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Bott–Chern Formality and Massey Products on Strong Kähler with Torsion and Kähler Solvmanifolds

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Abstract

We study the interplay between geometrically-Bott–Chern-formal metrics and SKT metrics. We prove that a 6-dimensional nilmanifold endowed with a invariant complex structure admits an SKT metric if and only if it is geometrically-Bott–Chern-formal. We also provide some partial results in higher dimensions for nilmanifolds endowed with a class of suitable complex structures. Furthermore, we prove that any Kähler solvmanifold is geometrically formal. Finally, we explicitly construct lattices for a complex solvable Lie group in the list of Nakamura (J Differ Geom 10:85–112, 1975) on which we provide a non vanishing quadruple ABC-Massey product.

Keywords SKT metric · ABC-Massey product · Geometrically formal · Nilmanifold · Solvmanifold

Mathematics Subject Classification 53C55 · 53B35 · 22E25

1 Introduction

Let (M, J) be a compact complex manifold of real dimension $2n$. Dolbeault, Bott–Chern, and Aeppli cohomologies play a relevant role in the study of the holomorphic invariants and provide obstructions for the existence of further structures. For example, the existence of a Kähler metric on (M, J) implies that the (p, p)-Hodge numbers are positive, the $\partial\overline{\partial}$-Lemma holds, the de Rham complex is a formal DGA [10, 33, 34] and M satisfies the Hard Lefschetz condition. When (M, J) admits a Kähler metric,

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all these cohomology groups are isomorphic. This fact is no longer true for compact non Kähler manifolds; in such a case, Bott–Chern cohomology may yield further information on the complex geometry of the manifold.

Compact quotients of simply connected nilpotent Lie groups by a lattice, namely *nilmanifolds*, endowed with an invariant complex structure, are one of the main sources of concrete examples on which explicit cohomological computations can be carried out. Due to Benson and Gordon [7], such manifolds never admit a Kähler structure unless they are tori. Nevertheless, nilmanifolds can admit special Hermitian metrics, e.g., *strong Kähler with torsion metrics*, shortly *SKT metrics* (see [4, 8, 14, 18, 19, 21] for general results on SKT geometry and generalized Kähler geometry and [13, 15–17, 28] for existence results on nilmanifolds).

Bott–Chern cohomology provides an obstruction to the existence of curves of SKT metrics starting from a SKT metric (see [26]). Another relation between Bott–Chern cohomology and metric properties is given by the notion of *Bott–Chern formality*. More precisely, following Kotschick [23], in [3] the notion of *geometrically-Bott–Chern-formal metrics*, shortly *geometrically-BC-formal metrics*, is defined. In the same work, as an obstruction to the existence of geometrically-BC-formal metrics, the authors introduced the *triple Aeppli–Bott–Chern–Massey products*, shortly *triple ABC-Massey products*; such products are defined by three cohomology Bott–Chern classes satisfying suitable conditions which produce a coholomogy class living in a quotient of Aeppli cohomology modulo an ideal of indeterminacy. Recently, Milivojevic and Stelzig [24] introduced the *n-fold Aeppli–Bott–Chern–Massey products*, generalizing the triple ABC-Massey products in [3].

In the first part of the present paper, we investigate possible further interplays between SKT metrics and geometrically-BC-formal metrics on nilmanifolds endowed with an invariant complex structure. We prove the following.

Theorem (see Theorem 3.1) *Let $(\Gamma\backslash G := M, J)$ be a 6-dimensional nilmanifold endowed with an invariant complex structure J. Then, (M, J) is SKT if, and only if, it is geometrically-BC-formal. In particular, every Hermitian invariant metric is SKT if, and only if, it is geometrically-BC-formal.*

For complex dimension $n > 3$, there exists SKT nilmanifolds with no geometrically-BC-formal metrics (see Proposition 4.8). Consequently, for $n > 3$, it is natural to look for special classes of complex structures, in order to extend Theorem 3.1. In Proposition 4.2 we prove that, under suitable assumptions on the complex structure, see (2.4), if the nilmanifold has an invariant geometrically-BC-formal metric, then it also admits an SKT metric. This is a partial converse of [32, Theorem 7.4].

The second part of the paper is devoted to the investigation of cohomological properties of solvmanifolds endowed with an invariant complex structure. By a *solvmanifold*, we mean a compact quotient of a simply connected solvable Lie group by a lattice. In [20], Hasegawa fully characterized solvmanifolds admitting a Kähler structure. Namely, he proved that solvmanifolds admit a Kähler structure if and only if they are finite quotients of a tori with a structure of a holomorphic bundle over a torus with fiber a torus. Following the explicit description by Hasegawa, we prove

Theorem (see Theorem 5.10 and Corollary 5.11) *Let X be any Kähler solvmanifold. Then, X is geometrically-BC-formal and, consequently, every ABC-Massey on X vanishes.*

Therefore, the vanishing of ABC-Massey products provides an obstruction to the Kählerianity of solvmanifolds. It is useful to observe that in [31], the authors construct an explicit example of a compact complex manifold satisfying the $\partial\overline{\partial}$-lemma and admitting a triple non vanishing ABC-Massey product. As already asked by Milivojevic and Stelzig in [24], it is natural to rise the following

Question Is the non vanishing of an ABC-Massey product an obstruction to the existence of a Kähler metric on a given compact complex manifold?

Concerning the explicit construction of higher ABC-Massey products, we provide a non vanishing quadruple ABC-Massey product on a complex solvmanifold of complex dimension 4.

The work is organized as follows. In Sect. 2 we start by fixing the notation and recalling the basic facts on ABC-Massey products. Section 3 is devoted to the proof of Theorem 3.1. The argument relies on the characterization, due to Ugarte [37], of global frames for which J can be written in a convenient way, following the paper of Salamon [29]. In Sect. 4, complex structures of *special type* on nilmanifolds are introduced. For such complex structures, Lemma 4.3 gives a concrete description of their Bott–Chern cohomology and Theorem 4.5 gives a sufficient condition for the existence of a non vanishing ABC-Massey product on nilmanifolds endowed with complex structures of special type. Theorem 4.5 is then applied explicitly in Example 4.6 and Example 4.7. Section 5 contains the proof of Theorem 5.10. Such a result follows from the full description of the cohomology of Kähler solvmanifolds proved in Theorem 5.8. Finally, in Sect. 6 we construct a non vanishing quadruple ABC-Massey product of the quotient of a complex Lie group of complex dimension 4 in the list of Nakamura [25, IV 6., p. 108]. We explicitly describe two families of lattices (Lemma 6.1) giving rise to two complex parallelizable manifolds with different Dolbeault and Bott–Chern cohomologies (Tables 1, 2) which have a structure of holomorphic fiber bundle over a complex torus with non Kähler typical fiber, i.e., the *Iwasawa* manifold.

2 Notation and Preliminary

We denote by $M = \Gamma\backslash G$ the quotient of a simply connected solvable Lie group G with Lie algebra $\mathfrak{g}$ by a lattice Γ and by J an invariant complex structure, i.e., a complex structure on $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$ and let J define a complex structure as usual on $\mathfrak{g}^*$ by $J\alpha(X) = \alpha(JX)$, for every $\alpha \in \mathfrak{g}^*$, $X \in \mathfrak{g}$. Then, the space $\mathfrak{g}^*_{\mathbb{C}} := \mathfrak{g}^* \otimes \mathbb{C}$ decomposes as

$$\mathfrak{g}^*_{\mathbb{C}} = (\mathfrak{g}^*_{\mathbb{C}})^{1,0} \oplus (\mathfrak{g}^*_{\mathbb{C}})^{0,1}$$

where $(\mathfrak{g}^*_{\mathbb{C}})^{1,0} = \{\alpha - iJ\alpha : \alpha \in \mathfrak{g}^*\}$ and $(\mathfrak{g}^*_{\mathbb{C}})^{0,1} = \{\alpha + iJ\alpha : \alpha \in \mathfrak{g}^*\}$ are, respectively, the $\pm i$-eigenspaces of the $\mathbb{C}$-linear extension of J on $\mathfrak{g}^*_{\mathbb{C}}$.

Table 1 Bott-Chern cohomology of the manifold $M_{\pi/2}$

(p,q)	$H^{p,q}_{BC}(M_{\frac{\pi}{2}})$
(0,0)	$\mathbb{C}\langle 1 \rangle$
(1,0)	$\mathbb{C}\langle dz_1 \rangle$
(0,1)	$\mathbb{C}\langle dz_{\overline{1}} \rangle$
(2,0)	$\mathbb{C}\langle e^{z_1} dz_{12},\ e^{-z_1} dz_{13},\ dz_{23} \rangle$
(1,1)	$\mathbb{C}\langle dz_{1\overline{1}} \rangle$
(0,2)	$\mathbb{C}\langle e^{\overline{z}_1} dz_{\overline{12}},\ e^{-\overline{z}_1} dz_{\overline{13}},\ dz_{\overline{23}} \rangle$
(3,0)	$\mathbb{C}\langle dz_{123},\ e^{z_1} dz_{12} \wedge \varphi^4, e^{-z_1} dz_{13} \wedge \varphi^4, dz_{23} \wedge \varphi^4 \rangle$
(2,1)	$\mathbb{C}\langle e^{z_1} dz_{12\overline{1}},\ e^{-z_1} dz_{13\overline{1}},\ dz_{23\overline{1}} \rangle$
(1,2)	$\mathbb{C}\langle e^{\overline{z}_1} dz_{1\overline{12}},\ e^{-\overline{z}_1} dz_{1\overline{13}},\ dz_{1\overline{23}},\ \rangle$
(0,3)	$\mathbb{C}\langle dz_{\overline{123}},\ e^{\overline{z}_1} dz_{\overline{12}} \wedge \varphi^{\overline{4}}, e^{-\overline{z}_1} dz_{\overline{13}} \wedge \varphi^{\overline{4}},\ dz_{\overline{23}} \wedge \varphi^{\overline{4}} \rangle$
(4,0)	$\mathbb{C}\langle dz_{123} \wedge \varphi^4 \rangle$
(3,1)	$\mathbb{C}\langle dz_{123\overline{1}},\ e^{z_1} dz_{12} \wedge \varphi^4 \wedge dz_{\overline{1}},\ e^{-z_1} dz_{13} \wedge \varphi^4 \wedge dz_{\overline{1}},\ dz_{23} \wedge \varphi^4 \wedge dz_{\overline{1}} \rangle$
(2,2)	0

Table 1 continued

(p, q)	$H^{p,q}_{BC}(M_{\frac{\pi}{2}})$
(1,3)	$\mathbb{C}\langle dz_{1\overline{123}},\ e^{\bar{z}_1}dz_{1\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-\bar{z}_1}dz_{1\overline{13}}\wedge\varphi^{\overline{4}},\ dz_{1\overline{23}}\wedge\varphi^{\overline{4}}\rangle$
(0,4)	$\mathbb{C}\langle dz_{\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(4,1)	$\mathbb{C}\langle dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{1}}\rangle$
(3,2)	$\mathbb{C}\langle e^{\bar{z}_1}dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{12}},\ e^{-\bar{z}_1}dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{13}},\ dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{23}}\rangle$
(2,3)	$\mathbb{C}\langle e^{z_1}dz_{12\overline{23}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{13\overline{23}}\wedge\varphi^{\overline{4}},\ dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{23}}\rangle$
(1,4)	$\mathbb{C}\langle dz_{1\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(4,2)	$\mathbb{C}\langle e^{\bar{z}_1}dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{12}},\ e^{-\bar{z}_1}dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{13}},\ dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{123}}\rangle$
(3,3)	$\mathbb{C}\langle dz_{123\overline{23}}\wedge\varphi^{\overline{4}},\ e^{z_1}dz_{12}\wedge\varphi^{4}\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{13}\wedge\varphi^{4}\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}},$ $e^{\bar{z}_1}dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-\bar{z}_1}dz_{23}\wedge\varphi^{4}\wedge dz_{13}\wedge\varphi^{\overline{4}},\ dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}}\rangle$
(2,4)	$\mathbb{C}\langle e^{z_1}dz_{12\overline{123}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{13\overline{123}}\wedge\varphi^{\overline{4}},\ dz_{23\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(4,3)	$\mathbb{C}\langle dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{123}},\ e^{\bar{z}_1}dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-\bar{z}_1}dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}},$ $dz_{23\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(3,4)	$\mathbb{C}\langle dz_{123\overline{123}}\wedge\varphi^{\overline{4}},\ e^{z_1}dz_{12}\wedge\varphi^{4}\wedge dz_{\overline{123}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{13}\wedge\varphi^{4}\wedge dz_{\overline{123}}\wedge\varphi^{\overline{4}},$ $dz_{23}\wedge\varphi^{4}\wedge dz_{\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(4,4)	$\mathbb{C}\langle dz_{123}\wedge\varphi^{4}\wedge dz_{\overline{123}}\wedge\varphi^{\overline{4}}\rangle$

Table 2 Bott-Chern cohomology of the manifold M_π

(p,q)	$H^{p,q}_{BC}(M_\pi)$
(0,0)	$\mathbb{C}\langle 1\rangle$
(1,0)	$\mathbb{C}\langle dz_1\rangle$
(0,1)	$\mathbb{C}\langle dz_{\bar 1}\rangle$
(2,0)	$\mathbb{C}\langle e^{z_1}dz_{12},\ e^{-z_1}dz_{13},\ dz_{23}\rangle$
(1,1)	$\mathbb{C}\langle dz_{1\bar 1},\ e^{z_1}dz_{1\bar 2},\ e^{-z_1}dz_{1\bar 3},\ e^{\bar z_1}dz_{2\bar 1},\ dz_{2\bar 3},\ e^{-\bar z_1}dz_{3\bar 1},\ dz_{3\bar 2}\rangle$
(0,2)	$\mathbb{C}\langle e^{\bar z_1}dz_{\bar 1\bar 2},\ e^{-\bar z_1}dz_{\bar 1\bar 3},\ dz_{\bar 2\bar 3}\rangle$
(3,0)	$\mathbb{C}\langle dz_{123},\ e^{z_1}dz_{12}\wedge\varphi^4,\ e^{-z_1}dz_{13}\wedge\varphi^4, dz_{23}\wedge\varphi^4\rangle$
(2,1)	$\mathbb{C}\langle e^{z_1}dz_{12\bar 1},\ e^{\bar z_1}dz_{12\bar 1},\ e^{2z_1}dz_{12\bar 2},\ dz_{12\bar 3},\ e^{-z_1}dz_{13\bar 1},\ e^{-\bar z_1}dz_{13\bar 1},\ dz_{13\bar 2},\ e^{-2z_1}dz_{13\bar 3},\ dz_{23\bar 1},$ $e^{\bar z_1}dz_2\wedge\varphi^4\wedge dz_{\bar 1},\ dz_2\wedge\varphi^4\wedge dz_{\bar 3},\ e^{-\bar z_1}dz_3\wedge\varphi^4\wedge dz_{\bar 1},\ dz_3\wedge\varphi^4\wedge dz_{\bar 2}\rangle$
(1,2)	$\mathbb{C}\langle e^{z_1}dz_{1\bar 1\bar 2},\ e^{\bar z_1}dz_{1\bar 1\bar 2},\ e^{-z_1}dz_{1\bar 1\bar 3},\ e^{-\bar z_1}dz_{1\bar 1\bar 3},\ dz_{1\bar 2\bar 3},\ e^{z_1}dz_{1\bar 2}\wedge\varphi^{\bar 4},\ e^{-z_1}dz_{1\bar 3}\wedge\varphi^{\bar 4},\ e^{2\bar z_1}dz_{2\bar 1\bar 2},$ $dz_{2\bar 1\bar 3},\ dz_{2\bar 3}\wedge\varphi^{\bar 4},\ dz_{3\bar 1\bar 2},\ e^{-2\bar z_1}dz_{3\bar 1\bar 3},\ dz_{3\bar 2}\wedge\varphi^{\bar 4}\rangle$
(0,3)	$\mathbb{C}\langle dz_{\bar 1\bar 2\bar 3},\ e^{\bar z_1}dz_{\bar 1\bar 2}\wedge\varphi^{\bar 4},\ e^{-\bar z_1}dz_{\bar 1\bar 3}\wedge\varphi^{\bar 4}, dz_{\bar 2\bar 3}\wedge\varphi^{\bar 4}\rangle$
(4,0)	$\mathbb{C}\langle dz_{123}\wedge\varphi^4\rangle$
(3,1)	$\mathbb{C}\langle dz_{123\bar 1},\ e^{z_1}dz_{123\bar 2},\ e^{-z_1}dz_{123\bar 3},\ e^{z_1}dz_{12}\wedge\varphi^4\wedge dz_{\bar 1},\ e^{\bar z_1}dz_{12}\wedge\varphi^4\wedge dz_{\bar 1},\ e^{2z_1}dz_{12}\wedge\varphi^4\wedge dz_{\bar 2},$ $dz_{12}\wedge\varphi^4\wedge dz_{\bar 3},\ e^{-z_1}dz_{13}\wedge\varphi^4\wedge dz_{\bar 1},\ e^{-\bar z_1}dz_{13}\wedge\varphi^4\wedge dz_{\bar 1},\ dz_{13}\wedge\varphi^4\wedge dz_{\bar 2},$ $e^{-2z_1}dz_{13}\wedge\varphi^4\wedge dz_{\bar 3},\ dz_{23}\wedge\varphi^4\wedge dz_{\bar 1}\rangle$

Table 2 continued

(p, q)	$H^{p,q}_{BC}(M_\pi)$
(2,2)	$\mathbb{C}\langle e^{2z_1} dz_{12\overline{12}},\ e^{2\bar{z}_1} dz_{12\overline{12}},\ dz_{12\overline{13}},\ e^{2z_1} dz_{12\bar{2}} \wedge \varphi^{\bar{4}},\ dz_{12\bar{3}} \wedge \varphi^{\bar{4}},\ dz_{13\overline{12}},\ e^{-2z_1} dz_{13\overline{13}},\ e^{-2\bar{z}_1} dz_{13\overline{13}}$ $dz_{13\bar{2}} \wedge \varphi^{\bar{4}},\ e^{-2z_1} dz_{13\bar{3}} \wedge \varphi^{\bar{4}},\ e^{2\overline{z_1}} dz_2 \wedge \varphi^4 \wedge dz_{\overline{12}},\ dz_2 \wedge \varphi^4 \wedge dz_{\overline{13}},\ dz_2 \wedge \varphi^4 \wedge dz_{\bar{3}} \wedge \varphi^{\bar{4}},$ $dz_3 \wedge \varphi^4 \wedge dz_{\overline{12}},\ e^{-\bar{z}_1} dz_3 \wedge \varphi^4 \wedge dz_{\bar{2}} \wedge \varphi^{\bar{4}}\rangle$
(1,3)	$\mathbb{C}\langle dz_{1\overline{123}},\ e^{z_1} dz_{1\overline{12}} \wedge \varphi^{\bar{4}},\ e^{\bar{z}_1} dz_{1\overline{12}} \wedge \varphi^{\bar{4}},\ e^{-z_1} dz_{1\overline{13}} \wedge \varphi^{\bar{4}},\ e^{-\bar{z}_1} dz_{1\overline{13}} \wedge \varphi^{\bar{4}}, dz_{\overline{123}} \wedge \varphi^{\bar{4}}, e^{\bar{z}_1} dz_{2\overline{123}},$ $e^{2\bar{z}_1} dz_{2\overline{12}} \wedge \varphi^{\bar{4}},\ dz_{2\overline{13}} \wedge \varphi^{\bar{4}},\ e^{-\bar{z}_1} dz_{3\overline{123}},\ dz_{3\overline{12}} \wedge \varphi^{\bar{4}},\ e^{-2\bar{z}_1} dz_{3\overline{13}} \wedge \varphi^{\bar{4}}\rangle$
(0,4)	$\mathbb{C}\langle dz_{\overline{123}} \wedge \varphi^{\bar{4}}\rangle$
(4,1)	$\mathbb{C}\langle dz_{123} \wedge \varphi^4 \wedge dz_{\bar{1}},\ e^{z_1} dz_{123} \wedge \varphi^4 \wedge dz_{\bar{2}},\ e^{-z_1} dz_{123} \wedge \varphi^4 \wedge dz_{\bar{3}}\rangle$
(3,2)	$\mathbb{C}\langle e^{z_1} dz_{123\overline{12}},\ e^{-z_1} dz_{123\overline{13}},\ e^{z_1} dz_{123\bar{2}} \wedge \varphi^{\bar{4}},\ e^{-z_1} dz_{123\bar{3}} \wedge \varphi^{\bar{4}},\ e^{2z_1} dz_{12} \wedge \varphi^4 \wedge dz_{\overline{12}},$ $e^{2\bar{z}_1} dz_{12} \wedge \varphi^4 \wedge dz_{\overline{12}},\ dz_{12} \wedge \varphi^4 \wedge dz_{\overline{13}},\ e^{2z_1} dz_{12} \wedge \varphi^4 \wedge dz_{\bar{2}} \wedge \varphi^{\bar{4}},\ dz_{12} \wedge \varphi^4 \wedge dz_{\bar{3}} \wedge \varphi^{\bar{4}},$ $dz_{13} \wedge \varphi^4 \wedge dz_{\overline{12}},\ e^{-2z_1} dz_{13} \wedge \varphi^4 \wedge dz_{\overline{13}},\ e^{-2\bar{z}_1} dz_{13} \wedge \varphi^4 \wedge dz_{\overline{13}},\ dz_{13} \wedge \varphi^4 \wedge dz_{\bar{2}} \wedge \varphi^{\bar{4}},$ $e^{-2z_1} dz_{12} \wedge \varphi^4 \wedge dz_{\bar{3}} \wedge \varphi^{\bar{4}},\ e^{\bar{z}_1} dz_{23} \wedge \varphi^4 \wedge dz_{\overline{12}},\ e^{-\bar{z}_1} dz_{23} \wedge \varphi^4 \wedge dz_{\overline{13}},\ dz_{23} \wedge \varphi^4 \wedge dz_{\overline{23}}\rangle$
(2,3)	$\mathbb{C}\langle e^{\bar{z}_1} dz_{12\overline{123}},\ e^{2z_1} dz_{12\overline{12}} \wedge \varphi^{\bar{4}},\ e^{2\bar{z}_1} dz_{12\overline{12}} \wedge \varphi^{\bar{4}},\ dz_{12\overline{13}} \wedge \varphi^{\bar{4}},\ e^{z_1} dz_{12\overline{23}} \wedge \varphi^{\bar{4}},\ e^{-\bar{z}_1} dz_{12\overline{123}},$ $dz_{13\overline{12}} \wedge \varphi^{\bar{4}},\ e^{-2z_1} dz_{13\overline{13}} \wedge \varphi^{\bar{4}},\ e^{-2\bar{z}_1} dz_{13\overline{13}} \wedge \varphi^{\bar{4}},\ e^{-z_1} dz_{13\overline{23}} \wedge \varphi^{\bar{4}},\ dz_{23\overline{23}} \wedge \varphi^{\bar{4}},$ $e^{\bar{z}_1} dz_2 \wedge \varphi^4 \wedge dz_{\overline{123}},\ e^{2\bar{z}_1} dz_2 \wedge \varphi^4 \wedge dz_{\overline{12}} \wedge \varphi^{\bar{4}},\ dz_2 \wedge \varphi^4 \wedge dz_{\overline{13}} \wedge \varphi^{\bar{4}},\ e^{-\bar{z}_1} dz_3 \wedge \varphi^4 \wedge dz_{\overline{123}},$ $dz_3 \wedge \varphi^4 \wedge dz_{\overline{12}} \wedge \varphi^{\bar{4}},\ e^{-2\bar{z}_1} dz_3 \wedge \varphi^4 \wedge dz_{\overline{13}} \wedge \varphi^{\bar{4}}\rangle$

Table 2 continued

(p,q)	$H^{p,q}_{BC}(M_\pi)$
(1,4)	$\mathbb{C}\langle dz_{1\overline{123}}\wedge\varphi^{\overline{4}},\ e^{\bar z_1}dz_{2\overline{123}}\wedge\varphi^{\overline{4}},\ e^{-\bar z_1}dz_{3\overline{123}}\wedge\varphi^{\overline{4}}\rangle$
(4,2)	$\mathbb{C}\langle e^{z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{12}},\ e^{\bar z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{12}},\ e^{-z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{13}},\ e^{-\bar z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{13}}\rangle$ $dz_{123}\wedge\varphi^4\wedge dz_{\overline{23}},\ e^{z_1}dz_{123}\wedge\varphi^4\wedge dz_{\bar 2}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{123}\wedge\varphi^4\wedge dz_{\bar 3}\wedge\varphi^{\overline{4}}\rangle$
(3,3)	$\mathbb{C}\langle e^{z_1}dz_{123\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{123\overline{13}}\wedge\varphi^{\overline{4}},\ dz_{123\overline{234}},\ e^{\bar z_1}dz_{12}\wedge\varphi^4\wedge dz_{\overline{123}},\ e^{2z_1}dz_{12}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},$ $e^{2\bar z_1}dz_{12}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}}, dz_{12}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}}, e^{z_1}dz_{12}\wedge\varphi^4\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}},\ e^{-\bar z_1}dz_{13}\wedge\varphi^4\wedge dz_{\overline{123}},$ $dz_{13}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-2z_1}dz_{13}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}},\ e^{-2\bar z_1}dz_{13}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}},$ $e^{-z_1}dz_{13}\wedge\varphi^4\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}},\ dz_{23}\wedge\varphi^4\wedge dz_{\overline{123}},\ e^{\bar z_1}dz_{23}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-\bar z_1}dz_{23}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}},$ $dz_{23}\wedge\varphi^4\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}}\rangle$
(2,4)	$\mathbb{C}\langle e^{z_1}dz_{12\overline{123}}\wedge\varphi^{\overline{4}},\ e^{\bar z_1}dz_{12\overline{123}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{13\overline{123}}\wedge\varphi^{\overline{4}},\ e^{-\bar z_1}dz_{13\overline{123}}\wedge\varphi^{\overline{4}},\ dz_{23\overline{123}}\wedge\varphi^{\overline{4}},$ $e^{\bar z_1}dz_2\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}},\ e^{-\bar z_1}dz_3\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}}\rangle$
(4,3)	$\mathbb{C}\langle dz_{123}\wedge\varphi^4\wedge dz_{\overline{123}},\ e^{z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{\bar z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{12}}\wedge\varphi^{\overline{4}},\ e^{-z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}},$ $e^{-\bar z_1}dz_{123}\wedge\varphi^4\wedge dz_{\overline{13}}\wedge\varphi^{\overline{4}}, dz_{123}\wedge\varphi^4\wedge dz_{\overline{23}}\wedge\varphi^{\overline{4}}\rangle$
(3,4)	$\mathbb{C}\langle dz_{123\overline{123}}\wedge\varphi^{\overline{4}},\ e^{z_1}dz_{12}\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}},\ e^{\bar z_1}dz_{12}\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}}, e^{-z_1}dz_{13}\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}},$ $e^{-\bar z_1}dz_{13}\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}},\ dz_{23}\wedge\varphi^4\wedge dz_{123}\wedge\varphi^{\overline{4}}\rangle$
(4,4)	$\mathbb{C}\langle dz_{123}\wedge\varphi^4\wedge dz_{\overline{123}}\wedge\varphi^{\overline{4}}\rangle$

For a compact complex manifold (M, J), we denote by $\mathcal{A}^\bullet_{\mathbb{C}}(M)$ and $\mathcal{A}^{\bullet,\bullet}(M)$ respectively, the space of complex-valued differential forms on (M, J), and the space of (p, q)-forms on (M, J). Set, as usual,

$$\partial = \Pi^{1,0} \circ d : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M) \quad \overline{\partial} = \Pi^{0,1} \circ d : \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M).$$

Accordingly, the exterior differential decomposes as $d = \partial + \overline{\partial}$. Then,

$$H^{\bullet,\bullet}_{BC}(M) := \frac{\ker \partial \cap \ker \overline{\partial}}{\operatorname{Im} \partial\overline{\partial}} \cap \mathcal{A}^{\bullet,\bullet}(M)$$

is the *Bott–Chern cohomology* of (M, J) and we denote by

$$H^{\bullet,\bullet}_{A}(M) := \frac{\ker \partial\overline{\partial}}{\operatorname{Im} \partial + \operatorname{Im} \overline{\partial}} \cap \mathcal{A}^{\bullet,\bullet}(M)$$

the *Aeppli cohomology* of (M, J).

According to [30], once fixed an Hermitian metric g on (M, J), setting the *Bott–Chern Laplacian* and the *Aeppli Laplacian* respectively as

$$\Delta_{BC} := \partial\overline{\partial}\,\overline{\partial}\overline{\partial}^{*}\partial^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \overline{\partial}^*\partial\partial^*\overline{\partial} + \partial^*\overline{\partial}\,\overline{\partial}^*\partial + \overline{\partial}^*\overline{\partial} + \partial^*\partial,$$
$$\Delta_A := \partial\partial^* + \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\partial^*\partial\overline{\partial} + \partial\overline{\partial}\,\overline{\partial}^*\partial^* + \partial\overline{\partial}^*\overline{\partial}\partial^* + \overline{\partial}\partial^*\partial\overline{\partial}^*,$$

it turns out that Δ_{BC} and Δ_A are fourth order self-adjonint elliptic operators and, consequently, the following isomorphisms of vector spaces hold

$$\ker \Delta_{BC} \cong H_{BC}(M), \qquad \ker \Delta_A \cong H_A(M).$$

In particular, a (p, q)-form is *Bott–Chern harmonic* (respectively, *Aeppli harmonic*) with respect to a fixed a Hermitian metric g on (M, J) if it holds

$$d\alpha = 0, \qquad \partial\overline{\partial} *_g \alpha = 0,$$

respectively,

$$\partial\overline{\partial}\alpha = 0, \qquad \partial *_g \alpha = 0, \qquad \overline{\partial} *_g \alpha = 0.$$

The Bott–Chern (respectively, Aeppli) cohomology of a compact complex manifold has a structure of algebra (respectively, H_{BC}-module) induced by the $\cup$ product, whereas this is no longer true for their harmonic representatives. Therefore, along the lines of [23] and [36], the authors in [3] define a Hermitian metric g on compact complex manifold (M, J) to be *geometrically-Bott–Chern-formal* (shortly, *geometrically-BC-formal*) if the space of *Bott–Chern harmonic forms* $\mathcal{H}^{\bullet,\bullet}_{\Delta_{BC}}(M) :=$ $\ker \Delta_{BC}$ has a structure of algebra induced by the $\wedge$ product.

We recall the construction of *triple ABC-Massey products* and *quadruple ABC-Massey products* (see, respectively, [3] and [24] for further details). Let $[\alpha] \in H^{p,q}_{BC}(M)$, $[\beta] \in H^{r,s}_{BC}(M)$, and $[\gamma] \in H^{u,v}_{BC}(M)$ such that

$$\alpha \wedge \beta = (-1)^{p+q}\partial\overline{\partial} f_{\alpha\beta}, \qquad \beta \wedge \gamma = (-1)^{r+s}\partial\overline{\partial} f_{\beta\gamma}.$$

Then, the *triple ABC-Massey product* $\langle[\alpha], [\beta], [\gamma]\rangle_{ABC}$ is represented by

$$\begin{aligned}\langle[\alpha], [\beta], [\gamma]\rangle_{ABC} &= [(-1)^{p+q}\alpha \wedge f_{\beta\gamma} - (-1)^{r+s} f_{\alpha\beta} \wedge \gamma] \\ &\in \frac{H_A^{p+r+u-1,q+s+v-1}(M)}{[\alpha] \cup H_A^{r+u-1,s+v-1}(M) + [\gamma] \cup H_A^{p+r-1,q+s-1}(M)}.\end{aligned}$$

Lemma 2.1 *Let $(M = \Gamma\backslash G, J)$ be the compact quotient of simply connected Lie group G by a discrete co-compact subgroup Γ and let J be an invariant complex structure on M, i.e., induced by an invariant structure J on the Lie algebra $\mathfrak{g}$ of G. Then, the map*

$$i : \{\text{triple ABC-Massey products on } (\mathfrak{g}, J)\} \hookrightarrow \{\text{triple ABC-Massey products on } (M, J)\}$$

is injective.

Proof Let $[\alpha], [\beta], [\gamma] \in H_{BC}(\mathfrak{g})$ so that $\mathcal{P} := \langle[\alpha], [\beta], [\gamma]\rangle_{ABC}$ is a non vanishing triple ABC-Massey product on $\mathfrak{g}$ with $f_{\alpha\beta}$ and $f_{\beta\gamma}$ its associated invariant $\partial\overline{\partial}$-primitives. Being $\mathcal{P}$ non vanishing, for the representative of $\mathcal{P}$ it holds that

$$(-1)^{|\alpha|}\alpha \wedge f_{\beta\gamma} - (-1)^{|\beta|} f_{\alpha\beta} \wedge \gamma \notin [\alpha] \cup H_A(\mathfrak{g}) + [\gamma] \cup H_A(\mathfrak{g}). \tag{2.1}$$

By contradiction let us assume that $i(\mathcal{P})$ is trivial as a triple ABC-Massey product on (M, J). We can choose again $f_{\alpha\beta}$ and $f_{\beta\gamma}$ as $\partial\overline{\partial}$-primitives, as any other choice of (not necessarily invariant) primitives, by definition of ABC-Massey product, would yield terms of the representative that still belong to the indeterminacy ideal. Since $i(\mathcal{P})$ is trivial on (M, J), there exist $\xi, \eta, R, S \in \mathcal{A}(M)$ such that

$$0 \neq (-1)^{|\alpha|}\alpha \wedge f_{\beta\gamma} - (-1)^{|\beta|} f_{\alpha\beta} \wedge \gamma = \alpha \wedge \xi + \gamma \wedge \eta + \partial R + \overline{\partial} S.$$

Note that at least one form in $\{\xi, \eta, R, S\}$ is not left-invariant. However, by applying a symmetrization process (see, e.g., [6, 12, 37]) on both sides of the above equation, we obtain

$$0 \neq (-1)^{|\alpha|}\alpha \wedge f_{\beta\gamma} - (-1)^{|\beta|} f_{\alpha\beta} \wedge \gamma = \alpha \wedge \tilde{\xi} + \gamma \wedge \tilde{\eta} + \partial \tilde{R} + \overline{\partial} \tilde{S}.$$

The right-hand side of the equations, however, has to be non vanishing and it belongs to $[\alpha] \cup H_A(\mathfrak{g}) + [\gamma] \cup H_A(\mathfrak{g})$, contraddicting (2.1), which concludes the proof. □

We now recall the construction of quadruple ABC-Massey products. The *Schweitzer complex* $S^k_{p,q}$ is defined for every pair (p, q) as

$$\begin{aligned}\dots \xrightarrow{\partial\overline{\partial}} S^{-2}_{p,q} = \mathcal{A}^{p-2,q-2}(M) \xrightarrow{\partial+\overline{\partial}} \quad & S^{-1}_{p,q} = \mathcal{A}^{p-1,q-2}(M) \oplus \mathcal{A}^{p-2,q-1}(M)\\ & \xrightarrow{\mathrm{pr}\circ d} S^{0}_{p,q} = \mathcal{A}^{p-1,q-1}\\ & \xrightarrow{\partial\overline{\partial}} S^{1}_{p,q} = \mathcal{A}^{p,q}(M) \xrightarrow{\partial+\overline{\partial}} \dots\end{aligned}$$

so that

$$H^1_{S_{p,q}} = H^{p,q}_{BC}(M), \qquad H^0_{S_{p,q}} = H^{p-1,q-1}_{A}(M).$$

Let $[\alpha] \in H_{BC}(M)$, $[\beta] \in H_{BC}(M)$, $[\gamma] \in H_{BC}(M)$, and $[\delta] \in H_{BC}(M)$ such that $\alpha \wedge \beta \wedge \gamma \wedge \delta \in \mathcal{A}^{p,q}(M)$. Let $x, y, z, \eta, \eta', \xi, \xi' \in \mathcal{A}^{\bullet,\bullet}(M)$ such that

$$\partial\overline{\partial}x = \alpha \wedge \beta, \quad \partial\overline{\partial}y = \beta \wedge \gamma, \quad \partial\overline{\partial}z = \gamma \wedge \delta$$

and

$$x\gamma - \alpha y = \partial\eta + \overline{\partial}\eta', \quad y\delta - \beta z = \partial\xi + \overline{\partial}\xi'.$$

Then the quadruple ABC-Massey product $\langle[\alpha], [\beta], [\gamma], [\delta]\rangle_{ABC}$ is represented by

$$[(-1)^{|\alpha|}\alpha \wedge (\xi + \xi') - (\partial x) \wedge z + (-1)^{|x|+1}x \wedge \overline{\partial}z + (\eta + \eta') \wedge \delta] \tag{2.2}$$

in

$$\begin{aligned}&H^{-1}_{S_{p,q}}(\mathcal{A}^{\bullet,\bullet}(M))\\ &= \frac{\ker(\mathrm{pr}\circ d\colon \mathcal{A}^{p-2,q-1}(M) \oplus \mathcal{A}^{p-1,q-2}(M) \to \mathcal{A}^{p-1,q-1}(M))}{\mathrm{Im}(\mathrm{pr}\circ d\colon \mathcal{A}^{p-3,q-1}(M) \oplus \mathcal{A}^{p-2,q-2}(M) \oplus \mathcal{A}^{p-1,q-3}(M) \to \mathcal{A}^{p-2,q-1}(M) \oplus \mathcal{A}^{p-1,q-2}(M))}.\end{aligned} \tag{2.3}$$

Let $(M = \Gamma\backslash G, J)$ be a $2k$-dimensional nilmanifold. An invariant almost complex structure J on M is said to be *nilpotent* [9] if there exists a frame $\{\eta^1, \dots, \eta^k\}$ of $(\mathfrak{g}^*_{\mathbb{C}})^{1,0}$ such that $d\eta^l \in \mathrm{Span}_{\mathbb{C}}\langle\eta^{ij}, \eta^{i\overline{j}}\rangle_{i,j=1,\dots,l-1}$. If the coframe $\{\eta^1, \dots, \eta^k\}$ satisfies

$$\begin{cases} d\eta^j = 0, \quad j \in \{1, \dots, k-1\} \\ d\eta^k \in \mathrm{Span}_{\mathbb{C}}\langle\eta^{ij}, \eta^{i\overline{j}}\rangle_{i,j=1,\dots,k-1}, \end{cases} \tag{2.4}$$

we will call J a *special type complex structure* on M. Finally, we recall that a Hermitian metric g on a complex manifold (M, J) is said to be *strong Kähler with torsion*, shortly SKT, if its fundamental form ω satisfies $\partial\overline{\partial}\omega = 0$.

3 SKT Metrics and Geometrically-*BC*-Formal Metrics in Complex Dimension 3

In this section, we prove that the property of admitting a geometrically-BC-formal metric is equivalent to the property of admitting a SKT metric on any six-dimensional nilmanifold endowed with an invariant complex structure.

Theorem 3.1 *Let* $(\Gamma\backslash G := M, J)$ *be a* 6*-dimensional nilmanifold endowed with an invariant complex structure* J*. Then,* (M, J) *is SKT if, and only if, it is geometrically-*BC*-formal. In particular, every Hermitian invariant metric is SKT if, and only if, it is geometrically-*BC*-formal.*

Proof Let us first assume that (M, J) admits a SKT metric, which we can assume to be invariant by the symmetrization process. By [16, Theorem 1.2], then (M, J) admits a basis $\{\alpha^1, \alpha^2, \alpha^3\}$ of invariant $(1, 0)$-forms such that

$$d\alpha^1 = 0, \qquad d\alpha^2 = 0, \qquad d\alpha^3 = A\alpha^{12} + B\alpha^{1\bar{1}} + C\alpha^{1\bar{2}} + D\alpha^{2\bar{1}} + E\alpha^{2\bar{2}},$$

with $A, B, C, D, E \in \mathbb{C}$ satisfying $|A|^2 + |C|^2 + |D|^2 = 2\mathfrak{Re}(\overline{B}E)$. In particular, every invariant metric on (M, J) is SKT. But then, in [32, Theorem 7.3] it was shown that every invariant Hermitian metric on (M, J) is also geometrically-BC-formal.

Viceversa, let us assume that the nilmanifold $(M = \Gamma\backslash G, J)$ is geometrically-BC-formal. Since the nilmanifold $M = \Gamma\backslash G$ admits an invariant complex structure J, by [37, Proposition 2], there exists a invariant $(1, 0)$-coframe $\{\omega^1, \omega^2, \omega^3\}$ such that it satisfies the following structure equations

(a) if J is nonnilpotent:

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = E\omega^{13} + \omega^{1\bar{3}}, \\ d\omega^3 = A\omega^{1\bar{1}} + ib\omega^{1\bar{2}} - ib\overline{E}\omega^{2\bar{1}}, \end{cases} \tag{3.1}$$

with $A, E \in \mathbb{C}$, $|E| = 1$, and $b \in \mathbb{R}\backslash\{0\}$,

(b) if J is nilpotent:

$$\begin{cases} d\omega^1 = 0, \\ d\omega^2 = \epsilon\omega^{1\bar{1}}, \\ d\omega^3 = \rho\omega^{12} + (1-\epsilon)A\omega^{1\bar{1}} + B\omega^{1\bar{2}} + C\omega^{2\bar{1}} + (1-\epsilon)D\omega^{2\bar{2}}, \end{cases} \tag{3.2}$$

with $A, B, C, D \in \mathbb{C}$ and $\epsilon, \rho \in \{0, 1\}$.

We claim that J is nilpotent. By contradiction, let us assume that J is nonnilpotent. But then, by (3.1),

$$\partial\overline{\partial}\omega^{3\bar{3}} = -2b^2\omega^{12\bar{1}\bar{2}}.$$

and the Bott–Chern cohomology classes $[\omega^{12}] \in H^{2,0}_{BC}(\mathfrak{g})$ and $[\omega^{\overline{12}}] \in H^{0,2}_{BC}(\mathfrak{g})$ are well defined. Then, the following ABC-Massey product on $(\mathfrak{g}, J)$ is well defined

$$\left\langle [\omega^{12}], [\omega^{\overline{12}}], [\omega^{\overline{12}}] \right\rangle_{ABC} = \left[\frac{1}{2b^2} \omega^{3\overline{123}} \right]_A \in \frac{H^{2,2}_A(\mathfrak{g})}{[\omega^{\overline{12}}] \cup H^{1,1}_A(\mathfrak{g})}.$$

Moreover, $\operatorname{Im} \overline{\partial}|_{\bigwedge^{1,2}\mathfrak{g}^*_{\mathbb{C}}} + \operatorname{Im} \partial|_{\bigwedge^{0,3}\mathfrak{g}^*_{\mathbb{C}}} \subset \operatorname{Span}\langle \omega^{1\overline{123}}, \omega^{2\overline{123}} \rangle$, hence $[\frac{1}{2b^2}\omega^{3\overline{123}}]_A \neq 0$ as an Aeppli cohomology class.

Suppose that $\left[\frac{1}{2b^2}\omega^{3\overline{123}}\right]_A \in [\omega^{\overline{12}}] \cup H^{1,1}_A(\mathfrak{g})$, i.e., there exist $\mu_j \in \mathbb{C}$, $R \in \bigwedge^{0,3}\mathfrak{g}^*_{\mathbb{C}}$, and $S \in \bigwedge^{1,2}\mathfrak{g}^*_{\mathbb{C}}$ such that

$$\frac{1}{2b^2}\omega^{3\overline{123}} = \sum_{j=1}^{h^{1,1}_A} \mu_j \omega^{\overline{12}} \wedge \xi^j + \partial R + \overline{\partial} S, \tag{3.3}$$

where $\{\xi^j\}_{j=1}^{h^{1,1}_A}$ is a basis of $\mathcal{H}^{1,1}_{\Delta_A}(\mathfrak{g})$. We take the usual left-invariant extension to (M, J) of the forms and the metric on $(\mathfrak{g}, J)$. By multiplying by ω^{12} and integrating over M each side of equation (3.3), we obtain

$$0 \neq \frac{1}{2b^2} ||\omega^{3\overline{123}}||^2_{L^2(M)} = \int_M \sum_{j=1}^{h^{1,1}_A} \mu_j \omega^{12\overline{12}} \wedge \xi^j + \int_M \partial R \wedge \omega^{12} + \int_M \overline{\partial} S \wedge \omega^{12}.$$

Since $d\omega^{12} = 0$, the last two terms on the right hand side of the equation vanish by Stokes' theorem, yielding

$$\begin{aligned} 0 \neq \int_M \sum_{j=1}^{h^{1,1}_A} \mu_j \omega^{12\overline{12}} \wedge \xi^j &= \sum_{j=1}^{h^{1,1}_A} \mu_j \int_M \omega^{12\overline{12}} \wedge *(*\xi^j) \\ &= \sum_{j=1}^{h^{1,1}_A} \mu_j \, \langle\!\langle \omega^{12\overline{12}}, *\xi^j \rangle\!\rangle_{L^2(M)} = 0, \end{aligned}$$

where the last equation holds since $\{*\xi^j\}_{j=1}^{h^{1,1}_A}$ are Bott–Chern harmonic and $\omega^{12\overline{12}}$ is $\partial\overline{\partial}$-exact. This is absurd, therefore, $\langle [\omega^{12}], [\omega^{\overline{12}}], [\omega^{\overline{12}}] \rangle_{ABC}$ gives rise via Lemma 2.1 to a non vanishing ABC-Massey product on (M, J), which contradicts the assumption that (M, J) is geometrically-BC-formal. Hence, if (M, J) is geometrically-BC-formal, J is nilpotent and we can assume the existence of a $(1, 0)$-coframe $\{\omega^1, \omega^2, \omega^3\}$ as in (3.2).

By assumption, there exists an invariant geometrically-BC-formal metric g on (M, J) whose fundamental form F can be written as

$$F = \frac{i}{2}\sum_{j=1}^{3} F_{j\bar{j}}\omega^{j\bar{j}} + \frac{1}{2}\sum_{1\leq i<j\leq k}(F_{i\bar{j}}\,\omega^{i\bar{j}} - \overline{F}_{i\bar{j}}\,\omega^{j\bar{i}}).$$

From (3.2), we immediately compute

$$\partial\overline{\partial}F = -\frac{iF_{3\bar{3}}}{2}(\rho^2 + |B|^2 + |C|^2 - 2(1-\epsilon)^2\mathfrak{Re}(A\overline{D}))\omega^{12\bar{1}\bar{2}}.$$

By contradiction, if $\partial\overline{\partial}F \neq 0$, setting $L := \rho^2 + |B|^2 + |C|^2 - 2(1-\epsilon)^2\mathfrak{Re}(A\overline{D}) \neq 0$, then one can show, by using the same argument as above, that

$$\left\langle[\omega^{12}], [\omega^{\overline{12}}], [\omega^{\overline{12}}]\right\rangle_{ABC} = \left[\frac{1}{L}\omega^{3\overline{3}1\overline{2}}\right]_A \in \frac{H_A^{2,2}(\mathfrak{g})}{[\omega^{\overline{12}}] \cup H_A^{1,1}(\mathfrak{g})}$$

gives rise via Lemma 2.1 to a non vanishing ABC-Massey product on (M, J). This is absurd, therefore $\partial\overline{\partial}F = 0$ and g is SKT. The theorem is proved. □

Remark 3.2 The SKT nilmanifolds are at most 2-step by Arroyo and Nicolini in [5], as conjectured by Enrietti, Fino, and Vezzoni in [11]. Moreover, as proved by Rollenske in [27, Proposition 3.3], every complex structure on a 2-step nilpotent Lie algebra is nilpotent. So, every invariant complex structure on SKT nilmanifolds is nilpotent.

Remark 3.3 If $(\Gamma\backslash G = M, J)$ is either a 6-dimensional nilmanifold M with an SKT invariant complex structure J or, more generally, a $2n$-dimensional nilmanifold endowed with a SKT special complex structure, then from Theorem 3.1 (see also [32, Theorem 7.2] and [32, Theorem 7.4]) we know that (M, J) is geometrically-BC-formal. Since J is nipotent, by [1], the Aeppli cohomology and the Bott–Chern cohomology of (M, J) are computed, respectively, by the space of invariant Aeppli (respectively, Bott–Chern) harmonic forms with respect to a fixed invariant SKT metric g on (M, J). Moreover, the space $\mathcal{H} := \mathcal{H}_{\Delta_A}(\mathfrak{g}) + \mathcal{H}_{\Delta_{BC}}(\mathfrak{g})$ is closed under $\star$ and since [32, Lemma 7.3] $\partial\overline{\partial} \equiv 0$ on left-invariant forms on G, then $\partial\overline{\partial}|_{\mathcal{H}} \equiv 0$. Hence, any invariant SKT is, in particular, an *ABC-geometrically formal metric* in the sense of [24] and (M, J) is *weakly formal* in the sense of [24].

4 SKT Metrics and Bott–Cher-Geometrically Formal Metrics in Higher Dimensions

In this section we further study the relation between SKT metrics and geometrically-BC-formal metrics for nilmanifolds of complex dimension strictly higher than 3.

As proved in [32], for the class of nilmanifolds endowed with special type complex structures, in any complex dimension the existence of a SKT metric implies the existence of a geometrically-BC-formal metric. In Proposition 4.2, we prove a partial converse of this result by showing that, if a nilmanifold endowed with a special type complex structure admits no SKT metric, then it does not admit any invariant geometrically-BC-formal metric. However, this does not suffice to show that there exist no (not necessarily invariant) geometrically-BC-formal metrics. Thus, in Theorem 4.5, we give explicit sufficient conditions on the Bott–Chern cohomology for the existence of a non vanishing ABC-Massey product; hence, under such conditions, there cannot exist geometrically-BC-formal metrics on such a class of manifolds.

Finally, in Proposition 4.8 we show that by dropping the hypothesis of special type complex structure, we can construct a nilmanifold with invariant nilpotent complex structure which is SKT but admits a non trivial ABC-Massey product.

Let now (M, J) be a $2n$-dimensional nilmanifold endowed with a complex structure of special type, see (2.4). Then, the complex structure J is nilpotent [9] on M. By [1, Theorem 2.8], the Aeppli cohomology and the Bott–Chern cohomology of (M, J) are computed via the complex of left-invariant forms on G. As a consequence, by Lemma 2.1, ABC-Massey products on $(\mathfrak{g}, J)$ corresponds bijectively to ABC-Massey products on (M, J).

Let g be an invariant Hermitian metric on (M, J) and denote by

$$\omega = \frac{i}{2}\sum_{j=1}^{k} \omega_{j\bar{j}}\eta^{j\bar{j}} + \frac{1}{2}\sum_{1\leq i<j\leq k} (\omega_{i\bar{j}}\,\eta^{i\bar{j}} - \overline{\omega}_{i\bar{j}}\,\eta^{j\bar{i}}).$$

its fundamental form.

For this family of nilmanifolds, by structure equations (2.4), the metric g is SKT if and only if $\partial\overline{\partial}\eta^{k\bar{k}} = 0$, i.e., the SKT condition does not depend on the choice of the Hermitian metric.

In that case, it was shown in [32] that if J is such that every invariant Hermitian metric on (M, J) is SKT, the manifold (M, J) is geometrically-BC-formal, hence [35] every ABC-Massey product vanishes.

Remark 4.1 As a consequence of [32, Theorem 7.4], a necessary condition for the existence of non vanishing triple ABC-Massey products is that (M, J) does not admit any SKT metric.

Furthermore, we observe the following result.

Proposition 4.2 *For nilmanifolds endowed with complex structure of special type, if there exists an invariant geometrically-BC-formal metric, then it is also SKT.*

Proof Let us fix $\{\eta^1, \ldots, \eta^k\}$ a coframe of invariant $(1, 0)$-forms on (M, J) satisfying (2.4). By Gram-Schmidt, we can assume that the coframe $\{\eta^1, \ldots, \eta^k\}$ is unitary and still satisfy structure equations of type (2.4). In this situation, the fundamental form of g can be written as $\omega = \frac{i}{2}\sum_{j=1}^{k}\eta^{j\bar{j}}$. Let us assume that g is not SKT, i.e.,

$$0 \neq \partial\overline{\partial}\eta^{k\bar{k}} = \sum_{1\leq r,u<s,v\leq k-1} L_{rs\bar{u}\bar{v}}\eta^{rs\bar{u}\bar{v}}.$$

We can then choose $r_0, s_0, u_0.v_0$ such that $L_{r_0 s_0 \overline{u}_0 \overline{v}_0} \neq 0$. By structure equations and bidegree reasons, the forms $\alpha := \eta^{r_0 s_0}$ and $\beta := \eta^{\overline{u}_0 \overline{v}_0}$ are both d-closed and $\partial\overline{\partial}*$-closed, so they are Bott–Chern harmonic. We check whether this holds also for the wedge product $\alpha \wedge \beta$. By Leibniz rule, the form $\alpha \wedge \beta$ is d-closed. However, $*(\alpha \wedge \beta) = \varepsilon \eta^{1 \dots r_0^* \dots s_0^* \dots k \overline{1} \dots \overline{u}_0^* \dots \overline{v}_0^* \dots \overline{k}}$, with ε a sign constant, so that

$$\begin{aligned} \partial\overline{\partial} * (\alpha \wedge \beta) &= (-1)^{k-3} \varepsilon \eta^{1 \dots r_0^* \dots s_0^* \dots k-1 \overline{1} \dots \overline{u}_0^* \dots \overline{v}_0^* \dots \overline{k-1}} \wedge \partial\overline{\partial}(\eta^{k\overline{k}}) \\ &= \varepsilon' L_{r_0 s_0 \overline{u}_0 \overline{v}_0} \eta^{1 \dots k-1 \overline{1} \dots \overline{k-1}} \neq 0, \end{aligned}$$

with ε' a sign constant. As a consequence, the form $\alpha \wedge \beta$ is not Bott–Chern harmonic, i.e., the invariant metric g is not geometrically-BC-formal. □

We now prove a sufficient condition for the existence of a non trivial ABC-Massey product on nilmanifolds endowed with a special type complex structure.

Recall that, in our notation, a real $2k$-dimensional nilmanifold $M = \Gamma \backslash G$ admits a special type complex structure J if it admits a coframe of invariant $(1, 0)$-forms $\{\eta^1, \dots, \eta^k\}$ such that

$$\begin{cases} d\eta^j = 0, \quad j \in \{1, \dots, k-1\} \\ d\eta^k \in \mathrm{Span}_{\mathbb{C}} \langle \eta^{ij}, \eta^{i\overline{j}} \rangle_{i,j=1,\dots,k-1} \end{cases} . \tag{4.1}$$

Let g be a fixed invariant Hermitian metric on (M, J) (which we can always assume to be diagonal) and let $\omega = \frac{i}{2} \sum_{j=1}^{k} \eta^{j\overline{j}}$ be its fundamental form.

By Remark 4.1, we assume that (M, J) is not SKT, i.e., $\partial\overline{\partial}\eta^{k\overline{k}} \neq 0$. In this case, the $\partial\overline{\partial}$-operator on invariant forms is not zero only on $\mathcal{I}(\eta^{k\overline{k}})$. Moreover, structure equations (4.1) imply that

$$\mathrm{Im}\, \partial\overline{\partial} \subset \bigwedge\nolimits^{\bullet,\bullet} \left(\mathrm{Span}_{\mathbb{C}} \langle \eta^1, \dots, \eta^{k-1}, \eta^{\overline{1}}, \dots, \eta^{\overline{k-1}} \rangle \right). \tag{4.2}$$

Since the Bott–Chern cohomology and the Aeppli cohomology of (M, J) are computed via the complex of left-invariant forms on G, by further exploiting the structure equations (4.1), straightforward computations allow us to describe the structures of Bott–Chern and Aeppli cohomologies of (M, J).

Lemma 4.3 *The Bott–Chern cohomology of (M, J) of any bidegree (p, q) admits the following decomposition*

$$H^{p,q}_{BC}(M) \cong \frac{H_1}{I_1} \oplus H_2 \oplus H_3,$$

where

$$\begin{aligned} H_1 &= \mathrm{Span}_{\mathbb{C}} \langle \eta^{i_1 \dots i_p} \wedge \eta^{\overline{j}_1 \dots \overline{j}_q} \rangle_{i_l, j_l \neq k} \\ I_1 &= \mathrm{Im}\, \partial\overline{\partial} \left(\mathrm{Span}_{\mathbb{C}} \langle \eta^{k\overline{k}} \wedge \eta^{i_1 \dots i_{p-2}} \wedge \eta^{\overline{j}_1 \dots \overline{j}_{q-2}} \rangle \right) \end{aligned}$$

$$H_2 = \operatorname{Ker} d \cap \left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{k} \wedge \eta^{i_1\ldots i_{p-1}} \wedge \eta^{\bar{j}_1\ldots\bar{j}_q}\rangle \oplus \operatorname{Span}_{\mathbb{C}}\langle \eta^{\bar{k}} \wedge \eta^{i_1\ldots i_p} \wedge \eta^{\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\right)$$

$$H_3 = \operatorname{Ker} d \cap \operatorname{Span}_{\mathbb{C}}\langle \eta^{k\bar{k}} \wedge \eta^{i_1\ldots i_{p-1}} \wedge \eta^{\bar{j}_1\ldots\bar{j}_{q-1}}\rangle.$$

In particular, if either $p < 2$ or $q < 2$, we have that

$$H^{p,q}_{BC}(M) \cong H_1 \oplus H_2 \oplus H_3.$$

Proof By [1, Theorem 2.8], we have that $H^{p,q}_{BC}(M) \cong H^{p,q}_{BC}(\mathfrak{g})$, where

$$H^{p,q}_{BC}(\mathfrak{g}) = \frac{\operatorname{Ker}(d\colon \bigwedge^{p,q} \mathfrak{g}^*_{\mathbb{C}} \to \bigwedge^{p+1,q} \mathfrak{g}^*_{\mathbb{C}} \oplus \bigwedge^{p,q+1} \mathfrak{g}^*_{\mathbb{C}})}{\operatorname{Im}(\partial\bar{\partial}\colon \bigwedge^{p-1,q-1} \mathfrak{g}^*_{\mathbb{C}} \to \bigwedge^{p,q} \mathfrak{g}^*_{\mathbb{C}})}.$$

But then the following decomposition holds

$$\operatorname{Ker}\left(d\colon \textstyle\bigwedge^{p,q} \mathfrak{g}^*_{\mathbb{C}} \to \bigwedge^{p+1,q} \mathfrak{g}^*_{\mathbb{C}} \oplus \bigwedge^{p,q+1} \mathfrak{g}^*_{\mathbb{C}}\right) = H_1 \oplus H_2 \oplus H_3.$$

By (4.2), we have that

$$\operatorname{Im}\left(\partial\bar{\partial}\colon \textstyle\bigwedge^{p-1,q-1} \mathfrak{g}^*_{\mathbb{C}} \to \bigwedge^{p,q} \mathfrak{g}^*_{\mathbb{C}}\right) = I_1.$$

Then, since $I_1 \cap H_2 = \{0\}$ and $I_1 \cap H_3 = \{0\}$, we can conclude that

$$H^{p,q}_{BC}(\mathfrak{g}) = \frac{H_1 \oplus H_2 \oplus H_3}{I_1} \cong \frac{H_1}{I_1} \oplus H_2 \oplus H_3.$$

Note that if either $p < 2$ or $q < 2$, then $I_1 = \{0\}$, which concludes the proof. □

Lemma 4.4 *The Aeppli cohomology of (M, J) of any bidegree (p, q) admits the following decomposition*

$$H^{p,q}_{A}(M) \cong \frac{K_1}{L_1} \oplus \frac{K_2}{L_2} \oplus K_3,$$

where

$$\begin{aligned}
K_1 &= \operatorname{Span}_{\mathbb{C}}\langle \eta^{i_1\ldots i_p\bar{j}_1\ldots\bar{j}_q}\rangle_{i_l, j_l \neq k}\\
L_1 &= \operatorname{Im}\partial\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{ki_1\ldots i_{p-2}\bar{j}_1\ldots\bar{j}_q}\rangle\right) \oplus \operatorname{Im}\bar{\partial}\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{ki_1\ldots i_{p-1}\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\right)\\
&\quad \oplus \operatorname{Im}\partial\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{\bar{k}i_1\ldots i_{p-1}\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\right) \oplus \operatorname{Im}\bar{\partial}\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{\bar{k}i_1\ldots i_p\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\right)\\
K_2 &= \operatorname{Span}_{\mathbb{C}}\langle \eta^{ki_1\ldots i_{p-1}\bar{j}_1\ldots\bar{j}_q}, \eta^{\bar{k}i_1\ldots i_p\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\\
L_2 &= \operatorname{Im}\partial\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{k\bar{k}i_1\ldots i_{p-2}\bar{j}_1\ldots\bar{j}_{q-1}}\rangle\right) \oplus \operatorname{Im}\bar{\partial}\left(\operatorname{Span}_{\mathbb{C}}\langle \eta^{k\bar{k}i_1\ldots i_{p-1}\bar{j}_1\ldots\bar{j}_{q-2}}\rangle\right)\\
K_3 &= \operatorname{Ker}\partial\bar{\partial} \cap \operatorname{Span}_{\mathbb{C}}\langle \eta^{k\bar{k}i_1\ldots i_{p-1}\bar{j}_1\ldots\bar{j}_{q-1}}\rangle.
\end{aligned}$$

Proof The proof is analogous to the proof of Lemma 4.3. □

We will now work at the level of the Lie algebra $\mathfrak{g}$ of the universal cover G of M. We note that $\mathfrak{g}$ is endowed with a special type complex structure inherited by (M, J), which we will still denote by J. Therefore, there exists a basis $\{\eta^1, \dots, \eta^k\}$ of $(1,0)$-forms on $(\mathfrak{g}, J)$ satisfying (4.1) and the fixed invariant Hermitian metric g on (M, J) descends to a Hermitian metric on $(\mathfrak{g}, J)$ with fundamental form $\omega = \frac{i}{2}\sum_{j=1}^k \eta^{j\bar{j}}$.

Let us then consider a ABC-Massey product $\mathcal{P}$ on $(\mathfrak{g}, J)$ given by the following expression

$$\mathcal{P} = \Big\langle [\alpha], [\beta], [\beta] \Big\rangle_{ABC},$$

where $[\alpha] \in H^{p,q}_{BC}(\mathfrak{g})$, $[\beta] \in H^{r,s}_{BC}(\mathfrak{g})$. We assume that the representatives α and β satisfy the following properties

$$\alpha = \eta^{i_1 \dots i_p} \wedge \eta^{\bar{j}_1 \dots \bar{j}_q}, \qquad \beta = \eta^{l_1 \dots l_r} \wedge \eta^{\bar{m}_1 \dots \bar{m}_s}, \qquad \alpha \wedge \beta \neq 0.$$

In order for $\mathcal{P}$ to be a well defined non trivial ABC-Massey product, by (4.2), it must hold that

- $\alpha, \beta \in H_1$ of Lemma 4.3, i.e.,

$$\alpha \in \bigwedge^{p,q}\left(\mathrm{Span}_{\mathbb{C}}\langle \eta^1, \dots, \eta^{k-1}, \eta^{\bar{1}}, \dots, \eta^{\overline{k-1}}\rangle\right),$$
$$\beta \in \bigwedge^{r,s}\left(\mathrm{Span}_{\mathbb{C}}\langle \eta^1, \dots, \eta^{k-1}, \eta^{\bar{1}}, \dots, \eta^{\overline{k-1}}\rangle\right),$$

- there exists a form $f = \eta^{k\bar{k}} \wedge \tilde{\gamma} \in \bigwedge^{p+r-1, q+s-1} \mathfrak{g}^*_{\mathbb{C}}$ with $\tilde{\gamma} \in \bigwedge^{p+r-2, q+s-2} \mathfrak{g}^*_{\mathbb{C}}$ such that

$$(-1)^{p+q} \partial\bar{\partial} f = \alpha \wedge \beta.$$

We assume that $\tilde{\gamma}$ can be written as $\tilde{\gamma} := \eta^{t_1 \dots t_{p+r-2}} \wedge \eta^{\bar{v}_1 \dots \bar{v}_{q+s-2}}$ and, up to a constant, it is the only $(p+r-2, q+s-2)$-form φ which satisfies

$$0 \neq \partial\bar{\partial}(\eta^{k\bar{k}}) \wedge \varphi \in \mathrm{Span}_{\mathbb{C}}\langle \alpha \wedge \beta \rangle.$$

Then the ABC-Massey product $\mathcal{P}$ is represented, up to constant, by the Aeppli cohomology class

$$[\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta] \in H_A^{p+2r-1, q+2s-1}(\mathfrak{g}).$$

Note that, since the form $*(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta)$ does not contain η^k nor $\bar{\eta}^k$, then by structure equations $*(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta)$ is ∂-closed and $\bar{\partial}$-closed; hence, the form $\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta$ is Aeppli harmonic and, as a Aeppli cohomology class, $[\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta]_A \neq 0$.

As a ABC-Massey product, $\mathcal{P}$ is non vanishing if and only if $[\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta]_A \notin \mathcal{J}$, where $\mathcal{J}$ is the ideal

$$[\alpha]_{BC} \cup H_A^{2r-1,2s-1}(\mathfrak{g}) + [\beta]_{BC} \cup H_A^{p+r-1,q+s-1}(\mathfrak{g}).$$

Let us suppose by contradiction that this is the case, i.e., there exists $\lambda_j, \mu_l \in \mathbb{C}$ and $R \in \bigwedge^{p+2r-2,q+2s-1} \mathfrak{g}_{\mathbb{C}}^*$, $S \in \bigwedge^{p+2r-1,q+2s-2} \mathfrak{g}_{\mathbb{C}}^*$ such that

$$\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta = \sum_{j=1}^{h_A^{2r-1,2s-1}} \lambda_j \, \alpha \wedge \xi^j + \sum_{l=1}^{h_A^{p+r-1,q+s-1}} \mu_l \, \beta \wedge \psi^l + \partial R + \bar{\partial} S, \quad (4.3)$$

with $\{\xi^j\}_{j=1}^{h_A^{2r-1,2s-1}}$ and $\{\psi^l\}_{l=1}^{h_A^{p+r-1,q+s-1}}$ bases of $\mathcal{H}_A^{2r-1,2s-1}(\mathfrak{g})$ and, respectively, of $\mathcal{H}_A^{p+r-1,q+s-1}(\mathfrak{g})$.

Considering now the usual extension to left-invariant tensors of the forms and the metric on $(\mathfrak{g}, J)$, by taking the L^2-product of equation (4.3) with $\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta$, we obtain

$$\begin{aligned} 0 \neq ||\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta||_{L^2}^2 &= \int_M \left(\sum_j \lambda_j \alpha \wedge \xi^j \right) \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \\ &+ \int_M \left(\sum_l \mu_l \beta \wedge \psi^l \right) \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \\ &+ \int_M \partial R \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) + \int_M \bar{\partial} S \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta). \end{aligned}$$

Being $*\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta$ a d-closed form, by Stokes' theorem the last two terms vanish, hence yielding

$$0 \neq \int_M \left(\sum_j \lambda_j \alpha \wedge \xi^j \right) \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) + \int_M \left(\sum_l \mu_l \beta \wedge \psi^l \right) \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta).$$

Then, it must exist $j_0 \in \{1, \ldots, h_A^{2r-1,2s-1}\}$ (or $l_0 \in \{1, \ldots, h_A^{p+r-1,q+s-1}\}$), such that either

$$\alpha \wedge \xi^{j_0} \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0$$

or

$$\beta \wedge \psi^{l_0} \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0.$$

Suppose the first case holds. By definition of Hodge $*$-operator, this is equivalent to

$$g(\alpha \wedge \xi^{j_0}, \eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0$$

Since the product g is diagonal with respect to the basis $\{\eta^1, \ldots, \eta^k\}$, by Lemma 4.4 this forces $\xi^{j_0} \in K_3$, i.e., $\xi^{j_0} = \eta^{k\bar{k}} \wedge \tilde{\omega}$, with $\tilde{\omega} \in \bigwedge^{2r-2,2s-2} \mathfrak{g}^*_{\mathbb{C}}$, which yields

$$g(\eta^{k\bar{k}} \wedge \alpha \wedge \tilde{\omega}, \eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0.$$

Since $\alpha \wedge \beta \neq 0$ and $\alpha \notin \mathcal{I}(\eta^k, \overline{\eta}^k)$, then $\gamma = \alpha \wedge \zeta$, for some $\zeta \in \bigwedge^{r-2,s-2} \mathfrak{g}^*_{\mathbb{C}}$. Note that if either $r < 2$ or $s < 2$, we obtain a contradiction. Let us assume the opposite, i.e., $r, s \geq 2$. Then,

$$(-1)^{p+q} \alpha \wedge \beta = \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge \tilde{\gamma} = \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge \alpha \wedge \zeta,$$

which implies that $\beta = \partial\overline{\partial}\left((-1)^{p+q+1}\eta^{k\bar{k}} \wedge \zeta\right)$, which leads to a contradiction, since β is a Bott–Chern harmonic form, i.e., β cannot be $\partial\overline{\partial}$-exact.

On the other hand, let us assume that $\beta \wedge \psi^{l_0} \wedge *(\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0$, which is equivalent to

$$g(\beta \wedge \psi^{l_0}, \eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) \neq 0.$$

Since β does not contain $\eta^{k\bar{k}}$, again by Lemma 4.4 the form $\psi^{l_0} \in K_3$, i.e., $\psi^{l_0} = \eta^{k\bar{k}} \wedge \tilde{\omega}$, with $\tilde{\omega} = \tilde{\omega}_1 + \cdots + \tilde{\omega}_d \in \bigwedge^{p+r-2,q+s-2} \mathfrak{g}^*_{\mathbb{C}}$ and each form $\tilde{\omega}_i = C_i \eta^{I\bar{J}}$ for $C_i \in \mathbb{C}$ and $I, J \subset \{1, \ldots, k\}$, $|I| = p + r - 2$, $|J| = q + s - 2$. We obtain

$$0 \neq g(\beta \wedge \eta^{k\bar{k}} \wedge \tilde{\omega}, \eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta) = (-1)^{|\beta| \cdot |\tilde{\omega}|} g(\tilde{\omega}, \tilde{\gamma})$$

which then implies that there exists $1 \leq d_0 \leq d$ and $0 \neq C \in \mathbb{C}$ such that $\tilde{\omega}_{d_0} = C\tilde{\gamma}$. So, we have that

$$\begin{aligned} 0 = \partial\overline{\partial}\psi^{l_0} &= \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge (\tilde{\omega}_1 + \cdots + \tilde{\omega}_d - \tilde{\omega}_{d_0}) + \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge \tilde{\omega}_{d_0} \\ &= \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge (\tilde{\omega} - \tilde{\omega}_{d_0}) + (-1)^{p+q} C \alpha \wedge \beta. \end{aligned}$$

Consequently, $-\frac{1}{C}\eta^{k\bar{k}} \wedge (\tilde{\omega} - \tilde{\gamma})$ is a $\partial\overline{\partial}$-primitive of $(-1)^{p+q}\alpha \wedge \beta$. However, by unicity of the $\partial\overline{\partial}$-primitive $\tilde{\gamma}$, we obtain a contradiction.

As a result, the Aeppli cohomology class $[\eta^{k\bar{k}} \wedge \tilde{\gamma} \wedge \beta] \notin \mathcal{J}$, hence the ABC-Massey product on $(\mathfrak{g}, J)$

$$\mathcal{P} = \left\langle [\alpha], [\beta], [\beta] \right\rangle_{ABC} \in \frac{H_A^{p+2r-1,q+2s-1}(\mathfrak{g})}{[\alpha] \cup H_A^{2r-1,2s-1}(\mathfrak{g}) + [\beta] \cup H_A^{p+r-1,q+s-1}(\mathfrak{g})}$$

is not vanishing. Finally, by Lemma 2.1, we have that $\mathcal{P}$ corresponds to a non trivial ABC-Massey product on (M, J).

To summarize, we have proved the following.

Theorem 4.5 *Let $(M = \Gamma\backslash G, J)$ be a nilmanifold of complex dimension k with a invariant complex structure J determined by a basis $\{\eta^1, \dots, \eta^k\}$ of $(1,0)$-forms such that*

$$\begin{cases} d\eta^j = 0, & j \in \{1, \dots, k-1\}, \\ d\eta^k \in \mathrm{Span}_{\mathbb{C}}\langle \eta^{jl}, \eta^{j\bar{l}}\rangle_{j,l=1}^{k-1}. \end{cases}$$

If there exist forms $\alpha := \eta^{i_1\dots i_p\bar{j}_1\dots\bar{j}_q} \in \bigwedge^{p,q}\mathfrak{g}^*_{\mathbb{C}}$, $\beta := \eta^{l_1\dots l_r\overline{m}_1\dots\overline{m}_s} \in \bigwedge^{r,s}\mathfrak{g}^*_{\mathbb{C}}$ and a unique form (up to constant) $\tilde{\gamma} = \eta^{t_1\dots t_{p+r-2}\overline{v}_1\dots\overline{v}_{q+s-2}} \in \bigwedge^{p+r-2,q+s-2}\mathfrak{g}^*_{\mathbb{C}}$, such that

(1) *the forms α and β are Bott–Chern harmonic and $\alpha \wedge \beta \neq 0$,*
(2) *the form $\eta^{k\bar{k}} \wedge \tilde{\gamma}$ is a $\partial\overline{\partial}$-primitive of $\alpha \wedge \beta$, i.e., $(-1)^{p+q}\alpha \wedge \beta = \partial\overline{\partial}(\eta^{k\bar{k}}) \wedge \tilde{\gamma}$, and $\tilde{\gamma} \wedge \beta \neq 0$,*

then (M, J) admits a non vanishing ABC-Massey product given by

$$\Big\langle [\alpha]_{BC}, [\beta]_{BC}, [\beta]_{BC} \Big\rangle_{ABC}.$$

As an immediate application of Theorem 4.5 we provide the explicit examples of two families of nilmanifolds admitting non trivial ABC-Massey products.

Example 4.6 Consider the family of nilmanifolds $(\Gamma\backslash G = M, J)$ endowed with invariant complex structure J characterized by a basis $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ of left-invariant $(1,0)$-forms on G such that

$$\begin{cases} d\eta^1 = d\eta^2 = d\eta^3 = 0, \\ d\eta^4 = A\eta^{12} + B\eta^{1\bar{3}} + C\eta^{2\bar{1}}, \end{cases}$$

with $A, B, C \in \mathbb{Q}[i] \setminus \{0\}$. Note that

$$0 \neq \partial\overline{\partial}(\eta^{4\bar{4}}) = -(|A|^2 + |C|^2)\eta^{12\bar{1}\bar{2}} - |B|^2\eta^{13\bar{1}\bar{3}}.$$

Then

$$\Big\langle [\eta^{12\bar{2}}]_{BC}, [\eta^{3\bar{1}}]_{BC}, [\eta^{3\bar{1}}]_{BC} \Big\rangle_{ABC}$$

is a non vanishing triple ABC-Massey product. It suffices to set $\tilde{\gamma} = -\frac{1}{|B|^2}\eta^2$ and then Theorem 4.5 directly applies.

Example 4.7 Consider the family of nilmanifolds $(\Gamma\backslash G = M, J)$ endowed with invariant complex structure J characterized by a basis $\{\eta^1, \eta^2, \eta^3, \eta^4, \eta^5\}$ of left-invariant $(1, 0)$-forms on G such that

$$\begin{cases} d\eta^1 = d\eta^2 = d\eta^3 = d\eta^4 = 0, \\ d\eta^5 = D\eta^{12} + E\eta^{3\overline{2}} + F\eta^{4\overline{3}}, \end{cases}$$

with $E, F, G \in \mathbb{Q}[i] \setminus \{0\}$. Note that

$$0 \neq \partial\overline{\partial}(\eta^{5\overline{5}}) = -|D|^2\eta^{12\overline{12}} - |E|^2\eta^{23\overline{23}} - |F|^2\eta^{34\overline{34}}.$$

Then,

$$\left\langle [\eta^{34\overline{1}}]_{BC}, [\eta^{1\overline{12}}]_{BC}, [\eta^{1\overline{12}}]_{BC} \right\rangle_{ABC}$$

is a non vanishing triple ABC-Massey product. In this case, it suffices to set $\tilde{\gamma} = -\frac{1}{|D|^2}\eta^{34}$ and Theorem 4.5 directly follows.

However, by dropping the hypotesis of special type complex structure, we show that there exist families of nilmanifolds endowed with nilpotent complex structures which admit SKT metrics but are never geometrically-BC-formal.

Proposition 4.8 *There exist SKT nilmanifolds with nilpotent complex structures admitting no geometrically-BC-formal metrics.*

Proof Let $(M = \Gamma\backslash G, J)$ be a family of nilmanifolds of complex dimension 4 endowed with an invariant complex structure J characterized by a basis $\{\eta^1, \eta^2, \eta^3, \eta^4\}$ of $(\mathfrak{g}^{(1,0)})^*$ such that

$$\begin{cases} d\eta^1 = 0 \\ d\eta^2 = 0 \\ d\eta^3 = B_1\eta^{2\overline{1}} \\ d\eta^4 = G_1\eta^{12} + D_1\eta^{1\overline{1}} + D_2\eta^{1\overline{2}} + E_2\eta^{2\overline{2}}, \end{cases} \tag{4.4}$$

with $B_1, G_1, D_1, D_2, D_3 \in \mathbb{Q}[i] \setminus \{0\}$. Let $\omega = \frac{i}{2}(\eta^{1\overline{1}} + \eta^{2\overline{2}} + \eta^{3\overline{3}} + \eta^{4\overline{4}})$ be the fundamental form of the diagonal Hermitian metric g on $\mathfrak{g}$. Then, g is SKT if, and only if, it holds $\partial\overline{\partial}\omega = 0$, i.e.,

$$|B_1|^2 + |G_1|^2 + |D_2|^2 - 2\mathfrak{Re}(D_1\overline{E}_2) = 0. \tag{4.5}$$

Assume that (4.5) holds. We now construct a non trivial ABC-Massey product.

Consider the forms $\eta^{124\overline{1}}$ and $\eta^{\overline{2}}$. Note that, by structure equations, $d\eta^{124\overline{1}} = d\eta^{\overline{2}} = 0$, hence

$$[\eta^{124\overline{1}}]_{BC} \in H^{3,1}_{BC}(\mathfrak{g}), \qquad [\eta^{\overline{2}}] \in H^{0,1}_{BC}(\mathfrak{g})$$

are well defined Bott–Chern cohomology classes. Moreover, the forms $*\eta^{124\bar{1}} = \eta^{3\bar{2}\bar{3}\bar{4}}$ and $*\eta^{\bar{2}} = -\eta^{1234\bar{1}\bar{3}\bar{4}}$ are $\partial\bar{\partial}$-closed, thus the forms $\eta^{124\bar{1}}$ and $\eta^{\bar{2}}$ are Bott–Chern harmonic forms and they define non vanishing Bott–Chern cohomology classes, i.e.,

$$[\eta^{124\bar{1}}]_{BC} \neq 0, \qquad [\eta^{\bar{2}}]_{BC} \neq 0.$$

Now, $\eta^{124\bar{1}} \wedge \eta^{\bar{2}} = \partial\bar{\partial}(-\frac{1}{|B_1|^2}\eta^{34\bar{3}})$, so that the ABC-Massey product

$$\mathcal{P} = \left\langle [\eta^{124\bar{1}}], [\eta^{\bar{2}}], [\eta^{\bar{2}}] \right\rangle_{ABC} \in \frac{H_A^{2,2}(\mathfrak{g})}{[\eta^{\bar{2}}] \cup H_A^{2,1}(\mathfrak{g})}$$

is well defined and represented by the Aeppli cohomology class

$$\left[\frac{1}{|B_1|^2}\eta^{34\bar{2}\bar{3}}\right]_A \in H_A^{2,2}(\mathfrak{g}).$$

Note that $*\eta^{34\bar{2}\bar{3}} = \eta^{12\bar{1}\bar{4}}$, which is d-closed, hence, as a Aeppli-cohomology class, $\left[\frac{1}{|B_1|^2}\eta^{34\bar{2}\bar{3}}\right]_A \neq 0$. We claim that $\left[\frac{1}{|B_1|^2}\eta^{34\bar{2}\bar{3}}\right]_A$ does not belong to the ideal $[\eta^{\bar{2}}] \cup H_A^{2,1}(\mathfrak{g})$.

Assume, by contradiction, that this is the case, i.e., there exist constants $\lambda_j \in \mathbb{C}$ and forms $R \in \bigwedge^{1,2} \mathfrak{g}_{\mathbb{C}}^*$, $S \in \bigwedge^{2,1} \mathfrak{g}_{\mathbb{C}}^*$ such that

$$\frac{1}{|B_1|^2}\eta^{34\bar{2}\bar{3}} = \sum_{j=1}^{h_A^{2,1}} \lambda_j \eta^{\bar{2}} \wedge \xi^j + \partial R + \bar{\partial} S, \tag{4.6}$$

where $\{\xi^j\}_{j=1}^{h_A^{2,1}}$ is a basis of $\mathcal{H}_A^{2,1}(\mathfrak{g})$. By multiplying both sides of (4.6) by $\eta^{12\bar{1}\bar{4}}$, integrating over M, and observing that $\eta^{12\bar{1}\bar{4}}$ is d-closed, we obtain

$$\begin{aligned} 0 \neq \frac{1}{|B_1|^2}||\eta^{34\bar{2}\bar{3}}||^2_{L^2(M)} &= \sum_{j=1}^{h_A^{2,1}} \int_M \eta^{\bar{2}} \wedge \xi^j \wedge \eta^{12\bar{1}\bar{4}} \\ &= -\sum_{j=1}^{h_A^{2,1}} \lambda_j \langle\langle \eta^{12\bar{1}\bar{2}\bar{4}}, *\xi^j \rangle\rangle_{L^2(M)} = 0, \end{aligned}$$

where the last equality holds being $\eta^{12\bar{1}\bar{2}\bar{4}}$ a $\partial\bar{\partial}$-exact form and $\{*\xi^j\}_{j=1}^{h_A^{2,1}}$ Bott–Chern harmonic forms. This is absurd.

Therefore, the ABC-Massey product on $(\mathfrak{g}, J)$

$$\mathcal{P} := \left\langle [\eta^{124\bar{1}}]_{BC}, [\eta^{\bar{2}}]_{BC}, [\eta^{\bar{2}}]_{BC} \right\rangle_{ABC} \neq 0,$$

is not vanishing. Thus, by Lemma 2.1, the product $\mathcal{P}$ corresponds to non vanishing ABC-Massey product on (M, J), which, consequently, is never geometrically-BC-formal.

As a result, each element of the family of manifolds (4.4) satisfying assumption (4.5) admits a SKT metric, i.e., the diagonal metric with respect to the coframe $\{\eta^1, \eta^2, \eta^3, \eta^4\}$, but it admits a non vanishing ABC-Massey product, so it is never geometrically-BC-formal. □

5 Kähler Solvmanifolds and Bott–Chern Formality

In this section, we prove that every Kähler solvmanifold is geometrically-BC-formal, hence answering a question in [24]: despite the existence of a manifold satisfying the $\partial\overline{\partial}$-lemma and yet admitting a non vanishing ABC-Massey product [31], it remains unclear whether ABC-Massey product provide an obstruction to stronger cohomological properties on a compact complex manifold. As a starting point, the authors in [24] suggest that one may look at the class of Kähler solvmanifolds, although they seem inclined to think that this class of manifolds might not be decisive. We will indeed prove that Kähler solvmanifolds are metrically formal, so that every ABC-Massey product vanishes.

We start by recalling the fundamental result of Hasegawa on the structure of Kähler solvmanifolds.

Theorem 5.1 *A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. In particular, a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.*

In [20, Example 4], the author gives a characterization of the geometric structure of a Kähler non toric solvmanifold, which we now recall.

A Kähler solvmanifold X arises as the compact quotient $\Gamma\backslash G$ of a simply connected solvable Lie group $G = \mathbb{C}^l \rtimes_\phi \mathbb{R}^{2k}$ defined by a map

$$\phi\colon \mathbb{R}^{2k} \to \mathrm{Aut}(\mathbb{C}^l)$$
$$\phi(t_i E_i) \cdot {}^t(z_1, \ldots, z_l) := {}^t\left(e^{\sqrt{-1}\gamma^i t_i} z_1, \ldots, e^{\sqrt{-1}\gamma^i t_i} z_l\right) = e^{\sqrt{-1}\gamma^i t_i}\mathbb{I} \cdot {}^t(z_1, \ldots, z_l),$$

where E_i is the i-th unit vector of $\mathbb{R}^{2k}$ and $e^{\sqrt{-1}\gamma^i}$ is the s_i-th root of unity, for $i = 1, \ldots, 2k$. The discrete subgroup $\Gamma := \mathbb{Z}^{2l} \rtimes_\phi \mathbb{Z}^{2k}$ is defined so that $\phi(\mathbb{Z}^{2k})(\mathbb{Z}^{2l}) \subset \mathbb{Z}^{2l}$ and, hence, Γ a lattice of G.

Under the natural identification $\mathbb{R}^{2k} \simeq \mathbb{C}^k$, let $W = (z_{l+1}, \ldots, z_{l+k}) \in \mathbb{C}^k$ be the coordinates of a point in $\mathbb{C}^k$ with respect to the standard basis of $\mathbb{C}^k$. Then, the map ϕ can be extended to

$$\phi(W) \cdot {}^t(z_1, \ldots, z_l) = e^{\sqrt{-1}(\sum_{i=1}^k \mathfrak{Re}(z_{l+i})\gamma^{2l+2i-1} - \mathfrak{Im}(z_{l+i})\gamma^{2l+2i})}\mathbb{I} \cdot {}^t(z_1, \ldots, z_l). \tag{5.1}$$

In particular, if $z = {}^t(z_1, \ldots, z_l, z_{l+1}, \ldots, z_{l+k})$, $z' = {}^t(z'_1, \ldots, z'_l, z'_{l+1}, \ldots, z'_{l+k}) \in G$, then the product rule $*$ of G is defined as

$$z' * z = \big(\phi(z'_{l+1}, \ldots, z'_{l+k}) \cdot {}^t(z_1, \ldots, z_l) \\ + {}^t(z'_1, \ldots, z'_l), {}^t(z_{l+i} + z'_{l+1}, \ldots, z_{l+k} + z'_{l+k})\big).$$

The Lie algebra $\mathfrak{g}$ of G is spanned by the set of G-left-invariant vector fields

$$\{X_1, \ldots X_{2l}, X_{2l+1} \ldots, X_{2l+2k}\}$$

which satisfy the bracket relations

$$[X_{2l+2i}, X_{2j-1}] = -X_{2j}, \qquad [X_{2l+2i}, X_{2j}] = X_{2j-1}, \tag{5.2}$$

for $i = 1, \ldots, k$, $j = 1, \ldots, l$, whereas any other bracket vanishes. The standard G-left-invariant almost complex structure J on G is defined by

$$\begin{aligned} JX_{2j-1} &= X_{2j}, \quad JX_{2j} = -X_{2j-1}, \\ JX_{2l+2i-1} &= X_{2l+2i}, \quad JX_{2l+2i} = -X_{2l+2i-1}, \end{aligned} \tag{5.3}$$

for $i = 1, \ldots, k$, $j = 1, \ldots, k$, so that $\mathfrak{g}_{\mathbb{C}}$ decomposes as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1},$$

where $\mathfrak{g}^{1,0} = \langle Z_1, \ldots, Z_l, Z_{l+1}, \ldots, Z_{l+k}\rangle$ and $\mathfrak{g}^{0,1} = \langle \overline{Z_1}, \ldots, \overline{Z_l}, \overline{Z_{l+1}}, \ldots, \overline{Z_{l+k}}\rangle$ are, respectively, the $\pm i$-eigenspaces of J and

$$Z_j := \frac{1}{2}\left(X_{2j-1} - iX_{2j}\right), \quad Z_{l+i} := \frac{1}{2}\left(X_{2l+2i-1} - iX_{2l+2i}\right)$$

for $i = 1, \ldots, k$, $j = 1, \ldots, l$.

If we denote by $\left\{e^1, \ldots, e^{2l}, e^{2l+1}, \ldots, e^{2l+2k}\right\}$ the coframe of G-left-invariant forms on G dual to $\{X_1, \ldots, X_{2l}, X_{2l+1}, \ldots, X_{2l+2k}\}$, dualizing the relations (5.2), we obtain the structure equations

$$\begin{cases} de^{2j-1} &= -\sum_{i=1}^{k} e^{2j} \wedge e^{2l+2i}, \quad j = 1, \ldots, l, \\ de^{2j} &= \sum_{i=1}^{k} e^{2j-1} \wedge e^{2l+2i}, \quad j = 1, \ldots, l, \\ de^{2l+2i-1} &= 0, \quad i = 1, \ldots, k \\ de^{2l+2i} &= 0, \quad i = 1, \ldots, k. \end{cases}$$

The complex structure J defined in (5.3) determines the basis of $(\mathfrak{g}^*_{\mathbb{C}})^{1,0}$ which we denote by $\{\varphi^1, \dots, \varphi^l, \varphi^{l+1}, \dots, \varphi^{l+k}\}$, where

$$\varphi^j := e^{2j-1} + ie^{2j}, \qquad \varphi^{l+i} := e^{2l+2i-1} + ie^{2l+2i}, \tag{5.4}$$

for $j = 1, \dots, l$, $i = 1, \dots, k$, which satisfy the complex structure equations

$$\begin{cases} d\varphi^j &= \dfrac{1}{2}\displaystyle\sum_{i=1}^{k} \varphi^j \wedge \varphi^{l+i} - \varphi^j \wedge \varphi^{\overline{l+i}}, \quad j = 1, \dots, l \\ d\varphi^{l+i} &= 0, \quad i = 1, \dots, k. \end{cases} \tag{5.5}$$

As a result, the manifold $X = (\Gamma\backslash G, J)$ is a Kähler solvmanifold endowed with a G-invariant complex structure, since J is G-left-invariant and hence descends to $\Gamma\backslash G$. An invariant Kähler metric on X is, for example, the canonical diagonal with respect to the coframe $\{\varphi^1, \dots, \varphi^{l+k}\}$, as we will prove in Corollary 5.4.

We now give an explicit expression for the differential of any G-left-invariant (p, q)-form on X. At this level, we consider the elements of the natural basis

$$\Big\{\varphi^{a_1} \wedge \cdots \wedge \varphi^{a_p} \wedge \varphi^{\overline{b_1}} \wedge \cdots \wedge \varphi^{\overline{b_q}} : 1 \le a_1 < \cdots < a_p \le l + k, \\ 1 \le b_1 < \cdots < b_q \le l + k\Big\}. \tag{5.6}$$

Notation From now on, we will write the holomorphic and the anti-holomorphic components of any element of the basis (5.6) by listing first the $(1, 0)$-forms (respectively, $(0, 1)$-forms) belonging to $\{\varphi^j\}_{j=1}^l$ (respectively, $\{\varphi^{\overline{j}}\}_{j=1}^l$) and then the $(1, 0)$-forms (respectively, the $(0, 1)$-forms) belonging to $\{\varphi^{l+i}\}_{i=1}^k$ (respectively, to $\{\varphi^{\overline{l+i}}\}_{i=1}^k$), i.e., we will adopt the following notation

$$\alpha = \varphi^{j_1 \dots j_r} \wedge \varphi^{l+i_1 \dots l+i_{p-r}} \wedge \varphi^{\overline{m}_1 \dots \overline{m}_s} \wedge \varphi^{\overline{l+n_1} \dots \overline{l+n_{q-s}}}, \tag{5.7}$$

with $r \in \{\max(0, p-k), \dots, p\}$, $s \in \{\max(0, q-k), \dots, q\}$, $j_1, \dots, j_r \in \{1, \dots, l\}$, $i_1, \dots, i_{p-r} \in \{1, \dots, k\}$, $m_1, \dots, m_s \in \{1, \dots, l\}$, $n_1, \dots, n_{q-s} \in \{1, \dots, k\}$.

With respect to this notation the differential acts as in the following lemma.

Lemma 5.2 *For any invariant (p, q)-form $\alpha \in X$ written as in (5.7), it holds*

$$d\alpha = \frac{s-r}{2}\left(\sum_{i=1}^{k} \varphi^{l+i}\right) \wedge \alpha + \frac{r-s}{2}\left(\sum_{i=1}^{k} \varphi^{\overline{l+i}}\right) \wedge \alpha.$$

In particular,

$$\partial\alpha = \frac{s-r}{2}\left(\sum_{i=1}^{k} \varphi^{l+i}\right) \wedge \alpha, \qquad \overline{\partial}\alpha = \frac{r-s}{2}\left(\sum_{i=1}^{k} \varphi^{\overline{l+i}}\right) \wedge \alpha.$$

Proof We see that, by Leibniz rule, we have

$$\begin{aligned} d\alpha &= d\left(\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\right)\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}} \\ &\quad+(-1)^p\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge d\left(\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}\right) \end{aligned}$$

Let us first focus on the first term $\Omega_1 = d\left(\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\right)\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}$. By the structure equations and graded-commutativity of the wedge product, it holds that

$$\Omega_1 = -\frac{r}{2}\left(\sum_{i=1}^k\varphi^{l+i}-\varphi^{\overline{l+i}}\right)\wedge\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}.$$

Analogously, for the other summand $\Omega_2 = \varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge d\left(\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}\right)$ we obtain the explicit expression

$$\Omega_2 = (-1)^p\frac{s}{2}\left(\sum_{i=1}^k\varphi^{l+i}-\varphi^{\overline{l+i}}\right)\wedge\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}.$$

Hence, by adding Ω_1 and $(-1)^p\Omega_2$, we obtain

$$\begin{aligned} d\alpha &= \frac{s-r}{2}\left(\sum_{i=1}^k\varphi^{l+i}\right)\wedge\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}} \\ &\quad+\frac{r-s}{2}\left(\sum_{i=1}^k\varphi^{\overline{l+i}}\right)\wedge\varphi^{j_1\dots j_r}\wedge\varphi^{l+i_1\dots l+i_{p-r}}\wedge\varphi^{\overline{m}_1\dots\overline{m}_s}\wedge\varphi^{\overline{l+n_1}\dots\overline{l+n_{q-s}}}, \end{aligned}$$

which yields the thesis. □

As an immediate application of Lemma 5.2, we observe the following.

Corollary 5.3 *Let α be any invariant (p, q)-form on X, written as in (5.7). If $r = s$, then $d\alpha = 0$.*

We have then an explicit expression for an invariant Kähler metric on X.

Corollary 5.4 *The canonical Hermitian metric g on X with fundamental form*

$$\omega = \frac{i}{2}\sum_{j=1}^l\varphi^{j\overline{j}}+\frac{i}{2}\sum_{i=1}^k\varphi^{l+i\,\overline{l+i}}$$

is Kähler.

Proof By structure equations (5.3) and Lemma 5.2, we have that $d\varphi^{l+i\,\overline{l+i}} = 0$, for every $i \in \{1, \dots, k\}$, and

$$d\varphi^{j\bar{j}} = \frac{1-1}{2}\sum_{i=1}^{k} \varphi^{l+i} \wedge \varphi^{j\bar{j}} + \frac{1-1}{2}\sum_{i=1}^{k} \varphi^{\overline{l+i}} \wedge \varphi^{j\bar{j}} = 0,$$

for every $j \in \{1, \dots, l\}$. Hence, $d\omega = 0$. □

We now adapt the techniques in [2, Section 2.5] to compute the cohomology of X. We preliminarly observe the following.

Lemma 5.5 *X is a complex solvmanifold of splitting type, i.e., X is a solvmanifold $X = \Gamma\backslash G$ endowed with a G-left-invariant complex structure J such that G is the semidirect product $\mathbb{C}^n \rtimes_\phi N$ so that:*

- *(i) N is a simply connected $2m$-dimensional nilpotent Lie group endowed with an N-left-invariant complex structure J_N; (denote the Lie algebras of $\mathbb{C}^n$ and N by $\mathfrak{a}$ and, respectively, $\mathfrak{n}$),*
- *(ii) For any $t \in \mathbb{C}^n$, it holds that $\phi(t) \in GL(N)$ is a holomorphic automorphism of N with respect to J_N;*
- *(iii) ϕ induces a semisimple action on $\mathfrak{n}$;*
- *(iv) G has a lattice Γ; (then Γ can be written as $\Gamma = \Gamma_{\mathbb{C}^n} \ltimes_\phi \Gamma_N$ such that $\Gamma_{\mathbb{C}^n}$ and Γ_N are lattices of $\mathbb{C}^n$ and, respectively, N, and, for any $t \in \Gamma_{\mathbb{C}^n}$, it holds $\phi(t)(\Gamma_N) \subseteq \Gamma_N$);*
- *(v) The inclusion $\bigwedge^{\bullet,\bullet} \mathfrak{n}^*_{\mathbb{C}} \hookrightarrow \mathcal{A}^{\bullet,\bullet}(\Gamma_N\backslash N)$ induces the isomorphism*

$$H^\bullet(\textstyle\bigwedge^{\bullet,\bullet} \mathfrak{n}^*_{\mathbb{C}}, \bar{\partial}) \simeq H^{\bullet,\bullet}_{\bar{\partial}}(\Gamma_N\backslash N).$$

Proof We recall that X is a solvmanifold $\Gamma\backslash G$ endowed with a G-invariant complex structure J. The universal cover of X is the Lie group $G = \mathbb{C}^l \rtimes_\phi \mathbb{C}^k$ which admits the lattice $\Gamma = \mathbb{Z}^l \rtimes_\phi \mathbb{Z}^k$. Note that $\mathbb{C}^l$ is the nilradical of G with Lie algebra $\mathfrak{n}$ and $\mathbb{C}^k$ is the factor with Lie algebra $\mathfrak{a}$, so that the Lie algebra of G decomposes as

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}.$$

- *(i)* The factor $\mathbb{C}^l$ is a simply connected abelian Lie group of real dimension $2l$, i.e., it is clearly a simply connected nilpotent Lie group of real dimension $2l$. The standard complex structure $J_{\mathbb{C}^l}$ of $\mathbb{C}^l$ is determined by the standard frame $\{\frac{\partial}{\partial z_j}\}_{j=1}^l$ and its dual frame $\{dz_j\}_{j=1}^l$ and it is clearly invariant under the action of $\mathbb{C}^l$ on itself by traslation, i.e., $J_{\mathbb{C}^l}$ is $\mathbb{C}^l$-left-invariant.
- *(ii)* For any $W = (z_{l+1}, \dots, z_{l+k}) \in \mathbb{C}^k$, the map $\phi(W)$ with expression (5.1) is clearly a holomorphic automorphism of $\mathbb{C}^l$ with respect to $J_{\mathbb{C}^l}$.
- *(iii)* For every $W = (z_{l+1}, \dots, z_{l+k}) \in \mathbb{C}^k$ map ϕ induces a semisimple action on $\mathfrak{n}$ by

$$(d\phi(W))_e = e^{\sqrt{-1}(\sum_{i=1}^k \mathfrak{Re}(z_{l+i})\gamma^{2l+2i-1} - \mathfrak{Im}(z_{l+i})\gamma^{2l+2i})}\mathbb{I} \in \mathrm{Gl}(\mathfrak{n}). \tag{5.8}$$

(iv) The universal cover G admits a lattice Γ by hypotesis.

(v) Since $\mathbb{Z}^l \backslash \mathbb{C}^l$ is biholomorphic to a complex l-dimensional torus $\mathbb{T}^l$, the inclusion $\bigwedge^{\bullet,\bullet} \mathfrak{n}^*_{\mathbb{C}} \hookrightarrow \mathcal{A}^{\bullet,\bullet}(\mathbb{T}^l)$ induces an isomorphism in the Dolbeault cohomology, i.e.,

$$H^{\bullet}(\bigwedge^{\bullet,\bullet} \mathfrak{n}^*_{\mathbb{C}}, \overline{\partial}) = (\bigwedge^{\bullet,\bullet} \mathfrak{n}^*_{\mathbb{C}}, \overline{\partial} \equiv 0) \simeq H^{\bullet,\bullet}_{\overline{\partial}}(\mathbb{T}^l).$$

□

We are in the position to apply the results for complex solvmanifolds of splitting type as in [2, Section 2.5]. With respect to the standard frame $\frac{\partial}{\partial z_j}$ of $\mathbb{C}^l$-left-invariant $(1,0)$-vector fields, the induced action (5.8) of ϕ on $\mathfrak{n}^{1,0}$ is given by

$$\mathbb{C}^l \ni W = (z_{l+1}, \dots, z_{l+k}) \mapsto \chi(W)\mathbb{I},$$

with χ the character of $\mathbb{C}^k$ determined by $\chi(W) = e^{\sqrt{-1}(\sum_{i=1}^k \mathfrak{Re}(z_{l+i})\gamma^{2l+2i-1} - \mathfrak{Im}(z_{l+i})\gamma^{2l+2i})}$. Then, χ can be extended to a character of G and the set

$$\{\chi^{-1}dz^1, \dots, \chi^{-1}dz^l, \sqrt{-1}(\gamma^{2l+1} + \sqrt{-1}\gamma^{2l+2})dz^{l+1}, \\ \dots, \sqrt{-1}(\gamma^{2l+2k-1} + \sqrt{-1}\gamma^{2l+2k})dz^{l+k}\} \tag{5.9}$$

is a coframe of G-left-invariant $(1,0)$-forms on G. More precisely, the coframe (5.9) coincides with the coframe of G-left-invariant $(1,0)$-forms defined in (5.4) with structure equations (5.5), i.e.,

$$\begin{cases} \varphi^j = \chi^{-1}dz^j, & j \in \{1, \dots, k\} \\ \varphi^{l+i} = \sqrt{-1}(\gamma^{2l+2i-1} + \sqrt{-1}\gamma^{2l+2i})dz^{l+i}, & i \in \{1, \dots, k\}. \end{cases}$$

By Lemma [2, Lemma 2.12] (see also the proof of [22, Lemma 2.2]), the unique unitary characters $\beta_1, \beta_2 \in \mathrm{Hom}(\mathbb{C}^k, \mathbb{C}^*)$ such that $\chi \cdot \beta_1^{-1}$ and $\overline{\chi} \cdot \beta_2^{-1}$ are holomorphic characters are exactly

$$\beta_1 = \chi, \qquad \beta_2 = \overline{\chi} = \chi^{-1},$$

since both χ and $\overline{\chi}$ are unitary and holomorphic characters. For any multi-index $J = (j_1, \dots, j_p) \subset \{1, \dots, l\}$ of length $|J| = p$ and set of characters $\{\delta_1, \dots, \delta_l\}$, we set $\delta_J := \delta_{j_1} \cdot \dots \cdot \delta_{j_p}$.

In the adapted notation, the bi-complex $B^{\bullet,\bullet}_{\Gamma}$ [2, Theorem 2.13] which computes the Dolbeault cohomology is

$$B^{p,q}_{\Gamma} = \mathbb{C}\left\langle \varphi^J \wedge \varphi^{l+I} \wedge \varphi^{\overline{M}} \wedge \varphi^{\overline{l+N}} : |J| + |I| = p,\ |M| + |N| = q,\ \chi_J \cdot \chi_M^{-1}|_{\Gamma} \equiv 1 \right\rangle \\ \subset \bigwedge^{p,q} \mathfrak{g}^*_{\mathbb{C}}. \tag{5.10}$$

Note that $\chi_J = \chi^{|J|}$ and $\chi_M^{-1} = \chi^{-|M|}$, so the condition $\chi_J \cdot \chi_M^{-1}|_\Gamma \equiv 1$ reduces to

$$\chi^{|J|-|M|}|_\Gamma \equiv 1,$$

which is satisfied if and only if $|J| = |M|$.

Remark 5.6 Note that the defining property of the complex $B_\Gamma^{\bullet,\bullet}$ actually does not depend on the choice of the lattice Γ, therefore we will omit the dependency on Γ from the notation and we will write only $B^{\bullet,\bullet}$.

The complex $B^{\bullet,\bullet}$ is given by

$$B^{p,q} = \mathbb{C}\left\langle \varphi^J \wedge \varphi^{l+I} \wedge \varphi^{\overline{M}} \wedge \varphi^{\overline{l+N}} : |J| + |I| = p,\ |M| + |N| = q,\ |J| = |M| \right\rangle.$$

Note that the invariant diagonal Kähler metric g on X defines, by restriction, a metric for the complex $B^{\bullet,\bullet}$, which we still denote by g.

Lemma 5.7 *The exterior differential on the complex $B^{\bullet,\bullet}$ is identically zero, i.e, $d|_{B^{\bullet,\bullet}} \equiv 0$. Moreover, the complex $B^{\bullet,\bullet}$ is closed under conjugation, i.e., $\overline{B^{p,q}} = B^{q,p}$, and it is closed with respect to $*_g$ and has a structure of an algebra with respect to $\wedge$.*

Proof Let α be an element of the natural basis $B^{p,q}$, i.e.,

$$\alpha = \varphi^{j_1 \ldots j_r} \wedge \varphi^{l+i_1 \ldots l+i_{p-r}} \wedge \varphi^{\overline{m}_1 \ldots \overline{m}_r} \wedge \varphi^{\overline{l+n_1} \ldots \overline{l+n_{q-r}}}, \tag{5.11}$$

with respect to the usual decomposition (5.7). By Lemma 5.2, it is immediate that $d\alpha = 0$, hence, by linearity of the exterior differential, we obtain $d|_{B^{\bullet,\bullet}} \equiv 0$.

Note that for any α of the form (5.11), the complex conjugation acts as

$$\overline{\alpha} = \varphi^{\overline{j}_1 \ldots \overline{j}_r} \wedge \varphi^{\overline{l+i_1} \ldots \overline{l+i_{p-r}}} \wedge \varphi^{m_1 \ldots m_r} \wedge \varphi^{l+n_1 \ldots l+n_{q-r}}.$$

Hence $\overline{\alpha} \in B^{q,p}$ and $\overline{B^{p,q}} = B^{q,p}$.

If g is the invariant diagonal Kähler metric for $B^{\bullet,\bullet}$, the coframe $\{\varphi^1, \ldots, \varphi^{l+k}\}$ is orthonormal with respect to g. Let us denote by $* := *_g$ the $\mathbb{C}$-antilinear Hodge operator

$$*\colon \textstyle\bigwedge^{p,q} \mathfrak{g}_{\mathbb{C}}^* \to \bigwedge^{l+k-p,l+k-q} \mathfrak{g}_{\mathbb{C}}^*.$$

In particular, if $\{j_{r+1}, \ldots, j_l\}$ and $\{m_{r+1}, \ldots, m_l\}$ are the complementary sets in $\{1, \ldots, l\}$ of $\{j_1, \ldots, j_r\}$ and, respectively, $\{m_1, \ldots, m_r\}$, and $\{i_{p-r+1}, \ldots, i_k\}$ and $\{n_{q-r+1}, \ldots, n_k\}$ are the complementary sets in $\{1, \ldots, k\}$ of $\{i_1, \ldots, i_k\}$ and, respectively, of $\{n_1, \ldots, n_{q-r}\}$, then for any element of the natural basis of $B^{\bullet,\bullet}$ with expression as in (5.11), up to a sign, the Hodge $*$ operator operates as

$$*\alpha = \varphi^{j_{r+1} \ldots j_l} \wedge \varphi^{l+i_{p-r+1} \ldots l+i_k} \wedge \varphi^{\overline{m}_{r+1} \ldots \overline{m}_l} \wedge \varphi^{\overline{l+n_{q-r+1}} \ldots \overline{l+n_k}},$$

i.e., for every $\alpha \in B^{\bullet,\bullet}$, also $*\alpha \in B^{\bullet,\bullet}$. Moreover, taking any two elements α and β of the natural basis of $B^{\bullet,\bullet}$, i.e.,

$$\alpha = \varphi^J \wedge \varphi^{l+I} \wedge \varphi^{\overline{M}} \wedge \varphi^{\overline{l+N}}, \qquad \beta = \varphi^{J'} \wedge \varphi^{l+I'} \wedge \varphi^{\overline{M}'} \wedge \varphi^{\overline{l+N'}}$$

with $|J|+|I| = p$, $|M|+|N| = q$, and $|J| = |M|$, and $|J'|+|I'| = r$, $|M'|+|N'| = s$, and $|J'| = |M'|$, then the expression of the wedge product of α and β is given by

$$\begin{aligned}\alpha \wedge \beta = (-1)^{|J'|(|M|+|I|)+|I|'(|M|+|N|)} \varphi^J \wedge \varphi^{J'} \wedge \varphi^{l+I} \wedge \varphi^{l+I'} \wedge \varphi^{\overline{M}} \wedge \varphi^{\overline{M}'} \\ \wedge \varphi^{\overline{l+N}} \wedge \varphi^{\overline{l+N'}}.\end{aligned}$$

Since $|J| + |J'| = |M| + |M|'$, the form $\alpha \wedge \beta \in B^{\bullet,\bullet}$, i.e, $(B^{\bullet,\bullet}, \wedge)$ is an algebra. □

We are now finally ready to describe explicitly the Dolbeault cohomology of any Kähler solvmanifold and, hence, by the Hodge decomposition and the Kähler identities, also its de Rham cohomology, the Bott–Chern cohomology and the Aeppli cohomology.

Theorem 5.8 *Let $X = (\Gamma \backslash G, J)$ be a Kähler solvmanifold with $G = \mathbb{C}^l \rtimes \mathbb{C}^k$ and J the associated invariant complex structure. Let also $\{\varphi^1, \dots, \varphi^l, \varphi^{l+1}, \dots, \varphi^{l+k}\}$ be the coframe of G-invariant forms on X determined by the structure equations*

$$\begin{cases} d\varphi^j = \dfrac{1}{2}\varphi^j \wedge \left(\displaystyle\sum_{i=1}^{k} \varphi^{l+i} - \varphi^{\overline{l+i}}\right), & j = 1, \dots, l, \\ d\varphi^{l+i} = 0, \quad i = 1, \dots, k. \end{cases}$$

Then, the Dolbeault cohomology spaces of X are

$$H^{p,q}_{\overline{\partial}}(X) \cong \mathbb{C}\left\langle \left[\varphi^{j_1 \dots j_r} \wedge \varphi^{l+i_1 \dots l+i_{p-r}} \wedge \varphi^{\overline{m}_1 \dots \overline{m}_r} \wedge \varphi^{\overline{l+n_1} \dots \overline{l+n_{q-r}}}\right]\right\rangle_{\max(0,p-k,q-k) \le r \le \min(p,q)}$$

and the de Rham cohomology spaces of X

$$\begin{aligned} &H^s_{dR}(X; \mathbb{C}) \\ &\cong \bigoplus_{p+q=s} \mathbb{C}\left\langle \left[\varphi^{j_1 \dots j_r} \wedge \varphi^{l+i_1 \dots l+i_{p-r}} \wedge \varphi^{\overline{m}_1 \dots \overline{m}_r} \wedge \varphi^{\overline{l+n_1} \dots \overline{l+n_{q-r}}}\right]\right\rangle_{\max(0,p-k,q-k) \le r \le \min(p,q)}. \end{aligned}$$

In particular, the Hodge numbers of X are

$$h^{p,q}_{\overline{\partial}} = \sum_{r=\max(0,p-k,q-k)}^{\min(p,q)} \binom{l}{r} \left[\binom{k}{p-r} + \binom{k}{q-r}\right]. \tag{5.12}$$

Proof Let us start by fixing g the invariant diagonal Kähler metric on $B^{\bullet,\bullet}$. Then, by Lemma 5.7, if $\alpha \in B^{\bullet,\bullet}$, then $\overline{\partial}\alpha = 0$ and $\overline{\partial} * \alpha = 0$, hence

$$B^{\bullet,\bullet} = \mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(B^{\bullet,\bullet}) \cong H^{\bullet,\bullet}(B^{\bullet,\bullet}, \overline{\partial}), \tag{5.13}$$

where the isomorphism is given by the projection onto the Dolbeault cohomology of the complex $B^{\bullet,\bullet}$. By [2, Theorem 2.14], the inclusion $B^{\bullet,\bullet} \hookrightarrow \mathcal{A}^{\bullet,\bullet}X$ induces the isomorphism in Dolbeault cohomology

$$H^{\bullet,\bullet}(B^{\bullet,\bullet}, \overline{\partial}) \cong H^{\bullet,\bullet}_{\overline{\partial}}(X), \tag{5.14}$$

so we obtain that each for every p, q, for the Dolbeault cohomology spaces $H^{p,q}_{\overline{\partial}}(X)$ of X satisfy

$$\begin{aligned} &H^{p,q}_{\overline{\partial}}(X) \\ &\cong \mathbb{C}\left\langle\left[\varphi^{j_1\ldots j_r} \wedge \varphi^{l+i_1\ldots l+i_{p-r}} \wedge \varphi^{\overline{m}_1\ldots\overline{m}_r} \wedge \varphi^{\overline{l+n_1}\ldots\overline{l+n_{q-r}}}\right]\right\rangle_{\max(0,p-k,q-k)\le r\le \min(p,q)}, \end{aligned} \tag{5.15}$$

and, by the Hodge decomposition, for every k, the de Rham cohomology of X is

$$\begin{aligned} &H^k_{dR}(X;\mathbb{C}) \\ &= \bigoplus_{p+q=k} \left\langle\left[\varphi^{j_1\ldots j_r} \wedge \varphi^{l+i_1\ldots l+i_{p-r}} \wedge \varphi^{\overline{m}_1\ldots\overline{m}_r} \wedge \varphi^{\overline{l+n_1}\ldots\overline{l+n_{q-r}}}\right]\right\rangle_{\max(0,p-k,q-k)\le r\le \min(p,q)}. \end{aligned}$$

As a straightforward consequence, the explicit description (5.15) shows that the Hodge numbers of X are given by

$$h^{p,q}_{\overline{\partial}} = \sum_{r=\max(0,p-k,q-k)}^{\min(p,q)} \binom{l}{r}\binom{k}{p-r} \cdot \sum_{r=\max(0,p-k,q-k)}^{\max(p,q)} \binom{l}{r}\binom{k}{q-r},$$

which concludes the proof. □

Since X is compact Kähler manifold, X satisfies the $\partial\overline{\partial}$-lemma. Therefore, we have the following isomorphisms between the complex cohomologies of X, i.e.,

$$H^{\bullet,\bullet}_{\overline{\partial}}(M) \cong H^{\bullet,\bullet}_{BC}(X) \cong H^{\bullet,\bullet}_{A}(X).$$

Thus, from Theorem 5.8 we immediately obtain the following.

Corollary 5.9

$$\begin{aligned} H^{p,q}_{BC}(X) = H^{p,q}_{A}(X) \\ \cong \mathbb{C}\left\langle\left[\varphi^{j_1\ldots j_r} \wedge \varphi^{l+i_1\ldots l+i_{p-r}} \wedge \varphi^{\overline{m}_1\ldots\overline{m}_r} \wedge \varphi^{\overline{l+n_1}\ldots\overline{l+n_{q-r}}}\right]\right\rangle_{\max(0,p-k,q-k)\le r\le \min(p,q)}. \end{aligned}$$

Moreover, by the Kähler identities, it turns out that the spaces of harmonic forms with respect to the complex Laplacians coincide, i.e.,

$$\mathcal{H}^{\bullet,\bullet}_{\Delta}(X) = \mathcal{H}^{\bullet,\bullet}_{\Delta_{\overline{\partial}}}(X) = \mathcal{H}^{\bullet,\bullet}_{\Delta_{BC}}(X) = \mathcal{H}^{\bullet,\bullet}_{\Delta_{A}}(X).$$

Therefore, the following result holds for any type of geometric formality (Kotschick, Dolbeault, Bott–Chern, and ABC).

Theorem 5.10 *Let X be any Kähler solvmanifold. Then X is geometrically formal.*

Proof Since every map of (5.13) and (5.14) is induced by either an inclusion or a projection in cohomology, we obtain that the space of harmonic forms of X coincides with $B^{\bullet,\bullet}$, i.e.,

$$\mathcal{H}^{\bullet,\bullet}_{\square}(X) = B^{\bullet,\bullet}, \qquad \square \in \{\Delta, \Delta_{\overline{\partial}}, \Delta_{BC}, \Delta_{A}\}.$$

By Lemma 5.7, the pair $(B^{\bullet,\bullet}, \wedge)$ is an algebra, so that also the spaces

$$(\mathcal{H}^{\bullet,\bullet}_{\square}(X), \wedge), \qquad \square \in \{\Delta, \Delta_{\overline{\partial}}, \Delta_{BC}, \Delta_{A}\}$$

are an algebra, which proves that X is geometrically formal. □

As a direct consequence of Theorem 5.10, we are able to prove that the non vanishing of ABC-Massey products is an obstruction for the existence of a Kähler metric on solvmanifolds.

Corollary 5.11 *Let X be any Kähler solvmanifold. Then every ABC-Massey product vanishes.*

Proof By Theorem 5.10 and [24, Remark 4.7], X is also weakly formal in the sense of [24]. Since X satisfies the $\partial\overline{\partial}$-lemma, then X is also strongly formal. But by [24, Proposition 4.4], we can conclude that every ABC-Massey product on X vanishes. In particular, every triple ABC-Massey product on X vanishes (see also [3, Theorem 2.4]). □

6 Quadruple ABC-Massey Product

In this section, we explicitly construct two families of solvmanifolds admitting a non vanishing quadruple ABC-Massey product.

Let $G' := (\mathbb{C}^4, *')$ be the complex Lie group with operation $*' : \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$ defined by

$$\begin{aligned} &{}^t(y_1, y_2, y_3, y_4) *' {}^t(z_1, z_2, z_3, z_4) \\ &\quad :=^t \left(z_1 + y_1, e^{-y_1} z_2 + y_2, e^{y_1} z_3 + y_3, z_4 + w_4 + e^{y_1} y_2 z_3\right). \end{aligned}$$

More precisely, G' is the element 6 of the characterization of 4-dimensional complex Lie groups by Nakamura in [25, Section 6]. Let now $G = (\mathbb{C}^4, *)$ be the complex Lie group endowed with the operation $*\colon \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$ defined by

$$
{}^t(y_1, y_2, y_3, y_4) * {}^t(z_1, z_2, z_3, z_4) := {}^t\left(z_1 + y_1, e^{-y_1} z_2 + y_2, e^{y_1} z_3 + y_3, z_4 + y_4 \right.
$$
$$
\left. + \frac{1}{2} e^{y_1} y_2 z_3 - \frac{1}{2} e^{-y_1} y_3 z_2 \right).
$$

Then, one can see that G and G' are isomorphic as complex Lie groups via the invertible Lie group homomorphism $F\colon G' \to G$ given by

$$
F(z_1, z_2, z_3, z_4) := \left(z_1, z_2, z_3, z_4 - \frac{1}{2} z_2 z_3\right). \tag{6.1}
$$

From now on, we choose to use the presentation $G = (\mathbb{C}^4, *)$. Note that the operation $*$ of G can be written as

$$
\begin{aligned}
&{}^t(y_1, y_2, y_3, y_4) * {}^t(z_1, z_2, z_3, z_4) \\
&:= {}^t\left(z_1 + y_1, {}^t\left(\begin{pmatrix} e^{-y_1} & 0 \\ 0 & e^{y_1} \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}\right),\right. \\
&\qquad \left. y_4 + \frac{1}{2}\,(y_2, y_3) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{-y_1} & 0 \\ 0 & e^{y_1} \end{pmatrix} \begin{pmatrix} z_2 \\ z_3 \end{pmatrix} + z_4 \right).
\end{aligned} \tag{6.2}
$$

It is straightforward to check that a coframe of left-invariant holomorphic forms on G is given in the standard coordinates $\{z_1, z_2, z_3, z_4\}$ of $\mathbb{C}^4$ by

$$
\varphi^1 = dz_1, \quad \varphi^2 = e^{z_1} dz_2, \quad \varphi^3 = e^{-z_1} dz_3, \quad \varphi^4 = dz_4 - \frac{1}{2} z_2 dz_3 + \frac{1}{2} z_3 dz_2, \tag{6.3}
$$

and they satisfy the structure equations

$$
d\varphi^1 = 0, \qquad d\varphi^2 = \varphi^{12}, \qquad d\varphi^3 = -\varphi^{13}, \qquad d\varphi^4 = -\varphi^{23}.
$$

Note that the dual $\mathfrak{g}_{\mathbb{C}}^*$ of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of G decomposes as

$$
\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}_+^* \oplus \mathfrak{g}_-^*,
$$

where we denote by $\mathfrak{g}_+^* = \langle \varphi^1, \varphi^2, \varphi^3, \varphi^4 \rangle$ the subspace of holomorphic left-invariant 1-forms on G and by $\mathfrak{g}_-^* = \langle \varphi^{\bar{1}}, \varphi^{\bar{2}}, \varphi^{\bar{3}}, \varphi^{\bar{4}}, \rangle$ the subspace of anti-holomorphic left-invariant 1-forms on G.

The frame $\{Z_1, Z_2, Z_3, Z_4\}$ of $\mathfrak{g}_+$ dual to (6.3) is then given in holomorphic coordinates by

$$Z_1 = \frac{\partial}{\partial z_1}, \quad Z_2 = e^{-z_1}\frac{\partial}{\partial z_2} + e^{-z_1} z_3 \frac{\partial}{\partial z_4}, \quad Z_3 = e^{z_1}\frac{\partial}{\partial z_3} + e^{z_1} z_2 \frac{\partial}{\partial z_4}, \quad Z_4 = \frac{\partial}{\partial z_4},$$

and by dualizing (6.3), the only non vanishing brackets are

$$[Z_1, Z_2] = -Z_2, \quad [Z_1, Z_3] = Z_3, \quad [Z_2, Z_3] = Z_4. \tag{6.4}$$

By complex conjugation, we obtain the frame $\{\overline{Z}_1, \overline{Z}_2, \overline{Z}_3, \overline{Z}_4\}$ for $\mathfrak{g}_-$ and their brackets, so to obtain the analogous decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_+ \oplus \mathfrak{g}_-.$$

We recover the dual $\mathfrak{g}^* = \langle e^1, \dots, e^8 \rangle_{\mathbb{R}}$ of the underying real Lie algebra of G by defining the complex structure J as $Je^{2j} := e^{2j-1}$, for $j \in \{1, \dots, 4\}$, so that

$$\varphi^j =: e^{2j-1} + ie^{2j},$$

for $j \in \{1, \dots, 4\}$. In particular, the real structure equations of $\mathfrak{g}$ are

$$\begin{aligned} &de^1 = 0, && de^2 = 0, && de^3 = e^{13} - e^{24}, && de^4 = e^{14} + e^{23}, \\ &de^5 = -e^{15} + e^{26}, && de^6 = -e^{16} - e^{25}, && de^7 = -e^{35} + e^{46}, && de^8 = -e^{36} - e^{45}, \end{aligned}$$

and the real Lie algebra $\mathfrak{g}$ of G is then spanned by the left-invariant vector fields $e_1, \dots, e_8$ on G which satisfy the bracket relations

$$\begin{aligned} &[e_1, e_3] = -e_3, && [e_1, e_4] = -e_4, && [e_1, e_5] = e_5, \\ &[e_1, e_6] = e_6, && [e_2, e_3] = -e_4, && [e_2, e_4] = e_3, \\ &[e_2, e_5] = e_6, && [e_2, e_6] = -e_5, && [e_3, e_5] = e_7, \\ &[e_3, e_6] = e_8, && [e_4, e_5] = e_8, && [e_4, e_6] = -e_7. \end{aligned}$$

It follows immediately that $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \langle e_3, e_4, e_5, e_6, e_7, e_8 \rangle$ and $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1] = \langle e^7, e^8 \rangle$ and $\mathfrak{g}^3 = 0$. Hence, G is a 3-step solvable non nilpotent complex Lie group.

We point out that G has a structure of semidirect product, i.e., G can be presented as

$$G \cong \mathbb{C} \ltimes_{\phi} N,$$

where the factor $\mathbb{C}$ is simply-connected abelian (hence, nilpotent) and corresponds to the subalgebra $\mathfrak{c} = \langle Z_1 \rangle \subset \mathfrak{g}_+^*$, whereas $N = \{(0, z_2, z_3, z_4)\} \leq G$ is the nilradical of G and corresponds to the ideal $\mathfrak{n} = \langle Z_2, Z_3, Z_4 \rangle \subset \mathfrak{g}_+^*$. Note that, via the isomorphism (6.1), N is isomorphic to the 3-dimensional complex Heisenberg group i.e., $N \cong \mathbb{H}(3; \mathbb{C})$.

The semidirect product map $\phi : \mathbb{C} \to \mathrm{Aut}(N)$ is defined, according to (6.2), by

$$\phi(y_1)\,{}^t(z_2, z_3, z_4) = {}^t\,(e^{-y_1}y_2, e^{y_1}z_3, z_4),$$

for every $z_1 \in \mathbb{C}$, ${}^t(z_2, z_3, z_4) \in N$, where we have identified N with $(\mathbb{C}^3, \star)$ and

$${}^t(y_2, y_3, y_4)\star{}^t(z_2, z_3, z_4) := {}^t\,(z_2 + y_2, z_3 + y_3, z_4 + y_4 + \frac{1}{2}y_2z_3 - \frac{1}{2}z_2y_3)$$

for every ${}^t(y_2, y_3, y_4)$, ${}^t(z_2, z_3, z_4) \in \mathbb{C}^3$.

We now construct a lattice for the Lie group G. Let us fix $A \in SL(2;\mathbb{Z})$ and $e^{-\lambda}, e^{\lambda} \in \mathbb{R}$ its real eigenvalues, with $\lambda > 0$. In particular, there exists a matrix $P \in GL(2;\mathbb{R})$ such that

$$P \cdot A \cdot P^{-1} = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^{\lambda} \end{pmatrix} =: \Lambda. \tag{6.5}$$

Set $|P| := \det(P)$. Along the lines of [38, Example 3.4], we define the discrete subset Γ of $\mathbb{C}^4$ as

$$\Gamma_\mu := \left\{ \begin{pmatrix} \lambda a + i\mu b \\ P(m_1 + im_2) \\ \frac{1}{2}|P|(h + ik) \end{pmatrix} : a, b, h, k \in \mathbb{Z}, m_1, m_2 \in \mathbb{Z}^2 \right\}, \qquad \mu \in \mathbb{R}.$$

Lemma 6.1 *For $\mu = \pi$ and $\mu = \frac{\pi}{2}$, Γ_μ is a lattice of G.*

Proof First of all we prove that Γ_π is a subgroup of G. Clearly the identity $e = {}^t(0, 0, 0, 0)$ of G belongs to Γ by choosing $a = b = h = k = 0$ and $m_1 = m_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Taking $\gamma, \gamma' \in \Gamma$, then by using the formula (6.2) for $*$, we obtain

$$\gamma * \gamma' = \begin{pmatrix} \lambda a + i2\pi b \\ P(m_1 + im_2) \\ \frac{1}{2}|P|(h + ik) \end{pmatrix} * \begin{pmatrix} \lambda a' + i2\pi b' \\ P(m_1' + im_2') \\ \frac{1}{2}|P|(h' + ik') \end{pmatrix}$$
$$= \begin{pmatrix} \lambda(a + a') + i2\pi(b + b') \\ \begin{pmatrix} e^{-\lambda a - 2\pi ib} & 0 \\ 0 & e^{\lambda a + i2\pi b} \end{pmatrix} P(m_1' + im_2') + P(m_1 + im_2) \\ \frac{1}{2}|P|(h + ik) + \frac{1}{2}\,{}^t(P(m_1 + im_2)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} e^{-\lambda a - 2\pi ib} & 0 \\ 0 & e^{\lambda a + i2\pi b} \end{pmatrix} P(m_1' + im_2') + \frac{1}{2}|P|(h' + ik') \end{pmatrix}.$$

Since $e^{i2\pi b} = 1$ for every $b \in \mathbb{Z}$, and by (6.5),

$$\begin{pmatrix} e^{-\lambda a - i2\pi b} & 0 \\ 0 & e^{\lambda a + i2\pi b} \end{pmatrix} \cdot P = \begin{pmatrix} (e^{-\lambda})^a & 0 \\ 0 & (e^{\lambda})^a \end{pmatrix} \cdot P = \Lambda^a \cdot P = P \cdot A^a,$$

then

$$\begin{pmatrix} e^{-\lambda a - i2\pi b} & 0 \\ 0 & e^{\lambda a + i2\pi b} \end{pmatrix} P(m_1 + im_2) + P(m_1' + im_2')$$
$$= PA^a(m_1 + im_2) + P(m_1' + im_2') \in P(\mathbb{Z} + i\mathbb{Z})$$

and

$$\begin{aligned} &\frac{1}{2}\, {}^t(P(m_1 + im_2)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Lambda^a P(m_1' + im_2') \\ &= \frac{1}{2}\, {}^t(m_1 + im_2)\, {}^tP \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} PA^a(m_1' + im_2') \\ &= \frac{1}{2}\, {}^t(m_1 + im_2) \begin{pmatrix} 0 & |P| \\ -|P| & 0 \end{pmatrix} A^a(m_1' + im_2') \in \frac{1}{2}|P|(\mathbb{Z} + i\mathbb{Z}). \end{aligned}$$

Hence, $\gamma * \gamma' \in \Gamma$. Finally, for any given $z =^t (z_1, z_2, z_3, z_4) \in G$, the inverse with respect to $*$ is

$$z^{-1} =^t \left(-z_1, -^t\left(\begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix}\begin{pmatrix} z_2 \\ z_3 \end{pmatrix}\right), -z_4\right).$$

Then, if $\gamma = \begin{pmatrix} \lambda a + i2\pi b \\ P(m_1 + im_2) \\ \frac{1}{2}|P|(h + ik) \end{pmatrix} \in \Gamma$, its inverse with respect to $*$ is

$$\begin{aligned} \gamma^{-1} &= \begin{pmatrix} -\lambda a - i2\pi b \\ -\begin{pmatrix} e^{\lambda a + i2\pi b} & 0 \\ 0 & e^{-\lambda a - i2\pi b} \end{pmatrix} P(m_1 + im_2) \\ -\frac{1}{2}|P|(h + ik) \end{pmatrix} \\ &= \begin{pmatrix} -\lambda a - i2\pi b \\ -\Lambda^{-a} P(m_1 + im_2) \\ -\frac{1}{2}|P|(h + ik) \end{pmatrix} \begin{pmatrix} -\lambda a - i2\pi b \\ -PA^{-a}(m_1 + im_2) \\ -\frac{1}{2}|P|(h + ik) \end{pmatrix} \in \Gamma. \end{aligned}$$

Therefore Γ_π is a discrete subgroup of G and it is straightforward to prove that $M_\pi := \Gamma_\pi \backslash G$ is a compact manifold, that is, Γ_π is a lattice of G. The proof that $\Gamma_{\frac{\pi}{2}}$ is a lattice of G is similar and it is omitted. □

As a consequence, $M_\mu := \Gamma_\mu \backslash G$ is a compact complex parallelizable manifold, for $\mu \in \{\pi, \frac{\pi}{2}\}$.

Remark 6.2 The map $\pi : G \to \mathbb{C}$ defined that $\pi(z_1, \ldots, z_4) = z_1$ induces a holomorphic map

$$\pi : M_\mu \to (\lambda\mathbb{Z} + i\mu\mathbb{Z})\backslash\mathbb{C}$$

with fiber biholomorphic to the Iwasawa manifold $(H(3; \mathbb{C}) \cap GL(3; \mathbb{Z}[i])) \backslash H(3; \mathbb{C})$.

We now make us of the results in [2] to construct the subcomplex $C^{\bullet,\bullet}_{\Gamma_\mu}$ of $\mathcal{A}^{\bullet,\bullet}(M_\mu)$ which computes the Bott–Chern cohomology of M_μ. We recall that for every $g \in G$, the adjoint respresentation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ of G is defined as $\mathrm{Ad}_g := d(L_g \circ R_{g^{-1}})_e$, where $e = (0, 0, 0, 0)$ is the identity element of G. By restricting Ad to $\mathbb{C}$, for every $y_1 \in \mathbb{C}$ we have that $(L_{y_1} \circ R_{y_1^{-1}})(z_1, z_2, z_3, z_4) = (z_1, e^{-y_1} z_2, e^{y_1} z_3, z_4)$, so that

$$\mathrm{Ad}_{y_1} = d(\mathrm{Id}_{\mathbb{C}}, \phi(y_1)).$$

Hence, for any $y_1 \in \mathbb{C}$, we obtain the following expression of the semisimple part of Ad_{y_1} with respect to the frame $\{Z_1, Z_2, Z_3, Z_4\}$

$$(\mathrm{Ad}_{y_1})_s = \mathrm{diag}(1, e^{-y_1}, e^{y_1}, 1),$$

where $\alpha_1 \equiv \alpha_4 \equiv 1$, $\alpha_3(y_1) = e^{-y_1}$, and $\alpha_4(y_1) = e^{y_1}$ are characters of $\mathbb{C}$. Let us denote by the same symbols $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ their trivial extentions to characters of G.

Let us then define, as in [2, Theorem 2.20], the subcomplex of $\mathcal{A}^{0,\bullet}(M_\mu)$ given by

$$B^{\bullet}_{\Gamma_\mu} := \left\langle \left(\frac{\bar{\alpha}_I}{\alpha_I}\right) \varphi^{\bar{I}} : I = \{i_1, \ldots, i_k\} \subset \{1, \ldots 4\} \text{ such that } \left(\frac{\bar{\alpha}_I}{\alpha_I}\right)|_{\Gamma_\mu} = 1 \right\rangle,$$

where $\alpha_I := \alpha_{i_1} \cdots \alpha_{i_k}$.

6.1 Case: $\mu = \pi$

Note that $\frac{\bar{\alpha}_2}{\alpha_2}|_{\Gamma_\pi} = e^{z_1 - \overline{z_1}}|_{\Gamma_\pi} = e^{2\pi i b} = 1$ and, analogously, $\frac{\bar{\alpha}_2}{\alpha_2}|_{\Gamma_\pi} = e^{\bar{z}_1 - z_1}|_{\Gamma_\pi} = e^{-2\pi i b} = 1$. Hence, we can describe explicity $B^{\bullet}_{\Gamma_\pi}$ as

$$\begin{aligned} B^0_{\Gamma_\pi} &= \langle 1 \rangle, \qquad B^1_{\Gamma_\pi} = \langle \varphi^{\bar{1}}, e^{z_1 - \bar{z}_1} \varphi^{\bar{2}}, e^{\bar{z}_1 - z_1} \varphi^{\bar{3}}, \varphi^{\bar{4}} \rangle, \\ B^2_{\Gamma_\pi} &= \langle e^{z_1 - \bar{z}_1} \varphi^{\overline{12}}, e^{\bar{z}_1 - z_1} \varphi^{\overline{13}}, \varphi^{\overline{14}}, \varphi^{\overline{23}}, e^{z_1 - \bar{z}_1} \varphi^{\overline{24}}, e^{\bar{z}_1 - z_1} \varphi^{\overline{34}} \rangle, \\ B^3_{\Gamma_\pi} &= \langle \varphi^{\overline{123}}, e^{z_1 - \bar{z}_1} \varphi^{\overline{124}}, e^{\bar{z}_1 - z_1} \varphi^{\overline{134}}, \varphi^{\overline{234}} \rangle, \qquad B^4_{\Gamma_\pi} = \langle \varphi^{\overline{1234}} \rangle. \end{aligned}$$

6.2 Case: $\mu = \frac{\pi}{2}$

In this case, $\frac{\bar{\alpha}_2}{\alpha_2}|_{\Gamma_{\frac{\pi}{2}}} = e^{z_1 - \overline{z_1}}|_{\Gamma_{\frac{\pi}{2}}} = e^{i\pi b} \not\equiv 1$ and, analogously, $\frac{\bar{\alpha}_2}{\alpha_2}|_{\Gamma_\pi} = e^{-\pi i b} \not\equiv 1$. In particular, unlike the previous case, the function $e^{z_1 - \bar{z}_1}$ and its inverse are not defined on $M_{\frac{\pi}{2}}$. Hence, we can describe explicity $B^{\bullet}_{\Gamma_{\frac{\pi}{2}}}$ as

$$\begin{aligned} &B^0_{\Gamma_{\frac{\pi}{2}}} = \langle 1 \rangle, \quad B^1_{\Gamma_{\frac{\pi}{2}}} = \langle \varphi^{\bar{1}}, \varphi^{\bar{4}} \rangle, \quad B^2_{\Gamma_{\frac{\pi}{2}}} = \langle \varphi^{\overline{14}}, \varphi^{\overline{23}} \rangle, \\ &B^3_{\Gamma_{\frac{\pi}{2}}} = \langle \varphi^{\overline{123}}, \varphi^{\overline{234}} \rangle \quad B^4_{\Gamma_{\frac{\pi}{2}}} = \langle \varphi^{\overline{1234}} \rangle. \end{aligned}$$

Furthermore, let us define the subcomplex of $\mathcal{A}^{\bullet,\bullet}(M_\mu)$ given by

$$C^{\bullet_1,\bullet_2}_{\Gamma_\mu} := \bigwedge^{\bullet_1} \mathfrak{g}^*_+ \otimes_{\mathbb{C}} B^{\bullet_2}_{\Gamma_\mu} + \overline{B}^{\bullet_1}_{\Gamma_\mu} \otimes_{\mathbb{C}} \bigwedge^{\bullet_2} \mathfrak{g}^*_- .$$

From [2, Theorem 2.22], the inclusion

$$C^{\bullet,\bullet}_{\Gamma_\mu} \hookrightarrow \mathcal{A}^{\bullet,\bullet}(M_\mu)$$

induces the isomorphism

$$H(C^{\bullet-1,\bullet-1}_{\Gamma_\mu} \xrightarrow{\partial\overline{\partial}} C^{\bullet,\bullet}_{\Gamma_\mu} \xrightarrow{\partial+\overline{\partial}} C^{\bullet+1,\bullet}_{\Gamma_\mu} \oplus C^{\bullet,\bullet+1}_{\Gamma_\mu}) \xrightarrow{\cong} H^{\bullet,\bullet}_{BC}(M_\mu).$$

As a result, we are able to describe the Bott–Chern cohomology of M_μ for $\mu = \pi$ and $\mu = \frac{\pi}{2}$, see Tables 1 and 2.

Theorem 6.3 *The solvmanifold M_μ admits a non vanishing quadruple ABC-Massey product for $\mu = \pi$ and $\mu = \frac{\pi}{2}$.*

Proof We will exhibit explictly the non vanishing quadruple ABC-Massey product on M_π and $M_{\frac{\pi}{2}}$. The construction will not depend on the choice of the lattice Γ_π or $\Gamma_{\frac{\pi}{2}}$, so that we will use the generic notation M_μ. Let us consider the following invariant forms on M_μ

$$\alpha = \varphi^{12}, \quad \beta = \varphi^{\overline{23}}, \quad \gamma = \varphi^{\overline{13}}, \quad \delta = \varphi^{\overline{12}}.$$

By structure equations (6.3), they are all d-closed forms and with respect to any Hermitian metric g on $\Gamma\backslash G$, they are all $\partial\overline{\partial}*_g$-closed by bidegree reasons. Hence, the forms α, β, γ, and δ are all Bott–Chern harmonic and define non vanishing Bott–Chern cohomology classes

$$[\alpha] \in H^{2,0}_{BC}(M_\mu), \quad [\beta], [\gamma], [\delta] \in H^{0,2}_{BC}(M_\mu).$$

From now on, we will fix g to be the invariant diagonal Hermitian metric on (M_μ, J) with fundamental form $\omega = \frac{i}{2}\sum_{j=1}^4 \varphi^{j\overline{j}}$. Let us then consider the quadruple ABC-Massey product on M_μ

$$\mathcal{P} := \left\langle [\varphi^{12}], [\varphi^{\overline{23}}], [\varphi^{\overline{13}}], [\varphi^{\overline{12}}] \right\rangle_{ABC}.$$

We claim that $\mathcal{P}$ is a well defined non vanishing quadruple ABC-Massey product. In fact, since

$$\alpha \wedge \beta = \partial\overline{\partial}\varphi^{2\overline{4}}, \quad \beta \wedge \gamma = 0, \quad \gamma \wedge \delta = 0,$$

we can set $x := \varphi^{2\overline{4}}$, and $y := 0$, $z := 0$. Note that

$$x \wedge \gamma - \alpha \wedge y = x \wedge \gamma = \varphi^{2\overline{134}}, \quad y\delta - \beta z = 0,$$

hence, we can fix

$$\eta = 0, \quad \eta' = \varphi^{2\overline{34}}, \quad \xi = \xi' = 0.$$

Thus, by the formula (2.2), $\mathcal{P}$ is we defined and represented by the form $\varphi^{2\overline{1234}}$. Moreover, with respect to g, the form $*\varphi^{2\overline{1234}} = \varphi^{134}$ is d-closed, i.e., the form $\varphi^{2\overline{1234}}$ is Δ_A-harmonic and its Aeppli cohomology class $[\varphi^{2\overline{1234}}]_A$ is non vanishing. Furthermore, the space $H_{S^{-1}_{p,q}}$ defined by (2.3) restricts in our case to

$$H^{-1}_{S_{2,6}}(\mathcal{A}^{\bullet,\bullet}(M_\mu)) = \frac{\ker(\mathrm{pr} \circ d \colon \mathcal{A}^{1,4}(M_\mu) \to \{0\})}{\mathrm{Im}(\mathrm{pr} \circ d \colon \mathcal{A}^{0,4}(M_\mu) \oplus \mathcal{A}^{1,3}(M_\mu) \to \mathcal{A}^{1,4}(M_\mu))},$$

and hence coincides with the Aeppli cohomology space

$$H^{1,4}_A(M_\mu) = \frac{\ker(\partial\overline{\partial}\mathcal{A}^{1,4}(M_\mu) \to \{0\})}{\mathrm{Im}\,\partial|_{\mathcal{A}^{0,4}(M_\mu)} + \mathrm{Im}\,\overline{\partial}|_{\mathcal{A}^{1,3}(M_\mu)}}.$$

As a consequence, the representative $\varphi^{1\overline{1234}}$ defines a non vanishing element in $H^{-1}_{S_{2,6}}(\mathcal{A}^{\bullet,\bullet}(M_\mu))$. Furthermore, no other choice of primitives for $x \wedge \gamma$ yields the trivial class, proving that $[\varphi^{1\overline{1234}}]$ is not trivial also as a equivalence class of $H^{-1}_{S_{2,6}}(\mathcal{A}^{\bullet,\bullet}(M_\mu))$. As a result, the product $\mathcal{P}$ is a well defined non vanishing quadruple ABC-Massey product on M_μ. □

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