357 (2005), 837–850.

[17] Bass R. F. & Kassmann M.: Hölder continuity of harmonic functions with respect to operators of variable order. *Comm. Partial Differential Equations* 30 (2005), 1249–1259.

[18] Beck L. & Mingione G.: Lipschitz bounds and non-uniform ellipticty. *Comm. Pure Appl. Math.*, to appear.

[19] Bildhauer M. & Fuchs M.: $C^{1,\alpha}$-solutions to non-autonomous anisotropic variational problems. *Calc. Var. Partial Differential Equations* 24 (2005) 309–340.

[20] Boccardo L. & Gallouët T.: Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.* 87 (1989),149–169.

[21] Bousquet P. & Brasco L.: C^1-regularity of orthotropic p-harmonic functions in the plane. *Anal. PDE* 11 (2018), 813–854.

[22] Brasco L. & Lindgren E. & Schikorra A.: Hölder regularity for the fractional p-Laplacian in the superquadratic case. *Adv. Math.* 338 (2018), 782–846.

[23] Breit D. & Cianchi A. & Diening L. & Kuusi T. & Schwarzacher S.: Pointwise Calderón-Zygmund gradient estimates for the p-Laplace system. *J. Math. Pures Appl. (IX)* 114 (2018), 146–190.

[24] Byun S. S.: Elliptic equations with BMO coefficients in Lipschitz domains. *Trans. Amer. Math. Soc.* 357 (2005), 1025–1046.

[25] Byun S. S.: Parabolic equations with BMO coefficients in Lipschitz domains. *J. Differential Equations* 209 (2005), 229–265.

[26] Byun S. S. & Oh J.: Global gradient estimates for non-uniformly elliptic equations. *Calc. Var. Partial Differential Equations* 56:46 (2017), 36 pp.

[27] Byun S.S. & Ok J. & Ryu S.: Global gradient estimates for elliptic equations of $p(x)$-Laplacian type with BMO nonlinearity. *J. Reine Angew. Math. (Crelle's J.)* 715 (2016), 1–38.

[28] Caffarelli L.: Interior a priori estimates for solutions of fully nonlinear equations. *Ann. of Math. (II)* 130 (1989), 189–213.

[29] Caffarelli L. & Chan C. H. & Vasseur A.: Regularity theory for parabolic nonlinear integral operators. *J. Amer. Math. Soc.* 24 (2011), 849–869.

[30] Caffarelli L. & Peral I.: On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.* 51 (1998), 1–21.

[31] Caffarelli L. & Silvestre L.: An extension problem related to the fractional Laplacian. Comm PDE 32 (2007). 1245–1260.

[32] Caffarelli L. & Vasseur A.: Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. *Ann. of Math. (II)* 171 (2010), 1903–1930.

[33] Calderón A. P. & Zygmund A.: On the existence of certain singular integrals. *Acta Math.* 88 (1952), 85–139.

[34] Calderón A. P. & Zygmund A.: On singular integrals. *Amer. J. Math.* 78 (1956), 289–309.

[35] Cencelj M. & Radulescu V. & Repovs D.: Double phase problems with variable growth. *Nonlinear Analysis* 177 (part a) (2018), 270–287.

[36] Chlebicka I.: Gradient estimates for problems with Orlicz growth. *Nonlinear Analysis*, to appear.

[37] Chlebicka I. & De Filippis C.: Removable sets in non-uniformly elliptic problems. *Preprint* (2018). `https://arxiv.org/abs/1901.03412`

[38] Cianchi A.: Maximizing the L^∞ norm of the gradient of solutions to the Poisson equation. *J. Geom. Anal.* 2 (1992), 499–515.

[39] Cianchi A.: Nonlinear potentials, local solutions to elliptic equations and rearrangements. *Ann. Scu. Norm. Sup. Cl. Sci. (V)* 10 (2011), 335–361

[40] Cianchi A. & Maz'ya V.: Global Lipschitz regularity for a class of quasilinear equations. *Comm. Partial Differential Equations* 36 (2011), 100–133.

[41] Cianchi A. & Maz'ya V.: Global boundedness of the gradient for a class of nonlinear elliptic systems. *Arch. Rat. Mech. Anal.* 212 (2014), 129–177.

[42] Cianchi A. & Maz'ya V.: Gradient regularity via rearrangements for p-Laplacian type elliptic boundary value problems. *J. Eur. Math. Soc.* 16 (2014), 571–595.

[43] Colombo M. & Mingione G.: Calderón-Zygmund estimates and non-uniformly elliptic operators. *J. Funct. Anal.* 270 (2016), 1416–1478.

[44] Cozzi M.: Regularity results and Harnack inequalities for minimizers and solutions of nonlocal problems: a unified approach via fractional De Giorgi classes. *J. Funct. Anal.* 272 (2017), 4762–4837

[45] Cruz-Uribe D. & Hästö P.: Extrapolation and interpolation in generalized Orlicz spaces. *Trans. Amer. Math. Soc.* 370 (2018), 4323–4349.

[46] Cupini G. & Marcellini P. & Mascolo E.: Nonuniformly elliptic energy integrals with p, q-growth. *Nonlinear Anal.* 177 (part a) (2018), 312–324.

[47] Cupini G. & Marcellini P. & Mascolo E.: Regularity of minimizers under limit growth conditions. *Nonlinear Anal.* 153 (2017), 294–310.

[48] De Filippis C.: Higher integrability for constrained minimizers of integral functionals with (p, q)-growth in low dimension. *Nonlinear Anal.* 170 (2018), 1–20.

[49] De Filippis C.: Partial regularity for manifold constrained $p(x)$-harmonic maps. *Calc. Var. Partial Differential Equations*, `https://doi.org/10.1007/s00526-019-1483-6`

[50] De Filippis C. & Mingione G.: A borderline case of Calderón-Zygmund estimates for non-uniformly elliptic problems. *St. Petersburg Math. J.*, to appear.

[51] De Filippis C. & Palatucci G.: Hölder regularity for nonlocal double phase equations. *J. Differential Equations*, `https://doi.org/10.1016/j.jde.2019.01.017`

[52] De Giorgi E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (III)* 125 3 (1957), 25–43.

[53] DiBenedetto E.: *Degenerate parabolic equations.* Universitext. Springer-Verlag, New York, 1993.

[54] DiBenedetto E. & Manfredi J.J.: On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems. *Amer. J. Math.* 115 (1993), 1107–1134.

[55] Di Castro A. & Kuusi T. & Palatucci G.: Local behavior of fractional p-minimizers. *Ann. Inst. H. Poincaré, Anal. Non Linèaire* 33 (2016), 1279–1299.

[56] Di Castro A. & Kuusi T. & Palatucci G.: Nonlocal Harnack inequalities. *J. Funct. Anal.* 267 (2014), 1807–1836.

[57] Diening L. & Harjulehto P. & Hasto P. & Ruzicka M.: *Lebesgue and Sobolev spaces with a variable growth exponent.* Springer Lecture notes Math. 2017 (2011).

[58] Diening L. & Kaplický P. & Schwarzacher S.: BMO estimates for the p-Laplacian. *Nonlinear Anal.* 75 (2012), 637–650.

[59] Diening L. & Schwarzacher S.: Global gradient estimates for the $p(\cdot)$-Laplacian. *Nonlinear Anal.* 106 (2014), 70–85.

[60] Di Nezza E. & Palatucci G. & Valdinoci E.: Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. math.* 136 (2012), 521–573.

[61] Dolzmann G. & Hungerbühler N. & Müller S.: The p-harmonic system with measure-valued right-hand side. *Ann. Inst. H. Poincaré Anal. Non Linèaire* 14 (1997), 353–364.

[62] Dolzmann G. & Hungerbühler N. & Müller S.: Uniqueness and maximal regularity for nonlinear elliptic systems of n-Laplace type with measure valued right-hand side. *J. Reine Angew. Math. (Crelle's J.)* 520 (2000), 1–35.

[63] Duzaar F. & Kristensen J. & Mingione G.: The existence of regular boundary points for non-linear elliptic systems. *J. Reine Angew. Math. (Crelle's J.)* 602 (2007), 17–58.

[64] Duzaar F. & Mingione G.: Local Lipschitz regularity for degenerate elliptic systems. *Ann. Inst. H. Poincaré Anal. Non Linèaire* 27 (2010), 1361–1396.

[65] Duzaar F. & Mingione G.: Gradient estimates via linear and nonlinear potentials. *J. Funct. Anal.* 259 (2010), 2961–2998.

[66] Duzaar F. & Mingione G.: Gradient estimates via non-linear potentials. *Amer. J. Math.* 133 (2011), 1093–1149.

[67] Eleuteri M. & Marcellini P. & Mascolo E.: Lipschitz estimates for systems with ellipticity conditions at infinity. *Ann. Mat. Pura Appl. (IV)* 195 (2016), 1575–1603.

[68] Eleuteri M. & Marcellini P. & Mascolo E.: Regularity for scalar integrals without structure conditions. *Adv. Calc. Var.*, to appear.

[69] Giaquinta M. & Giusti E.: On the regularity of the minima of variational integrals. *Acta Math.* 148 (1982), 31–46.

[70] Harjulehto P. & Hästö P. & Toivanen O.: Hölder regularity of quasiminimizers under generalized growth conditions. *Calc. Var. Partial Differential Equations* 56:22 (2017), 26 pp.

[71] Harjulehto P. & Hästö P.: The Riesz potential in generalized Orlicz spaces. *Forum Math.* 29 (2017), 229–244.

[72] Hästö P.: The maximal operator on generalized Orlicz spaces. *J. Funct. Anal.* 269 (2015), 4038–4048.

[73] Hästö P. & Ok J.: Maximal regularity for local minimizers of non-autonomous functionals. *Preprint 2019.*

[74] Hästö P. & Ok J.: Calderón-Zygmund estimates in generalized Orlicz spaces. *Preprint* 2018.

[75] Havin M. & Maz'ja V. G.: Nonlinear potential theory. *Russ. Math. Surveys* 27 (1972), 71–148.

[76] Hedberg L. & Wolff Th. H.: Thin sets in Nonlinear Potential Theory. *Ann. Inst. Fourier (Grenoble)* 33 (1983), 161–187.

[77] Heinonen J. & Kilpeläinen T. & Martio O.: *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs., New York, 1993.

[78] Iwaniec T.: Projections onto gradient fields and L^p-estimates for degenerated elliptic operators. *Studia Math.* 75 (1983), 293–312.

[79] Iwaniec T. & Sbordone C.: Weak minima of variational integrals. *J. Reine Angew. Math. (Crelle's J.)* 454 (1994), 143–161.

[80] Kassmann M.: A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations* 34 (2009), 1–21.

[81] Kilpeläinen T. & Kuusi T. & Tuhola-Kujanpää A.: Superharmonic functions are locally renormalized solutions. *Ann. Inst. H. Poincaré, Anal. Non Linèaire*, 28 (2011), 775–795.

[82] Kilpeläinen T. & Malý J.: Degenerate elliptic equations with measure data and nonlinear potentials. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV)* 19 (1992), 591–613.

[83] Kilpeläinen T. & Malý J.: The Wiener test and potential estimates for quasilinear elliptic equations. *Acta Math.* 172 (1994), 137–161.

[84] Korvenpää J. & Kuusi T. & Palatucci G.: The obstacle problem for nonlinear integro-differential operators. *Calc. Var. Partial Differential Equations* 55 (2016), no. 3, Art. 63, 29 pp.

[85] Korvenpää J. & Kuusi T. & Palatucci G.: Fractional superharmonic functions and the Perron method for nonlinear integro-differential equations. *Math. Ann.* 369 (2017), 1443–1489.

[86] Kuusi T. & Mingione G.: Universal potential estimates. *J. Funct. Anal.* 262 (2012), 4205–4638.

[87] Kuusi T. & Mingione G.: Linear potentials in Nonlinear Potential Theory. *Arch. Ration. Mech. Anal.* 207 (2013), 215–246.

[88] Kuusi T. & Mingione G.: Gradient regularity for nonlinear parabolic equations. *Ann. Scu. Norm. Sup. Pisa Cl. Sci. (V)* 12 (2013), 755–822.

[89] Kuusi T. & Mingione G.: Riesz potentials and nonlinear parabolic equations. *Arch. Ration. Mech. Anal.* 212 (2014), 727–780.

[90] Kuusi T. & Mingione G.: Guide to nonlinear potential estimates. *Bull. Math. Sci.* 4 (2014), 1–82.

[91] Kuusi T. & Mingione G.: The Wolff gradient bound for degenerate parabolic equations. *J. Eur. Math. Soc. (JEMS)* 16 (2014), 835–892.

[92] Kuusi T. & Mingione G.: A nonlinear stein theorem. *Calc. Var. Partial Differential Equations* 51 (2014), 45–86.

[93] Kuusi T. & Mingione G.: Vectorial Nonlinear Potential Theory. *J. Eur. Math. Soc. (JEMS)* 20 (2018), 929–1004.

[94] Kuusi T. & Mingione G. & Sire Y.: Nonlocal equations with measure data. *Comm. Math. Phyis.* 337 (2015), 1317–1368.

[95] Kuusi T. & Mingione G. & Sire Y.: Nonlocal self-improving properties. *Anal. PDE* 8 (2015), 57-114.

[96] Ladyzhenskaya O.A. & Ural'tseva N.N.: *Linear and quasilinear elliptic equations.* Academic Press, New York-London 1968

[97] Lewis J. L.: On very weak solutions of certain elliptic systems. *Comm. Partial Differential Equations* 18 (1993), 1515–1537.

[98] Lindgren E.: Hölder estimates for viscosity solutions of equations of fractional p-Laplace type. *NoDEA Nonlinear Differential Equations Appl.* 23:55 (2016), 18 pp.

[99] Lindgren E. & Lindqvist P.: Perron's method and Wiener's theorem for a nonlocal equation. *Potential Anal.* 46 (2017), 705–737.

[100] Lindqvist P. & Martio O.: Two theorems of N. Wiener for solutions of quasilinear elliptic equations. *Acta Math.* 155 (1985), 153–171.

[101] Littman W. & Stampacchia G. & Weinberger H. F.: Regular points for elliptic equations with discontinuous coefficients. *Ann. Scu. Norm. Sup. Pisa (III)* 17 (1963), 43–77.

[102] Malý J. & Ziemer W.P.: *Fine regularity of solutions of elliptic partial differential equations.* Mathematical Surveys and Monographs, 51. Am. Math. Soc., Providence, RI, 1997.

[103] Manfredi J.J.: Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations. *Ph.D. Thesis.* University of Washington, St. Louis.

[104] Marcellini P.: Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions *Arch. Ration. Mech. Anal.* 105 (1989), 267–284.

[105] Marcellini P.: Regularity and existence of solutions of elliptic equations with p, q-growth conditions. *J. Differential Equations* 90 (1991), 1–30.

[106] Marcellini P.: Regularity under general and p, q-growth conditions. *Preprint* 2018

[107] Maz'ya V.: The continuity at a boundary point of the solutions of quasi-linear elliptic equations. (Russian) *Vestnik Leningrad. Univ.* 25 (1970), 42–55.

[108] Mingione G.: Bounds for the singular set of solutions to non linear elliptic systems. *Calc. Var. Partial Differential Equations* 18 (2003), 373–400.

[109] Mingione G.: Regularity of minima: an invitation to the Dark side of the Calculus of Variations. *Applications of Mathematics* 51 (2006), 355–425.

[110] Mingione G.: The Calderón-Zygmund theory for elliptic problems with measure data. *Ann. Scu. Norm. Sup. Pisa Cl. Sci. (V)* 6 (2007), 195–261.

[111] Mingione G.: Gradient estimates below the duality exponent. *Math. Ann.* 346 (2010), 571-627.

[112] Mingione G.: Gradient potential estimates. *J. Eur. Math. Soc. (JEMS)* 13 (2011), 459–486.

[113] Nguyen Q.H. & Phuc N.C.: Good-λ and Muckenhoupt-Wheeden type bounds in quasilinear measure datum problems, with applications. *Math. Ann.* `https://doi.org/10.1007/s00208-018-1744-2`.

[114] Ok J.: Gradient estimates for elliptic equations with $L^{p(\cdot)} \log L$ growth. *Calc. Var. Partial Differential Equations* 55 (2016), no. 2, Art. 26, 30 pp.

[115] Ok J.: Regularity of ω-minimizers for a class of functionals with non-standard growth. *Calc. Var. Partial Differential Equations* 56:48 (2017), 31 pp.

[116] Ok J.: Partial Hölder regularity for elliptic systems with non-standard growth. *J. Funct. Anal.* 274 (2018), 723–768.

[117] Palatucci G.: The Dirichlet problem for the p-fractional Laplace equation. *Nonlinear Anal.* 177 (2018), 699–732.

[118] Phuc N.C.: Global integral gradient bounds for quasilinear equations below or near the natural exponent. *Arkiv för Matematik*, 52 (2014), 329–354.

[119] Phuc N.C.: Morrey global bounds and quasilinear Riccati type equations below the natural exponent. *J. Math. Pures Appl. (IX)* 102 (2014), 99–123.

[120] Phuc N. C. & Verbitsky I. E.: Quasilinear and Hessian equations of Lane-Emden type. *Ann. of Math. (II)* 168 (2008), 859–914.

[121] Phuc N. C. & Verbitsky I. E.: Singular quasilinear and Hessian equations and inequalities. *J. Funct. Anal.* 256 (2009), 1875–1906.

[122] Radulescu V. & Repovs D.:*Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis.* Chapman & Hall/CRC Monographs and Research Notes in Mathematics. 2015

[123] Schikorra A.: Nonlinear commutators for the fractional p-Laplacian and applications. *Math. Ann.* 366 (2016), 695–720.

[124] Serrin J.: Local behavior of solutions of quasi-linear equations. *Acta Math.* 111 (1964), 247–302.

[125] Serrin J.: Isolated singularities of solutions of quasi-linear equations. *Acta Math.* 113 (1965), 219–240.

[126] Stein E. M.: Editor's note: the differentiability of functions in $\mathbb{R}^n$. *Ann. of Math. (II)* 113 (1981), 383–385.

[127] Šverák V. & Yan X.: Non-Lipschitz minimizers of smooth uniformly convex variational integrals. *Proc. Natl. Acad. Sci. USA* 99/24 (2002), 15269-15276.

[128] Tran M.P.: Good-λ type bounds of quasilinear elliptic equations for the singular case. *Nonlinear Anal.* 178 (2019), 266–281.

[129] Trudinger N.S. & Wang X.J.: On the weak continuity of elliptic operators and applications to potential theory. *Amer. J. Math.* 124 (2002), 369–410.

[130] Uhlenbeck K.: Regularity for a class of nonlinear elliptic systems. *Acta Math.* 138 (1977), 219–240.

[131] Ural'tseva N.N.: Degenerate quasilinear elliptic systems. *Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 7 (1968), 184–222.

[132] Wang L.: A geometric approach to the Calderón-Zygmund estimates. *Acta Math. Sin. (Engl. Ser.)* 19 (2003), 381–396.

[133] Zhang Q. & Radulescu V.: Double phase anisotropic variational problems and combined effects of reaction and absorption terms. *J. Math. Pures Appl. (IX)* 118 (2018), 159–203.

[134] Zhikov V.V.: Averaging of functionals of the calculus of variations and elasticity theory. *Izv. Akad. Nauk SSSR Ser. Mat.* 50 (1986), 675–710.

[135] Zhikov V.V.: On Lavrentiev's Phenomenon. *Russian J. Math. Phys.* 3 (1995), 249–269.

[136] Zhikov V.V.: On some variational problems. *Russian J. Math. Phys.* 5 (1997), 105–116.

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