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Constraints from Superconformal Symmetry on Higher Point Functions

Coordinatore:
Chiar.mo Prof. Raffaella Burioni

Tutore:
Chiar.mo Prof. Carlo Meneghelli

Dottorando : Sophie Müller

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Abstract

This thesis investigates the superconformal Ward identities (SCWI) for higher-point correlation functions within the maximally symmetric four-dimensional superconformal field theory of $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. As a cornerstone of theoretical physics, $\mathcal{N} = 4$ SYM is notable for its high degree of symmetry, properties of integrability, and its duality through the AdS/CFT correspondence, providing remarkable insights also into strongly coupled physics. The constraints imposed by SCWI are essential for understanding the behavior of key objects in this theory.

Studying SCWI for higher points, particularly five-point functions and beyond, is crucial for extracting additional conformal field theory (CFT) data via the bootstrap program and for gaining deeper understandings of phenomena such as integrability, hidden symmetries and the AdS/CFT correspondence. This work develops a systematic method utilising analytic superspace to derive these higher point superconformal Ward identities for half-BPS multiplets. Although exemplified through stress-tensor multiplets, the method is adaptable and applicable to various superconformal setups. In particular, it is designed to work independently of the number of inserted operators, addressing limitations of previous techniques used for lower-point functions.

To further illustrate the adaptability of our approach, we also apply it to the six-point function of displacement multiplets on the supersymmetric Wilson line defect in $\mathcal{N} = 4$ SYM.

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Sophie Müller

Advisor : Prof. Carlo Meneghelli
Coordinator : Prof. Raffaella Burioni

The material presented in this thesis is based on two publications that are to appear or under preparation, respectively.

- C. Meneghelli, S. Müller, *Superconformal Ward Identities and Their Solutions for Five-Point Functions in $\mathcal{N} = 4$ SYM* (to appear)
- J. Barrat, C. Meneghelli, S. Müller, *Multipoint correlators on the Wilson line in $\mathcal{N} = 4$ SYM* (under preparation)

Specifically, Part II of this thesis expands upon the findings of the first paper, while Part III addresses the foundational developments of the second paper. As a co-author of each work, I have permission to incorporate these materials, in full or in part, into this dissertation.

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Contents

Part I.

Preliminaries

1. Introduction

Symmetry, as wide or as narrow as you may define its meaning, is one idea by which man through the ages has tried to comprehend and create order, beauty and perfection. *Hermann Weyl*

Modern day theoretical physics is built upon two profound frameworks, Quantum Field Theory (QFT) and General Relativity (GR).
On the one hand, there is Quantum Field Theory: a powerful mathematical framework that arose to unify classical field theory, special relativity, and quantum mechanics.
Classical field theories from Maxwell and Einstein replaced instantaneous forces, typical of Coulomb force and Newton's laws, with local interactions mediated by fields. This emphasis on locality motivated the development of a quantum field approach, providing a unification between field theory and quantum mechanics. This unification allows for instance to resolve the problem of identical particles in quantum mechanics: emerging as excitations of the same quantum fields, identical particles -such as two electrons- are truly and indistinguishably identical.
Special relativity, established in 1905, asserts that physical laws are covariant across inertial frames and that the speed of light remains constant. Integrating special relativity with quantum mechanics introduces for instance particle creation and annihilation, a core feature that can only be explained within the framework of QFT.
Accounting for these processes at the quantum level though often introduces divergences that require careful regularisation and renormalisation. Yet, despite these challenges, QFTs provide the most precise description of the microscopic world at our disposal.

The key principle in constructing QFTs is *symmetry*. Particles and forces are given as just different representations of the same underlying symmetry group.
Moreover, different theories are characterised by the different symmetry groups. For instance, Quantum Electrodynamics (QED), the first fully consistent QFT, that describes light-matter-interactions, is a gauge theory (i.e. it is invariant under

local transformations) with gauge symmetry group $U(1)$. In contrast, Quantum Chromodynamics (QCD), which describes the strong interactions between quarks, is a gauge theory with symmetry group $SU(3)$. This particular symmetry group $SU(3)$ introduces remarkable features like confinement and asymptotic freedom, ensuring that quarks, which carry charge under this group, cannot be observed as asymptotic states but only in bound states.
The theories of electromagnetism, and the strong and weak interactions are combined into the Standard Model of Particle Physics, providing a framework that not only predicts a variety of non-trivial effects but also describes the microscopic world with exceptional accuracy.

A special class of Quantum Field Theories is represented by *conformal field theories* (CFTs). These theories feature an extended spacetime symmetry group that goes beyond the Poincaré symmetry group -comprising translations and Lorentz transformations, foundational to any physically meaningful theory- to include scaling and special conformal transformations, the latter being all angle-preserving transformations.
The mathematical roots of CFTs trace back to the 1850s [1], and they were formally introduced into physics in 1910 with the realization that Maxwell's equations exhibit conformal invariance [2, 3].

CFTs find applications across a wide spectrum of theoretical physics. One notable application lies in the study of second-order phase transitions between distinct phases of matter, which are themselves characterised by symmetry properties. For instance, the crystalline structure of solids breaks the translational and rotational symmetries present in liquids and gases. The transition from one symmetry class to another defines a *phase transition.*
Approaching the critical point of a second-order phase transition, the correlation lengths diverge, resulting in fluctuations of arbitrary large size. This absence of a characteristic length scale signifies *scale invariance*, making the critical points of second order phase transitions describable by a CFT.[1]
A classic example of this phenomenon is the liquid-gas transition observed, for instance, in water. As the liquid approaches its critical point -here defined by the critical temperature-, density fluctuations increase in length scale. When these lengths exceed the wavelength of visible light, light scattering produces the observed "cloudiness" of the fluid, known as *critical opalescence.*

[1] The question of whether scale invariance extends to conformal invariance is subtle; however, in most local, unitary theories, this is the case. In all examples considered in this thesis, scale invariance is enhanced to conformal invariance.

Interestingly, the CFT describing this critical opalescence is the same CFT that governs second-order phase transitions in ferromagnetic systems: the 3d Ising model. Thus, two microscopically very different systems exhibit identical behavior and share the same CFT at the phase transition, leading to identical measurable quantities such as the critical exponents. This equivalence can be interpreted from both systems belonging to the same *universality class* under the Renormalization Group (RG) flow.
The concept of RG flow, developed for QFTs in the 1960s and 1970s [4, 5] and subsequently applied to second-order phase transitions [6, 7], describes how theories evolve as their parameters vary. By altering these parameters, theories either diverge to infinity in theory space (i.e. the couplings become arbitrarily large when following the RG flow) or converge towards a fixed point. Such a fixed point is inherently scale invariant, thus describable by a CFT.
In the context of the above outlined second-order phase transitions, RG flow can be interpreted as successively "zooming out" on microscopic degrees of freedom until the fixed point, represented by a CFT, is reached. In more general terms, this concept bridges high-energy theories (in the ultraviolet, or UV regime), encompassing for instance fundamental interactions and string theory, with low-energy theories (in the infrared, or IR regime), such as those in condensed matter physics. The fixed points of these bridges being described by CFTs highlights the prevalence of conformal symmetry across the landscape of physical theories.

The additional symmetries of the conformal group impose stringent constraints on the fundamental observables of CFTs, namely the correlation functions. Conformal symmetry fully determines the form of two- and three-point functions, while all higher-point correlators can be systematically related to these lower-point functions through the *Operator Product Expansion* (OPE). This concept, developed by Wilson and Kadanoff in 1969 [8, 9], expresses products of two local operators as a series of single local operators, thereby expressing multi-point correlators as a sum of lower-point functions. In a CFT, the OPE has a finite radius of convergence, meaning it is exact. Hence, we can interpret a CFT as solved once all two- and three-point functions -or more specifically, their *CFT data*- are specified.
It should be noted that this simplification holds uniquely in CFTs. In general QFTs, where the OPE is not exact but approximate, a full determination of the theory would still require calculating all correlators.

This realisation has inspired an entire new research approach to compute these correlators in a CFT using symmetry and consistency conditions alone, the *conformal bootstrap program*.
Unlike traditional perturbative methods, which are effective only in limited regimes

-such as weak coupling or large charge- and rely on a Lagrangian formulation, the conformal bootstrap provides a non-perturbative method intended to be valid across all regimes of the theory and grounded solely in fundamental principles. By leveraging the Operator Product Expansion (OPE) and the constraints on two- and three-point functions imposed by conformal symmetry, the bootstrap seeks to "solve" CFTs with minimal external assumptions, relying instead on intrinsic symmetry properties and consistency requirements such as unitarity and OPE associativity.
Initially formulated in the 1970s for two-dimensional CFTs [10, 11], where the conformal algebra corresponds to the infinite-dimensional Virasoro algebra, the bootstrap was successfully applied to solve 2d minimal models, a series of CFTs describing the critical points of the 2d Ising model, through purely symmetry-based arguments [12].
The first significant breakthrough in higher-dimensional CFTs occurred in 2008, when numerical bounds on scaling dimensions in a four-dimensional CFT were computed [13]. Since then, numerous "islands" and bounds on CFT data have been established, culminating in the determination of some of the most precise and numerically accurate critical exponents for the 3d Ising model [14–16].
While the numerical bootstrap provides highly accurate bounds on CFT data, recent developments in the *analytic bootstrap*, initiated in 2012 [17, 18], aim to uncover more universal results for general CFTs. Following this initial framework of the large-spin bootstrap, various other methods, such as the large-charge bootstrap [19, 20], Mellin-space bootstrap [21–23], and more, have emerged. For an overview of these methodologies, see [24, 25].

Our journey from the development of Quantum Field Theory (QFT) to the modern conformal bootstrap program reveals not only remarkable advancements in theoretical physics, but also a recurring theme: many of these techniques remain deeply rooted in symmetry principles.

However, despite the remarkable outlined successes of QFTs, and particularly Conformal Field Theories (CFTs), they do not yet provide a definitive description of all fundamental forces in nature. This brings us to the second pillar of modern theoretical physics: General Relativity (GR).
Formulated by Einstein in 1915, GR remains the most accurate theory describing the gravitational force -the weakest but longest-ranging among the four fundamental interactions. GR revolutionised our understanding of space and time by merging them into a unified concept of spacetime and relating its curvature directly to the energy and momentum of matter. Like QFT, GR is formulated as a local field theory, with gravitational interactions being mediated by a field. Fur-

thermore, it is as well grounded in symmetry principles: GR is invariant under diffeomorphisms, meaning that the theory's laws are independent of a specific labelling of spacetime points.

Thus, we have two foundational pillars in theoretical physics: QFTs, with the special cases of CFTs, which describe the strong and electroweak forces at atomic scales, and GR, which describes the gravitational force at larger distances where significant masses are involved. Both theories predict their respective domains with unparalleled accuracy. However, one of the major unresolved challenges in theoretical physics is to describe scenarios involving large masses concentrated in small regions of spacetime, such as in the vicinity of black hole singularities or in the early moments following the Big Bang. These situations require the simultaneous consideration of both quantum and gravitational effects. A naive combination of QFT and GR, such as attempting to quantise GR, is not possible as GR is non-renormalisable, meaning that we cannot remove the divergencies emerging within a quantum field theoretic framework.
Decades of research have yielded several approaches to address these challenges, with one of the most promising being *superstring theory.* String theory, formulated in the late 1960s, naturally encompasses both the gravitational theory of Einstein at ordinary scales and a modified version at high energies or small distances, as well as gauge theories of the type used to construct the Standard Model. However, to make string theory consistent and to incorporate fermions, a new class of symmetry, *supersymmetry*, was introduced in this context [26], leading to superstring theory.
Supersymmetry links the bosons and fermions of the theory, thus treating the equations governing forces and matter on equal terms. It can also be incorporated into field theories, resulting in *superconformal field theories* (SCFTs).

There are various superconformal field theories across different dimensions, with one of the foremost examples being $\mathcal{N} = 4$ *Super Yang-Mills (SYM) theory*, the maximally symmetric SCFT in four dimensions [27, 28]. $\mathcal{N} = 4$ SYM is a remarkable theory with many unique features. It is UV-complete, meaning it lacks the divergences typical of other QFTs, and it has a vanishing β-function, so that its superconformal symmetry survives at the quantum level. This symmetry, the highest achievable in four dimensions without including gravity, imposes stringent constraints on the theory's elements, leading to highly structured, computable mathematical forms.

Interest in $\mathcal{N} = 4$ SYM was raised tremendously by one of modern theoretical physics' most exciting conjectures: Maldacena's gauge-gravity duality [29]. Inspired by 't Hooft's work and the holographic principle (which has emerged from black hole entropy being described by the surface area, rather than the volume), this duality posits a correspondence between a gravity theory in $d+1$ dimensions and a theory without gravity on its d-dimensional asymptotic boundary. Known also as the AdS/CFT correspondence, this duality often involves a CFT on the boundary that reflects properties of a dual string theory on an Anti-de Sitter (AdS) background, underscoring once again the special role of conformal field theories in theoretical physics. Notably, the only known UV-complete examples of this duality involve the higher symmetric superconformal theories. The most successful and well-studied case of this correspondence relates $\mathcal{N} = 4$ SYM to type IIB superstring theory on $AdS_5 \times S^5$, as formulated in [29].

One particularly powerful feature of the AdS/CFT correspondence is its role as a *weak-strong duality*. When fields on the CFT side are weakly coupled, the corresponding fields in the dual string theory are strongly coupled, and vice versa. This duality allows to apply perturbative methods on one side to gain insights into the strongly coupled regime of the dual theory -an otherwise notoriously challenging domain to analyse. This utility is a major factor driving the extensive studies and breakthroughs achieved in $\mathcal{N} = 4$ SYM, standing out as the best understood example of AdS/CFT.

However, many challenges are still present such as intermediate coupling regimes that remain challenging to study, necessitating non-perturbative methods for a full understanding of these theories.
As we have seen, conformal symmetry imposes strong constraints on observables, inspiring the conformal bootstrap program. Given that superconformal field theories possess even greater symmetry, thereby imposing stronger constraints, the natural question arises of extending the conformal bootstrap to a superconformal bootstrap program? With $\mathcal{N} = 4$ SYM serving as the prime candidate for such an endeavor, efforts in this direction have been initiated for instance in [30–34], mostly for four-point correlators. Like all attempts of this nature, these efforts are grounded in one essential question:

How does the underlying symmetry group constrain the observables of the theory?

The constraints emerging from this analysis are referred to as *Superconformal Ward Identities (SCWI)*, and are the primary focus of this thesis. Deriving these identities not only serves as a foundational step in potential bootstrap programs (as did e.g. the works of [35–37] for the above mentioned attempts in $\mathcal{N} = 4$ SYM), but also provides valuable insights on its own.

Examining the implications of superconformal symmetry on the correlation functions in $\mathcal{N} = 4$ SYM has yielded, for instance, numerous non-renormalization theorems concerning three- and four-point functions of so-called short multiplets -multiplets that trivialise under a subset of the fermionic algebra. For relevant studies, see [38–44].

These short multiplets have remained central to much of the research on superconformal Ward identities. Specifically, correlators with up to four multiplet insertions are well-understood, and SCWI have been derived through various approaches (see [35, 36, 43, 45, 46] for some notable examples).

However, for five-point correlators involving half-BPS multiplets, substantially less is known. The objective of this thesis is to further deepening this understanding by deriving the corresponding superconformal Ward identities.

Defects

The understanding of (S)CFTs can be broadened by introducing defects into the theory that break the underlying symmetries in a controlled manner. In the context of conformal field theories, we focus particularly on *conformal defects*, which still preserve part of the original conformal symmetry.

Consider a conformal theory in d dimensions, which is described by the symmetry group $SO(d, 2)$. When a p-dimensional defect is introduced, this symmetry is partially broken as follows:

$$SO(d, 2) \to SO(p, 2) \times SO(q), \quad q \equiv d - p.$$

This p-dimensional defect retains a p-dimensional conformal field theory on the defect itself, even though it lacks a local stress tensor and thus corresponds to a non-local CFT. Furthermore, the defect preserves a q-dimensional rotational symmetry around it, referred to as *transverse spin* from the defect perspective. Here, q is known as the codimension, and allows for a characterisation of defects in terms of its dimension and codimension.

Defects with codimension one are either *boundaries*, if they exhibit a CFT on one side, or *interfaces*, if there is a non-trivial CFT on each side of the interface. These defects are notable for not having a transverse spin, i.e. rotations around the defects are absent. The foundational work on such defects within conformal field theories can be traced back to Cardy [47, 48].
Higher codimension defects, which allow for non-trivial transverse spin, can further be classified by their dimensionality. For example, $p = 1$-dimensional defects, termed *line defects*, may manifest either as time-extended point defects or as spatial lines, which may either remain open or form closed *loops*. A prominent example of this category is the *Wilson line* or *Wilson loop*, originally introduced by [49]. Similarly, $p = 2$-dimensional defects are referred to as *surface defects*, and so forth.

Given their considerable variety, including also non-conformal defects, they find application across a plethora of physical setups. Being extended objects, they serve as probes in gauge theories, providing insights into the properties of the associated gauge groups, such as those related to confinement [49]. They also facilitate the study of generalized symmetries (e.g., see [50]) and contribute additional structure in the AdS/CFT correspondence (e.g., see [51]). Moreover, defects hold relevance in experimental contexts, such as magnetic line defects in the Ising model, which are used to model magnetic impurities [52, 53].

From a bootstrap perspective, defects introduce novel CFT data and probe unique dynamics, thereby enhancing the overall understanding of the theory. In addition to modified bulk CFT data -such as non-vanishing one-point functions in the presence of a defect- one can analyse correlation functions of operators inserted solely on the defect -which retain information about the original theory, for example, through transverse spin. Interactions between bulk and defect operators provide another rich area for exploration.
This interest in incorporating defects into the bootstrap framework has motivated substantial research, culminating in the development of the *defect bootstrap*. Foundational works and advances in this field for the various types of defects include [54–58]. Since these early contributions, research on the defect bootstrap has expanded significantly.

Conformal defects can also be extended to include supersymmetry, yielding *superconformal defects* that preserve portions of the corresponding bulk superconformal algebra.
The various types of superconformal line defects have been classified in [59]. A particularly notable example is the supersymmetric Wilson line in $\mathcal{N} = 4$ Super

Yang-Mills theory [60]. This line defect is dual to a two-dimensional string worldsheet in AdS that terminates on the Wilson line in the boundary CFT [51]. Its high degree of symmetry, coupled with its one-dimensional nature, makes this defect especially well-suited to study it based on symmetry arguments only, as performed in this thesis.

Outline of the thesis

The aim of this thesis is to derive the constraints imposed by superconformal symmetry on higher-point functions of half-BPS multiplets, known as Superconformal Ward identities (SCWI). We will primarily focus on the five-point function of stress-tensor multiplets in $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. However, the approach developed herein is adaptable and can be applied to various setups, as will be demonstrated through toy models and the application of the same methodology to the six-point function of displacement multiplets on the supersymmetric line defect in $\mathcal{N} = 4$ SYM.

This thesis is organized as follows:
Chapter 2 introduces the foundational aspects necessary for understanding the analysis presented later. We discuss key features of four-dimensional superconformal theories, focusing on the principles of superconformal symmetry and its implications for correlation functions. We will particularly highlight the role of analytic superspace, our primary tool for deriving SCWI.
Part II represents the main body of this thesis, centering on the SCWI for five-point function of the stress-tensor multiplet in $\mathcal{N} = 4$ SYM. In Chapter 3, we develop a systematic method for deriving the SCWI. We detail the process of combining the constraints of superconformal invariance on the correlator with constraints imposed by general group-theoretic properties of the half-BPS multiplets.
Along the way, we will derive the SCWI also for other cases, such as a five-point function in the chiral algebra $\mathfrak{psu}(1,1|2)$ and the four-point function of $\mathfrak{psu}(2,2|4)$ to build an understanding in a gradual way.
In Chapter 4, we analyse the derived SCWI. While the SCWI for the simpler cases, such as the one-dimensional setup of $\mathfrak{psu}(1,1|2)$ and the four-point functions, can be solved straightforwardly, the five-point functions present greater complexity. Nonetheless, we provide evidence of the correctness of the derived equations and make initial strides toward their simplification.
Part III explores other setups and further directions of the developed methodology.
In Chapter 5, we illustrate the adaptability of our method by applying it to the six-point function of displacement multiplets on the supersymmetric Wilson line defect in $\mathcal{N} = 4$ SYM.
The thesis concludes with an outlook in Chapter 6 on potential future directions.

2. $\mathcal{N} = 4$ SYM

We begin by introducing the central theoretical framework of our study: $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, with particular focus on its underlying symmetry algebra, $\mathfrak{psu}(2, 2|4)$. We will establish the necessary foundational concepts required for the analyses presented in this thesis.

In Section 2.1, we present the complete superconformal algebra $\mathfrak{psu}(2, 2|4)$, which serves as the foundation for our discussion, before exploring the representations of this algebra in Section 2.2. We first review conformal representations and the Cartan classification before discussing the full superconformal representations.
The resulting objects, along with their correlation functions, will be studied within the framework of *analytic superspace*, which is introduced in Section 2.3.
In Section 2.4, we then turn to those correlation functions on analytic superspace, particularly examining the kinematic properties of correlation functions involving up to four operator insertions.
Further insights into the structure of these correlation functions can be obtained via the (Super)Conformal Bootstrap approach, whose key concepts are outlined in Section 2.5.
Finally, in Section 2.6, we introduce the superconformal algebra $\mathfrak{psu}(1, 1|2)$, which, as will be demonstrated, shares a close connection with the $\mathfrak{psu}(2, 2|4)$ algebra and serves as a useful toy model for further exploration.

It should be noted that the first two sections, 2.1 and 2.2, intend to be a summary of well-established results that are extensively covered in several detailed and comprehensive lecture notes and reviews. In particular, these sections closely follow the material presented in [61–66], to which the reader is referred for further details.

2.1. Superconformal Algebra

We begin by introducing the main theory - *the main stage* - of this thesis, the maximally symmetric superconformal field theory (SCFT) in four dimensions: $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory.

This theory comprises the following field content:

$$\begin{aligned} &\{\Phi^{\mathrm{I}},\, \Psi_{\alpha i},\, \bar{\Psi}_{i\dot{\alpha}},\, F_{\mu\nu}\}; \\ \text{where } &\mathrm{I} = 1, \ldots, 6,\ i = 1, \ldots, 4, \\ &\alpha, \dot{\alpha} = 1, 2 \quad (\mathfrak{sl}(2) - \text{indices}), \\ &\mu = 0, 1, 2, 3 \quad (\mathfrak{so}(4) - \text{indices}), \end{aligned} \tag{2.1}$$

which consists of six scalar fields, Φ^{I}, transforming in the fundamental representation of $SO(6) \sim SU(4)$, four complex spinors $\Psi_{\alpha i}$ and their conjugates $\bar{\Psi}_{i\dot{\alpha}}$, and the field strength tensor $F_{\mu\nu}$.

The dynamics of these fields are described by the Lagrangian:

$$\begin{aligned} \mathcal{L} =&\mathrm{tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mathcal{D}^{\mu}\Phi^{\mathrm{I}}\mathcal{D}_{\mu}\Phi_{\mathrm{I}} + \bar{\Psi}^{i}{}_{\dot{\alpha}}(\sigma_{\mu})^{\dot{\alpha}\beta}\mathcal{D}^{\mu}\Psi_{\beta i} - \frac{g^2}{4}[\Phi^{\mathrm{I}}, \Phi^{\mathrm{J}}][\Phi_{\mathrm{I}}, \Phi_{\mathrm{J}}]\right. \\ &\left. -\frac{ig}{2}\Psi_{\alpha i}(\sigma_{\mathrm{I}})^{ij}\epsilon^{\alpha\beta}[\Phi^{\mathrm{I}}, \Psi_{\beta j}] - \frac{ig}{2}\bar{\Psi}^{i}{}_{\dot{\alpha}}(\sigma^{\mathrm{I}})_{ij}\epsilon^{\dot{\alpha}\dot{\beta}}[\Phi^{\mathrm{I}}, \bar{\Psi}^{j}_{\dot{\beta}}]\right). \end{aligned} \tag{2.2}$$

Here, $\mathcal{D}_{\mu}$ denotes the covariant derivative involving the gauge field, $\epsilon^{\alpha\beta}$ is the antisymmetric $\mathfrak{su}(2)$-tensor, and the σ-matrices are related to the Dirac matrices. The Lagrangian incorporates the standard kinetic terms for each field, along with a quartic scalar coupling and a Yukawa interaction between the scalar fields and the spinors.[1]

By construction, this Lagrangian is invariant under the $\mathcal{N} = 4$ Super-Poincaré group, which is the supersymmetric extension of the Poincaré group, consisting of translations and Lorentz transformations.
Moreover, with the field dimensions given by

$$[\Phi] = [\mathcal{D}_{\mu}] = 1,\ [\Psi_{\alpha}] = \frac{3}{2},\ [F_{\mu\nu}] = 2, \tag{2.3}$$

[1] For a more detailed treatment of the field-theoretical content presented here, the reader is referred to [64].

it follows that the Lagrangian has dimension $[\mathcal{L}] = 4$, ensuring that the action is classically scale-invariant. As is the case for any local, unitary theory, this scale invariance is enhanced to full conformal invariance. The combination of $\mathcal{N} = 4$ supersymmetry and conformal symmetry leads to $\mathcal{N} = 4$ superconformal symmetry, as evidenced by the corresponding algebra (see Equation 2.10 below). This superconformal symmetry is described by the group $PSU(2, 2|4)$.
In the case of $\mathcal{N} = 4$ SYM theory, this full superconformal symmetry is exact also at the quantum level (i.e. the β-function vanishes at any loop order).

Having established the full symmetry group of the $\mathcal{N} = 4$ SYM Lagrangian, we will now follow the bootstrap philosophy and forget about a specific Lagrangian description alltogether.

Indeed, the subsequent analysis will focus solely on the group $PSU(2, 2|4)$, or rather its algebra $\mathfrak{psu}(2, 2|4)$. Thus, the results derived in this thesis are in fact valid for any maximally symmetric superconformal field theory in 4 dimensions, irrespective of the gauge group.

2.1.1. $\mathfrak{psu}(2, 2|4)$

The superconformal algebra $\mathfrak{psu}(2, 2|4)$ consists of the following subalgebras:

- The conformal algebra in four dimensions, $\mathfrak{so}(4, 2) \cong \mathfrak{su}(2, 2)$.
  This algebra consists of the generators for *translations* P_μ, *Lorentz transformations* or *rotations* $M_{\mu\nu}$, *dilatations* D, and *special conformal transformations* K_μ. The special conformal transformations can be viewed as an inversion, followed by a translation, and another inversion.
- The bosonic subalgebra representing the internal or *R-symmetry transformations*, $\mathfrak{su}(4)_R$, which is generated by the elements $R^i{}_j$, where $i, j = 1, \dots, \mathcal{N}$, with $\mathcal{N} = 4$ in this case.
- Lastly, there are 32 fermionic generators, consisting of the 16 supersymmetric translations $Q^i{}_\alpha, \bar{Q}_{i\dot{\alpha}}$ where $i = 1, \dots, 4$ and $\alpha, \dot{\alpha} = 1, 2$, and the 16 superconformal generators $S_i{}^\alpha, \bar{S}^{i\dot{\alpha}}$.

The generators of $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(4)_R$ commute with each other, allowing us to organize the full superalgebra into the following block structure:

$$\begin{pmatrix} P_\mu, M_{\mu\nu}, K_\mu, D & Q^i{}_\alpha, \bar{S}^{i\dot{\alpha}} \\ \bar{Q}_{i\dot{\alpha}}, S_i{}^\alpha & R^i{}_j \end{pmatrix}. \tag{2.4}$$

The commutation relations between these generators, i.e., the full superalgebra, are as follows, following the conventions of [66]. To begin, the superalgebra includes the standard conformal algebra, given by

$$
\begin{aligned}
[M_{\mu\nu}, P_\rho] &= i\left(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\right), \\
[M_{\mu\nu}, K_\rho] &= i\left(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu\right), \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}\right), \\
[D, P_\mu] &= iP_\mu, \\
[D, K_\mu] &= -iK_\mu, \\
[K_\mu, P_\nu] &= -2iM_{\mu\nu} - 2i\eta_{\mu\nu}D,
\end{aligned}
\tag{2.5}
$$

where any commutators not explicitly mentioned are assumed to vanish.

The above generators of $\mathfrak{su}(2,2)$ can be rewritten as elements of the algebra M_{MN}, with M,N $= 0, 1, \ldots, d+1 = 5$, as follows [66]

$$
M_{\mathrm{MN}} = \begin{pmatrix} M_{\mu\nu} & -\frac{1}{2}(P_\mu - K_\mu) & -\frac{1}{2}(P_\mu + K_\mu) \\ \frac{1}{2}(P_\nu - K_\nu) & 0 & D \\ \frac{1}{2}(P_\mu + K_\mu) & -D & 0 \end{pmatrix}. \tag{2.6}
$$

With this convention, the commutation relations in Eqn. (2.5) can be rewritten in the form of the standard Lorentz algebra in $d+2$ dimensions, $\mathfrak{so}(d,2)$. In four dimensions, we thus indeed recover $\mathfrak{so}(4,2)$.

One way to derive the conformal algebra in Eqn. (2.5) is through explicit computation from the representation of the generators acting as differential operators on Minkowski spacetime coordinates, given by [63]

$$
\begin{aligned}
P_\mu &= -i\partial_\mu, \\
M_{\mu\nu} &= -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right), \\
D &= -ix^\mu\partial_\mu, \\
K_\mu &= -i\left(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu\right).
\end{aligned}
\tag{2.7}
$$

Thus far, the conformal group and Lorentz group have been described using vector indices $\mathfrak{so}(4)$, i.e., $\mu = 0, 1, 2, 3$. However, when extending to the supersymmetric case with spinor fields (described by $\mathfrak{su}(2)$ indices $\alpha, \dot{\alpha}$), it is often more convenient to express Minkowski spacetime in spinor notation: $x^{\alpha\dot{\alpha}} = (\sigma_\mu x^\mu)^{\alpha\dot{\alpha}}$, exploiting the fact that $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The translation of the conformal generators into $\mathfrak{su}(2)$ notation is as follows [66]

$$
\begin{aligned}
P_{\alpha\dot{\alpha}} &= (\sigma^{\mu})_{\alpha\dot{\alpha}} P_{\mu}, \\
K^{\alpha\dot{\alpha}} &= (\bar{\sigma}^{\mu})^{\alpha\dot{\alpha}} K_{\mu}, \\
M_{\alpha}{}^{\beta} &= -\frac{1}{4} i (\sigma^{\mu}\bar{\sigma}^{\nu})_{\alpha}{}^{\beta} M_{\mu\nu}, \\
\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} &= -\frac{1}{4} i (\bar{\sigma}^{\mu}\sigma^{\nu})^{\dot{\alpha}}{}_{\dot{\beta}} M_{\mu\nu},
\end{aligned}
\tag{2.8}
$$

where σ^{μ} are the standard Pauli matrices.[2]

In addition to the ordinary conformal algebra 2.5, the completion to the full superconformal algebra is given by [66]

$$
\begin{aligned}
&\{Q^{i}{}_{\alpha}, \bar{Q}_{j\dot{\alpha}}\} = 2\delta^{i}_{j} P_{\alpha\dot{\alpha}}, \quad \{Q^{i}{}_{\alpha}, Q^{j}{}_{\beta}\} = \{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\} = 0 \\
&\{\bar{S}^{i\dot{\alpha}}, S_{j}{}^{\alpha}\} = 2\delta^{i}_{j} K_{\alpha\dot{\alpha}}, \quad \{\bar{S}^{i\dot{\alpha}}, \bar{S}^{j\dot{\beta}}\} = \{S_{i}{}^{\alpha}, S_{j}{}^{\beta}\} = 0 \\
&\{Q^{i}{}_{\alpha}, \bar{S}^{j\dot{\alpha}}\} = 0, \; \{S_{i}{}^{\alpha}, \bar{Q}_{j\dot{\alpha}}\} = 0
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
[M_{\alpha}{}^{\beta}, Q^{i}{}_{\gamma}] &= \delta^{\beta}_{\gamma} Q^{i}{}_{\alpha} - \frac{1}{2}\delta^{\beta}_{\alpha} Q^{i}{}_{\gamma}, \; [M_{\alpha}{}^{\beta}, S_{i}{}^{\gamma}] = -\delta^{\gamma}_{\alpha} S_{i}{}^{\beta} + \frac{1}{2}\delta^{\beta}_{\alpha} S_{i}{}^{\gamma} \\
[\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \bar{Q}_{i\dot{\gamma}}] &= -\delta^{\dot{\alpha}}_{\dot{\gamma}} \bar{Q}_{i\dot{\beta}} + \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} \bar{Q}_{i\dot{\gamma}}, \; [\bar{M}^{\dot{\alpha}}{}_{\dot{\beta}}, \bar{S}^{i\dot{\gamma}}] = \delta^{\dot{\gamma}}_{\dot{\beta}} \bar{S}^{i\dot{\alpha}} - \frac{1}{2}\delta^{\dot{\alpha}}_{\dot{\beta}} \bar{S}^{i\dot{\gamma}} \\
[D, Q^{i}{}_{\alpha}] &= \frac{1}{2} i Q^{i}{}_{\alpha}, \; [D, \bar{Q}_{i\alpha}] = \frac{1}{2} i \bar{Q}_{i\alpha} \\
[D, S_{i}{}^{\alpha}] &= -\frac{1}{2} i S_{i}{}^{\alpha}, \; [D, \bar{S}^{i\alpha}] = -\frac{1}{2} i \bar{S}^{i\dot{\alpha}} \\
[K_{\mu}, Q^{i}{}_{\alpha}] &= -(\sigma_{\mu})_{\alpha\dot{\alpha}} \bar{S}^{i\dot{\alpha}}, \; [K_{\mu}, \bar{Q}_{i\alpha}] = S_{i}{}^{\alpha} (\sigma_{\mu})_{\alpha\dot{\alpha}} \\
[P_{\mu}, \bar{S}^{i\dot{\alpha}}] &= -(\sigma_{\mu})^{\dot{\alpha}\alpha} Q^{i}{}_{\alpha}, \; [P_{\mu}, S_{i}{}^{\alpha}] = \bar{Q}_{i\alpha} (\sigma_{\mu})^{\dot{\alpha}\alpha}
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
[R^{i}_{j}, R^{k}{}_{l}] &= \delta^{k}_{j} R^{i}{}_{l} - \delta^{i}_{l} R^{k}{}_{j} \\
[R^{i}_{j}, Q^{k}{}_{\alpha}] &= \delta^{k}_{j} Q^{i}{}_{\alpha} - \frac{1}{4}\delta^{i}_{j} Q^{k}{}_{\alpha}, \; [R^{i}_{j}, S_{k}{}^{\alpha}] = -\delta^{k}_{j} S_{i}{}^{\alpha} + \frac{1}{4}\delta^{i}_{j} S_{k}{}^{\alpha} \\
[R^{i}_{j}, \bar{Q}_{k\alpha}] &= -\delta^{i}_{k} \bar{Q}_{j\dot{\alpha}} + \frac{1}{4}\delta^{i}_{j} \bar{Q}_{k\alpha}, \; [R^{i}_{j}, \bar{S}^{k\dot{\alpha}}] = \delta^{k}_{j} \bar{S}^{i\dot{\alpha}} - \frac{1}{4}\delta^{i}_{j} \bar{S}^{k\dot{\alpha}}.
\end{aligned}
\tag{2.11}
$$

[2]In this chapter, which is intended as an overview of the fundamental concepts, both $\mathfrak{so}(4)$ and $\mathfrak{su}(2)$ indices are used to describe Minkowski spacetime and the generators. The choice of notation depends on the particular intention of each section. In subsequent parts of the thesis, only spinor notation will be used. For the conventions employed, see Appendix A.1.

Note that the structure of this algebra justifies the existence of the superconformal generators $S_i{}^{\alpha}$ and $\bar{S}^{i\dot{\alpha}}$. Their inclusion arises naturally due to the non-commutative behavior between the conformal generators K_μ and the supersymmetric translations $Q^i{}_{\alpha}$ and $\bar{Q}_{i\dot{\alpha}}$.

Additionally, from the commutation relations involving the R-symmetry generators, shown in Eqn. (2.11), it follows that for $\mathcal{N} = 4$, the trace $R^i{}_i$ (for $i = 1, \ldots, 4$) is central, i.e., it commutes with all other generators. This allows us, without loss of generality, to set $R^i{}_i = 0$, reducing the R-symmetry algebra $\mathfrak{u}(\mathcal{N})$ to $\mathfrak{su}(4)$ in the case of $\mathcal{N} = 4$. Consequently, this reduction leads to the irreducible superconformal algebra $\mathfrak{psu}(2, 2|4)$, as opposed to $\mathfrak{su}(2, 2|\mathcal{N})$ for a general $\mathcal{N}$.

In this work, we will only consider unitary theories under $\mathfrak{psu}(2, 2|4)$. Therefore, it is necessary to establish the appropriate Hermiticity conditions. Following the conventions of [66], the Hermiticity conditions are given by:

$$\begin{aligned} &D^\dagger = D, \quad P^\dagger_{\alpha\dot{\alpha}} = K_{\alpha\dot{\alpha}}, \quad (M_\alpha{}^\beta)^\dagger = \bar{M}^{\dot{\beta}}{}_{\dot{\alpha}}, \\ &(Q^i{}_\alpha)^\dagger = \bar{Q}_{i\dot{\alpha}}, \quad (S_i{}^\alpha)^\dagger = \bar{S}^{i\dot{\alpha}}, \quad (R^i{}_j)^\dagger = R^j{}_i. \end{aligned} \tag{2.12}$$

2.2. Representations

In this section, we review the classification and characterisation of the possible representations of the algebra $\mathfrak{psu}(2, 2|4)$.

The discussion is divided into three parts. First, we will review the representations of the ordinary conformal algebra $\mathfrak{su}(2, 2)$. The presented discussion shall be viewed as a concise summary, with more comprehensive treatments available in [61–63].

Next, we will briefly discuss the representation theory of $\mathfrak{su}(4)$, or simple Lie algebras in general. Our review will closely follow [65, 66].

Finally, we present the representation theory of the full superconformal algebra $\mathfrak{psu}(2, 2|4)$, based on the works of [66] and [67].

2.2.1. Representations of the conformal group

For simplicity and brevity, we will consider the representations of the conformal group in vector notation, using spacetime indices $\mu = 0, \ldots, 3$.

Starting with the conformal algebra in four dimensions, $\mathfrak{su}(2,2)$, the action of the generators on local operators is given by

$$\begin{aligned} [P_\mu, \mathcal{O}(x)] &= i\partial_\mu \mathcal{O}(x), \\ [M_{\mu\nu}, \mathcal{O}(x)] &= i(x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{O}(x) + \mathcal{O}(x) s_{\mu\nu}, \\ [D, \mathcal{O}(x)] &= i(x^\mu \partial_\mu + \Delta)\mathcal{O}(x), \\ [K_\mu, \mathcal{O}(x)] &= i\left(x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu - 2\Delta x_\mu\right)\mathcal{O}(x) - 2\mathcal{O}(x) s_{\mu\nu} x^\nu, \end{aligned} \tag{2.13}$$

where Δ is the scaling dimension of the operator $\mathcal{O}(x)$ inserted at spacetime point x, and $s_{\mu\nu}$ is the finite-dimensional matrix specifying the representation under Lorentz transformations $\mathfrak{so}(4)$. Each operator is thus characterized by its quantum numbers (Δ, s).

The above relations can be derived from the classical field transformations, as they remain valid through any quantisation scheme. One particularly useful quantisation method for CFTs, which illustrates how operators are characterised by (Δ, s), is *radial quantisation*.

Quantisation generally involves choosing a suitable foliation of spacetime and associating a Hilbert space to each leaf, where suitable means respecting the generators of the underlying symmetry group. In a conformal field theory, this foliation is over spheres of varying radii, rather than time slices as in Poincaré-invariant theories. The Hamiltonian, which can be viewed as translating between different Hilbert spaces, then corresponds to the dilatation operator D, whose eigenvalue is the scaling dimension Δ.

In radial quantisation, there is a one-to-one correspondence between local operators and states. Inserting an operator $\mathcal{O}_\Delta(0)$ of scaling dimension Δ at the origin creates an eigenstate of the dilatation operator:

$$D\left|\Delta\right\rangle \equiv D\left(\mathcal{O}_\Delta(0)\left|0\right\rangle\right) = i\Delta\left|\Delta\right\rangle, \tag{2.14}$$

where the action of D on $\mathcal{O}_\Delta(0)$ is understood as in Eq. 2.13, and $|0\rangle$ denotes the vacuum state, which is invariant under all transformations.

States at other points in spacetime can similarly be created by

$$\left|\Psi\right\rangle = e^{iP_\mu x^\mu}\mathcal{O}_\Delta(0)e^{-iP_\mu x^\mu}\left|0\right\rangle. \tag{2.15}$$

Note that this is not an eigenstate of D, but a superposition of states with different eigenvalues.

Conversely, given a state, the corresponding operator can always be extracted. This establishes a synonymy between local operators and states. For more detailed discussions on *radial quantisation* and the *operator-state correspondence*, we refer to the aforementioned lecture notes and reviews, such as [61, 62].

We then denote an eigenstates of D in a spin representation s as

$$|\Delta, s\rangle \,. \tag{2.16}$$

The action of the other generators on these states can then be studied.
While the dilatation operator D and Lorentz transformations $M_{\mu\nu}$ commute, translations and special conformal transformations raise and lower the scaling dimension by one unit, respectively, as

$$DP_\mu |\Delta, s\rangle = i(\Delta + 1)P_\mu |\Delta, s\rangle \,, \tag{2.17}$$
$$DK_\mu |\Delta, s\rangle = i(\Delta - 1)K_\mu |\Delta, s\rangle \,. \tag{2.18}$$

Thus, towers of states can be constructed by acting with P_μ and K_μ, which act as ladder operators.

In any unitary theory, energies must be bounded from below. Since in radial quantisation, energy corresponds to the scaling dimension, there must exist a state with the lowest scaling dimension, for which

$$K_\mu |\Delta, s\rangle_{\text{prim}} = 0. \tag{2.19}$$

This condition defines a *conformal primary.*
States of higher scaling dimensions are then generated by acting successively with the raising operator P_μ on the primary state:

$$\prod_n (P_{\mu_n})^n |\Delta, s\rangle_{\text{prim}} \,, \tag{2.20}$$

giving the so-called *conformal descendants* with scaling dimensions $\Delta + n$, where $n \in \mathbb{Z}_{\geq 0}$.
Representations of the conformal algebra $\mathfrak{su}(2,2)$ therefore organise into multiplets, characterised by the respective conformal primary of scaling dimension Δ and spin representation s.[3]

[3] This creates what is known as an infinite highest-weight module. More details on modules, particularly Verma modules, are provided in the section on representations of the internal group.

As already mentioned, in unitary theories, the scaling dimensions of operators must be bounded from below. This condition for primary operators has already been determined above, specifically in Eqn. 2.19. However, stronger bounds can be derived by directly enforcing unitarity, i.e., allowing only states with non-negative norms. By computing the norms of all primaries and descendants for each possible representation of the Lorentz group, one can derive a bound on the scaling dimension for each representation. These bounds are referred to as the *unitarity bounds.*

The norms are computed using the Hermitian conjugates defined in Eqn. 2.8. Translating these into vector notation yields:

$$M^{\dagger}_{\mu\nu} = -M_{\nu\mu} = M_{\mu\nu},\ P^{\dagger}_{\mu} = K_{\mu},\ D^{\dagger} = D. \tag{2.21}$$

As an example, we will compute the unitarity bounds for scalars $|\Delta, 0\rangle$, following the discussion in [68]. A more detailed treatment of unitarity bounds for spinning operators can also be found in that reference.

We begin by declaring the highest weight state, the primary, to have unit norm: $\| |\Delta, 0\rangle \| = |\langle \Delta, 0 | \Delta, 0\rangle| = 1$. The norm of the first-level descendant is then computed as

$$\begin{aligned} \Big\| a^{\mu} P_{\mu} |\Delta, 0\rangle \Big\| &= \bar{a}^{\mu} a^{\nu} \langle \Delta, 0| K_{\mu} P_{\nu} |\Delta, 0\rangle \\ &= \bar{a}^{\mu} a^{\nu} \langle \Delta, 0| \Big[K_{\mu}, P_{\nu}\Big] + P_{\nu} K_{\mu} |\Delta, 0\rangle \\ &= 2\bar{a}^{\mu} a^{\nu} \langle \Delta, 0| \Big(iM_{\mu\nu} - i\eta_{\mu\nu} D\Big) |\Delta, 0\rangle \\ &= 2|a|^2 \Delta \qquad \text{for any complex vector } a^{\mu}. \end{aligned} \tag{2.22}$$

Demanding the norm to be non-negative, we derive the bound $\Delta \geq 0$ for scalar representations of the Lorentz group from this first-level descendant.

Proceeding to the second-level descendant, a similar computation shows:

$$\Delta \geq \frac{d}{2} - 1 \quad \text{or} \quad \Delta = 0, \tag{2.23}$$

where $\Delta = 0$ corresponds to the identity operator.

In principle, one would compute the norms of higher-level descendants to determine if stronger bounds exist. However, it can be shown that no stronger bounds appear, and $\Delta \geq \frac{d}{2} - 1$ is the correct bound for scalar operators.

Similar computations yield the unitarity bounds for all other possible spin representations. In four dimensions, where $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and the spin representation is given by $(j, \bar{\jmath})$, the unitarity bounds are [68]:

$$\begin{aligned} &\Delta = 0 \quad \text{identity}, \\ &\Delta \geq \begin{cases} 1 & \text{for} \quad j = \bar{\jmath} = 0, \\ j+1 & \text{for} \quad j > 0,\, \bar{\jmath} = 0, \\ \bar{\jmath}+1 & \text{for} \quad j = 0,\, \bar{\jmath} > 0, \\ j+\bar{\jmath}+2 & \text{for} \quad j > 0,\, \bar{\jmath} > 0. \end{cases} \end{aligned} \tag{2.24}$$

2.2.2. Representations of the internal group

Before delving into the full superconformal case, a brief review of the representations under the second bosonic subalgebra, $\mathfrak{su}(4)_R$, is provided.
$SU(4)$ is the compact, simply connected Lie group of special unitary transformations, represented by 4×4 matrices with determinant equal to 1. For simply connected Lie groups, there is a one-to-one correspondence between representations of the group and those of its algebra. Therefore, we can equivalently consider the Lie algebra $\mathfrak{su}(4)$. $\mathfrak{su}(4)$ is a finite-dimensional, simple Lie algebra, and its *complex representations* are best studied via the *Cartan classification*. The complex representations of the real algebra $\mathfrak{su}(4)$ can be directly obtained from those of the complexified algebra $\mathfrak{su}_{\mathbb{C}}(4)$, to which the Cartan classification applies.[4] [5]

We will now briefly review the concepts of Cartan classification and apply them to $\mathfrak{su}(4)$. For a more detailed exposition of the general concepts, see, for instance, [65]. The details specific to $\mathfrak{su}(4)$ follow [66].

A Cartan decomposition of a semi-simple Lie algebra of rank r in the standard *Cartan-Weyl basis* starts by identifying the maximal set of commuting Hermitian

[4] It is noteworthy that both $SU(2,2)$ and $SU(4)$ complexify to $SL(4;\mathbb{C})$, such that the complex algebras are given by $\mathfrak{sl}(4;\mathbb{C})$. To obtain the correct representation of the real groups, one must take the appropriate real form. However, for the purposes of this thesis, it is sufficient to treat the representations presented in the previous section as those of the conformal group, while using the material in this section for the internal group, without delving into the details of real forms.

[5] To maintain brevity and avoid possible confusion regarding the issue of real forms, in this section we will use the notation $\mathfrak{su}(4)$ even when referring to the complexified algebra.

generators, which form the *Cartan subalgebra*:

$$[h^i, h^j] = 0 \qquad \forall i, j = 1, \ldots, r. \tag{2.25}$$

Next, linear combinations e^α of the remaining generators are formed such that

$$[h^i, e^\alpha] = \alpha^i e^\alpha, \quad [e^\alpha, e^\beta] = \begin{cases} N_{\alpha,\beta} e^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi, \\ \frac{2}{|\alpha|^2}\alpha \cdot h & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \tag{2.26}$$

where e^α act as ladder operators, $N_{\alpha,\beta}$ is a normalisation constant, and $\alpha = (\alpha^1, \ldots, \alpha^r)$ is called a *root*, corresponding to the non-zero eigenvalues of h^i in this adjoint representation. Φ denotes the set of all roots. Since the commutation relations are determined by the roots, the roots encode all the essential information about the algebra. The scalar product of the roots, which is properly defined using the *Killing form* (see [65] for details), can be used to construct the *Cartan matrix*. The Cartan matrix is thus a single object that encodes all the information about the algebra. It is defined as

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2}. \tag{2.27}$$

As can be derived from above, there is a natural correspondence between the roots α and the generators e^α, implying that the number of roots equals the dimension of the algebra minus its rank.
Since the maximum number of independent roots, or *simple roots* (i.e., those not expressible as a sum of two positive roots), equals the rank of the algebra, this means that in most cases, linear dependencies arise. To focus solely on the simple roots and simultaneously highlight the importance of the Cartan matrix, it is useful to work in the *Chevalley basis*. In this basis, we assign to each simple root the three generators:

$$E^{+\alpha_i}, E^{-\alpha_i}, H^i = \frac{2\alpha_i \cdot h}{|\alpha_i|^2}, \quad i = 1, \ldots, r. \tag{2.28}$$

Each such set of three generators forms an $\mathfrak{su}(2)$ subalgebra with the standard commutation relations:

$$[H^i, H^j] = 0, \quad [E^{+\alpha_i}, E^{-\alpha_j}] = \delta_{ij} H^j, \quad [H^i, E^{\pm\alpha_j}] = \pm A_{ji} E^{\pm\alpha_j}, \tag{2.29}$$

where A_{ji} are the entries of the Cartan matrix.
The remaining generators can be obtained unambiguously thanks to the *Serre relations*:

$$[\mathrm{ad}(E^{+\alpha_i})]^{1-A_{ji}} E^{+\alpha_j} = 0, \quad [\mathrm{ad}(E^{-\alpha_i})]^{1-A_{ji}} E^{-\alpha_j} = 0, \tag{2.30}$$

where "ad" refers to the adjoint map. These relations provide a procedure for reconstructing the non-simple roots from the simple roots.
All the commutation relations in this basis, and thus all the information about the algebra, are encoded in terms of the Cartan matrix. This underscores that the Cartan matrix contains all the essential information about the respective algebra. (This structure can be represented diagrammatically using *Dynkin diagrams*; for more details, see [65].)

Thus far, everything has been defined in terms of the adjoint representation to specify the algebra. For general representations, however, one can also find a basis $\{|\lambda\rangle\}$ such that

$$H^i |\lambda\rangle = \lambda^i |\lambda\rangle , \tag{2.31}$$

where H^i are the generators defined in 2.29, and $\lambda = (\lambda^1, \ldots, \lambda^r)$ is called the *weight of the representation.* For detailed relations between the weights and roots, refer to [65].
The generators $E^{\pm\alpha_i}$ from 2.29 act as ladder operators on general representation states, and hence most representations can be characterised by a *highest-weight state*, defined by:

$$E^{+\alpha_i} |\lambda\rangle^{\text{hw}} = 0 \quad \forall i. \tag{2.32}$$

The full (infinite-dimensional) *Verma module* is then spanned by the successive action of various lowering generators

$$\prod_i \left(E^{-\alpha_i}\right)^{n_i} |\lambda\rangle^{\text{hw}} , \tag{2.33}$$

producing states of lower weight.
For simple Lie algebras, all irreducible representations (irreps) are necessarily finite-dimensional. This implies the existence of such a highest-weight state. Furthermore, it tells us that the infinite Verma module corresponding to a simple Lie algebra is reducible (i.e., it contains multiple highest-weight states). The finite-dimensional irrep is obtained by taking a suitable quotient of the Verma module, i.e., by removing all the states that belong to the next sub-highest-weight Verma module. (The standard example for $\mathfrak{su}(2)$ is $V_j = \mathcal{V}_j/\mathcal{V}_{-j-1}$, yielding a $2j+1$-dimensional representation on V_j, where $\mathcal{V}$ denotes the Verma module.)
To summarise, the finite-dimensional irreps of simple Lie algebras are fully characterised by the highest-weight state $|\lambda\rangle = |\lambda^1, \ldots, \lambda^r\rangle$, or more concisely, by its Dynkin labels $[\lambda^1, \ldots, \lambda^r]$.

Let us make these concepts more concrete by applying them to the $\mathfrak{su}(4)$ algebra. The Cartan matrix of $\mathfrak{su}(4)$, which has rank 3, is given by

$$[A_{ij}] = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{2.34}$$

The generators of $\mathfrak{su}(4)_R$, R^i_j, can be expressed in the Chevalley basis. The commutation relations in this basis, as indicated in equation (2.29), yield the established algebra relations presented in equation 2.11. The specific matrix that expresses the 16 generators in the Chevalley basis is given by [66]

$$[R^i{}_j] = \begin{pmatrix} \frac{1}{4}(3H_1+2H_2+H_3) & E_1^+ & [E_1^+,E_2^+] & [E_1^+,[E_2^+,E_3^+]] \\ E_1^- & \frac{1}{4}(-H_1+2H_2+H_3) & E_2^+ & [E_2^+,E_3^+] \\ -[E_1^-,E_2^-] & E_2^- & -\frac{1}{4}(H_1+2H_2-H_3) & E_3^+ \\ [E_1^-,[E_2^-,E_3^-]] & -[E_2^-,E_3^-] & E_3^- & -\frac{1}{4}(H_1+2H_2+3H_3) \end{pmatrix}. \tag{2.35}$$

The various representations are specified by their respective highest weight states $|\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}}$, which satisfy the following conditions:

$$E_i^+ |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}} = R^i{}_{i+1} |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}} = 0 \qquad \forall i = 1,2,3, \tag{2.36}$$

$$(H_1,H_2,H_3) |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}} = (\lambda_1,\lambda_2,\lambda_3) |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}}. \tag{2.37}$$

The remaining module is spanned as follows:

$$\prod_i (E_i^-)^{n_i} |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}} = \prod_i (R^{i+1}{}_i)^{n_i} |\lambda_1,\lambda_2,\lambda_3\rangle^{\text{hw}} \qquad \forall i = 1,2,3. \tag{2.38}$$

Upon restricting to the finite-dimensional irreducible representation, the dimension of this representation, denoted by $[\lambda_1,\lambda_2,\lambda_3]$, is given by the formula

$$\dim([\lambda_1,\lambda_2,\lambda_3]) = \frac{1}{12}(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2+2)(\lambda_2+\lambda_3+2)(\lambda_1+1)(\lambda_2+1)(\lambda_3+1). \tag{2.39}$$

It is customary to refer to the Dynkin labels for $\mathfrak{su}(4)$ as $[k,p,q]$ rather than $[\lambda_1,\lambda_2,\lambda_3]$. Henceforth, we will adopt this notation.

2.2.3. Superconformal Representations

The concepts developed previously can be straightforwardly extended to the supersymmetric case, which will be outlined below. The following discussion closely follows references [66] and [67], where further details and derivations of the statements presented herein can be found.

In addition to the bosonic subalgebras $\mathfrak{su}(2,2)$ and $\mathfrak{su}(4)_R$, there are the 32 fermionic charges $Q^i{}_\alpha$, $\bar{Q}_{i\dot{\alpha}}$, $S_i{}^\alpha$, and $\bar{S}^{i\dot{\alpha}}$, which are utilized to construct the supermultiplet.

Analogous to ordinary conformal field theory, a supermultiplet under $\mathfrak{psu}(2,2|4)$ is characterized by its respective superprimary state, denoted as $|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}$. This state satisfies the following conditions:

$$\begin{aligned}&(K_{\alpha\dot{\alpha}}, R_i^{i+1}, J_+, \bar{J}_+, S_k{}^\alpha, \bar{S}^{k\dot{\alpha}})\,|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}} = 0, \qquad \forall i = 1,2,3;\ k = 1,\ldots,4,\\ &(D; H_1, H_2, H_3; J_3, \bar{J}_3)\,|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}} = (\Delta; k, p, q; j, \bar{\jmath})\,|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}\,.\end{aligned} \tag{2.40}$$

It is noteworthy that $|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}$ not only is a conformal primary state and a highest weight state in an $\mathfrak{su}(4)$ representation but also defines a highest weight state under the four-dimensional Lorentz transformations $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, represented as

$$[M_\alpha{}^\beta] = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}, \quad [\bar{M}^{\dot{\beta}}{}_{\dot{\alpha}}] = \begin{pmatrix} \bar{J}_3 & \bar{J}_+ \\ \bar{J}_- & -\bar{J}_3 \end{pmatrix}, \tag{2.41}$$

where the sets $\{J_3, J_\pm\}$ and $\{\bar{J}_3, \bar{J}_\pm\}$ satisfy the standard $\mathfrak{su}(2)$ commutation relations.
The state is rendered a full superconformal primary by imposing additionally the equivalence of the primary condition in the supersymmetric case; specifically, the requirement that it vanishes under the action of the superconformal generators $S_i{}^\alpha$ and $\bar{S}_{i\dot{\alpha}}$.
The highest-weight state $|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}$ can also be compactly represented as

$$[k, p, q]^{\Delta}_{(j,\bar{\jmath})}. \tag{2.42}$$

The full Verma module is then spanned by

$$\prod_{\substack{i,j,k,l=1,2,3,4;k>l\\ \alpha,\dot{\alpha},\beta,\dot{\beta}=1,2}} (P_{\alpha\dot{\alpha}})^{N_{\alpha\dot{\alpha}}} (Q^i{}_\beta)^{n_{i\beta}} (\bar{Q}_{j\dot{\beta}})^{\bar{n}_{j\dot{\beta}}} (J_-)^N (\bar{J}_-)^{\bar{N}} (R^k{}_l)^{N_{kl}}\,|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}\,, \tag{2.43}$$

where $N_{\alpha\dot{\alpha}}, N, \bar{N}, N_{kl} = 0, 1, 2, \ldots$ and $n_{i\beta}, \bar{n}_{j\dot{\beta}} = 0, 1$. It is important to note that only a finite number of applications of Q and $\bar{Q}$ charges are allowed due to their

fermionic nature.
In particular, we can separate the fermionic and bosonic generators above by rendering the bosonic generators implicit. Consequently, a supermultiplet can be expressed as

$$\prod_{\substack{i,j=1,2,3,4;\\ \beta,\dot\beta=1,2}} (Q^i{}_\beta)^{n_{i\beta}} (\bar{Q}_{j\dot\beta})^{\bar{n}_{j\dot\beta}} \,|\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}}, \quad n_{i\beta}, \bar{n}_{j\dot\beta} = 0, 1, \tag{2.44}$$

where each application of $Q^i{}_\beta$ and $\bar{Q}_{j\dot\beta}$ creates a distinct module under $\mathfrak{su}(2,2) \times \mathfrak{su}(4)_R$, identified by their primary highest weight state. Given the fermionic nature of the supercharges, the issue of infinite-dimensional Verma modules affects only the bosonic representations. Thus, finite-dimensional irreducible representations can be obtained by taking quotients as described above.[6]

The Dynkin labels of the bosonic multiplets constructed by acting with $Q^i{}_\alpha$ and $\bar{Q}_{i\dot\alpha}$ on the respective superprimary states are derived by adding the weights of the supercharges to those of the superprimary.
The weights of the supercharges can be readily inferred from the commutation relations given in equation 2.11 and the matrix in equation 2.35. For instance, we have:

$$\begin{aligned} \left[H_1, Q^1{}_\alpha\right] &= \left[R^1{}_1 - R^2{}_2, Q^1{}_\alpha\right] = Q^1{}_\alpha,\ \left[H_i, Q^1{}_\alpha\right] = 0 \text{ for } i = 2,3,\ \left[E_i^+, Q^1{}_\alpha\right] = 0,\\ \left[H_3, \bar{Q}_{4\dot\alpha}\right] &= \left[R^3{}_3 - R^4{}_4, \bar{Q}_{4\dot\alpha}\right] = \bar{Q}_{4\dot\alpha},\ \left[H_i, \bar{Q}_{4\dot\alpha}\right] = 0 \text{ for } i = 1,2,\ \left[E_i^+, \bar{Q}_{4\dot\alpha}\right] = 0, \end{aligned} \tag{2.45}$$

where $H_1 = R^1{}_1 - R^2{}_2$ and $H_3 = R^3{}_3 - R^4{}_4$ are not unique. Furthermore, we can observe

$$Q^1{}_\alpha \overset{E_1^-}{\to} Q^2{}_\alpha \overset{E_2^-}{\to} Q^3{}_\alpha \overset{E_3^-}{\to} Q^4{}_\alpha, \tag{2.46}$$

$$\bar{Q}_{4\dot\alpha} \overset{E_3^-}{\to} \bar{Q}_{3\dot\alpha} \overset{E_2^-}{\to} \bar{Q}_{2\dot\alpha} \overset{E_1^-}{\to} \bar{Q}_{1\dot\alpha}. \tag{2.47}$$

Thus, $Q^1{}_\alpha$ and $\bar{Q}_{4\dot\alpha}$ correspond to the highest weight states in the $[1,0,0]$ and $[0,0,1]$ representations, respectively.

[6] With this consideration, we will henceforth omit the notion of modules and refer directly to the finite-dimensional multiplets.

The weights of the complete set of supercharges can be computed similarly:

$$\begin{aligned} &{Q^1}_\alpha \sim [1,0,0]_{(\pm\frac{1}{2},0)}, \quad {Q^2}_\alpha \sim [-1,1,0]_{(\pm\frac{1}{2},0)}, \\ &{Q^3}_\alpha \sim [0,-1,1]_{(\pm\frac{1}{2},0)}, \quad {Q^4}_\alpha \sim [0,0,-1]_{(\pm\frac{1}{2},0)}, \\ &\bar{Q}_{4\dot\alpha} \sim [0,0,1]_{(0,\pm\frac{1}{2})}, \quad \bar{Q}_{3\dot\alpha} \sim [0,1,-1]_{(0,\pm\frac{1}{2})}, \\ &\bar{Q}_{2\dot\alpha} \sim [1,-1,0]_{(0,\pm\frac{1}{2})}, \quad \bar{Q}_{1\dot\alpha} \sim [-1,0,0]_{(0,\pm\frac{1}{2})}. \end{aligned} \tag{2.48}$$

The correct construction of the full supermultiplet, especially when adding the above weights to the respective state produces negative quantum numbers, is given for instance by the *Racah-Speiser algorithm* (for a detailed description see [66]). However, for the supermultiplets considered in this thesis, it suffices to consider all possible applications of the Q and $\bar{Q}$ charges that yield non-negative Dynkin labels, while “ignoring” the rest.

The resulting supermultiplets can be categorised into three distinct classes, as derived in [66, 69].
The full, unconstrained action of all 16 supercharges on the superprimary state $[k,p,q]^{\Delta}_{(j,\bar{\jmath})}$ gives rise to a so-called *long supermultiplet*, denoted as the $\mathcal{A}$-series:

$$\mathcal{A}_{[k,p,q]^{\Delta}_{(j,\bar{\jmath})}}. \tag{2.49}$$

The dimension of a long supermultiplet is given by:

$$\dim(\mathcal{A}) = 2^8 \dim(k,p,q)(2j+1)(2\bar{\jmath}+1), \tag{2.50}$$

where $\dim(k,p,q)$ represents the dimension of the $\mathfrak{su}(4)_R$ representation described in equation 2.39.
In a manner analogous to the bosonic case, unitarity imposes a lower bound on the scaling dimensions of long multiplets. This unitarity bound has been calculated in reference [69] to be:

$$\Delta \geq 2 + 2j + \frac{1}{2}(2k+2p+q) \quad \& \quad \Delta \geq 2 + 2\bar{\jmath} + \frac{1}{2}(k+2p+3q). \tag{2.51}$$

Furthermore, there are the so-called *shortened* or *BPS-supermultiplets.*
The *short supermultiplet* is characterised by additional BPS-shortening conditions of the form

$$ {Q^i}_\alpha \left|\Delta; k,p,q; j,\bar{\jmath}\right\rangle^{\text{hw}} = 0 \quad \text{for } \alpha = 1,2, \tag{2.52}$$

$$\bar{Q}_{\bar{\imath}\dot{\alpha}} \, |\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}} = 0 \quad \text{for } \dot{\alpha} = 1, 2, \tag{2.53}$$

which can be demanded for various values of i. From the superconformal algebra 2.10, one can see that imposing these equations automatically leads to $j = 0$ or $\bar{\jmath} = 0$, respectively. Furthermore, it automatically leads to the additional constraint

$$(\delta^i_l \frac{1}{2}\Delta - R^i{}_l) \, |\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}} = 0, \quad l = 1, 2, 3, 4, \tag{2.54}$$

$$(\delta^l_{\bar{\imath}} \frac{1}{2}\Delta - R^l{}_{\bar{\imath}}) \, |\Delta; k, p, q; j, \bar{\jmath}\rangle^{\text{hw}} = 0, \quad l = 1, 2, 3, 4, \tag{2.55}$$

which allows us to compute the scaling dimensions for these multiplets exactly. In particular, this means that those scaling dimensions do not receive quantum corrections, but are *protected*.

The various BPS-shortenings are then denoted by t and $\bar{t}$, which are the fraction of charges for which the above condition holds, as

$$\begin{cases} i = 1 & \leftrightarrow t = \frac{1}{4} \Leftrightarrow & \Delta = \frac{1}{2}(3k + 2p + q) \\ i = 1, 2 & \leftrightarrow t = \frac{1}{2} \Leftrightarrow & \Delta = \frac{1}{2}(2p + q), \; k = 0 \\ i = 1, 2, 3, 4 & \leftrightarrow t = 1 \Leftrightarrow & \Delta = 0, \; (\text{Identity}) \end{cases} \tag{2.56}$$

$$\begin{cases} \bar{\imath} = 4 & \leftrightarrow \bar{t} = \frac{1}{4} \Leftrightarrow & \Delta = \frac{1}{2}(k + 2p + 3q) \\ \bar{\imath} = 3, 4 & \leftrightarrow \bar{t} = \frac{1}{2} \Leftrightarrow & \Delta = \frac{1}{2}(k + 2p), \; q = 0 \\ \bar{\imath} = 1, 2, 3, 4 & \leftrightarrow \bar{t} = 1 \Leftrightarrow & \Delta = 0, \; (\text{Identity}) \end{cases} \tag{2.57}$$

[7] In this thesis, we will only consider multiplets for which $t = \bar{t}$.

Those relevant *short multiplets* or $\mathcal{B}$-series can be summarized as follows:

$$\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0](0,0)} \text{ with } \Delta = p \qquad (\text{half-BPS}) \tag{2.58}$$

$$\mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q](0,0)} \text{ with } \Delta = p + 2q \qquad (\text{quarter-BPS}). \tag{2.59}$$

The dimensions of these multiplets are given by:

$$\begin{aligned} \dim(\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0](0,0)}) &= 2^8 \dim(0, p-2, 0), \\ \dim(\mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q](0,0)}) &= 2^{12} \dim(q-2, p, q). \end{aligned} \tag{2.60}$$

[7] For further details on $t = \frac{3}{4}, \bar{t} = \frac{3}{4}$, see [66]. Due to no relevance for this thesis, they have been omitted.

For the sake of brevity, we will denote the half-BPS multiplets as $\mathbb{O}_p$, where p represents the R-symmetry charge, given in $[0, p, 0]$.

Additionally, we consider the *semi-short supermultiplets*, defined by shortening conditions applicable for $j > 0$ and $\bar{j} > 0$. Specifically, these conditions are

$$\left(Q_2^i - \frac{1}{2j+1} J_- Q_1^i\right) |\Delta; k, p, q; j, \bar{j}\rangle^{\text{hw}} = 0, \tag{2.61}$$

$$\left(\bar{Q}_{\bar{\imath}1} - \frac{1}{2\bar{j}+1} \bar{J}_- \bar{Q}_{\bar{\imath}2}\right) |\Delta; k, p, q; j, \bar{j}\rangle^{\text{hw}} = 0. \tag{2.62}$$

These constraints can be applied for different portions of the supersymmetric generators, yielding exactly computable scaling dimensions:

$$\begin{cases} i = 1 & \leftrightarrow t = \frac{1}{4} \Leftrightarrow & \Delta = 2 + 2j + \frac{1}{2}(3k + 2p + q) \\ i = 1,2 & \leftrightarrow t = \frac{1}{2} \Leftrightarrow & \Delta = 2 + 2j + \frac{1}{2}(2p + q),\ k = 0 \\ i = 1,2,3,4 & \leftrightarrow t = 1 \Leftrightarrow & \Delta = 2 + 2j \end{cases} \tag{2.63}$$

$$\begin{cases} \bar{\imath} = 4 & \leftrightarrow \bar{t} = \frac{1}{4} \Leftrightarrow & \Delta = 2 + 2\bar{j} + \frac{1}{2}(k + 2p + 3q) \\ \bar{\imath} = 3,4 & \leftrightarrow \bar{t} = \frac{1}{2} \Leftrightarrow & \Delta = 2 + 2\bar{j} + \frac{1}{2}(k + 2p),\ q = 0 \\ \bar{\imath} = 1,2,3,4 & \leftrightarrow \bar{t} = 1 \Leftrightarrow & \Delta = 2 + 2\bar{j},\ \text{(Identity)} \end{cases} \tag{2.64}$$

It is important to note that, while the scaling dimensions are precisely computable, semi-short multiplets are not necessarily protected against quantum corrections. Under certain circumstances, they may recombine (also with quarter-BPS multiplets) into long multiplets, for which the acquisition of an anomalous dimension can occur.

In this thesis, we focus on the following *semi-short supermultiplets*, which are denoted as the $\mathcal{C}$-series:

$$\mathcal{C}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0]_{(j,j)}} \text{ with } \Delta = 2 + 2j + p, \tag{2.65}$$

$$\mathcal{C}^{\frac{1}{4},\frac{1}{4}}_{[k,p,q]_{(j,\bar{j})}} \text{ with } k - q = 2(\bar{j} - j) \text{ and } \Delta = 2 + j + \bar{j} + p + 2q, \tag{2.66}$$

$$\mathcal{C}^{1,1}_{[0,0,0]_{(j,j)}} \text{ with } \Delta = 2 + 2j. \tag{2.67}$$

Those $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ series characterise the principal representations under the superconformal group in four dimensions, $\mathfrak{psu}(2,2|4)$, which will be the focus of our study.
A special emphasis of this thesis will be placed on the so-called *stress-tensor supermultiplet*, which will be introduced in the subsequent section.

2.2.4. The Stress-Tensor Supermultiplet

Among the possible representations of the superconformal algebra $\mathfrak{psu}(2,2|4)$, a particularly important role is played by the *stress-tensor supermultiplet*, which will be elaborated in detail in this section.

The stress-tensor supermultiplet is a half-BPS multiplet, characterised by the superprimary in the $[0,2,0]^{\Delta=2}_{(0,0)}$ representation. Specifically, we define the multiplet as

$$\mathcal{T} \equiv \mathbb{O}_2 = \mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,2,0]_{(0,0)}}, \tag{2.68}$$

which will be denoted as $\mathcal{T}$ for clarity and distinction from other BPS multiplets.

The full structure of this supermultiplet is depicted in Figure 2.1 [66]. In the diagram, arrows pointing to the left represent the action of $Q^i{}_\alpha$ with $i = 3,4$, while arrows pointing to the right indicate the action of $\bar{Q}_{i\dot{\alpha}}$ with $i = 1,2$, such that the resulting quantum numbers remain non-negative.

The stress-tensor supermultiplet is the simplest non-trivial gauge-invariant multiplet in $\mathcal{N} = 4$ Super-Yang-Mills theory (SYM) [70]. Its field content makes it of significant physical interest, and we introduce its components using the notation employed throughout this thesis.
To begin, the multiplet is spanned by the $\mathbf{20'}$-operator, which corresponds to the $[0,2,0]_{(0,0)}$ representation and is denoted as $\mathcal{O}_{20'}$.
Additionally, the multiplet contains the stress tensor T, after which it is named. The stress tensor T is associated with the $[0,0,0]^{\Delta=4}_{(1,1)}$ representation.
Furthermore, the Lagrangian $\mathcal{L}$, represented by $[0,0,0]^{\Delta=4}_{(0,0)}$, is included in the multiplet. This Lagrangian can be obtained by applying four Q-charges to the superprimary (i.e., $\mathcal{L}$ corresponds to the $[0,0,0]^{\Delta=4}_{(0,0)}$ representation on the left side of Figure 2.1). Its conjugate, $\bar{\mathcal{L}} \sim [0,0,0]_{(0,0)}$, is located on the right side of Figure 2.1, obtained through the action of four $\bar{Q}$-charges on the superprimary.
Finally, the stress-tensor supermultiplet incorporates all the conserved currents. These include the $SU(4)_R$ current $\mathcal{J}_{\alpha\dot{\alpha}}(x)$, which transforms as $[1,0,1]_{(\frac{1}{2},\frac{1}{2})}$, as well as the spinor currents in the representations $[0,0,1]_{(\frac{1}{2},1)}$ and $[1,0,0]_{(1,\frac{1}{2})}$, respectively.

The negative representations in Figure 2.1 are there to ensure finite-dimensional irreducible representations of the currents.
However, they get eliminated when the respective conservation equations are ap-

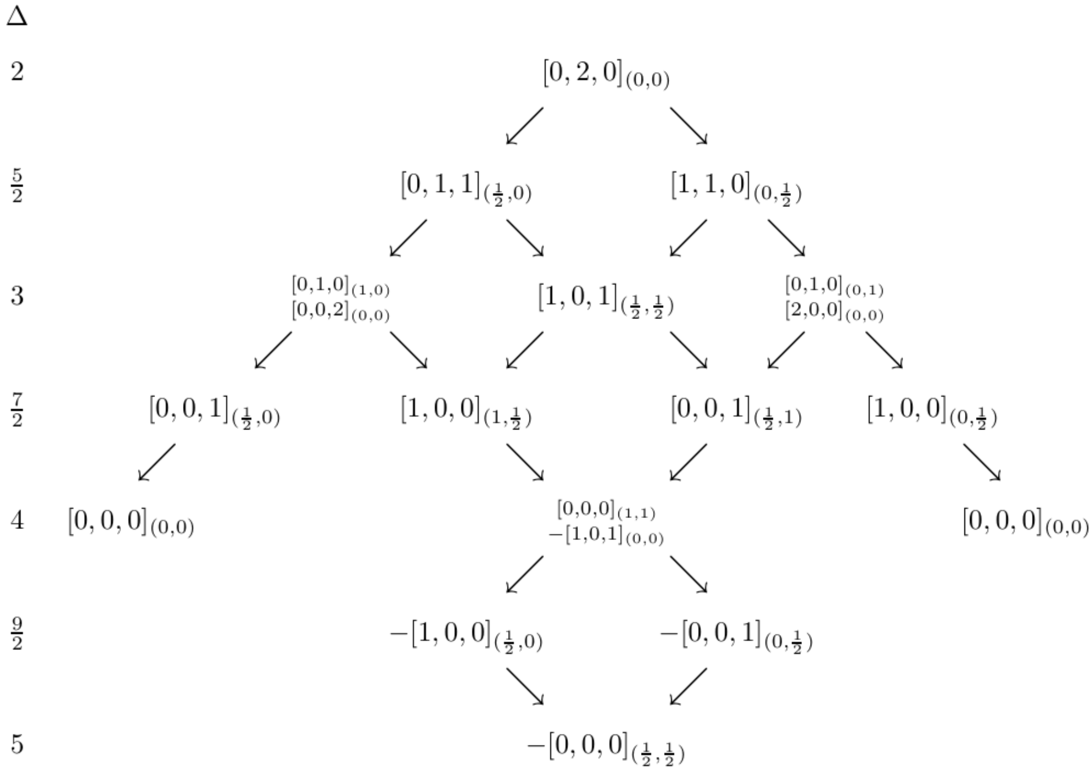

Figure 2.1.: The structure of the stress-tensor supermultiplet $\mathcal{T}$. The superprimary is a scalar transforming as the $\mathbf{20'}$-operator, corresponding to the $[0, 2, 0]$ representation of $\mathfrak{su}(4)_R$. The successive actions of Q-charges (left arrows) and $\bar{Q}$-charges (right arrows) on the superprimary generate the full multiplet. Each entry in the figure represents a complete bosonic multiplet under $\mathfrak{su}(2, 2) \times \mathfrak{su}(4)_R$. Representations with a negative overall sign are eliminated upon imposing the conservation equations on the currents. The figure is adapted from [66].

plied on the fields. For instance, for the $SU(4)_R$ current, the conservation equation takes the form

$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\mathcal{J}_{\alpha\dot{\alpha}} = 0,$$

which eliminates the representations $-[1, 0, 1]_{(0,0)}$ in Figure 2.1.

This elimination can be understood through the relationship between the currents and these negative-sign representations. For example, the representation $-[1, 0, 1]_{(0,0)}$ at $\Delta = 4$ is related to the $SU(4)_R$ current $[1, 0, 1]_{(\frac{1}{2},\frac{1}{2})}$ as follows (schematically):

$$\begin{aligned} \mathcal{J}_{\alpha\dot{\alpha}} &\equiv [1,0,1]_{(\frac{1}{2},\frac{1}{2})}, \\ &\downarrow \quad \text{(via the action of } P_{\beta\dot{\beta}} \sim \partial_{\beta\dot{\beta}}), \\ J &\equiv [1,0,1]_{(0,0)}. \end{aligned} \tag{2.69}$$

Thus, the conservation equation implies that J vanishes, as $\partial\mathcal{J}_{\alpha\dot{\alpha}} \sim J$. Similarly, the remaining representations with negative signs in Figure 2.1 are eliminated [66].

2.2.5. Fundamental Multiplet

Throughout this thesis, we will also encounter the so-called *fundamental supermultiplet*. Thus, a brief summary of this multiplet is given.

The fundamental multiplet of $\mathcal{N} = 4$ SYM is a half-BPS multiplet under the superconformal algebra $\mathfrak{psu}(2,2|4)$, generated by a superprimary of charge $p = 1$. This can be expressed as

$$\mathbb{O}_1 = \mathcal{B}^{\frac{1}{2}}_{[0,1,0]_{(0,0)}}. \tag{2.70}$$

The structure of this supermultiplet is depicted in Figure 2.2 [66], where the graphical representation follows the same rules as the stress-tensor supermultiplet (see Figure 2.1).

From a field-theoretic perspective, this multiplet contains all of the fundamental fields of $\mathcal{N} = 4$ SYM. At the bosonic level, represented at the top of Figure 2.2, we find the six scalar fields Φ^I, where $I = 1, \ldots, 6$, corresponding to the superprimary $\mathcal{O}_6 \sim [0,1,0]_{(0,0)}$. Recall that $[0,1,0]$ is the $\mathbf{6}$-dimensional representation of $\mathfrak{su}(4)_R$.
At level one, after a single action of Q or $\bar{Q}$, we encounter the representations $[0,0,1]_{(\frac{1}{2},0)}$ and $[1,0,0]_{(0,\frac{1}{2})}$. These correspond to the four complex fermions $\bar{\Psi}^i_{\dot{\alpha}}$ and their conjugates Ψ^i_α, where $i = 1, \ldots, 4$.
At level two, the field strength tensor $F_{\mu\nu} = -F_{\nu\mu}$, (split into $\mathfrak{su}(2)$-indices) emerges in the structure of the multiplet.

It is important to note that this fundamental multiplet is the simplest and shortest multiplet under $\mathfrak{psu}(2,2|4)$. However, it is not gauge-invariant on its own. Therefore, when moving from the algebraic analysis of $\mathfrak{psu}(2,2|4)$ to the full theoretical framework of $\mathcal{N} = 4$ SYM, the lowest gauge-invariant multiplet encountered is the stress-tensor supermultiplet, as discussed earlier.

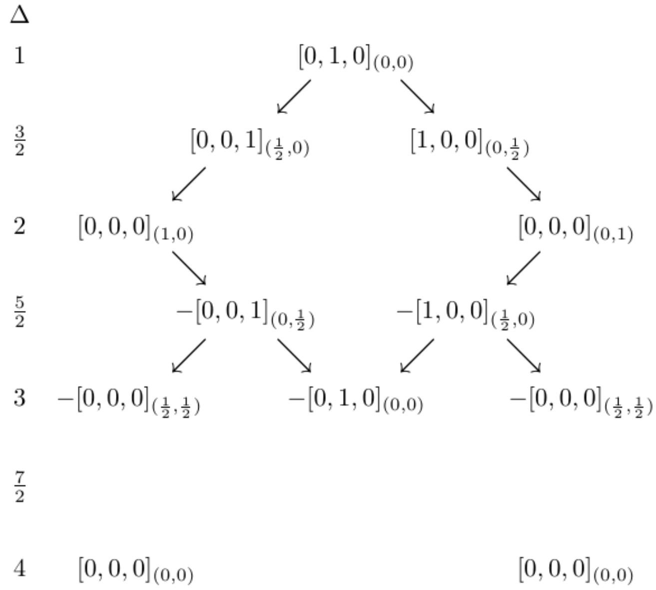

Figure 2.2.: Representations in the fundamental supermultiplet $\mathbb{O}_1$ in $\mathfrak{psu}(2,2|4)$. The superprimary is the $\mathcal{O}_6$-operator: a scalar transforming in the $[0,1,0]$ (or the $\mathbf{6}$) representation of $\mathfrak{su}(4)_R$. Successive actions of Q-charges (left arrows) and $\bar{Q}$-charges (right arrows) span the multiplet. Each representation represents a full bosonic multiplet under $\mathfrak{su}(2,2)\times\mathfrak{su}(4)_R$. Representations with a negative overall sign are eliminated by imposing the equations of motion on the fundamental fields. The figure is adapted from [66].

2.3. Analytic Superspace

In the previous sections, we introduced the algebra $\mathfrak{psu}(2,2|4)$ and its possible representations, with a special focus on the half-BPS stress-tensor multiplet. The next step is to relate these representations to fields that are supported on a specific space to study their kinematics. In this section, we introduce such a space, on which the corresponding fields are supported: the *analytic superspace*, a special form of *harmonic superspace*.

Harmonic superspaces were introduced in 1984 in the context of $\mathcal{N}=2$ superconformal field theories [71]. The idea behind harmonic superspace is to extend Minkowski superspacetime by an internal manifold, typically taken to be a coset space of the internal symmetry group, to facilitate a more efficient study of the corresponding superconformal field theory. The resulting superspace not only makes

superconformal symmetry manifest but also allows the theory's field content to be encapsulated into a single superfield living on the harmonic superspace.
When a field is Grassmann-analytic (a generalization of chirality or, in the notation of this thesis, half-BPS), the corresponding analyticity constraints must be imposed on the field on harmonic superspace. These constraints can be solved in terms of the coordinates of harmonic superspace, leading to a reduction in the number of fermionic coordinates. For example, in $\mathcal{N} = 2$ superspace, this reduction is from 4 fermionic coordinates $(\theta^i, \bar{\theta}_i)$, $i = 1, 2$ to 2 fermionic coordinates $(\theta^2, \bar{\theta}_1)$ [72]. The resulting space is referred to as *analytic superspace*.
Fields representing half-BPS multiplets can then be described by simple scalar superfields on this analytic superspace [72, 73], making it the most natural framework for studying these objects. The concepts of $\mathcal{N} = 2$ harmonic superspace have been further generalized to different $\mathcal{N}$-superspaces associated with $4d$ complex Minkowski spacetime, including $\mathcal{N} = 4$ SYM [74, 75]. In particular, these works relate analytic superspace to a coset space of the (complexified) superconformal group, enabling simple superconformal transformations of this space.

In this section, we will provide a complete definition and description of the analytic superspace used for $\mathcal{N} = 4$ SYM, as developed in a series of papers by Heslop, Howe, and West [42, 73, 76–78]. We will then show how this space is derived from a coset space construction of the complexified superconformal group $SL(4; \mathbb{C})$ and further examine the advantages of using this space in more detail.[8]

2.3.1. Definition of Analytic Superspace

Following the conventions of [79], analytic superspace for $\mathcal{N} = 4$ superconformal field theories can be defined in terms of local coordinates as a $(4|4) \times (4|4)$-supermatrix:

$$X^{A\dot{A}} = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \rho^{\alpha\dot{a}} \\ \bar{\rho}^{a\dot{\alpha}} & y^{a\dot{a}} \end{pmatrix}, \qquad \text{with } A = (\alpha|a),\ \dot{A} = (\dot{\alpha}|\dot{a}) \text{ and } \alpha, \dot{\alpha} = 1, 2;\ a, \dot{a} = 1, 2, \tag{2.71}$$

where $x^{\alpha\dot{\alpha}} = (x^\mu \sigma_\mu)^{\alpha\dot{\alpha}}$ are the coordinates of complexified Minkowski spacetime $\mathbb{R}^{1,3} \overset{\mathbb{C}}{\to} \mathbb{C}^4$. The coordinates $y^{a\dot{a}}$ provide a second set of bosonic coordinates that parametrise the internal manifold associated with the complexified internal group

[8] The coset space construction involves working with complexified groups and spaces. For the purposes of this thesis, the distinction between real and complex vector spaces is not critical, so the established results will be used without further comments.

$SL(4;\mathbb{C})$[9].
The off-diagonal terms, $\rho^{\alpha\dot{a}}$ and $\bar{\rho}^{a\dot{\alpha}}$, represent fermionic (i.e. Grassmann-odd) coordinates. Thus, $\mathcal{N}=4$ analytic superspace has a total of 8 bosonic and 8 fermionic coordinates. The reduction from 16 fermionic coordinates in $\mathcal{N}=4$ superspace to 8 fermionic coordinates in analytic superspace can be understood as generalized chirality, as described above.

The manifolds described by the bosonic coordinates $x^{\alpha\dot{\alpha}}$ and $y^{a\dot{a}}$ share the same complexification: both $SU(2,2)$ and $SU(4)$ complexify to the group $SL(4;\mathbb{C})$. However, while $x^{\alpha\dot{\alpha}}$ parametrise a non-compact spacetime, the $y^{a\dot{a}}$ coordinates parametrise a compact group. This introduces strong analyticity constraints on the internal coordinates, the effects of which will be examined later.

Further details regarding the conventions and basic definitions of supermatrices used in this thesis can be found in Appendix A.1.

2.3.2. Coset Space Construction

This analytic superspace makes superconformal symmetry manifest, allowing superconformal transformations to act in a straightforward manner, which for instance greatly simplifies constructing invariants (see Section 2.4). This simplicity becomes evident when examining the origin of this space, which follows the discussion presented in [70].

Analytic superspace in this form is equivalent to the big cell of the super Grassmannian $\mathrm{Gr}(2|2,4|4)$, which is defined as the space of $(2|2)$-planes in a $(4|4)$-dimensional complex vector space [70]. The concept of super Grassmannians is a straightforward supersymmetric generalization of the “ordinary” Grassmannian $\mathrm{Gr}(2,4)$, which describes the space of 2-planes in a 4-dimensional complex vector space, $\mathbb{C}^4$.

In the context of conformal field theories (CFTs), $\mathrm{Gr}(2,4)$ provides not only a suitable compactification of Minkowski spacetime but also serves as a homogeneous space for the complexified conformal group $SL(4;\mathbb{C})$. This means that $SL(4;\mathbb{C})$

[9] These coordinates are closely related to the null coordinates y^I, $I=1,\ldots,6$, with $y^I y_I=0$, which are often used to describe internal symmetry (see Section 2.4). They are connected via the Plücker embedding. For more details, see [70] or [80].

acts transitively on Gr(2, 4), and we can define the diffeomorphism:

$$\mathrm{Gr}(2,4) \cong {SL(4;\mathbb{C})}/{P_0}, \tag{2.72}$$

where P_0 is the isotropy group.
In local coordinates, this diffeomorphism can be understood as follows: Gr(2, 4) is the space of 2-planes in $\mathbb{C}^4$, so a point in Gr(2, 4) is determined by two vectors spanning the corresponding plane, i.e., $x^{\alpha}{}_{\mathrm{B}}$ with $\alpha = 1, 2$ and $\mathrm{B} = 1, \ldots, 4$. On the right-hand side of Eqn. 2.72, elements of $SL(4;\mathbb{C})$ are matrices $x^{\mathrm{A}}{}_{\mathrm{B}}$, $\mathrm{A}, \mathrm{B} = 1, \ldots, 4$, and the coset space is defined as:

$$\text{coset space: } \left\{ x^{\mathrm{A}}{}_{\mathrm{B}} \sim h^{\mathrm{A}}{}_{\mathrm{C}} x^{\mathrm{C}}{}_{\mathrm{B}} \,:\, x^{\mathrm{A}}{}_{\mathrm{B}} \in SL(4;\mathbb{C}),\, h \in P_0 = \left\{ \begin{pmatrix} m & 0 \\ n & p \end{pmatrix} \right\} \right\}, \tag{2.73}$$

where $m, p \in GL(2)$.
This coset space can be restricted to the Grassmannian space, yielding Eqn. 2.72 as

$$\mathrm{Gr}(2,4) = \left\{ x^{\alpha}{}_{\mathrm{B}} \sim m^{\alpha}{}_{\gamma} x^{\gamma}{}_{\mathrm{B}} \,:\, x \in SL(4;\mathbb{C}),\, m^{\alpha}{}_{\gamma} \in GL(2) \right\}. \tag{2.74}$$

We can fix the equivalence relation by choosing a representative class. Following [70], we define the Grassmannians as:

$$x^{\alpha}{}_{\mathrm{B}} = \left(\delta^{\alpha}{}_{\beta}, x^{\alpha\dot{\beta}} \right), \tag{2.75}$$

where splitting the $SL(4)$-index B into $\mathrm{B} = (\beta, \dot{\beta})$ does not break any symmetry. It can be shown that the isometry group of the matrix $x^{\alpha\dot{\beta}}$ in the big cell is the Poincaré group, and thus $x^{\alpha\dot{\beta}}$ corresponds to Minkowski spacetime.

This logic extends to the supersymmetric case, allowing us to define the super Grassmannian as a quotient space of the superconformal group:

$$\mathrm{Gr}(2|2,4|4) \cong {SL(4|4;\mathbb{C})}/{P_0}, \text{ with } P_0 = \left\{ \begin{pmatrix} M & 0 \\ N & P \end{pmatrix} \,:\, M, N, P \in GL(2|2) \right\}. \tag{2.76}$$

In local coordinates, this defines the super Grassmannian as:

$$\mathrm{Gr}(2|2,4|4) = \left\{ x^{A}{}_{\mathcal{B}} \sim M^{A}{}_{C} x^{C}{}_{\mathcal{B}} \,:\, M^{A}{}_{C} \in GL(2|2) \right\}, \tag{2.77}$$

where $A = (\alpha|a)$ and $V_{\mathcal{A}} = (V_A, V^{\dot{A}}) = (V_\alpha|V^a, V^{\dot{a}}|V_{\dot{\alpha}})$ (describing a (2|4|2) vector). Fixing a representative, the super Grassmannian is expressed as:

$$x^{A}{}_{\mathcal{B}} \sim (\delta^{A}{}_{B}, X^{A\dot{B}}), \tag{2.78}$$

where the space in the right cell corresponds to the analytic superspace defined in Eqn. 2.71 [70].

One key advantage of the Grassmannian formulation is the simplicity of superconformal transformations, which, in these conventions, act as matrix multiplications from the right. The transformation of analytic superspace under superconformal transformations (SCTs) can be deduced as follows.
Let

$$G = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \in SL(4|4;\mathbb{C}). \tag{2.79}$$

Then, the transformation of $x^{A}{}_{\mathcal{B}}$ under G is given by:

$$\begin{aligned} x^{A}{}_{\mathcal{B}} \overset{G}{\mapsto} x^{A}{}_{\mathcal{C}}\, G^{\mathcal{C}}{}_{\mathcal{B}} &= (\delta^{A}{}_{C}, X^{A\dot{C}}) \begin{pmatrix} \mathrm{A}^{C}{}_{B} & \mathrm{B}^{C\dot{B}} \\ \mathrm{C}_{\dot{C}B} & \mathrm{D}_{\dot{C}}{}^{\dot{B}} \end{pmatrix} \\ &= \left(\mathrm{A}^{C}{}_{B} + X^{A\dot{C}}\mathrm{C}_{\dot{C}B}, \mathrm{B}^{A\dot{B}} + X^{A\dot{C}}\mathrm{D}_{\dot{C}}{}^{\dot{B}}\right) \\ &\mapsto \left(\delta^{A}{}_{B}, ((\mathrm{A} + X\cdot\mathrm{C})^{-1}(\mathrm{B} + X\cdot\mathrm{D}))^{A\dot{B}}\right), \end{aligned} \tag{2.80}$$

where the last step uses the equivalence relation to rescale back to the chosen representative class. Thus, the transformation of analytic superspace under SCTs can be extracted to be

$$X \overset{G}{\mapsto} (\mathrm{A} + X\cdot\mathrm{C})^{-1}(\mathrm{B} + X\cdot\mathrm{D}) \quad \forall G \in SL(4|4;\mathbb{C}). \tag{2.81}$$

[10] In its infinitesimal form, this transformation defines the action of the superconformal generators, as introduced in Eqn. 2.4, on analytic superspace. These are listed in Appendix A.2.

2.3.3. Half-BPS Multiplets in Analytic Superspace

Analytic superspace provides the most natural setting for studying half-BPS multiplets, which are central to this thesis. Due to the reduced number of fermionic

[10] In some literature, this transformation is written as:

$$X \overset{G}{\mapsto} (\mathrm{A}\cdot X + \mathrm{B})(\mathrm{C}\cdot X + \mathrm{D})^{-1} \quad \forall G \in SL(4|4;\mathbb{C}), \tag{2.82}$$

which is equivalent to the transformation described here and depends only on the conventions used to define the coset space.

coordinates, half-BPS multiplets are expressed as simple scalar superfields in analytic superspace [70].

These superfields can be intuitively derived by the consecutive action of supercharges on the superprimary field. Following [81], we can represent the half-BPS superfield as:

$$\begin{aligned} \mathbb{O}_p(X) &= \exp\left(\rho^{\alpha\dot{a}} Q_{\alpha\dot{a}} + \bar{\rho}^{a\dot{\alpha}} \bar{Q}_{a\dot{\alpha}}\right) \mathcal{O}_p(x,y) \\ &= \mathcal{O}_p(x,y) + \rho^{\alpha\dot{a}} \Psi^{(p)}_{\alpha\dot{a}} + \bar{\rho}^{a\dot{\alpha}} \bar{\Psi}^{(p)}_{a\dot{\alpha}} + \ldots, \end{aligned} \tag{2.83}$$

where the fields corresponding to different components of the half-BPS multiplet appear at successive orders in the fermionic expansion variables ρ and $\bar{\rho}$.
In this notation, $\mathcal{O}_p(x,y)$ represents the superprimary field of the full half-BPS multiplet $\mathbb{O}_p(X)$ with charge p. (Recall, that $\Delta = p$ for half-BPS multiplets). The detailed structure of these expansions, especially for the stress tensor multiplet, will be developed in Section 3.3.

Under superconformal transformations $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(4|4;\mathbb{C})$, half-BPS multiplets transform as [41]:

$$\begin{aligned} &\delta\mathbb{O}_p(X) = (\mathcal{V} + p\Delta)\, \mathbb{O}_p(X), \\ &\text{where } \Delta = \operatorname{str}(A + XC)\,, \end{aligned} \tag{2.84}$$

with $\mathcal{V}$ being the vector field that generates all superconformal transformations, given in Appendix A.2.

2.4. Correlation Functions

A significant advantage of analytic superspace, or equivalently the Grassmannian formalism, is the relative simplicity in constructing invariants for correlation functions involving half-BPS multiplets.

As outlined in [70], the n-point correlation functions of half-BPS multiplets -which are denoted by $\mathbb{O}_p(X)$, where p corresponds to the $\mathfrak{su}(4)_R$-representation $[0,p,0]$ and $p = \Delta$- are expressed as

$$\langle \mathbb{O}_{p_1}(X_1) \ldots \mathbb{O}_{p_n}(X_n) \rangle\,. \tag{2.85}$$

These correlation functions depend on the n insertion points $X_1^{A\dot{A}}, \ldots, X_n^{A\dot{A}}$, which in the Grassmannian formalism become $(x_1)^A{}_{\mathcal{B}}, \ldots, (x_n)^A{}_{\mathcal{B}}$. In this context, superconformal transformations in $SL(4|4;\mathbb{C})$ act as simple matrix multiplications from the right. Consequently, invariants can be readily formed by contracting the indices as follows:

$$(x_i)^A{}_{\mathcal{B}}(x_j^{\perp})^{\mathcal{B}\dot{C}}, \tag{2.86}$$

where $(x_j^{\perp})^A{}_{\mathcal{B}}$ is defined as the plane perpendicular to $(x_j)^A{}_{\mathcal{B}}$. A straightforward calculation gives:

$$(x_j^{\perp})^{\mathcal{B}\dot{C}} = \begin{pmatrix} -(X_j)^{B\dot{C}} \\ \delta_{\dot{B}}^{\ \dot{C}} \end{pmatrix} \tag{2.87}$$

Thus, the invariant in (2.86) evaluates to:

$$(x_i)^A{}_{\mathcal{B}}(x_j^{\perp})^{\mathcal{B}\dot{C}} = \begin{pmatrix} \delta^A{}_B, (X_i)^{A\dot{B}} \end{pmatrix} \begin{pmatrix} -(X_j)^{B\dot{C}} \\ \delta_{\dot{B}}^{\ \dot{C}} \end{pmatrix} = -X_j^{A\dot{C}} + X_i^{A\dot{C}} \equiv X_{ij}^{A\dot{C}}. \tag{2.88}$$

The object $X_{ij}^{A\dot{C}}$ is superconformally invariant. However, it must also be verified that this object is defined on a valid Grassmannian, meaning it must transform correctly under the $GL(2|2)$ matrix $M^A{}_C$ (and the corresponding matrix for the perpendicular plane) as defined in Eqn. 2.77.
For half-BPS operators, which transform under $GL(2|2)$ transformations by only a scaling factor, this requirement is ensured by taking the superdeterminant as [70]:

$$g_{ij} \equiv \text{sdet}\left(X_{ij}\right). \tag{2.89}$$

The objects g_{ij} are then proper superconformal invariants of the theory, whose inverse are defining the *superpropagators*, which can be utilized to express correlation functions.

Studies of the correlation functions of half-BPS multiplets on analytic superspace have been initiated soon after its introduction; for the case of $\mathcal{N} = 4$ SYM for instance in [82, 83].
In the case of 2- and 3-pt functions of half-BPS multiplets, superconformal symmetry entirely fixes those correlation functions. They can be computed exactly and are protected against any quantum corrections.
The 2pt function can be constrained to be proportional to the above constructed superpropagator as

$$\langle \mathbb{O}_{p_1}(X_1)\mathbb{O}_{p_2}(X_2)\rangle = \left(\hat{d}_{12}\right)^p \quad \text{with } \hat{d}_{12} \equiv g_{12}^{-1} = \frac{1}{\text{sdet}\,(X_{12})} \text{ and } p \equiv p_1 = p_2. \tag{2.90}$$

The propagator $\hat{d}_{ij} = \text{sdet}\,(X)_{ij}^{-1}$ is, with the conventions given in Appendix A.1, given by

$$\hat{d}_{ij} = \frac{\hat{y}_{ij}^2}{x_{ij}^2} \quad \text{with} \;\; \hat{y_{ij}}^{a\dot{a}} = y_{ij}^{a\dot{a}} - \bar{\rho}_{ij}^{a\dot{\alpha}} (x_{ij}^{-1})_{\dot{\alpha}\alpha} \rho_{ij}^{\alpha\dot{a}} \tag{2.91}$$

and equals thus the supersymmetrisation of the bosonic propagators $\frac{y_{ij}^2}{x_{ij}^2}$.

Also the 3pt function is by pure symmetry arguments constrained to be of the simple form [83]

$$\langle \mathbb{O}_{p_1}(X_1)\mathbb{O}_{p_2}(X_2)\mathbb{O}_{p_3}(X_3)\rangle \propto (\hat{d}_{12})^{\frac{p_1+p_2-p_3}{2}} (\hat{d}_{13})^{\frac{p_1+p_3-p_2}{2}} (\hat{d}_{23})^{\frac{p_2+p_3-p_1}{2}}, \tag{2.92}$$

where the proportionality constant has been shown to be independent of the coupling such that also 3pt functions do not receive quantum corrections [83].

The four-point functions of half-BPS operators are not entirely fixed by the superconformal symmetry group $PSU(2,2|4)$ anymore; rather, they are proportional to functions of the cross ratios, which encapsulate the dynamical information of the theory. This can be understood by observing that each operator insertion in analytic superspace introduces eight fermionic degrees of freedom, resulting in a total of 16, 24, and 32 degrees of freedom for two-, three-, and four-point functions, respectively. Conversely, there exist 32 fermionic charges that impose constraints on these degrees of freedom. Consequently, after the constraints on the two- and three-point functions have been applied to eliminate all fermionic degrees of freedom, the operators retain a shared amount of supersymmetry that constrains the correlator and prevents quantum corrections. In the case of four-point functions, it is as well possible to eliminate all 32 fermionic degrees of freedom. However, the resulting correlator "does not possess any remaining supersymmetry", leading to the emergence of quantum corrections, which are captured by unknown functions of the cross ratios.

The fully constrained form of the four-point functions of half-BPS operators, derived from pure symmetry arguments, is given by [46, 70]

$$\langle \mathbb{O}_{p_1}(X_1)\mathbb{O}_{p_2}(X_2)\mathbb{O}_{p_3}(X_3)\mathbb{O}_{p_4}(X_4)\rangle = \text{prefactor} \times f\,(z,\bar{z}|y,\bar{y})$$

$$\text{with } f\,(z,\bar{z}|y,\bar{y}) = a + \left[\left(\frac{(z-y)(z-\bar{y})(\bar{z}-y)(\bar{z}-\bar{y})}{(z-\bar{z})(y-\bar{y})} b(z,y) + z \leftrightarrow \bar{z}\right) + y \leftrightarrow \bar{y}\right] + (z-y)(\bar{z}-y)(z-\bar{y})(\bar{z}-\bar{y}) c(z,\bar{z}|y,\bar{y}), \tag{2.93}$$

where the prefactor captures the transformation behavior of the correlator and can be expressed in terms of the superpropagators $\hat{d}_{ij}$.
The variables $\{z, \bar{z}, y, \bar{y}\}$ refer to the conformal and internal cross ratios, respectively, and are defined as follows [36]

$$\begin{aligned} z\bar{z} =& \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \\ y\bar{y} =& \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \qquad (1-y)(1-\bar{y}) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}\,. \end{aligned} \tag{2.94}$$

As the primary objective of this thesis is to derive the equations leading to such expressions, the superconformal Ward identities (SCWI), for higher-point functions, notably five-point functions, it is beneficial to briefly examine the key ideas and strategies that resulted in the four-point expression 2.93[11].

The exploration of superconformal Ward identities (SCWI) for four-point functions of half-BPS operators with charge 2 commenced in 2000 for $\mathcal{N} = 2$ Superconformal Field Theories [84]. Utilizing analytic superspace, an ansatz for the superprimary correlation function, compatible with the bosonic symmetries, was formulated, followed by the application of the remaining supersymmetry transformations to establish a frame with all fermionic coordinates set to zero.[12] Thus, all relevant information is encapsulated in the superprimary correlator, which, through analyticity arguments, has been shown to depend solely on a single unknown function of the conformal cross ratios [84].

The first complete set of SCWI for four-point functions of stress tensor multiplets in $\mathcal{N} = 4$ SYM was derived shortly thereafter in [43]. By employing field theoretical input and the point-permutation symmetry of the four identical operators, the authors demonstrated that this four-point correlator is also proportional to a

[11] The studies of the 4pt function of half-BPS operators have been deep and numerous. Thus, this summary is by no means complete but should be rather viewed as a collection of some major landmarks in the development of the expression 2.93.

[12] For four-point functions, such a purely bosonic frame is possible, as illustrated by the counting argument provided above. For instance, the equivalence of the standard conformal frame offers such a frame, expressed as:

$$X_1^{A\dot{A}} \to \delta^{A\dot{A}}, \quad X_2^{A\dot{A}} \to \infty, \quad X_3^{A\dot{A}} \to 0, \quad X_4^{A\dot{A}} \to \begin{pmatrix} z & & & \\ & \bar{z} & & \\ & & y & \\ & & & \bar{y} \end{pmatrix}. \tag{2.95}$$

Here, only a $GL(2|2)$ symmetry, which exchanges the eigenvalues, remains.

single function of the two conformal cross ratios, in addition to a single-variable function of only one cross ratio. In particular, they showed that this single-variable function does not receive quantum corrections and is thus protected. This result is known as the *partial non-renormalization theorem* [43]. This theorem has been confirmed perturbatively (e.g., [85]) as well as on the SUGRA side [86, 87].

In 2002, this result was re-derived without the assumption of permutation symmetry for general half-BPS operators in $\mathcal{N} = 4$ SYM [36]. By employing a component field approach (entirely within Minkowski spacetime), the authors showed that, without imposing permutation symmetry, there exist in fact two single-variable functions that remain protected against quantum corrections. Furthermore, these single-variable functions encode only information about short and semi-short multiplets [36].[13]
Throughout this derivation, the variables $z, \bar{z}$ for the conformal cross ratios (see Eqn 2.94) were introduced, significantly simplifying the SCWI.

In the same year, four-point functions of identical operators with arbitrary charge $p \geq 2$ were studied using analytic superspace methods [45]. In a manner analogous to the construction of two-point invariants on analytic superspace presented above, the four-point invariant matrix

$$Z = X_{21}\, X_{13}^{-1}\, X_{34}\, X_{42}^{-1} \tag{2.96}$$

was constructed, such that that the complete superconformal invariants are functions of Z under the adjoint action of the remaining $GL(2|2)$, i.e., [45]

$$f(Z) = f(G^{-1}ZG) \quad \text{with } G \in GL(2|2). \tag{2.97}$$

Functions that satisfy these properties include supertraces, thus allowing the full four-point function to be expressed in terms of the Schur polynomials of Z, defined as $S_{\mathcal{R}}(Z) \equiv \mathrm{str}\,(\mathcal{R}(Z))$, where $\mathcal{R}$ denotes a finite-dimensional representation of $GL(2|2)$. For more details, see [45]. Moreover, by rotating Z into a diagonal form (similar to 2.95), it was demonstrated that the eigenvalues of the resulting rotated matrix correspond exactly to the $z, \bar{z}$ variables introduced in [36].

In 2004, Nirschl and Osborn revisited their previous results to enhance the understanding of the various contributions to the operator product expansion (OPE)

[13] The removal of permutation symmetry allows the usage of the derived results for the operator product expansion (OPE) and the resulting superconformal blocks, which are superconformally invariant but not crossing symmetric. In this manner, it was demonstrated that the single-variable functions allow for an OPE over only short and semi-short multiplets. The topic of OPEs and superconformal block expansions will be outlined in 2.5.

[46]. In this process, utilizing polarisation vectors for the internal symmetry group (see 2.99 below), the SCWI were re-derived, leading to the expression for four-point functions stated in 2.93.

Finally, these results were generalized to various dimensions $d = 3, 4, 5, 6$ in [35]. The authors employed the appropriate analytic superspace formalism for each dimension. Once again, supersymmetry was used to rotate into a frame where all fermionic coordinates were set to zero, a process feasible for four-point functions. In this context, the full supercorrelator is encoded in the superprimary function, as previously established. The invariance under bosonic transformations can then be applied to this superprimary function, demonstrating that in $d = 4$, it is parametrized by six functions of the conformal cross ratios.
To derive a valid and general expression for the full four-point supercorrelator, it is essential to revert from this frame, effectively extending the constructed covariants to full supersymmetric invariants. In the case of four-point functions, there are no nilpotent invariants arising (as can be seen by the aforementioned counting). Thus, it suffices to supersymmetrise the bosonic covariants, as for instance the propagator in Equation 2.91. However, it is crucial to ensure that the analyticity conditions in the y-variables are satisfied. Ensuring this results in the superconformal Ward identities. These Ward identities effectively reduce the six functions of the cross ratios to a single unknown function of the cross ratios, providing consistency with previous derived results.

These equations, leading to the result in 2.93, are given by [35]

$$\left(\partial_{\bar{z}} + \partial_{\bar{y}}\right) \mathcal{G}_4\left(z, \bar{z}; y, \bar{y}\right)\Big|_{\bar{y}\to\bar{z}} = 0, \tag{2.98}$$

where $\{z, \bar{z}, y, \bar{y}\}$ are the cross ratios defined in 2.94, and $\mathcal{G}_4\left(z, \bar{z}; y, \bar{y}\right)$ denotes the four-point function of half-BPS operators of arbitrary charge.

These results concerning the four-point functions of half-BPS multiplets in $\mathcal{N} = 4$ SYM can be reformulated in terms of two topological twists derived in [88] and [89]. The first twist, which we will refer to as the *Drukker-Plefka twist*, aligns the internal polarization vectors y_I, defined as

$$\Phi^I(x) \to y_I \Phi^I(x) \equiv \Phi(x, y), \quad y_I y^I = 0, \tag{2.99}$$

with the Minkowski spacetime coordinates as follows:

$$\bar{y}_I = \left(ix_I^1, ix_I^2, ix_I^3, ix_I^4, \frac{i}{2}(1 - (x^\mu)), \frac{i}{2}(1 + (x^\mu))\right). \tag{2.100}$$

The resulting twisted correlator is shown to be topological, meaning it is independent of the insertion points, or, in other words, it is a constant. This constraint arises from supersymmetry and holds for any n-point function. In analytic superspace coordinates, the twist can be straightforwardly rewritten as $y_i = x_i$.

The second twist is termed the *chiral algebra twist* and is defined when inserting all operators on a plane. For four-point functions, this can be easily accomplished by rotating into the frame defined in 2.95. [14]
This configuration is described by the coordinates of the respective plane, $z, \bar{z}$ (and $y, \bar{y}$ for the corresponding internal plane), with the twist amounting to setting $\bar{y} = \bar{z}$ (or equivalently, $y = z$). The resulting twisted correlator is demonstrated to be a meromorphic function of the variable z (or $\bar{z}$) only [89], encoding the chiral algebra data.
This can be formulated as a differential constraint:

$$\partial_z \mathcal{G}(z, \bar{z}|z, \bar{y}) = 0 \quad \text{and} \quad \partial_{\bar{z}} \mathcal{G}(z, \bar{z}|y, \bar{z}) = 0. \tag{2.101}$$

The resulting meromorphic functions capturing the chiral algebra data correspond exactly to the single-variable functions derived in [36].

These two constraints, arising from twisting the correlator, lead to the expression in 2.93 and are thus equivalent to the full superconformal Ward identities for four-point functions.

The chiral algebra of $\mathfrak{psu}(2, 2|4)$ is introduced in Section 2.6.

2.5. (Super)Conformal Bootstrap

The utility of (super-)conformal symmetry lies in its ability to relate the above four-point as well as higher-point functions to the fully determined two- and three-point functions by leveraging the Operator Product Expansion (OPE). This principle is the foundational motivation for the bootstrap program, which aims to use further the associativity of the OPE to constrain the higher-point function further, thereby enabling the extraction of Conformal Field Theory (CFT) data -namely, the scaling dimensions and three-point coefficients- associated with the constituent two- and three-point functions.

[14] It is important to note that for higher-point functions, a plane configuration cannot be achieved using superconformal transformations. Consequently, this twist will only impose constraints on a restricted part of the full higher-point function.

In this section, we will provide the conceptual kinematical preliminaries necessary to perform further bootstrap analyses.

2.5.1. Operator Product Expansion (OPE)

The Operator Product Expansion (OPE) posits that two operators, when brought infinitesimally close together (specifically, with no other operator inserted between them), can be expressed as a sum over local operators. In a conformal field theory, this expansion has a non-zero radius of convergence, enabling it to serve as an exact analytical tool. Of simplest instance is the OPE between scalar operators, that is schematically expressed as

$$\mathcal{O}_{\Delta_1,l_1}(x)\mathcal{O}_{\Delta_2,l_2}(0) \sim \sum_k c_{12k} \left[C_\mu(x)\mathcal{O}_{\Delta_k,l_k}(0) + \dots\right], \tag{2.102}$$

where the sum spans all conformal primaries in the theory. The function $C_\mu(x)$ is known, and the terms represented by $\dots$ correspond to the descendants of the conformal primary $\mathcal{O}_{\Delta_k,l_k}$, with spin indicated by l_k.
The coefficients c_{12k}, known as *OPE coefficients* or *structure constants*, are proportional to the coefficients of the three-point functions, thus allowing direct access to the CFT data through the OPE.

2.5.2. (Super)Conformal Blocks

The OPE then enables higher-point functions to be expanded in terms of conformal blocks. For example, a four-point function can be written as

$$\langle\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle = \sum_k c_{12k}c_{34k}\, g(u,v), \tag{2.103}$$

where u and v are the standard conformal cross ratios. The function $g(u,v)$, known as the *conformal block*, is fully determined by conformal symmetry and resums the contributions from a conformal multiplet in the OPE expansion, here taken between the operator pairs $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)$ and $\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)$.

These conformal blocks can be derived explicitly, for instance, as eigenfunctions of the quadratic conformal Casimir operator, as originally developed by Dolan and Osborn [90, 91].

The conformal bootstrap framework then proceeds from the observation that the OPE could alternatively be performed between different pairs of operators, such

as $\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_3}(x_3)$ and $\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_4}(x_4)$. This imposes additional constraints on the CFT data, known as *crossing equations*, which the bootstrap program seeks to solve.
The methods used to solve these equations are, for instance, reviewed in [24, 25] and are not subject of this thesis.

The expansion of four-point and higher-point functions in conformal blocks can also be adapted for short supermultiplets within a superconformal field theory. Dolan and Osborn pioneered this approach in [36, 37], leveraging superconformal Ward identities as formulated in [46].

For example, the four-point function of half-BPS multiplets with charges p_i can be expanded as [81]

$$\langle \mathbb{O}_{p_1}(X_1)\mathbb{O}_{p_2}(X_2)\mathbb{O}_{p_3}(X_3)\mathbb{O}_{p_4}(X_4)\rangle = \mathcal{L}(\{x_i,y_i\}) \sum_{\Delta,l,\mathcal{R}} c^{p_1p_2}_{\Delta,l,\mathcal{R}} c^{p_3p_4}_{\Delta,l,\mathcal{R}}\, g^{\{p_i\}}_{\Delta,l,\mathcal{R}}(x_i,y_i,\rho_i,\bar{\rho}_i), \tag{2.104}$$

where $g^{\{p_i\}}_{\Delta,l,\mathcal{R}}(x_i,y_i,\rho_i,\bar{\rho}_i)$ defines a superblock encompassing the entire supermultiplet. The sum runs over all superprimaries in the OPE of the external operators in (12) and (34), characterised by their scaling dimension Δ, spin representation l, and R-symmetry representation $\mathcal{R}$. The term $\mathcal{L}(\{x_i,y_i\})$ denotes a so-called *leg factor* encapsulating the correct transformation behavior of the correlator under the bosonic subgroups.
These four-point superblocks can for instance also be studied via the two-point quadratic Casimir of the complete superconformal algebra, as elaborated in [81].

Likewise, higher-point functions can be expanded in superconformal blocks.
However, the study of superconformal blocks presents certain complexities, and thus we will limit further details to the specific example examined in section 4.2, where we study in particular how these superblocks are constrained by the superconformal Ward identities.

2.6. $\mathfrak{psu}(1,1|2)$

The chiral algebra twist introduced in Section 2.4 restricts $\mathfrak{psu}(2,2|4)$ to its chiral algebra $\mathfrak{psu}(1,1|2)$, which is also occasionally referred to as *small* $\mathcal{N}=4$ *SYM*. Given that this algebra will also be explored throughout this thesis, this section provides a brief overview of the algebra and its representations, the corresponding analytic superspace, and the precise connections to the full four-dimensional maximally symmetric algebra $\mathfrak{psu}(2,2|4)$.

$\mathfrak{psu}(1,1|2)$ consists of the following subalgebras:

- The conformal algebra in one dimensions $\mathfrak{su}(1,1)$, which is generated by translations P, dilatations D and special conformal transformations K. Note that there are no rotations in one dimension.
- The 3-dimensional internal or R-symmetry algebra $\mathfrak{su}(2)_R$, which is generated by the R-symmetry generators $R^a{}_b = R^b{}_a$, $a = 1,2$.
- Furthermore, there are the 4 supersymmetric translations $Q^a, \bar{Q}_a$, $a = 1,2$ and the 4 superconformal generators $S_a, \bar{S}^a$, $a = 1,2$.

The complexification of $\mathfrak{psu}(1,1|2)$ is given by

$$\mathfrak{psu}(1,1|2) \overset{\mathbb{C}}{\to} \mathfrak{psl}(2|2;\mathbb{C}) \tag{2.105}$$

$$\text{with } \mathfrak{su}(1,1) \overset{\mathbb{C}}{\to} \mathfrak{sl}(2;\mathbb{C}) \quad \& \quad \mathfrak{su}(2)_R \overset{\mathbb{C}}{\to} \mathfrak{sl}(2;\mathbb{C}), \tag{2.106}$$

such that we can package the 6 bosonic generators into a single object $K^{ab} = -K^{ba}$, $a,b = 1,\dots 4$ obeying the standard algebra relations of $\mathfrak{so}(4;\mathbb{C}) \cong \mathfrak{su}(2;\mathbb{C}) \oplus \mathfrak{su}(2;\mathbb{C})$ [92].[15]

2.6.1. Representations of $\mathfrak{psu}(1,1|2)$

Rather than directly studying all possible representations of $\mathfrak{psu}(1,1|2)$ (or its complexification) from the superalgebra, we will focus on the representations relevant to this thesis by utilizing the chiral algebra twist constructed in [89], which has also been applied to representations in [93].[16]

In [89], the authors identified subsectors of four-dimensional $\mathcal{N} = 2$ SCFTs that form a two-dimensional infinite chiral algebra. This chiral map is expressed as follows:

$$\begin{array}{rccc} \chi: & \text{4d } \mathcal{N}=2 \text{ SCFTs} & \to & \text{2d chiral algebra} \\ & \mathfrak{sl}(4|2) & \to & \mathfrak{sl}(2) \times \mathfrak{sl}(2|2) \end{array} \tag{2.107}$$

[15]The full superalgebra can be found in [92] as well. We will not make explicit use of the algebra and hence do not state it here.

[16]We will utilize the results established here for $\mathfrak{psl}(2|2;\mathbb{C})$ in relation to $\mathfrak{psu}(1,1|2)$ without delving into the specifics of taking real forms. Additionally, we will omit the notation $\mathbb{C}$ for complex algebras henceforth.

This map can be extended to $\mathfrak{psl}(4|4)$ (the complexification of $\mathfrak{psu}(2,2|4)$) since the $\mathcal{N}=2$ algebra is embedded within the $\mathcal{N}=4$ algebra as

$$\mathfrak{psl}(4|4) \supset \mathfrak{sl}(4|2) \oplus \mathfrak{sl}(2). \tag{2.108}$$

Consequently, the map acts on the irreducible representations of $\mathfrak{psl}(4|4)$, reducing them to irreducible representations of $\mathfrak{psl}(2|2)$ as demonstrated in [93]:

$$\chi : \text{Irreps of } \mathfrak{psl}(4|4) \quad \rightarrow \quad \text{Irreps of } \mathfrak{psl}(2|2) \tag{2.109}$$

$$\begin{aligned} \chi\left(\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,p,0](0,0)}\right) &= \mathcal{S}_{\left(h=\frac{1}{2}p\right)} \\ \chi\left(\mathcal{B}^{\frac{1}{4},\frac{1}{4}}_{[q,p,q](0,0)}\right) &= \mathcal{L}_{(h,j)=\left(q+\frac{1}{2}p,\frac{1}{2}p\right)} \\ \chi\left(\mathcal{C}_{[k,p,q](j,\bar{j})}\right) &= \mathcal{L}_{(h,j)=\left(\frac{1}{2}(k+p+q)+j+\bar{j}+2,\frac{1}{2}p\right)} \\ \chi\left(\mathcal{A}\right) &= 0, \end{aligned} \tag{2.110}$$

where j denotes the Dynkin label (i.e., the highest weight) of the $\mathfrak{sl}(2)_R$ representation, and h labels the representation of the complexified $\mathfrak{su}(1,1)$.
Note that this convention allows for half-integer representation labels.

The full construction of the respective supermultiplets under $\mathfrak{psu}(1,1|2)$, as listed above, adheres to the same principles described in the previous section of superconformal representations.
$\mathcal{L}_{(h,j)}$ refers to long multiplets of $\mathfrak{psu}(1,1|2)$, i.e., supermultiplets spanned by the complete action of all four Q- and $\bar{Q}$-charges on the superprimary.
Conversely, $\mathcal{S}_{\left(h=\frac{1}{2}p,j=\frac{1}{2}p\right)}$ denotes half-BPS or short multiplets of $\mathfrak{psu}(1,1|2)$. The superprimaries within these multiplets satisfy shortening conditions of the form

$$Q^1 \left|h,j\right\rangle^{\text{hw}} = 0 \quad \& \quad \bar{Q}_2 \left|h,j\right\rangle^{\text{hw}} = 0, \tag{2.111}$$

such that only Q^2 and $\bar{Q}_1$ act non-trivially.
It is important to note that under the chiral algebra map, the long representations of $\mathfrak{psu}(2,2|4)$ vanish, implying that the chiral algebra captures only the short and semi-short contributions of $\mathfrak{psu}(2,2|4)$. This reinforces the idea that the single-variable functions of the four-point function in $PSU(2,2|4)$ (introduced in 2.4), which admit an operator product expansion only over short and semi-short multiplets [36], indeed encapsulate the chiral algebra data.

As an explicit example, consider the projection of the stress-tensor multiplet under the chiral map. This projection is given by:

$$\chi\left(\mathcal{T}\right) = \mathcal{S}_{(h=1,j=1)}. \tag{2.112}$$

This supermultiplet can be easily constructed from the superprimary $(j=1)_{(h=1)}$ as illustrated in Figure 2.3 [66]. The multiplet depicted in this figure is constructed from left to right. The potential negative contributions to this supermultiplet are

$$\begin{array}{ccccc} & & (\frac{1}{2})_{\frac{1}{2}} & & \\ & \nearrow & & \searrow & \\ (1)_1 & & & & (0)_0. \\ & \searrow & & \nearrow & \\ & & (\frac{1}{2})_{\frac{1}{2}} & & \end{array}$$

Figure 2.3.: Short supermultiplet of charge $j=1$ in $\mathfrak{psu}(1,1|2)$. The superprimary is a scalar transforming in the (1) (or the **3**-dimensional) representation of $\mathfrak{su}(2)_R$. Successive actions of Q-charges (indicated by downward arrows) and $\bar{Q}$-charges (indicated by upward arrows) on the superprimary span the multiplet. Representations with a negative overall sign are eliminated when imposing conservation equations and are thus omitted.

eliminated when enforcing the conservation equations, following analogous reasoning as presented in the case of $\mathfrak{psu}(2,2|4)$. Therefore, these representations have been directly excluded in Figure 2.3.

There are different notations for $\mathfrak{su}(2)$-multiplets and their respective representations. In this discussion, the highest-weight notation for half-integer spins has been employed, wherein the label (1) corresponds to the three-dimensional representation **3** of $SU(2)$. This notation clarifies particularly the chiral algebra map. However, we will now transition to a highest-weight notation that permits only integer spins, such that representations are labeled by $p=2j$. For a better distinction, we will denote half-BPS multiplets under $\mathfrak{psu}(1,1|2)$ using this notation as

$$\mathcal{S}_{h,j} \to \mathcal{W}_p \quad \text{with } p = 2j. \tag{2.113}$$

This notation facilitates a more straightforward and intuitive application of superspace methods, as will be elaborated below.

To elucidate this, we present the following representations in the $\mathcal{W}_2$ supermultiplet in all various notations including the corresponding field notation:

$$J(z,y) \sim \mathbf{3}\text{-dim. rep. of } su(2)_R \leftrightarrow \text{Dynkin l. } (2) \leftrightarrow \frac{1}{2}\text{integer spin: } (1) \tag{2.114}$$

$$T(z) \sim \mathbf{1}\text{-dim. rep. of } su(2)_R \leftrightarrow \text{Dynkin l. } (0) \leftrightarrow \frac{1}{2}\text{integer spin: } (0) \tag{2.115}$$

$$G(z,y) \sim \mathbf{2}\text{-dim. rep. of } su(2)_R \leftrightarrow \text{Dynkin l. } (1) \leftrightarrow \frac{1}{2}\text{integer spin: } (\frac{1}{2}) \tag{2.116}$$

$$\tilde{G}(z,y) \sim \mathbf{2}\text{-dim. rep. of } su(2)_R \leftrightarrow \text{Dynkin l. } (1) \leftrightarrow \frac{1}{2}\text{integer spin: } (\frac{1}{2})\,, \tag{2.117}$$

with the half-integer spin notation being omitted from now own.
$G(x)$ correspond to the supercurrent and $T(x)$ is the Stress-Energy Tensor in 1d.

2.6.2. Analytic superspace of $\mathfrak{psu}(1,1|2)$

In a manner analogous to the full four-dimensional case, the half-BPS multiplets of $\mathfrak{psu}(1,1|2)$ are most effectively described on analytic superspace.
For $\mathfrak{psu}(1,1|2)$, the analytic superspace takes the following form:

$$X^{A\dot{A}} = \begin{pmatrix} x & \rho \\ \bar{\rho} & y \end{pmatrix}, \tag{2.118}$$

where x denotes the one-dimensional spacetime coordinate, y represents the single bosonic coordinate that parametrizes the internal symmetry group $SU(2)_R$, and ρ and $\bar{\rho}$ correspond to the two non-trivial supersymmetry charges Q^2 and $\bar{Q}_1$, as outlined in Equation 2.111.
Each subspace of this analytic superspace is one-dimensional, rendering $\mathfrak{psu}(1,1|2)$, in addition to its significance as the chiral algebra of $\mathfrak{psu}(2,2|4)$, a particularly simple toy model.

Again, half-BPS supermultiplets are represented as simple scalar superfields within the analytic superspace framework, that can be derived from the following expression:

$$\mathcal{W}_p(X) = \exp\left(\rho Q + \bar{\rho}\bar{Q}\right) \mathcal{O}_p(x,y), \tag{2.119}$$

where $\mathcal{O}_p$ denotes the superprimary of the short multiplet $\mathcal{W}_p$ with charge p.
The precise fermionic expansions of the multiplets under consideration are developed in Section 3.2.

The two-point functions of the half-BPS multiplets of $\mathfrak{psu}(1,1|2)$ can be computed using the superpropagator discussed in Section 2.4. Utilizing the analytic superspace of $\mathfrak{psu}(1,1|2)$ from Equation 2.118, the two-point function thus takes the form

$$\langle \mathcal{W}_{p_1}(X_1)\mathcal{W}_{p_2}(X_2)\rangle = \frac{1}{\mathrm{sdet}\,(X_{12})^p} = \left(\frac{y_{12} - \bar{\rho}_{12}x_{12}^{-1}\rho_{12}}{x_{12}}\right)^p, \quad \text{with } p_1 = p_2 \equiv p. \tag{2.120}$$

Note that in this one-dimensional context, the determinants of the subspaces correspond directly to the coordinates.

Consequently, covariants such as $\dfrac{y^2}{x^2}$ represent the literal squares of the respective variables.

Part II.

5pt Function in $\mathcal{N} = 4$ SYM

3. Constraining the Correlator

3.1. Introduction

The previous chapter outlined the extensive research that has been performed on the kinematics of correlation functions of half-BPS operators and provided the necessary background to understand these studies. In particular, correlation functions with up to four operator insertions are by now well understood from the kinematical point of view.
However, much less is known about five- and higher-point correlation functions, despite their significant importance. The study of higher-point correlators is crucial for several reasons.

One main reason is the wealth of CFT data encoded within higher-point (scalar) functions, which cannot be extracted from four-point functions of scalar operators. The CFT data, comprising the scaling dimensions Δ_i of all operators in the spectrum and their three-point structure constants λ_{ijk}, contains the complete information about the theory. In principle, this data is fully determined by the two- and three-point functions, as shown in Equations 2.90 and 2.92. However, in practice, extracting this data from those functions is extremely challenging, as it requires solving an infinite amount of increasingly difficult equations.
As outlined in section 2.5, an alternative approach is to utilize the operator product expansion (OPE) to express a single multi-point function as an infinite sum of three-point functions, thereby accessing all the respective CFT data of those three-point functions from the single multi-point function. For example, the four-point function of external half-BPS multiplets studied in Section 2.4 is equivalent to infinitely many three-point functions involving two half-BPS operators and one unprotected (non-BPS) operator when taking the OPE limit. However, three-point functions involving multiple unprotected operators cannot be accessed through four-point functions of external half-BPS multiplets. This type of information about the theory is most efficiently obtained through the study of higher-point functions. For instance, a five-point function of half-BPS operators in the double-OPE limit encapsulates an infinite sum of three-point functions that involve multiple non-BPS operators.

This reasoning similarly applies to the study of higher-point functions in general CFTs, where it is necessary to study higher-point functions in order to access CFT data of multiple spinning operators.

The study of higher-point functions is further motivated by numerous other reasons. For instance, there exists a two-fold duality involving half-BPS correlators and scattering amplitudes. Through the AdS/CFT correspondence, correlation functions of half-BPS multiplets in $\mathcal{N} = 4$ SYM are dual to on-shell scattering amplitudes of type IIB supergravitons in AdS.
Additionally, there is a correlator/amplitude duality between half-BPS correlators and scattering amplitudes within $\mathcal{N} = 4$ SYM. In the planar polygonal lightlike limit, half-BPS correlators can be directly related to scattering amplitudes in the planar limit, even directly at the level of the integrand [94–96]. For instance, the integrand of the four-point function of stress tensor multiplets in the lightlike limit is dual to the square of the integrand of a planar four-gluon scattering amplitude. These dualities have not only provided efficient computational tools for scattering amplitudes, but they have also offered new insights into general aspects of the theory. To further investigate multi-particle scattering and explore these dualities in greater depth, it is crucial to develop a more comprehensive understanding of higher-point functions.

Moreover, a hidden symmetry originating from the ten-dimensional theory, discovered in [97], can be used to organize all four-point tree-level correlators in $\mathrm{AdS}_5 \times S_5$. This phenomenon has also been observed in the planar limit on the dual CFT side, where the integrands of four-point correlators of half-BPS operators with arbitrary charges can be derived from the stress tensor correlator, i.e. the four-point correlator of lowest charge (see, for example, [98,99]). Whether this symmetry is specific to four points or extends to higher-point functions can only be answered by studying higher-point correlators.

For these and more reasons, the primary aim of this thesis is to advance the understanding of higher-point functions by focusing on the fundamental step in the analysis of superconformal correlation functions: the derivation and examination of superconformal Ward identities (SCWI).
The SCWI encode the full constraints imposed by the underlying superconformal symmetry group. When applied to correlation functions, they significantly reduce the complexity of the correlators, enabling a more efficient and systematic analysis.
Moreover, the SCWI provide a pathway to important theoretical insights, such

as the partial non-renormalization theorem that has been derived for four-point functions [43], and play a central role in an efficient computation of holographic correlators in AdS (see for instance [100] for a recent study of higher point correlators). These and more reason make superconformal Ward identities essential for a thorough exploration of five-point functions and beyond.

In Chapter 2, it was established that two- and three-point correlation functions are completely fixed by superconformal symmetry. However, starting from 4-point functions, this is no longer the case, and additional considerations and strategies are needed. Various methods for deriving the SCWI for four-point functions, stated in Equation 2.98, were discussed in Section 2.4. Many of these approaches, however, leverage the fact that four-point functions of half-BPS multiplets do not involve nilpotent invariants (in other words, we can turn off all fermionic dependence with the superconformal transformations, showing that the correlator is entirely determined by the superprimary correlator). Extending these methods to five-point functions and beyond, where nilpotent invariants are present, introduces considerable challenges.
The construction of these higher-point nilpotent invariants has been initiated in [101, 102], where the invariants for the chiral half of general n-point functions have been constructed. This has been utilised in [70] to construct the superprimary contribution of the five-point correlator of stress-tensor multiplets, which represents the current state-of-the art. [1]
However, this construction has not been generalised to the full fermionic and non-perturbative dependence, emphasising the necessity of developing new strategies to derive the SCWI for higher-point functions.

Two subclasses of such identities have been established in earlier studies: those arising from the Drukker-Plefka twist [88] and from the chiral algebra twist [89]. The Drukker-Plefka twist imposes conditions from supersymmetry that hold for any n-point function and should therefore be included in the complete set of superconformal Ward identities. In contrast, the chiral algebra twist is only defined when all operators are inserted on plane. Consequently, since higher-point functions are not generally restricted to planar configurations, the chiral algebra twist impacts only a subset of the correlator. This naturally leads to the expectation that the full superconformal Ward identities are stronger than these two constraints.

In this chapter, we begin addressing these questions by introducing a new ap-

[1] A generalization of the construction proposed in [101,102] to the full non-chiral n-point correlation functions of stress-tensor multiplets in the Born approximation was developed in [103].

proach to derive the superconformal Ward identities for half-BPS correlators in a safe and reliable way, independent of the number of insertion points and thus especially suited for higher point functions. Specifically, we will examine the five-point function of stress-tensor multiplets, as introduced in Section 2.2.4, and denoted by

$$\mathcal{G}_{22222}(\{X_i\}) = \langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle = \langle 22222\rangle \,, \tag{3.1}$$

where we use the shorthand notation $\langle 22222\rangle$ to indicate that the stress-tensor multiplet is a half-BPS multiplet with charge $p = 2$.

Considerations of this correlator include expressions from the SUGRA side [104] as well as perturbative expressions from weak coupling [105]. Recently, the integrand of the superprimary contribution to this correlator has been computed up to three loops [106]. These expressions can be used for cross-checks with the derived identities.

Before proceeding with the strategy and derivation of the superconformal Ward identities, it is worth completing this introduction to five- and higher-point functions by highlighting several efforts undertaken to study these objects in general d-dimensional (S)CFTs from a bootstrap perspective. These efforts include both analytic approaches [100,107–110] and numerical studies [111,112]. However, bootstrapping higher-point functions remains a challenging endeavour, as explicit results for five-point conformal blocks are generally unavailable. An exception is the conformal block for scalar exchange, which was computed in [113]. Additionally, a series expansion of five-point conformal blocks with exchanged spinning operators was presented in [104]. Beyond these results, higher-point conformal blocks have been linked to mathematical frameworks such as Gaudin models [114–117] and more.

Strategy to obtain higher-point SCWI.

In order to derive the SCWI for the above five-point correlation function 3.1, additional constraints or strategies must be employed beyond the basic invariance under the generators of the superconformal group, as was the case for the four-point functions. To this end, we combine superconformal invariance with general properties of half-BPS multiplets in analytic superspace.

As introduced in Section 2.3, half-BPS supermultiplets are most naturally studied in the framework of Analytic Superspace, where they can be systematically described through an expansion in the fermionic coordinates ρ and $\bar{\rho}$. The complete expansion of a half-BPS multiplet is governed by the differential constraint[2]

$$\left(\frac{\partial}{\partial X}\right)^{p+1} \mathbb{O}_p(X) = 0 \quad \text{w. graded symmetrisation,} \tag{3.2}$$

where $\mathbb{O}_p(X)$ denotes a half-BPS multiplet of charge p, and X represents the analytic superspace corresponding to the superconformal group under consideration. This constraint should be understood with graded symmetrisation. Specifically, it involves symmetrising the latin indices $(a, \dot{a})$, while anti-symmetrising the greek indices $(\alpha, \dot{\alpha})$. This convention ensures unitary representations and is preferred over the anti-symmetric representation (where $(\alpha, \dot{\alpha})$ are symmetrised and $(a, \dot{a})$ are anti-symmetrised), which would yield non-unitary representations.
The origin and details of this constraint will be further elaborated in Section 3.3.

By combining this constraint on the half-BPS multiplet, that extends to hold on the correlator (see A.3), with the requirement of full superconformal invariance, we have the following system of equations at our disposal:

$$\left(\frac{\partial}{\partial X_i}\right)^{p_i+1} \mathcal{G}_{p_1,\dots,p_n}(\{X_i\}) = 0, \tag{3.3}$$

$$\mathcal{J}\, \mathcal{G}_{p_1,\dots,p_n}(\{X_i\}) = 0 \quad \forall\, \mathcal{J} \in \mathfrak{psu}(2,2|4), \tag{3.4}$$

where $\mathcal{J}$ represents the generators of the superconformal group $\mathfrak{psu}(2,2|4)$.
These two types of constraints will lead to the full superconformal Ward identities.

The general procedure for deriving the SCWI through the constraints 3.3 and 3.4 can be outlined as follows. Solving the constraint on the half-BPS multiplet provides the complete and fully specified expansion of the multiplet in fermionic coordinates, with the superprimary appearing at zero order and the various descendants emerging at higher orders. This expansion can then be substituted into the correlator defined in Equation 3.1, resulting in an expansion of the correlation function itself in terms of the fermionic coordinates ρ and $\bar{\rho}$. At zero order, this expansion contains the superprimary correlator

$$\langle \mathcal{O}_{20'}(x_1, y_1)\mathcal{O}_{20'}(x_2, y_2)\mathcal{O}_{20'}(x_3, y_3)\mathcal{O}_{20'}(x_4, y_4)\mathcal{O}_{20'}(x_5, y_5)\rangle\,, \tag{3.5}$$

and at higher orders, it includes correlators involving descendant operators.
Importantly, all the correlators appearing in this expansion are fully bosonic, with

[2]It should be noted that this is not any *external constraint* in the common sense, but simply reflects the analyticity properties of half-BPS multiplets. For more details see [78].

the fermionic dependence entirely captured by the expansion variables ρ and $\bar{\rho}$. We will denote these correlators appearing in the full expansion of the supercorrelator $\mathcal{G}_{\{p_i\}}$, that involve only conformal and R-symmetry primaries, as $G^{\mathcal{O}_1 \dots \mathcal{O}_n}$, where $\mathcal{O}_i$ refers to the respective conformal primary and R-symmetry highest weight operator.
Once this expansion is obtained, the constraints from superconformal invariance, as schematically expressed in Equation 3.4, can be applied to the correlation function.

Invariance under the bosonic subgroups allows us to factorize the correlators (of conformal and R-symmetry primaries) into a basis of prefactors, which carry the correct transformation properties of the correlator, and unknown functions of the theory's invariants, the cross ratios. We refer to these conformal and R-symmetry prefactors as *structures*, and can thus schematically express the correlator as

$$G^{\mathcal{O}_1 \dots \mathcal{O}_n}(\{x_i, y_i\}) = \sum_{i=1}^{N} (\text{y-structure} \cdot \text{x-structure})_i \cdot f_i(\{u\}), \tag{3.6}$$

where N is the dimension of the basis, i.e. the maximal number of independent structures, and $\{u\}$ denotes the respective set of conformal cross ratios. It is important to note that the functions $f_i(\{u\})$ depend only on the conformal cross ratios and not on the equivalent R-symmetry cross ratios. This is because, in this ansatz, the analyticity conditions on the internal coordinates, originating from the R-symmetry group being compact, are directly implemented by excluding possible singularities, which would arise when having arbitrary factors of cross ratios. Instead, the entire y-dependence is polynomial and is fully encoded within the R-symmetry structures.
Writing the bosonic correlators in this form - in a basis of known structures multiplying unknown functions of the cross ratios - exhausts all the bosonic symmetries, such that only the supersymmetry constraints remain to be applied.

Supersymmetric invariance on the full supercorrelator in analytic superspace translates into the following simple condition:

$$\sum_{i=1}^{n} \frac{\partial}{\partial \rho_i} \mathcal{G}_{\{p_i\}}(\{x_i, y_i, \rho_i, \bar{\rho}_i\}) = 0 \tag{3.7}$$

$$\sum_{i=1}^{n} \frac{\partial}{\partial \bar{\rho}_i} \mathcal{G}_{\{p_i\}}(\{x_i, y_i, \rho_i, \bar{\rho}_i\}) = 0, \tag{3.8}$$

since the only non-trivial supercharges act as simple shifts in these fermionic coordinates (see Appendix A.2 for further reference). Note that it suffices to impose the invariance under the above fermionic charges. Invariance under the remaining

fermionic charges follows then from the superalgebra 2.9.
Imposing these constraints on the correlator expansion of $\mathcal{G}_{\{p_i\}}$ affects only the fermionic variables, yielding δ-functions that establish relations among the various bosonic correlators $G^{\mathcal{O}_1\ldots\mathcal{O}_n}$. By writing the bosonic correlators in a manner that is consistent with the bosonic symmetries, as outlined above (Eqn. 3.6), we can compare coefficients of each independent structure, ultimately deriving relations that involve only the unknown functions of the conformal cross ratios. These relations culminate in the formulation of the superconformal Ward identities.

The outline of this chapter is as follows. We will begin in Section 3.2 by detailing the procedure using the simpler case of the correlator

$$\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)} = \langle \mathcal{W}_2(X_1)\mathcal{W}_2(X_2)\mathcal{W}_2(X_3)\mathcal{W}_2(X_4)\mathcal{W}_2(X_5)\rangle \tag{3.9}$$

invariant under $\mathfrak{psu}(1,1|2)$. The purpose of this section is to elucidate the outlined main strategy and to demonstrate its efficacy before transitioning to the more complex $\mathfrak{psu}(2,2|4)$ case. Due to the emphasis on the practical strategy, certain details, such as the origin of the multiplet constraint in Eqn. 3.2, will be deferred to section 3.3.
Subsequently, we will focus on the $\mathfrak{psu}(2,2|4)$ case for the remainder of the chapter. In Section 3.3, we will examine the multiplet constraint 3.3 for the stress tensor multiplet, providing a thorough explanation of its origin and all relevant details.
In Section 3.4, we will construct the non-trivial conformal and R-symmetry structures required to express all the bosonic correlators present in the full supercorrelator 3.1.
Next, in Section 3.5, we will apply the constraints from supersymmetry to derive the superconformal Ward identities (SCWI). This section will focus however momentarily on the four-point function of stress tensors again, as the implementation and the resulting equations are considerably simpler than those for the five-point functions. Thus, the four-point function serves as an excellent starting point to illustrate how the supersymmetric invariance can be implemented in the four-dimensional case while also confirming that it aligns with known results from the literature.
Finally, in Section 3.6, we will obtain the SCWI for the five-point function of stress tensor multiplets invariant under $\mathfrak{psu}(2,2|4)$.

3.2. psu$(1, 1|2)$ as a toy model

To illustrate the methodology, we begin by examining correlation functions constrained by the symmetry group $PSU(1, 1|2)$, which can be interpreted via a chiral algebra twist of the full superconformal group in four dimensions (see Section 2.6). Specifically, we consider the five-point function of the half-BPS multiplet with charge $p = 2$, denoted as $\mathcal{W}_2(X)$, which is obtained by performing the chiral algebra twist to the stress tensor multiplet of $PSU(2, 2|4)$. The considered correlation function is thus

$$\mathcal{G}_{22222}^{psu(1,1|2)}(\{X_i\}) = \langle \mathcal{W}_2(X_1)\mathcal{W}_2(X_2)\mathcal{W}_2(X_3)\mathcal{W}_2(X_4)\mathcal{W}_2(X_5)\rangle. \tag{3.10}$$

For details on this multiplet and the relevant conventions, refer to Section 2.6.

This section serves as an outline of the main strategy to aid in understanding the procedure before delving into the full five-point function in $PSU(2, 2|4)$. Thus, many details will be omitted here, focusing instead on the general procedure.

3.2.1. Multiplet Field Expansion in Analytic Superspace

The first set of constraints used to derive the five-point function $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ arise from general properties of half-BPS multiplets in analytic superspace. These are encoded in the following equation:

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^{p+1} \mathcal{W}_p(X^{A\dot{A}}) = 0 \quad \text{w. graded sym.} \tag{3.11}$$

The origin and detailed discussion of this constraint are provided in Section 3.3. For the purposes of this section, we will present only the results and focus on the implications of the constraint.

For the multiplet with charge $p = 2$, we obtain the specific constraint:

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^{3} \mathcal{W}_2(X^{A\dot{A}}) = 0 \quad \text{w. graded sym.} \tag{3.12}$$

This constraint should be understood with graded symmetrisation, meaning it decomposes into various derivatives on the subspaces, with a symmetrisation over the latin indices $(a, \dot{a})$ and an antisymmetrisation over the greek indices $(\alpha, \dot{\alpha})$. For the one-dimensional subspaces present here, this results in the absence of higher-order derivatives of x. Additionally, no higher-order derivatives of the coordinates

ρ and $\bar{\rho}$ are present due to their fermionic nature.
The relevant components of 3.12 are:

$$\begin{aligned}
&\frac{\partial^3}{\partial y^3}, \quad \frac{\partial^2}{\partial y^2}\frac{\partial}{\partial \rho}, \quad \frac{\partial^2}{\partial y^2}\frac{\partial}{\partial \bar{\rho}}, \quad \left(\frac{\partial}{\partial y}\frac{\partial}{\partial \bar{\rho}}\frac{\partial}{\partial \rho} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}\right), \\
&\frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial \rho}, \quad \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial \bar{\rho}}.
\end{aligned} \tag{3.13}$$

Details on how these derivatives are extracted from the constraint in equation 3.12 will be provided in the discussion of the four-dimensional case.
As the constraints hold however in particular on the correlators, these equations can be confirmed through the two-point function.

These constraints act on the general ansatz for the half-BPS multiplet $\mathcal{W}_2(X)$ in analytic superspace. Since the multiplet is constructed via the successive action of the Q and $\bar{Q}$-charges, which are parameterized by ρ and $\bar{\rho}$, the ansatz can be formulated as an expansion in these fermionic coordinates as

$$\mathcal{W}_2(X) = J(x, y) + \rho G(x, y) + \bar{\rho}\bar{G}(x, y) + \rho\bar{\rho}A(x, y). \tag{3.14}$$

The constraints apply at each power of ρ and $\bar{\rho}$, specifying the corresponding field content.

For example, at the order zero (level zero), the only non-trivial constraint is:

$$\frac{\partial^3}{\partial y^3}\left(\mathcal{W}_2(X)\Big|_{\rho=\bar{\rho}=0}\right) = \frac{\partial^3}{\partial y^3}J(x, y) = 0. \tag{3.15}$$

Expanding $J(x, y)$ in the internal coordinates, this constraint truncates the Taylor expansion at cubic order, implying:

$$J(x, y) = J(x, 0) + y\frac{\partial J(x, y)}{\partial y}\Big|_{y=0} + y^2\frac{\partial^2 J(x, y)}{\partial y^2}\Big|_{y=0}. \tag{3.16}$$

Since each order in y corresponds to an internal degree of freedom, $J(x, y)$ describes thus indeed the **3**-dimensional representation of $\mathfrak{su}(2)_R$, corresponding to the label (2), which is indeed the superprimary of $\mathcal{W}_2(X)$.

Applying this constraint $\frac{\partial^3}{\partial y^3}(\ldots) = 0$ at higher orders in the fermionic expansion restricts the respective fields very similarly, limiting their Taylor expansions to quadratic order in y. However, those fields get further constrained by:

$$\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial \rho} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\frac{\partial}{\partial \bar{\rho}}. \tag{3.17}$$

At orders ρ and $\bar{\rho}$ (level 1), these constraints 3.17 ensure that the fields $G(x, y)$ and $\bar{G}(x, y)$ indeed correspond to the **2**-dimensional representations (1) of $SU(2)_R$.

At order $\rho\bar{\rho}$, the constraint

$$0 = \left(\frac{\partial}{\partial y}\frac{\partial}{\partial \bar{\rho}}\frac{\partial}{\partial \rho} + \frac{1}{2}\frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x}\right)\mathcal{W}_2(X) \tag{3.18}$$

yields:

$$\frac{\partial}{\partial y}\left(A(x, y) + \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial}{\partial x}J(x, y)\right) = 0. \tag{3.19}$$

This constraint affects only the y-dependent part of $A(x, y)$. Thus, expanding $A(x, y)$ as

$$A(x, y) = A(x, 0) + y\frac{\partial A(x, y)}{\partial y}\Big|_{y=0}, \tag{3.20}$$

where we have used the aforementioned constraints 3.17 to truncate after the linear order, we observe that the y-dependent term at order $\rho\bar{\rho}$ is given by the derivative of the superprimary, i.e.

$$A(x, y) = A(x, 0) - \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial}{\partial x}J(x, y). \tag{3.21}$$

The term $A(x, 0)$ can be identified with the stress tensor representation $(0)_0$, yielding

$$A(x, y) = T(x) - \frac{1}{2}\frac{\partial}{\partial y}\frac{\partial}{\partial x}J(x, y). \tag{3.22}$$

Solving all constraints in a similar fashion, we obtain the full field expansion:

$$\mathcal{W}_2(X) = \left(1 - \frac{1}{2}\rho\bar{\rho}\frac{\partial^2}{\partial x \partial y}\right)J(x, y) + \rho G(x, y) + \bar{\rho}\bar{G}(x, y) + \rho\bar{\rho}T(x), \tag{3.23}$$

where the fields correspond to the representations in the supermultiplet as derived above. The field expansion is now in correspondence with the notation used in 2.117, with $J(x, y), G(x, y), \bar{G}(x, y), T(x)$ being the conformal primaries and highest weights under $\mathfrak{su}(2)_R$ as discussed.
A special emphasis should be placed on the fact that the superprimary reappears at higher fermionic orders. This will prove to be crucial in the derivation of the Superconformal Ward identities.

3.2.2. Constraining the Correlator

The objective is to derive the correlator $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ in its most constrained form, where the constraints are purely kinematical, i.e. being determined solely by the symmetry group.
The properties established above for the half-BPS multiplet $\mathcal{W}_2(X)$ on Analytic Superspace can be utilized by inserting the multiplet field expansion 3.23 into the correlation function $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$. This yields the following fermionic expansion of the full supercorrelator itself:

$$
\begin{aligned}
\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) = &\langle \mathcal{W}_2(X_1)\mathcal{W}_2(X_2)\mathcal{W}_2(X_3)\mathcal{W}_2(X_4)\mathcal{W}_2(X_5)\rangle \\
= &\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \sum_{i=1}^{5} \rho_i\bar{\rho}_i \langle T(x_i)J(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \\
&- \frac{1}{2}\sum_{i=1}^{5} \rho_i\bar{\rho}_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_i} \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&- \sum_{i=1}^{5}\sum_{j=1,j\neq i}^{5} \rho_i\bar{\rho}_j \langle G(x_i,y_i)\bar{G}(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \\
&+ \dots
\end{aligned}
\tag{3.24}
$$

where $i \neq j \neq k \neq l \neq m \in \{1,2,3,4,5\}$. The correlators at order $\mathcal{O}(\rho), \mathcal{O}(\bar{\rho})$ are excluded due to symmetry constraints. We will initiate our analysis by considering everything up to order $\mathcal{O}(\rho\bar{\rho})$ and thus, higher order terms in the expansion have been suppressed.
In this form, the supercorrelator comprises the superprimary correlator at zero order, in addition to three contributions at order $\mathcal{O}(\rho\bar{\rho})$. These contributions include two superdescendant correlators, i.e. correlators with at least one superdescendent. Moreover, due to the correction term at order $\rho\bar{\rho}$ in the multiplet expansion, the superprimary correlator reappears at this order.
With the correlator expressed in this expanded form, we can impose the constraints arising from invariance under the superconformal symmetry group $\mathfrak{psu}(1,1|2)$.

In this section, we will commence by applying the conditions for supersymmetric invariance, before turning to the bosonic subgroups. This order demonstrates how supersymmetric invariance leads to relations between the purely bosonic correlators only, and thus allows the generators of the remaining bosonic subgroups on analytic superspace to be reduced to the conventional bosonic generators of $\mathfrak{su}(1,1) \times \mathfrak{su}(2)_R$.

3.2.3. Invariance under supersymmetry

Invariance under supersymmetry dictates that the correlator remains unchanged under the action of the fermionic generators of $\mathfrak{psu}(1,1|2)$. The charges, that act non-trivially on the half-BPS multiplets, act as simple shifts in the coordinates ρ and $\bar{\rho}$ and thus, the non-trivial constraints arising from this supersymmetric invariance in the context of $\mathfrak{psu}(1,1|2)$ can be expressed as:

$$0 = \sum_{i=1}^{5} \frac{\partial}{\partial \rho_i} \mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) \tag{3.25}$$

$$0 = \sum_{i=1}^{5} \frac{\partial}{\partial \bar{\rho}_i} \mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) \,. \tag{3.26}$$

By inserting the fermionic expansion of the correlator derived in 3.24 into the constraints (3.25) and (3.26), it becomes evident that these constraints act solely on the fermionic expansion variables, leaving the purely bosonic correlators unaffected.

Specifically, applying the first constraint (3.25) to $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ (up to the order considered in our analysis) yields

$$\begin{aligned} 0 =& -\frac{1}{2}\sum_{i=1}^{5} \bar{\rho}_i \frac{\partial}{\partial x_i}\frac{\partial}{\partial y_i} \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \sum_{i=1}^{5} \bar{\rho}_i \langle T(x_i)J(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \\ &- \sum_{i=1}^{5}\sum_{j=1,j\neq i}^{5} \bar{\rho}_j \langle G(x_i,y_i)\bar{G}(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \,. \end{aligned} \tag{3.27}$$

This constraint must hold for all values of $\bar{\rho}_i$. Consequently, we can perform a comparison of coefficients for the various $\bar{\rho}_i$, demonstrating that Equation (3.27) in fact provides five constraints. For instance, for the terms proportional to $\bar{\rho}_1$, we obtain:

$$\begin{aligned} 0 =& -\frac{1}{2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1} \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle T(x_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle \bar{G}(x_1,y_1)G(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle \bar{G}(x_1,y_1)J(x_2,y_2)G(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle \bar{G}(x_1,y_1)J(x_2,y_2)J(x_3,y_3)G(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle \bar{G}(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)G(x_5,y_5)\rangle \end{aligned} \tag{3.28}$$

Analogously, constraints for terms proportional to $\bar{\rho}_2$, $\bar{\rho}_3$, $\bar{\rho}_4$, and $\bar{\rho}_5$ can be derived. Further, five additional constraints can be obtained in an identical manner from Equation 3.26. The complete list of all ten constraints imposed by supersymmetry on $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ is provided in Appendix B.1. With these ten equations, we have fully exhausted the implications of invariance under the fermionic part of $\mathfrak{psu}(1,1|2)$ (up to order $(\rho\bar{\rho})$), leaving only the bosonic symmetry to be applied.

3.2.4. Invariance under the bosonic subgroups

To obtain the most constrained expression for the correlator, there are further bosonic symmetries to be applied. These symmetries act on each of the ten constraints derived from supersymmetry, imposing additional restrictions on the bosonic correlators, which can be considered independently. Since the correlators in question are purely bosonic, the generators of the bosonic subgroups on analytic superspace reduce to the ordinary generators of $\mathfrak{su}(1,1)$ and $\mathfrak{su}(2)_R$, respectively.

Both conformal symmetry and R-symmetry dictate the specific form of the correlators. In particular, expressing the correlator in a manner consistent with these bosonic symmetries is equivalent to decomposing it into a basis of structures that exhibit the appropriate transformation properties and weights, along with arbitrary invariant functions of the cross-ratios, as outlined in Section 3.1. Specifically, the conformal×R-symmetry correlators can be written as

$$G_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) = \sum_{i=1}^{N} (\text{y-structure} \cdot \text{x-structure})_i \cdot f_i(u,v)\,, \tag{3.29}$$

where N represents the maximum number of independent structures, and u and v are the two independent one-dimensional conformal cross-ratios, defined as

$$u = \frac{x_{12}x_{34}}{x_{13}x_{24}},\, v = \frac{x_{23}\,x_{45}}{x_{24}x_{35}}, \quad x_{ij} = \left(x_i - x_j\right)\,. \tag{3.30}$$

We emphasise again that the functions $f_i(u,v)$ do not depend on the analogous R-symmetry cross-ratios due to the analyticity condition imposed by the internal group $SU(2)_R$ being compact. Instead, the full dependence on the y-variables will be encoded in the R-symmetry prefactor.
In principle, any R-symmetry structure can multiply any conformal structure, thus $N = m_y{\cdot}m_x$, where m_y denotes the number of independent R-symmetry structures, and m_x the number of independent spacetime structures, that in one-dimension is $m_x = 1$, as we will demonstrate shortly.

The objective is to derive relations involving only the arbitrary functions of the cross-ratios, yielding the superconformal Ward identities (SCWI). To achieve this, we construct the structures for both the R-symmetry and conformal coordinates for each correlator involved in Eqn. 3.28 and the remaining nine constraints in Appendix B.1. By inserting these expressions into the constraints, we obtain a system of ten equations that fully exhausts the superconformal symmetry. To solve those constraints, we perform a comparison of coefficients in all the independent structures. This procedure reduces the ten equations to a system of equations that involves only the unknown functions of the cross-ratios. These equations correspond to the SCWI, and their solution will greatly simplify the full correlator $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$.

Conformal Structures

The construction of the conformal, or *spacetime structures*, highlights why $\mathfrak{psu}(1,1|2)$ serves as a particularly simple toy model. Since the spacetime is only one-dimensional, the conformal structures of all correlators are straightforward invariant scalars.

These scalars can be constructed in the standard way. Invariance under translations requires the correlator to depend solely on distances, i.e.,

$$\begin{aligned} G^{\mathcal{O}_1 \dots \mathcal{O}_n}(\{x_i\}) &= g\left(x_{ij}\right) \quad \forall i,j = 1,\dots,5\,, \\ \text{where} \quad x_{ij} &\equiv x_i - x_j, \end{aligned} \tag{3.31}$$

where g is an arbitrary function. (For the moment, we omit the dependence on the y-coordinates in order to focus fully on the construction of the conformal structures.)

Further, invariance under conformal transformations dictates the correct powers in which these distances appear, specifically as $x^{-2\Delta}$, where the scaling dimensions are given by

$$\Delta_J = 1, \quad \Delta_G = \Delta_{\bar{G}} = \frac{2}{3}, \quad \Delta_T = 2. \tag{3.32}$$

To finally determine the number of independent conformal structures, we note that we are in fact considering structures that are independent over the space of functions of the cross-ratios u and v. Since each structure multiplies an arbitrary function of u and v, we can factor out powers of u and v. These powers will transform different structures into each other. In the one-dimensional case, or for any scalar structure, all structures can be transformed into each other this way, resulting in only one independent structure for each correlator.

With these principles in mind, we can now easily construct the conformal structures for each type of correlator. We have three different types appearing in the constraints: the superprimary correlator, the correlator involving the stress tensor $T(x)$, and the correlator involving the currents $G(x)$ and $\bar{G}(x)$. We change to a shortened notation of those correlators as:

$$\text{superprimary} \ : \ G_5^J \equiv \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \tag{3.33}$$
$$\text{stress tensor} \ : \ G_5^{\mathrm{T}} \equiv \langle T(x_i,y_i)J(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \tag{3.34}$$
$$\text{currents} \ : \ G_5^{\mathrm{G}} \equiv \langle G(x_i,y_i)\bar{G}(x_j,y_j)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \,, \tag{3.35}$$

where the subscript 5 refers to the number of inserted operators. All permutations of insertions in these correlators, as they appear in the ten supersymmetry constraints, must be taken into account.

For the superprimary correlator in Eq. 3.33, the possible scalar structure can be of the form

$$\langle J(i)J(j)J(k)J(l)J(m)\rangle \ : \ \frac{1}{x_{ij}^2 x_{kl}x_{lm}x_{mk}} \quad \text{or} \quad \frac{1}{x_{ij}x_{jk}x_{kl}x_{lm}x_{mi}}, \tag{3.36}$$

where, in principle, all possible permutations of $i,j,k,l,m \in \{1,2,3,4,5\}$ can be considered to produce a valid structure. As mentioned, those various resulting structures are related by cross ratios such that only one structure has to be chosen.

Similarly, the spacetime structures of the descendant correlators can be (for instance) of the following form

$$\langle T(i)J(j)J(k)J(l)J(m)\rangle \ : \frac{1}{x_{ij}x_{ik}x_{il}x_{im}x_{jk}x_{lm}} \tag{3.37}$$
$$\langle G(i)\bar{G}(j)J(k)J(l)J(m)\rangle \ : \frac{1}{x_{ij}^3 x_{kl}x_{lm}x_{mk}}, \tag{3.38}$$

for different permutations of $i,j,k,l,m \in \{1,2,3,4,5\}$, depending on where the stress tensor or the currents are inserted.

Note that once the position of, for instance, the stress tensor is fixed, all permutations of j,k,l,m yield valid structures, similar to the superprimary case, since the remaining operators are identical. The same reasoning applies to the correlator involving G and $\bar{G}$.

By choosing one structure from each of these possibilities and multiplying it by an unknown function of the conformal cross-ratios, we ensure that full conformal symmetry is imposed.

R-Symmetry Structures

We must also express the correlator in a manner that is consistent with R-symmetry. Since there are no arbitrary functions of the R-symmetry cross-ratios multiplying the R-symmetry structures, we need to find truly linearly independent structures to form the basis of each correlator.

The number of linearly independent R-symmetry structures can be determined, for instance, by counting the number of singlets in the tensor product representation of the correlator. Ignoring spacetime dependence for now and focusing purely on the internal (R-symmetry) dependence, we have the following:

$$\langle J(y_1)J(y_2)J(y_3)J(y_4)J(y_5)\rangle \sim (2)^{\otimes_5} : 6 \text{ singlets/structures}, \tag{3.39}$$

$$\langle T(y_i)J(y_j)J(y_k)J(y_l)J(y_m)\rangle \sim (0) \otimes (2)^{\otimes_4} : 3 \text{ singlets/structures}, \tag{3.40}$$

$$\langle G(y_i)\bar{G}(y_j)J(y_k)J(y_l)J(y_m)\rangle \sim (1)^{\otimes_2} \otimes (2)^{\otimes_3} : 4 \text{ singlets/structures}, \tag{3.41}$$

where $(j)^{\otimes_n}$ represents the n-fold tensor product of the operator corresponding to the representation with Dynkin label $(p = 2j)$.
This reasoning holds for all possible permutations of the operator insertions.

To construct the explicit R-symmetry structures, recall that $\mathfrak{su}(2) \sim \mathfrak{su}(1,1)$. Indeed, the generators of these two algebras take very similar forms, leading to the same transformations of the respective spacetime points. Therefore, we can utilize the same structures as in the conformal case. The only difference lies in the powers of the y variables, which follow the form y^{+2j} (as opposed to x^{-2h}), ensuring a polynomial dependence, where j represents the half-integer Dynkin labels.
Thus, the possible general R-symmetry structures are

$$\langle J(1)J(2)J(3)J(4)J(5)\rangle : y_{ij}^2 y_{kl} y_{lm} y_{mk} \quad \text{and} \quad y_{ij} y_{jk} y_{kl} y_{lm} y_{mi}, \tag{3.42}$$

$$\langle T(i)J(j)J(k)J(l)J(m)\rangle : y_{jk}^2 y_{lm}^2 \quad \text{and} \quad y_{jk} y_{kl} y_{lm} y_{mj}, \tag{3.43}$$

$$\langle G(i)\bar{G}(j)J(k)J(l)J(m)\rangle : y_{ij} \cdot y_{kl} y_{lm} y_{mk}, \quad y_{ik} y_{jk} y_{lm}^2, \quad \text{and} \quad y_{ik} y_{jl} y_{km} y_{lm}. \tag{3.44}$$

All reasonable permutations of i, j, k, l, m from the above *generating structures* yield valid structures for representing the corresponding correlator. However, these structures are not all linearly independent. Therefore, it is essential to restrict the set of structures to a linearly independent basis to effectively represent the correlator. Our choice for the linear independent structures to represent the respective correlators can be found in Eqn. 3.45 and Appendix B.2.

Bosonic Correlators

Following the principles outlined above for constructing the conformal and R-symmetry structures, we choose the following expressions for the relevant correlators, writing them in a form consistent with bosonic symmetries. Starting with the superprimary correlator, we define it as follows:

$$\begin{aligned}
&\langle J\left(x_1, y_1\right) J\left(x_2, y_2\right) J\left(x_3, y_3\right) J\left(x_4, y_4\right) J\left(x_5, y_5\right)\rangle \\
=&\frac{y_{12}y_{13}y_{23}y_{45}^2}{x_{12}x_{13}x_{23}x_{45}^2} f_1(u,v) + \frac{y_{12}y_{13}y_{43}y_{25}y_{45}}{x_{12}x_{13}x_{43}x_{25}x_{45}} f_2(u,v) + \frac{y_{12}y_{23}y_{43}y_{15}y_{45}}{x_{12}x_{23}x_{43}x_{15}x_{45}} f_3(u,v) \\
&+ \frac{y_{12}y_{24}y_{43}y_{35}y_{51}}{x_{12}x_{24}x_{43}x_{35}x_{51}} f_4(u,v) + \frac{y_{31}y_{32}y_{41}y_{25}y_{45}}{x_{31}x_{32}x_{41}x_{25}x_{45}} f_5(u,v) + \frac{y_{13}y_{34}y_{42}y_{25}y_{51}}{x_{13}x_{34}x_{42}x_{25}x_{51}} f_6(u,v).
\end{aligned} \tag{3.45}$$

We have 6 independent R-symmetry structures, multiplying one independent conformal structure, resulting in 6 independent bosonic structures. Each of these bosonic structures is multiplied with an unknown function of the cross ratios, yielding the 6 functions $f_i(u,v)$.

These six structures, particularly the six R-symmetry structures, are linearly independent. For the spacetime structures, we have not restricted ourselves to a single structure for the entire correlator. Instead, we use a slightly different notation in which the spacetime structure mirrors the form of each respective R-symmetry structure. This reflects the typical manner in which bosonic correlators are represented in the literature and can be visualised through Wick contractions.

However, it is important to note again that these six spacetime structures are indeed just one independent structure, which we could pull out, introducing factors of the cross ratios. For example,

$$\frac{1}{x_{12}x_{13}x_{43}x_{25}x_{45}} \frac{(1-v)}{v} = \frac{1}{x_{12}x_{13}x_{43}x_{25}x_{45}} \frac{x_{34}x_{25}}{x_{23}x_{45}} = \frac{-1}{x_{12}x_{13}x_{23}x_{45}^2}. \tag{3.46}$$

The rationale behind choosing the above basis in equation (3.45) will become apparent in Section 4.2.

The bases chosen for the descendant correlators are provided in Appendix B.2.

When considering all the correlators that appear in the ten supersymmetry constraints, there are 95 unknown functions of the cross ratios arising from the descendant correlators, as detailed in Appendix B.1.

This can also be quickly computed as

$$\langle T(y_i)J(y_j)J(y_k)J(y_l)J(y_m)\rangle \ : m_y \cdot m_x = 3 \cdot 1 = 3 \tag{3.47}$$

$$\langle G(y_i)\bar{G}(y_j)J(y_k)J(y_l)J(y_m)\rangle \ : m_y \cdot m_x = 4 \cdot 1 = 4, \tag{3.48}$$

As we get this number of independent structures for each correlator of different insertions of $T(y)$ and $G(y), \bar{G}(y)$, respectively, we obtain in total $5 \cdot 3 = 15$ plus $5 \cdot 4 \cdot 4 = 80$ independent structures.
Additionally, there are the six unknown functions $f_i(u,v)$ corresponding to the superprimary correlator.

3.2.5. Superconformal Ward identities

These unknown functions get related via the supersymmetry constraints, derived in Section 3.2.3, hence reducing the number of undefined degrees of freedom. In particular, we substitute the expressions for the bosonic correlators, which are compatible with bosonic symmetries, such as equation (3.45), into the ten supersymmetry constraints. We can then organize the constraints according to the independent structures. Since these structures are independent, the constraints must hold for each of their coefficients individually, leading to equations that relate only the different unknown functions of the involved correlators. In particular, the unknown functions from the descendant correlators become related to the unknown functions of the superprimary correlator. This is due to the fact that the superprimary correlator reappears at order $\mathcal{O}(\rho\bar{\rho})$ in the expansion of the supercorrelator, a result traceable to the correction term in the multiplet field expansion 3.23.

Carrying out this procedure for the 5-point correlation function of the multiplet $\mathcal{W}_2(X)$ under $\mathfrak{psu}(1,1|2)$, for all ten constraints, we arrive at the following conclusions:

Out of the 95 unknown functions from the superdescendant correlators, 94 can be expressed in terms of the six superprimary functions $f_i(u,v)$. This leaves only one undetermined descendant function at the order $\mathcal{O}(\rho\bar{\rho})$ in the correlator expansion. This demonstrates that, as expected, the 5-point half-BPS correlation function in $\mathfrak{psu}(1,1|2)$ is not entirely determined by the superprimary function.

Additionally, the functions $f_i(u,v)$ of the superprimary correlator are subject to relations among themselves. These are first-order differential equations, as the correlator (3.45) appears in differential form within the constraints, such as those in equation (3.28). The derivatives with respect to the spacetime coordinates have been translated throughout the process into derivatives with respect to the cross ratios as follows:

$$\frac{\partial}{\partial x_j} f_i(u,v) = \frac{\partial u}{\partial x_j}\frac{\partial}{\partial u} f_i(u,v) + \frac{\partial v}{\partial x_j}\frac{\partial}{\partial v} f_i(u,v). \tag{3.49}$$

The resulting constraints are known as the conditions arising from performing the Drukker-Plefka twist, plus an additional condition, namely:

$$\begin{aligned} 0 &= \sum_{i=1}^{6} f_i^{(u)}(u,v) \\ 0 &= \sum_{i=1}^{6} f_i^{(v)}(u,v) \end{aligned} \tag{3.50}$$

$$\begin{aligned} 0 =& \frac{(u-1)(u(v-2)-2v+2)}{(v-1)v^2} f_1^{(u)}(u,v) + \frac{(u-1)^2}{v^2}\left(f_2^{(u)}(u,v) + f_5^{(u)}(u,v)\right) \\ &- \frac{(u-1)(u+v-1)}{(v-1)v^2}\left(f_3^{(u)}(u,v) + f_4^{(u)}(u,v)\right) \\ &- \frac{\left(u^2-2u-v+1\right)}{uv}\left(f_3^{(v)}(u,v) + f_4^{(v)}(u,v)\right) \\ &+ \frac{(u+v-1)}{uv}\left(f_1^{(v)}(u,v) + f_2^{(v)}(u,v)\right) + f_6^{(v)}(u,v) \end{aligned} \tag{3.51}$$

$$\text{with } u = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad v = \frac{x_{23}x_{45}}{x_{24}x_{35}}.$$

The above equations 3.50 and 3.51 represent the full set of superconformal Ward identities for the correlator $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$. In particular, the equations above are stronger than the ones proposed in [118], which are equivalent only to Equations 3.50.

These full set of SCWI will be further analysed in Section 4.2.[3]

[3]The equations involving the descendent correlators can be provided in a Mathematica notebook.

3.3. Half-BPS multiplets on analytic superspace

In this section, we initiate the derivation of the superconformal Ward identities (SCWI) for the five-point function of stress tensor multiplets in the $\mathfrak{psu}(2,2|4)$ superalgebra, i.e.,

$$\mathcal{G}_{22222}(\{X_i\}) = \langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle \,. \tag{3.52}$$

The approach for obtaining these SCWI follows the exact same methodology as outlined for the $\mathfrak{psu}(1,1|2)$ case in Section 3.2. Accordingly, we begin by considering the first set of constraints, which arise from the properties of the half-BPS multiplet on analytic superspace.

To do so, we recall that half-BPS supermultiplets in $\mathfrak{psu}(2,2|4)$ can be represented on analytic superspace through a straightforward expansion in the fermionic coordinates $\rho^{\alpha\dot{a}}, \bar{\rho}^{a\dot{\alpha}}$ as [81]:

$$\begin{aligned}\mathbb{O}_p(X) =& \exp\left(\rho^{\alpha\dot{a}}Q_{\alpha\dot{a}} + \bar{\rho}^{a\dot{\alpha}}\bar{Q}_{a\dot{\alpha}}\right)\mathcal{O}_p(x,y)\\ =&\mathcal{O}_p(x,y) + \rho^{\alpha\dot{a}}\Psi^{(p)}_{\alpha\dot{a}} + \bar{\rho}^{a\dot{\alpha}}\bar{\Psi}^{(p)}_{a\dot{\alpha}} + \ldots\end{aligned} \tag{3.53}$$

Here, $\mathcal{O}_p$ denotes the superprimary of the full superconformal multiplet $\mathbb{O}_p$. Given the fermionic nature of the expansion parameters, this series is naturally truncated at $\mathcal{O}(\rho^4\bar{\rho}^4)$.

This expansion is fully determined by the differential constraint:

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^{p+1}\mathbb{O}_p(X) = 0, \quad \text{w. graded symmetrisation} \tag{3.54}$$

In this section, we introduce this constraint as the covariantised version of the condition that isolates the $[0,p,0]$-representation of the $\mathfrak{su}(4)_R$ symmetry. We will provide the full definition and discuss the consequences of graded symmetrisation, and demonstrate how this constraint yields the most precise formulation of the field expansion for the stress tensor multiplet in analytic superspace.

It is important to note that, although the discussion is centered on the example of the $\mathfrak{psu}(2,2|4)$ superalgebra and its analytic superspace, the same principles apply to various superconformal groups, as previously illustrated in the context of $\mathfrak{psu}(1,1|2)$.

3.3.1. Origin of the Constraint

To understand the origin of the constraint in Eq. 3.54, it is important to note that the full supermultiplet can be expanded not only in the fermionic coordinates, but also through a Taylor expansion in the bosonic coordinates. While an expansion in the Minkowski coordinates $x^{\alpha\dot{\alpha}}$ is as well possible, it does not yield significant insights. However, expanding in the internal coordinates $y^{a\dot{a}}$ reveals the necessity for additional constraints on the multiplet.

Focusing momentarily on the superprimary, this expansion is given by

$$\begin{aligned}\mathcal{O}_{20'}(x,y) =& \mathcal{O}_{20'}(x,0) + y^{a\dot{a}} \left(\frac{\partial}{\partial y^{a\dot{a}}} \mathcal{O}_{20'}(x,y) \right) \Bigg|_{y=0} \\ &+ \frac{1}{2} y^{a\dot{a}} y^{b\dot{b}} \left(\frac{\partial}{\partial y^{a\dot{a}}} \frac{\partial}{\partial y^{b\dot{b}}} \mathcal{O}_{20'}(x,y) \right) \Bigg|_{y=0} + \dots .\end{aligned} \tag{3.55}$$

The ... denote all possible higher-order terms, which, in principle, could extend indefinitely.

However, it is well-known that the $20'$-operator, corresponding to the $\mathfrak{su}(4)$ representation $[0,2,0]$, is only 20-dimensional. Consequently, some form of truncation must occur. These truncation conditions can be derived through representation theory.
To relate group-theoretical representations to the physical coordinates $y^{a\dot{a}}$, we begin by considering the subgroup

$$H = SU(2) \times SU(2) \times U(1) \subset SU(4). \tag{3.56}$$

The branching of the $SU(4)$ representation $[0,2,0]$ into irreducible representations of H is given by [119]

$$[0,2,0] \to (\mathbf{1},\mathbf{1})_{+2} \oplus (\mathbf{2},\mathbf{2})_{+1} \oplus ((\mathbf{3},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0) \oplus (\mathbf{2},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-2}, \tag{3.57}$$

where $(d_1, d_2)_c$ represents the dimensionality of the irreducible representations under the two $SU(2)$ factors, and c denotes the $U(1)$ charge.

This can be interpreted in terms of the coordinates by associating the first $SU(2)$ dimension label with the undotted index a, while the second label specifies the dotted $SU(2)$ index $\dot{a}$. The $U(1)$ charge c is related to the power of y as

$$c = (p - y\partial_y) = (2 - y\partial_y). \tag{3.58}$$

Thus, we obtain the following identifications:

$$\left[\begin{array}{c||c|c|c|c|c|c} \text{irrep.} & (\mathbf{1},\mathbf{1})_{+2} & (\mathbf{2},\mathbf{2})_{+1} & (\mathbf{3},\mathbf{3})_{0} & (\mathbf{1},\mathbf{1})_{0} & (\mathbf{2},\mathbf{2})_{-1} & (\mathbf{1},\mathbf{1})_{-2} \\ \text{coord.} & y^0 = 1 & y^{a\dot{a}} & y^{(a\dot{a}}y^{b\dot{b})} & y^{a\dot{a}}y^{b\dot{b}}\epsilon_{ba}\epsilon_{\dot{a}\dot{b}} = \det y & \det y \cdot y^{a\dot{a}} & (\det y)^2 \\ \text{dim.} & \mathbf{1} & \mathbf{4} & \mathbf{9} & \mathbf{1} & \mathbf{4} & \mathbf{1} \end{array}\right]. \tag{3.59}$$

The branching ratios in Eqn. 3.59 determine the allowed powers of the internal coordinates. In particular, they indicate that the fully symmetrised contribution at order y^3, corresponding to the representation $(\mathbf{4},\mathbf{4})$, must be absent. (The representation $(\mathbf{1},\mathbf{1})$ at this order is absent as a full anti-symmetrisation of three indices, with only two possible values for each index, is impossible.) At order y^4, only fully contracted y-indices remain, while all other contributions and higher-order terms are truncated. These conditions yield the 20-dimensional representation.

These constraints, given by the branching ratios, can be translated into a differential constraint acting directly on the internal coordinates. This constraint leads precisely to the aforementioned truncation. It is expressed as:

$$\left(\frac{\partial}{\partial y^{a\dot{a}}}\right)^3 \mathcal{O}_{20'}\bigg|_{(abc),(\dot{a}\dot{b}\dot{c})} = 0, \tag{3.60}$$

which implies full symmetrisation over both the dotted and undotted indices.

This constraint for the expansion of $SU(4)$ representations can be generalized to describe the expansion of full half-BPS superconformal multiplets on analytic superspace as

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^{p+1} \mathbb{O}_p(X) = 0, \quad \text{w. graded symmetrisation} \tag{3.61}$$

where the differential operator $\frac{\partial}{\partial X^{A\dot{A}}}$ is understood with graded symmetrisation, as super(conformal) algebras are $\mathbb{Z}_2$-graded algebras. The details of this graded symmetrisation will be clarified through the example of the stress tensor multiplet constraint.

Specifically, for the stress tensor multiplet $\mathcal{T}$, which has charge $p = 2$, the constraint becomes:

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^3 \mathcal{T}(X) = 0, \quad \text{w. graded sym.} \tag{3.62}$$

with the convention that each factor (i.e., the one denoted by undotted indices and the one denoted by dotted indices, respectively) in the three-fold tensor product in Eqn. 3.62 transforms in the graded symmetric representation of $SU(2|2)$, as represented by the diagram:

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^3 \sim (\boxslash\boxslash\boxslash\,, \boxslash\boxslash\boxslash)\,. \tag{3.63}$$

In this context, selecting the graded symmetric representation corresponds to symmetrising the indices $(a, \dot{a})$ (as in the $SU(4)$ constraint 3.60), while anti-symmetrising the $(\alpha, \dot{\alpha})$ indices. This convention is preferred because it leads to unitary representations. In contrast, adopting the anti-symmetric representation would produce non-unitary representations.[4]

This choice of symmetrisation can be explicitly implemented by introducing the auxiliary vector

$$\xi^A = \begin{pmatrix} \eta^\alpha \\ \omega^a \end{pmatrix}, \tag{3.64}$$

where η^α is fermionic (leading to an anti-symmetrisation of the coordinates $x^{\alpha\dot{\alpha}}$) and ω^a is bosonic (leading to a symmetrisation of the coordinates $y^{a\dot{a}}$). Working with the contracted differential

$$\partial \sim \bar{\xi}^{\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} \xi^A, \tag{3.65}$$

the correct signs respecting the grading are automatically produced.

The conventions for the grading and a sketch of the proof for the condition presented in Eqn.3.62 are provided in Appendix A.3.
There, it also is demonstrated again that the constraint in fact holds also on the two-point function of half-BPS multiplets.

[4] For further details on super Young tableaux in representations of supergroups, see for example [120].

3.3.2. Obtaining the explicit coordinate constraints.

Analytic superspace is defined with the superindex $A = (\alpha|a)$, where α and a are $SU(2)$ indices (cf. Eqn. 2.71). The decomposition of the corresponding full differential under $SU(2|2) \to SU(2) \otimes SU(2)$ is given by

$$\boxslash\boxslash\boxslash \to (\square\square\square\,, \cdot) \oplus (\square\square\,, \square) \oplus \left(\square\,, \begin{array}{c}\square\\[-4pt]\square\end{array}\right) . \tag{3.66}$$

Due to the chosen graded symmetrisation, the representations in this decomposition are associated as (a, α). Note that the representation

$$\left(\cdot\,, \begin{array}{c}\square\\[-4pt]\square\\[-4pt]\square\end{array}\right)$$

is not realizable, as a full anti-symmetrisation of three $SU(2)$ indices is not possible.

The above decomposition can also be expressed in terms of the respective dimensions as

$$\boxslash\boxslash\boxslash \to (\mathbf{4}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{2}, \underline{\mathbf{1}}) \; . \tag{3.67}$$

Here, the $\underline{\mathbf{1}}$ indicates that the singlet is obtained from anti-symmetrisation.

By combining the dotted and undotted legs, i.e., considering

$$(\boxslash\boxslash\boxslash\,, \boxslash\boxslash\boxslash) \,,$$

the full differential $(\frac{\partial}{\partial X^{A\dot{A}}})^3$ decomposes as

$$((\mathbf{4}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{2}, \underline{\mathbf{1}})) \otimes ((\mathbf{4}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{2}) \oplus (\mathbf{2}, \underline{\mathbf{1}})) \, . \tag{3.68}$$

From a practical perspective, this means that the constraint in 3.62 decomposes into multiple equations involving different coordinates $(x, y, \rho, \bar{\rho})$, where the possible combinations are specified in 3.68.

To express the above decomposition of 3.62 explicitly in terms of the coordinates $(x, y, \rho, \bar{\rho})$, recall how these dimensions are achieved in terms of $SU(2)$ indices:

$$\begin{array}{l} \mathbf{4} = \text{sym. t.p. of 3 indices: } (abc), (\alpha\beta\gamma) \\ \mathbf{3} = \text{sym. t.p. of 2 indices: } (ab), (\alpha\beta) \\ \mathbf{2} = \text{open index } a, \alpha \\ \underline{\mathbf{1}} = \text{fully anti-symmetrised t.p.: } \{ab\}, \{\alpha\beta\} \\ \mathbf{1} = \text{trivial rep.} \end{array} \; . \tag{3.69}$$

Thus, the combinations of the form $(a,\alpha)\times(\dot{a},\dot{\alpha})$ listed in Eqn. 3.68 correspond to the following equations:

$$(\mathbf{4},\mathbf{1})\times(\mathbf{4},\mathbf{1}):\quad \frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial y^{c\dot{c}}}\mathcal{T}(X)\Big|_{(abc),(\dot{a}\dot{b}\dot{c})}=0 \tag{3.70}$$

$$(\mathbf{3},\mathbf{2})\times(\mathbf{4},\mathbf{1}):\quad \frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}\mathcal{T}(X)\Big|_{(ab),(\dot{a}\dot{b}\dot{c})}=0 \tag{3.71}$$

$$(\mathbf{4},\mathbf{1})\times(\mathbf{3},\mathbf{2}):\quad \frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial \bar{\rho}^{c\dot{\gamma}}}\mathcal{T}(X)\Big|_{(abc),(\dot{a}\dot{b})}=0 \tag{3.72}$$

$$(\mathbf{2},\underline{\mathbf{1}})\times(\mathbf{4},\mathbf{1}):\quad \epsilon^{\beta\gamma}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial \rho^{\beta\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}\mathcal{T}(X)\Big|_{(\dot{a}\dot{b}\dot{c})}=0 \tag{3.73}$$

$$(\mathbf{4},\mathbf{1})\times(\mathbf{2},\underline{\mathbf{1}}):\quad \epsilon^{\dot{\beta}\dot{\gamma}}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial \bar{\rho}^{b\dot{\beta}}}\frac{\partial}{\partial \bar{\rho}^{c\dot{\gamma}}}\mathcal{T}(X)\Big|_{(abc)}=0 \tag{3.74}$$

$$(\mathbf{3},\mathbf{2})\times(\mathbf{3},\mathbf{2}):\quad \left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial y^{c\dot{c}}}+2\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial \bar{\rho}^{c\dot{\alpha}}}\frac{\partial}{\partial \rho^{\alpha\dot{c}}}\right)\mathcal{T}(X)\Big|_{(bc),(\dot{b}\dot{c})}=0 \tag{3.75}$$

$$(\mathbf{2},\underline{\mathbf{1}})\times(\mathbf{3},\mathbf{2}):\quad \epsilon^{\beta\gamma}\left(\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}\frac{\partial}{\partial \rho^{\beta\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}+2\frac{\partial}{\partial x^{\beta\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}\right)\mathcal{T}(X)\Big|_{(\dot{b}\dot{c})}=0 \tag{3.76}$$

$$(\mathbf{3},\mathbf{2})\times(\mathbf{2},\underline{\mathbf{1}}):\quad \epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}\frac{\partial}{\partial \bar{\rho}^{b\dot{\beta}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}+2\frac{\partial}{\partial x^{\gamma\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{c}}}\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}\right)\mathcal{T}(X)\Big|_{(ab)}=0 \tag{3.77}$$

$$(\mathbf{2},\underline{\mathbf{1}})\times(\mathbf{2},\underline{\mathbf{1}}):\quad \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}+2\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial \bar{\rho}^{c\dot{\beta}}}\frac{\partial}{\partial \rho^{\beta\dot{c}}}\right)\mathcal{T}(X)=0. \tag{3.78}$$

The coefficients in the last four equations containing multiple terms can be determined using the auxiliary vectors introduced in 3.64.
To summarise, the constraint in 3.62 is equivalent to all the above equations, which must be obeyed by the stress-tensor multiplet at each order in the fermionic expansion.

For the remainder of this chapter, we will use the notation of dimensions to label internal representations while denoting spin representations with highest weights for better distinction.

3.3.3. Application of Constraints on the Stress-Tensor Multiplets

The stress-tensor multiplet must satisfy the constraint

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^3 \mathcal{T}(X) = 0, \tag{3.79}$$

which is equivalent to enforcing the conditions delineated in equations 3.70-3.78.

To determine the fully constrained form of the stress-tensor multiplet, we begin by considering the most general ansatz for the supermultiplet, expanded in terms of the fermionic coordinates:

$$\begin{aligned}\mathcal{T}(X) =& \mathcal{O}_{20'}(x,y) + \rho^{\alpha\dot{a}}\Psi_{\alpha\dot{a}}(x,y) + \bar{\rho}^{a\dot{\alpha}}\bar{\Psi}_{a\dot{\alpha}}(x,y) + \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}A_{\alpha\dot{a};a\dot{\alpha}}(x,y)\\ &+ \bar{\rho}^{a\dot{\alpha}}\bar{\rho}^{b\dot{\beta}}\bar{F}_{a\dot{\alpha},b\dot{\beta}}(x,y) + \rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}F_{\alpha\dot{a},\beta\dot{b}}(x,y) + \bar{\rho}^{a\dot{\alpha}}\bar{\rho}^{b\dot{\beta}}\rho^{\alpha\dot{a}}\bar{B}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x,y)\\ &+ \rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}\bar{\rho}^{a\dot{\alpha}}B_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y) + \rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}\bar{\rho}^{a\dot{\alpha}}\bar{\rho}^{b\dot{\beta}}C_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha},b\dot{\beta}}(x,y) + \ldots,\end{aligned} \tag{3.80}$$

where the fields at each order in ρ and $\bar{\rho}$ are understood to encompass the entire field content.
The dots indicate higher-order terms in the expansion. For the purposes of this analysis, we will restrict our consideration to terms up to the order of $(\rho\bar{\rho})^2$; however, the constraints similarly dictate the higher-order fields.
The conditions expressed in equations 3.70-3.78 must be satisfied at every order of the fermionic expansion, thereby defining the field content at each level.

In this section, we will develop the most insightful equations in detail. In particular, we will develop the precise expansion up to $\mathcal{O}(\rho\bar{\rho})$. The full results, encompassing the solutions of all possible constraints at all various orders in the expansion, will be presented without proof. However, the solutions to the remaining equations can be straightforwardly derived using the same methodology applied to the constraints presented here.

We have already previously established that the condition given in equation 3.70 specifies the field at the bosonic order in the expansion 3.80 as the $20'$-operator associated with the $[0,2,0]$-representation.
Applying the same constraint at other orders in the $\rho, \bar{\rho}$ expansion yields analogous internal expansions for the remaining fields. Consequently, no field within the fermionic expansion can possess a dimension greater than that of the superprimary.

However, the field content at higher orders in $\rho, \bar{\rho}$ is further restricted by the additional constraints specified in equations 3.71-3.78, which results in even smaller expansions in the internal coordinates.

Field content at level 1.

To elucidate this, we begin by applying the second constraint, Eqn. 3.71, at order ρ (the lowest non-trivial fermionic order, also referred to as *level 1*), specifically:

$$\left(\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}\right)\left(\mathcal{T}\left(X\right)|_{\mathcal{O}(\rho)}\right) = 0 \quad \text{with } (ab),\ (\dot{a}\dot{b}\dot{c}). \tag{3.81}$$

This leads to the condition:

$$\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\Psi_{\gamma\dot{c}}\left(x,y\right) = 0 \;\text{ with } (ab),\ (\dot{a}\dot{b}\dot{c}), \tag{3.82}$$

To fully understand the implications of this constraint, we expand $\rho\Psi\left(x,y\right)$ in terms of the internal coordinates as follows:

$$\begin{aligned}\rho^{\alpha\dot{a}}\Psi_{\alpha\dot{a}}\left(x,y\right) =& \rho^{\alpha\dot{a}}\Psi_{\alpha\dot{a}}\left(x,0\right) + \rho^{\alpha\dot{a}}y^{b\dot{b}}\left(\frac{\partial}{\partial y^{b\dot{b}}}\Psi_{\alpha\dot{a}}\left(x,0\right)\right)\Bigg|_{y=0} \\ &+ \frac{1}{2}\rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}}\left(\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial y^{c\dot{c}}}\Psi_{\alpha\dot{a}}\left(x,0\right)\right)\Bigg|_{y=0} + \ldots,\end{aligned} \tag{3.83}$$

where contributions at non-trivial orders in y can be separated into symmetrised and anti-symmetrised terms with respect to the dotted and undotten indices.

The constraint expressed in equation 3.82 does not impose restrictions on the orders $\mathcal{O}(1)$ and $\mathcal{O}(y)$. Thus, we obtain the complete contributions from these orders as follows:

$$\rho^{\alpha\dot{a}}\cdot y^0 \sim \left(\mathbf{1},\mathbf{2}\right)_3, \tag{3.84}$$

$$\rho^{\alpha\dot{a}}y^{b\dot{b}} = \rho^{\alpha\dot{a}}y^{b\dot{b}}\Big|_{(\dot{a}\dot{b})} + \rho^{\alpha\dot{a}}y^{b\dot{b}}\Big|_{\{\dot{a}\dot{b}\}} \sim \left(\mathbf{2},\mathbf{3}\right)_1 \oplus \left(\mathbf{2},\mathbf{1}\right)_1, \tag{3.85}$$

where the $SU(2)\times SU(2)\times U(1)$ representations are labeled as $(d_1,d_2)_c$, with d_1 corresponding to y^a and d_2 to $y^{\dot{a}}$. The $U(1)$-charge c for the complete superconformal case is given by

$$c = 2\left(p - y\partial_y - \frac{1}{2}(\rho\partial_\rho + \bar{\rho}\partial_{\bar{\rho}})\right). \tag{3.86}$$

In contrast, the order y^2 in the internal expansion is affected non-trivially. The bosonic coordinates $y^{a\dot{a}}y^{b\dot{b}}$ can be symmetrically decomposed as follows:

$$(\mathbf{2},\mathbf{2}) \otimes (\mathbf{2},\mathbf{2}) = (\mathbf{2} \otimes_S \mathbf{2}, \mathbf{2} \otimes_S \mathbf{2}) \oplus (\mathbf{2} \otimes_A \mathbf{2}, \mathbf{2} \otimes_A \mathbf{2}) = (\mathbf{3},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1}).$$

In terms of the coordinates, and multiplied and (anti-)symmetrised further with $\rho^{\alpha\dot{a}}$, this decomposition results in:

$$\rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}} = \rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}}\Big|_{(\dot{a}\dot{b}\dot{c}),(bc)} + \rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}}\Big|_{\{\dot{a}\dot{b}\}} + \rho^{\alpha\dot{a}} \det y \tag{3.87}$$

$$\sim (\mathbf{1},\mathbf{2}) \otimes ((\mathbf{3},\mathbf{3}) \oplus (\mathbf{1},\mathbf{1})) = (\mathbf{3},\mathbf{4}) \oplus (\mathbf{3},\mathbf{2}) \oplus (\mathbf{1},\mathbf{2}). \tag{3.88}$$

We observe that the differential constraint given by equation 3.82 necessitates the absence of the first, fully symmetrized term. Therefore, we are left with:

$$\rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}} \to \rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}}\Big|_{\{\dot{a}\dot{b}\}} + \rho^{\alpha\dot{a}} \det y \tag{3.89}$$

$$\sim (\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_{-1}\,. \tag{3.90}$$

At order y^3, only the maximally anti-symmetrized term survives the constraint, given by

$$\rho^{\alpha\dot{a}}y^{b\dot{b}}y^{c\dot{c}}y^{d\dot{d}} \to \epsilon_{\dot{a}\dot{b}}\rho^{\alpha\dot{a}}y^{b\dot{b}} \det y \sim (\mathbf{2},\mathbf{1})_{-3}\,, \tag{3.91}$$

while all other terms vanish. Further, all higher-order terms vanish identically.

Thus, the field $\Psi_{\alpha\dot{a}}(x,y)$, which appears at order ρ in the fermionic expansion of the stress-tensor multiplet, consists of the following terms:

$$\Psi_{\alpha\dot{a}}(x,y) \to (\mathbf{2},\mathbf{1})_{-3} \oplus (\mathbf{3},\mathbf{2})_{-1} \oplus (\mathbf{1},\mathbf{2})_{-1} \oplus (\mathbf{2},\mathbf{3})_{1} \oplus (\mathbf{2},\mathbf{1})_{1} \oplus (\mathbf{1},\mathbf{2})_{3}\,. \tag{3.92}$$

These contributions on the right-hand side correspond precisely to the branching ratio of $[0,1,1]$ under $SU(4) \to SU(2) \times SU(2) \times U(1)$ [119]. Hence, we can identify:

$$\Psi_{\alpha\dot{a}}(x,y) \sim \mathbf{20} \sim [0,1,1]_{(\frac{1}{2},0)}. \tag{3.93}$$

The spin representation can be directly inferred from the Minkowski indices present in each contribution. Each term is proportional to $\rho^{\alpha\dot{a}}$, giving one open undotted index α, corresponding to the two-dimensional or $j_1 = \frac{1}{2}$ representation, while there are no open dotted indices $\dot{\alpha}$, resulting in $j_2 = 0$.

In a similar manner, the constraint given by equation 3.72 specifies the field $\bar{\Psi}_{a\dot{\alpha}}(x,y)$ as

$$\begin{aligned}\bar{\Psi}_{a\dot{\alpha}}\,(x,y) &\rightarrow (\mathbf{2},\mathbf{1})_3 \oplus (\mathbf{3},\mathbf{2})_1 \oplus (\mathbf{1},\mathbf{2})_1 \oplus (\mathbf{2},\mathbf{3})_{-1} \oplus (\mathbf{2},\mathbf{1})_{-1} \oplus (\mathbf{1},\mathbf{2})_{-3} \\ &\sim \overline{\mathbf{20}} \sim [1,1,0]_{(0,\frac{1}{2})}.\end{aligned} \tag{3.94}$$

Applying these two constraints, 3.71 and 3.72, at higher orders in the fermionic expansions yields similar results. Thus, all higher fields can have at most this expansion.

Field Content at Level 2 : First.

The field content at order ρ^2 and $\bar{\rho}^2$ are further constrained by Eqs. 3.73 and 3.74, respectively. Considering ρ^2 and Eqn. 3.73, note that, since the coordinate ρ is fermionic, the term ρ^2 can be decomposed into two distinct contributions:

$$\begin{aligned}\rho^2 = \rho^{\alpha\dot{a}}\rho^{\beta\dot{b}} &= \left(\frac{1}{2}\,\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}\Big|_{(\dot{a}\dot{b}),\{\alpha\beta\}} + \frac{1}{2}\,\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}\Big|_{(\alpha\beta),\{\dot{a}\dot{b}\}}\right) \\ &\equiv \frac{1}{2}\left(\epsilon^{\alpha\beta}(\rho^2)^{\dot{a}\dot{b}} + \epsilon^{\dot{a}\dot{b}}(\rho^2)^{\alpha\beta}\right) \\ &\sim (\mathbf{1},\mathbf{3}) \oplus (\mathbf{1},\underline{\mathbf{1}}),\end{aligned} \tag{3.95}$$

which correspond to the spin representations $(j_1,j_2) = (\underline{0},0)$ (first term) and $(j_1,j_2) = (1,0)$ (second term), respectively.

Expanding $F_{\alpha\dot{a},\beta\dot{b}}$ as a Taylor series up to order y (for the moment), we can express the product $\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}F_{\alpha\dot{a},\beta\dot{b}}(x,y)$ as follows:

$$\begin{aligned}\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}F_{\alpha\dot{a},\beta\dot{b}}(x,y) \sim\, &\frac{1}{2}\epsilon^{\alpha\beta}(\rho^2)^{\dot{a}\dot{b}}F_{\alpha\dot{a},\beta\dot{b}}(x,0) + \frac{1}{2}\epsilon^{\alpha\beta}(\rho^2)^{\dot{a}\dot{b}}y^{c\dot{c}}\left(\frac{\partial}{\partial y^{c\dot{c}}}F_{\alpha\dot{a},\beta\dot{b}}(x,y)\right)\Bigg|_{y=0} \\ &+\frac{1}{2}\epsilon^{\dot{a}\dot{b}}(\rho^2)^{\alpha\beta}F_{\alpha\dot{a},\beta\dot{b}}(x,0) + \frac{1}{2}\epsilon^{\dot{a}\dot{b}}(\rho^2)^{\alpha\beta}y^{c\dot{c}}\left(\frac{\partial}{\partial y^{c\dot{c}}}F_{\alpha\dot{a},\beta\dot{b}}(x,y)\right)\Bigg|_{y=0} \\ \sim\, &(\mathbf{1},\mathbf{3})_2 \oplus (\mathbf{2},\mathbf{4})_0 \oplus (\mathbf{2},\mathbf{2})_0 \\ &+ (\mathbf{1},\mathbf{1})_2 \oplus (\mathbf{2},\mathbf{2})_0.\end{aligned} \tag{3.96}$$

where at order y, in the first line, we can either symmetrize over all dotted indices $(\dot{a}\dot{b}\dot{c})$, or antisymmetrize the y-index with the ρ^2 indices $\{\dot{a}\dot{b}\}$, yielding two distinct contributions: $(\mathbf{2},\mathbf{4})_0$ and $(\mathbf{2},\mathbf{2})_0$, respectively.

The constraint given by Eq. 3.73, applied at order $\mathcal{O}(\rho\rho)$, gives

$$\epsilon^{\beta\gamma}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial \rho^{\beta\dot{b}}}\frac{\partial}{\partial \rho^{\gamma\dot{c}}}\left(\rho^{\delta\dot{d}}\rho^{\epsilon\dot{e}}F_{\delta\dot{d},\epsilon\dot{e}}(x,y)\right)\Big|_{(\dot{a}\dot{b}\dot{c})}=0 \tag{3.97}$$

This equation eliminates exactly the contribution $(\mathbf{2},\mathbf{4})_0$ from the expansion discussed earlier.

Proceeding similarly for the terms at order y^2 in the Taylor expansion, and employing the results derived from the previous constraints, we obtain the final contributions:

$$\begin{aligned} F_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y) \sim\, &(\mathbf{1},\mathbf{3})_2\oplus(\mathbf{2},\mathbf{2})_0\oplus(\mathbf{3},\mathbf{1})_{-2} \qquad \text{for } (\alpha,\dot{\alpha})=(0,0)\\ &\oplus(\mathbf{1},\mathbf{1})_2\oplus(\mathbf{2},\mathbf{2})_0\oplus(\mathbf{1},\mathbf{1})_{-2} \qquad \text{for } (\alpha,\dot{\alpha})=(1,0). \end{aligned} \tag{3.98}$$

From this, we can identify the representations as follows [119]:

$$\begin{aligned} F_{\alpha\dot{a},\beta\dot{b}}(x,y) &\sim \bar{\mathbf{10}}\oplus\mathbf{6}\\ &\sim [0,0,2]_{(0,0)}\oplus[0,1,0]_{(1,0)}. \end{aligned} \tag{3.99}$$

Similarly, applying the constraint from Eq. 3.74 at order $\bar{\rho}^2$ yields:

$$\begin{aligned} \bar{F}_{a\dot{\alpha},b\dot{\beta}}(x,y) &\sim \mathbf{10}\oplus\mathbf{6}\\ &\sim [2,0,0]_{(0,0)}\oplus[0,1,0]_{(0,1)}. \end{aligned} \tag{3.100}$$

Field content at level 2: Second.

The second class of constraints, Eqns. 3.75-3.78, involves multiple terms and, as outlined below, thus incorporates various contributions from the fermionic expansion.

We begin by examining Eq. 3.75 at bosonic order, which is given by:

$$\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial y^{c\dot{c}}}+2\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial \bar{\rho}^{c\dot{\alpha}}}\frac{\partial}{\partial \rho^{\alpha\dot{c}}}\right)\mathcal{T}(X)\Big|_{\rho=\bar{\rho}=0}=0 \quad \text{with } (bc),(\dot{b}\dot{c}). \tag{3.101}$$

This constraint links the superprimary field $\mathcal{O}_{20'}(x,y)$ with the fields at order $\rho\bar{\rho}$ in Eq. 3.80, as follows:

$$\frac{\partial}{\partial y^{b\dot{b}}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}_{20'}(x,y)+2A_{\alpha\dot{c},c\dot{\alpha}}(x,y)\right)=0 \quad \text{with } (bc),(\dot{b}\dot{c}). \tag{3.102}$$

To fully grasp the implications of this equation, we begin by expanding $A(x,y)$, representing the field content at order $\rho\bar{\rho}$ in the internal coordinates, as follows:

$$\begin{aligned}\rho\bar{\rho}A(x,y) = \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}A_{\alpha\dot{a},a\dot{\alpha}}(x,0) + \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}y^{b\dot{b}}\left(\frac{\partial}{\partial y^{b\dot{b}}}A_{\alpha\dot{a},a\dot{\alpha}}(x,y)\right)\Bigg|_{y=0} \\ + \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}y^{b\dot{b}}y^{c\dot{c}}\left(\frac{\partial}{\partial y^{b\dot{b}}}\frac{\partial}{\partial y^{c\dot{c}}}A_{\alpha\dot{a},a\dot{\alpha}}(x,y)\right)\Bigg|_{y=0} + \dots\end{aligned} \tag{3.103}$$

Taking into account all possible symmetrisations and antisymmetrisations, this results in the following contributions of $SU(2)\times SU(2)\times U(1)$ contributions:

$$\begin{aligned}&\mathcal{O}(y^0) : (\mathbf{2},\mathbf{2})_2,\\ &\mathcal{O}(y^1) : (\mathbf{3},\mathbf{3})_0 \oplus (\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0,\\ &\mathcal{O}(y^2) : (\mathbf{4},\mathbf{4})_{-2} \oplus (\mathbf{2},\mathbf{4})_{-2} \oplus (\mathbf{4},\mathbf{2})_{-2} \oplus (\mathbf{2},\mathbf{2})_{-2}.\end{aligned} \tag{3.104}$$

The constraint expressed in Eq. 3.102 singles out the contribution $(\mathbf{3},\mathbf{3})_0$ from $A_{\alpha\dot{a},a\dot{\alpha}}$ and indicates that this contribution is, in fact, given by the derivative of the superprimary field, as follows:

$$(\mathbf{3},\mathbf{3})_0 \sim -\frac{1}{2}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}}\mathcal{O}_{20'}(x,y). \tag{3.105}$$

At order y^2, only the $(\mathbf{2},\mathbf{2})_{-2}$ term survives, while all other contributions are eliminated by the previously studied constraints.

Thus, at order $\rho\bar{\rho}$ in the multiplet expansion, we obtain the following contributions:

$$\begin{aligned}\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}A_{\alpha\dot{a},a\dot{\alpha}}(x,y) \sim (\mathbf{2},\mathbf{2})_2 \oplus (\mathbf{3},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{2},\mathbf{2})_{-2} \\ - \frac{1}{2}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}}\mathcal{O}_{20'}(x,y).\end{aligned} \tag{3.106}$$

The first line corresponds to the branching ratios of the $SU(4)_R$ current $\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x,y)$, transforming in the representation $[1,0,1]_{(\frac{1}{2},\frac{1}{2})}$ [119]. Therefore, we can identify

$$\boxed{\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}A_{\alpha\dot{a},a\dot{\alpha}}(x,y) \sim \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\left(\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x,y) - \frac{1}{2}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}}\mathcal{O}_{20'}(x,y)\right).} \tag{3.107}$$

This relation is essential for the derivation of the SCWI.

In a similar manner, the higher-order terms in the multiplet field expansion can be identified. Since our analysis is restricted to the $\rho\bar{\rho}$ order, we will conclude here and, without proof, only state the results for the remaining levels below (cf. Eqn. 3.113).

Conservation Equations

The final constraint, given by Eqn. 3.78, determines the spacetime dependence of specific fields within the multiplet expansion. In particular, it yields the conservation equations for the currents.

The constraint is expressed as follows:

$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}+2\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial\bar{\rho}^{c\dot{\beta}}}\frac{\partial}{\partial\rho^{\beta\dot{c}}}\right)\mathcal{T}(X)=0. \tag{3.108}$$

To illustrate its consequences, we consider the lowest non-trivial order in ρ and $\bar{\rho}$. The constraint can then be rewritten as

$$\begin{aligned}0&=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}+2\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial\bar{\rho}^{c\dot{\beta}}}\frac{\partial}{\partial\rho^{\beta\dot{c}}}\right)\mathcal{T}(X)\Bigg|_{\rho=\bar{\rho}=0}\\
&=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}_{20'}(x,y)+2\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}A_{\beta\dot{c},c\dot{\beta}}(x,y)\right)\\
&=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}_{20'}(x,y)+2\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\left(-\frac{1}{2}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}_{20'}(x,y)+\mathcal{J}_{\beta\dot{c},c\dot{\beta}}(x,y)\right)\right)\\
&=2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathcal{J}_{\beta\dot{c};c\dot{\beta}}(x,y).\end{aligned}$$

We have thus recovered the conservation equation for the $SU(4)$ current:

$$0=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathcal{J}_{\beta\dot{c};c\dot{\beta}}(x,y). \tag{3.109}$$

Similarly, at orders $\mathcal{O}(\rho)$ and $\mathcal{O}(\bar{\rho})$, one obtains:

$$0=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathcal{O}^{[1,0,0]_{(1,\frac{1}{2})}}_{\beta\dot{b},\gamma\dot{c};b\dot{\beta}}(x,y), \tag{3.110}$$

$$0=\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathcal{O}^{[0,0,1]_{(\frac{1}{2},1)}}_{b\dot{\beta},c\dot{\gamma};\beta\dot{b}}(x,y). \tag{3.111}$$

Here, $\mathcal{O}^{[1,0,0]_{(1,\frac{1}{2})}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y)$ and $\mathcal{O}^{[0,0,1]_{(\frac{1}{2},1)}}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x,y)$ represent the fields corresponding to the spinor currents. (For clarity, the fields are designated by the representation to which they correspond).

Finally, from $\mathcal{O}(\rho\bar{\rho})$, we obtain:

$$0 = \epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\mathcal{O}^{[0,0,0]_{(1,1)}}_{\beta\dot{b},\gamma\dot{c};b\dot{\beta},c\dot{\gamma}}(x,y), \tag{3.112}$$

where $\mathcal{O}^{[0,0,0]_{(1,1)}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha},b\dot{\beta}}(x) = T_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha},b\dot{\beta}}(x)$ represents the stress tensor.

Stress Tensor Multiplet Field Expansion

Summarising all results and applying them on the initial expansion Eqn. 3.80 (up to order $\mathcal{O}((\rho\bar{\rho})^2)$, the most general, yet constrained expression of the stress tensor multiplet on analytic superspace is given by

$$\begin{aligned}
\mathcal{T}(X) = &\left(1 - \frac{1}{2}\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}} + \frac{1}{16}\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\rho^{\beta\dot{b}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)\mathcal{O}_{20'}(x,y)\\
&+\rho^{\alpha\dot{a}}\left(1 - \frac{1}{4}\rho^{\beta\dot{b}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)\Psi_{\alpha\dot{a}}(x,y)\\
&+\bar{\rho}^{a\dot{\alpha}}\left(1 - \frac{1}{4}\rho^{\beta\dot{b}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)\bar{\Psi}_{a\dot{\alpha}}(x,y)\\
&+\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x,y)\\
&+\frac{1}{2}\epsilon^{\alpha\beta}\left(\rho^2\right)^{\dot{a}\dot{b}}\mathcal{O}^{[0,0,2]_{(0,0)}}_{\alpha\dot{a},\beta\dot{b}}(x,y) + \frac{1}{2}\epsilon^{\dot{a}\dot{b}}\left(\rho^2\right)^{\alpha\beta}\mathcal{O}^{[0,1,0]_{(1,0)}}_{\alpha\dot{a},\beta\dot{b}}(x,y)\\
&+\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\left(\bar{\rho}^2\right)^{ab}\mathcal{O}^{[2,0,0]_{(0,0)}}_{a\dot{\alpha},b\dot{\beta}}(x,y) + \frac{1}{2}\epsilon^{ab}\left(\rho^2\right)^{\dot{\alpha}\dot{\beta}}\mathcal{O}^{[0,1,0]_{(0,1)}}_{a\dot{\alpha},b\dot{\beta}}(x,y)\\
&+\frac{1}{2}\epsilon^{\dot{a}\dot{b}}\left(\rho^2\right)^{\alpha\beta}\bar{\rho}^{a\dot{\alpha}}\mathcal{O}^{[1,0,0]_{(1,\frac{1}{2})}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y) + \frac{1}{2}\epsilon^{ab}\left(\rho^2\right)^{\dot{\alpha}\dot{\beta}}\rho^{\alpha\dot{a}}\mathcal{O}^{[0,0,1]_{(\frac{1}{2},1)}}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x,y)\\
&+\frac{1}{4}\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}(\rho^2)^{\alpha\beta}(\bar{\rho}^2)^{\dot{\alpha}\dot{\beta}}\mathcal{O}^{[0,0,0]_{(1,1)}}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x) + \ldots
\end{aligned} \tag{3.113}$$

where ... refer to higher order terms. The presented fields obey further

$$\begin{aligned}
&\mathcal{O}_{20'}(x,y) = \mathcal{O}^{[0,2,0]_{(0,0)}}(x,y) && 0 = \tfrac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}^{[0,0,1]_{(\frac{1}{2},1)}}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x,y)\,|_{(ac),(\dot{a}\dot{c})} \\
&\Psi_{\alpha\dot{a}}(x,y) = \mathcal{O}^{[0,1,1]_{(\frac{1}{2},0)}}_{\alpha\dot{a}}(x,y)\,, && 0 = \epsilon^{\beta\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}\tfrac{\partial}{\partial x^{\beta\dot{\beta}}}\mathcal{O}^{[1,0,1]_{(\frac{1}{2},\frac{1}{2})}}_{\alpha\dot{a},a\dot{\alpha}}(x,y) \\
&\bar{\Psi}_{a\dot{\alpha}}(x,y) = \mathcal{O}^{[1,1,0]_{(0,\frac{1}{2})}}_{a\dot{\alpha}}(x,y) && 0 = \epsilon^{\gamma\alpha}\epsilon^{\dot{\gamma}\dot{\alpha}}\tfrac{\partial}{\partial x^{\gamma\dot{\gamma}}}\mathcal{O}^{[0,0,1]_{(\frac{1}{2},1)}}_{a\dot{\alpha},b\dot{\beta};\alpha\dot{a}}(x,y) \\
&\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x,y) = \mathcal{O}^{[1,0,1]_{(\frac{1}{2},\frac{1}{2})}}_{\alpha\dot{a},a\dot{\alpha}}(x,y)\,, && 0 = \epsilon^{\gamma\alpha}\epsilon^{\dot{\gamma}\dot{\alpha}}\tfrac{\partial}{\partial x^{\gamma\dot{\gamma}}}\mathcal{O}^{[1,0,0]_{(1,\frac{1}{2})}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y) \\
&0 = \tfrac{\partial}{\partial y^{b\dot{b}}}\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x,y)\,|_{(ab),(\dot{a}\dot{b})} && 0 = \epsilon^{\gamma\alpha}\epsilon^{\dot{\gamma}\dot{\alpha}}\tfrac{\partial}{\partial x^{\gamma\dot{\gamma}}}\mathcal{O}^{[0,0,0]_{(1,1)}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha},b\dot{\beta}}(x,y) \\
&0 = \tfrac{\partial}{\partial y^{c\dot{c}}}\mathcal{O}^{[1,0,0]_{(1,\frac{1}{2})}}_{\alpha\dot{a},\beta\dot{b};a\dot{\alpha}}(x,y)\,|_{(ac),(\dot{a}\dot{c})}\,, &&
\end{aligned} \tag{3.114}$$

In the following sections and chapters, we will use the expression 3.113 up to order $\mathcal{O}(\rho\bar{\rho})$.

Similarly, multiplet field expansions of other multiplets can be obtained. The result for the fundamental multiplet, $\mathbb{O}_1(X)$, can be found in Appendix A.3.2.

3.4. (Bosonic) Structures in Higher Point Correlators

In the previous section 3.3, we derived the full supermultiplet expansion of the stress tensor multiplet on analytic superspace. Thus, we can now finally proceed to examine the five-point correlation function given by

$$\mathcal{G}_{22222}(\{X_i\}) = \langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle \,. \tag{3.115}$$

The remainder of this chapter is dedicated to investigating the constraints imposed on this correlation function by the requirement of invariance under the superconformal group $\mathfrak{psu}(2,2|4)$.

To effectively analyze these constraints, we utilize the multiplet field expansion of the stress tensor multiplet, as outlined in 3.113, to derive a fermionic expansion for the supercorrelator 3.115 itself as:

$$\begin{aligned}
\mathcal{G}_5(\{X_i\}) &= \langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle \\
&= \langle \mathcal{O}_{20'}\left(x_1,y_1\right)\mathcal{O}_{20'}\left(x_2,y_2\right)\mathcal{O}_{20'}\left(x_3,y_3\right)\mathcal{O}_{20'}\left(x_4,y_4\right)\mathcal{O}_{20'}\left(x_5,y_5\right)\rangle \\
&\quad -\frac{1}{2}\sum_{i=1}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y_i^{a\dot{a}}}\langle \mathcal{O}_{20'}\left(x_1,y_1\right)\dots\mathcal{O}_{20'}\left(x_5,y_5\right)\rangle \\
&\quad +\sum_{i=1}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\langle \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}\left(x_i,y_i\right)\prod_{k=1,k\neq i}^{5}\mathcal{O}_{20'}\left(x_k,y_k\right)\rangle \\
&\quad -\sum_{i=1}^{5}\sum_{j=1,j\neq i}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_j^{a\dot{\alpha}}\langle \Psi_{\alpha\dot{a}}\left(x_i,y_i\right)\bar{\Psi}_{a\dot{\alpha}}\left(x_j,y_j\right)\prod_{k=1,k\neq i\neq j}^{5}\mathcal{O}_{20'}\left(x_k,y_k\right)\rangle \\
&\quad +\text{ higher orders.}
\end{aligned} \tag{3.116}$$

Note that there are no terms linear in ρ or $\bar{\rho}$ as the corresponding correlators are forbidden by symmetries.
For the same reasons as outlined in the toy model case $\mathfrak{psu}(1,1|2)$, we can consider the constraints arising from the bosonic subgroups separately from those associated with the fermionic generators.
In this section, we will begin by examining the invariance under the bosonic subgroups. In particular, we will construct the bases of R-symmetry and conformal spacetime structures for the correlators appearing in the expression 3.116. Specifically, we will consider the following three correlators (and all the relevant permutations):

$$G_5^{\mathcal{O}_{20'}} \equiv \langle \mathcal{O}_2(x_1,y_1)\mathcal{O}_2(x_2,y_2)\mathcal{O}_2(x_3,y_3)\mathcal{O}_2(x_4,y_4)\mathcal{O}_2(x_5,y_5)\rangle \tag{3.117}$$

$$G_5^{\mathcal{J}} \equiv \langle \mathcal{J}_{\alpha\dot{\alpha}}(x_1,y_1)\mathcal{O}_2(x_2,y_2)\mathcal{O}_2(x_3,y_3)\mathcal{O}_2(x_4,y_4)\mathcal{O}_2(x_5,y_5)\rangle \tag{3.118}$$

$$G_5^{\Psi\bar{\Psi}} \equiv \langle \bar{\Psi}_{\dot{\alpha}}(x_1,y_1)\Psi_{\alpha}(x_2,y_2)\mathcal{O}_2(x_3,y_3)\mathcal{O}_2(x_4,y_4)\mathcal{O}_2(x_5,y_5)\rangle \tag{3.119}$$

We will begin by constructing the conformal spacetime structures. By focusing solely on the spacetime dependence of the correlators, this problem becomes equivalent to the discussion of spacetime structures in ordinary conformal field theories (CFTs).
Afterwards, we will construct the R-symmetry structures. By leveraging the similarities between $SU(2,2)$ and $SU(4)$, we can utilize the same form of structures as those established in the conformal case.
Finally, we will present the complete expressions of the relevant correlators, that respect the bosonic symmetries.

3.4.1. Spacetime Structures

Recall that a conformal correlator can be expressed as a linear combination of a basis of spacetime structures, each carrying the correct transformation behavior of the correlator, and unknown functions of the invariants of the theory, i.e., functions of the conformal cross ratios. Schematically, this can be represented as

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \sum_{i=1}^{m}(\text{structures})_i \cdot f_i(\{u\})\,, \tag{3.120}$$

where $\mathcal{O}(x_n)$ is a general operator inserted at position x_n, and $\{u\}$ refers to the cross ratios. i labels the structures from 1 to the number of maximal independent structures m. The spacetime structures $\{s_1,\ldots,s_l\}$ are said to be independent, if there is no choice of functions of cross ratios $f_1(\{u\}),\ldots,f_l(\{u\})$ such that

$$\sum{}_{i=1}^{l} s_i f_i(\{u\}) = 0.$$

When all inserted operators are scalars, we have $m=1$ and the single spacetime structure is a simple scalar prefactor. For all other cases, we have $m \neq 1$.

For five-point functions in a four-dimensional conformal theory, we will see below that there are five cross ratios. Following the conventions of [121], we define these cross ratios as

$$\begin{aligned}
u_1 &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z_1\bar{z}_1, \quad u_2 = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z_1)(1-\bar{z}_1),\\
u_3 &= \frac{x_{23}^2 x_{45}^2}{x_{24}^2 x_{35}^2} = z_2\bar{z}_2, \quad u_4 = \frac{x_{25}^2 x_{34}^2}{x_{24}^2 x_{35}^2} = (1-z_2)(1-\bar{z}_1),\\
u_5 &= \frac{x_{15}^2 x_{23}^2 x_{34}^2}{x_{24}^2 x_{13}^2 x_{35}^2} = w(z_1-\bar{z}_1)(z_2-\bar{z}_2) + (1-z_1-z_2)(1-\bar{z}_1-\bar{z}_2).
\end{aligned} \tag{3.121}$$

Constructing the tensorial structures for non-scalar correlators presents a non-trivial challenge.
These tensor structures must exhibit the correct transformation behavior of the respective correlator under the conformal algebra in four dimensions, $\mathfrak{su}(2,2)$. As can be inferred from the generators outlined in Appendix A.2, this algebra, when acting on bosonic correlators only, operates solely on the upper left cell of analytic superspace, which corresponds to 4d complex Minkowski spacetime expressed in spinorial form as $x^{\alpha\dot{\alpha}} = (x^\mu \sigma_\mu)^{\alpha\dot{\alpha}}$. Thus, we can restrict our attention to Minkowski spacetime, rendering the problem equivalent to the ordinary CFT discussion regarding spinning conformal correlators.

Understanding the space of these conformal structures is crucial for comprehending correlation functions, as it is essential for any use of bootstrap techniques and considerations of conformal blocks, among others. Consequently, this topic has garnered significant interest throughout the years; see, for instance, [122–127] for just a few references.
Despite the explicit construction of many tensor structures for various types of spinning correlators, a complete characterisation of such structures remains missing [126].

Our approach follows the most straightforward method of first constructing the basic building blocks of the structures. For instance, we will identify the correct tensorial components that transform under $\mathfrak{so}(4)$, and the conformal prefactor that specifies the transformations under dilatations. These building blocks will then be combined in all possible ways, considering all inequivalent permutations. From this extensive set of tensor structures, we will select a basis of independent structures to characterise the respective correlator.
This choice is arbitrary, as long as the selected structures are independent, as the superconformal Ward identities do not uniquely determine the tensor structures (they remain invariant when multiplied by functions of the cross ratios). The choices made below aim for simplicity.

The first step in this procedure, however, is to count how many structures a basis should consist of.

3.4.1.1. Counting Spacetime Structures

Determining the dimension of the space of tensorial spacetime structures for a given correlator is a non-trivial task due to the structures obeying non-trivial al-

gebraic relations. In fact, as of now, there is no complete classification of tensorial structures available. [126]
However, a group-theory-based approach toward such a classification of tensor structures in general dimensions was introduced in [126] and will be outlined and utilized below.

Consider an arbitrary n-point function in a d-dimensional conformal field theory:

$$\langle \mathcal{O}_1^{\rho_1}(x_1)\mathcal{O}_2^{\rho_2}(x_2)\cdots\mathcal{O}_n^{\rho_n}(x_n)\rangle \ , \tag{3.122}$$

where ρ_i specifies the representation of the operator $\mathcal{O}_i$ inserted at x_i under the rotation group $SO(d)$. For now, we will consider only the rotational part, as it is the non-trivial component of the conformal tensor structures.

The idea is to *gauge-fix* the conformal symmetry to eliminate the symmetry relations between the structures. A natural starting point is the standard CFT configuration where we translate $x_1 \to 0$, scale $x_2 \to e$ (with e being a unit vector), and use special conformal transformations to send $x_3 \to \infty$, resulting in

$$\langle \mathcal{O}_{\rho_1}(0)\mathcal{O}_{\rho_2}(e)\mathcal{O}_{\rho_3}(\infty)\cdots\mathcal{O}_{\rho_n}(x_n)\rangle \ .$$

The resulting configuration must be invariant under the stabilizer group $SO(d-1)$, which represents the remaining rotations around the line through 0, e, and ∞.
This $SO(d-1)$ can be further utilized to fix additional points. In particular, the i-th point can be fixed to an $(i-2)$-dimensional subspace by employing $SO(d-i+3)$. For example, this procedure fixes the fourth point in an $n \geq 4$-point correlation function to the well-known two-dimensional plane characterized by the two variables z and $\bar{z}$ [36].
This process can be iteratively applied until all symmetry has been utilized or, equivalently, for all points $i \leq m = \min(n, d+2)$. The ultimate stabilizer group for the final configuration is $SO(d+2-m)$, and the conformally invariant tensor structures are given by

$$\left(\mathrm{Res}^{SO(d)}_{SO(d+2-m)} \otimes_{i=1}^{n} \rho_i\right)^{SO(d+2-m)} \ , \tag{3.123}$$

where Res^G_H denotes the restriction of a representation of G to a representation of the subgroup $H \subseteq G$, ρ_i are the $SO(d)$ representations of the operator at x_i, and $(\rho)^H$ denotes a H-singlet in ρ_i.

This procedure can be made more specific by applying it to correlators, specifically the relevant 5-point correlators, 3.118 and 3.119, in a four-dimensional conformal

theory. (Note that the superprimary correlator transforms trivially under rotations.)
For this part, we will focus only on the spacetime aspect, characterized by the spacetime indices α and $\dot{\alpha}$, and will temporarily omit the internal dependence.

As for any 5-point function, we start with the standard conformal transformations sending $x_1 \to 0$, $x_2 \to \mathbb{1}$, and $x_3 \to \infty$. In 4 dimensions, the remaining symmetry group of this configuration is $SO(3)$. We can use these three-dimensional rotations around the line further to bring x_4 into a two-dimensional plane, leaving $SO(2)$, the rotations around the plane. This can be used finally to fix x_5 to a three-dimensional plane, leaving no further symmetry.
Consequently, any 5-point function in a 4-dimensional conformal field theory is described by 5 independent coordinates, which are equivalent to the 5 conformal cross ratios.
The number of independent tensor structures for the 5-point functions 3.118 and 3.119 is, however, as described above, given by the number of singlets under the remaining group $SO(d+2-m) = SO(1) = \{\text{Id}\}$:

$$\left(\text{Res}^{SO(4)}_{SO(1)} \otimes_{i=1}^{n} \rho_i\right)^{SO(1)} . \tag{3.124}$$

Since $SO(1)$ does not impose any non-trivial constraints, the number of tensor structures for five-point functions in four dimensions is simply given by the dimensions of the representations in the tensor product:

$$\prod_{i=1}^{n} \dim\left(\rho_i\right) . \tag{3.125}$$

For the correlators involving the $SU(4)_R$-current $\mathcal{J}_{\alpha\dot{\alpha}}$ (3.118) and the spinors $\Psi_\alpha, \bar{\Psi}_{\dot{\alpha}}$ (3.119), this amounts to:

$$\langle \mathcal{J}2222\rangle \ : \ \dim\left(\mathbf{4}\right) \cdot \dim\left(\mathbf{1}\right)^4 = 4\,, \tag{3.126}$$

$$\langle \bar{\Psi}\Psi 222\rangle \ : \ \dim\left(\mathbf{2}\right)^2 \cdot \dim\left(\mathbf{1}\right)^3 = 4\,, \tag{3.127}$$

which means that both correlators can be described, with respect to the rotational group $SO(4)$, by 4 independent tensorial spacetime structures.
As for any scalar correlator, there is just one independent scalar structure for the superprimary correlator.

3.4.1.2. Constructing Spacetime Structures

Having determined the dimension of the space of tensorial spacetime structures, i.e., the explicit number of tensor structures required to describe the correlators

(3.118) and (3.119), the goal of this section is to construct them explicitly.

The structures are defined by reflecting the transformation behaviour of the correlator under the conformal algebra in 4 dimensions, $\mathfrak{su}(2,2)$. This algebra consists of translations $P_{\dot{\alpha}\alpha}$, rotations $L_\alpha{}^\beta$, $\bar{L}_{\dot{\alpha}}{}^{\dot{\beta}}$, dilatations D, and special conformal transformations $K_{\dot{\alpha}\alpha}$. The group transformations on complex Minkowski spacetime in matrix form follow from chapter 2

$$x_{ij}^{\alpha\dot{\alpha}} \xrightarrow{g} (x'_{ij})^{\alpha\dot{\alpha}} = (g_i)^\alpha{}_\beta \, x_{ij}^{\beta\dot{\beta}} \, (g_j^{-1})_{\dot{\beta}}{}^{\dot{\alpha}} \quad \forall g \in GL(2). \tag{3.128}$$

As outlined earlier, the explicit spacetime structures representing the conformal part of the correlator are constructed by first forming basic building blocks with the correct transformation behaviour under different parts of the conformal group. We then consider all possibilities of multiplying these building blocks together, and finally, we select a concrete basis. We will work separately for each correlator 3.117, 3.118 and 3.119.

$\langle \mathcal{O}_{20'}(x_1)\mathcal{O}_{20'}(x_2)\mathcal{O}_{20'}(x_3)\mathcal{O}_{20'}(x_4)\mathcal{O}_{20'}(x_5)\rangle$

The structure for the superprimary correlator is the simplest to consider. Since the superprimary $\mathcal{O}_{20'}$ is a scalar under the rotational group $SO(4)$, the 5-point function must also be a scalar, i.e., there is no tensorial part. Thus, there is just one independent scalar structure, which is constructed from the standard constraints of translational and rotational invariance on scalars, requiring that the 5-point function depends only on absolute values of distances between points. This is commonly ensured by considering squares of distances:

$$\begin{aligned} \langle \mathcal{O}_{20'}(x_1)\mathcal{O}_{20'}(x_2)\mathcal{O}_{20'}(x_3)\mathcal{O}_{20'}(x_4)\mathcal{O}_{20'}(x_5)\rangle &= f\left(x_{ij}^2\right) \; \forall i,j = 1,\dots,5\,, \\ \text{with } x_{ij} &\equiv x_i - x_j\,, \\ \text{and } x_{ij}^2 &\equiv \det x_{ij}^{\alpha\dot{\alpha}}. \end{aligned} \tag{3.129}$$

Finally, the structure must reflect the correct conformal weight of the operators in the 5-point function by being proportional to $x_i^{-2\Delta_i}$. Since $\Delta_{\mathcal{O}_{20'}} = 2$, the structure must be proportional to x_i^{-4} for each $i = 1,\dots,5$.

The spacetime structure for the superprimary correlator can be obtained from the equivalent of the 22 Wick contractions of the 5-point function in free theory, depicted in Figure 3.1. The vertices 1 to 5 represent the insertion points of the superprimary operators, while an edge connecting two vertices represents a propagator of the form $\frac{1}{x_{ij}^2}$. A thick line corresponds to a double contraction of the form $\frac{1}{x_{ij}^4}$.

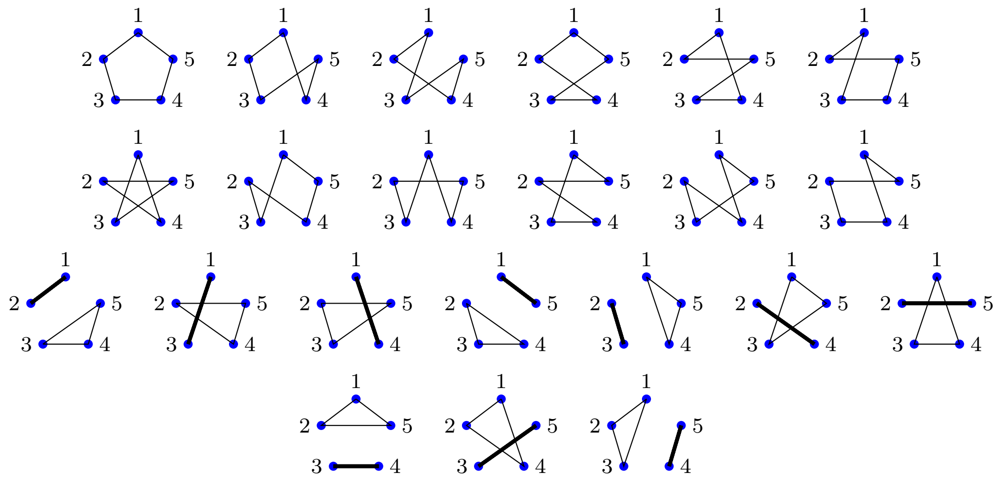

Figure 3.1.: 22 structures in the superprimary correlator $\langle 22222 \rangle$. The correlator decomposes into 10 *intrinsic* 5pt structures of the pentagon-type (first two lines) and 12 products of 3pt structures (triangles) times 2pt structures (double-lines) (third and fourth line). The blue nodes each mark an insertion point of the superprimary, $\mathcal{O}_2$, while each edges illustrates a contraction, i.e. a propagator between the two involved points. A thick line represents a double contraction corresponding to a squared propagator.

The explicit spacetime structures corresponding to the various Wick contractions are given by:

$$\frac{1}{x_{1i}^2 x_{ij}^2 x_{jk}^2 x_{kl}^2 x_{l1}^2} \quad \text{for the pentagon type,} \tag{3.130}$$

$$\frac{1}{x_{ij}^4 x_{kl}^2 x_{lm}^2 x_{mk}^2} \quad \text{for the triangle plus double-line.} \tag{3.131}$$

Considering the possible permutations of $i, j, k, l = 2, 3, 4, 5$ in 3.130 and $i, j, k, l, m = 1, 2, 3, 4, 5$ in 3.131, we obtain the above Wick contractions.

These 22 possible spacetime structures illustrate that, over the space of functions of the cross ratios, there is just one independent structure. This is because all

22 structures can be transformed into each other by multiplying with powers of conformal cross ratios. For instance,

$$\frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} = \frac{x_{23}^2 x_{51}^2}{x_{12}^2 x_{53}^2} \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} = \frac{u_5}{u_1} \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} \,. \tag{3.132}$$

Thus, there is only one independent structure over the space of functions of the cross ratios. This result is consistent with formula 3.124, which holds for any scalar n-point function, as:

$$\dim \left(\mathbf{1}\right)^5 = 1 \,. \tag{3.133}$$

Therefore, any of the 22 spacetime structures can be chosen to express the superprimary correlator. For instance, we can write:

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} f_1(\{u\}) \,, \tag{3.134}$$

where $f_1(\{u\})$ is a single function of the five conformal cross ratios $\{u\}$.
The chosen expression of the conformal primary correlator for this thesis is given at the end of this section, in Eqn. 3.214, after also the R-symmetry structures have been considered.

In the field expansion of the correlator 3.116, it becomes evident that the superprimary correlator at order $\mathcal{O}(\rho\bar{\rho})$ does not appear in its original covariant form but rather in a differential form. Consequently, when studying the constraints from supersymmetry, we do not possess the covariant correlator directly. Instead, we are dealing with its differential form, which introduces additional tensorial structures.
For example, using the conventions presented in A.1, differentiating the structure above with respect to $x_1^{\alpha\dot{\alpha}}$, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \left(\frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} f_1(\{u\}) \right) = \\
&= \left(\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} \right) f_1(\{u\}) + \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} \left(\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} f_1(\{u\}) \right) \\
&= \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} \left(-(x_{12}^{-1} - x_{15}^{-1})_{\dot{\alpha}\alpha} f_1(\{u\}) + \sum_{a=1}^{5} \frac{\partial u_a}{\partial x_1^{\alpha\dot{\alpha}}} \frac{\partial}{\partial u_a} f_1(\{u\}) \right) .
\end{aligned} \tag{3.135}$$

These type of identities will be used when studying the full superconformal constraints on the correlation function.

$\langle \mathcal{J}_{\alpha\dot\alpha}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle$

The correlator $\langle \mathcal{J}_{\alpha\dot\alpha}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle$ involves spinning operators, necessitating the construction of more involved tensorial structures.

As with other correlators, translational invariance dictates that these structures must depend solely on distances between the points. While the scalar operators $\mathcal{O}_{20'}$ remain invariant under the action of the rotation group $SO(4)$, the current operator $\mathcal{J}_{\alpha\dot\alpha}$, for instance positioned at $x_1^{\alpha\dot\alpha}$, transforms non-trivially under this symmetry. Consequently, the corresponding tensor structures must also transform appropriately with respect to x_1.

A natural approach to ensure this transformation behaviour is to contract the distance matrices $x_{ij}^{\alpha\dot\alpha}$ such that the transformations act exclusively at the point x_1. This can be achieved, for example, through the well-established 3-point structures (see, for instance, [128]), which take the following form:

$$V_{1,ij} \equiv \left(x_{1i}^{-1}\,x_{ij}\,x_{j1}^{-1}\right)_{\dot\alpha\alpha} = (x_{1i}^{-1})_{\dot\alpha\beta}(x_{ij})^{\beta\dot\beta}(x_{j1}^{-1})_{\dot\beta\alpha}, \tag{3.136}$$

$$\text{where } (x^{-1})_{\dot\alpha\alpha} = \frac{\epsilon_{\dot\alpha\dot\beta}x^{\beta\dot\beta}\epsilon_{\beta\alpha}}{x^2}. \tag{3.137}$$

These 3-point structures can be extended straightforwardly to form intrinsic 5-point structures, such as:

$$V_{1,ijkl} = \left(x_{1i}^{-1}\,x_{ij}\,x_{jk}^{-1}\,x_{kl}\,x_{l1}^{-1}\right)_{\dot\alpha\alpha}. \tag{3.138}$$

The term *intrinsic* here refers to the fact that the structures in Eq. 3.138 represent the most natural description for a 5-point function, as they explicitly involve all five points. These structures can thus not be used to represent lower-point functions. In contrast, the structures given in Eq. 3.136 can be used to describe both 3-point and 5-point (and higher-point) functions.

Proof that the rotation group only transforms point 1: Use 3.128 to see

$$\begin{aligned} V_{1,ij} \xrightarrow{g} V'^{1,ij} =& (g_1)_{\dot\alpha}{}^{\dot\mu}(x_{1i}^{-1})_{\dot\mu\mu}(g_i^{-1})^{\mu}{}_{\beta}(g_i)^{\beta}{}_{\nu}(x_{ij})^{\nu\dot\nu}(g_j^{-1})_{\dot\nu}{}^{\dot\beta}(g_j)_{\dot\beta}{}^{\dot\rho}(x_{j1}^{-1})_{\dot\rho\rho}(g_1^{-1})^{\rho}{}_{\alpha} \\ =& (g_1)_{\dot\alpha}{}^{\dot\mu}(x_{1i}^{-1})_{\dot\mu\mu}(x_{ij})^{\mu\dot\rho}(x_{j1}^{-1})_{\dot\rho\rho}(g_1^{-1})^{\rho}{}_{\alpha} \end{aligned}$$

The $g \in SO(4)$ indeed only act in the first point. (Similarly for $V_{1,ijkl}$). □

Both the 3-point structures and the 5-point structures can be employed to describe the 5-point correlator. A useful perspective for illustrating this is that a 5-point function, as given by Eq. 3.118, can be decomposed similarly to the well-known 5-point functions of the superprimaries $\mathcal{O}_{20'}$. Specifically, it can be expressed as a pentagon as well as triangle-plus-double-line configuration, as depicted in Figure 3.2. In this figure, vertices connected by red lines are involved in tensorial contractions, and the points (a_1, a_2, a_3, a_4) correspond to any permutation of $2, 3, 4, 5$.

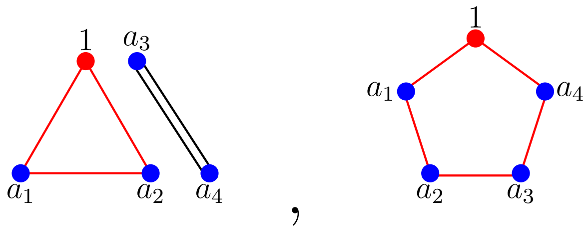

Figure 3.2.: Inequivalent structures in the correlator $\langle \mathcal{J}2222 \rangle$. The correlator decomposes into *intrinsic* 5-point structures of the pentagon type (right figure) and products of 3-point structures (triangles involving the current) multiplied by 2-point structures (double lines involving only superprimaries) (left figure). Note that 2-point functions of the type $\langle \mathcal{J}2 \rangle$ vanish due to the orthogonality of the correlator.
The red node denotes the insertion of the current $\mathcal{J}_{\alpha\dot{\alpha}}$, in this case at x_1, while the blue nodes represent the superprimary $\mathcal{O}_{20'}$, inserted at $a_1, \dots, a_4 = 2, 3, 4, 5$.
Moreover, vertices connected by red lines are contracted within a tensorial structure.

After imposing all the constraints of the Poincaré group within $SU(2,2)$ —namely, translations and rotations— the resulting structures must also be made conformally covariant. This is achieved by multiplying the structures with conformal prefactors that ensure the correct scaling weight at each point, i.e., $x_i^{-2\Delta}$.
The conformal weights of the inserted operators are

$$\Delta_{\mathcal{O}_{20'}} = 2 \quad \& \quad \Delta_{\mathcal{J}} = 3. \tag{3.139}$$

It is important to note that the tensorial components, as given by Eq. 3.136 and Eq. 3.138, are already proportional to x_1^{-2} and remain of order $\mathcal{O}(1)$ at any other point, regardless of the permutation of indices i, j, k, l.
Thus, to ensure the correct conformal weight, the conformal prefactor should be proportional to x_i^{-4} at each point $i = 1, 2, 3, 4, 5$. Any conformal prefactor that satisfies this requirement is valid; however, without loss of generality, the following prefactor is chosen:

$$\frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}. \tag{3.140}$$

As in the case of scalar correlators, any other conformal prefactor can be obtained by multiplying the expression with powers of the conformal cross-ratios.

By combining the tensorial parts with the conformal prefactors, the correlator in Eq. 3.118 can be expressed in terms of spacetime structures of the following form:

$$\frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} V_{1,ij} \quad \& \quad \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} V_{1,ijkl}. \tag{3.141}$$

Considering all possible permutations of the points $i, j = 2, 3, 4, 5$, there are 12 distinct candidates for the *short* type structures $V_{1,ij}$, and 24 candidates for the *long* type structures $V_{1,ijkl}$. However, these structures are not all independent over the space of functions of $\{u\}$, but they are related.

Focusing first on the tensorial part (since the relation between the conformal prefactors has already been established through cross ratios), we observe that the 3-point structures obey simple linear relations. This can be seen by considering:

$$\begin{aligned} V_{1,ij} = \left(x_{1i}^{-1}\, x_{ij}\, x_{j1}^{-1}\right)_{\dot{\alpha}\alpha} &= \left(x_{1i}^{-1}\,(x_{i1} - x_{j1})\, x_{j1}^{-1}\right)_{\dot{\alpha}\alpha} \\ &= \left(-x_{j1}^{-1} - x_{1i}^{-1}\right)_{\dot{\alpha}\alpha} = -\left((x_{1i}^{-1})_{\dot{\alpha}\alpha} - (x_{1j}^{-1})_{\dot{\alpha}\alpha}\right). \end{aligned} \tag{3.142}$$

As a result, all twelve forms of $V_{1,ij}$ can be reduced to only three independent structures, as demonstrated by the identities:

$$V_{1,34} = -V_{1,23} + V_{1,24}, \tag{3.143}$$

$$V_{1,35} = -V_{1,23} + V_{1,25}, \tag{3.144}$$

$$V_{1,45} = -V_{1,24} + V_{1,25}. \tag{3.145}$$

Thus, from the four independent spacetime structures required to form a basis for the correlator in Eq. 3.118, three can be chosen from the *short* or *simple 3-point function* type.

This leaves one structure that must be of the intrinsic 5-point function, or the *long* type. Any of the 24 candidates for this type can be selected, as all other candidates can be expressed as linear combinations of the chosen one and the three short structures. For example, we can write:

$$V_{1,3425} = -V_{1,35} + u \cdot V_{1,2345} + v \cdot V_{1,45}, \tag{3.146}$$

where a 4-point identity has been used (see below), and u and v represent the two 4-point cross ratios, which in this specific case are given by:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{3.147}$$

Proof of the identity in Eq. 3.146:

$$\begin{aligned}
&(x_{13}^{-1} x_{34} x_{42}^{-1} x_{25} x_{51}^{-1})_{\dot\alpha\alpha} = (x_{13}^{-1} x_{34} x_{42}^{-1} (x_{24} + x_{45}) x_{51}^{-1})_{\dot\alpha\alpha} \\
&\quad = -(x_{13}^{-1} x_{34} x_{51}^{-1})_{\dot\alpha\alpha} + (x_{13}^{-1} x_{34} x_{42}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} \\
&\quad = -(x_{13}^{-1} x_{34} x_{51}^{-1})_{\dot\alpha\alpha} + (x_{13}^{-1} x_{34} x_{42}^{-1})_{\dot\alpha\beta} (x_{45})^{\beta\dot\beta} (x_{51}^{-1})_{\dot\beta\alpha} \\
&\quad \overset{\text{4pt id.}}{=} -(x_{13}^{-1} x_{34} x_{51}^{-1})_{\dot\alpha\alpha} + (-(x_{13}) + u \cdot (x_{12}^{-1} x_{23} x_{34}^{-1}) + v \cdot (x_{14}^{-1}))_{\dot\alpha\beta} (x_{45})^{\beta\dot\beta} (x_{51}^{-1})_{\dot\beta\alpha} \\
&\quad = -(x_{13}^{-1} x_{34} x_{51}^{-1})_{\dot\alpha\alpha} - (x_{13}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} + u \cdot (x_{12}^{-1} x_{23} x_{34}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} \\
&\qquad + v \cdot (x_{14}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} \\
&\quad = -(x_{13}^{-1} (x_{34} + x_{45}) x_{51}^{-1})_{\dot\alpha\alpha} + u \cdot (x_{12}^{-1} x_{23} x_{34}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} + v \cdot (x_{14}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} \\
&\quad = -(x_{13}^{-1} x_{35} x_{51}^{-1})_{\dot\alpha\alpha} + u \cdot (x_{12}^{-1} x_{23} x_{34}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha} + v \cdot (x_{14}^{-1} x_{45} x_{51}^{-1})_{\dot\alpha\alpha}.
\end{aligned}$$

The identity for four points used within this proof is given by

$$(x_{ij}^{-1} x_{jk} x_{kl}^{-1})_{\dot\alpha\alpha} = (-(x_{ij}) + \frac{x_{il}^2 x_{jk}^2}{x_{ij}^2 x_{lk}^2} \cdot (x_{il}^{-1} x_{lj} x_{jk}^{-1}) + \frac{x_{ik}^2 x_{lj}^2}{x_{ij}^2 x_{lk}^2} \cdot (x_{ik}^{-1}))_{\dot\alpha\alpha}. \tag{3.148}$$

In the same way, all other *long* 5-point structures can similarly be expressed in terms of $V_{1,2345}$.

Without loss of generality, but with arguments for simplicity preferring the *short* structures, the correlator in Eq. 3.118 will be expressed using the four tensor structures as follows:

$$\begin{aligned}
&\langle \mathcal{J}_{\alpha\dot\alpha}(x_1) \mathcal{O}_2(x_2) \mathcal{O}_2(x_3) \mathcal{O}_2(x_4) \mathcal{O}_2(x_5) \rangle = \\
&= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} \Big(-(x_{12}^{-1} - x_{13}^{-1})_{\dot\alpha\alpha}\, g_1(\{u\}) - (x_{12}^{-1} - x_{14}^{-1})_{\dot\alpha\alpha}\, g_2(\{u\}) \\
&\qquad -(x_{12}^{-1} - x_{15}^{-1})_{\dot\alpha\alpha}\, g_3(\{u\}) + (x_{12}^{-1}\, x_{23}\, x_{34}^{-1}\, x_{45}\, x_{51}^{-1})_{\dot\alpha\alpha}\, g_4(\{u\}) \Big),
\end{aligned} \tag{3.149}$$

where the $g_i(\{u\})$ are arbitrary, invariant functions of the cross ratios.

In a similar manner, the correlators involving different permutations of the current $\mathcal{J}_{\alpha\dot\alpha}(x_i)$, where $i = 2, 3, 4, 5$, are constructed by applying the appropriate transformation at the respective insertion points.

The prefactors are chosen to be identical for each correlator in order to facilitate the establishment of relations between the various correlators as efficiently as possible.

The explicit forms of these corresponding expressions, including the notation that will be used in further steps of the analysis, are provided in Appendix C.1.

$\langle \bar{\Psi}_{\dot\alpha}(x_1)\Psi_\alpha(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle$

The structures for the five-point correlators involving the spinor fields $\bar{\Psi}_{\dot\alpha}$ and Ψ_α can be constructed using analogous arguments to those applied in the previous section.

As in the case above, the tensorial structures that govern the transformation properties under $SO(4)$ rotations are constructed by contracting matrices of distances x_{ij} in such a way that the transformations align with the behavior of the five-point correlator.

With $\bar{\Psi}_{\dot\alpha}(x_1)$ and $\Psi_\alpha(x_2)$, there are now two non-trivial transformation points. Specifically, the group action on x_1 affects $\dot\alpha$, while the action on x_2 acts on α. This constrains the tensorial components under $SO(4)$ to the following two forms:

$$(x_{12}^{-1})_{\dot\alpha\alpha}, \tag{3.150}$$

$$\left(x_{1i}^{-1}\, x_{ij}\, x_{j2}^{-1}\right)_{\dot\alpha\alpha}. \tag{3.151}$$

In this setup, the $SO(4)$ rotations act on x_1 and x_2 at the respective indices α and $\dot\alpha$, while leaving the remaining points invariant.
This can be proven in the same way as 3.138.

These tensorial components are of order x_i^{-1} in x_1 and x_2, and of order $\mathcal{O}(1)$ at the other points. With

$$\Delta_\Psi = \Delta_{\bar\Psi} = \frac{5}{2}, \quad \Delta_{\mathcal{O}_{20'}} = 2, \tag{3.152}$$

a conformal prefactor proportional to x_i^{-4} for each $i = 1, ..., 5$ must be included to ensure the correct covariance of the correlator. For simplicity and consistency with the previous correlator involving the $SU(4)_R$-current, see Eq. 3.118, the same prefactor is employed. Thus, the structures take the following form:

$$\frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}(x_{12}^{-1})_{\dot\alpha\alpha} \quad \text{and} \quad \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}(x_{1i}^{-1} x_{ij} x_{j2}^{-1})_{\dot\alpha\alpha}. \tag{3.153}$$

Considering all possible permutations of the tensorial components, we find that there is only one short two-point structure but six (3×2) possible longer structures, namely:

$$\begin{aligned}&(x_{13}^{-1}x_{34}x_{42}^{-1})_{\dot\alpha\alpha}, \quad (x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot\alpha\alpha},\\ &(x_{13}^{-1}x_{35}x_{52}^{-1})_{\dot\alpha\alpha}, \quad (x_{15}^{-1}x_{53}x_{32}^{-1})_{\dot\alpha\alpha},\\ &(x_{14}^{-1}x_{45}x_{52}^{-1})_{\dot\alpha\alpha}, \quad (x_{15}^{-1}x_{54}x_{42}^{-1})_{\dot\alpha\alpha}.\end{aligned} \tag{3.154}$$

However, the structures that involve the same points, i.e., two structures within the same line above, can be easily related to each other, along with $(x_{12}^{-1})_{\dot\alpha\alpha}$, by the following relation:

$$(x_{1j}^{-1}x_{ji}x_{i2}^{-1})_{\dot\alpha\alpha} = -\frac{x_{12}^2x_{ji}^2}{x_{1j}^2x_{2i}^2}(x_{12}^{-1})_{\dot\alpha\alpha} + \frac{x_{1i}^2x_{2j}^2}{x_{1j}^2x_{2i}^2}(x_{1i}^{-1}x_{ij}x_{j2}^{-1})_{\dot\alpha\alpha}. \tag{3.155}$$

Proof of above relation along example $j=3, i=4$:

$$\begin{aligned}
&(x_{13}^{-1}x_{34}x_{42}^{-1})_{\dot\alpha\alpha}\\
=&\frac{1}{x_{13}^2x_{24}^2}(x_{13}x_{34}x_{42})_{\dot\alpha\alpha} = \frac{1}{x_{13}^2x_{24}^2}((x_{14}-x_{34})x_{34}(x_{43}-x_{23}))_{\dot\alpha\alpha}\\
=&\frac{1}{x_{13}^2x_{24}^2}((x_{14}x_{34}x_{43}) - (x_{34}x_{34}x_{43}) - (x_{14}x_{34}x_{23}) + (x_{34}x_{34}x_{23}))_{\dot\alpha\alpha}\\
=&\frac{x_{14}^2x_{34}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{43}^{-1})_{\dot\alpha\alpha} - \frac{x_{34}^2x_{34}^2}{x_{13}^2x_{24}^2}(x_{34}^{-1}x_{34}x_{43}^{-1})_{\dot\alpha\alpha}\\
&-\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha} + \frac{x_{34}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{34}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha}\\
=&-\frac{x_{14}^2x_{34}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1})_{\dot\alpha\alpha} - \frac{x_{34}^2x_{34}^2}{x_{13}^2x_{24}^2}(x_{43}^{-1})_{\dot\alpha\alpha} - \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha} + \frac{x_{34}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{23}^{-1})_{\dot\alpha\alpha}\\
=&\frac{x_{34}^2}{x_{13}^2x_{24}^2}(-x_{14}-x_{43}+x_{23})_{\dot\alpha\alpha} - \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha}\\
=&\frac{x_{34}^2}{x_{13}^2x_{24}^2}(-x_{12})_{\dot\alpha\alpha} - \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha} = -\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}(x_{12}^{-1})_{\dot\alpha\alpha} - \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}(x_{14}^{-1}x_{34}x_{23}^{-1})_{\dot\alpha\alpha}\square
\end{aligned}$$

Similar relations hold true when considering other permutations of $\Psi_{\alpha\dot a}(x)$ and $\bar\Psi_{a\dot\alpha}(x)$.

Thus, the basis for the four tensorial structures of the correlator in Eq. 3.119 is straightforwardly selected by choosing one structure from each line in Eq. 3.154, along with the short two-point structure.

$$\begin{aligned}&\langle\bar\Psi_{\dot\alpha}(x_1)\Psi_\alpha(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle =\\ &= \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{14}^{-1}\,x_{43}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_1(\{u\}) + (x_{15}^{-1}\,x_{53}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_2(\{u\})\\ &\qquad + (x_{15}^{-1}\,x_{54}\,x_{42}^{-1})_{\dot\alpha\alpha}\,h_3(\{u\}) + (x_{12}^{-1})_{\dot\alpha\alpha}\,h_4(\{u\})\Big)\end{aligned} \tag{3.156}$$

$$\begin{aligned}&\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\Psi_\alpha(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\ &= \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{14}^{-1}\,x_{42}\,x_{23}^{-1})_{\dot{\alpha}\alpha}\,h_1(\{u\}) + (x_{15}^{-1}\,x_{52}\,x_{23}^{-1})_{\dot{\alpha}\alpha}\,h_2(\{u\}) \\ &\qquad +(x_{15}^{-1}\,x_{54}\,x_{43}^{-1})_{\dot{\alpha}\alpha}\,h_3(\{u\}) + (x_{13}^{-1})_{\dot{\alpha}\alpha}\,h_4(\{u\})\Big)\end{aligned} \tag{3.157}$$

$$\begin{aligned}&\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\Psi_\alpha(x_4)\mathcal{O}_2(x_5)\rangle = \\ &= \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{13}^{-1}\,x_{32}\,x_{24}^{-1})_{\dot{\alpha}\alpha}\,h_1(\{u\}) + (x_{15}^{-1}\,x_{52}\,x_{24}^{-1})_{\dot{\alpha}\alpha}\,h_2(\{u\}) \\ &\qquad +(x_{15}^{-1}\,x_{53}\,x_{34}^{-1})_{\dot{\alpha}\alpha}\,h_3(\{u\}) + (x_{14}^{-1})_{\dot{\alpha}\alpha}\,h_4(\{u\})\Big)\end{aligned} \tag{3.158}$$

$$\begin{aligned}&\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\Psi_\alpha(x_5)\rangle = \\ &= \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{13}^{-1}\,x_{32}\,x_{25}^{-1})_{\dot{\alpha}\alpha}\,h_1(\{u\}) + (x_{14}^{-1}\,x_{42}\,x_{25}^{-1})_{\dot{\alpha}\alpha}\,h_2(\{u\}) \\ &\qquad +(x_{14}^{-1}\,x_{43}\,x_{35}^{-1})_{\dot{\alpha}\alpha}\,h_3(\{u\}) + (x_{15}^{-1})_{\dot{\alpha}\alpha}\,h_4(\{u\})\Big)\end{aligned} \tag{3.159}$$

The unknown functions of the cross ratios, which correspond to correlators involving the spinor currents, are denoted by $h(\{u\})$.

Similarly, correlators with $\bar{\Psi}_{\dot{\alpha}}$ inserted at other points can be constructed using the same approach. A comprehensive list of all spacetime structures for the relevant correlators including the notation used in the further steps of the analysis is provided in Appendix C.1.

3.4.1.3. Relations between spacetime structures

When the bosonic correlation functions are inserted into the constraints derived from supersymmetric invariance, a single such constraint will decompose into several constraints —one for each coefficient corresponding to an independent bosonic structure. (Note that the application of supersymmetric invariance to the full super-correlator results in objects (the constraints) that are not conformally covariant. Instead, these constraints are proportional to the tensorial structures.) In the case of spacetime structures, linear independence is defined over the space of functions of the cross ratios, similar to the situation in the $\mathfrak{psu}(1,1|2)$ case.
Unlike the simpler toy model case, however, the relations between the tensorial structures in the current setup are significantly more complex and require a more detailed study.

We will prepare this discussion here while the spacetime structures are still fresh in mind and use the results in the further sections, when the full superconformal constraints are derived.

As in the $\mathfrak{psu}(1,1|2)$ case (and elaborated in detail in the coming sections), supersymmetric invariance relates all correlation functions proportional to a $\rho_i^{\alpha\dot{a}}$ or a $\bar{\rho}_i^{a\dot{\alpha}}$, respectively.
Consequently, it becomes necessary to study the relations between the tensorial structures of all correlators proportional to a $\rho_i^{\alpha\dot{a}}$ (or $\bar{\rho}_i^{a\dot{\alpha}}$).

Given the complexity of these relations, we lack a straightforward method for counting how many independent structures will emerge. Instead, we take advantage of the fact that these structures are valued in a four-dimensional space. Thus, in a first step, we express the structures in a four-dimensional basis with coefficients, which are not necessarily functions of the conformal cross ratios. Independence with respect to the cross ratios will be studied in a subsequent step.

We consider the following bases of four fundamental structures for correlators proportional to each ρ_i or $\bar{\rho}_i$:

$$\rho_1, \bar{\rho}_1 \;:\; (x_{12}^{-1})_{\dot{\alpha}\alpha},\; (x_{13}^{-1})_{\dot{\alpha}\alpha},\; (x_{14}^{-1})_{\dot{\alpha}\alpha},\; (x_{15}^{-1})_{\dot{\alpha}\alpha}, \tag{3.160}$$

$$\rho_2, \bar{\rho}_2 \;:\; (x_{21}^{-1})_{\dot{\alpha}\alpha},\; (x_{23}^{-1})_{\dot{\alpha}\alpha},\; (x_{24}^{-1})_{\dot{\alpha}\alpha},\; (x_{25}^{-1})_{\dot{\alpha}\alpha}, \tag{3.161}$$

$$\rho_3, \bar{\rho}_3 \;:\; (x_{31}^{-1})_{\dot{\alpha}\alpha},\; (x_{32}^{-1})_{\dot{\alpha}\alpha},\; (x_{34}^{-1})_{\dot{\alpha}\alpha},\; (x_{35}^{-1})_{\dot{\alpha}\alpha}, \tag{3.162}$$

$$\rho_4, \bar{\rho}_4 \;:\; (x_{41}^{-1})_{\dot{\alpha}\alpha},\; (x_{42}^{-1})_{\dot{\alpha}\alpha},\; (x_{43}^{-1})_{\dot{\alpha}\alpha},\; (x_{45}^{-1})_{\dot{\alpha}\alpha}, \tag{3.163}$$

$$\rho_5, \bar{\rho}_5 \;:\; (x_{51}^{-1})_{\dot{\alpha}\alpha},\; (x_{52}^{-1})_{\dot{\alpha}\alpha},\; (x_{53}^{-1})_{\dot{\alpha}\alpha},\; (x_{54}^{-1})_{\dot{\alpha}\alpha}. \tag{3.164}$$

The procedure of expressing the various tensor structures in terms of the 4-dimensional basis will be outlined using the example of correlators proportional to $\bar{\rho}_1{}^{\dot{\alpha}}$, and hence structures proportional to "$(x_{1i}^{-1})_{\dot{\alpha}\alpha}$".

The correlators or terms proportional to $\bar{\rho}_1{}^{\dot{\alpha}}$ are:

$$\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \langle \mathcal{O}_{20'}(x_1)\mathcal{O}_{20'}(x_2)\mathcal{O}_{20'}(x_3)\mathcal{O}_{20'}(x_4)\mathcal{O}_{20'}(x_5)\rangle \tag{3.165}$$

$$\langle \mathcal{J}_{\alpha\dot{\alpha}}(x_1)\mathcal{O}_{20'}(x_2)\mathcal{O}_{20'}(x_3)\mathcal{O}_{20'}(x_4)\mathcal{O}_{20'}(x_5)\rangle \tag{3.166}$$

$$\langle \bar{\Psi}_{\dot{\alpha}}(x_1)\Psi_{\alpha}(x_i)\mathcal{O}_{20'}(x_k)\mathcal{O}_{20'}(x_l)\mathcal{O}_{20'}(x_m)\rangle \tag{3.167}$$

for all permutations of $\Psi_\alpha(x_i), i = 2,3,4,5$. Any internal dependence has been omitted for the duration of this argument.

These correlators involve the following structures:

$$(x_{12}^{-1})_{\dot\alpha\alpha}, (x_{13}^{-1})_{\dot\alpha\alpha}, (x_{14}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1})_{\dot\alpha\alpha}, \tag{3.168}$$

$$\begin{aligned}
&(x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{53}x_{32}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{54}x_{42}^{-1})_{\dot\alpha\alpha},\\
&(x_{14}^{-1}x_{42}x_{23}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{52}x_{23}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{54}x_{43}^{-1})_{\dot\alpha\alpha},\\
&(x_{13}^{-1}x_{32}x_{24}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{52}x_{24}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{53}x_{34}^{-1})_{\dot\alpha\alpha},\\
&(x_{13}^{-1}x_{32}x_{25}^{-1})_{\dot\alpha\alpha}, (x_{14}^{-1}x_{42}x_{25}^{-1})_{\dot\alpha\alpha}, (x_{15}^{-1}x_{43}x_{35}^{-1})_{\dot\alpha\alpha},
\end{aligned} \tag{3.169}$$

$$(x_{12}^{-1}x_{23}x_{34}^{-1}x_{45}x_{51}^{-1})_{\dot\alpha\alpha} \tag{3.170}$$

All these structures will be expressed in the basis of the first four structures, 3.168.
Note that the short structures of the correlator involving the $SU(4)_R$-current, 3.166, are already expanded in terms of the short 2-pt structures, Eqn. 3.168, as written in Eqn. 3.142. Therefore, only the long structure of this correlator-type has to be considered.

In a preliminary step, the structures of correlators involving $\bar\Psi_{\dot\alpha}(x_1)$ and $\Psi_\alpha(x_i)$ for all $i = 2, 3, 4, 5$ can be reduced to only 4 independent structures (over the space of functions of cross ratios) by the four-point identities:

$$(x_{il}^{-1}x_{lj}x_{jk}^{-1})_{\dot\alpha\alpha} = -(x_{il}^{-1}x_{lk}x_{kj}^{-1})_{\dot\alpha\alpha} - (x_{il}^{-1})_{\dot\alpha\alpha} \tag{3.171}$$

$$(x_{ik}^{-1}x_{kj}x_{jl}^{-1})_{\dot\alpha\alpha} = \frac{x_{il}^2 x_{jk}^2}{x_{ik}^2 x_{jl}^2}(x_{il}^{-1}x_{lk}x_{kj}^{-1})_{\dot\alpha\alpha} - (x_{ik}^{-1})_{\dot\alpha\alpha} + \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}(x_{ij}^{-1})_{\dot\alpha\alpha}. \tag{3.172}$$

This can be utilized for the above structures 3.169 by taking $i = 1$. This then yields the following identities:

$$(x_{14}^{-1}x_{42}x_{23}^{-1})_{\dot\alpha\alpha} = -(x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot\alpha\alpha} - (x_{14}^{-1})_{\dot\alpha\alpha} \tag{3.173}$$

$$(x_{13}^{-1}x_{32}x_{24}^{-1})_{\dot\alpha\alpha} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}(x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot\alpha\alpha} - (x_{13}^{-1})_{\dot\alpha\alpha} + \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}(x_{12}^{-1})_{\dot\alpha\alpha} \tag{3.174}$$

$$(x_{15}^{-1}x_{52}x_{23}^{-1})_{\dot\alpha\alpha} = -(x_{15}^{-1}x_{53}x_{32}^{-1})_{\dot\alpha\alpha} - (x_{15}^{-1})_{\dot\alpha\alpha} \tag{3.175}$$

$$(x_{13}^{-1}x_{32}x_{25}^{-1})_{\dot\alpha\alpha} = \frac{x_{15}^2 x_{23}^2}{x_{13}^2 x_{25}^2}(x_{15}^{-1}x_{53}x_{32}^{-1})_{\dot\alpha\alpha} - (x_{13}^{-1})_{\dot\alpha\alpha} + \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}(x_{12}^{-1})_{\dot\alpha\alpha} \tag{3.176}$$

$$(x_{15}^{-1}x_{52}x_{24}^{-1})_{\dot\alpha\alpha} = -(x_{15}^{-1}x_{54}x_{42}^{-1})_{\dot\alpha\alpha} - (x_{15}^{-1})_{\dot\alpha\alpha} \tag{3.177}$$

$$(x_{14}^{-1}x_{42}x_{25}^{-1})_{\dot\alpha\alpha} = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2}(x_{15}^{-1}x_{54}x_{42}^{-1})_{\dot\alpha\alpha} - (x_{14}^{-1})_{\dot\alpha\alpha} + \frac{x_{12}^2 x_{45}^2}{x_{14}^2 x_{25}^2}(x_{12}^{-1})_{\dot\alpha\alpha} \tag{3.178}$$

$$(x_{15}^{-1}x_{53}x_{34}^{-1})_{\dot\alpha\alpha} = -(x_{15}^{-1}x_{54}x_{43}^{-1})_{\dot\alpha\alpha} - (x_{15}^{-1})_{\dot\alpha\alpha} \tag{3.179}$$

$$(x_{14}^{-1}x_{43}x_{35}^{-1})_{\dot{\alpha}\alpha} = \frac{x_{15}^2 x_{34}^2}{x_{14}^2 x_{35}^2}(x_{15}^{-1}x_{54}x_{43}^{-1})_{\dot{\alpha}\alpha} - (x_{14}^{-1})_{\dot{\alpha}\alpha} + \frac{x_{13}^2 x_{45}^2}{x_{14}^2 x_{35}^2}(x_{13}^{-1})_{\dot{\alpha}\alpha} \tag{3.180}$$

Note that all the coefficients are conformal cross ratios (or combinations thereof) such that up to this point, no additional complexity has been introduced.

To summarise, within this preliminary step, we have reduced the 27 structures 3.168-3.170 to 9 structures of the form

$$\begin{aligned}&(x_{12}^{-1})_{\dot{\alpha}\alpha}, (x_{13}^{-1})_{\dot{\alpha}\alpha}, (x_{14}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1})_{\dot{\alpha}\alpha},\\ &(x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1}x_{53}x_{32}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1}x_{54}x_{42}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1}x_{54}x_{43}^{-1})_{\dot{\alpha}\alpha},\\ &(x_{12}^{-1}x_{23}x_{34}^{-1}x_{45}x_{51}^{-1})_{\dot{\alpha}\alpha}.\end{aligned} \tag{3.181}$$

Now, all 5 remaining structures in the second and third lines get expressed in terms of the short ones in the first line. As the structures are valued in a 4-dimensional space, only 4 structures can be entirely independent. The remaining structures will exhibit dependencies. The rest of this section is devoted to studying those dependencies.

To outline how the remaining five structures are expressed in the chosen basis, we study the example of

$$V_{\dot{\alpha}\alpha} \equiv (x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot{\alpha}\alpha}. \tag{3.182}$$

Expressing this structure V in terms of the basis

$$e = \{(x_{12}^{-1})_{\dot{\alpha}\alpha}, (x_{13}^{-1})_{\dot{\alpha}\alpha}, (x_{14}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1})_{\dot{\alpha}\alpha}\} \tag{3.183}$$

involves solving an equation of the form

$$V_{\dot{\alpha}\alpha} = a \cdot (e_1)_{\dot{\alpha}\alpha} + b \cdot (e_2)_{\dot{\alpha}\alpha} + c \cdot (e_3)_{\dot{\alpha}\alpha} + d \cdot (e_4)_{\dot{\alpha}\alpha} \tag{3.184}$$

for the coefficients a, b, c, d.

By construction, the coefficients a, b, c, d must be polynomial in the Lorentz invariants. The space of Lorentz invariants for five points is ten-dimensional. In this case, we must consider however the eleven invariants

$$\begin{aligned}&x_{12}^2,\ x_{13}^2,\ x_{14}^2,\ x_{15}^2,\ x_{23}^2,\ x_{24}^2,\ x_{25}^2,\ x_{34}^2,\ x_{35}^2,\ x_{45}^2,\\ &\mathcal{I} = \epsilon_{\mu\nu\rho\sigma}x_{15}^{\mu}x_{25}^{\nu}x_{35}^{\rho}x_{45}^{\sigma} = x_{15}^{\alpha\dot{\alpha}}x_{25}^{\beta\dot{\beta}}x_{35}^{\gamma\dot{\gamma}}x_{45}^{\delta\dot{\delta}}(\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\delta}}\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon_{\gamma\delta} - \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\gamma\delta}\epsilon_{\dot{\gamma}\dot{\delta}} + \epsilon_{\alpha\gamma}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\beta\delta}\epsilon_{\dot{\gamma}\dot{\delta}})\end{aligned} \tag{3.185}$$

(or any permutation of the last invariant).
This is because the last invariant does not depend polynomially on the ten Lorentz squares in the first line (only its square does) and hence, in this case, must be considered as well. To summarize, the coefficients a, b, c, d are polynomials in the 11 invariants 3.185.

To obtain the expressions for the coefficients in terms of these Lorentz invariants, we make a polynomial ansatz

$$\text{ansatz} = \sum cc_n (x_{ij}^2)^n + ... \tag{3.186}$$

and equate it with the explicit coefficients obtained by solving equation 3.184 (that could be only solved in component form). In practice, this resulting equation is transformed into a system of four equations (one for each component), by contracting the indices as follows:

$$(x_{12})^{\alpha\dot{\alpha}} V_{\dot{\alpha}\alpha} = a(x_{12})^{\alpha\dot{\alpha}}(x_{12}^{-1})_{\dot{\alpha}\alpha} + b(x_{12})^{\alpha\dot{\alpha}}(x_{13}^{-1})_{\dot{\alpha}\alpha} + c(x_{12})^{\alpha\dot{\alpha}}(x_{14}^{-1})_{\dot{\alpha}\alpha} + d(x_{12})^{\alpha\dot{\alpha}}(x_{15}^{-1})_{\dot{\alpha}\alpha}, \tag{3.187}$$

$$(x_{13})^{\alpha\dot{\alpha}} V_{\dot{\alpha}\alpha} = a(x_{13})^{\alpha\dot{\alpha}}(x_{12}^{-1})_{\dot{\alpha}\alpha} + b(x_{13})^{\alpha\dot{\alpha}}(x_{13}^{-1})_{\dot{\alpha}\alpha} + c(x_{13})^{\alpha\dot{\alpha}}(x_{14}^{-1})_{\dot{\alpha}\alpha} + d(x_{13})^{\alpha\dot{\alpha}}(x_{15}^{-1})_{\dot{\alpha}\alpha}, \tag{3.188}$$

$$(x_{14})^{\alpha\dot{\alpha}} V_{\dot{\alpha}\alpha} = a(x_{14})^{\alpha\dot{\alpha}}(x_{12}^{-1})_{\dot{\alpha}\alpha} + b(x_{14})^{\alpha\dot{\alpha}}(x_{13}^{-1})_{\dot{\alpha}\alpha} + c(x_{14})^{\alpha\dot{\alpha}}(x_{14}^{-1})_{\dot{\alpha}\alpha} + d(x_{14})^{\alpha\dot{\alpha}}(x_{15}^{-1})_{\dot{\alpha}\alpha}, \tag{3.189}$$

$$(x_{15})^{\alpha\dot{\alpha}} V_{\dot{\alpha}\alpha} = a(x_{15})^{\alpha\dot{\alpha}}(x_{12}^{-1})_{\dot{\alpha}\alpha} + b(x_{15})^{\alpha\dot{\alpha}}(x_{13}^{-1})_{\dot{\alpha}\alpha} + c(x_{15})^{\alpha\dot{\alpha}}(x_{14}^{-1})_{\dot{\alpha}\alpha} + d(x_{15})^{\alpha\dot{\alpha}}(x_{15}^{-1})_{\dot{\alpha}\alpha}. \tag{3.190}$$

This system of four equations fixes the four coefficients a, b, c, d.

To solve for the coefficients in terms of the Lorentz scalars, we use the following trace identities:

$$\operatorname{tr}\left(x_{1i}\, x_{1j}^{-1}\right) = \frac{x_{1i}^2 + x_{1j}^2 - x_{ij}^2}{x_{1j}^2}, \tag{3.191}$$

$$\begin{aligned}\operatorname{tr}\left(x_{1i}\, x_{1j}^{-1} x_{jk} x_{kl}^{-1}\right) = \frac{1}{2x_{1j}^2 x_{kl}^2} \Big(& x_{il}^2 x_{jk}^2 - x_{ik}^2 x_{jl}^2 + x_{ij}^2 x_{lk}^2 - x_{1i}(x_{lk}^2 - x_{lj}^2 + x_{kj}^2) \\ & - x_{1j}(x_{lk}^2 - x_{li}^2 + x_{ki}^2) + x_{1k}(x_{lj}^2 - x_{li}^2 + x_{ij}^2) \\ & - x_{1l}(x_{kj}^2 - x_{ki}^2 + x_{ji}^2) - \epsilon_{\mu\nu\rho\sigma} x_{1i}^{\mu} x_{li}^{\nu} x_{ki}^{\rho} x_{ji}^{\sigma} \Big),\end{aligned} \tag{3.192}$$

where the last term must be translated into $\mathfrak{su}(2)$-indices. It is written in terms of 4-vector indices here for simplicity of the expression.

A similar expression holds for $\mathrm{tr}\left(x_{1i}\, x_{1j}^{-1} x_{jk} x_{kl}^{-1} x_{lm} x_{m1}^{-1}\right)$, but is too large to display here.[5]

By applying this procedure to each of the five remaining structures, we can express them all in terms of a basis of the four short structures, with coefficients that are functions of the eleven Lorentz invariants. The "eleventh" invariant $\mathcal{I}$ appears linearly only.

Inserting the expressions for the coefficients into the supersymmetry constraint from $\bar{\rho}_1^{a\dot{\alpha}}$ will yield four equations, one for each coefficient corresponding to the independent basis structures $(x_{12}^{-1})_{\dot{\alpha}\alpha}, (x_{13}^{-1})_{\dot{\alpha}\alpha}, (x_{14}^{-1})_{\dot{\alpha}\alpha}, (x_{15}^{-1})_{\dot{\alpha}\alpha}$.
However, this does not complete the process. These four structures are independent when considering functions of Lorentz invariants, but if we consider the space of cross ratios, further independencies are introduced. This splits up the equations even further.

To address this, we express five of the Lorentz scalars in terms of the five conformal cross ratios. For example,

$$\begin{aligned} &x_{12}^2 \to u_1 \frac{x_{13}^2 x_{24}^2}{x_{34}^2}, \quad x_{14}^2 \to u_2 \frac{x_{13}^2 x_{24}^2}{x_{23}^2}, \quad x_{15}^2 \to u_5 \frac{x_{13}^2 x_{24}^2 x_{35}^2}{x_{23}^2 x_{34}^2}, \\ &x_{25}^2 \to u_4 \frac{x_{24}^2 x_{35}^2}{x_{34}^2}, \quad x_{45}^2 \to u_3 \frac{x_{24}^2 x_{35}^2}{x_{23}^2}. \end{aligned} \tag{3.193}$$

By extracting the coefficients of the remaining $(5+1)$ Lorentz invariants within the four constraints, we obtain the desired equations reflecting the independence over the space of functions of cross ratios.
In particular, the final system of equations will only involve the five conformal cross ratios.

This procedure will be used in the further sections when deriving the superconformal Ward identities.

3.4.2. R-symmetry structures

In addition to the spacetime structures, we must now consider the second bosonic subgroup of $PSU(2,2|4)$, namely the R-symmetry group $SU(4)_R$, which, within

[5] A Mathematica notebook containing the mentioned identity can provided upon request.

this procedure of deriving the SCWI, acts exclusively on the internal coordinates $y^{a\dot{a}}$. The group $SU(4)_R$ transforms the internal coordinates in a way analogous to the way the conformal group $SU(2,2)$ acts on the spacetime coordinates $x^{\alpha\dot{\alpha}}$, with the key difference that $SU(4)$ is compact, whereas $SU(2,2)$ is non-compact. This distinction introduces additional analyticity constraints on the R-symmetry structures.
In practical terms, this means that arbitrary functions of internal cross ratios, which could lead to singularities, are not allowed. Instead, the structures must exhibit only polynomial dependence on the internal coordinates.
This significantly impacts the representation of correlators in terms of independent structures, as will be discussed in more detail below.

3.4.2.1. Counting R-symmetry structures

Correlation functions that are covariant under $SU(4)_R$ are required to depend polynomially on the internal coordinates $y^{a\dot{a}}$. Consequently, it is not possible to multiply these correlators by arbitrary functions of the R-symmetry cross ratios, and only truly linearly independent structures must be considered.
In other words, a set of R-symmetry structures $\{r_1, \ldots, r_l\}$ is considered independent if there exist no $\{c_1, \ldots, c_l\} \in \mathbb{C}$ such that

$$\sum{}_{i=1}^{l} c_i r_i = 0.$$

This restriction simplifies the process of counting the independent structures but simultaneously leads to a higher number of such structures.

A useful method for determining the number of independent R-symmetry structures in a given correlator is to count the number of singlets in the relevant tensor product of R-symmetry representations.
For instance, consider the 5-point function of superprimaries

$$\langle \mathcal{O}_{20'}(y_1)\mathcal{O}_{20'}(y_2)\mathcal{O}_{20'}(y_3)\mathcal{O}_{20'}(y_4)\mathcal{O}_{20'}(y_5)\rangle\,, \tag{3.194}$$

where, for the moment, we focus exclusively on the internal dependence y_i.
In the language of Dynkin labels, the superprimary $\mathcal{O}_{20'}$ transforms in the representation $[0,2,0]$ under $SU(4)_R$. Thus, from the perspective of R-symmetry, the 5-point function corresponds to the tensor product

$$[0,2,0]^{\otimes_5}. \tag{3.195}$$

This tensor product decomposes into irreducible representations as

$$[0,2,0]^{\otimes_5} = \mathbf{22}[0,0,0] \oplus 130[1,0,1] \oplus 145[0,2,0] \oplus \ldots, \tag{3.196}$$

indicating that there are 22 singlets. Indeed, there are 22 linearly independent R-symmetry structures, which can be represented in terms of Wick contractions (see Figure 3.1).

Applying the same reasoning to different correlators, the number of singlets in the relevant tensor product provides the number of independent R-symmetry structures for correlators involving currents and spinors. Specifically, we find:

$$\begin{aligned}&\langle \mathcal{J}_{a\dot{a}}(y_1)\mathcal{O}_2(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle : \\ &\qquad [1,0,1]\otimes[0,2,0]^{\otimes 4} = \mathbf{21}[0,0,0]\oplus 142[1,0,1]\oplus 130[0,2,0]\oplus\dots\end{aligned} \tag{3.197}$$

$$\begin{aligned}&\langle \bar{\Psi}_a(y_1)\Psi_{\dot{a}}(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle : \\ &\qquad [1,1,0]\otimes[0,1,1]\otimes[0,2,0]^{\otimes 3} = \mathbf{28}[0,0,0]\oplus 193[1,0,1]\oplus 187[0,2,0]\oplus\dots\end{aligned} \tag{3.198}$$

Thus, the correlator involving the current $\mathcal{J}_{a\dot{a}}$ must be expressed in terms of 21 linearly independent R-symmetry structures, while the correlators involving the spinor fields Ψ_a and $\bar{\Psi}_{\dot{a}}$ must be represented by 28 linearly independent R-symmetry structures.

3.4.2.2. Constructing R-symmetry structures

$\langle \mathcal{O}_2(y_1)\mathcal{O}_2(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle$

A systematic method for deriving the 22 R-symmetry structures associated with the superprimary 5-point function is to employ the Wick contractions, as illustrated in Figure 3.1.
Interpreting Figure 3.1 in the context of R-symmetry amounts to assigning each edge a propagator in the internal variables, represented as $y_{ij}^2 = \det y_{ij}$, where i and j denote the vertices that the edge connects. A thick edge corresponds to a double contraction, which, in this context, is expressed as y_{ij}^4.
All 22 structures are linearly independent, leading to the conclusion that the internal part of the superprimary correlator can be expressed by those structures

$$\begin{aligned}&\langle \mathcal{O}_{20'}(y_1)\,\mathcal{O}_{20'}(y_2)\,\mathcal{O}_{20'}(y_3)\,\mathcal{O}_{20'}(y_4)\,\mathcal{O}_{20'}(y_5)\rangle = \\ &\quad = \Big\{ y_{12}^4y_{34}^2y_{45}^2y_{53}^2,\ y_{13}^4y_{24}^2y_{45}^2y_{52}^2,\ y_{14}^4y_{23}^2y_{35}^2y_{52}^2,\ y_{15}^4y_{23}^2y_{34}^2y_{42}^2,\ y_{23}^4y_{14}^2y_{45}^2y_{51}^2, \\ &\qquad y_{24}^4y_{13}^2y_{35}^2y_{51}^2,\ y_{25}^4y_{13}^2y_{34}^2y_{41}^2,\ y_{34}^4y_{12}^2y_{25}^2y_{51}^2,\ y_{35}^4y_{12}^2y_{24}^2y_{41}^2,\ y_{45}^4y_{12}^2y_{23}^2y_{31}^2, \\ &\qquad y_{12}^2y_{23}^2y_{34}^2y_{45}^2y_{51}^2,\ y_{12}^2y_{23}^2y_{35}^2y_{54}^2y_{41}^2,\ y_{12}^2y_{24}^2y_{45}^2y_{53}^2y_{31}^2,\ y_{12}^2y_{24}^2y_{43}^2y_{35}^2y_{51}^2\ , \\ &\qquad y_{12}^2y_{25}^2y_{53}^2y_{34}^2y_{41}^2,\ y_{12}^2y_{25}^2y_{54}^2y_{43}^2y_{31}^2,\ y_{13}^2y_{35}^2y_{52}^2y_{24}^2y_{41}^2,\ y_{13}^2y_{32}^2y_{24}^2y_{45}^2y_{51}^2, \\ &\qquad y_{13}^2y_{32}^2y_{25}^2y_{54}^2y_{41}^2,\ y_{13}^2y_{34}^2y_{42}^2y_{25}^2y_{51}^2,\ y_{14}^2y_{42}^2y_{23}^2y_{35}^2y_{51}^2,\ y_{15}^2y_{52}^2y_{23}^2y_{34}^2y_{41}^2 \Big\}\,.\end{aligned} \tag{3.199}$$

The full bosonic correlator, with R-symmetry and spacetime structures combined, is provided at the end of this section.

$\langle \mathcal{J}_{a\dot{a}}(y_1)\mathcal{O}_2(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle$

The correlator $\langle \mathcal{J}_{a\dot{a}}(y_1)\mathcal{O}_2(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle$ involves the current $\mathcal{J}_{a\dot{a}}(y)$, which transforms non-trivially under the $SU(4)_R$ symmetry group.
Consequently, the correlator is represented by tensorial structures with transformations acting specifically on the internal coordinates $y_1^{a\dot{a}}$.
Since both the conformal group $SU(2,2)$ and the R-symmetry group $SU(4)_R$ act in a similar manner, the construction of the tensorial structures follows the same reasoning as in the spinning conformal case. Thus, we identify the candidate structures as follows:

$$(y_{1i}^{-1} y_{ij} y_{j1}^{-1})_{a\dot{a}}, \quad (y_{1i}^{-1} y_{il} y_{jl}^{-1} y_{kj} y_{j1}^{-1})_{a\dot{a}}. \tag{3.200}$$

However, to ensure analyticity and avoid potential singularities, especially in the limits $i \to 1$ or $j \to 1$, polynomial dependence must be guaranteed. This is achieved by multiplying the structures with appropriate prefactors.
Thus, the R-symmetry tensorial components can be expressed as:

$$R_{1,ij} = y_{1i}^2 y_{1j}^2 (y_{1i}^{-1} y_{ij} y_{j1}^{-1})_{a\dot{a}}, \tag{3.201}$$
$$R_{1,ijkl} = y_{1i}^2 y_{jk}^2 y_{l1}^2 (y_{1i}^{-1} y_{ij} y_{jk}^{-1} y_{kl} y_{l1}^{-1})_{a\dot{a}}. \tag{3.202}$$

As in the case of spacetime structures, we also require an invariant prefactor that carries the correct R-symmetry weight of the correlator at each point.
The $SU(4)_R$-symmetry weight p is reflected in the internal coordinates as $\sim y^{2\cdot p}$, where

$$p_{\mathcal{O}_{20'}} = 2, \quad p_{\mathcal{J}} = 1. \tag{3.203}$$

Since full linear independence must be taken into account, the first step involves constructing all such prefactors that multiply each tensorial component. The general structures can be expressed as follows:

$$y_{1i}^2 y_{jk}^2 y_{l1}^2 (y_{1i}^{-1} y_{ij} y_{jk}^{-1} y_{kl} y_{l1}^{-1})_{a\dot{a}} \cdot \begin{cases} y_{ij}^2 y_{kl}^2 \\ y_{ik}^2 y_{jl}^2 \\ y_{il}^2 y_{kl}^2 \end{cases}, \quad y_{1i}^2 y_{1j}^2 (y_{1i}^{-1} y_{ij} y_{j1}^{-1})_{a\dot{a}} \cdot \begin{cases} y_{ij}^2 y_{kl}^4 \\ y_{ik}^2 y_{jl}^2 y_{kl}^2 \\ y_{il}^2 y_{kl}^2 y_{kl}^2 \end{cases} \tag{3.204}$$

Considering all permutations of indices $i, j, k, l \in \{2, 3, 4, 5\}$, there are a total of $72 = (3 \cdot 4!)$ candidate structures of the first type and $36 = (3 \cdot 12)$ of the second type. However, only 21 of these structures are linearly independent.

From the *short, three-point-like* structures represented in Eqn 3.204, there are 18 linearly independent structures, which can be taken as follows:

$$\left.\begin{array}{l} y_{12}^2 y_{13}^2 \left(y_{12}^{-1} - y_{13}^{-1}\right)_{\dot{a}a} \cdot y_{45}^2 \\ y_{12}^2 y_{14}^2 \left(y_{12}^{-1} - y_{14}^{-1}\right)_{\dot{a}a} \cdot y_{35}^2 \\ y_{12}^2 y_{15}^2 \left(y_{12}^{-1} - y_{15}^{-1}\right)_{\dot{a}a} \cdot y_{34}^2 \\ y_{13}^2 y_{14}^2 \left(y_{13}^{-1} - y_{14}^{-1}\right)_{\dot{a}a} \cdot y_{25}^2 \\ y_{13}^2 y_{15}^2 \left(y_{13}^{-1} - y_{15}^{-1}\right)_{\dot{a}a} \cdot y_{24}^2 \\ y_{14}^2 y_{15}^2 \left(y_{14}^{-1} - y_{15}^{-1}\right)_{\dot{a}a} \cdot y_{23}^2 \end{array}\right\} \left\{\begin{array}{l} y_{23}^2 y_{45}^2 \\ y_{24}^2 y_{35}^2 \\ y_{25}^2 y_{34}^2 \end{array}\right. \tag{3.205}$$

Here, we have directly rewritten those short structures in terms of the two-point structures, by the same identity as used in the spacetime case.
For simplicity, akin to the spacetime case, all 18 structures will be considered as part of the basis representing the full correlator. Consequently, three additional structures must be of the *long* type.

To maintain simplicity, we select only one tensorial part with each of the three different scalar factors, resulting in three independent structures. Without loss of generality, the remaining three structures are chosen to be of the form:

$$y_{12}^2 y_{34}^2 y_{51}^2 \left(y_{12}^{-1} y_{23} y_{34}^{-1} y_{45} y_{51}^{-1}\right)_{\dot{a}a} \cdot \left\{\begin{array}{l} y_{23}^2 y_{45}^2 \\ y_{24}^2 y_{35}^2 \\ y_{25}^2 y_{34}^2 \end{array}\right. \tag{3.206}$$

Summarising, the correlator involving the current $\mathcal{J}_{a\dot{a}}(y_1)$ can be written in terms of the structures

$$\begin{aligned} &\langle \mathcal{J}(y_1) \mathcal{O}_{20'}(y_2) \mathcal{O}_{20'}(y_3) \mathcal{O}_{20'}(y_4) \mathcal{O}_{20'}(y_5)\rangle \ : \\ &\Big\{ y_{12}^2 y_{14}^2 y_{24}^2 y_{35}^4 \left(y_{12}^{-1}\, y_{24}\, y_{41}^{-1}\right)_{\dot{a}a},\ y_{12}^2 y_{14}^2 y_{25}^2 y_{35}^2\, y_{34}^2 \left(y_{12}^{-1}\, y_{24}\, y_{41}^{-1}\right)_{\dot{a}a}, \\ &\quad y_{12}^2 y_{15}^2 y_{25}^2 y_{34}^4 \left(y_{12}^{-1}\, y_{25}\, y_{51}^{-1}\right)_{\dot{a}a},\ y_{12}^2 y_{15}^2 y_{24}^2 y_{34}^2\, y_{35}^2 \left(y_{12}^{-1}\, y_{25}\, y_{51}^{-1}\right)_{\dot{a}a}, \\ &\quad y_{13}^2 y_{14}^2 y_{25}^4 y_{34}^2 \left(y_{13}^{-1}\, y_{34}\, y_{41}^{-1}\right)_{\dot{a}a},\ y_{13}^2 y_{14}^2 y_{25}^2 y_{35}^2\, y_{24}^2 \left(y_{13}^{-1}\, y_{34}\, y_{41}^{-1}\right)_{\dot{a}a}, \\ &\quad y_{13}^2 y_{15}^2 y_{24}^4 y_{35}^2 \left(y_{13}^{-1}\, y_{35}\, y_{51}^{-1}\right)_{\dot{a}a},\ y_{13}^2 y_{15}^2 y_{24}^2 y_{34}^2\, y_{25}^2 \left(y_{13}^{-1}\, y_{35}\, y_{51}^{-1}\right)_{\dot{a}a}, \\ &\quad y_{12}^2 y_{13}^2 y_{23}^2 y_{45}^4 \left(y_{12}^{-1}\, y_{23}\, y_{31}^{-1}\right)_{\dot{a}a},\ y_{12}^2 y_{13}^2 y_{24}^2 y_{35}^2\, y_{45}^2 \left(y_{12}^{-1}\, y_{23}\, y_{31}^{-1}\right)_{\dot{a}a}, \\ &\quad y_{12}^2 y_{14}^2 y_{23}^2 y_{35}^2\, y_{45}^2 \left(y_{12}^{-1}\, y_{24}\, y_{41}^{-1}\right)_{\dot{a}a},\ y_{13}^2 y_{15}^2 y_{24}^2 y_{23}^2\, y_{45}^2 \left(y_{13}^{-1}\, y_{35}\, y_{51}^{-1}\right)_{\dot{a}a}, \end{aligned} \tag{3.207}$$

$$y_{14}^2 y_{15}^2 y_{23}^4 y_{45}^2 \left(y_{14}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a},\ y_{14}^2 y_{15}^2 y_{23}^2 y_{24}^2\, y_{35}^2 \left(y_{14}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a},$$
$$y_{13}^2 y_{14}^2 y_{25}^2 y_{23}^2\, y_{45}^2 \left(y_{13}^{-1}\, y_{34}\, y_{41}^{-1}\right)_{\dot{a}a},\ y_{12}^2 y_{13}^2 y_{25}^2 y_{34}^2\, y_{45}^2 \left(y_{12}^{-1}\, y_{23}\, y_{31}^{-1}\right)_{\dot{a}a},$$
$$y_{12}^2 y_{15}^2 y_{23}^2 y_{34}^2\, y_{45}^2 \left(y_{12}^{-1}\, y_{25}\, y_{51}^{-1}\right)_{\dot{a}a},\ y_{14}^2 y_{15}^2 y_{23}^2 y_{34}^2\, y_{25}^2 \left(y_{14}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a},$$
$$y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2\, y_{15}^2 \left(y_{12}^{-1}\, y_{23}\, y_{34}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a}, \quad y_{12}^2 y_{24}^2 y_{34}^2 y_{35}^2\, y_{51}^2 \left(y_{12}^{-1}\, y_{23}\, y_{34}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a},$$
$$y_{12}^2 y_{25}^2 y_{34}^4 y_{15}^2 \left(y_{12}^{-1}\, y_{23}\, y_{34}^{-1}\, y_{45}\, y_{51}^{-1}\right)_{\dot{a}a} \Bigg\}\ .$$

The correlators where $\mathcal{J}_{a\dot{a}}(x_i)$ is inserted at a different point can be constructed in the very same way.[6]

$\langle \bar{\Psi}_a(y_1)\Psi_{\dot{a}}(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle$

In the same manner as described above, the tensorial part of the R-symmetry structures for the correlator $\langle \bar{\Psi}_a(y_1)\Psi_{\dot{a}}(y_2)\mathcal{O}_2(y_3)\mathcal{O}_2(y_4)\mathcal{O}_2(y_5)\rangle$ can be constructed by *copying* the spacetime part, utilizing the analogous transformation group actions of $SU(4)_R$ and $SU(2,2)$.
To ensure polynomial dependence, we multiply by the respective prefactors, resulting in tensor structures of the following form:

$$y_{12}^2 (y_{12}^{-1})_{\dot{a}a}, \quad y_{1i}^2 y_{2j}^2 (y_{1i}^{-1} y_{ij} y_{j2}^{-1})_{\dot{a}a}. \tag{3.208}$$

Furthermore, an internal prefactor of the correct weight must be included. Recall that those R-symmetry weights are

$$p_{\mathcal{O}_2} = 2, \quad p_\Psi = p_{\bar{\Psi}} = \frac{3}{2}. \tag{3.209}$$

Thus, the following structures are possible:

$$y_{12}^2 (y_{12}^{-1})_{\dot{a}a} \cdot \begin{cases} y_{12}^2 y_{ij}^2 y_{jk}^2 y_{ki}^2 \\ y_{1i}^2 y_{2j}^2 y_{ik}^2 y_{jk}^2 \\ y_{1i}^2 y_{2i}^2 y_{jk}^2 y_{jk}^2 \end{cases}, \qquad y_{1i}^2 y_{2j}^2 (y_{1i}^{-1} y_{ij} y_{j2}^{-1})_{\dot{a}a} \cdot \begin{cases} y_{12}^2 y_{ik}^2 y_{jk}^2 \\ y_{1k}^2 y_{2k}^2 y_{ij}^2 \\ y_{1k}^2 y_{2i}^2 y_{jk}^2 \\ y_{1k}^2 y_{2j}^2 y_{ik}^2 \\ y_{1j}^2 y_{2k}^2 y_{ik}^2 \\ y_{1i}^2 y_{2k}^2 y_{jk}^2 \end{cases}, \tag{3.210}$$

where all permutations of indices i, j, k in $\{3,4,5\}$ must be considered.
It is evident that the above structures with all permutations are not all linearly

[6] A notebook including all those correlators can be requested from the author.

dependent. In fact, all candidates reduce to 28 linearly independent structures, as desired.

Without loss of generality, we choose the correlator to be represented by the following R-symmetry structures:

$$\begin{aligned}
&\langle\bar{\Psi}_{a\dot{\alpha}}(y_1)\Psi_{\alpha\dot{a}}(y_2)\mathcal{O}_{20'}(y_3)\mathcal{O}_{20'}(y_4)\mathcal{O}_{20'}(y_5)\rangle\ : \\
&\quad\Big\{y_{14}^4y_{23}^2y_{25}^2y_{35}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{14}^2y_{15}^2y_{23}^2y_{25}^2y_{34}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{14}^2y_{23}^2y_{25}^2y_{45}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{13}^2y_{14}^2y_{25}^4y_{34}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{14}^2y_{24}^2y_{25}^2y_{35}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{14}^2y_{25}^2y_{34}^2y_{35}^2\left(y_{14}^{-1}\,y_{45}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{14}^2y_{15}^2y_{23}^2y_{25}^2y_{34}^2\left(y_{14}^{-1}\,y_{43}\,y_{32}^{-1}\right)_{\dot{a}a},\ y_{14}^2y_{15}^2y_{23}^4y_{45}^2\left(y_{14}^{-1}\,y_{43}\,y_{32}^{-1}\right)_{\dot{a}a},\\
&\quad y_{14}^2y_{15}^2y_{23}^2y_{24}^2y_{35}^2\left(y_{14}^{-1}\,y_{43}\,y_{32}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{14}^2y_{24}^2y_{35}^4\left(y_{12}^{-1}\right)_{\dot{a}a},\\
&\quad y_{12}^2y_{14}^2y_{25}^2y_{35}^2y_{34}^2\left(y_{12}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{14}^2y_{23}^2y_{35}^2y_{45}^2\left(y_{12}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^4y_{24}^2y_{25}^2y_{45}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{13}^2y_{15}^2y_{24}^2y_{25}^2y_{34}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{14}^2y_{24}^2y_{25}^2y_{35}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{13}^2y_{14}^2y_{25}^4y_{34}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{14}^2y_{23}^2y_{25}^2y_{45}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{13}^2y_{25}^2y_{34}^2y_{45}^2\left(y_{13}^{-1}\,y_{35}\,y_{52}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^4y_{24}^2y_{25}^2y_{45}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\ y_{13}^2y_{14}^2y_{24}^2y_{25}^2y_{35}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{15}^2y_{24}^2y_{25}^2y_{34}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\ y_{13}^2y_{15}^2y_{24}^4y_{35}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\\
&\quad y_{13}^2y_{15}^2y_{23}^2y_{24}^2y_{45}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{13}^2y_{24}^2y_{35}^2y_{45}^2\left(y_{13}^{-1}\,y_{34}\,y_{42}^{-1}\right)_{\dot{a}a},\\
&\quad y_{12}^2y_{13}^2y_{23}^2y_{45}^4\left(y_{12}^{-1}\right)_{\dot{a}a},\ y_{12}^2y_{13}^2y_{25}^2y_{34}^2y_{45}^2\left(y_{12}^{-1}\right)_{\dot{a}a},\\
&\quad y_{12}^2y_{13}^2y_{24}^2y_{35}^2y_{45}^2\left(y_{12}^{-1}\right)_{\dot{a}a},\ y_{12}^4y_{34}^2y_{45}^2y_{53}^2\left(y_{12}^{-1}\right)_{\dot{a}a}\Big\}\,.
\end{aligned} \tag{3.211}$$

Similarly, all correlators with various different insertion points of $\bar{\Psi}_a$ and Ψ_a are constructed.[7]

3.4.3. Full bosonic correlators

To obtain the complete expressions for the bosonic correlators, it is necessary to combine the constructed R-symmetry and conformal structures. In principle, each

[7]Again, a full list of those correlators can be provided.

R-symmetry structure multiplies any spacetime structure, which can be expressed schematically as:

$$G_n(\{x_i, y_i\}) = \sum_{i=1}^{N} (\text{y-structure} \cdot \text{x-structure})_i \cdot f_i(\{u\}), \tag{3.212}$$

where $\{u\}$ denotes the five cross ratios defined earlier in Eqn. 3.121 [121]:

$$\begin{aligned}
u_1 &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z_1 \bar{z}_1, \quad u_2 = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z_1)(1-\bar{z}_1), \\
u_3 &= \frac{x_{23}^2 x_{45}^2}{x_{24}^2 x_{35}^2} = z_2 \bar{z}_2, \quad u_4 = \frac{x_{25}^2 x_{34}^2}{x_{24}^2 x_{35}^2} = (1-z_2)(1-\bar{z}_1), \\
u_5 &= \frac{x_{15}^2 x_{23}^2 x_{34}^2}{x_{24}^2 x_{13}^2 x_{35}^2} = w\left(z_1 - \bar{z}_1\right)\left(z_2 - \bar{z}_2\right) + \left(1 - z_1 - z_2\right)\left(1 - \bar{z}_1 - \bar{z}_2\right).
\end{aligned} \tag{3.213}$$

For the case of the 5-point function of superprimaries $\mathcal{O}_{20'}(x,y)$, the correlator takes the form:

$$\begin{aligned}
&\langle \mathcal{O}_{20'}\left(x_1,y_1\right) \mathcal{O}_{20'}\left(x_2,y_2\right) \mathcal{O}_{20'}\left(x_3,y_3\right) \mathcal{O}_{20'}\left(x_4,y_4\right) \mathcal{O}_{20'}\left(x_5,y_5\right)\rangle = \\
&\quad = \frac{y_{12}^4 y_{34}^2 y_{45}^2 y_{53}^2}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} f_1(\{u\}) + \frac{y_{13}^4 y_{24}^2 y_{45}^2 y_{52}^2}{x_{13}^4 x_{24}^2 x_{45}^2 x_{52}^2} f_2(\{u\}) + \frac{y_{14}^4 y_{23}^2 y_{35}^2 y_{52}^2}{x_{14}^4 x_{23}^2 x_{35}^2 x_{52}^2} f_3(\{u\}) \\
&\quad + \frac{y_{15}^4 y_{23}^2 y_{34}^2 y_{42}^2}{x_{15}^4 x_{23}^2 x_{34}^2 x_{42}^2} f_4(\{u\}) + \frac{y_{23}^4 y_{14}^2 y_{45}^2 y_{51}^2}{x_{23}^4 x_{14}^2 x_{45}^2 x_{51}^2} f_5(\{u\}) + \frac{y_{24}^4 y_{13}^2 y_{35}^2 y_{51}^2}{x_{24}^4 x_{13}^2 x_{35}^2 x_{51}^2} f_6(\{u\}) \\
&\quad + \frac{y_{25}^4 y_{13}^2 y_{34}^2 y_{41}^2}{x_{25}^4 x_{13}^2 x_{34}^2 x_{41}^2} f_7(\{u\}) + \frac{y_{34}^4 y_{12}^2 y_{25}^2 y_{51}^2}{x_{34}^4 x_{12}^2 x_{25}^2 x_{51}^2} f_8(\{u\}) + \frac{y_{35}^4 y_{12}^2 y_{24}^2 y_{41}^2}{x_{35}^4 x_{12}^2 x_{24}^2 x_{41}^2} f_9(\{u\}) \\
&\quad + \frac{y_{45}^4 y_{12}^2 y_{23}^2 y_{31}^2}{x_{45}^4 x_{12}^2 x_{23}^2 x_{31}^2} f_{10}(\{u\}) + \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{51}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} f_{11}(\{u\}) + \frac{y_{12}^2 y_{23}^2 y_{35}^2 y_{54}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{35}^2 x_{54}^2 x_{41}^2} f_{12}(\{u\}) \\
&\quad + \frac{y_{12}^2 y_{24}^2 y_{45}^2 y_{53}^2 y_{31}^2}{x_{12}^2 x_{24}^2 x_{45}^2 x_{53}^2 x_{31}^2} f_{13}(\{u\}) + \frac{y_{12}^2 y_{24}^2 y_{43}^2 y_{35}^2 y_{51}^2}{x_{12}^2 x_{24}^2 x_{43}^2 x_{35}^2 x_{51}^2} f_{14}(\{u\}) + \frac{y_{12}^2 y_{25}^2 y_{53}^2 y_{34}^2 y_{41}^2}{x_{12}^2 x_{25}^2 x_{53}^2 x_{34}^2 x_{41}^2} f_{15}(\{u\}) \\
&\quad + \frac{y_{12}^2 y_{25}^2 y_{54}^2 y_{43}^2 y_{31}^2}{x_{12}^2 x_{25}^2 x_{54}^2 x_{43}^2 x_{31}^2} f_{16}(\{u\}) + \frac{y_{13}^2 y_{35}^2 y_{52}^2 y_{24}^2 y_{41}^2}{x_{13}^2 x_{35}^2 x_{52}^2 x_{24}^2 x_{41}^2} f_{17}(\{u\}) + \frac{y_{13}^2 y_{32}^2 y_{24}^2 y_{45}^2 y_{51}^2}{x_{13}^2 x_{32}^2 x_{24}^2 x_{45}^2 x_{51}^2} f_{18}(\{u\}) \\
&\quad + \frac{y_{13}^2 y_{32}^2 y_{25}^2 y_{54}^2 y_{41}^2}{x_{13}^2 x_{32}^2 x_{25}^2 x_{54}^2 x_{41}^2} f_{19}(\{u\}) + \frac{y_{13}^2 y_{34}^2 y_{42}^2 y_{25}^2 y_{51}^2}{x_{13}^2 x_{34}^2 x_{42}^2 x_{25}^2 x_{51}^2} f_{20}(\{u\}) + \frac{y_{14}^2 y_{42}^2 y_{23}^2 y_{35}^2 y_{51}^2}{x_{14}^2 x_{42}^2 x_{23}^2 x_{35}^2 x_{51}^2} f_{21}(\{u\}) \\
&\quad + \frac{y_{15}^2 y_{52}^2 y_{23}^2 y_{34}^2 y_{41}^2}{x_{15}^2 x_{52}^2 x_{23}^2 x_{34}^2 x_{41}^2} f_{22}(\{u\})
\end{aligned} \tag{3.214}$$

where $f_i(\{u\})$ are functions of the conformal cross ratios $\{u\}$.

We adopted a convention similar to that used in the $\mathfrak{psu}(1,1|2)$ case, where the conformal structures are chosen to mirror the R-symmetry structures. This choice reflects the origin of those structures in Wick contractions and aligns with conventions commonly found in the literature. Each function $f_i(\{u\})$ is an arbitrary, conformally invariant function of the five cross ratios $\{u\}$.

Similarly, the descendant correlators are constructed by multiplying each R-symmetry structure with each conformal structure, along with the associated unknown functions of the cross ratios. We do not present the full form of the correlators here[8], but we provide some remarks regarding the notation and key details.

The descendant correlators involving the $SU(4)_R$ current are proportional to a set of functions denoted by

$$g_{i,kl}(\{u\}),$$

where the index i corresponds to the insertion point of the $SU(4)_R$ current, and the indices k and l refer to the R-symmetry and spacetime structures, respectively, that this function is multiplying. Specifically, the function $g_{i,kl}(\{u\})$ multiplies the k-th R-symmetry structure and the l-th spacetime structure. Consequently, the total number of unknown functions $g_{i,kl}(\{u\})$ in this context is given by

$$\#(\text{inequiv. insertions of } \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}) \cdot k \cdot l = 5 \cdot 21 \cdot 4 = 420.$$

Moreover, the descendant correlators involving the spinor fields $\Psi_{\alpha\dot{a}}(x)$ and $\bar{\Psi}_{a\dot{\alpha}}(x)$ are proportional to functions denoted as

$$h_{i,j,kl}(\{u\}),$$

where i denotes the insertion point of $\bar{\Psi}_{a\dot{\alpha}}(x_i)$ and j refers to the insertion point of $\Psi_{\alpha\dot{a}}(x_j)$. As before, the index k labels the R-symmetry structure and l labels the conformal structure associated with each function. Therefore, the total number of unknown functions $h_{ij,kl}(\{u\})$, with $i \neq j$, is given by

$$\#(\text{inequiv. ins. of } \bar{\Psi}_{a\dot{\alpha}}) \cdot \#(\text{inequiv. ins. of } \Psi_{\alpha\dot{a}}) \cdot k \cdot l = 5 \cdot 4 \cdot 28 \cdot 4 = 2240.$$

In summary, we can express all correlators appearing in the fermionic expansion of the five-point function of stress tensor multiplets (up to $\mathcal{O}(\rho\bar{\rho})$) in a form that is consistent with the bosonic symmetries. This expansion introduces

$$22 + 2660$$

[8] A Mathematica notebook containing all those correlators can be provided.

unknown functions of the cross ratios; 22 of the superprimary correlator and 2660 of the various descendent correlators. The total number of unknowns is subsequently reduced by imposing supersymmetric invariance constraints, which will be the objective of the next sections.

3.5. SCWI for the Four-Point Correlator

In the remaining two sections of this chapter, we will derive the superconformal Ward identities (SCWI) by integrating the three key steps discussed and developed previously:

1. Exact multiplet expressions in analytic superspace,
2. Constraints from bosonic subgroups, and
3. Constraints from supersymmetry.

Before finalising the analysis of the five-point function, however, we shall revisit and rederive the SCWI for the four-point function of stress-tensor multiplets:

$$\mathcal{G}_{2222}(\{X_i\}) = \langle \mathcal{T}(X_1)\, \mathcal{T}(X_2)\, \mathcal{T}(X_3)\, \mathcal{T}(X_4)\rangle \ . \tag{3.215}$$

This rederivation is beneficial for several reasons. First, the four-point function involves a considerably smaller number of tensor structures and, consequently, fewer unknown functions of the cross ratios. This reduction simplifies the presentation and facilitates a clear understanding of the method in the context of four-dimensional theories. In contrast, the five-point case introduces significantly more complexity, making it challenging to present all technical details comprehensively.
Thus, this derivation also serves to illustrate the power of the superconformal Ward identities in a clear and rigorous manner.
Moreover, the four-point case allows us to verify the consistency of our approach by reproducing well-established results from the literature, which were discussed in Section 2.4.

The method for deriving the SCWI remains the same as in the five-point case. Moreover, the fundamental building blocks constructed in the previous sections are identical or, at the very least, closely related.

Since we are focusing on the four-point function of stress-tensor multiplets, we can employ the multiplet field expansion developed in Section 3.3, specifically the expression given by equation 3.113.
Substituting this field expansion into the four-point function yields an expansion of the correlator analogous in structure to that of the five-point function, as presented

in equation 3.116. The four-point expansion is as follows:

$$\begin{aligned}
\mathcal{G}_{2222}(\{X_i\}) &= \langle \mathcal{T}\,(X_1)\,\mathcal{T}\,(X_2)\,\mathcal{T}\,(X_3)\,\mathcal{T}\,(X_4)\rangle \\
&= \langle \mathcal{O}_{20'}\,(x_1,y_1)\,\mathcal{O}_{20'}\,(x_2,y_2)\,\mathcal{O}_{20'}\,(x_3,y_3)\,\mathcal{O}_{20'}\,(x_4,y_4)\rangle \\
&- \frac{1}{2}\sum_{i=1}^{4}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y_i^{a\dot{a}}}\langle \mathcal{O}_{20'}\,(x_1,y_1)\,\mathcal{O}_{20'}\,(x_2,y_2)\,\mathcal{O}_{20'}\,(x_3,y_3)\,\mathcal{O}_{20'}\,(x_4,y_4)\rangle \\
&+ \sum_{i=1}^{4}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\,\langle \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}\,(x_i,y_i)\prod_{k=1,k\neq i}^{4}\mathcal{O}_{20'}\,(x_k,y_k)\rangle \\
&- \sum_{i=1}^{4}\sum_{j\neq i}^{4}\rho_i^{\alpha\dot{a}}\bar{\rho}_j^{a\dot{\alpha}}\,\langle \Psi_{\alpha\dot{a}}(x_i,y_i)\bar{\Psi}_{a\dot{\alpha}}(x_j,y_j)\mathcal{O}_{20'}\,(x_k,y_k)\,\mathcal{O}_{20'}\,(x_l,y_l)\rangle \\
&+ \text{higher-order terms},
\end{aligned} \tag{3.216}$$

where $i \neq j \neq k \neq l \in \{1,2,3,4\}$. As in the five-point case, correlators of order ρ or $\bar{\rho}$ are ruled out by symmetry, and higher-order terms are suppressed, as we perform our analysis for the moment only up to order $\rho\bar{\rho}$.

The bosonic structures for each of the correlators are constructed in an identical manner as in Section 3.4. In fact, we can directly use the previously derived structures, provided they involve no more than four points. The numbers of independent structures (str.) for each correlator, based on similar arguments as developed in section 3.4, are

	R-sym. str.	Conf. str.	Total nr. of bosonic str.
$\langle \mathcal{O}_{20'}(1)\mathcal{O}_{20'}(2)\mathcal{O}_{20'}(3)\mathcal{O}_{20'}(4)\rangle$	6	1	6
$\langle \mathcal{J}(i)\mathcal{O}_{20'}(j)\mathcal{O}_{20'}(k)\mathcal{O}_{20'}(l)\rangle$	3	2	6
$\langle \bar{\Psi}(i)\Psi(j)\mathcal{O}_{20'}(k)\mathcal{O}_{20'}(l)\rangle$	6	2	12

(3.217)

where each independent bosonic structure introduces an unknown function of the two conformal cross-ratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}). \tag{3.218}$$

We will not rederive the possible structures here, but proceed by stating and using the final correlator expressions below.

Let us pause for a moment to emphasize the significance of accounting for the full superconformal invariance of the full supercorrelator. The most commonly used approaches for the four-point function, summarised in Section 2.4, exploit the fact that one can rotate into a frame where all fermionic coordinates are set to zero. In other words, by fully utilizing supersymmetry, the full supercorrelator reduces to:

$$\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\rangle \,. \tag{3.219}$$

Employing bosonic invariance, the correlator above corresponds to six independent structures, each introducing an unknown function, as stated in Eqn. 3.217. Using an analogue of Wick contractions, the four-point correlator takes the form:

$$\begin{aligned}
&\langle \mathcal{O}_{20'}\left(x_1,y_1\right)\mathcal{O}_{20'}\left(x_2,y_2\right)\mathcal{O}_{20'}\left(x_3,y_3\right)\mathcal{O}_{20'}\left(x_4,y_4\right)\rangle \\
=&\frac{y_{12}^4 y_{34}^4}{x_{12}^4 x_{34}^4} f_1(u,v) + \frac{y_{13}^4 y_{24}^4}{x_{13}^4 x_{24}^4} f_2(u,v) + \frac{y_{14}^4 y_{23}^4}{x_{14}^4 x_{23}^4} f_3(u,v) \\
&+ \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2} f_4(u,v) + \frac{y_{12}^2 y_{14}^2 y_{23}^2 y_{34}^2}{x_{12}^2 x_{14}^2 x_{23}^2 x_{34}^2} f_5(u,v) + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2} f_6(u,v)
\end{aligned} \tag{3.220}$$

where $f_i(u,v)$ are the unknown functions. From the perspective of the symmetry group, all constraints have been exhausted, and thus the complete supercorrelator should be parametrised by six functions.
However, it is known that the four-point function is ultimately parametrised by a single function of the cross-ratios, as established in equation 2.93 in Section 2.4. This reduction can be achieved, for instance, by imposing analyticity conditions on the supersymmetric extension of the invariants, as discussed in [35]. In our approach, this reduction will be obtained through additional constraints on the superprimary correlator, coming from supersymmetry invariance imposed on the higher-order terms in the expansion. (Similar to the final equations presented for the case of the $\mathfrak{psu}(1,1|2)$ algebra in Eqn. 3.50,3.51). This demonstrates the crucial importance of considering the full superconformal invariance of the full supercorrelator.

To implement this invariance, we begin by enforcing supersymmetry. As before, this requires imposing invariance under the non-trivial fermionic charges, parametrised as (see Appendix A.2):

$$Q_{\alpha\dot{a}}\left\langle \ldots \right\rangle = \sum_{i=1}^{4} \frac{\partial}{\partial \rho_i^{\alpha\dot{a}}} \left\langle \ldots \right\rangle = 0\,, \quad \bar{Q}_{a\dot{\alpha}}\left\langle \ldots \right\rangle = \sum_{i=1}^{4} \frac{\partial}{\partial \bar{\rho}_i^{a\dot{\alpha}}} \left\langle \ldots \right\rangle = 0\,. \tag{3.221}$$

Once these equations hold, invariance under the remaining fermionic charges follows from the superalgebra.

To start, we apply the first constraint, namely invariance under $Q_{\alpha\dot{a}}$, to the correlator expansion in equation 3.216. This yields the following expression:

$$0 = \sum_{i=1}^{4} \Bigg\{ -\frac{1}{2} \bar{\rho}_i^{a\dot{\alpha}} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} \frac{\partial}{\partial y_i^{a\dot{a}}} \left\langle \mathcal{O}_{20'}\left(x_1, y_1\right) \mathcal{O}_{20'}\left(x_2, y_2\right) \mathcal{O}_{20'}\left(x_3, y_3\right) \mathcal{O}_{20'}\left(x_4, y_4\right) \right\rangle \\ + \bar{\rho}_i^{a\dot{\alpha}} \left\langle \mathcal{J}_{\alpha\dot{\alpha}, a\dot{a}}\left(x_i, y_i\right) \prod_{j=1, j\neq i}^{4} \mathcal{O}_{20'}\left(x_j, y_j\right) \right\rangle \\ - \sum_{k=1, k\neq i}^{4} \bar{\rho}_k^{a\dot{\alpha}} \left\langle \Psi_{\alpha\dot{a}}\left(x_i, y_i\right) \bar{\Psi}_{a\dot{\alpha}}\left(x_k, y_k\right) \mathcal{O}_{20'}\left(x_l, y_l\right) \mathcal{O}_{20'}\left(x_m, y_m\right) \right\rangle \Bigg\} . \tag{3.222}$$

Since this equation must hold for any $\bar{\rho}_i^{a\dot{\alpha}}$, we can isolate four distinct equations corresponding to terms proportional to each $\bar{\rho}_i^{a\dot{\alpha}}$. For example, for $i = 1$, we obtain:

$$\begin{aligned} 0 = &-\frac{1}{2} \frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \frac{\partial}{\partial y_1^{a\dot{a}}} \left\langle \mathcal{O}_{20'}\left(x_1, y_1\right) \mathcal{O}_{20'}\left(x_2, y_2\right) \mathcal{O}_{20'}\left(x_3, y_3\right) \mathcal{O}_{20'}\left(x_4, y_4\right) \right\rangle \\ &+ \left\langle \mathcal{J}_{\alpha\dot{\alpha}, a\dot{a}}\left(x_1, y_1\right) \mathcal{O}_{20'}\left(x_2, y_2\right) \mathcal{O}_{20'}\left(x_3, y_3\right) \mathcal{O}_{20'}\left(x_4, y_4\right) \right\rangle \\ &+ \sum_{j=2}^{4} \left\langle \bar{\Psi}_{a\dot{\alpha}}(x_1, y_1) \Psi_{\alpha\dot{a}}(x_j, y_j) \mathcal{O}_{20'}\left(x_k, y_k\right) \mathcal{O}_{20'}\left(x_l, y_l\right) \right\rangle . \end{aligned} \tag{3.223}$$

We will proceed with our analysis along the example of this constraint 3.223.

In the next step, the expressions for the correlators, consistent with bosonic symmetries, will be inserted into this constraint. In practice, we first consider the internal dependence, writing the correlators consisting of conformal×R-symmetry primaries schematically as:

$$G^{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4}(\{x_i, y_i\}) = \sum_{i=1}^{m_y} (\text{R-symmetry structure } i)_{\dot{a}a} \cdot C_i(\{x\}), \tag{3.224}$$

where m_y is the number of independent R-symmetry structures, and $C_i(\{x\})$ represents the spacetime dependence, including the tensorial structures as well as the unknown functions of the conformal cross-ratios.

We select the following basis to express the correlators:

$$\begin{aligned}
&\frac{\partial}{\partial y_1^{a\dot{a}}}\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}}\langle \mathcal{O}_{20'}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\rangle\\
&= 2y_{12}^4\,y_{34}^4\,(y_{12}^{-1})_{\dot{a}a}F_1(\{x\}) + 2y_{13}^4\,y_{24}^4\,(y_{13}^{-1})_{\dot{a}a}F_2(\{x\}) + 2y_{14}^4\,y_{23}^4\,(y_{14}^{-1})_{\dot{a}a}F_3(\{x\})\\
&\quad + y_{12}^2\,y_{13}^2\,y_{24}^2\,y_{34}^2\,(y_{12}^{-1}+y_{13}^{-1})_{\dot{a}a}F_4(\{x\}) + y_{12}^2\,y_{14}^2\,y_{23}^2\,y_{34}^2\,(y_{12}^{-1}+y_{14}^{-1})_{\dot{a}a}F_5(\{x\})\\
&\quad + y_{13}^2\,y_{14}^2\,y_{23}^2\,y_{24}^2(y_{13}^{-1}+y_{14}^{-1})_{\dot{a}a}F_6(\{x\})
\end{aligned} \tag{3.225}$$

$$\begin{aligned}
&\langle \mathcal{J}_{\alpha\dot{\alpha}}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\rangle\\
=&-y_{12}^2\,y_{13}^2\,y_{24}^2\,y_{34}^2\,(y_{12}^{-1}-y_{13}^{-1})_{\dot{a}a}G_1(\{x\}) - y_{12}^2\,y_{14}^2\,y_{23}^2\,y_{34}^2\,(y_{12}^{-1}-y_{14}^{-1})_{\dot{a}a}G_2(\{x\})\\
&- y_{13}^2\,y_{14}^2\,y_{23}^2\,y_{24}^2(y_{13}^{-1}-y_{14}^{-1})_{\dot{a}a}G_3(\{x\})
\end{aligned} \tag{3.226}$$

$$\begin{aligned}
&\langle \bar{\Psi}_{a\dot{\alpha}}(x_1,y_1)\,\Psi_{\alpha\dot{a}}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\rangle\\
=&(y_{14}^{-1}\,y_{43}\,y_{32}^{-1})_{\dot{a}a}\,y_{14}^2\,y_{23}^2\,\Big(y_{14}^2\,y_{23}^2H_1(\{x\}) + y_{12}^2\,y_{34}^2H_2(\{x\}) + y_{13}^2\,y_{24}^2H_3(\{x\})\Big)\\
&+ (y_{12}^{-1})_{\dot{a}a}\,y_{12}^2\,y_{34}^2\,\Big(y_{12}^2\,y_{34}^2H_4(\{x\}) + y_{13}^2\,y_{24}^2H_5(\{x\}) + y_{14}^2\,y_{23}^2H_6(\{x\})\Big)
\end{aligned} \tag{3.227}$$

$$\begin{aligned}
&\langle \bar{\Psi}_{a\dot{\alpha}}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\Psi_{\alpha\dot{a}}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\rangle\\
=&(y_{12}^{-1}\,y_{24}\,y_{43}^{-1})_{\dot{a}a}\,y_{12}^2\,y_{34}^2\,\Big(y_{12}^2\,y_{34}^2K_1(\{x\}) + y_{13}^2\,y_{24}^2K_2(\{x\}) + y_{14}^2\,y_{23}^2K_3(\{x\})\Big)\\
&+ (y_{13}^{-1})_{\dot{a}a}\,y_{13}^2\,y_{24}^2\,\Big(y_{13}^2\,y_{24}^2K_4(\{x\}) + y_{14}^2\,y_{23}^2K_5(\{x\}) + y_{12}^2\,y_{34}^2K_6(\{x\})\Big)
\end{aligned} \tag{3.228}$$

$$\begin{aligned}
&\langle \bar{\Psi}_{a\dot{\alpha}}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\Psi_{\alpha\dot{a}}(x_4,y_4)\rangle\\
=&(y_{13}^{-1}\,y_{32}\,y_{24}^{-1})_{\dot{a}a}\,y_{13}^2\,y_{24}^2\,\Big(y_{13}^2\,y_{24}^2L_1(\{x\}) + y_{14}^2\,y_{23}^2L_2(\{x\}) + y_{12}^2\,y_{34}^2L_3(\{x\})\Big)\\
&+ (y_{14}^{-1})_{\dot{a}a}\,y_{14}^2\,y_{23}^2\,\Big(y_{14}^2\,y_{23}^2L_4(\{x\}) + y_{12}^2\,y_{34}^2L_5(\{x\}) + y_{13}^2\,y_{24}^2L_6(\{x\})\Big)
\end{aligned} \tag{3.229}$$

Note that for simplicity, the derivative with respect to $x_1^{\alpha\dot{\alpha}}$ has been absorbed into the F-functions. It will be reintroduced in subsequent calculations.

Expressing the correlators in this manner allows to straightforwardly solve the equation 3.223 in terms of the spacetime functions, as for the R-symmetry structures, only trivial linear dependencies needs to be considered.
Substituting the expressions into constraint 3.223, we derive the following twelve

relations between the spacetime-dependent components:

$$
\begin{aligned}
H_2(\{x\}) &= -\,2G_1(\{x\}) + H_5(\{x\}) - K_6(\{x\}) + L_1(\{x\}) \\
H_3(\{x\}) &= 2G_3(\{x\}) - H_1(\{x\}) - K_5(\{x\}) - L_1(\{x\}) + L_6(\{x\}) \\
K_2(\{x\}) &= -\,2G_3(\{x\}) + H_1(\{x\}) + K_5(\{x\}) - L_6(\{x\}) \\
K_3(\{x\}) &= -\,2G_2(\{x\}) - H_1(\{x\}) + H_6(\{x\}) - K_1(\{x\}) - L_5(\{x\}) \\
L_2(\{x\}) &= 2G_2(\{x\}) - H_6(\{x\}) + K_1(\{x\}) + L_5(\{x\}) \\
L_3(\{x\}) &= 2G_1(\{x\}) - H_5(\{x\}) - K_1(\{x\}) + K_6(\{x\}) - L_1(\{x\}) \\
F_1(\{x\}) &= 2G_1(\{x\}) + H_4(\{x\}) - H_5(\{x\}) - K_1(\{x\}) + K_6(\{x\}) - L_1(\{x\}) \\
F_2(\{x\}) &= 2G_3(\{x\}) - H_1(\{x\}) + K_4(\{x\}) - K_5(\{x\}) - L_1(\{x\}) + L_6(\{x\}) \\
F_3(\{x\}) &= -\,2G_2(\{x\}) - H_1(\{x\}) + H_6(\{x\}) - K_1(\{x\}) + L_4(\{x\}) - L_5(\{x\}) \\
F_4(\{x\}) &= -\,2\left(G_1(\{x\}) - H_5(\{x\}) - L_1(\{x\})\right) \\
F_5(\{x\}) &= 2\left(G_2(\{x\}) + K_1(\{x\}) + L_5(\{x\})\right) \\
F_6(\{x\}) &= -\,2\left(G_3(\{x\}) - H_1(\{x\}) - K_5(\{x\})\right)
\end{aligned}
\tag{3.230}
$$

Next, we will re-express the functions of $\{x\}$ in terms of their explicit spacetime structures, multiplied by the unknown coefficient functions, which depend on the spacetime cross-ratios. The equations above will then provide relations between these cross-ratio-dependent functions.

Recall that the functions $F_i(\{x\})$ had the differential absorbed as

$$
F_i(\{x\}) = \frac{\partial}{\partial x_1^{\alpha\dot\alpha}} \tilde F_i(\{x\}), \tag{3.231}
$$

where $\tilde F_i(\{x\})$ consists of the covariant spacetime structure and an unknown function. Thus, we have in this four-point case

$$
\begin{aligned}
F_1(\{x\}) \equiv \frac{\partial}{\partial x_1^{\alpha\dot\alpha}} F_1(\{x\}) &= \frac{\partial}{\partial x_1^{\alpha\dot\alpha}} \left(\frac{1}{x_{12}^4 x_{34}^4} f_1(u,v) \right) \\
&= \frac{-2}{x_{12}^4\, x_{34}^4} (x_{12}^{-1})_{\dot\alpha\alpha}\, \tilde f_1(u,v) + \frac{1}{x_{12}^4\, x_{34}^4} \left(\frac{\partial u}{\partial x_1^{\alpha\dot\alpha}} \frac{\partial}{\partial u} \tilde f_1(u,v) + \frac{\partial v}{\partial x_1^{\alpha\dot\alpha}} \frac{\partial}{\partial v} \tilde f_1(u,v) \right) \\
\text{with }\ \frac{\partial u}{\partial x_1^{\alpha\dot\alpha}} &= u(x_{12}^{-1} - x_{13}^{-1})_{\dot\alpha\alpha}, \quad \frac{\partial v}{\partial x_1^{\alpha\dot\alpha}} = v(x_{14}^{-1} - x_{13}^{-1})_{\dot\alpha\alpha},
\end{aligned}
\tag{3.232}
$$

and similarly for the other five spacetime structures present in the superprimary correlator 3.220.

We further have to consider the spacetime dependence of the correlator involving the $SU(4)$-current $J_{\alpha\dot{\alpha},a\dot{a}}(x_1)$, inserted at x_1. Its spacetime dependence is encoded in the functions $G_i(\{x\})$, where i labels the R-symmetry structure with which this function is associated. The possible conformal structures can be reduced to only two independent forms, which we express as

$$G_i\left(\{x\}\right) = -\frac{1}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{41}^2}\left\{(x_{12}^{-1} - x_{13}^{-1})_{\dot{\alpha}\alpha} g_{i,1}\left(u,v\right) + (x_{12}^{-1} - x_{14}^{-1})_{\dot{\alpha}\alpha} g_{i,2}\left(u,v\right)\right\}. \tag{3.233}$$

The functions $g_{i,j}(u,v)$ depend only on the conformal cross ratios, with j indexing the corresponding conformal structure.

The correlator involving $\bar{\Psi}_{a\dot{\alpha}}(x_1)$ and $\Psi_{\alpha\dot{a}}(x_i)$ has six possible tensor structures, which can similarly be reduced to only two independent spacetime structures. We express the spacetime parts of the three different correlators as follows:

$$\begin{aligned}&\langle\bar{\Psi}_{a\dot{\alpha}}\left(x_1,y_1\right)\Psi_{\alpha\dot{a}}\left(x_2,y_2\right)\mathcal{O}_{20'}\left(x_3,y_3\right)\mathcal{O}_{20'}\left(x_4,y_4\right)\rangle : \\ H_i(\{x\}) =&\frac{1}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{41}^2}(x_{14}^{-1}\, x_{43}\, x_{32}^{-1})_{\dot{\alpha}\alpha} h_{i,1}(u,v) + \frac{1}{x_{12}^4\, x_{34}^4}(x_{12}^{-1})_{\dot{\alpha}\alpha} h_{i,2}(u,v)\end{aligned} \tag{3.234}$$

$$\begin{aligned}&\langle\bar{\Psi}_{a\dot{\alpha}}\left(x_1,y_1\right)\mathcal{O}_{20'}\left(x_2,y_2\right)\Psi_{\alpha\dot{a}}\left(x_3,y_3\right)\mathcal{O}_{20'}\left(x_4,y_4\right)\rangle : \\ K_i(\{x\}) =&\frac{1}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{41}^2}(x_{12}^{-1}\, x_{24}\, x_{43}^{-1})_{\dot{\alpha}\alpha} k_{i,1}(u,v) + \frac{1}{x_{13}^4\, x_{24}^4}(x_{13}^{-1})_{\dot{\alpha}\alpha} k_{i,2}(u,v)\end{aligned} \tag{3.235}$$

$$\begin{aligned}&\langle\bar{\Psi}_{a\dot{\alpha}}\left(x_1,y_1\right)\mathcal{O}_{20'}\left(x_2,y_2\right)\mathcal{O}_{20'}\left(x_3,y_3\right)\Psi_{\alpha\dot{a}}\left(x_4,y_4\right)\rangle : \\ L_i(\{x\}) =&\frac{1}{x_{12}^2\, x_{23}^2\, x_{34}^2\, x_{41}^2}(x_{13}^{-1}\, x_{32}\, x_{24}^{-1})_{\dot{\alpha}\alpha} l_{i,1}(u,v) + \frac{1}{x_{14}^4\, x_{23}^4}(x_{14}^{-1})_{\dot{\alpha}\alpha} l_{i,2}(u,v)\end{aligned} \tag{3.236}$$

The *intrinsic four-point* structures involved in the last three correlators are not independent over the space of functions of u, v. Instead, they are related as follows:

$$\left(x_{13}^{-1}\, x_{32}\, x_{24}^{-1}\right)_{\dot{\alpha}\alpha} = -\left(x_{13}^{-1}\right)_{\dot{\alpha}\alpha} + v(x_{14}^{-1}\, x_{43}\, x_{32}^{-1})_{\dot{\alpha}\alpha} + u(x_{12}^{-1})_{\dot{\alpha}\alpha}, \tag{3.237}$$

$$\left(x_{12}^{-1}\, x_{24}\, x_{43}^{-1}\right)_{\dot{\alpha}\alpha} = -\frac{1}{u}(x_{13}^{-1})_{\dot{\alpha}\alpha} + \frac{v}{u}(x_{14}^{-1}\, x_{43}\, x_{32}^{-1})_{\dot{\alpha}\alpha} + \frac{v}{u}(x_{14}^{-1})_{\dot{\alpha}\alpha}. \tag{3.238}$$

By inserting the expressions for $F_i(\{x\})$, $G_i(\{x\})$, $H_i(\{x\})$, $K_i(\{x\})$, and $L_i(\{x\})$ into the equations given in 3.230, we can extract an equation for each coefficient

of the independent structures, given by $(x_{12}^{-1})_{\dot{\alpha}\alpha}, (x_{13}^{-1})_{\dot{\alpha}\alpha}, (x_{14}^{-1})_{\dot{\alpha}\alpha}, (x_{14}^{-1}x_{43}x_{32}^{-1})_{\dot{\alpha}\alpha}$. This choice of four independent structures is not unique. However, they can be proven to be entirely independent. This way, we derive equations between the unknown functions of the cross ratios only. In particular, we obtain algebraic relations that enable us to express all possible descendant functions, $g_{i,j}(u,v)$, $h_{i,j}(u,v)$, $k_{i,j}(u,v)$ and $l_{i,j}(u,v)$ in terms of the superprimary functions. Those equations reflect the fact that for four-point functions, indeed the descendent correlators are fully determined by the superprimary correlator.
The algebraic relations for the descendent functions from correlators proportional to $\bar{\rho}^1_{a\dot{\alpha}}$ are given in Appendix D.

Furthermore, we obtain equations that solely involve the unknown functions of the superprimary correlator. These equations are first-order partial differential equations expressed as follows:

$$
\begin{aligned}
f_4^{(u)}(u,v) &= -\frac{f_1^{(u)}(u,v)u + f_2^{(u)}(u,v)u - f_2^{(u)}(u,v)vu - f_2^{(v)}(u,v)vu + f_1^{(v)}(u,v)v}{u},\\
f_4^{(v)}(u,v) &= -\frac{f_2^{(u)}(u,v)u^2 + f_2^{(v)}(u,v)u^2 - f_1^{(u)}(u,v)u + f_1^{(v)}(u,v) - f_1^{(v)}(u,v)v}{u},\\
f_5^{(u)}(u,v) &= -\frac{f_3^{(u)}(u,v)u^2 - f_3^{(u)}(u,v)u + f_3^{(u)}(u,v)vu + f_3^{(v)}(u,v)vu - f_1^{(v)}(u,v)v^2}{uv},\\
f_5^{(v)}(u,v) &= -\frac{f_1^{(v)}(u,v)(u-1)}{u} - \frac{f_1^{(u)}(u,v)u + f_1^{(v)}(u,v)v}{u} + \frac{f_3^{(u)}(u,v)u}{v},\\
f_6^{(u)}(u,v) &= -\frac{f_2^{(u)}(u,v)v^2 + f_2^{(v)}(u,v)v^2 - f_3^{(v)}(u,v)v + f_3^{(u)}(u,v) - f_3^{(u)}(u,v)u}{v},\\
f_6^{(v)}(u,v) &= -\frac{f_3^{(u)}(u,v)u - f_2^{(u)}(u,v)vu - f_2^{(v)}(u,v)vu + f_2^{(v)}(u,v)v + f_3^{(v)}(u,v)v}{v},
\end{aligned}
\tag{3.239}
$$

where $f_i^{(u)}(u,v) = \frac{\partial}{\partial u} f_i(u,v)$ and $f_i^{(v)}(u,v) = \frac{\partial}{\partial v} f_i(u,v)$.
These equations will further constrain the expression of the superprimary correlators, as outlined above. The study and analysis of these equations will be the subject of Section 4.3.

The same analysis has been performed for the other nine constraints obtained from supersymmetric invariance. Note, however, that for the four point functions of stress tensor multiplets, the consideration of one such equation (as performed here) is sufficient to derive the above equations 3.239.

3.6. SCWI for the Five-Point Correlator

The purpose of this section is to derive the superconformal Ward identities for the five-point correlation function of stress tensor multiplets in $\mathcal{N} = 4$ SYM, thereby obtaining the maximally constrained form of this correlator in adherence to superconformal symmetry.

The approach used to derive these identities parallels the method applied to the four-point function, and indeed, this procedure can be extended to any n-point function. To emphasize the general applicability of this approach, we will derive the five-point Ward identities in Section 3.6.1, reperforming and underscoring the key steps and highlighting their similarity to those previously employed in deriving the four-point function identities in Section 3.5.

3.6.1. Deriving the SCWI

1. Expanding the Five-Point Correlator

The first step utilizes the solutions to the differential constraint satisfied by half-BPS multiplets in analytic superspace,

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^{p+1} \mathbb{O}_p = 0 \quad \text{w. graded symmetrisation.} \tag{3.240}$$

This constraint was developed and solved for the stress tensor multiplet in Section 3.3, resulting in a fermionic expansion for this multiplet in analytic superspace. This expansion, given in Equation 3.113, up until order $\mathcal{O}(\rho\bar{\rho})$ can be written as:

$$\begin{aligned} \mathcal{T}(X) \,=\, & \left(1 - \frac{1}{2}\rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}}\right)\mathcal{O}_{20'}\left(x,y\right) \\ & + \rho^{\alpha\dot{a}}\Psi_{\alpha\dot{a}}\left(x,y\right) + \bar{\rho}^{a\dot{\alpha}}\bar{\Psi}_{a\dot{\alpha}}\left(x,y\right) \\ & + \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}\left(x,y\right) \\ & + \dots . \end{aligned} \tag{3.241}$$

This expansion can then be inserted into the five-point function, yielding:

$$\begin{aligned}
&\langle \mathcal{T}(X_1)\,\mathcal{T}(X_2)\,\mathcal{T}(X_3)\,\mathcal{T}(X_4)\,\mathcal{T}(X_5)\rangle \\
=&\langle \mathcal{O}_{20'}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\,\mathcal{O}_{20'}(x_5,y_5)\rangle \\
&+\frac{1}{2}\sum_{i=1}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y_i^{a\dot{a}}}\langle \mathcal{O}_{20'}(x_1,y_1)\ldots\mathcal{O}_{20'}(x_5,y_5)\rangle \\
&+\sum_{i=1}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_i^{a\dot{\alpha}}\langle \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}(x_i,y_i)\prod_{k=1,k\neq i}^{5}\mathcal{O}_{20'}(x_k,y_k)\rangle \\
&-\sum_{i=1}^{5}\sum_{j=1,j\neq i}^{5}\rho_i^{\alpha\dot{a}}\bar{\rho}_j^{a\dot{\alpha}}\langle \Psi_{\alpha\dot{a}}(x_i,y_i)\,\bar{\Psi}_{a\dot{\alpha}}\left(x_j,y_j\right)\mathcal{O}_{20'}(x_k,y_k)\,\mathcal{O}_{20'}(x_l,y_l)\,\mathcal{O}_{20'}(x_m,y_m)\rangle \\
&+\ldots
\end{aligned} \tag{3.242}$$

where $i \neq j \neq k \neq l \neq m \in \{1,2,3,4,5\}$.
Similarly to the four-point case, correlators of order $\mathcal{O}(\rho)$ or $\mathcal{O}(\bar{\rho})$ are prohibited by symmetry. Additionally, the above expression is truncated at the fermionic order $\mathcal{O}(\rho\bar{\rho})$ to provide a starting point for our analysis.

This expansion allows us to analyse the first level of descendent correlators. In principle, one would have to study all fermionic expansion orders to achieve the full set of superconformal Ward identities, and indeed, this analysis is structurally identical at higher orders.
However, higher orders present increased technical challenges, so our focus will remain on the constraints derived from order $\mathcal{O}(\rho\bar{\rho})$. Moreover, it may not be necessary to consider higher orders to obtain the full constraints on the superprimary correlator.[9]

Note that, in the same way, the field expansion of the stress tensor multiplet could be inserted into any n-point function, yielding a n-point correlator expansion of the same kind as Eqn. 3.242.

2. Supersymmetry Constraints

Supersymmetry invariance requires that the correlator remains unchanged under the action of the fermionic component of the symmetry group. The relevant supersymmetry charges, acting on analytic superspace, are defined as (see Section

[9] As we will show in Section 4.3, this order was sufficient to obtain the full superconformal Ward identities for the superprimary correlator of four stress tensors.

A.2)

$$Q_{\alpha\dot{a}} = \frac{\partial}{\partial \rho^{\alpha\dot{a}}}, \quad \bar{Q}_{a\dot{\alpha}} = \frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}} \tag{3.243}$$

so that this invariance condition translates into the following requirements:

$$0 = \sum_{i=1}^{5} \frac{\partial}{\partial \rho_i^{\alpha\dot{a}}} \langle \mathcal{T}(X_1)\, \mathcal{T}(X_2)\, \mathcal{T}(X_3)\, \mathcal{T}(X_4)\, \mathcal{T}(X_5) \rangle \tag{3.244}$$

$$0 = \sum_{i=1}^{5} \frac{\partial}{\partial \bar{\rho}_i^{a\dot{\alpha}}} \langle \mathcal{T}(X_1)\, \mathcal{T}(X_2)\, \mathcal{T}(X_3)\, \mathcal{T}(X_4)\, \mathcal{T}(X_5) \rangle\ . \tag{3.245}$$

Again, those are the very same requirements as in the four-point case or in any n-point case, with the remaining fermionic charges acting trivially .

Expanding the fermionic components of the correlator as in Equation 3.242, the first equation, for example, becomes

$$\begin{aligned} 0 = \sum_{i=1}^{5} \Bigg(& -\frac{1}{2} \bar{\rho}_i^{a\dot{\alpha}} \frac{\partial}{\partial x_i^{\alpha\dot{\alpha}}} \frac{\partial}{\partial y_i^{a\dot{a}}} \langle \mathcal{O}_{20'}(1)\, \mathcal{O}_{20'}(2)\, \mathcal{O}_{20'}(3)\, \mathcal{O}_{20'}(4)\, \mathcal{O}_{20'}(5) \rangle \\ & + \bar{\rho}_i^{a\dot{\alpha}} \langle \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}(x_i, y_i) \prod_{k \neq i} \mathcal{O}_{20'}(k) \rangle \\ & + \sum_{j \neq i} \bar{\rho}_j^{a\dot{\alpha}} \langle \bar{\Psi}_{a\dot{\alpha}}(x_j, y_j) \Psi_{\alpha\dot{a}}(x_i, y_i) \mathcal{O}_{20'}(k)\, \mathcal{O}_{20'}(l)\, \mathcal{O}_{20'}(m) \rangle \Bigg)\ , \end{aligned} \tag{3.246}$$

where the short-hand notation $\mathcal{O}_{20'}(x_i, y_i) = \mathcal{O}_{20'}(i)$ has been introduced.

Since this equation must hold for all values of $\bar{\rho}_i$, we derive five distinct constraints from Equation (3.246). For instance, by isolating terms proportional to $\bar{\rho}_1$, we obtain:

$$\begin{aligned} 0 = & -\frac{1}{2} \frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} \frac{\partial}{\partial y_1^{a\dot{a}}} \langle \mathcal{O}_{20'}(x_1, y_1)\, \mathcal{O}_{20'}(x_2, y_2)\, \mathcal{O}_{20'}(x_3, y_3)\, \mathcal{O}_{20'}(x_4, y_4)\, \mathcal{O}_{20'}(x_5, y_5) \rangle \\ & + \langle \mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}(x_1, y_1)\, \mathcal{O}_{20'}(x_2, y_2)\, \mathcal{O}_{20'}(x_3, y_3)\, \mathcal{O}_{20'}(x_4, y_4)\, \mathcal{O}_{20'}(x_5, y_5) \rangle \\ & + \sum_{i=2}^{5} \langle \bar{\Psi}_{a\dot{\alpha}}(x_1, y_1)\, \Psi_{\alpha\dot{a}}(x_i, y_i)\, \mathcal{O}_{20'}(x_k, y_k)\, \mathcal{O}_{20'}(x_l, y_l)\, \mathcal{O}_{20'}(x_m, y_m) \rangle \end{aligned} \tag{3.247}$$

In a similar fashion, we obtain five further constraints by differentiating with respect to ρ. Together, these ten equations fully encapsulate the implications of invariance under the fermionic sector of the $\mathfrak{psu}(2,2|4)$ algebra, up to order $(\rho\bar{\rho})$.

This procedure, as illustrated here, is structurally analogous to the four-point function case, demonstrating that supersymmetry imposes similar constraints on also higher-point functions in a systematic manner.

3. Bosonic Symmetries

In addition to supersymmetry, we also apply the constraints of bosonic symmetries.
We have established all the necessary bosonic structures and derived the expressions for the correlators appearing in Equation 3.247 (and the remaining nine equations) in Section 3.4.

As with the four-point case, we start by rephrasing the bosonic correlators in the form

$$\sum_{i=1}^{m_y} (\text{R-symmetry structure})_i \cdot C_i(\{x\}) \tag{3.248}$$

where all spacetime dependencies are implicit. This representation allows us to exploit the (straightforward implementable) linear independence of the R-symmetry structures, resulting in a set of algebraic equations solely for $C_i(\{x\})$.

We denote by $F_i(\{x\}), i = 1, \dots, 22$ the spacetime structures and unknown functions multiplying the i-th R-symmetry structure of the superprimary five-point function. The functions $G_{i,k}(\{x\})$ multiply the k-th R-symmetry structure of the correlator involving the $SU(4)_R$-current at X_i, while $H_{i,j,k}(\{x\})$ corresponds to the k-th R-symmetry structure for the correlator with $\bar{\Psi}_{a\dot{\alpha}}(x,y)$ at X_i and $\Psi_{\alpha\dot{a}}(x,y)$ at X_j.

Using the expressions given in Equations 3.199, 3.207, and 3.211 (along with their permutations as detailed in Section 3.4), and inserting those for example into Equation 3.247, we can solve this equation for the spacetime parts, arriving at algebraic equations of the form

$$
\begin{aligned}
&H_{1,3,5}(\{x\}) = -\frac{1}{2}\frac{\partial}{\partial x_1}F_{20}(\{x\}) - G_{1,8}(\{x\}) - H_{12,21}(\{x\}) - H_{12,23}(\{x\}) + H_{12,24}(\{x\}),\\
&H_{1,3,10}(\{x\}) = -\frac{1}{2}\frac{\partial}{\partial x_1^{\alpha\dot\alpha}}F_{17}(\{x\}) - G_{1,6}(\{x\}) - H_{1,2,4}(\{x\}) - H_{1,2,9}(\{x\}) + H_{1,2,15}(\{x\}),\\
&\dots\\
&H_{1,5,28}(\{x\}) = -\frac{1}{2}\frac{\partial}{\partial x_1^{\alpha\dot\alpha}}F_{8}(\{x\}) + G_{1,3}(\{x\}) + H_{1,3,28}(\{x\}),
\end{aligned}
\tag{3.249}
$$

From Equation 3.247, we derive 64 equations constraining the spacetime components. Analogously, each additional equation for $\rho_i^{\alpha\dot a}$ and $\bar\rho_i^{a\dot\alpha}$ provides 64 further constraints, resulting in a total of 640 equations governing the spacetime components.[10]

The following step entails substituting the explicit spacetime structures and unknown functions of the cross ratios as established in Section 3.4. Specifically, for the descendent correlators, we find from the expressions 3.149 and 3.156

$$
\begin{aligned}
G_{1,k}(\{x\}) =&\frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big(-(x_{12}^{-1}-x_{13}^{-1})_{\dot\alpha\alpha}\,g_{1,k,1}(\{u\}) - (x_{12}^{-1}-x_{14}^{-1})_{\dot\alpha\alpha}\,g_{1,k,2}(\{u\})\\
&-(x_{12}^{-1}-x_{15}^{-1})_{\dot\alpha\alpha}\,g_{1,k,3}(\{u\}) + (x_{12}^{-1}\,x_{23}\,x_{34}^{-1}\,x_{45}\,x_{51}^{-1})_{\dot\alpha\alpha}\,g_{1,k,4}(\{u\})\Big)\,,
\end{aligned}
\tag{3.250}
$$

$$
\begin{aligned}
H_{1,2,k}(\{x\}) =&\frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{14}^{-1}\,x_{43}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_{1,2,k,1}(\{u\}) + (x_{15}^{-1}\,x_{53}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_{1,2,k,2}(\{u\})\\
&+(x_{15}^{-1}\,x_{54}\,x_{42}^{-1})_{\dot\alpha\alpha}\,h_{1,2,k,3}(\{u\}) + (x_{12}^{-1})_{\dot\alpha\alpha}\,h_{1,2,k,4}(\{u\})\Big)\,,
\end{aligned}
\tag{3.251}
$$

with analogous expressions for other permutations of $H_{1,j,k}(\{x\})$, that can be deduced from the spacetime expressions given in Appendix C.1.

Since our main focus in the further analysis will be the equations for the unknown functions parametrising the superprimary five-point functions, we restate explicitly all its 22 functions. These functions can be deduced from expression 3.214 as follows:

$$
F_1(\{x\}) = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}f_1(\{u\}),\quad F_2(\{x\}) = \frac{1}{x_{13}^4x_{24}^2x_{45}^2x_{52}^2}f_2(\{u\})
$$

[10] These 640 equations can be provided in a Mathematica notebook upon request.

$$
\begin{aligned}
F_3(\{x\}) &= \frac{1}{x_{14}^4 x_{23}^2 x_{35}^2 x_{52}^2} f_3(\{u\}), \quad F_4(\{x\}) = \frac{1}{x_{15}^4 x_{23}^2 x_{34}^2 x_{42}^2} f_4(\{u\}) \\
F_5(\{x\}) &= \frac{1}{x_{23}^4 x_{14}^2 x_{45}^2 x_{51}^2} f_5(\{u\}), \quad F_6(\{x\}) = \frac{1}{x_{24}^4 x_{13}^2 x_{35}^2 x_{51}^2} f_6(\{u\}) \\
F_7(\{x\}) &= \frac{1}{x_{25}^4 x_{13}^2 x_{34}^2 x_{41}^2} f_7(\{u\}), \quad F_8(\{x\}) = \frac{1}{x_{34}^4 x_{12}^2 x_{25}^2 x_{51}^2} f_8(\{u\}) \\
F_9(\{x\}) &= \frac{1}{x_{35}^4 x_{12}^2 x_{24}^2 x_{41}^2} f_9(\{u\}), \quad F_{10}(\{x\}) = \frac{1}{x_{45}^4 x_{12}^2 x_{23}^2 x_{31}^2} f_{10}(\{u\}) \\
F_{11}(\{x\}) &= \frac{1}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} f_{11}(\{u\}), \quad F_{12}(\{x\}) = \frac{1}{x_{12}^2 x_{23}^2 x_{35}^2 x_{54}^2 x_{41}^2} f_{12}(\{u\}), \\
F_{13}(\{x\}) &= \frac{1}{x_{12}^2 x_{13}^2 x_{24}^2 x_{35}^2 x_{45}^2} f_{13}(\{u\}), \quad F_{14}(\{x\}) = \frac{1}{x_{12}^2 x_{15}^2 x_{24}^2 x_{34}^2 x_{35}^2} f_{14}(\{u\}), \\
F_{15}(\{x\}) &= \frac{1}{x_{12}^2 x_{14}^2 x_{25}^2 x_{34}^2 x_{35}^2} f_{15}(\{u\}), \quad F_{16}(\{x\}) = \frac{1}{x_{12}^2 x_{13}^2 x_{25}^2 x_{34}^2 x_{45}^2} f_{16}(\{u\}), \\
F_{17}(\{x\}) &= \frac{1}{x_{13}^2 x_{14}^2 x_{24}^2 x_{25}^2 x_{35}^2} f_{17}(\{u\}), \quad F_{18}(\{x\}) = \frac{1}{x_{13}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{45}^2} f_{18}(\{u\}), \\
F_{19}(\{x\}) &= \frac{1}{x_{13}^2 x_{14}^2 x_{23}^2 x_{25}^2 x_{45}^2} f_{19}(\{u\}), \quad F_{20}(\{x\}) = \frac{1}{x_{13}^2 x_{15}^2 x_{24}^2 x_{25}^2 x_{34}^2} f_{20}(\{u\}), \\
F_{21}(\{x\}) &= \frac{1}{x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{35}^2} f_{21}(\{u\}), \quad F_{22}(\{x\}) = \frac{1}{x_{14}^2 x_{15}^2 x_{23}^2 x_{25}^2 x_{34}^2} f_{22}(\{u\}).
\end{aligned}
$$

The functions $F_i(\{x\})$ enter the equations in 3.249 through derivatives with respect to spacetime coordinates. In particular, we have as developed in section 3.4

$$
\begin{aligned}
&\frac{\partial}{\partial x_1^{\alpha\dot{\alpha}}} F_1(\{x\}) = \\
&= -\frac{2}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} (x_{12}^{-1})_{\dot{\alpha}\alpha} f_1(\{u\}) + \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} \sum_{a=1}^{5} \frac{\partial u_a}{\partial x_1^{\alpha\dot{\alpha}}} \frac{\partial}{\partial u_a} f_1(\{u\}),
\end{aligned}
\tag{3.252}
$$

and similarly for the remaining functions.
The derivatives with respect to other points, entering the additional nine constraints from supersymmetry, can be constructed analogously.

Recalling the computation in Section 3.4.3, we categorise the unknown functions as follows:

- **22 superprimary functions.**
  These functions appear in the equations as derivatives with respect to the cross ratios. Since there are five cross ratios, we have $5 \cdot 22 = 110$ such derivatives $f_i^{(u_a)}(\{u\})$.

- **2660 descendant functions.**
  These arise from the following sources:
  - $\mathcal{J}$-correlators: $5 \cdot 21 \cdot 4 = 420$ channels,
  - $\bar{\Psi}, \Psi$-correlators: $5 \cdot 4 \cdot 28 \cdot 4 = 2240$ channels.
- In total, there are thus **2682 unknown functions** (plus 110 derivatives) entering the equations from supersymmetry.

This initial number of unknowns reduces upon substituting the above expressions into the equations.

Let us compare this unreduced count with the number of unknown functions expected in the standard conformal frame. Exploiting the 32 supersymmetry charges, we can set 32 fermionic degrees of freedom to zero, effectively reducing to the correlator

$$\begin{aligned}&\langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle \\ &\quad \to \langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{T}(X_5)\rangle \,.\end{aligned} \tag{3.253}$$

Expanding this correlator in the remaining fermionic coordinates, we find

$$\begin{aligned}&\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{T}(X_5)\rangle \\ &\quad = \langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{O}_{20'}(x_5,y_5)\rangle \\ &\qquad + \rho_5^{\alpha\dot{a}}\bar{\rho}_5^{a\dot{\alpha}}\,\langle \mathcal{O}_{20'}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}(x_5,y_5)\rangle \\ &\qquad + \ldots\end{aligned} \tag{3.254}$$

Applying the remaining bosonic symmetries, we arrive at

$$22 + 21 \cdot 4 = 106 \ \text{ unknown functions} \tag{3.255}$$

parametrising the supercorrelator up to order $\mathcal{O}(\rho\bar{\rho})$. Here, the 22 functions stem from the superprimary correlator, and the term $21 \cdot 4$ reflects the independent R-symmetry structures times the independent conformal structures for the correlator including $\mathcal{J}_{\alpha\dot{\alpha},a\dot{a}}$. Note that we do not account for additional possible insertion points of the $SU(4)_R$-current, as the remaining four points are fixed to be the superprimary operator $\mathcal{O}_{20'}$. Thus, the conformal frame estimate suggests a total of 106 unknown functions parametrising the supercorrelator. However, as we observed in the four-point case (where we reduced from six unknowns in this conformal frame to only a single unknown function of the cross ratios), this number can reduce further: higher-order terms in the fermionic expansion (and hence the supersymmetry constraints) can decrease the number of functions needed to parametrise both the descendant correlators and the superprimary functions. Hence, the 106 unknowns derived in the conformal frame serve primarily as an upper bound.

To derive the relations reducing the number of unknown functions, we substitute the expressions for $F_i(\{x\}), G_{1,k}(\{x\}), H_{1,j,k}(\{x\})$ into the equations 3.249. By comparing coefficients of each independent spacetime tensor structure over the space of functions of conformal cross ratios, ensuring independence as outlined in Section 3.4, we obtain equations purely in terms of the unknown functions of the cross ratios.
From Equation 3.249, this method yields 511 independent equations involving the unknown functions parametrising correlators proportional to $\bar{\rho}_1^{a\dot{\alpha}}$ in the expansion 3.242.
Similarly, each constraint from supersymmetry generates 511 equations, leading to $10 \cdot 511 = 5110$ equations governing the 2682 unknown functions of the five-point function of stress tensor multiplets up to order $\mathcal{O}(\rho\bar{\rho})$.
This system is not overconstrained, as not all 5110 equations are independent.

From these 5110 equations, we conclude:

- Of the 2660 descendant functions, 2654 are determined algebraically in terms of the superprimary functions. Thus, at order $\mathcal{O}(\rho\bar{\rho})$, there remain **6 undetermined descendant functions of the cross ratios**. These functions, indicating the existence of nilpotent invariants, must be incorporated when analysing the five-point correlator of stress tensor multiplets. As seen in the $\mathfrak{psu}(1,1|2)$ case as well, encoding the supercorrelator solely in terms of the superprimary function is not feasible anymore when five or more operators are inserted.
- Additionally, we obtain 35 partial differential equations for the unknown functions $f_i(\{u\})$ of the superprimary correlator. Due to their complexity, these equations are not fully displayed here.[11]
- However, these equations include the conditions of Drukker-Plefka and chiral algebra twist, as we will demonstrate shortly. In fact, they include more constraints and are thus **stronger than Drukker-Plefka twist and chiral algebra twist alone.**

The cross-check of the equations against known expressions from the literature, the

[11] A Mathematica notebook containing these equations can be provided upon request.

demonstration of the last point and the analysis of these 35 equations in general will be the subject of section 4.4 in the following chapter.

Note that, as opposed to the four-point case, all ten constraints from supersymmetry had to be taken into account to arrive at the above conclusions.

4. Analysing the Constraints

4.1. Introduction

In this chapter, we analyse the derived equations governing superprimary correlators within each case under consideration.

In Section 4.2, we begin by examining the equations obtained in Section 3.2 for the five-point function of $\mathcal{W}_2$-multiplets within the chiral algebra of $\mathfrak{psu}(2,2|4)$, i.e. $\mathfrak{psu}(1,1|2)$. These equations are significantly simplified compared to the four-dimensional case, involving fewer cross ratios and unknown functions, which makes them more tractable for initial analysis. The structural insights gained here may guide us in approaching the more complex four-dimensional case. Furthermore, the results from this chiral algebra analysis can serve as direct inputs for potential solutions to the four-dimensional five-point equations, given the close relation facilitated by the chiral algebra map.

In Section 4.3, we address the equations derived for the four-point functions of stress-tensor multiplets in $\mathfrak{psu}(2,2|4)$. Here, we cross-check our derived equations against well-established results from the literature. Further, we verify that the four-point SCWI are indeed obtained from both the Drukker-Plefka twist and chiral algebra twist.

Finally, in Section 4.4, we explore the equations formulated for the five-point functions of $\mathcal{O}_{20'}$-operators. Due to the complexity of these equations, this analysis is ongoing. However, within this thesis, we provide strong evidence supporting the correctness of our derived equations by validating them against known results in both the weak coupling regime and the strong coupling regime, obtained from the SUGRA side. Moreover, we show that the five-point equations impose constraints beyond those of the Drukker-Plefka and chiral algebra twist conditions. While these conditions are embedded within the derived equations, they do not encompass the full set of SCWI for higher-point functions.
We conclude this analysis by presenting a partial simplification of the 35 equations, allowing to take first steps towards a potential parametrisation solving the SCWI.

4.2. Analysis of the $\mathfrak{psu}(1,1|2)$ SCWI

In this section, we undertake a detailed analysis of the superconformal Ward identities derived in Section 3.2 for the five-point correlator

$$\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) = \langle \mathcal{W}_2(X_1)\mathcal{W}_2(X_2)\mathcal{W}_2(X_3)\mathcal{W}_2(X_4)\mathcal{W}_2(X_5)\rangle\,, \tag{4.1}$$

which is invariant under $\mathfrak{psu}(1,1|2)$ transformations.

Recall that the five-point function of the superprimary operator $J(x,y)$, associated with the $\mathbf{3}$-dimensional representation of $\mathfrak{su}(2)$, can be expressed as

$$\begin{aligned}
&\langle J(x_1,y_1)\, J(x_2,y_2)\, J(x_3,y_3)\, J(x_4,y_4)\, J(x_5,y_5)\rangle\\
=&\frac{y_{12}y_{13}y_{23}y_{45}^2}{x_{12}x_{13}x_{23}x_{45}^2}f_1(u,v)+\frac{y_{12}y_{13}y_{43}y_{25}y_{45}}{x_{12}x_{13}x_{43}x_{25}x_{45}}f_2(u,v)+\frac{y_{12}y_{23}y_{43}y_{15}y_{45}}{x_{12}x_{23}x_{43}x_{15}x_{45}}f_3(u,v)\\
&+\frac{y_{12}y_{24}y_{43}y_{35}y_{51}}{x_{12}x_{24}x_{43}x_{35}x_{51}}f_4(u,v)+\frac{y_{31}y_{32}y_{41}y_{25}y_{45}}{x_{31}x_{32}x_{41}x_{25}x_{45}}f_5(u,v)+\frac{y_{13}y_{34}y_{42}y_{25}y_{51}}{x_{13}x_{34}x_{42}x_{25}x_{51}}f_6(u,v),
\end{aligned} \tag{4.2}$$

where the two cross-ratios are defined as:

$$u=\frac{x_{12}x_{34}}{x_{13}x_{24}},\quad v=\frac{x_{23}x_{45}}{x_{24}x_{35}}. \tag{4.3}$$

Referring to the superconformal Ward identities obtained in section 3.2, we observed that in expanding the five-point correlation function $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ in terms of the fermionic coordinates of analytic superspace, all but one of the unknown functions characterising the descendant correlators are algebraically determined by the superprimary functions.
Thus, we can schematically express the full correlator $\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\})$ as

$$\begin{aligned}
\mathcal{G}_{22222}^{\mathfrak{psu}(1,1|2)}(\{X_i\}) =&\sum_{j=1}^{6} F_j\left(\{x_i,y_i,\rho_i,\bar\rho_i\},u,v,\partial_u,\partial_v\right) f_j(\{u,v\})\\
&+\bar\rho_1\rho_2\frac{y_{13}y_{23}y_{45}^2}{x_{12}^3x_{34}x_{45}x_{53}}H(\{u,v\}),
\end{aligned} \tag{4.4}$$

where the F_i encompass all relevant bosonic structures and correlator expansions developed in Section 3.2. Additionally, it includes also the relations between descendant functions and superprimary functions as dictated by the Ward identities. The function $H(\{u,v\})$ is the only independent function arising from all descendant correlators at order $\mathcal{O}(\rho\bar\rho)$. Notably, the specific structure that multiplies

$H(\{u,v\})$ is not unique and could have been selected differently.

Additionally, three independent relations involving only the six superprimary correlator functions $f_i(u,v)$ emerge, specified by two Drukker-Plefka twist-type conditions and one additional constraint, as stated in Eqs. 3.51.
For convenience and to facilitate comparison with and utility for the results derived for the full $\mathfrak{psu}(2,2|4)$ case, we reformulate the correlator and these three equations in terms of the variables (z_1, z_2) as

$$u = \frac{x_{12}x_{34}}{x_{13}x_{24}} = z_1, \quad v = \frac{x_{23}x_{45}}{x_{24}x_{35}} = z_2. \tag{4.5}$$

Accordingly, the primary equations of interest are

$$0 = \sum_{i=1}^{6} f_i^{(1,0)}(z_1,z_2), \quad 0 = \sum_{i=1}^{6} f_i^{(0,1)}(z_1,z_2), \tag{4.6}$$

$$\begin{aligned} 0 =& \frac{(z_1-1)(z_1(z_2-2)-2z_2+2)}{(z_2-1)z_2^2} f_1^{(1,0)}(z_1,z_2) \\ &+ \frac{(z_1-1)^2}{z_2^2}\left(f_2^{(1,0)}(z_1,z_2)+f_5^{(1,0)}(z_1,z_2)\right) \\ &- \frac{(z_1-1)(z_1+z_2-1)}{(z_2-1)z_2^2}\left(f_3^{(1,0)}(z_1,z_2)+f_4^{(1,0)}(z_1,z_2)\right) \\ &- \frac{\left(z_1^2-2z_1-z_2+1\right)}{z_1z_2}\left(f_3^{(0,1)}(z_1,z_2)+f_4^{(0,1)}(z_1,z_2)\right) \\ &+ \frac{(z_1+z_2-1)}{z_1z_2}\left(f_1^{(0,1)}(z_1,z_2)+f_2^{(0,1)}(z_1,z_2)\right)+f_6^{(0,1)}(z_1,z_2). \end{aligned} \tag{4.7}$$

The focus of this section is an in-depth examination of these three equations for the superprimary correlator, which we will refer to, with slight abuse of notation, as *the SCWI*, even though the complete superconformal Ward identities contain the constraints imposed on the descendant correlators as well.
In Section 4.2.1, we will demonstrate how the superconformal Ward identities can be systematically employed to constrain the superconformal blocks encoding the OPE content of the five-point function. Specifically, we analyse the case where a double-OPE is taken in the channels (12) and (45), reducing the system to three-point functions.
Finally, Section 4.2.2 is devoted to outlining the approach for solving the above equations by identifying a parametrisation of the correlator that explicitly satisfies the superconformal Ward identities.

4.2.1. Conformal blocks and solving the constraints

In this section, we examine the superconformal blocks in the decomposition of the five-point superprimary correlator

$$\mathcal{G}^{11111}(\{X_i\}) = \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle\,, \tag{4.8}$$

and determine how they are constrained by the superconformal Ward identities, as given in Eqs. 4.6 and 4.7. Here, the notation $\mathcal{G}^{11111}(\{X_i\})$ denotes that $J(x,y)$ corresponds to a highest weight state of weight 1. This notation of weight 1 is chosen to align with the conventions used in the literature, that are used below, as well as to distinguish from the full five-point supercorrelator. The letter $\mathcal{G}$ is used to distinguish the correlator from the conformal blocks, which commonly get denoted by the letter G (see below).
The concepts of the superconformal block decomposition have been introduced in 2.5. However, note that here we consider only the decomposition of the superprimary correlator, rather than the full supercorrelator.

By taking the OPE in the points (12) and (45), we can expand the five-point superprimary correlator in terms of superconformal blocks as follows:

$$\mathcal{G}^{11111}(\{X_i\}) = \mathcal{L}(x_i,y_i)\sum_{(\mathbb{h}_1,l_1)}\sum_{(\mathbb{h}_2,l_2)}\lambda^{h_1=1,h_2=1}_{\mathbb{h}_1}\lambda^{h_4=1,h_5=1}_{\mathbb{h}_2}\mathfrak{g}_{\mathbb{h}_1,\mathbb{h}_2;l_1,l_2}(z_i,\mathrm{y}_i). \tag{4.9}$$

Here, the sums run over all superconformal primaries in the OPE of $J\otimes^{\mathfrak{psu}(1,1|2)}_{\mathrm{OPE}} J$. These exchanged superprimaries are labelled by the conformal weight $\mathbb{h}_i$ and R-symmetry weight l_i. The coefficients λ correspond to the (properly normalised) three-point functions of $\mathcal{J},\mathcal{J}$, and the respective intermediate superconformal primary with dimension $\mathbb{h}_i$ and spin l_i; $\mathcal{L}$ denotes the leg factor, a prefactor that carries the correct transformation behaviour in both spacetime coordinates x and internal coordinates y. This prefactor can be factored out such that the remaining expression depends solely on the conformal and internal cross ratios $\{z_i,\mathrm{y}_i\}$, defined as

$$z_1 = \frac{x_{12}x_{34}}{x_{13}x_{24}},\quad z_2 = \frac{x_{23}x_{45}}{x_{24}x_{35}}, \tag{4.10}$$

$$\mathrm{y}_1 = \frac{y_{12}y_{34}}{y_{13}y_{24}},\quad \mathrm{y}_2 = \frac{y_{23}y_{45}}{y_{24}y_{35}}. \tag{4.11}$$

Explicit expressions for conformal higher-point functions in one dimension are provided in [113]. Following these conventions, we define the conformal leg factor as

$$\mathcal{L}^{h_1,h_2,h_3,h_4,h_5}(\{x_i\}) = \left(\frac{x_{23}}{x_{12}x_{13}}\right)^{h_1}\left(\frac{x_{34}}{x_{35}x_{45}}\right)^{h_5}\left(\frac{x_{13}}{x_{12}x_{23}}\right)^{h_2}\left(\frac{x_{24}}{x_{23}x_{34}}\right)^{h_3}\left(\frac{x_{35}}{x_{34}x_{45}}\right)^{h_4}. \tag{4.12}$$

For the correlator $\mathcal{G}^{11111}(\{X_i\})$ this simplifies to

$$\mathcal{L}^{1,1,1,1,1}(x_1,x_2,x_3,x_4,x_5) = \frac{x_{24}}{x_{12}^2 x_{23} x_{34} x_{45}^2}\,. \tag{4.13}$$

Similarly, we express the R-symmetry leg factor, using that the R-symmetry weight is $j = h$, but realised via inverse power in the coordinates:

$$\mathcal{L}^{-1,-1,-1,-1,-1}(y_1,y_2,y_3,y_4,y_5) = \frac{y_{12}^2 y_{23} y_{34} y_{45}^2}{y_{24}}\,. \tag{4.14}$$

Each superblock $\mathfrak{g}_{\mathbb{h};l}(z_i,\mathrm{y}_i)$ can be (schematically) decomposed into a conformal part and an R-symmetry part as follows:

$$\mathfrak{g}_{\mathbb{h};l}(z_i,\mathrm{y}_i) = G_{\mathbb{h}}(z_i)R_l(\mathrm{y}_i), \tag{4.15}$$

where $G(z_i)$ comprises a sum of the contributions from all descendants of the exchanged superprimary, organized according to the various R-symmetry representations of $\mathfrak{psu}(1,1|2)$.

In particular, the spacetime conformal blocks for five-point correlators with a double-OPE in the comb channel in one dimension have been determined in [113] to be

$$G^{h_1,h_2,h_3,h_4,h_5}_{\mathbb{h}_1,\mathbb{h}_2}(z_1,z_2) = z_1^{\mathbb{h}_1} z_2^{\mathbb{h}_2}\, F_K\left[\begin{matrix} h_1+\mathbb{h}_1-h_2, \mathbb{h}_1+\mathbb{h}_2-h_3, \mathbb{h}_2+h_5-h_4 \\ 2\mathbb{h}_1, 2\mathbb{h}_2 \end{matrix}; z_1 z_2\right], \tag{4.16}$$

where F_K is a multivariable hypergeometric function, also called the comb function, defined by

$$F_K\left[\begin{matrix} a_1, b_1, a_2 \\ c_1, c_2 \end{matrix}; x_1, x_2\right] = \sum_{n_1,n_2=0}^{\infty} \frac{(a_1)_{n_1}(b_1)_{n_1+n_2}(a_2)_{n_2}}{(c_1)_{n_1}(c_2)_{n_2}} \frac{x_1^{n_1}}{n_1!}\frac{x_2^{n_2}}{n_2!}\,, \tag{4.17}$$

with $(a)_n = \Gamma(a+n)/\Gamma(a)$ denoting the Pochhammer symbol.

The R-symmetry blocks can be computed as eigenfunctions of the two-point quadratic Casimir as

$$\begin{aligned} &[C_2(1,2) - l_1(l_1+1)]\, R_l(\mathrm{y}_1, \mathrm{y}_2) = 0, \\ &[C_2(4,5) - l_1(l_1+1)]\, R_l(\mathrm{y}_1, \mathrm{y}_2) = 0, \end{aligned} \tag{4.18}$$

where $C_2(i,j)$ is the two-point quadratic Casimir of $\mathfrak{su}(2)_R$, acting on the points (i,j) as

$$\begin{aligned} C_2(i,j) &= \left(l^{(i)} + l^{(j)}\right)^2, \\ \text{where} \quad l_0^{(i)} &= y_i \partial_{y_i} + p_i, \quad l_{-1}^{(i)} = \partial_{y_i}, \quad l_1^{(i)} = y_i^2 \partial_{y_i} + 2 p_i y_i. \end{aligned} \tag{4.19}$$

To explicitly construct the full superblock expansion (4.9), we start by examining the OPE structure. The OPE content of $J \otimes_{\text{OPE}}^{\mathfrak{psu}(1,1|2)} J$ determines the supermultiplets exchanged in the OPE of two $J(x_i, y_i)$ operators, and thereby the supermultiplets contributing to the sum in Eq. (4.9). We have

$$J \otimes_{\text{OPE}}^{\mathfrak{psu}(1,1|2)} J = \mathbb{1} + J_{l=1}^{\mathbb{h}=1} + W_{l=2}^{\mathbb{h}=2} + \mathcal{A}_{l=0}^{\mathbb{h}} + \mathcal{A}_{l=1}^{\mathbb{h}}\,, \tag{4.20}$$

which can be derived from the OPE of two $\mathcal{O}_{20'}$-operators of $\mathfrak{psu}(2,2|4)$ (see, for example, [104]) by applying the chiral algebra twist on the representations, as specified in [93]. Here, W denotes the operator obtained from $[0,4,0]$ in $\mathfrak{psu}(2,2|4)$ via the chiral algebra twist, and $\mathcal{A}_j^{\mathbb{h}}$ represents long multiplets with conformal weight $\mathbb{h}$ and R-symmetry weight l, obtained from semi-short multiplets in $\mathfrak{psu}(2,2|4)$.

Since we consider a double OPE in $(J(1)J(2))$ and $(J(4)J(5))$, additional selection rules apply. The valid combinations of multiplets in the OPE (4.20) that can be exchanged are determined by solving the simultaneous Casimir equations (4.18) and identifying the permissible eigenvalues. For the correlator (4.8), we find the following allowed combinations of $\mathfrak{su}(2)_R$ representations:

$$R_{10}, \quad R_{11}, \quad R_{12}, \quad R_{21}, \quad R_{22}, \quad R_{01}, \tag{4.21}$$

where the first index indicates the R-symmetry weight l_1 of the exchanged operator in the first OPE, and the second index denotes the weight l_2 of the operator in the second OPE.

The corresponding R-symmetry blocks can be obtained from the Casimir equations to be of the form (with a leg factor as given in 4.14)

$$\begin{aligned}
R_{10}(\mathrm{y}_1,\mathrm{y}_2) &= \frac{1}{\mathrm{y}_1}\\
R_{11}(\mathrm{y}_1,\mathrm{y}_2) &= \frac{-2+\mathrm{y}_1+\mathrm{y}_2}{\mathrm{y}_1\mathrm{y}_2}\\
R_{12}(\mathrm{y}_1,\mathrm{y}_2) &= \frac{6+3\mathrm{y}_1(-2+\mathrm{y}_2)-6\mathrm{y}_2+\mathrm{y}_2^2}{3\mathrm{y}_1\mathrm{y}_2^2}\\
R_{21}(\mathrm{y}_1,\mathrm{y}_2) &= -\frac{6+\mathrm{y}_1^2+3\mathrm{y}_1(-2+\mathrm{y}_2)-6\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2}\\
R_{22}(\mathrm{y}_1,\mathrm{y}_2) &= -\frac{\mathrm{y}_1^2(-2+\mathrm{y}_2)+\mathrm{y}_1(6-6\mathrm{y}_2+\mathrm{y}_2^2)-2(2-3\mathrm{y}_2+\mathrm{y}_2^2)}{3\mathrm{y}_1^2\mathrm{y}_2^2}\\
R_{01}(\mathrm{y}_1,\mathrm{y}_2) &= -\frac{1}{\mathrm{y}_2}\,.
\end{aligned} \tag{4.22}$$

The above blocks can be easily seen to be linear combinations of the base blocks

$$\left\{\frac{1}{\mathrm{y}_1},\frac{1}{\mathrm{y}_2},\frac{1}{\mathrm{y}_1\mathrm{y}_2},\frac{1-\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2},\frac{1-\mathrm{y}_1}{\mathrm{y}_1\mathrm{y}_2^2},\frac{-1+\mathrm{y}_1+\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2^2}\right\}. \tag{4.23}$$

In terms of the superblocks 4.15, this means that for each combination of exchanged supermultiplet in the correlator 4.8, we can make the following ansatz

$$\begin{aligned}
&\mathfrak{g}_{\mathbb{h}_1,\mathbb{h}_2;l_1,l_2}(z_i,\mathrm{y}_i)\\
=&G^{(1,0)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{1,0}(\mathrm{y}_1,\mathrm{y}_2)+G^{(1,1)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{1,1}(\mathrm{y}_1,\mathrm{y}_2)+G^{(1,2)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{1,2}(\mathrm{y}_1,\mathrm{y}_2)\\
&+G^{(2,1)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{2,1}(\mathrm{y}_1,\mathrm{y}_2)+G^{(2,2)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{2,2}(\mathrm{y}_1,\mathrm{y}_2)+G^{(0,1)}_{\mathbb{h}_1,\mathbb{h}_2}(z_i)R_{0,1}(\mathrm{y}_1,\mathrm{y}_2)\,,
\end{aligned} \tag{4.24}$$

where $G^{(l_1,l_2)}_{\mathbb{h}_1,\mathbb{h}_2}$ sums the spacetime blocks of all the possible combinations of the respective exchanged supermultiplets that yield the R-symmetry representation l_1, l_2.

To clarify this decomposition, consider as an example the exchange of the long supermultiplet $\mathcal{A}^{\mathbb{h}}_{j=0}$ in both of the channels, (12) and (45). The relevant operators of consideration in this multiplet are

$$\begin{array}{ccccc}
 & & \mathcal{O}^{\mathbb{h}}_{j=0} & & \\
 & \swarrow & & \searrow & \\
 & \searrow & & \swarrow & \\
 & & \mathcal{O}^{\mathbb{h}+1}_{j=0},\ \mathcal{O}^{\mathbb{h}+1}_{j=1} & & \\
 & \swarrow & & \searrow & \\
 & \searrow & & \swarrow & \\
 & & \mathcal{O}^{\mathbb{h}+2}_{j=0} & &
\end{array} \tag{4.25}$$

Note that operators not proportional to products of $\rho\bar{\rho}$, i.e., not being *central* in the above multiplet, cannot contribute to the blocks as these contributions would violate the $U(1)$ bonus symmetry (acting as $\rho \to \lambda\rho$ and $\bar{\rho} \to \lambda^{-1}\bar{\rho}$). Thus, these representations have not been listed in the above supermultiplet.
The above ansatz 4.24 becomes

$$\begin{aligned}\mathfrak{g}_{\mathcal{L}^{\mathbb{h}_1}_{j=0},\mathcal{L}^{\mathbb{h}_1}_{j=0}}(z_1,z_2;\mathrm{y}_1,\mathrm{y}_2) &= c_1 G_{\mathbb{h}_1+1,\mathbb{h}_2+1}(z_1,z_2)\cdot R_{1,1}(\mathrm{y}_1,\mathrm{y}_2)\\ &+\left\{c_2 G_{\mathbb{h}_1+1,\mathbb{h}_2}(z_1,z_2)+c_3 G_{\mathbb{h}_1+1,\mathbb{h}_2+1}(z_1,z_2)+c_4 G_{\mathbb{h}_1+1,\mathbb{h}_2+2}(z_1,z_2)\right\} R_{1,0}(\mathrm{y}_1,\mathrm{y}_2)\\ &+\left\{c_5 G_{\mathbb{h}_1,\mathbb{h}_2+1}(z_1,z_2)+c_6 G_{\mathbb{h}_1+1,\mathbb{h}_2+1}(z_1,z_2)+c_7 G_{\mathbb{h}_1+2,\mathbb{h}_2+1}(z_1,z_2)\right\} R_{1,0}(\mathrm{y}_1,\mathrm{y}_2)\,,\end{aligned} \tag{4.26}$$

with R-symmetry blocks given in (4.22), and spacetime blocks in (4.16). In this ansatz, the coefficients c_i are undetermined coefficients.
There are seven undetermined coefficients $c_1, ..., c_7$ in this ansatz for the superblock. However, each superconformal block, and hence the ansatz above, satisfies the superconformal Ward identities. These include Drukker-Plefka twist-type conditions, along with the additional constraint equation derived in our analysis, as in Equation 3.51. By imposing these identities, we can place further constraints on each superconformal block, fixing the unknown coefficients.

Applying these constraints, the final superblock for the exchange of two long multiplets $\mathcal{A}^{\mathbb{h}}_{j=0}$ becomes

$$\begin{aligned}&\mathfrak{g}_{\mathcal{L}^{\mathbb{h}_1}_{j=0},\mathcal{L}^{\mathbb{h}_2}_{j=0}}(z_1,z_2;\mathrm{y}_1,\mathrm{y}_2) =\\ &= \frac{1}{2}c_1 z_1^{\mathbb{h}_1} z_2^{\mathbb{h}_2}\Bigg(2z_1z_2\frac{(\mathrm{y}_1+\mathrm{y}_2-2)}{\mathrm{y}_1\mathrm{y}_2}F_2\left[\begin{matrix}\mathbb{h}_1+1,\mathbb{h}_1+\mathbb{h}_2+1,\mathbb{h}_2+1\\ 2(\mathbb{h}_1+1),2(\mathbb{h}_2+1)\end{matrix},z_1,z_2\right]\\ &\quad+\frac{(\mathbb{h}_2+1)(\mathbb{h}_1+\mathbb{h}_2+1)}{(2\mathbb{h}_2+1)(2\mathbb{h}_2+3)}\frac{z_1z_2^2}{\mathrm{y}_1}F_2\left[\begin{matrix}\mathbb{h}_1+1,\mathbb{h}_1+\mathbb{h}_2+2,\mathbb{h}_2+2\\ 2(\mathbb{h}_1+1),2(\mathbb{h}_2+2)\end{matrix},z_1,z_2\right]\\ &\quad+\frac{4\mathbb{h}_2}{(\mathbb{h}_1+\mathbb{h}_2)}\frac{z_1}{\mathrm{y}_1}F_2\left[\begin{matrix}\mathbb{h}_1+1,\mathbb{h}_1+\mathbb{h}_2,\mathbb{h}_2\\ 2(\mathbb{h}_1+1),2\mathbb{h}_2\end{matrix},z_1,z_2\right]\\ &\quad+\frac{(\mathbb{h}_1+1)(\mathbb{h}_1+\mathbb{h}_2+1)}{(2\mathbb{h}_1+1)(2\mathbb{h}_1+3)}\frac{z_1^2z_2}{\mathrm{y}_2}F_2\left[\begin{matrix}\mathbb{h}_1+2,\mathbb{h}_1+\mathbb{h}_2+2,\mathbb{h}_2+1\\ 2(\mathbb{h}_1+2),2(\mathbb{h}_2+1)\end{matrix},z_1,z_2\right]\\ &\quad+\frac{4\mathbb{h}_1}{(\mathbb{h}_1+\mathbb{h}_2)}\frac{z_2}{\mathrm{y}_2}F_2\left[\begin{matrix}\mathbb{h}_1,\mathbb{h}_1+\mathbb{h}_2,\mathbb{h}_2+1\\ 2\mathbb{h}_1,2(\mathbb{h}_2+1)\end{matrix},z_1,z_2\right]\Bigg)\end{aligned} \tag{4.27}$$

with all relations obtained from the SCWI imposed. As can be seen, the entire superconformal block is determined up to a single constant c_1.

In a similar manner, all the other possible combinations of exchanged supermultiplets can be analysed, leading to analogous results.[1]
In particular, the superconformal blocks of exchanged short multiplets are fixed up to a constant by imposing the conditions from Drukker-Plefka twist alone.

4.2.2. Parametrization for the Blocks

Each superblock, as represented in 4.27, satisfies the superconformal Ward identities by construction. However, the manner in which these blocks are presented may not immediately convey this property.
The objective of this section is to derive a parametrisation for the superblocks $\mathfrak{g}_{\mathbb{h}_1,\mathbb{h}_2;l_1,l_2}(z_i,\mathrm{y}_i)$, or equivalently of the correlator

$$\mathcal{G}^{11111}(\{X_i\}) = \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle\,,$$

that manifestly solves the superconformal Ward identities.

To achieve this, we start from the ansatz proposed in equation 3.45 for the superprimary correlator,

$$\begin{aligned}
&\langle J\left(x_1,y_1\right) J\left(x_2,y_2\right) J\left(x_3,y_3\right) J\left(x_4,y_4\right) J\left(x_5,y_5\right)\rangle\\
=&\frac{y_{12}y_{13}y_{23}y_{45}^2}{x_{12}x_{13}x_{23}x_{45}^2}f_1(u,v)+\frac{y_{12}y_{13}y_{43}y_{25}y_{45}}{x_{12}x_{13}x_{43}x_{25}x_{45}}f_2(u,v)+\frac{y_{12}y_{23}y_{43}y_{15}y_{45}}{x_{12}x_{23}x_{43}x_{15}x_{45}}f_3(u,v)\\
&+\frac{y_{12}y_{24}y_{43}y_{35}y_{51}}{x_{12}x_{24}x_{43}x_{35}x_{51}}f_4(u,v)+\frac{y_{31}y_{32}y_{41}y_{25}y_{45}}{x_{31}x_{32}x_{41}x_{25}x_{45}}f_5(u,v)+\frac{y_{13}y_{34}y_{42}y_{25}y_{51}}{x_{13}x_{34}x_{42}x_{25}x_{51}}f_6(u,v).
\end{aligned} \tag{4.28}$$

This can also be interpreted as

$$\begin{aligned}
&\langle J\left(x_1,y_1\right) J\left(x_2,y_2\right) J\left(x_3,y_3\right) J\left(x_4,y_4\right) J\left(x_5,y_5\right)\rangle\\
&=\langle\phi(1)\phi(2)\rangle\,\langle\phi(4)\phi(5)\rangle\,\langle\phi\left(1\right)\phi\left(2\right)J\left(3\right)\phi\left(4\right)\phi\left(5\right)\rangle\\
&\quad+\ \text{(one extra structure of)}\ \ \langle\phi(4)\phi(5)\rangle\,\langle J\left(1\right)J\left(2\right)J\left(3\right)\phi\left(4\right)\phi\left(5\right)\rangle\\
&\quad+\ \text{(one extra structure of)}\ \ \langle\phi(1)\phi(2)\rangle\,\langle\phi\left(1\right)\phi\left(2\right)J\left(3\right)J\left(4\right)J\left(5\right)\rangle\\
&\quad+\ \text{(one extra structure of)}\ \ \langle J\left(1\right)J\left(2\right)J\left(3\right)J\left(4\right)J\left(5\right)\rangle\,,
\end{aligned} \tag{4.29}$$

where $\phi(x_i,y_i)\equiv\phi(i)$ represents the field obtained by applying the chiral algebra twist on the $[0,1,0]_{(0,0)}$ representation of $\mathfrak{psu}(2,2|4)$.

[1] A Mathematica notebook with the fully constrained superconformal blocks is available upon request.

To verify this formulation, we will outline the bosonic structures required to express the aforementioned correlators; which are products of 2pt functions and 5pt functions of reduced complexity. These are:

$$\begin{aligned}&\langle\phi(1)\phi(2)\rangle\langle\phi(4)\phi(5)\rangle\langle\phi(1)\phi(2)J(3)\phi(4)\phi(5)\rangle\\&=\frac{y_{12}}{x_{12}}\frac{y_{45}}{x_{45}}\frac{y_{13}y_{23}y_{45}}{x_{13}x_{23}x_{45}}f_1(u,v)+\frac{y_{12}}{x_{12}}\frac{y_{45}}{x_{45}}\frac{y_{13}y_{43}y_{25}}{x_{13}x_{43}x_{25}}f_2(u,v)+\frac{y_{12}}{x_{12}}\frac{y_{45}}{x_{45}}\frac{y_{23}y_{43}y_{15}}{x_{23}x_{43}x_{15}}f_3(u,v)\end{aligned} \tag{4.30}$$

$$\begin{aligned}&\langle\phi(4)\phi(5)\rangle\langle J(1)J(2)J(3)\phi(4)\phi(5)\rangle\\&=\frac{y_{45}}{x_{45}}\frac{y_{12}y_{13}y_{23}y_{45}}{x_{12}x_{13}x_{23}x_{45}}f_1(u,v)+\frac{y_{45}}{x_{45}}\frac{y_{12}y_{13}y_{43}y_{25}}{x_{12}x_{13}x_{43}x_{25}}f_2(u,v)+\frac{y_{45}}{x_{45}}\frac{y_{12}y_{23}y_{43}y_{15}}{x_{12}x_{23}x_{43}x_{15}}f_3(u,v)\\&\qquad+\frac{y_{45}}{x_{45}}\frac{y_{31}y_{32}y_{41}y_{25}}{x_{31}x_{32}x_{41}x_{25}}f_5(u,v)\end{aligned} \tag{4.31}$$

$$\begin{aligned}&\langle\phi(1)\phi(2)\rangle\langle\phi(1)\phi(2)J(3)J(4)J(5)\rangle\\&=\frac{y_{12}}{x_{12}}\frac{y_{13}y_{23}y_{45}^2}{x_{13}x_{23}x_{45}^2}f_1(u,v)+\frac{y_{12}}{x_{12}}\frac{y_{13}y_{43}y_{25}y_{45}}{x_{13}x_{43}x_{25}x_{45}}f_2(u,v)+\frac{y_{12}}{x_{12}}\frac{y_{23}y_{43}y_{15}y_{45}}{x_{23}x_{43}x_{15}x_{45}}f_3(u,v)\\&\qquad+\frac{y_{12}}{x_{12}}\frac{y_{24}y_{43}y_{35}y_{51}}{x_{24}x_{43}x_{35}x_{51}}f_4(u,v)\end{aligned} \tag{4.32}$$

Indeed, they are the same structures as the ones used in the expression 4.28.

The superconformal Ward identities (SCWI) for each of these correlators, which we refer to as *subcorrelators*, can be derived by performing a similar analysis for each individual 5-point component. Alternatively, we may deactivate all non-relevant functions in equations 4.6 and 4.7, yielding the equations for each specific subcorrelator. This approach yields the SCWI outlined as follows.
For each subcorrelator, we obtain two conditions of the Drukker-Plefka twist type, encompassing all relevant channels.
This then can be used to eliminate one of the unknown functions as

$$f_j(z_1,z_2)=-\sum_{i=1,i\neq j}^{n}f_i(z_1,z_2)+\text{constant}, \tag{4.33}$$

with the sum running over all the remaining unknown functions present in the respective correlator.
Additionally, in each case, there is one extra condition. Substituting already the

solution obtained from the Drukker-Plefka twist condition, these are given by

$$
\begin{aligned}
&\langle\phi(1)\phi(2)\rangle\,\langle\phi(4)\phi(5)\rangle\,\langle\phi(1)\phi(2)J(3)\phi(4)\phi(5)\rangle\ :\\
&\qquad 0 = f_1^{(0,1)}(z_1,z_2) + f_2^{(0,1)}(z_1,z_2)\\
&\qquad\quad + \frac{z_1-1}{z_2} f_1^{(1,0)}(z_1,z_2) + \frac{z_1}{z_2-1} f_2^{(1,0)}(z_1,z_2)
\end{aligned}
\tag{4.34}
$$

$$
\begin{aligned}
&\langle\phi(4)\phi(5)\rangle\,\langle J(1)J(2)J(3)\phi(4)\phi(5)\rangle\ :\\
&\qquad 0 = f_1^{(0,1)}(z_1,z_2) + f_2^{(0,1)}(z_1,z_2) + \frac{1-2z_1-z_1^2+z_2}{-1+z_1+z_2} f_3^{(0,1)}(z_1,z_2)\\
&\qquad\quad + \frac{(-1+z_1)z_1}{(-1+z_2)z_2} f_1^{(1,0)}(z_1,z_2) - \frac{(-1+z_1)z_1^2}{(-1+z_2)(-1+z_1+z_2)} f_3^{(1,0)}(z_1,z_2)
\end{aligned}
\tag{4.35}
$$

$$
\begin{aligned}
&\langle\phi(1)\phi(2)\rangle\,\langle\phi(1)\phi(2)J(3)J(4)J(5)\rangle\ :\\
&\qquad 0 = f_1^{(0,1)}(z_1,z_2) + f_2^{(0,1)}(z_1,z_2) + \frac{-2+2z_1+z_2}{z_2} f_1^{(1,0)}(z_1,z_2)\\
&\qquad\quad + \frac{(-1+z_2)^2+z_1(-1+2z_2)}{(-1+z_2)z_2} f_2^{(1,0)}(z_1,z_2) + \frac{-1+z_1+z_2}{z_2} f_3^{(1,0)}(z_1,z_2)
\end{aligned}
\tag{4.36}
$$

Thus, rather than solving the SCWI directly for the 5-point correlator 4.28 involving the fields $J(x,y)$, we will proceed incrementally, beginning with solutions for the simpler correlator $\langle\phi\phi J\phi\phi\rangle$.

In this correlator, given by

$$
\langle\phi(1)\phi(2)J(3)\phi(4)\phi(5)\rangle\ , \tag{4.37}
$$

when performing the OPEs $\overbracket{\phi(1)\phi(2)}$ and $\overbracket{\phi(4)\phi(5)}$, the only possible R-symmetry representations that can be exchanged are

$$
(l_1,l_2) = \{(1,0),(0,1),(1,1)\}\ . \tag{4.38}
$$

The corresponding R-symmetry blocks can be determined to be of the form

$$
\left\{\frac{1}{\mathrm{y}_1},\frac{1}{\mathrm{y}_2},\frac{1}{\mathrm{y}_2\mathrm{y}_2}\right\}. \tag{4.39}
$$

Consequently, the ansatz for the correlator $\langle\phi\phi J\phi\phi\rangle$ (which we consider with an appropriate leg factor already being separated) must be a linear combination of these three R-symmetry structures, and, by analyticity, no additional factors of the y-cross ratios are permitted.

Regarding the spacetime dependence, we initiated from three unknown functions of the cross-ratios, as indicated in ansatz 4.30, and eliminated one function via the Drukker-Plefka twist type as outlined above. Thus, the ansart has to incorporate two unknown functions of the cross ratios.
The twist itself imposes the condition

$$\langle \phi\phi J\phi\phi\rangle\,|_{\mathrm{y}_1\to z_1,\mathrm{y}_2\to z_2} = \text{const.} \tag{4.40}$$

To satisfy this condition explicitly within the parametrisation of the correlator, we assign the two remaining unknown functions to be proportional to $(z_1 - \mathrm{y}_1)$ and $(z_2 - \mathrm{y}_2)$, respectively.

Thus, a preliminary ansatz for this correlator would be

$$\sim \frac{1}{\mathrm{y}_1\mathrm{y}_2}\left((z_1 - \mathrm{y}_1)A_1(z_1, z_2) + (z_2 - \mathrm{y}_2)A_2(z_1, z_2)\right). \tag{4.41}$$

It is important to note that the analyticity conditions on the y-variables remain applicable. Expanding the above expression, we recover only the allowed R-symmetry blocks specified in 4.39.

For further convenience, we adopt a slightly modified ansatz:

$$\langle \phi\phi J\phi\phi\rangle \sim \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left((z_1 - \mathrm{y}_1)A_1(z_1, z_2) + (z_2 - \mathrm{y}_2)A_2(z_1, z_2)\right). \tag{4.42}$$

By matching this ansatz with the initial form given in 4.28, we can express $f_1(z_1, z_2)$, $f_2(z_1, z_2)$, and $f_3(z_1, z_2)$ in terms of $A_1(z_1, z_2)$ and $A_2(z_1, z_2)$. Substituting these expressions into the original SCWI of the Drukker-Plefka condition for this correlator, along with Equation 4.34, we verify that the Drukker-Plefka twist condition is manifestly satisfied, while the additional condition reduces to[2]

$$\partial_{z_2} A_1(z_1, z_2) = \partial_{z_1} A_2(z_1, z_2). \tag{4.43}$$

The most general solutions to this equation take the form

$$\begin{aligned} A_1(z_1, z_2) &= g_2(z_1) + \partial_{z_1} H\,(z_1, z_2) + \text{ const.} \\ A_2(z_1, z_2) &= g_1(z_2) + \partial_{z_2} H\,(z_1, z_2) + \text{ const.} \end{aligned} \tag{4.44}$$

[2]This form is achieved thanks to the additional factor $z_1 z_2$. If we had proceeded with the preliminary ansatz, we would have obtained equivalent, but slightly modified equations.

where $H(z_1, z_2)$ is a function of two variables, and $g_2(z_1)$ and $g_1(z_2)$ are functions of a single variable.

Substituting these solutions into the correlator ansatz 4.42, we obtain

$$\begin{aligned}\langle \phi\phi J\phi\phi\rangle \,|_{\text{bare}} =& \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\kappa + \frac{z_1 z_2(z_1-\mathrm{y}_1)g_2(z_1) + z_1 z_2(z_2-\mathrm{y}_2)g_1(z_2)}{\mathrm{y}_1\mathrm{y}_2} \\ &+ \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\Big((z_1-\mathrm{y}_1)H^{(1,0)}(z_1,z_2) + (z_2-\mathrm{y}_2)H^{(0,1)}(z_1,z_2)\Big) \\ =& \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\kappa + \frac{z_2(z_1-\mathrm{y}_1)g_2(z_1) + z_1(z_2-\mathrm{y}_2)g_1(z_2)}{\mathrm{y}_1\mathrm{y}_2} \\ &+ \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\Big((z_1-\mathrm{y}_1)H^{(1,0)}(z_1,z_2) + (z_2-\mathrm{y}_2)H^{(0,1)}(z_1,z_2)\Big),\end{aligned} \tag{4.45}$$

where the two constants have been combined into a single term, κ. In the transition from the first to the second line, the factor z_1 is absorbed into $g_2(z_1)$ and z_2 into $g_1(z_2)$, respectively.
The subscript *bare* denotes that we are considering the correlator with the external leg factor removed.

We can apply this result to analyse in a next step the more complex correlator of the type $\langle JJJ\phi\phi\rangle$.
As before, the R-symmetry blocks can be expressed as linear combinations of the four fundamental blocks:

$$\left\{\frac{1}{\mathrm{y}_1}, \frac{1}{\mathrm{y}_2}, \frac{1}{\mathrm{y}_2\mathrm{y}_2}, \frac{(1-\mathrm{y}_2)}{\mathrm{y}_1^2\mathrm{y}_2}\right\}. \tag{4.46}$$

The additional fourth block reflects that the OPE between $J(1)J(2)$ allows also for an exchange of operators of weight $l_1 = 2$ (which are not present in the OPE of $\phi(1)\phi(2)$).
In a manner similar to previous cases, we construct an ansatz using these four R-symmetry blocks, ensuring it satisfies both the conditions imposed by the Drukker-Plefka twist and the additional constraint in Eq. 4.35 in the most manifest way.

After imposing the Drukker-Plefka twist, we proceed here with three unknown functions of the cross ratios. Since this correlator can be viewed as an extension of the previously examined correlator $\langle\phi\phi J\phi\phi\rangle$, we leverage the previously developed parametrisation and formulate an ansatz schematically as

$$\langle\phi\phi J\phi\phi\rangle + \left(\text{one additional term} \sim \frac{(1-\mathrm{y}_2)}{\mathrm{y}_1^2\mathrm{y}_2}C(z_1,z_2)\right).$$

By exploiting the freedom to include additional factors of the conformal cross ratios, we construct the ansatz

$$\begin{aligned}\langle JJJ\phi\phi\rangle \sim & \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left((z_1-\mathrm{y}_1)A_1(z_1,z_2)+(z_2-\mathrm{y}_2)A_2(z_1,z_2)\right)\\ &+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_2}{\mathrm{y}_1}-\frac{1-z_2}{z_1}\right)\frac{z_1-\mathrm{y}_1}{z_1}A_3(z_1,z_2).\end{aligned} \tag{4.47}$$

The final term includes a component

$$+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_2}{\mathrm{y}_1}-\frac{1-z_2}{z_1}\right) \tag{4.48}$$

which directly resembles the fourth R-symmetry block, with the conformal cross ratios chosen in the simplest way to manifest the Drukker-Plefka twist. The additional factor

$$\frac{z_1-\mathrm{y}_1}{z_1} \tag{4.49}$$

has been introduced to simplify the resulting third superconformal Ward identity equation. With this parametrisation, this equation takes again the simpler form

$$\partial_{z_2}A_1(z_1,z_2)=\partial_{z_1}A_2(z_1,z_2). \tag{4.50}$$

Note that the extra factor in Eq. 4.49 still yields a linear combination of the four R-symmetry base blocks.

Following the same reasoning, we can establish a parametrization for the correlator

$$\langle \phi\phi JJJ\rangle$$

as

$$\begin{aligned}\langle JJJ\phi\phi\rangle \sim & \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left((z_1-\mathrm{y}_1)A_1(z_1,z_2)+(z_2-\mathrm{y}_2)A_2(z_1,z_2)\right)\\ &+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_1}{\mathrm{y}_2}-\frac{1-z_1}{z_2}\right)\frac{z_2-\mathrm{y}_2}{z_2}A_4(z_1,z_2)\end{aligned} \tag{4.51}$$

which also satisfies the superconformal Ward identity $\partial_{z_2}A_1(z_1,z_2)=\partial_{z_1}A_2(z_1,z_2)$.

These subcorrelators enable us to construct an ansatz for the complete correlator $\langle JJJJJ\rangle$.

Referring to the observation made in Eq. 4.29, we can formulate an ansatz as follows:

$$
\begin{aligned}
\langle JJJJJ\rangle \sim & \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left((z_1-\mathrm{y}_1)A_1(z_1,z_2)+(z_2-\mathrm{y}_2)A_2(z_1,z_2)\right)\\
&+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_2}{\mathrm{y}_1}-\frac{1-z_2}{z_1}\right)\frac{z_1-\mathrm{y}_1}{z_1}A_3(z_1,z_2)\\
&+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_1}{\mathrm{y}_2}-\frac{1-z_1}{z_2}\right)\frac{z_2-\mathrm{y}_2}{z_2}A_4(z_1,z_2)\\
&+\text{one additional term.}
\end{aligned}
\tag{4.52}
$$

This additional term must be proportional to the sixth R-symmetry block, which represents an exchange of operators with $l_1 = 2, l_2 = 2$ - a term that appears only in the double-OPE of (JJ) and not in the sub-correlator configurations. Referring to Eq. 4.23, this final R-symmetry block is given by

$$
\frac{-1+\mathrm{y}_1+\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2^2}.
\tag{4.53}
$$

To ensure a manifest solution to the Drukker-Plefka twist condition, we construct the final ansatz as follows:

$$
\begin{aligned}
\langle JJJJJ\rangle \sim & \frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left((z_1-\mathrm{y}_1)A_1(z_1,z_2)+(z_2-\mathrm{y}_2)A_2(z_1,z_2)\right)\\
&+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_2}{\mathrm{y}_1}-\frac{1-z_2}{z_1}\right)\frac{z_1-\mathrm{y}_1}{z_1}A_3(z_1,z_2)\\
&+\frac{z_1 z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_1}{\mathrm{y}_2}-\frac{1-z_1}{z_2}\right)\frac{z_2-\mathrm{y}_2}{z_2}A_4(z_1,z_2)\\
&+\frac{-1+\mathrm{y}_1+\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2^2\, z_1 z_2}(y_1-y_2)(z_1-z_2)A_5(z_1,z_2).
\end{aligned}
\tag{4.54}
$$

This ansatz does not explicitly satisfy the superconformal Ward identities, yet, like the previous subcorrelators, it fulfills the additional condition

$$
\partial_{z_2}A_1(z_1,z_2)=\partial_{z_1}A_2(z_1,z_2).
\tag{4.55}
$$

This condition, however, can be solved as in Eq. 4.44. Substituting the solution into Eq. 4.54 yields a parametrisation of the correlator $\langle JJJJJ\rangle$ that explicitly satisfies the superconformal Ward identities.

The resulting parametrisation is given by

$$\begin{aligned}\langle JJJJJ\rangle\,|_{\rm bare} =&\frac{z_1z_2}{\mathrm{y}_1\mathrm{y}_2}\kappa+\frac{z_2(z_1-\mathrm{y}_1)g_2(z_1)+z_1(z_2-\mathrm{y}_2)g_1(z_2)}{\mathrm{y}_1\mathrm{y}_2}\\&+\frac{z_1z_2}{\mathrm{y}_1\mathrm{y}_2}\Big((z_1-\mathrm{y}_1)H^{(1,0)}(z_1,z_2)+(z_2-\mathrm{y}_2)H^{(0,1)}(z_1,z_2)\Big)\\&+\frac{z_1z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_2}{\mathrm{y}_1}-\frac{1-z_2}{z_1}\right)\frac{z_1-\mathrm{y}_1}{z_1}A_3(z_1,z_2)\\&+\frac{z_1z_2}{\mathrm{y}_1\mathrm{y}_2}\left(\frac{1-\mathrm{y}_1}{\mathrm{y}_2}-\frac{1-z_1}{z_2}\right)\frac{z_2-\mathrm{y}_2}{z_2}A_4(z_1,z_2)\\&+\frac{-1+\mathrm{y}_1+\mathrm{y}_2}{\mathrm{y}_1^2\mathrm{y}_2^2z_1z_2}(y_1-y_2)(z_1-z_2)A_5(z_1,z_2).\end{aligned}\tag{4.56}$$

The five-point function of the field $J(x,y)$ -obtained from the stress-tensor superprimary via the chiral algebra twist- is thus parametrised by four unknown functions of the two conformal cross ratios, of which one appears in differential form, along with two additional single-variable functions.

4.3. Analyse SCWI of Four-Point Function

In this section, we analyse the equations derived for the four-point functions of the stress-tensor multiplet.

We focus specifically on the six equations obtained for the superprimary correlator in Eq. 3.220, given by

$$\begin{aligned}&\langle\mathcal{O}_{20'}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\rangle\\&=\frac{y_{12}^4y_{34}^4}{x_{12}^4x_{34}^4}f_1(u,v)+\frac{y_{13}^4y_{24}^4}{x_{13}^4x_{24}^4}f_2(u,v)+\frac{y_{14}^4y_{23}^4}{x_{14}^4x_{23}^4}f_3(u,v)\\&\quad+\frac{y_{12}^2y_{13}^2y_{24}^2y_{34}^2}{x_{12}^2x_{13}^2x_{24}^2x_{34}^2}f_4(u,v)+\frac{y_{12}^2y_{14}^2y_{23}^2y_{34}^2}{x_{12}^2x_{14}^2x_{23}^2x_{34}^2}f_5(u,v)+\frac{y_{13}^2y_{14}^2y_{23}^2y_{24}^2}{x_{13}^2x_{14}^2x_{23}^2x_{24}^2}f_6(u,v).\end{aligned}\tag{4.57}$$

The six equations that constrain the six unknown functions of the correlator above are given by

$$\begin{aligned}f_4^{(u)}(u,v)&=-\frac{f_1^{(u)}(u,v)u+f_2^{(u)}(u,v)u-f_2^{(u)}(u,v)vu-f_2^{(v)}(u,v)vu+f_1^{(v)}(u,v)v}{u},\\f_4^{(v)}(u,v)&=-\frac{f_2^{(u)}(u,v)u^2+f_2^{(v)}(u,v)u^2-f_1^{(u)}(u,v)u+f_1^{(v)}(u,v)-f_1^{(v)}(u,v)v}{u},\\f_5^{(u)}(u,v)&=-\frac{f_3^{(u)}(u,v)u^2-f_3^{(u)}(u,v)u+f_3^{(u)}(u,v)vu+f_3^{(v)}(u,v)vu-f_1^{(v)}(u,v)v^2}{uv},\\f_5^{(v)}(u,v)&=-\frac{f_1^{(v)}(u,v)(u-1)}{u}-\frac{f_1^{(u)}(u,v)u+f_1^{(v)}(u,v)v}{u}+\frac{f_3^{(u)}(u,v)u}{v},\\f_6^{(u)}(u,v)&=-\frac{f_2^{(u)}(u,v)v^2+f_2^{(v)}(u,v)v^2-f_3^{(v)}(u,v)v+f_3^{(u)}(u,v)-f_3^{(u)}(u,v)u}{v},\\f_6^{(v)}(u,v)&=-\frac{f_3^{(u)}(u,v)u-f_2^{(u)}(u,v)vu-f_2^{(v)}(u,v)vu+f_2^{(v)}(u,v)v+f_3^{(v)}(u,v)v}{v},\end{aligned}\tag{4.58}$$

and we will refer to these equations, with a slight abuse of notation, as the four-point superconformal Ward identities (SCWI).

We begin this section by verifying the correctness of the derived equations.

4.3.1. Check the Equations.

As an initial check of equation 4.58, we utilise established results for the four-point correlation function from the literature to verify whether the corresponding coefficients satisfy the constraints derived in our analysis.

The four-point correlator is well known to take the (schematic) form [79]

$$\langle \mathcal{TTTT} \rangle = \mathrm{G}_{\text{free}} + \mathrm{G}_{\text{anom}} = \mathrm{G}_{\text{free}} + \mathcal{R}(\{x_i, y_i\}) \cdot \mathcal{H}(u,v), \tag{4.59}$$

where

$$\begin{aligned} \mathrm{G}_{\text{free}} = {} & \frac{y_{12}^4\, y_{34}^4}{x_{12}^4\, x_{34}^4} + \frac{y_{13}^4\, y_{24}^4}{x_{13}^4\, x_{24}^4} + \frac{y_{14}^4\, y_{23}^4}{x_{14}^4\, x_{23}^4} \\ & + \frac{4}{N^2-1}\left(\frac{y_{12}^2\, y_{13}^2\, y_{24}^2\, y_{34}^2}{x_{12}^2\, x_{13}^2\, x_{24}^2\, x_{34}^2} + \frac{y_{12}^2\, y_{14}^2\, y_{23}^2\, y_{34}^2}{x_{12}^2\, x_{14}^2\, x_{23}^2\, x_{34}^2} + \frac{y_{13}^2\, y_{14}^2\, y_{23}^2\, y_{24}^2}{x_{13}^2\, x_{14}^2\, x_{23}^2\, x_{24}^2} \right) \end{aligned} \tag{4.60}$$

$$\begin{aligned} \mathrm{G}_{\text{anom}} = {} & \frac{y_{12}^4\, y_{34}^4}{x_{12}^4\, x_{34}^4}\, v\, H\,(u,v) + \frac{y_{13}^4\, y_{24}^4}{x_{13}^4\, x_{24}^4}\, \frac{v}{u}\, H\,(u,v) + \frac{y_{14}^4\, y_{23}^4}{x_{14}^4\, x_{23}^4}\, \frac{v^2}{u}\, H\,(u,v) \\ & + \frac{y_{12}^2\, y_{13}^2\, y_{24}^2\, y_{34}^2}{x_{12}^2\, x_{13}^2\, x_{24}^2\, x_{34}^2}\, \frac{v}{u}\,(v-u-1)\, H\,(u,v) + \frac{y_{12}^2\, y_{14}^2\, y_{23}^2\, y_{34}^2}{x_{12}^2\, x_{14}^2\, x_{23}^2\, x_{34}^2}\, \frac{v}{u}\,(1-u-v)\; H\,(u,v) \\ & + \frac{y_{13}^2\, y_{14}^2\, y_{23}^2\, y_{24}^2}{x_{13}^2\, x_{14}^2\, x_{23}^2\, x_{24}^2}\, \frac{v}{u}\,(u-1-v)\; H\,(u,v) \end{aligned} \tag{4.61}$$

$$\text{with } u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{4.62}$$

Leaving the correlator unsupersymmetrised corresponds to verifying the relations for the superprimary correlator $\langle \mathcal{O}_{20'} \mathcal{O}_{20'} \mathcal{O}_{20'} \mathcal{O}_{20'} \rangle$ i.e. we focus on the consistency of the relations governing the derivatives of the functions $f_i(u,v)$, as given by the equation 4.58.

By comparing the expression 4.61 with our used ansatz for the four-point correlator in equation 4.57, we obtain the following identifications

$$\begin{aligned} & f_1\,(u,v) = 1 + v\, H_2\,(u,v)\,, \quad f_2\,(u,v) = 1 + \frac{v}{u}\, H_2\,(u,v)\,, \quad \ldots \\ \Rightarrow & f_1^{(u)}(u,v) = v\, H_2^{(u)}(u,v), \quad f_1^{(v)}(u,v) = H_2(u,v) + v\, H_2^{(v)}(u,v), \\ & f_2^{(u)}(u,v) = -\frac{v}{u^2} H_2(u,v) + \frac{v}{u} H_2^{(u)}(u,v), \quad f_2^{(v)}(u,v) = \frac{1}{u} H_2(u,v) + \frac{v}{u} H_2^{(v)}(u,v), \\ & \ldots \end{aligned}$$

It is straightforward to verify that these functions indeed satisfy the relations given in equation 4.58.

4.3.2. Solving the Equations

To demonstrate the utility of these equations and their ability to reproduce the result stated in equation 2.93 in Section 2.4, we proceed by solving the system of equations given in 4.58.

To solve these partial differential equations (PDEs) for the six functions $f_i(u,v)$ involved in the four-point superprimary correlator, it is convenient to first express them in terms of the variables z and $\bar{z}$, as introduced in [36]. Then, two equations can be shown to be equivalent to:

$$\sum_{i=1}^{6} f_i^{(0,1)}(z,\bar{z}) = 0, \qquad \sum_{i=1}^{6} f_i^{(1,0)}(z,\bar{z}) = 0, \tag{4.63}$$

which are equivalent to the constraints derived from performing the Drukker-Plefka twist. These two equations can be solved as follows:

$$f_6(z,\bar{z}) = -\sum_{i=1}^{5} f_i(z,\bar{z}) + \text{const.} \tag{4.64}$$

Substituting this expression into the remaining four equations, we obtain:

$$\begin{aligned}
\frac{{f_1}^{(1,0)}(z,\bar{z})}{\bar{z}} + \bar{z}{f_2}^{(1,0)}(z,\bar{z}) + {f_4}^{(1,0)}(z,\bar{z}) &= 0,\\
\frac{{f_1}^{(0,1)}(z,\bar{z})}{z} + z{f_2}^{(0,1)}(z,\bar{z}) + {f_4}^{(0,1)}(z,\bar{z}) &= 0,\\
\frac{(\bar{z}-1){f_1}^{(1,0)}(z,\bar{z})}{\bar{z}} + \frac{\bar{z}{f_3}^{(1,0)}(z,\bar{z})}{\bar{z}-1} + {f_5}^{(1,0)}(z,\bar{z}) &= 0,\\
\frac{(z-1){f_1}^{(0,1)}(z,\bar{z})}{z} + \frac{z{f_3}^{(0,1)}(z,\bar{z})}{z-1} + {f_5}^{(0,1)}(z,\bar{z}) &= 0.
\end{aligned} \tag{4.65}$$

This system of equations can be fully solved to give:

$$\begin{aligned}
f_2(z,\bar{z}) &= \frac{f_1(z,\bar{z})}{z\bar{z}} + \frac{c_1(z)-\bar{c}_1(\bar{z})}{z-\bar{z}},\\
f_3(z,\bar{z}) &= (z-1)(\bar{z}-1)\left(\frac{f_1(z,\bar{z})}{z\bar{z}} + \frac{\bar{c}_2(\bar{z})-c_2(z)}{z-\bar{z}}\right),\\
f_4(z,\bar{z}) &= \frac{z\bar{c}_1(\bar{z})-\bar{z}c_1(z)}{z-\bar{z}} - \frac{(z+\bar{z})f_1(z,\bar{z})}{z\bar{z}},\\
f_5(z,\bar{z}) &= \left(\frac{1}{z}+\frac{1}{\bar{z}}-2\right)f_1(z,\bar{z}) + \frac{(z-1)\bar{z}c_2(z)-z(\bar{z}-1)\bar{c}_2(\bar{z})}{z-\bar{z}},
\end{aligned} \tag{4.66}$$

where $c_1(z)$, $\bar{c}_1(\bar{z})$, $c_2(z)$, and $\bar{c}_2(\bar{z})$ are single-variable functions.
Indeed, the six functions $f_i(z, \bar{z})$ initially parametrising the superprimary correlator have been successfully reduced to a single function of the two conformal cross-ratios, $f_1(z, \bar{z})$ plus several single-variable functions.
Substituting the above solutions into the original ansatz and factoring out an appropriate leg factor reproduces an expression of the same kind as the form developed in Section 2.4 (see Eq. 2.93). Thus, the equations above indeed reconstruct the four-point superconformal Ward identities.

An alternative approach to deriving this result is to observe that the six equations in 4.63 and 4.65 are equivalent to the constraints imposed by the Drukker-Plefka twist and the chiral algebra twist introduced in Section 2.4. Applying these twists to the correlator ansatz in 3.220 yields equations equivalent to those derived above.

In other words, we have rederived that the Drukker-Plefka twist and the chiral algebra twist fully capture the superconformal symmetry constraints for four-point functions.

The next question to consider is whether this statement extends to five-point functions as well.

4.4. Analyse SCWI of Five-Point Function

Following the approaches taken in previous sections, we now analyse the SCWI obtained for the five-point correlator of stress-tensor multiplets in $\mathfrak{psu}(2,2|4)$. As before, we focus primarily on the partial differential equations that constrain the superprimary correlator:

$$G^{22222} = \langle \mathcal{O}_{20'}(X_1)\mathcal{O}_{20'}(X_2)\mathcal{O}_{20'}(X_3)\mathcal{O}_{20'}(X_4)\mathcal{O}_{20'}(X_5)\rangle\,. \tag{4.67}$$

This correlator is parametrized by 22 unknown functions of the cross ratios, as defined in 3.214,

$$\begin{aligned}
&\langle \mathcal{O}_{20'}(x_1,y_1)\,\mathcal{O}_{20'}(x_2,y_2)\,\mathcal{O}_{20'}(x_3,y_3)\,\mathcal{O}_{20'}(x_4,y_4)\,\mathcal{O}_{20'}(x_5,y_5)\rangle = \\
&= \frac{y_{12}^4y_{34}^2y_{45}^2y_{53}^2}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\,f_1(\{u\}) + \frac{y_{13}^4y_{24}^2y_{45}^2y_{52}^2}{x_{13}^4x_{24}^2x_{45}^2x_{52}^2}\,f_2(\{u\}) + \frac{y_{14}^4y_{23}^2y_{35}^2y_{52}^2}{x_{14}^4x_{23}^2x_{35}^2x_{52}^2}\,f_3(\{u\}) \\
&\quad + \frac{y_{15}^4y_{23}^2y_{34}^2y_{42}^2}{x_{15}^4x_{23}^2x_{34}^2x_{42}^2}\,f_4(\{u\}) + \frac{y_{23}^4y_{14}^2y_{45}^2y_{51}^2}{x_{23}^4x_{14}^2x_{45}^2x_{51}^2}\,f_5(\{u\}) + \frac{y_{24}^4y_{13}^2y_{35}^2y_{51}^2}{x_{24}^4x_{13}^2x_{35}^2x_{51}^2}\,f_6(\{u\}) \\
&\quad + \frac{y_{25}^4y_{13}^2y_{34}^2y_{41}^2}{x_{25}^4x_{13}^2x_{34}^2x_{41}^2}\,f_7(\{u\}) + \frac{y_{34}^4y_{12}^2y_{25}^2y_{51}^2}{x_{34}^4x_{12}^2x_{25}^2x_{51}^2}\,f_8(\{u\}) + \frac{y_{35}^4y_{12}^2y_{24}^2y_{41}^2}{x_{35}^4x_{12}^2x_{24}^2x_{41}^2}\,f_9(\{u\}) \\
&\quad + \frac{y_{45}^4y_{12}^2y_{23}^2y_{31}^2}{x_{45}^4x_{12}^2x_{23}^2x_{31}^2}\,f_{10}(\{u\}) + \frac{y_{12}^2y_{23}^2y_{34}^2y_{45}^2y_{51}^2}{x_{12}^2x_{23}^2x_{34}^2x_{45}^2x_{51}^2}\,f_{11}(\{u\}) + \frac{y_{12}^2y_{23}^2y_{35}^2y_{54}^2y_{41}^2}{x_{12}^2x_{23}^2x_{35}^2x_{54}^2x_{41}^2}\,f_{12}(\{u\}) \\
&\quad + \frac{y_{12}^2y_{24}^2y_{45}^2y_{53}^2y_{31}^2}{x_{12}^2x_{24}^2x_{45}^2x_{53}^2x_{31}^2}\,f_{13}(\{u\}) + \frac{y_{12}^2y_{24}^2y_{43}^2y_{35}^2y_{51}^2}{x_{12}^2x_{24}^2x_{43}^2x_{35}^2x_{51}^2}\,f_{14}(\{u\}) + \frac{y_{12}^2y_{25}^2y_{53}^2y_{34}^2y_{41}^2}{x_{12}^2x_{25}^2x_{53}^2x_{34}^2x_{41}^2}\,f_{15}(\{u\}) \\
&\quad + \frac{y_{12}^2y_{25}^2y_{54}^2y_{43}^2y_{31}^2}{x_{12}^2x_{25}^2x_{54}^2x_{43}^2x_{31}^2}\,f_{16}(\{u\}) + \frac{y_{13}^2y_{35}^2y_{52}^2y_{24}^2y_{41}^2}{x_{13}^2x_{35}^2x_{52}^2x_{24}^2x_{41}^2}\,f_{17}(\{u\}) + \frac{y_{13}^2y_{32}^2y_{24}^2y_{45}^2y_{51}^2}{x_{13}^2x_{32}^2x_{24}^2x_{45}^2x_{51}^2}\,f_{18}(\{u\}) \\
&\quad + \frac{y_{13}^2y_{32}^2y_{25}^2y_{54}^2y_{41}^2}{x_{13}^2x_{32}^2x_{25}^2x_{54}^2x_{41}^2}\,f_{19}(\{u\}) + \frac{y_{13}^2y_{34}^2y_{42}^2y_{25}^2y_{51}^2}{x_{13}^2x_{34}^2x_{42}^2x_{25}^2x_{51}^2}\,f_{20}(\{u\}) + \frac{y_{14}^2y_{42}^2y_{23}^2y_{35}^2y_{51}^2}{x_{14}^2x_{42}^2x_{23}^2x_{35}^2x_{51}^2}\,f_{21}(\{u\}) \\
&\quad + \frac{y_{15}^2y_{52}^2y_{23}^2y_{34}^2y_{41}^2}{x_{15}^2x_{52}^2x_{23}^2x_{34}^2x_{41}^2}\,f_{22}(\{u\})
\end{aligned} \tag{4.68}$$

where the conformal cross ratios are defined as in [121].

$$\begin{aligned}
u_1 &= \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2} = z_1\bar{z}_1, \quad u_2 = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2} = (1-z_1)(1-\bar{z}_1), \\
u_3 &= \frac{x_{23}^2x_{45}^2}{x_{24}^2x_{35}^2} = z_2\bar{z}_2, \quad u_4 = \frac{x_{25}^2x_{34}^2}{x_{24}^2x_{35}^2} = (1-z_2)(1-\bar{z}_1), \\
u_5 &= \frac{x_{15}^2x_{23}^2x_{34}^2}{x_{24}^2x_{13}^2x_{35}^2} = w(z_1-\bar{z}_1)(z_2-\bar{z}_2) + (1-z_1-z_2)(1-\bar{z}_1-\bar{z}_2).
\end{aligned} \tag{4.69}$$

As mentioned earlier, the 35 partial differential equations involving these 22 unknown functions are too complex to display here.
The complexity of these 35 equations arises not only from the involvement of 110 unknown functions, each of the form $f_i^{(u_j)}(\{u\})$ for $j = 1, \ldots, 5$ and $i = 1, \ldots, 22$, but each of those function also has intricate prefactors built from the five independent cross ratios, adding further complexity.
Schematically, the equations are of the form

$$0 = \sum_{i,j} c\,(u_1, u_2, u_3, u_4, u_5)\,\frac{\partial}{\partial u_j} f_i\,(u_1, u_2, u_3, u_4, u_5) \tag{4.70}$$

where $c\,(u_1, u_2, u_3, u_4, u_5)$ is a rational coefficient of the five cross ratios. The sum could run over $i = 1, \ldots, 22$ and $j = 1, \ldots, 5$. In general, not all of these 110 combinations are involved in one equation.
The 35 equations can be translated into equations involving instead the five variables $(z_1, \bar{z}_1, z_2, \bar{z}_2, w)$, thus being of the form

$$0 = \sum_{i,j} c\,(z_1, \bar{z}_1, z_2, \bar{z}_2, w)\,\frac{\partial}{\partial z_j} f_i\,(z_1, \bar{z}_1, z_2, \bar{z}_2, w)\,, \tag{4.71}$$

where $z_j \in \{z_1, \bar{z}_1, z_2, \bar{z}_2, w\}$.
At this point in the analysis, this does however neither simplify nor complicate drastically the situation. Therefore, both sets of variables are used, depending on the respective situation and goal. The statements presented in this thesis are true for both sets of variables, unless marked otherwise.

The complexity of the equations can be partially addressed by taking a numerical approach -that is, by assigning independent numerical values to each of the conformal cross ratios. This reduces the complexity of the prefactors considerably.

With this simplification, certain analyses become feasible. Similar to the four-point case, we begin by providing evidence that the derived equations are correct.

4.4.1. Check the Five-Point Equations.

To cross-check the accuracy of the 35 derived equations, we will compare them against two established expressions from the literature. First, we will examine a one-loop expression for $n = 5$ half-BPS correlators presented in [105] to confirm that this solution satisfies our equations. Second, we will use a strong-coupling expression derived from supergravity in [104] and verify its consistency with our equations. This two-fold comparison serves as a thorough validation of the correctness of our derived system across different physical regimes.

4.4.1.1. Check at Weak Coupling

In 2009, a closed-form expression for the one-loop correction to the correlation function of five half-BPS operators with dimension two in the planar limit of $\mathcal{N}=4$ SYM was presented in [105].

This expression, adapted to our notation, is given by [105]

$$\begin{aligned}
&\langle \mathcal{T}(1)\mathcal{T}(2)\mathcal{T}(3)\mathcal{T}(4)\mathcal{T}(5)\rangle_{\text{1-loop}} \\
=&-32\Big(D_{1234}\,[13,24|5]+D_{1324}\,[12,34|5]+D_{1243}\,[14,23|5]\Big)\\
&-32\Big(D_{1235}\,[13,25|4]+D_{1325}\,[12,53|4]+D_{1253}\,[15,23|4]\Big)\\
&-32\Big(D_{1254}\,[15,24|3]+D_{1524}\,[12,45|3]+D_{1245}\,[14,25|3]\Big)\\
&-32\Big(D_{1534}\,[13,54|2]+D_{1354}\,[15,34|2]+D_{1543}\,[14,53|2]\Big)\\
&-32\Big(D_{5234}\,[53,24|1]+D_{5324}\,[52,34|1]+D_{5243}\,[54,23|1]\Big)\,,
\end{aligned} \tag{4.72}$$

where the notation $[ij,kl|m]$ encodes the five-point bosonic structures in terms of two-point tree-level contractions:

$$[ij,kl|m]=[im]\,[jm]\,[kl] \tag{4.73}$$

$$[ij]:=\frac{u_i\cdot u_j}{(2\pi)^2\,x_{ij}^2}\to\frac{1}{(2\pi)^2}\frac{y_{ij}^2}{x_{ij}^2}\,, \tag{4.74}$$

where the u_i are the null vectors previously introduced as an alternative approach to address internal symmetry considerations (see Section 2.4). In the final expression, these are adapted to our conventions of analytic superspace.
The term D_{ijkl} refers to a planar four-point interaction, given by

$$D_{ijkl}=\frac{\lambda}{32\pi^2}\Phi(s,t)\left(2\,[ik]\,[jl]+(s-1-t)\,[il]\,[jk]+(t-1-s)\,[ij]\,[kl]\right), \tag{4.75}$$

where λ is the coupling constant, and

$$\begin{aligned}
\Phi(s,t)=&\frac{1}{A}\text{Im}\left(\text{Li}_2\frac{e^{i\varphi}\sqrt{s}}{\sqrt{t}}+\ln\frac{\sqrt{s}}{\sqrt{t}}\ln\frac{\sqrt{t}-e^{i\varphi}\sqrt{s}}{\sqrt{t}}\right)\\
e^{i\varphi}=&i\sqrt{\frac{1-s-t-4iA}{1-s-t+4iA}},\quad A=\frac{1}{4}\sqrt{4st-(1-s-t)^2}
\end{aligned} \tag{4.76}$$

with cross ratios defined by

$$s=\frac{x_{ij}^2\,x_{kl}^2}{x_{ik}^2\,x_{jl}^2},\ t=\frac{x_{il}^2\,x_{jk}^2}{x_{ik}^2\,x_{jl}^2}. \tag{4.77}$$

Using this expression, we can extract the coefficient functions that correspond to our unknown functions f_i by matching the terms in Eq. 4.72 to each of our 22 structures in 3.214. Then, we rewrite those expressions in terms of our chosen set of cross ratios, as defined in 3.121, yielding $f_i(\{u\})$ at one-loop order.

Inserting the resulting expressions into our 35 equations for the superprimary, and studying those numerically, this calculation shows that our equations are indeed satisfied, with an error margin in the range

$$(10^{-6} - 10^{-10})\lambda\,. \tag{4.78}$$

To summarise, the one-loop expression for the five-point correlator of stress tensor multiplets presented in [105] indeed satisfies our 35 equations.

4.4.1.2. Check at Strong Coupling

A similar consistency check can be performed using an expression for the five-point correlator of stress tensor multiplets at strong coupling, derived from the dual theory of type IIB supergravity on $AdS_5 \times S^5$. This expression was obtained in 2019 by [104].

Unlike the weak coupling result, but similar in complexity to our derived superconformal Ward identities, this expression is too lengthy to display in full here. However, the authors generously provided the full expression in a Mathematica file such that this check could be performed.[3]

Following a procedure similar to the weak coupling check, we rewrite the provided expression in terms of our defined set of cross ratios, extract the corresponding 22 unknown functions $f_i(\{u\})$, and substitute these into our 35 equations. We find numerically that the equations are indeed satisfied, with numerical accuracy up to an error of[4]

$$10^{-6}. \tag{4.79}$$

Having confirmed the accuracy of our equations, we will now outline the initial steps taken in order to control the superconformal Ward identities (SCWI) and to aim towards a parametrisation analogous to the discussion presented for the $\mathfrak{psu}(1,1|2)$ case.

[3] A special thank you to Vasco Gonçalves is here in order.

[4] Mathematica notebooks including both, the weak coupling and the strong coupling check, can be provided.

4.4.2. Content of the Five-Point SCWI.

Drukker-Plefka twist and chiral algebra twist.

Utilizing numerical simplifications further, we can cross-verify our 35 equations against two established subclasses of constraints identified in the literature: the Drukker-Plefka twist condition and the chiral algebra twist conditions.

We begin with the Drukker-Plefka twist condition. By aligning all five internal cross ratios with their corresponding conformal cross ratios, the resulting twisted correlator, which simplifies to a sum of the 22 unknown functions, must be a constant. This can be formally expressed as:

$$\text{DP twist:} \quad \sum_{i=1}^{22} \frac{\partial f_i}{\partial u_a}(u_1, \ldots, u_5) = 0 \;\text{ or }\; \sum_{i=1}^{22} \frac{\partial f_i}{\partial z_a}(z_1, \ldots, w) = 0, \tag{4.80}$$

where $z_a \in (z_1, \ldots, w)$. This constraint holds for each of the five cross ratios, thereby yielding five equations concerning the five-point functions.
Indeed, we demonstrate that five of our 35 equations are equivalent to the conditions imposed by the Drukker-Plefka twist. By taking linear combinations of our 35 numerical equations, we obtain exactly five equations of the Drukker-Plefka twist type, in addition to 30 independent, slightly modified equations. Consequently, our set of 35 equations encompasses the five independent equations characteristic of the Drukker-Plefka twist condition.

The chiral algebra twist condition must be addressed differently. It is only applicable when all points are positioned on a plane, which, unlike four-point functions, cannot be achieved using a standard conformal frame for five-point functions. The five-point configuration attainable through superconformal transformations is represented as follows:

$$\begin{gathered} X_1^{A\dot{A}} \to \begin{pmatrix} z_1 & 0 \\ 0 & \bar{z}_1 \end{pmatrix},\; X_2^{A\dot{A}} \to 0,\; X_3^{A\dot{A}} \to \infty, \\ X_4^{A\dot{A}} \to \mathbb{1},\; X_5^{A\dot{A}} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} z_2 - w(z_2 - \bar{z}_2) & i\sqrt{w-1}\sqrt{w}(z_2 - \bar{z}_2) \\ i\sqrt{w-1}\sqrt{w}(z_2 - \bar{z}_2) & \bar{z}_2 + w(z_2 - \bar{z}_2) \end{pmatrix} \end{gathered} \tag{4.81}$$

Thus, inserting the five operators in a planar configuration corresponds to nullifying the fifth cross ratio, denoted by w.
Indeed, when examining the graphical interpretation of these cross ratios, where each pair $(z_i, \bar{z}_i)$ defines a plane and w signifies the angle between these planes [117], the condition $w = 0$ precisely aligns the two planes represented by the remaining

four cross ratios, thus resulting in a single plane.

The chiral algebra twist then acts on this restricted five-point correlator as follows:

$$\partial_{z_1} G^{22222}(\{z_i\}, \{\mathrm{y}_i\})\Big|_{y_1 \to z_1} = 0 \tag{4.82}$$

where $\{z_i\} = \{z_1, \bar{z}_1; z_2, \bar{z}_2\}$ and similarly for the internal cross ratios. Analogous equations can be derived for twisting with respect to the other cross ratios. Numerically, we can demonstrate that these equations are included among our derived equations (when the fifth cross ratio w is set to zero). However, our equations extend beyond these two subclasses, providing stronger constraints.

Given that the chiral algebra constraint does not yield a general, everywhere-valid constraint for the five-point function, we cannot substitute some of our equations with those twists. Thus, the intermediate result for the 35 equations is as follows:

- $5\times$ DP type: $\sum_{i=1}^{22} \frac{\partial f_i}{\partial u_a}(u_1, \ldots, u_5) = 0$
- 30 additional equations.

Simplification of the SCWI and state-of-the-art.

However, by carefully analysing the 35 equations numerically and examining their resulting behaviour in depth, a more concise expression for some of the equations has been derived. The full set of 35 equations can be shown to be equivalent to:

$$\text{5 eqns} : \quad \sum_{i=1}^{22} \frac{\partial f_i}{\partial u_a}(u_1, \ldots, u_5) = 0 \text{ for each } a = 1, \ldots, 5 \tag{4.83}$$

$$\text{10 eqns} : \quad u_a \frac{\partial}{\partial u_a} G_b = u_b \frac{\partial}{\partial u_b} G_a \quad \forall a \neq b = 1, \ldots, 5 \tag{4.84}$$

20 eqns : which have thus far not been simplified.

Here, G_i represents a fixed linear combination of the unknown functions $f_i(\{u\})$: $G_i = \sum c_j f_j(\{u\})$ for $j \in \{1, \ldots, 22\}$, where the coefficients c_j are integer values. This form of the constraints, 4.84, is present for both sets of variables. For the cross-ratios being defined as $\{u_i\}$, we then in particular have:

$$G_5 = 2f_4 + f_5 + f_6 + f_8 + f_{11} + f_{14} + f_{18} + f_{20} + f_{21} + f_{22} \tag{4.85}$$

$$G_4 = f_2 + f_3 + 2f_7 + f_8 + f_{15} + f_{16} + f_{17} + f_{19} + f_{20} + f_{22} \tag{4.86}$$
$$G_3 = f_1 + f_2 + f_5 + 2f_{10} + f_{11} + f_{12} + f_{13} + f_{16} + f_{18} + f_{19} \tag{4.87}$$
$$G_2 = 2f_3 + f_5 + f_7 + f_9 + f_{12} + f_{15} + f_{17} + f_{19} + f_{21} + f_{22} \tag{4.88}$$
$$G_1 = 2f_1 + f_8 + f_9 + f_{10} + f_{11} + f_{12} + f_{13} + f_{14} + f_{15} + f_{16} \tag{4.89}$$

Here, f_i should be understood as depending on the cross ratios, $f_i(\{u\})$, and are the functions associated in the ansatz 4.68. Similarly, $G_a(\{u\})$.

These 15 “simplified” equations 4.83 and 4.84, along with the remaining 20 complex equations, represent the current state of progress.

4.4.3. Studying Subcorrelators

Following a similar approach as discussed for the $\mathfrak{psu}(1,1|2)$-case, these equations can be examined through *subcorrelators*. In particular, we observe that the ansatz for the $\mathcal{O}_{20'}$ five-point function also encodes the structures required to describe the correlator:

$$\langle 11211\rangle = \langle \mathcal{O}_1(1)\mathcal{O}_1(2)\rangle \, \langle \mathcal{O}_1(4)\mathcal{O}_1(5)\rangle \, \langle \mathcal{O}_1(1)\mathcal{O}_1(2)\mathcal{O}_{20'}(3) \, \langle \mathcal{O}_1(4)\mathcal{O}_1(5)\rangle\rangle \, , \tag{4.90}$$

where $\mathcal{O}_1$ denotes the superprimary of the half-BPS multiplet $\mathbb{O}_{p=1}$, transforming in the $\mathbf{6}$ of $\mathfrak{su}(4)_R$. We will refer to this correlator in shorthand notation as $\langle 11211\rangle$, reflecting the charges of the respective operators in the five-point function.

This correlator is represented in terms of six bosonic structures as follows:

$$\begin{aligned}
&\langle 11211\rangle \\
&= \frac{y_{12}^4 y_{34}^2 y_{45}^2 y_{53}^2}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2} f_1(\{u\}) + \frac{y_{45}^4 y_{12}^2 y_{23}^2 y_{31}^2}{x_{45}^4 x_{12}^2 x_{23}^2 x_{31}^2} f_{10}(\{u\}) + \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{45}^2 y_{51}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{45}^2 x_{51}^2} f_{11}(\{u\}) \\
&+ \frac{y_{12}^2 y_{23}^2 y_{35}^2 y_{54}^2 y_{41}^2}{x_{12}^2 x_{23}^2 x_{35}^2 x_{54}^2 x_{41}^2} f_{12}(\{u\}) + \frac{y_{12}^2 y_{24}^2 y_{45}^2 y_{53}^2 y_{31}^2}{x_{12}^2 x_{24}^2 x_{45}^2 x_{53}^2 x_{31}^2} f_{13}(\{u\}) + \frac{y_{12}^2 y_{25}^2 y_{54}^2 y_{43}^2 y_{31}^2}{x_{12}^2 x_{25}^2 x_{54}^2 x_{43}^2 x_{31}^2} f_{16}(\{u\})
\end{aligned} \tag{4.91}$$

For consistency, we retain the same labelling as for the unknown functions in the $\mathcal{O}_{20'}$ five-point function to facilitate comparison with those structures of $\langle 22222\rangle$. When restricting the derived 35 Ward identities to the above six functions (by turning off all the other functions), we are left with 20 independent equations governing the correlator $\langle 11211\rangle$.[5]
Specifically, they are of the form:

[5] Note that when rederiving the SCWI for $\langle 11211\rangle$ from scratch with the discussed method, there are in fact 21 independent equations of reduced complexity.

$$5 \text{ eqns } : \sum_{i=1}^{6} \frac{\partial u_a}{\partial f_u}(u_1, \ldots, u_5) = 0 \text{ for each } a = 1, \ldots, 5 \tag{4.92}$$

$$10 \text{ eqns} : u_a \frac{\partial}{\partial u_a} G_b = u_b \frac{\partial}{\partial u_b} G_a \quad \forall a \neq b = 1, \ldots, 5 \tag{4.93}$$

5 (or 6, respectively) other eqns.

For this simplified correlator, the linear combinations G_a reduce to:

$$\{G_5 = f_{11},\, G_4 = f_{16},\, G_3 = f_{10},\, G_2 = f_{12},\, G_1 = f_1\} \tag{4.94}$$

By eliminating $f_{13}(\{u\})$ using the five Drukker-Plefka twist conditions, all remaining channels are incorporated within the 10 simple equations 4.93.
A general solution to this system is given by:

$$G_a(\{u\}) = u_a \frac{\partial}{\partial u_a} H(\{u\}), \tag{4.95}$$

indicating that all five remaining functions of the cross ratios are, in fact, parametrised by derivatives of a single function of the cross ratios.
The investigation of this single function under the remaining five equations is currently ongoing.

Part III.

Other setups and Outlook

5. 6pt Function on the Wilson line

In this chapter, we apply the techniques developed in previous chapters to investigate the supersymmetric line defect in $\mathcal{N}=4$ Super Yang-Mills (SYM) theory. This study serves as an illustration of the versatility of our methods, demonstrating how they are applicable not only to arbitrary n-point correlation functions in $\mathcal{N}=4$ SYM, but can also be directly adapted to other configurations.

The supersymmetric line defect provides an ideal arena due to its retention of a significant portion of the original superconformal symmetry, specifically from the original $\mathcal{N}=4$ super algebra $\mathfrak{psu}(2,2|4)$. This preserved symmetry structure makes the analysis of constraints imposed by symmetry arguments particularly effective. Moreover, when considering operators inserted on the line defect, the system is effectively described by a one-dimensional conformal field theory (CFT), thereby enabling a simplified approach similar to the one observed in the $\mathfrak{psu}(1,1|2)$ scenario.

A commonly studied instance of the one-dimensional supersymmetric line defect is the Maldacena Wilson line, which is of substantial physical relevance. This line defect preserves many of the desired properties of the original theory: despite a high amount of supersymmetry, making it appealing for our analysis, it also possesses for instance a well-defined holographic dual: a two-dimensional string worldsheet in AdS, which terminates on the Wilson line in the boundary-CFT. Consequently, it holds significant interest in the context of the AdS/CFT correspondence.

We begin in Section 5.1 by reviewing the necessary preliminaries specific to this setup. Subsequently, in Section 3.2, we explicitly derive the superconformal Ward identities for the four-point function of half-BPS multiplets. This calculation not only illustrates the method but also serves as a cross-verification against established results in the literature. Finally, we extend our analysis to the six-point function of half-BPS multiplets in Section 5.3.

5.1. Preliminaries

In this section, we define the Maldacena-Wilson line in Subsection 5.1.1 and examine it from the perspective of the corresponding symmetry algebra in Subsection 5.1.2. Many of the underlying concepts, especially regarding the construction of representations, follow the principles outlined in Section 2.2; therefore, we will provide a concise summary of key elements rather than reproducing each derivation in full detail. In Subsection 5.1.3, we discuss the analytic superspace framework used to describe half-BPS operators associated with the line defect.

5.1.1. The Maldacena Wilson Line

Wilson lines, introduced in gauge theories by Kenneth Wilson [49], measure phase variations of gauge variables parallel transported around closed loops in spacetime. They probe global properties of the gauge group that would not be accessible through correlation functions of local operators. These Wilson lines can be extended supersymmetrically to construct analogues in supersymmetric gauge theories. The supersymmetric line defect, often referred to as the *Maldacena Wilson line*, inserted here as a straight line in $\mathcal{N} = 4$ SYM, is defined as

$$\mathcal{W} \equiv \frac{1}{N}\text{tr}\left(\mathcal{P}\exp\, i\int_{-\infty}^{\infty} \mathrm{d}\tau\, \left(i\dot{x}(\tau)^{\mu}A_{\mu}(x) + |\dot{x}^{\mu}(\tau)|\,\theta\cdot\Phi(x)\right)\right), \tag{5.1}$$

where τ is the one-dimensional coordinate parametrising the Wilson line. Here, $A_{\mu}(x)$ denotes the gauge field, and $\Phi^{I}(x)$ represent the six scalar fields of $\mathcal{N} = 4$ SYM.

The parameter θ^{I} is an $SO(6)_R$-vector satisfying the null-condition $\theta^{I}\theta_{I} = 0$, introduced similarly to the polarization vectors u^{I} to incorporate internal symmetry considerations. N indicates the rank of the gauge group $SU(N)$, and the entire expression is path-ordered.

With this explicit expression for the Maldacena Wilson line defined, we follow a similar spirit as with respect to the Lagrangian description in $\mathcal{N} = 4$ SYM to just abandon it alltogether and focusing exclusively on the underlying symmetry algebra.

5.1.2. Symmetry Group and Representations

The superconformal algebra of $\mathcal{N} = 4$ SYM, $\mathfrak{psu}(2, 2|4)$, consists of the four-dimensional conformal algebra $\mathfrak{su}(2, 2)$, the internal symmetry algebra $\mathfrak{so}(6)_R \cong$

$\mathfrak{su}(4)_R$, and 32 fermionic generators.
The introduction of a $\frac{1}{2}$-BPS supersymmetric line defect in $\mathcal{N} = 4$ SYM results in the breaking of this symmetry as

$$\mathfrak{psu}(2,2|4) \to \mathfrak{osp}(4^\star|4). \tag{5.2}$$

The algebra $\mathfrak{osp}(4^\star|4)$ contains the preserved bosonic subalgebra $\mathfrak{sl}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{sp}(4)$, where $\mathfrak{sl}(2) \cong \mathfrak{so}(2,1)$ corresponds to the one-dimensional conformal symmetry on the line, generated by translations P, dilations D, and special conformal transformations K. The algebra $\mathfrak{su}(2)$ captures the three-dimensional rotational symmetry around the line, generated by $L^\alpha{}_\beta$. Meanwhile, the R-symmetry algebra $\mathfrak{su}(4)_R$ is broken to $\mathfrak{sp}(4)_R \cong \mathfrak{so}(5)_R$. This reduction can be understood in field-theoretic terms, as only one out of the six scalar field couples to the supersymmetric Wilson line, leaving five fields uncoupled resulting in the residual internal symmetry $\mathfrak{so}(5)_R$ [129].
In addition, there are 16 preserved fermionic generators -eight supertranslations and eight superconformal transformations- which together form the algebra $\mathfrak{osp}(4^\star|4)$.

Following the approach outlined in Section 2.2, we construct the possible representations under this symmetry group. Focusing on operators inserted on the defect, we label the multiplets by its respective highest weight states denoted as

$$[a,b]^\Delta_s \tag{5.3}$$

$$\begin{aligned} \text{with }\ [a,b] &: \text{ Dynkin labels of } \mathfrak{so}(5)_R, \\ s &: \text{ transverse spin, associated with } \mathfrak{su}(2), \\ \Delta &: \text{ scaling dimension.} \end{aligned}$$

In this context, we use half-integer spin notation, i.e., $s \in \frac{1}{2}\mathbb{N}$. Additionally, we adopt the convention where $[0,1]$ labels the $\mathbf{5}$-dimensional representation of $\mathfrak{so}(5)$, while $[1,0]$ denotes the $\mathbf{4}$-dimensional representation.

Analogous to the full $\mathfrak{psu}(2,2|4)$ case, we can apply BPS conditions to obtain shortened multiplets. In this work, we focus on half-BPS multiplets, the shortest type, which annihilate under all eight superconformal charges and four supercharges. Following the conventions of [130], we denote the half-BPS multiplets as

$$\mathcal{B}_k \equiv [0,k]^\Delta_0\,, \quad \text{where } \Delta = k. \tag{5.4}$$

We refer to this as a half-BPS multiplet of charge k. Due to unitarity constraints, $\Delta = k$, and these multiplets are protected against quantum corrections.

Specifically, we examine the half-BPS multiplet with the lowest charge, $k = 1$. The full supermultiplet is constructed as

$$\mathcal{B}_1 \, : \, [0,1]^{\Delta=1}_{s=0} \to [1,0]^{\Delta=\frac{3}{2}}_{s=\frac{1}{2}} \to [0,0]^{\Delta=2}_{s=1}. \tag{5.5}$$

This multiplet holds physical significance as it includes the *displacement operator* $\mathcal{D} \sim [0,0]^{\Delta=2}_{s=1}$, which characterises the response to shifts in the defect's position. Accordingly, this supermultiplet is also referred to as the *displacement multiplet.*

5.1.3. Analytic Superspace

Much like $\mathfrak{psu}(2,2|4)$, the reduced algebra $\mathfrak{osp}(4^\star|4)$ also admits an analytic superspace that provides an efficient framework for describing half-BPS multiplets on the line defect. This analytic superspace is derived by isolating the coordinates relevant to the defect and bulk from the original $\mathfrak{psu}(2,2|4)$-superspace.

To review, the $\mathfrak{psu}(2,2|4)$-superspace is given by 2.71:

$$\begin{gathered} X^{A\dot{A}} = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \rho^{\alpha\dot{a}} \\ \bar{\rho}^{a\dot{\alpha}} & y^{a\dot{a}} \end{pmatrix}, \\ \text{with} \quad A = (\alpha|a),\ \dot{A} = (\dot{\alpha}|\dot{a}), \quad \alpha, \dot{\alpha} = 1,2;\ a, \dot{a} = 1,2. \end{gathered} \tag{5.6}$$

The analytic superspace for the reduced algebra $\mathfrak{osp}(4^\star|4)$ can then be constructed by categorising these coordinates according to defect and bulk components. This procedure, detailed in [130] for general defects, leads to a superspace for the supersymmetric line defect that has been explicitly stated in [131]:

$$X^{AB} = \begin{pmatrix} \tau\epsilon^{ab} & \theta^{a\beta} \\ \theta^{b\alpha} & y^{(\alpha\beta)} \end{pmatrix}, \tag{5.7}$$

where τ denotes the spacetime coordinate, following conventional notation in the literature, and the fermionic coordinates are denoted by θ to distinguish them from the original $\mathfrak{psu}(2,2|4)$-fermionic coordinates. Notably, the fermionic coordinates in the upper-right cell are the same as those in the lower-left cell, giving in total four fermionic coordinates that parametrise the four non-trivial fermionic charges relevant to half-BPS multiplets on the Wilson line.

The group $OSP(4^\star|4)$ acts projectively on this analytic superspace as follows [132]:

$$X \overset{g}{\mapsto} (A + BX)\,(CX + D)^{-1} \quad \forall g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in OSP(4^\star|4), \tag{5.8}$$

where this transformation can be derived similarly to the treatment in Section 2.3.

The analytic superspace framework is particularly effective for half-BPS multiplets, allowing these multiplets to be represented by a single, unconstrained superfield, in the same manner as for half-BPS multiplets in $\mathfrak{psu}(2,2|4)$ covered in the previous chapter. Since the multiplet is generated by the successive action of the non-trivial fermionic charges, a general half-BPS multiplet with charge k can be expanded in terms of the corresponding fermionic coordinates as

$$\mathcal{B}_k(X) = \phi\,(\tau, y) + \theta^{a\alpha}\Psi_{a\alpha}\,(\tau, y) + \theta^{a\alpha}\theta^{b\beta}A_{a\alpha;b\beta}(\tau, y) + \dots \tag{5.9}$$

The explicit field expansion of the multiplet is then determined by the differential constraint:

$$\left(\frac{\partial}{\partial X^{AB}}\right)^{k+1}\mathcal{B}_k(X) = 0 \quad \text{w. graded symmetrisation,} \tag{5.10}$$

which is formulated analogously to the approach in Chapter 3. This differential constraint can in fact be applied to any half-BPS multiplet in a superconformal field theory that supports an analytic superspace, as the derivation presented in section 3.3 is based on general considerations.

In particular, for the displacement multiplet, we impose the following differential constraint:

$$\left(\frac{\partial}{\partial X^{AB}}\right)^{2}\mathcal{B}_1(X) = 0 \quad \text{w. graded symmetrisation.} \tag{5.11}$$

Taking into account the inherited graded symmetrisation properties, the most relevant constraints are of the form:

$$\left.\frac{\partial}{\partial y^{\alpha\beta}}\frac{\partial}{\partial y^{\gamma\delta}}\right|_{(\alpha\gamma\beta\delta)}\mathcal{B}_1(X) = 0, \tag{5.12}$$

$$\left.\frac{\partial}{\partial \theta^{a\alpha}}\frac{\partial}{\partial y^{\beta\gamma}}\right|_{(\alpha\beta\gamma)}\mathcal{B}_1(X) = 0, \tag{5.13}$$

$$\left(\frac{1}{2}\epsilon^{ab}\frac{\partial}{\partial \theta^{a\alpha}}\frac{\partial}{\partial \theta^{b\beta}} - \frac{\partial}{\partial \tau}\frac{\partial}{\partial y^{\alpha\beta}}\right)\mathcal{B}_1(X) = 0. \tag{5.14}$$

Solving these constraints, as shown in Sections 3.2 and 3.3, yields the complete field expansion of the displacement multiplet in analytic superspace, also derived in [132]:

$$\begin{aligned}\mathcal{B}_1(X) = &\left(1 - \frac{1}{2}\theta^{a\alpha}\theta^{b\beta}\epsilon_{ab}\frac{\partial}{\partial\tau}\frac{\partial}{\partial y^{(\alpha\beta)}} + \frac{1}{4}\theta^4\frac{\partial^2}{\partial\tau^2}\det\left(\frac{\partial}{\partial y^{(\alpha\beta)}}\right)\right)\phi(\tau,y)\\ &+\theta^{a\alpha}\left(1-\frac{1}{3}\theta^{b\beta}\theta^{c\gamma}\epsilon_{bc}\frac{\partial}{\partial\tau}\frac{\partial}{\partial y^{(\beta\gamma)}}\right)\Psi_{a\alpha}(\tau,y)\\ &+\epsilon_{\alpha\beta}\theta^{a\alpha}\theta^{b\beta}\mathcal{D}_{(ab)}(\tau,y)+\ldots.\end{aligned} \tag{5.15}$$

Here, the field $\phi(\tau, y)$ represents the $[0,1]_{s=0}^{\Delta=1}$ multiplet component (see constraint 5.12), $\Psi_{a\alpha}(\tau, y)$ corresponds to the field of $[1,0]_{s=\frac{1}{2}}^{\Delta=\frac{3}{2}}$ (see constraint 5.13), and $\mathcal{D}_{(ab)}(\tau, y)$ expresses the displacement operator $[0,0]_{s=1}^{\Delta=2}$.

To initiate our analysis, we restrict to terms up to $\mathcal{O}(\theta^2)$, resulting in the reduced field expansion:

$$\begin{aligned}&\mathcal{B}_1(X)\\ &=\left(1-\frac{1}{2}\theta^{a\alpha}\theta^{b\beta}\epsilon_{ab}\frac{\partial}{\partial\tau}\frac{\partial}{\partial y^{(\alpha\beta)}}\right)\phi(\tau,y)+\theta^{a\alpha}\Psi_{a\alpha}(\tau,y)+\epsilon_{\alpha\beta}\theta^{a\alpha}\theta^{b\beta}\mathcal{D}_{(ab)}(\tau,y)+\ldots\\ &\equiv\left(1-\frac{1}{2}(\theta^2)^{\alpha\beta}\frac{\partial}{\partial\tau}\frac{\partial}{\partial y^{(\alpha\beta)}}\right)\phi(\tau,y)+\theta^{a\alpha}\Psi_{a\alpha}(\tau,y)+\epsilon_{\alpha\beta}\theta^{a\alpha}\theta^{b\beta}\mathcal{D}_{(ab)}(\tau,y)+\ldots\end{aligned} \tag{5.16}$$

where we have introduced the shorthand notation $(\theta^2)^{\alpha\beta} \equiv \epsilon_{ab}\theta^{a\alpha}\theta^{b\beta}$.

Observe that the superprimary field reappears at higher fermionic orders in the expansion, echoing the behaviour seen in the construction of superconformal Ward identities (SCWI) in Chapter 3. This recurrence establishes critical constraints in the SCWI.

5.1.4. Kinematics

With the expression for half-BPS multiplets on analytic superspace established, we can now proceed to examine their correlation functions.

Starting with the two- and three-point functions, superconformal symmetry entirely determines their forms, given by [132]

$$\langle\mathcal{B}_k(X_1)\mathcal{B}_l(X_2)\rangle = \delta_{kl}\,\hat{d}_{12}^k, \tag{5.17}$$

$$\langle \mathcal{B}_{k_1}(X_1)\mathcal{B}_{k_2}(X_2)\mathcal{B}_{k_3}(X_3)\rangle = \lambda_{123}\, \hat{d}_{12}^{\frac{k_1+k_2-k_3}{2}}\, \hat{d}_{13}^{\frac{k_1+k_3-k_2}{2}}\, \hat{d}_{23}^{\frac{k_2+k_3-k_1}{2}}, \tag{5.18}$$

where λ_{123} is the relevant three-point coefficient.
Here, the term $\hat{d}_{ij}$ denotes the superpropagator, defined in the standard way as

$$\hat{d}_{ij} = \text{sdet}\left(X_{ij}^{AB}\right) = \frac{\det\left(y_{ij}^{(\alpha\beta)} + \tau_{ij}^{-1}(\theta_{ij}^2)^{(\alpha\beta)}\right)}{\tau_{ij}^2}. \tag{5.19}$$

The superdeterminant is defined according to the conventions given in A.1, with the adjustment of identical off-diagonal elements, and with the spacetime coordinate restricted to one dimension.

The four-point correlator of half-BPS multiplets presents the lowest-point function with non-trivial dynamics, as it is not fully constrained by superconformal symmetry anymore. The superconformal Ward identities impose essential constraints on this four-point function, as derived in [130], which we will review here.

The four-point function's dependence compared to the $\mathfrak{psu}(2,2|4)$-case is reduced to a single conformal cross ratio, underscoring the notable simplifications relative to the $\mathfrak{psu}(2,2|4)$ case. This single cross ratio is defined as

$$\chi = \frac{\tau_{12}\tau_{34}}{\tau_{13}\tau_{24}}. \tag{5.20}$$

Additionally, the two R-symmetry cross ratios can be defined in line with the standard four-point cross ratios:

$$\zeta_1\zeta_2 = \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}, \quad (1-\zeta_1)(1-\zeta_2) = \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}, \tag{5.21}$$

where $y_{ij}^2 = \det y_{ij}$.

Then, superconformal symmetry restricts the four-point correlation function to the form

$$\begin{aligned} \mathcal{G}_{\{k_1,k_2,k_3,k_4\}} =& \langle \mathcal{B}_{k_1}(X_1)\mathcal{B}_{k_2}(X_2)\mathcal{B}_{k_3}(X_3)\mathcal{B}_{k_4}(X_4)\rangle \\ =& \mathcal{K}_{\{k_1,k_2,k_3,k_4\}}\mathcal{A}_{\{k_1,k_2,k_3,k_4\}}(\chi,\zeta_1,\zeta_2), \end{aligned} \tag{5.22}$$

where the expression is proportional to a single function, $\mathcal{A}_{\{k_1,k_2,k_3,k_4\}}(\chi,\zeta_1,\zeta_2)$, of the conformal and internal cross ratio. Here, $\mathcal{K}_{\{k_1,k_2,k_3,k_4\}}$ denotes an appropriate

bosonic prefactor.
The superconformal Ward identities can be written in the form [130]

$$\left(\partial_{\zeta_1}\mathcal{A} + \frac{1}{2}\partial_\chi\mathcal{A}\right)\Big|_{\zeta_1=\chi} = 0, \quad \left(\partial_{\zeta_2}\mathcal{A} + \frac{1}{2}\partial_\chi\mathcal{A}\right)\Big|_{\zeta_2=\chi} = 0. \tag{5.23}$$

Briefly, these identities follow from a construction similar to the superconformal Ward identities (SCWI) for the four-point function of half-BPS multiplets in $\mathfrak{psu}(2,2|4)$. We apply supersymmetry transformations to turn off all fermionic coordinates, allowing us to construct the bosonic invariants. By supersymmetrising these bosonic invariants, we obtain the full superconformal invariants, bypassing the need to consider nilpotent invariants. Ensuring analyticity in the internal coordinates then yields the equations above.

In the next section, we will apply our strategy to rederive these results for the displacement multiplet. With the multiplet field expansion specified in Equation 5.16, we proceed directly to the correlation function analysis. As the steps follow closely the previously discussed cases, we will focus on key intermediate results only.

5.2. SCWI of the Four-Point Function

Consider the four-point function of displacement multiplets:

$$\mathcal{G}_{1111} = \langle \mathcal{B}_1(X_1)\mathcal{B}_1(X_2)\mathcal{B}_1(X_3)\mathcal{B}_1(X_4)\rangle \,. \tag{5.24}$$

By inserting the multiplet field expansion from 5.16, we expand the correlator in terms of fermionic coordinates as follows:

$$\begin{aligned}
&\langle \mathcal{B}_1(X_1)\mathcal{B}_1(X_2)\mathcal{B}_1(X_3)\mathcal{B}_1(X_4)\rangle \\
&= \langle \phi(\tau_1,y_1)\phi(\tau_2,y_2)\phi(\tau_3,y_3)\phi(\tau_4,y_4)\rangle \\
&\quad - \frac{1}{2}\sum_{i=1}^{4}(\theta_i^2)^{\alpha\beta}\frac{\partial}{\partial\tau_i}\frac{\partial}{\partial y_i^{(\alpha\beta)}}\,\langle \phi(\tau_1,y_1)\phi(\tau_2,y_2)\phi(\tau_3,y_3)\phi(\tau_4,y_4)\rangle \\
&\quad + \sum_{i=1}^{4}\sum_{j=i+1}^{4}\theta_i^{a\alpha}\theta_j^{b\beta}\,\langle \Psi_{a\alpha}(\tau_i,y_i)\Psi_{b\beta}(\tau_j,y_j)\phi(\tau_k,y_k)\phi(\tau_l,y_l)\rangle \\
&\quad + \ldots \quad \text{(higher orders)}.
\end{aligned} \tag{5.25}$$

Note that up to this order in θ, there is no correlator involving the displacement operator, as a four-point function of three superprimaries and one displacement operator is forbidden by symmetry constraints associated with transverse spin.

Imposing supersymmetry invariance of this correlator in analytic superspace amounts to the equation

$$0 = \sum_{i=1}^{4}\frac{\partial}{\partial\theta_i^{c\gamma}}\,\langle \mathcal{B}_1(X_1)\mathcal{B}_1(X_2)\mathcal{B}_1(X_3)\mathcal{B}_1(X_4)\rangle \,. \tag{5.26}$$

In this setup, we only have a single equation, as only one type of fermionic coordinate appears.

Examining this constraint at the first non-trivial order, $\mathcal{O}(\theta^2)$, we derive the following condition from supersymmetry:

$$\begin{aligned}
0 = &-\sum_{n=1}^{4}\epsilon_{ca}\theta_n^{a\alpha}\frac{\partial}{\partial\tau_n}\frac{\partial}{\partial y_n^{\alpha\gamma}}\,\langle \phi(1)\phi(2)\phi(3)\phi(4)\rangle \\
&+ \sum_{n=1}^{4}\sum_{j>n}\theta_n^{a\alpha}\,\langle \Psi_{c\gamma}(j)\Psi_{a\alpha}(n)\phi(k)\phi(l)\rangle \\
&- \sum_{n=1}^{4}\sum_{j<n}\theta_n^{a\alpha}\,\langle \Psi_{a\alpha}(n)\Psi_{c\gamma}(j)\phi(k)\phi(l)\rangle \,,
\end{aligned} \tag{5.27}$$

where we introduced the shortened notation $\phi(i) = \phi(\tau_i, y_i)$ for the fields.

Since this equation must hold for all $\theta_n^{a\alpha}$, we obtain four distinct constraints:

$$\begin{aligned} 0 = &- \epsilon_{ca} \frac{\partial}{\partial \tau_1} \frac{\partial}{\partial y_1^{\alpha\gamma}} \langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle \\ &- \langle \Psi_{a\alpha}(1)\Psi_{c\gamma}(2)\phi(3)\phi(4) \rangle \\ &- \langle \Psi_{a\alpha}(1)\phi(2)\Psi_{c\gamma}(3)\phi(4) \rangle \\ &- \langle \Psi_{a\alpha}(1)\phi(2)\phi(3)\Psi_{c\gamma}(4) \rangle , \end{aligned} \tag{5.28}$$

and similarly for $n = 2, 3, 4$.

These constraints exhaust the entire fermionic sector of the algebra $\mathfrak{osp}(4^\star|4)$. Consequently, we can now turn our attention to the bosonic symmetries. To ensure covariance under the bosonic subgroup $SO(2,1) \times SO(3) \times SO(5)_R$, each correlator in the expansion is expressed in terms of a set of independent *structures* that encode the correct transformation properties and conformal weight, multiplied by an unknown function of the conformal cross ratio:

$$\chi = \frac{x_{12}x_{34}}{x_{13}x_{24}}, \quad \text{where } x_{ij} \equiv x_i - x_j. \tag{5.29}$$

Starting with the R-symmetry subgroup, the number of independent structures per correlator can be determined by counting the singlets in the respective tensor products. Specifically, we find:

$$\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle \, : \, \mathbf{5}^{\otimes_4} \supset 3 \text{ singlets/structures}, \tag{5.30}$$

$$\langle \Psi(i)\Psi(j)\phi(k)\phi(l) \rangle \, : \, \mathbf{4}^{\otimes_2} \otimes \mathbf{5}^{\otimes_2} \supset 2 \text{ singlets/structures}. \tag{5.31}$$

For the superprimary correlator, the structures can be constructed in association with the Wick contractions in the free field theory. We express the correlator as

$$\langle \phi(1)\phi(2)\phi(3)\phi(4) \rangle = y_{12}^2 y_{34}^2 F_1(\{\tau\}) + y_{13}^2 y_{24}^2 F_2(\{\tau\}) + y_{14}^2 y_{23}^2 F_3(\{\tau\}), \tag{5.32}$$

where all spacetime dependence is contained in the functions $F_i(\{\tau\})$ for now.
For the descendent correlators, we can form either a *short* or *2-point structure* by contracting the two tensorial operators or a full *intrinsic 4-point tensor structure* that involves all four points. Incorporating both possibilities, and using $k_\phi = 1$ and $k_\Psi = \frac{1}{2}$, we express the descendent correlators, following the construction principles in 3.4, as

$$\langle \Psi_{a\alpha}(i)\Psi_{b\beta}(j)\phi(k)\phi(l) \rangle = y_{kl}^2 y_{ij}^2 (y_{ij}^{-1})_{\alpha\beta} G_{ij,1}(\{\tau\}) + y_{il}^2 y_{jk}^2 (y_{il}^{-1} y_{lk} y_{kj}^{-1})_{\alpha\beta} G_{ij,2}(\{\tau\}), \tag{5.33}$$

where linear independence of the structures is ensured. The function $G_{ij,m}(\{\tau\})$ multiplies the k-th R-symmetry structure of the correlator with $\Psi_{a\alpha}$ inserted at (τ_i, y_i) and (τ_j, y_j).
We ensure analyticity in the y-variables by specifying all y-dependence explicitly within the structures, avoiding arbitrary functions of internal cross ratios.

The spacetime-dependent functions $F_i(\tau)$ and $G_i(\tau)$ are also expressible in terms of structures with the correct transformation properties and weights under the remaining $SO(2,1) \times SO(3)$, and are multiplied by functions of the spacetime cross ratio alone. In the one-dimensional spacetime, there is only one independent structure over the space of functions of the cross ratio per correlator. Given $\Delta_\phi = 1$ and $\Delta_\Psi = \frac{3}{2}$, we set

$$\tilde{F}_1(\{\tau\}) = \frac{1}{\tau_{12}^2\tau_{34}^2} f_1(\chi), \quad \tilde{F}_2(\{\tau\}) = \frac{1}{\tau_{13}^2\tau_{24}^2} f_2(\chi), \quad \tilde{F}_3(\{\tau\}) = \frac{1}{\tau_{14}^2\tau_{23}^2} f_3(\chi), \tag{5.34}$$

$$G_{ij,m}(\{\tau\}) = \frac{\epsilon_{ab}}{\tau_{ij}^3\tau_{kl}^2} g_{ij,m}(\chi), \tag{5.35}$$

where ϵ_{ab} ensures invariance under $SO(3)$.
Recall, that the full spacetime dependence was absorbed into the functions $F_i(\{\tau\})$, $G_i(\{\tau\})$, therefore in particular the spatial derivative of the superprimary correlator. We thus had defined

$$F_i(\{\tau\}) = \epsilon_{ab} \frac{\partial}{\partial\tau} \tilde{F}_i(\{\tau\}), \tag{5.36}$$

with $\tilde{F}_i(\{\tau\})$ given as above.

Next, we insert these bosonic expressions into the supersymmetry constraints, proceeding in two steps. First, we substitute the internal dependence, yielding equations involving only the spacetime components. Then, we introduce the spacetime structures, resulting in equations for the unknown functions alone, exactly as outlined in previous chapters. The results can be summarized as follows:

- All descendent functions $g_{ij,m}(\chi)$ are determined by the superprimary functions, as expected.
- Additionally, we obtain the following two relations for the superprimary functions $f_i(\chi)$:

$$0 = \frac{1}{\chi} f_1'(\chi) + f_2'(\chi), \quad 0 = \frac{-1+\chi}{\chi} f_1'(\chi) + f_3'(\chi). \tag{5.37}$$

These can be shown to be equivalent to the Ward identities:

$$\left(\frac{1}{2}\partial_\chi + \partial_{\xi_1}\right) \mathcal{A}_{1111}(\chi, \zeta_1, \zeta_2)\Big|_{\xi_1 \to \chi_1} = 0, \tag{5.38}$$

$$\left(\frac{1}{2}\partial_\chi + \partial_{\xi_2}\right) \mathcal{A}_{1111}(\chi, \zeta_1, \zeta_2)\Big|_{\xi_1 \to \chi_2} = 0. \tag{5.39}$$

5.3. SCWI for the Six-Point Function

In this section we derive, in a similar manner as before, the superconformal Ward identities for the six-point function of displacement multiplets:

$$\mathcal{G}_{111111} = \langle \mathcal{B}_1(X_1)\mathcal{B}_1(X_2)\mathcal{B}_1(X_3)\mathcal{B}_1(X_4)\mathcal{B}_1(X_5)\mathcal{B}_1(X_6)\rangle \,. \tag{5.40}$$

For the six-point function, it is no longer feasible to use a frame where all fermionic coordinates are turned off, which makes previous techniques for obtaining the superconformal Ward identities, like the one outlined four the four-point function, particularly challenging.

However, we note that in [129], the Ward identities for general n-point functions of half-BPS multiplets on the supersymmetric line defect have been conjectured to take the form

$$\sum_{k=1}^{n-3} \left(\frac{1}{2}\partial_{\chi_k} + \alpha_k \partial_{r_k} - (1-\alpha_k)\partial_{s_k} \right) \mathcal{A}_{k_1 \dots k_n} \Bigg|_{\substack{r_i \to \alpha_i \chi_i \\ s_i \to (1-\alpha_i)(1-\chi_i) \\ t_{ij} \to (\alpha_i - \alpha_j)(\chi_i - \chi_j)}} = 0, \tag{5.41}$$

for any arbitrary $\alpha_k \in \mathbb{R}$.

In the specific case of the $n = 6$ point function, the cross ratios χ_i, r_i, s_i, t_{ij} refer to the conformal and internal cross ratios. Following the conventions of [129], they can be defined as

$$\chi_1 = \frac{\tau_{12}\tau_{56}}{\tau_{15}\tau_{26}}, \quad \chi_2 = \frac{\tau_{13}\tau_{56}}{\tau_{15}\tau_{36}}, \quad \chi_3 = \frac{\tau_{14}\tau_{56}}{\tau_{15}\tau_{46}}, \tag{5.42}$$

$$\begin{aligned} r_1 &= \frac{y_{12}^2 y_{56}^2}{y_{15}^2 y_{26}^2}, \quad r_2 = \frac{y_{13}^2 y_{56}^2}{y_{15}^2 y_{36}^2}, \quad r_3 = \frac{y_{14}^2 y_{56}^2}{y_{15}^2 y_{46}^2}, \\ s_1 &= \frac{y_{16}^2 y_{25}^2}{y_{15}^2 y_{26}^2}, \quad s_2 = \frac{y_{16}^2 y_{35}^2}{y_{15}^2 y_{36}^2}, \quad s_3 = \frac{y_{16}^2 y_{45}^2}{y_{15}^2 y_{46}^2}, \\ t_{12} &= \frac{y_{16}^2 y_{23}^2 y_{56}^2}{y_{15}^2 y_{26}^2 y_{36}^2}, \quad t_{13} = \frac{y_{16}^2 y_{24}^2 y_{56}^2}{y_{15}^2 y_{26}^2 y_{46}^2}, \quad t_{23} = \frac{y_{16}^2 y_{34}^2 y_{56}^2}{y_{15}^2 y_{36}^2 y_{46}^2}. \end{aligned} \tag{5.43}$$

The conjectured Ward identities in Eq. 5.41 were extended in [118] to the form

$$0 = \lim_{\alpha_{i^\star}\to 1}\left(\frac{1}{2}\partial_{\chi_{i^\star}} + \alpha_{i^\star}\partial_{r_{i^\star}} - (1-\alpha_{i^\star})\partial_{s_{i^\star}} + \sum_j (\alpha_{i^\star} - \alpha_j)\partial_{t_{i^\star j}}\right)\mathcal{A}_{k_1\ldots k_n}\Bigg|_{\substack{r_i\to\alpha_i\chi_i \\ s_i\to(1-\alpha_i)(1-\chi_i) \\ t_{ij}\to(\alpha_i-\alpha_j)(\chi_i-\chi_j)}} \tag{5.44}$$

The goal of this section is to derive the superconformal Ward identities using the method developed throughout this thesis. This approach provides a systematic and reliable way to obtain the SCWI and can thus serve as a cross-check for the proposed Ward identities.

Inserting the multiplet field expansion in (5.16) into the six-point function

$$\mathcal{G}_{111111} = \langle \mathcal{B}_1(X_1)\mathcal{B}_1(X_2)\mathcal{B}_1(X_3)\mathcal{B}_1(X_4)\mathcal{B}_1(X_5)\mathcal{B}_1(X_6)\rangle\,, \tag{5.45}$$

and enforcing supersymmetry invariance with

$$0 = \sum_{i=1}^{6}\frac{\partial}{\partial\theta_i^{a\alpha}}\mathcal{G}_{111111} \tag{5.46}$$

leads to six constraints analogous to those derived previously. Each constraint takes the form:

$$\begin{aligned} 0 = &-\epsilon_{ca}\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial y_1^{(\alpha\gamma)}}\langle\phi(1)\phi(2)\phi(3)\phi(4)\phi(5)\phi(6)\rangle \\ &-\langle\Psi_{a\alpha}(1)\Psi_{c\gamma}(2)\phi(3)\phi(4)\phi(5)\phi(6)\rangle \\ &-\langle\Psi_{a\alpha}(1)\phi(2)\Psi_{c\gamma}(3)\phi(4)\phi(5)\phi(6)\rangle \\ &-\langle\Psi_{a\alpha}(1)\phi(2)\phi(3)\Psi_{c\gamma}(4)\phi(5)\phi(6)\rangle \\ &-\langle\Psi_{a\alpha}(1)\phi(2)\phi(3)\phi(4)\Psi_{c\gamma}(5)\phi(6)\rangle \\ &-\langle\Psi_{a\alpha}(1)\phi(2)\phi(3)\phi(4)\phi(5)\Psi_{c\gamma}(6)\rangle \end{aligned} \tag{5.47}$$

and similarly for $n = 2, 3, 4, 5, 6$, using arguments analogous to those in Section 5.2.

Turning our attention to the bosonic symmetries, with respect to R-symmetry, we determine the number of independent structures per correlator as follows:

$$\langle\phi(1)\phi(2)\phi(3)\phi(4)\phi(5)\phi(6)\rangle : \quad \mathbf{5}^{\otimes 6} \supset 15 \text{ singlets/structures}, \tag{5.48}$$

$$\langle\Psi(i)\Psi(j)\phi(k)\phi(l)\phi(m)\phi(n)\rangle : \quad \mathbf{4}^{\otimes 2}\otimes\mathbf{5}^{\otimes 4} \supset 10 \text{ singlets/structures}. \tag{5.49}$$

As we have a one-dimensional CFT, we find one independent scalar structure per correlator, constructed as before.

We can again express the superprimary correlator in terms of Wick contractions:

$$
\begin{aligned}
&\langle\phi(1)\phi(2)\phi(3)\phi(4)\phi(5)\phi(6)\rangle \\
=&\frac{y_{12}^2y_{56}^2y_{34}^2}{x_{12}^2x_{56}^2x_{34}^2}f_1(\{\chi_i\})+\frac{y_{12}^2y_{35}^2y_{46}^2}{x_{12}^2x_{35}^2x_{46}^2}f_2(\{\chi_i\})+\frac{y_{12}^2y_{36}^2y_{45}^2}{x_{12}^2x_{36}^2x_{45}^2}f_3(\{\chi_i\})+\frac{y_{13}^2y_{24}^2y_{56}^2}{x_{13}^2x_{24}^2x_{56}^2}f_4(\{\chi_i\}) \\
+&\frac{y_{13}^2y_{25}^2y_{46}^2}{x_{13}^2x_{25}^2x_{46}^2}f_5(\{\chi_i\})+\frac{y_{13}^2y_{26}^2y_{45}^2}{x_{13}^2x_{26}^2x_{45}^2}f_6(\{\chi_i\})+\frac{y_{14}^2y_{23}^2y_{56}^2}{x_{14}^2x_{23}^2x_{56}^2}f_7(\{\chi_i\})+\frac{y_{14}^2y_{25}^2y_{36}^2}{x_{14}^2x_{25}^2x_{36}^2}f_8(\{\chi_i\}) \\
+&\frac{y_{14}^2y_{26}^2y_{35}^2}{x_{14}^2x_{26}^2x_{35}^2}f_9(\{\chi_i\})+\frac{y_{15}^2y_{23}^2y_{46}^2}{x_{15}^2x_{23}^2x_{46}^2}f_{10}(\{\chi_i\})+\frac{y_{15}^2y_{24}^2y_{36}^2}{x_{15}^2x_{24}^2x_{36}^2}f_{11}(\{\chi_i\}) \\
+&\frac{y_{15}^2y_{26}^2y_{34}^2}{x_{15}^2x_{26}^2x_{34}^2}f_{12}(\{\chi_i\})+\frac{y_{16}^2y_{23}^2y_{45}^2}{x_{16}^2x_{23}^2x_{45}^2}f_{13}(\{\chi_i\})+\frac{y_{16}^2y_{24}^2y_{35}^2}{x_{16}^2x_{24}^2x_{35}^2}f_{14}(\{\chi_i\}) \\
+&\frac{y_{16}^2y_{25}^2y_{34}^2}{x_{16}^2x_{25}^2x_{34}^2}f_{15}(\{\chi_i\})
\end{aligned}
\tag{5.50}
$$

Following the same arguments as before, the descendent correlators can be represented via R-symmetry tensorial structures of the type

$$
\begin{aligned}
&\langle\Psi_{a\alpha}(i)\Psi_{b\beta}(j)\phi(k)\phi(l)\phi(m)\phi(n)\rangle: \\
&y_{kl}^2y_{mn}^2y_{ij}^2(y_{ij}^{-1})_{\alpha\beta},\ y_{ik}^2y_{jl}^2y_{mn}^2(y_{ik}^{-1}y_{kl}y_{lj}^{-1})_{\alpha\beta},\ y_{ik}^2y_{jl}^2y_{mn}^2(y_{ik}^{-1}y_{km}y_{mn}^{-1}y_{nl}y_{lj}^{-1})_{\alpha\beta}
\end{aligned}
\tag{5.51}
$$

For each correlator containing $\Psi_{a\alpha}$ at i and j, by considering all permutations of k,l,m,n over the remaining four points and then restricting to linearly independent structures, we obtain a complete expression.
The independent spacetime structure takes the generic form

$$
\langle\Psi_{a\alpha}(i)\Psi_{b\beta}(j)\phi(k)\phi(l)\phi(m)\phi(n)\rangle: \quad \frac{\epsilon_{ab}}{\tau_{ij}^3\tau_{kl}^2\tau_{mn}^2}. \tag{5.52}
$$

Permutations of k,l,m,n within the four points distinct from i,j yield valid structures for expressing this correlator, of which one can be chosen. We denote the associated functions of the cross ratios by $g_{ij,k}(\{\chi_i\})$, where i,j indicate the $\Psi_{a\alpha}(x,y)$ position, while $k=1,\dots,10$ labels the R-symmetry structures.

Inserting the respective correlators into the six supersymmetry constraints and following the solution steps from previous sections yields the following results.

We observe that the descendent correlators are entirely fixed in terms of the superprimary functions; all degrees of freedom are thus captured in the superprimary correlator.

Furthermore, we obtain a system of 26 equations corresponding to the 15 unknown functions $f_i(\{\chi_j\})$ associated with the superprimary correlator. These 26 equations can be separated into 15+11 equations, of which the first 15 are equivalent to:

$$\begin{aligned}\partial_{\chi_i} f_1(\{\chi\}) =& \frac{\chi_2-\chi_3}{-1+\chi_3}\partial_{\chi_i} f_3(\{\chi\}) + \frac{\chi_1(\chi_2-\chi_3)}{(\chi_1-\chi_2)\chi_3}\partial_{\chi_i} f_7(\{\chi\}) - \chi_1\partial_{\chi_i} f_{12}(\{\chi\}) \\ &+ \frac{\chi_1(-\chi_2+\chi_3)}{(-1+\chi_2)\chi_3}\partial_{\chi_i} f_9(\{\chi\}) + \frac{\chi_1(-\chi_2+\chi_3)}{(\chi_1-\chi_3)}\partial_{\chi_i} f_{11}(\{\chi\}) \\ &+ \frac{\chi_1(\chi_2-\chi_3)}{(\chi_1-\chi_2)(-1+\chi_3)}\partial_{\chi_i} f_{13}(\{\chi\}) + \frac{\chi_1(\chi_2-\chi_3)}{(\chi_1-\chi_3)(-1+\chi_2)}\partial_{\chi_i} f_{14}(\{\chi\})\end{aligned} \tag{5.53}$$

$$\begin{aligned}\partial_{\chi_i} f_2(\{\chi\}) =& \frac{(1-\chi_2)}{\chi_3-1}\partial_{\chi_i} f_3(\{\chi\}) + \frac{\chi_1(\chi_2-1)}{\chi_1-\chi_2}\partial_{\chi_i} f_{10}(\{\chi\}) + \frac{\chi_1(\chi_2-1)}{\chi_1-\chi_3}\partial_{\chi_i} f_{11}(\{\chi\}) \\ &- \frac{\chi_1}{\chi_1-\chi_3}\partial_{\chi_i} f_{14}(\{\chi\}) - \frac{\chi_1(\chi_2-1)}{(\chi_3-1)(\chi_1-\chi_2)}\partial_{\chi_i} f_{13}(\{\chi\})\end{aligned} \tag{5.54}$$

$$\begin{aligned}\partial_{\chi_i} f_4(\{\chi\}) =& \frac{(\chi_3-\chi_1)}{\chi_1-1}\partial_{\chi_i} f_5(\{\chi\}) + \frac{\chi_2(\chi_3-\chi_1)}{\chi_3(\chi_1-\chi_2)}\partial_{\chi_i} f_7(\{\chi\}) + \frac{\chi_2(\chi_1-\chi_3)}{(\chi_2-1)\chi_3}\partial_{\chi_i} f_9(\{\chi\}) \\ &+ \frac{\chi_2(\chi_3-\chi_1)}{\chi_1-\chi_2}\partial_{\chi_i} f_{10}(\{\chi\}) + \frac{\chi_2(\chi_1-\chi_3)}{\chi_2-\chi_3}\partial_{\chi_i} f_{12}(\{\chi\}) \\ &+ \frac{\chi_2}{1-\chi_2}\partial_{\chi_i} f_{14}(\{\chi\}) + \frac{\chi_2(\chi_3-\chi_1)}{(\chi_1-1)(\chi_2-\chi_3)}\partial_{\chi_i} f_{15}(\{\chi\})\end{aligned} \tag{5.55}$$

$$\begin{aligned}\partial_{\chi_i} f_6(\{\chi\}) =& \frac{(1-\chi_3)}{\chi_1-1}\partial_{\chi_i} f_5(\{\chi\}) + \frac{(\chi_2-\chi_2\chi_3)}{\chi_1-\chi_2}\partial_{\chi_i} f_{10}(\{\chi\}) + \frac{\chi_2(\chi_3-1)}{\chi_2-\chi_3}\partial_{\chi_i} f_{12}(\{\chi\}) \\ &+ \frac{\chi_2}{\chi_1-\chi_2}\partial_{\chi_i} f_{13}(\{\chi\}) - \frac{\chi_2(\chi_3-1)}{(\chi_1-1)(\chi_2-\chi_3)}\partial_{\chi_i} f_{15}(\{\chi\})\end{aligned} \tag{5.56}$$

$$\begin{aligned}\partial_{\chi_i} f_8(\{\chi\}) =& \frac{(1-\chi_1)}{\chi_2-1}\partial_{\chi_i} f_9(\{\chi\}) + \frac{(\chi_3-\chi_1\chi_3)}{\chi_1-\chi_3}\partial_{\chi_i} f_{11}(\{\chi\}) + \frac{(\chi_3-\chi_1\chi_3)}{\chi_2-\chi_3}\partial_{\chi_i} f_{12}(\{\chi\}) \\ &+ \frac{(\chi_1-1)\chi_3}{(\chi_2-1)(\chi_1-\chi_3)}\partial_{\chi_i} f_{14}(\{\chi\}) + \frac{\chi_3}{\chi_2-\chi_3}\partial_{\chi_i} f_{15}(\{\chi\})\end{aligned} \tag{5.57}$$

where each equation holds separately for each $i = 1, 2, 3$.

These 15 equations can be reformulated in the following compact form:

$$\left(\frac{1}{2}\partial_{\chi_i} + \alpha_i\partial_{r_i} - (1-\alpha_i)\partial_{s_i} + \sum_j(\alpha_i - \alpha_j)\partial_{t_{ij}}\right)\mathcal{A}_{k_1\ldots k_n}\Bigg|_{\substack{r_i\to\alpha_i\chi_i \\ s_i\to(1-\alpha_i)(1-\chi_i) \\ t_{ij}\to(\alpha_i-\alpha_j)(\chi_i-\chi_j)}} = 0, \tag{5.58}$$

where, as before, each $i = 1, 2, 3$ produces a distinct equation valid for any arbitrary $\alpha_i, \alpha_j \in \mathbb{R}$.
These reformulated equations encompass the superconformal Ward identities proposed in previous works [129] and [118] (refer to Eqs. 5.41 and 5.44). However, the set of equations derived here provides a stronger set of constraints.
Furthermore, there are other eleven equations which are subject to further analysis.[1]

[1]These eleven equations are too lengthy to be displayed here. However, a Mathematica notebook including those equations can be provided upon request.

6. Conclusions and Outlook

In this thesis, we developed a method to derive the superconformal Ward identities (SCWI) for higher-point correlation functions of half-BPS multiplets. This method works independently of the number of insertion points and can be applied to half-BPS correlators in any theory admitting an analytic superspace. Our main example, which demonstrates this method, is $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory, where we derived the superconformal Ward identities for the five-point function of stress tensor multiplets.

After introducing the main concepts necessary to understand this procedure in Chapter 2, we proceeded to derive the superconformal Ward identities in Chapter 3. To build complexity step by step, and to facilitate the application of the method to various setups, we began by deriving the Ward identities for five-point functions in the chiral algebra of $\mathfrak{psu}(2,2|4)$, given by $\mathfrak{psu}(1,1|2)$ (see Section 3.2). This one-dimensional setup is significantly simplified compared to the four-dimensional case, yet remains relevant due to its connection to the full four-dimensional case via the chiral algebra map. We then derived the SCWI for the four-point function of stress tensor multiplets in $\mathfrak{psu}(2,2|4)$, which provides a simplification relative to the five-point case due to a reduced number of parameters and unknowns. Additionally, these results can be used as a verification against the well-established expressions in the literature, affirming the validity of our approach. Finally, we derived the SCWI for the five-point functions in $\mathfrak{psu}(2,2|4)$, with particular focus on the stress tensor multiplets.

The various SCWI are derived using the same general method. We begin by examining a differential constraint that the half-BPS multiplets on the respective analytic superspace obey (see Section 3.3 for details). This constraint yields a fermionic expansion for the respective multiplet that can be utilized inside the correlator. We then impose superconformal invariance on this expanded correlator, which becomes a straightforward task in analytic superspace. Technical challenges arise from the bosonic structures, which reflect the transformation behaviour of the correlator under the bosonic parts of the superconformal algebra. These structures have been discussed in Section 3.4. Following this strategy, we derive the SCWI for four-point and five-point functions as shown in Sections 3.5 and 3.6, respectively.

In Chapter 4, we analyse the derived SCWI. For simpler cases such as $\mathfrak{psu}(1,1|2)$ and the four-point function, the equations can be solved straightforwardly, yielding a parametrization of the correlator that manifestly satisfies the respective identities. The five-point function SCWI, however, are more challenging to analyse. Nevertheless, we provide evidence of the correctness of our derived equations and present the first steps taken towards a simplified representation.
In Chapter 5, we further demonstrate the utility of our method by applying it to a different setup: the supersymmetric Wilson line defect in $\mathcal{N} = 4$ SYM. Here, we derived the SCWI for the six-point function of displacement multiplets on the Wilson line.

Outlook

Progress on the SCWI for the five-point function of stress tensor multiplets in $\mathcal{N} = 4$ SYM continues along three main directions. First, we aim to simplify the remaining complex equations similarly to the 15 "nice" equations presented in Section 4.4, potentially allowing a comprehensive exploration of solutions. Some of these solutions can already be examined for related correlators. The correlator $\langle 11211 \rangle$, where $p = 1$ or $p = 2$ refers to the charge of the superprimary of the inserted half-BPS multiplet, may be viewed as part of the full stress-tensor correlator. It is a candidate for which the equations simplify significantly and whose structure may provide insights into the stress-tensor correlator. Another candidate is the next-to-extremal correlator $\langle 22622 \rangle$, currently under investigation. Its superprimary, parametrised by six unknown functions, satisfies 21 equations, with 15 of the "nice" form and the remaining six taking a more straightforward form.
The third direction involves a direct application of the SCWI to superconformal blocks, as illustrated in the $\mathfrak{psu}(1,1|2)$ case. The superconformal blocks satisfy the SCWI regardless of their complexity, offering a direct application of the derived identities. This work is ongoing.
Further studies are also ongoing for the setup of the supersymmetric line defect in $\mathcal{N} = 4$ SYM. Work continues on solving the derived identities for the superprimary correlator. Additionally, higher-order terms in the fermionic expansion are under investigation to better understand the behavior of the descendant correlators.

Part IV.

Appendices

A. Analytic Superspace

A.1. Conventions

In the following section, we will list all the definitions and conventions related to the supermatrix representing analytic superspace,

$$X^{A\dot{A}} = \begin{pmatrix} x^{\alpha\dot{\alpha}} & \rho^{\alpha\dot{a}} \\ \bar{\rho}^{a\dot{\alpha}} & y^{a\dot{a}} \end{pmatrix}$$
$$\text{with } A = (\alpha, a),\ \dot{A} = (\dot{\alpha}, \dot{a}) \text{ and } \alpha, \dot{\alpha} = 1,2;\ a, \dot{a} = 1,2. \tag{A.1}$$

These conventions are the same used in [79].

This matrix is a (4|4)-supermatrix and thus, the supertrace and superdeterminant of this matrix are given by

$$\text{str}(X^{A\dot{A}}) = \text{tr}(x^{\alpha\dot{\alpha}}) - \text{tr}(y^{a\dot{a}}) \tag{A.2}$$

$$\text{sdet}(X^{A\dot{A}}) = \frac{\det\left(x^{\alpha\dot{\alpha}} - \rho^{\alpha\dot{a}} y^{-1}_{\dot{a}a} \bar{\rho}^{a\dot{\alpha}}\right)}{\det(y^{a\dot{a}})} = \frac{\det(x^{\alpha\dot{\alpha}})}{\det\left(y^{a\dot{a}} - \bar{\rho}^{a\dot{\alpha}} y^{-1}_{\dot{\alpha}\alpha} \rho^{\alpha\dot{a}}\right)}. \tag{A.3}$$

The indices are raised and lowered with the anti-symmetric $\mathfrak{su}(2)$-tensors $\epsilon^{\alpha\beta}, \epsilon^{ab}$ as

$$x^{\alpha\dot{\alpha}} = \epsilon^{\alpha\beta} x_{\dot{\beta}\beta} \epsilon^{\dot{\beta}\dot{\alpha}}, \quad y^{a\dot{a}} = \epsilon^{ab} x_{\dot{b}b} \epsilon^{\dot{b}\dot{a}}, \tag{A.4}$$

where $\epsilon^{12} = \epsilon_{12} = 1$.

Thus, we obtain the following identies for the scalar products and inverses

$$x^2 = x^{\alpha\dot{\alpha}} x_{\dot{\alpha}\alpha} = \det x^{\alpha\dot{\alpha}}, \qquad y^2 = y^{a\dot{a}} y_{\dot{a}a} = \det y^{a\dot{a}} \tag{A.5}$$

$$(x^{-1})_{\dot{\alpha}\alpha} = \frac{x_{\dot{\alpha}\alpha}}{x^2}, \qquad (y^{-1})_{\dot{a}a} = \frac{y_{\dot{a}a}}{y^2}. \tag{A.6}$$

Derivatives of the scalar product amount to

$$\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} x^2 = x_{\dot{\alpha}\alpha}, \quad \frac{\partial}{\partial y^{a\dot{a}}} y^2 = y_{\dot{a}a}. \tag{A.7}$$

With the above definitions, we introduce the following short-hand notation for the super propagators

$$\hat{d}_{ij} = \frac{1}{\mathrm{sdet}\left(X_{ij}\right)} = \frac{\hat{y}_{ij}^2}{x_{ij}^2} = \frac{y_{ij}^2}{\hat{x}_{ij}^2}, \quad \text{with} \quad \begin{aligned} \hat{y}^{a\dot{a}} &= y^{a\dot{a}} - \bar{\rho}^{a\dot{\alpha}}(x^{-1})_{\dot{\alpha}\alpha}\rho^{\alpha\dot{a}} \\ \hat{x}^{\alpha\dot{\alpha}} &= y^{\alpha\dot{\alpha}} - \rho^{\alpha\dot{a}}(y^{-1})_{\dot{a}a}\rho^{a\dot{\alpha}}. \end{aligned} \tag{A.8}$$

A.2. Superconformal generators

The form of the superconformal generators acting on analytic superspace can be inferred from the infinitesimal transformation

$$\delta X = \mathcal{V}X = \mathrm{B} + \mathrm{A}\cdot X + X\cdot D + X\cdot C\cdot X, \tag{A.9}$$

where A, B, C, D are the usual entries in $G = \begin{pmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{C} & \mathrm{D} \end{pmatrix} \in SL(4|4;\mathbb{C})$.

The explicit form of the generators in the superalgebra 2.4 can be read off to be [41, 81]:

$$\begin{aligned}
P_{\alpha\dot{\alpha}} =& \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \\
M_{\alpha}{}^{\beta} =& x^{\beta\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + \rho^{\beta\dot{a}}\frac{\partial}{\partial \rho^{\alpha\dot{a}}}, \quad \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} = x^{\beta\dot{\alpha}}\frac{\partial}{\partial x^{\beta\dot{\beta}}} + \bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial \bar{\rho}^{a\dot{\beta}}} \\
D =& x^{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + \frac{1}{2}\left(\rho^{\alpha\dot{a}}\frac{\partial}{\partial \rho^{\alpha\dot{a}}} + \bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}\right) \\
K^{\alpha\dot{\alpha}} =& x^{\beta\dot{\alpha}}x^{\alpha\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}} + x^{\beta\dot{\alpha}}\rho^{\alpha\dot{b}}\frac{\partial}{\partial \rho^{\beta\dot{b}}} + \bar{\rho}^{b\dot{\alpha}}x^{\alpha\dot{\beta}}\frac{\partial}{\partial \bar{\rho}^{b\dot{\beta}}} + \bar{\rho}^{b\dot{\alpha}}\rho^{\alpha\dot{b}}\frac{\partial}{\partial y^{b\dot{b}}}
\end{aligned} \tag{A.10}$$

$$\begin{aligned}
P'_{a\dot{a}} =& \frac{\partial}{\partial y^{a\dot{a}}} \\
R_{a}{}^{b} =& y^{b\dot{a}}\frac{\partial}{\partial y^{a\dot{a}}} + \bar{\rho}^{b\dot{\alpha}}\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}, \quad \bar{R}^{\dot{a}}{}_{\dot{b}} = y^{a\dot{a}}\frac{\partial}{\partial y^{a\dot{b}}} + \rho^{\alpha\dot{a}}\frac{\partial}{\partial \rho^{\alpha\dot{b}}} \\
D' =& y^{a\dot{a}}\frac{\partial}{\partial y^{a\dot{a}}} + \frac{1}{2}\left(\rho^{\alpha\dot{a}}\frac{\partial}{\partial \rho^{\alpha\dot{a}}} + \bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial \bar{\rho}^{a\dot{\alpha}}}\right) \\
K'^{a\dot{a}} =& \rho^{\beta\dot{a}}\bar{\rho}^{a\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}} + \rho^{\beta\dot{a}}y^{a\dot{b}}\frac{\partial}{\partial \rho^{\beta\dot{b}}} + y^{b\dot{a}}\bar{\rho}^{a\dot{\beta}}\frac{\partial}{\partial \bar{\rho}^{b\dot{\beta}}} + y^{b\dot{a}}y^{a\dot{b}}\frac{\partial}{\partial y^{b\dot{b}}}
\end{aligned} \tag{A.11}$$

$$
\begin{aligned}
Q_{\alpha\dot{a}} =& \frac{\partial}{\partial\rho^{\alpha\dot{a}}}, \quad \bar{Q}_{a\dot{\alpha}} = \frac{\partial}{\partial\bar{\rho}^{a\dot{\alpha}}} \\
Q_{\alpha}{}^{a} =& \bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + y^{a\dot{a}}\frac{\partial}{\partial\rho^{\alpha\dot{a}}}, \quad \bar{Q}^{\dot{a}}{}_{\dot{\alpha}} = \rho^{\alpha\dot{a}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}} - y^{a\dot{a}}\frac{\partial}{\partial\bar{\rho}^{a\dot{\alpha}}} \\
S_{a}{}^{\alpha} =& x^{\alpha\dot{\alpha}}\frac{\partial}{\partial\bar{\rho}^{a\dot{\alpha}}} + \rho^{\alpha\dot{a}}\frac{\partial}{\partial y^{a\dot{a}}}, \quad \bar{S}^{\dot{\alpha}}{}_{\dot{a}} = x^{\alpha\dot{\alpha}}\frac{\partial}{\partial\rho^{\alpha\dot{a}}} - \bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial y^{a\dot{a}}} \\
\bar{S}^{\dot{\alpha}a} =& x^{\beta\dot{\alpha}}\bar{\rho}^{a\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}} + x^{\beta\dot{\alpha}}y^{a\dot{b}}\frac{\partial}{\partial\rho^{\beta\dot{b}}} - \bar{\rho}^{b\dot{\alpha}}\bar{\rho}^{a\dot{\beta}}\frac{\partial}{\partial\bar{\rho}^{b\dot{\beta}}} - \bar{\rho}^{b\dot{\alpha}}y^{a\dot{b}}\frac{\partial}{\partial y^{b\dot{b}}} \\
S^{\dot{a}\alpha} =& y^{b\dot{a}}\rho^{\alpha\dot{b}}\frac{\partial}{\partial y^{b\dot{b}}} + y^{b\dot{a}}x^{\alpha\dot{\beta}}\frac{\partial}{\partial\bar{\rho}^{b\dot{\beta}}} - \rho^{\beta\dot{a}}\rho^{\alpha\dot{b}}\frac{\partial}{\partial\rho^{\beta\dot{b}}} - \rho^{\beta\dot{a}}x^{\alpha\dot{b}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}
\end{aligned}
\tag{A.12}
$$

A.3. Constraining Half-BPS Multiplets on Analytic Superspace

The algebra $\mathfrak{psu}(2,2|4)$ is a $\mathbb{Z}_2$-graded algebra, characterized by a decomposition of the form

$$A = A_0 \oplus A_1, \tag{A.13}$$

where the elements of A_0 are referred to as even, while those of A_1 are termed odd.
In the context of the analytic superspace that represents this superalgebra, we have $x^{\alpha\dot{\alpha}}, y^{a\dot{a}} \in A_0$, while $\rho^{\alpha\dot{a}}, \bar{\rho}^{a\dot{\alpha}} \in A_1$.

To translate this grading into single $\mathfrak{su}(2)$ indices α and a, we adopt the conventions outlined in [79], specifically setting $|a| = 1$ and $|\alpha| = 0$. This yields back the original grading:

$$\left|x^{\alpha\dot{\alpha}}\right| = |\alpha| + |\dot{\alpha}| = 0 + 0 = 0, \tag{A.14}$$

$$\left|y^{a\dot{a}}\right| = |a| + |\dot{a}| = 1 + 1 = 0, \tag{A.15}$$

$$\left|\rho^{\alpha\dot{a}}\right| = |\alpha| + |\dot{a}| = 0 + 1 = 1, \tag{A.16}$$

$$\left|\bar{\rho}^{a\dot{\alpha}}\right| = |a| + |\dot{\alpha}| = 1 + 0 = 1. \tag{A.17}$$

To implement the appropriate symmetrisation —specifically, the anti-symmetrisation of $(\alpha, \dot{\alpha})$ and the symmetrisation of (a, a)— as discussed in Section 3.3, we introduce the auxiliary vector

$$\xi^A = \begin{pmatrix} \eta^\alpha \\ \omega^a \end{pmatrix} \tag{A.18}$$

which has the grading $\left|\xi^A\right| = |A|+1$. This vector achieves the desired symmetrization by contracting with the differential operator as follows:

$$\partial \sim \bar{\xi}^{\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} \xi^A. \tag{A.19}$$

A.3.1. Proof of the Differential Constraint for Half-BPS Multiplets

The auxiliary vector can be employed to demonstrate the identity for half-BPS multiplets. This section provides a sketch of the proof for the specific case of the stress tensor multiplet in $\mathfrak{psu}(2,2|4)$, i.e. up to the third order differential constraint. The corresponding constraints for other half-BPS multiplets can be derived in the same manner.

We commence the proof of Eqn. 3.62 by examining the action of the differential operator on the two-point function of the fundamental multiplet, $\mathbb{O}_1(X)$, which is represented by the $[0,1,0]$ representation of $\mathfrak{su}(4)_R$. The two-point function is given by

$$\langle \mathbb{O}_1\left(X_1\right)\mathbb{O}_1\left(X_2\right)\rangle = \frac{1}{|\mathrm{sdet} X_{12}|}. \tag{A.20}$$

Using the identity presented in [79], we obtain

$$\left(\bar{\xi}^{\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} \xi^A\right) \langle \mathbb{O}_1\left(X_1\right)\mathbb{O}_1\left(X_2\right)\rangle = \frac{-\left(-1\right)^{|A|}\left(\bar{\xi}^{\dot{A}} X^{-1}_{\dot{A}A} \xi^A\right)}{|\mathrm{sdet} X|}. \tag{A.21}$$

This result can be generalized to two-point functions of arbitrary charge p as follows:

$$\langle \mathbb{O}_p(X_1)\mathbb{O}_p(X_2)\rangle = \frac{1}{|\mathrm{sdet} X_{12}|^p}, \tag{A.22}$$

which leads to

$$
\begin{aligned}
\left(\bar{\xi}^{\dot{A}}\frac{\partial}{\partial X^{A\dot{A}}}\xi^{A}\right)\frac{1}{|\mathrm{sdet}X|^{p}} =& \left(\bar{\xi}^{\dot{A}}\frac{\partial}{\partial X^{A\dot{A}}}\xi^{A}\right)\left(\frac{1}{|\mathrm{sdet}X|}\cdot\frac{1}{|\mathrm{sdet}X|}\cdots\frac{1}{|\mathrm{sdet}X|}\right)\\
=&\frac{-\left(-1\right)^{|A|}\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|}\cdot\frac{1}{|\mathrm{sdet}X|^{p-1}}+\\
&+\frac{1}{|\mathrm{sdet}X|}\frac{-\left(-1\right)^{|A|}\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|}\frac{1}{|\mathrm{sdet}X|^{p-2}}+\\
&+\cdots+\frac{1}{|\mathrm{sdet}X|^{p-1}}\cdot\frac{-\left(-1\right)^{|A|}\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|}\\
=&\frac{-p\cdot\left(-1\right)^{|A|}\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|^{p}}.
\end{aligned}
\tag{A.23}
$$

We can incorporate the recurring factor $(-1)^{|A|}$ into the definition of the differential operator, yielding

$$
\partial\equiv\left(-1\right)^{|A|}\bar{\xi}^{\dot{A}}\frac{\partial}{\partial X^{A\dot{A}}}\xi^{A}. \tag{A.24}
$$

Thus, the action of this differential operator on the general two-point functions of half-BPS multiplets is expressed as

$$
\partial\frac{1}{|\mathrm{sdet}X|^{p}}=\frac{-p\cdot\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|^{p}}, \tag{A.25}
$$

or, eliminating the auxiliary vector, as

$$
\frac{\partial}{\partial X^{A\dot{A}}}\frac{1}{|\mathrm{sdet}X|^{p}}=\frac{-p\left(-1\right)^{|A|}\cdot\left(X^{-1}_{\dot{A}A}\right)}{|\mathrm{sdet}X|^{p}}. \tag{A.26}
$$

With these results established, we proceed to investigate the second-order differential constraints. This can be computed as follows:

$$
\begin{aligned}
\partial^{2}\frac{1}{|\mathrm{sdet}X|^{p}}=&\partial\left(\partial\frac{1}{|\mathrm{sdet}X|^{p}}\right)=\partial\left(\frac{-p\cdot\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)}{|\mathrm{sdet}X|^{p}}\right)\\
=&-p\cdot\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)\left(\partial\frac{1}{|\mathrm{sdet}X|^{p}}\right)+\frac{1}{|\mathrm{sdet}X|^{p}}\partial\left(-p\cdot\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)\right)\\
=&\frac{p\left(p-1\right)\left(\bar{\xi}^{\dot{A}}X^{-1}_{\dot{A}A}\xi^{A}\right)^{2}}{|\mathrm{sdet}X|^{p}},
\end{aligned}
\tag{A.27}
$$

where we have utilized the second identity from [79] to arrive at the final expression. It is noteworthy that this second-order differential constraint becomes trivial when $p = 1$.

Similarly, the third-order differential constraint can be derived as follows:

$$\partial^3 \frac{1}{|\text{sdet}X|^p} = -p\,(p-1)\,(p-2)\,\frac{\left(\bar{\xi}^{\dot{A}} X^{-1}_{\dot{A}A} \xi^A\right)^3}{|\text{sdet}X|^p}. \tag{A.28}$$

This third-order differential constraint vanishes not only for $p = 1$, as obvious from the second-order constraint, but also for $p = 2$.

Continuing in this manner, we derive the general result:

$$\partial^{p+1} \frac{1}{|\text{sdet}X|^p} = 0 \quad \forall p. \tag{A.29}$$

This leads to a direct constraint on the multiplets $\mathbb{O}_p(X)$:

$$0 = \partial^{p+1} \frac{1}{|\text{sdet}X_{12}|^p} = \partial^{p+1} \left\langle \mathbb{O}_p(X_1) \mathbb{O}_p(X_2) \right\rangle, \tag{A.30}$$

from which we can conclude that

$$\begin{aligned} &\partial^{p+1} \mathbb{O}_p(X) = 0 \quad \forall p \\ \text{with } &\partial \equiv (-1)^{|A|}\, \bar{\xi}^{\dot{A}} \frac{\partial}{\partial X^{A\dot{A}}} \xi^A \quad \text{or} \quad \partial \equiv \frac{\partial}{\partial X^{A\dot{A}}}\Big|_{\text{with graded sym}}. \qquad \square \end{aligned} \tag{A.31}$$

A.3.2. Fundamental multiplet on Analytic Superspace

The fundamental multiplet, $\mathbb{O}_1(X)$ can be expanded in the fermionic coordinates of analytic superspace. The full expansion is then determined by

$$\left(\frac{\partial}{\partial X^{A\dot{A}}}\right)^2 \mathbb{O}_1(X) = 0. \tag{A.32}$$

Solving this constraint yields the following multiplet expansion

$$\begin{aligned}
\mathbb{O}_1(X) = &\left(1 - \rho^{\alpha\dot{a}}\bar{\rho}^{a\dot{\alpha}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial y^{a\dot{a}}} - \frac{1}{4}\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}\bar{\rho}^{a\dot{\alpha}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\alpha\dot{\alpha}}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{a\dot{a}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)|_{(\alpha\beta),(\dot{\alpha}\dot{\beta}),\{ab\},\{\dot{a}\dot{b}\}}\mathcal{O}_6\left(x,y\right)\\
&+\rho^{\alpha\dot{a}}\left(1-\frac{1}{2}\rho^{\beta\dot{b}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)\Psi_{\alpha\dot{a}}\left(x,y\right)|_{(\alpha\beta),\{\dot{a}\dot{b}\}}\\
&+\bar{\rho}^{a\dot{\alpha}}\left(1-\frac{1}{2}\rho^{\beta\dot{b}}\bar{\rho}^{b\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\frac{\partial}{\partial y^{b\dot{b}}}\right)\bar{\Psi}_{a\dot{\alpha}}\left(x,y\right)|_{(\dot{\alpha}\dot{\beta}),\{ab\}}\\
&+\rho^{\alpha\dot{a}}\rho^{\beta\dot{b}}F_{\alpha\dot{a},\beta\dot{b}}(x)|_{(\alpha\beta),\{\dot{a}\dot{b}\}}\\
&+\bar{\rho}^{a\dot{\alpha}}\bar{\rho}^{b\dot{\beta}}\bar{F}_{a\dot{\alpha},b\dot{\beta}}(x)|_{(\dot{\alpha}\dot{\beta}),\{ab\}}
\end{aligned}$$

$$\begin{aligned}
\text{with }&\mathcal{O}_6(x,y) \sim [0,0,0]_{(0,0)} \text{ and } \Box_x\mathcal{O}_6(x,y) = 0\\
&\Psi_{\alpha\dot{a}}(x,y) \sim [0,0,1]_{(\frac{1}{2},0)} \text{ and } \epsilon^{\alpha\beta}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\Psi_{\alpha\dot{a}}(x,y) = 0\\
&\bar{\Psi}_{a\dot{\alpha}}\left(x,y\right) \sim [1,0,0]_{(0,\frac{1}{2})} \text{ and } \epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial x^{\beta\dot{\beta}}}\bar{\Psi}_{a\dot{\alpha}}\left(x,y\right) = 0\\
&F_{\alpha\dot{a},\beta\dot{b}}(x)|_{(\alpha\beta),\{\dot{a}\dot{b}\}} \sim [0,0,0]_{(1,0)}\\
&\bar{F}_{a\dot{\alpha},b\dot{\beta}}(x)|_{(\dot{\alpha}\dot{\beta}),\{ab\}} \sim [0,0,0]_{(0,1)}
\end{aligned} \tag{A.33}$$

Note that for the fundamental multiplet, the masslessness equations as well as the conservation equations are implied by the constraint.

B. $PSU(1,1|2)$

We are studying the correlation function

$$\langle \mathcal{W}_2(X_1)\mathcal{W}_2(X_2)\mathcal{W}_2(X_3)\mathcal{W}_2(X_4)\mathcal{W}_2(X_5)\rangle \,. \tag{B.1}$$

B.1. Constraints from SUSY-Invariance

Imposing invariance of this correlator, expanded in fermionic coordinates, under supersymmetric transformations as $\sum \frac{\partial}{\partial \rho} \langle \ldots \rangle = 0$ leads to the following five constraints.

$$\begin{aligned}0 =& -\frac{1}{2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle T(x_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \sum_{i=2}^{5}\langle \bar{G}(x_1,y_1)G(x_i,y_i)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \\ &\text{with } i \neq k \neq l \neq m \in \{2,3,4,5\}\end{aligned} \tag{B.2}$$

$$\begin{aligned}0 =& -\frac{1}{2}\frac{\partial}{\partial x_2}\frac{\partial}{\partial y_2}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \langle J(x_1,y_1)T(x_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &- \langle G(x_1,y_1)\bar{G}(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\ &+ \sum_{i=3}^{5}\langle J(x_1,y_1)\bar{G}(x_2,y_2)G(x_i,y_i)J(x_l,y_l)J(x_m,y_m)\rangle \\ &\text{with } i \neq l \neq m \in \{3,4,5\}\end{aligned} \tag{B.3}$$

$$
\begin{aligned}
0 = &- \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial}{\partial y_3}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)T(x_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&- \langle G(x_1,y_1)J(x_2,y_2)\bar{G}(x_3,y_4)J(x_4,y_4)J(x_5,y_5)\rangle \\
&- \langle J(x_1,y_1)G(x_2,y_2)\bar{G}(x_3,y_4)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)\bar{G}(x_3,y_3)G(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)\bar{G}(x_3,y_3)J(x_4,y_4)G(x_5,y_5)\rangle
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
0 = &- \frac{1}{2}\frac{\partial}{\partial x_4}\frac{\partial}{\partial y_4}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)T(x_4)J(x_5,y_5)\rangle \\
&- \sum_{i=1}^{3}\langle G(x_i,y_i)J(x_l,y_l)J(x_m,y_m)\bar{G}(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)\bar{G}(x_4,y_4)G(x_5,y_5)\rangle \\
&\text{with } i \neq l \neq m \in \{1,2,3\}
\end{aligned}
\tag{B.5}
$$

$$
\begin{aligned}
0 = &- \frac{1}{2}\frac{\partial}{\partial x_5}\frac{\partial}{\partial y_5}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)T(x_5)\rangle \\
&- \sum_{i=1}^{4}\langle G(x_i,y_i)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\bar{G}(x_5,y_5)\rangle \\
&\text{with } i \neq k \neq l \neq m \in \{1,2,3,4\}.
\end{aligned}
\tag{B.6}
$$

The alternating sign in the correlators involving the currents $G(x)$ and $\bar{G}(x)$ is due to their fermionic nature.
Similarly, imposing invariance under the charge $\bar{Q}$, i.e. $\sum \frac{\partial}{\partial \bar{\rho}}\langle \ldots \rangle = 0$ leads to the five constraints

$$
\begin{aligned}
0 = &+ \frac{1}{2}\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&- \langle T(x_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\rangle \\
&+ \sum_{i=2}^{5}\langle G(x_1,y_1)\bar{G}(x_i,y_i)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)\rangle \\
&\text{with } i \neq k \neq l \neq m \in \{2,3,4,5\}
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
0 = &+ \frac{1}{2}\frac{\partial}{\partial x_2}\frac{\partial}{\partial y_2}\left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&+ \left\langle J(x_1,y_1)T(x_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle \bar{G}(x_1,y_1)G(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&+ \sum_{i=3}^{5}\left\langle J(x_1,y_1)G(x_2,y_2)\bar{G}(x_i,y_i)J(x_l,y_l)J(x_m,y_m)\right\rangle \\
&\text{with } i \neq l \neq m \in \{3,4,5\}
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
0 = &+ \frac{1}{2}\frac{\partial}{\partial x_3}\frac{\partial}{\partial y_3}\left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle J(x_1,y_1)J(x_2,y_2)T(x_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle \bar{G}(x_1,y_1)J(x_2,y_2)G(x_3,y_4)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle J(x_1,y_1)\bar{G}(x_2,y_2)G(x_3,y_4)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&+ \left\langle J(x_1,y_1)J(x_2,y_2)G(x_3,y_3)\bar{G}(x_4,y_4)J(x_5,y_5)\right\rangle \\
&+ \left\langle J(x_1,y_1)J(x_2,y_2)G(x_3,y_3)J(x_4,y_4)\bar{G}(x_5,y_5)\right\rangle
\end{aligned}
\tag{B.9}
$$

$$
\begin{aligned}
0 = &+ \frac{1}{2}\frac{\partial}{\partial x_4}\frac{\partial}{\partial y_4}\left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)T(x_4)J(x_5,y_5)\right\rangle \\
&- \sum_{i=1}^{3}\left\langle \bar{G}(x_i,y_i)J(x_l,y_l)J(x_m,y_m)G(x_4,y_4)J(x_5,y_5)\right\rangle \\
&+ \left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)G(x_4,y_4)\bar{G}(x_5,y_5)\right\rangle \\
&\text{with } i \neq l \neq m \in \{1,2,3\}
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
0 = &+ \frac{1}{2}\frac{\partial}{\partial x_5}\frac{\partial}{\partial y_5}\left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)J(x_5,y_5)\right\rangle \\
&- \left\langle J(x_1,y_1)J(x_2,y_2)J(x_3,y_3)J(x_4,y_4)T(x_5)\right\rangle \\
&- \sum_{i=1}^{4}\left\langle \bar{G}(x_i,y_i)J(x_k,y_k)J(x_l,y_l)J(x_m,y_m)G(x_5,y_5)\right\rangle \\
&\text{with } i \neq k \neq l \neq m \in \{1,2,3,4\}.
\end{aligned}
\tag{B.11}
$$

B.2. Bosonic correlators

This section contains the relevant descendent correlation functions for the analysis of the SCWI of $\mathfrak{psu}(1,1|2)$. In particular, this section lists the choices of bases used to express those correlators in a way compatible with the bosonic symmetry groups $SU(1,1)$ and $SU(2)_R$.

For instance, the correlator involving the 1d stress tensor $T(x)$ inserted at X_1 can be expressed by

$$\begin{aligned}&\langle T\left(x_1\right) J\left(x_2, y_2\right) J\left(x_3, y_3\right) J\left(x_4, y_4\right) J\left(x_5, y_5\right)\rangle \\ =&\frac{1}{x_{12}x_{13}x_{14}x_{15}x_{23}x_{45}}\left(y_{23}^2\, y_{45}^2 \cdot g_{1,1}(u,v) + y_{24}^2\, y_{35}^2 \cdot g_{1,2}(u,v)\right. \\ &\left. +y_{25}^2\, y_{34}^2 \cdot g_{1,3}(u,v)\right).\end{aligned} \tag{B.12}$$

The function $g_{i,j}(u,v)$ denotes the unknown function of the cross ratios multiplying the j^{th} R-symmetry structures of the correlator involving the stress tensor inserted at X_i.
The correlators involving the stress tensor inserted at other points $X_2, \ldots, X_5$ can be obtained from the one presented by the respective permutation in the points.

For the correlators involving the currents $G(x), \bar{G}(x)$, we will exemplary give the correlators proportional to $\bar{\rho}^1 \bar{G}_1(x)$. The remaining correlators are built following the same principles.

$$\begin{aligned}&\langle \bar{G}\left(x_1, y_1\right) G\left(x_2, y_2\right) J\left(x_3, y_3\right) J\left(x_4, y_4\right) J\left(x_5, y_5\right)\rangle \\ =&\frac{1}{x_{12}^2 x_{34}x_{45}x_{53}}\left(y_{12}\, y_{34}\, y_{45}\, y_{53}\, h_{12,1}(u,v) + y_{13}\, y_{23}\, y_{45}^2\, h_{12,2}(u,v)\right. \\ &\left. +y_{14}\, y_{24}\, y_{35}^2\, h_{12,3}(u,v) + y_{15}\, y_{25}\, y_{34}^2\, h_{12,4}(u,v)\right)\end{aligned} \tag{B.13}$$

$$\begin{aligned}&\langle \bar{G}\left(x_1, y_1\right) J\left(x_2, y_2\right) G\left(x_3, y_3\right) J\left(x_4, y_4\right) J\left(x_5, y_5\right)\rangle \\ =&\frac{1}{x_{13}^2 x_{24}x_{45}x_{52}}\left(y_{13}\, y_{24}\, y_{45}\, y_{52}\, h_{13,1}(u,v) + y_{12}\, y_{32}\, y_{45}^2\, h_{13,2}(u,v)\right. \\ &\left. +y_{14}\, y_{34}\, y_{25}^2\, h_{13,3}(u,v) + y_{15}\, y_{35}\, y_{24}^2\, h_{13,4}(u,v)\right)\end{aligned} \tag{B.14}$$

$$\begin{aligned}&\langle \bar{G}\left(x_1, y_1\right) J\left(x_2, y_2\right) J\left(x_3, y_3\right) G\left(x_4, y_4\right) J\left(x_5, y_5\right)\rangle \\ =&\frac{1}{x_{14}^2 x_{23}x_{35}x_{53}}\left(y_{14}\, y_{23}\, y_{35}\, y_{52}\, h_{14,1}(u,v) + y_{12}\, y_{42}\, y_{35}^2\, h_{14,2}(u,v)\right. \\ &\left. +y_{13}\, y_{43}\, y_{25}^2\, h_{14,3}(u,v) + y_{15}\, y_{45}\, y_{23}^2\, h_{14,4}(u,v)\right)\end{aligned} \tag{B.15}$$

$$\begin{aligned}
&\langle \bar{G}\left(x_1, y_1\right) J\left(x_2, y_2\right) J\left(x_3, y_3\right) J\left(x_4, y_4\right) G\left(x_5, y_5\right)\rangle \\
=&\frac{1}{x_{15}^2 x_{23} x_{34} x_{43}} \Big(y_{15}\, y_{23}\, y_{34}\, y_{42}\, h_{15,1}(u,v) + y_{12}\, y_{52}\, y_{34}^2 h_{15,2}(u,v) \\
&\qquad + y_{13}\, y_{53}\, y_{24}^2\, h_{15,3}(u,v) + y_{14}\, y_{54}\, y_{23}^2\, h_{15,4}(u,v) \Big)\,.
\end{aligned} \tag{B.16}$$

Here, the function $h_{ij,k}(u,v)$ is the unknown function of the cross ratios multiplying the k^{th} R-symmetry structure expressing the correlation function involving $\bar{G}(X_i)$ and $G(X_j)$.

C. Bosonic structures

In this chapter, we present the relevant correlators necessary for a comprehensive analysis of the five-point function, up to the order $\mathcal{O}(\rho\bar{\rho})$:

$$\mathcal{G}_{22222} = \langle \mathcal{T}(X_1)\mathcal{T}(X_2)\mathcal{T}(X_3)\mathcal{T}(X_4)\mathcal{T}(X_5)\rangle \,.$$

In Section C.1, we provide the expressions for the descendent correlators involving the $SU(4)_R$-current $\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x)$, focusing exclusively on the conformal or spacetime dependence. The basis chosen for these expressions is not unique, but reflects the particular choices made for the analysis carried out in this thesis.

C.1. Spacetime structures

This section provides a summary of the conformal spacetime structures employed to express the descendent correlators. We will start with the expressions for correlators involving $\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x_i)$ in a manner consistent with conformal symmetry. Without considering R-symmetry, there are four independent structures, each associated with an unknown function of the five cross ratios $\{u\}$. These functions, which correspond to the $\mathcal{J}$-correlators, are denoted by $g_{i,j,k}(\{u\})$ throughout this thesis. $g_{i,j,k}(\{u\})$ corresponds to the k^{th} spacetime structure, multiplying the j^{th} R-symmetry structure of the correlator with $\mathcal{J}_{\alpha\dot{a},a\dot{\alpha}}(x_i)$, k inserted at x_i.

$$\begin{aligned}
&\langle \mathcal{J}_{\alpha\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\
&= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big(-(x_{12}^{-1}-x_{13}^{-1})_{\dot{\alpha}\alpha}\, g_{1,j,1}(\{u\}) - (x_{12}^{-1}-x_{14}^{-1})_{\dot{\alpha}\alpha}\, g_{1,j,2}(\{u\}) \\
&\qquad -(x_{12}^{-1}-x_{15}^{-1})_{\dot{\alpha}\alpha}\, g_{1,j,3}(\{u\}) + (x_{12}^{-1}\, x_{23}\, x_{34}^{-1}\, x_{45}\, x_{51}^{-1})_{\dot{\alpha}\alpha}\, g_{1,j,4}(\{u\})\Big)
\end{aligned} \tag{C.1}$$

$$\begin{aligned}
&\langle \mathcal{O}_2(x_1)\mathcal{J}_{\alpha\dot{\alpha}}(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\
&\frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big(-(x_{21}^{-1}-x_{23}^{-1})_{\dot{\alpha}\alpha}\, g_{2,j,1}(\{u\}) - (x_{21}^{-1}-x_{24}^{-1})_{\dot{\alpha}\alpha}\, g_{2,j,2}(\{u\}) \\
&\qquad -(x_{21}^{-1}-x_{25}^{-1})_{\dot{\alpha}\alpha}\, g_{2,j,3}(\{u\}) + (x_{21}^{-1}\, x_{13}\, x_{34}^{-1}\, x_{45}\, x_{52}^{-1})_{\dot{\alpha}\alpha}\, g_{2,j,4}(\{u\})\Big)
\end{aligned} \tag{C.2}$$

$$\langle\mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{J}_{\alpha\dot{\alpha}}(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big(-(x_{34}^{-1}-x_{31}^{-1})_{\dot{\alpha}\alpha}\, g_{3,j,1}(\{u\}) - (x_{34}^{-1}-x_{32}^{-1})_{\dot{\alpha}\alpha}\, g_{3,j,2}(\{u\}) - (x_{34}^{-1}-x_{35}^{-1})_{\dot{\alpha}\alpha}\, g_{3,j,3}(\{u\}) + (x_{34}^{-1}\,x_{45}\,x_{51}^{-1}\,x_{12}\,x_{23}^{-1})_{\dot{\alpha}\alpha}\, g_{3,j,4}(\{u\})\Big) \tag{C.3}$$

$$\langle\mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{J}_{\alpha\dot{\alpha}}(x_4)\mathcal{O}_2(x_5)\rangle = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big(-(x_{45}^{-1}-x_{41}^{-1})_{\dot{\alpha}\alpha}\, g_{4,j,1}(\{u\}) - (x_{45}^{-1}-x_{42}^{-1})_{\dot{\alpha}\alpha}\, g_{4,j,2}(\{u\}) - (x_{45}^{-1}-x_{43}^{-1})_{\dot{\alpha}\alpha}\, g_{4,j,3}(\{u\}) + (x_{45}^{-1}\,x_{51}\,x_{12}^{-1}\,x_{23}\,x_{34}^{-1})_{\dot{\alpha}\alpha}\, g_{4,j,4}(\{u\})\Big) \tag{C.4}$$

$$\langle\mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{J}_{\alpha\dot{\alpha}}(x_5)\rangle = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big(-(x_{53}^{-1}-x_{51}^{-1})_{\dot{\alpha}\alpha}\, g_{5,j,1}(\{u\}) - (x_{53}^{-1}-x_{52}^{-1})_{\dot{\alpha}\alpha}\, g_{5,j,2}(\{u\}) - (x_{53}^{-1}-x_{54}^{-1})_{\dot{\alpha}\alpha}\, g_{5,j,3}(\{u\}) + (x_{51}^{-1}\,x_{12}\,x_{23}^{-1}\,x_{34}\,x_{45}^{-1})_{\dot{\alpha}\alpha}\, g_{5,j,4}(\{u\})\Big) \tag{C.5}$$

We further have the correlators involving the spinor fields $\bar{\Psi}_{a\dot{\alpha}}(x)$, $\Psi_{\alpha\dot{a}}(x)$. Without considering R-symmetry, there are as well four independent structures per correlator, each associated with an unknown function of the five cross ratios $\{u\}$. These functions, which correspond to the spinor-correlators, are denoted by $h_{ij,kl}(\{u\})$ throughout this thesis, referring to the l^{th} spacetime structure multiplying the k^{th} R-symmetry structure of the correlator with $\bar{\Psi}_{a\dot{\alpha}}(x_i)$ inserted at x_i and $\Psi_{\alpha\dot{a}}(x_j)$ inserted at x_j. We have

$$\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\Psi_{\alpha}(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{14}^{-1}\,x_{43}\,x_{32}^{-1})_{\dot{\alpha}\alpha}\, h_{12,j1}(\{u\}) + (x_{15}^{-1}\,x_{53}\,x_{32}^{-1})_{\dot{\alpha}\alpha}\, h_{12,j2}(\{u\}) + (x_{15}^{-1}\,x_{54}\,x_{42}^{-1})_{\dot{\alpha}\alpha}\, h_{12,j3}(\{u\}) + (x_{12}^{-1})_{\dot{\alpha}\alpha}\, h_{12,j4}(\{u\})\Big) \tag{C.6}$$

$$\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\Psi_{\alpha}(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = = \frac{1}{x_{12}^4x_{34}^2x_{45}^2x_{53}^2}\Big((x_{14}^{-1}\,x_{42}\,x_{23}^{-1})_{\dot{\alpha}\alpha}\, h_{13,j1}(\{u\}) + (x_{15}^{-1}\,x_{52}\,x_{23}^{-1})_{\dot{\alpha}\alpha}\, h_{13,j2}(\{u\}) + (x_{15}^{-1}\,x_{54}\,x_{43}^{-1})_{\dot{\alpha}\alpha}\, h_{13,j3}(\{u\}) + (x_{13}^{-1})_{\dot{\alpha}\alpha}\, h_{13,j4}(\{u\})\Big) \tag{C.7}$$

$$\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\Psi_\alpha(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{13}^{-1}\,x_{32}\,x_{24}^{-1})_{\dot{\alpha}\alpha}\,h_{14,j1}(\{u\}) + (x_{15}^{-1}\,x_{52}\,x_{24}^{-1})_{\dot{\alpha}\alpha}\,h_{14,j2}(\{u\}) \\ +(x_{15}^{-1}\,x_{53}\,x_{34}^{-1})_{\dot{\alpha}\alpha}\,h_{14,j3}(\{u\}) + (x_{14}^{-1})_{\dot{\alpha}\alpha}\,h_{14,j4}(\{u\})\Big) \tag{C.8}$$

$$\langle\bar{\Psi}_{\dot{\alpha}}(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\Psi_\alpha(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{13}^{-1}\,x_{32}\,x_{25}^{-1})_{\dot{\alpha}\alpha}\,h_{15,j1}(\{u\}) + (x_{14}^{-1}\,x_{42}\,x_{25}^{-1})_{\dot{\alpha}\alpha}\,h_{15,j2}(\{u\}) \\ +(x_{14}^{-1}\,x_{43}\,x_{35}^{-1})_{\dot{\alpha}\alpha}\,h_{15,j3}(\{u\}) + (x_{15}^{-1})_{\dot{\alpha}\alpha}\,h_{15,j4}(\{u\})\Big) \tag{C.9}$$

$$\langle\Psi_\alpha(x_1)\bar{\Psi}_{\dot{\alpha}}(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{24}^{-1}\,x_{43}\,x_{31}^{-1})_{\dot{\alpha}\alpha}\,h_{21,j1}(\{u\}) + (x_{25}^{-1}\,x_{53}\,x_{31}^{-1})_{\dot{\alpha}\alpha}\,h_{21,j2}(\{u\}) \\ +(x_{25}^{-1}\,x_{54}\,x_{41}^{-1})_{\dot{\alpha}\alpha}\,h_{21,j3}(\{u\}) + (x_{21}^{-1})_{\dot{\alpha}\alpha}\,h_{21,j4}(\{u\})\Big) \tag{C.10}$$

$$\langle\mathcal{O}_2(x_1)\bar{\Psi}_{\dot{\alpha}}(x_2)\Psi_\alpha(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{24}^{-1}\,x_{41}\,x_{13}^{-1})_{\dot{\alpha}\alpha}\,h_{23,j1}(\{u\}) + (x_{25}^{-1}\,x_{51}\,x_{13}^{-1})_{\dot{\alpha}\alpha}\,h_{23,j2}(\{u\}) \\ +(x_{25}^{-1}\,x_{54}\,x_{43}^{-1})_{\dot{\alpha}\alpha}\,h_{23,j3}(\{u\}) + (x_{23}^{-1})_{\dot{\alpha}\alpha}\,h_{23,j4}(\{u\})\Big) \tag{C.11}$$

$$\langle\mathcal{O}_2(x_1)\bar{\Psi}_{\dot{\alpha}}(x_2)\mathcal{O}_2(x_3)\Psi_\alpha(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{23}^{-1}\,x_{31}\,x_{14}^{-1})_{\dot{\alpha}\alpha}\,h_{24,j1}(\{u\}) + (x_{25}^{-1}\,x_{51}\,x_{14}^{-1})_{\dot{\alpha}\alpha}\,h_{24,j2}(\{u\}) \\ +(x_{25}^{-1}\,x_{53}\,x_{34}^{-1})_{\dot{\alpha}\alpha}\,h_{24,j3}(\{u\}) + (x_{24}^{-1})_{\dot{\alpha}\alpha}\,h_{24,j4}(\{u\})\Big) \tag{C.12}$$

$$\langle\mathcal{O}_2(x_1)\bar{\Psi}_{\dot{\alpha}}(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\Psi_\alpha(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{23}^{-1}\,x_{31}\,x_{15}^{-1})_{\dot{\alpha}\alpha}\,h_{25,j1}(\{u\}) + (x_{24}^{-1}\,x_{41}\,x_{15}^{-1})_{\dot{\alpha}\alpha}\,h_{25,j2}(\{u\}) \\ +(x_{24}^{-1}\,x_{43}\,x_{35}^{-1})_{\dot{\alpha}\alpha}\,h_{25,j3}(\{u\}) + (x_{25}^{-1})_{\dot{\alpha}\alpha}\,h_{25,j4}(\{u\})\Big) \tag{C.13}$$

$$\langle\Psi_\alpha(x_1)\mathcal{O}_2(x_2)\bar{\Psi}_{\dot{\alpha}}(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{34}^{-1}\,x_{42}\,x_{21}^{-1})_{\dot{\alpha}\alpha}\,h_{31,j1}(\{u\}) + (x_{35}^{-1}\,x_{52}\,x_{21}^{-1})_{\dot{\alpha}\alpha}\,h_{31,j2}(\{u\}) \\ +(x_{35}^{-1}\,x_{54}\,x_{41}^{-1})_{\dot{\alpha}\alpha}\,h_{31,j3}(\{u\}) + (x_{31}^{-1})_{\dot{\alpha}\alpha}\,h_{31,j4}(\{u\})\Big) \tag{C.14}$$

$$\langle \mathcal{O}_2(x_1)\Psi_\alpha(x_2)\bar{\Psi}_{\dot\alpha}(x_3)\mathcal{O}_2(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{34}^{-1}\,x_{41}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{32,j1}(\{u\}) + (x_{35}^{-1}\,x_{51}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{32,j2}(\{u\}) \\ +(x_{35}^{-1}\,x_{54}\,x_{42}^{-1})_{\dot\alpha\alpha}\,h_{32,j3}(\{u\}) + (x_{32}^{-1})_{\dot\alpha\alpha}\,h_{32,j4}(\{u\})\Big) \tag{C.15}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\bar{\Psi}_{\dot\alpha}(x_3)\Psi_\alpha(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{32}^{-1}\,x_{21}\,x_{14}^{-1})_{\dot\alpha\alpha}\,h_{34,j1}(\{u\}) + (x_{35}^{-1}\,x_{51}\,x_{14}^{-1})_{\dot\alpha\alpha}\,h_{34,j2}(\{u\}) \\ +(x_{35}^{-1}\,x_{52}\,x_{24}^{-1})_{\dot\alpha\alpha}\,h_{34,j3}(\{u\}) + (x_{34}^{-1})_{\dot\alpha\alpha}\,h_{34,j4}(\{u\})\Big) \tag{C.16}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\bar{\Psi}_{\dot\alpha}(x_3)\mathcal{O}_2(x_4)\Psi_\alpha(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{32}^{-1}\,x_{21}\,x_{15}^{-1})_{\dot\alpha\alpha}\,h_{35,j1}(\{u\}) + (x_{34}^{-1}\,x_{41}\,x_{15}^{-1})_{\dot\alpha\alpha}\,h_{35,j2}(\{u\}) \\ +(x_{34}^{-1}\,x_{42}\,x_{25}^{-1})_{\dot\alpha\alpha}\,h_{35,j3}(\{u\}) + (x_{35}^{-1})_{\dot\alpha\alpha}\,h_{35,j4}(\{u\})\Big) \tag{C.17}$$

$$\langle \Psi_\alpha(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\bar{\Psi}_{\dot\alpha}(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{43}^{-1}\,x_{32}\,x_{21}^{-1})_{\dot\alpha\alpha}\,h_{41,j1}(\{u\}) + (x_{45}^{-1}\,x_{52}\,x_{21}^{-1})_{\dot\alpha\alpha}\,h_{41,j2}(\{u\}) \\ +(x_{45}^{-1}\,x_{53}\,x_{31}^{-1})_{\dot\alpha\alpha}\,h_{41,j3}(\{u\}) + (x_{41}^{-1})_{\dot\alpha\alpha}\,h_{41,j4}(\{u\})\Big) \tag{C.18}$$

$$\langle \mathcal{O}_2(x_1)\Psi_\alpha(x_2)\mathcal{O}_2(x_3)\bar{\Psi}_{\dot\alpha}(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{43}^{-1}\,x_{31}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{42,j1}(\{u\}) + (x_{45}^{-1}\,x_{51}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{42,j2}(\{u\}) \\ +(x_{45}^{-1}\,x_{53}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_{42,j3}(\{u\}) + (x_{42}^{-1})_{\dot\alpha\alpha}\,h_{42,j4}(\{u\})\Big) \tag{C.19}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\Psi_\alpha(x_3)\bar{\Psi}_{\dot\alpha}(x_4)\mathcal{O}_2(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{42}^{-1}\,x_{21}\,x_{13}^{-1})_{\dot\alpha\alpha}\,h_{43,j1}(\{u\}) + (x_{45}^{-1}\,x_{51}\,x_{13}^{-1})_{\dot\alpha\alpha}\,h_{43,j2}(\{u\}) \\ +(x_{45}^{-1}\,x_{52}\,x_{23}^{-1})_{\dot\alpha\alpha}\,h_{43,j3}(\{u\}) + (x_{43}^{-1})_{\dot\alpha\alpha}\,h_{43,j4}(\{u\})\Big) \tag{C.20}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\bar{\Psi}_{\dot\alpha}(x_4)\Psi_\alpha(x_5)\rangle = \\ = \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{42}^{-1}\,x_{21}\,x_{15}^{-1})_{\dot\alpha\alpha}\,h_{45,j1}(\{u\}) + (x_{43}^{-1}\,x_{31}\,x_{15}^{-1})_{\dot\alpha\alpha}\,h_{45,j2}(\{u\}) \\ +(x_{43}^{-1}\,x_{32}\,x_{25}^{-1})_{\dot\alpha\alpha}\,h_{45,j3}(\{u\}) + (x_{45}^{-1})_{\dot\alpha\alpha}\,h_{45,j4}(\{u\})\Big) \tag{C.21}$$

$$\langle \Psi_\alpha(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\bar{\Psi}_{\dot\alpha}(x_5)\rangle =$$
$$= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{53}^{-1}\,x_{32}\,x_{21}^{-1})_{\dot\alpha\alpha}\,h_{51,j1}(\{u\}) + (x_{54}^{-1}\,x_{42}\,x_{21}^{-1})_{\dot\alpha\alpha}\,h_{51,j2}(\{u\})$$
$$+(x_{54}^{-1}\,x_{43}\,x_{31}^{-1})_{\dot\alpha\alpha}\,h_{51,j3}(\{u\}) + (x_{51}^{-1})_{\dot\alpha\alpha}\,h_{51,j4}(\{u\})\Big) \quad \text{(C.22)}$$

$$\langle \mathcal{O}_2(x_1)\Psi_\alpha(x_2)\mathcal{O}_2(x_3)\mathcal{O}_2(x_4)\bar{\Psi}_{\dot\alpha}(x_5)\rangle =$$
$$= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{53}^{-1}\,x_{31}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{52,j1}(\{u\}) + (x_{54}^{-1}\,x_{41}\,x_{12}^{-1})_{\dot\alpha\alpha}\,h_{52,j2}(\{u\})$$
$$+(x_{54}^{-1}\,x_{43}\,x_{32}^{-1})_{\dot\alpha\alpha}\,h_{52,j3}(\{u\}) + (x_{52}^{-1})_{\dot\alpha\alpha}\,h_{52,j4}(\{u\})\Big) \quad \text{(C.23)}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\Psi_\alpha(x_3)\mathcal{O}_2(x_4)\bar{\Psi}_{\dot\alpha}(x_5)\rangle =$$
$$= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{52}^{-1}\,x_{21}\,x_{13}^{-1})_{\dot\alpha\alpha}\,h_{53,j1}(\{u\}) + (x_{54}^{-1}\,x_{41}\,x_{13}^{-1})_{\dot\alpha\alpha}\,h_{53,j2}(\{u\})$$
$$+(x_{54}^{-1}\,x_{42}\,x_{23}^{-1})_{\dot\alpha\alpha}\,h_{53,j3}(\{u\}) + (x_{53}^{-1})_{\dot\alpha\alpha}\,h_{53,j4}(\{u\})\Big) \quad \text{(C.24)}$$

$$\langle \mathcal{O}_2(x_1)\mathcal{O}_2(x_2)\mathcal{O}_2(x_3)\Psi_\alpha(x_4)\bar{\Psi}_{\dot\alpha}(x_5)\rangle =$$
$$= \frac{1}{x_{12}^4 x_{34}^2 x_{45}^2 x_{53}^2}\Big((x_{52}^{-1}\,x_{21}\,x_{14}^{-1})_{\dot\alpha\alpha}\,h_{54,j1}(\{u\}) + (x_{53}^{-1}\,x_{31}\,x_{14}^{-1})_{\dot\alpha\alpha}\,h_{54,j2}(\{u\})$$
$$+(x_{53}^{-1}\,x_{32}\,x_{24}^{-1})_{\dot\alpha\alpha}\,h_{54,j3}(\{u\}) + (x_{54}^{-1})_{\dot\alpha\alpha}\,h_{54,j4}(\{u\})\Big) \quad \text{(C.25)}$$

D. Four-point functions

In this Appendix, we provide the relations obtained for the functions of cross ratios parametrising the descendent correlators

$$\langle \mathcal{J}_{\alpha\dot{\alpha};a\dot{a}}(x_1,y_1)\mathcal{O}_{20'}(x_2,y_2)\mathcal{O}_{20'}(x_3,y_3)\mathcal{O}_{20'}(x_4,y_4)\rangle$$
$$\langle \bar{\Psi}_{a\dot{\alpha}}(x_1,y_1)\Psi_{\alpha\dot{a}}(x_i,y_i)\mathcal{O}_{20'}(x_k,y_k)\mathcal{O}_{20'}(x_l,y_l)\rangle$$

for all permutations of $i,j,k=2,3,4$ and with the notations introduced in 3.5.

The equations, obtained from the constraint for terms proportional to $\bar{\rho}_1^{a\dot{\alpha}}$ after applying supersymmetry, are given by

$$\begin{aligned}
h_{1,1}(u,v) =& u^2 f_3^{(u)}(u,v) + vu f_3^{(v)}(u,v),\\
h_{1,2}(u,v) =& -\frac{-f_3^{(u)}(u,v)u^4 + f_3^{(u)}(u,v)u^3 - f_3^{(v)}(u,v)vu^3}{v^2},\\
h_{2,1}(u,v) =& -\frac{f_1^{(v)}(u,v)v^3 - f_1^{(v)}(u,v)v^2 + f_1^{(u)}(u,v)uv^2}{u},\\
h_{2,2}(u,v) =& -f_1^{(u)}(u,v)u^2 - f_1^{(v)}(u,v)vu,\\
h_{3,1}(u,v) =& -f_2^{(v)}(u,v)uv^2,\\
h_{3,2}(u,v) =& f_2^{(u)}(u,v)u^3,\\
h_{4,1}(u,v) =& -\frac{f_1^{(v)}(u,v)v^3 + f_1^{(u)}(u,v)uv^2}{u},\\
h_{4,2}(u,v) =& -f_1^{(u)}(u,v)u^2 + f_1^{(u)}(u,v)u - f_1^{(v)}(u,v)vu - 2f_1(u,v),\\
h_{5,1}(u,v) =& f_2^{(u)}(u,v)uv^2,\\
h_{5,2}(u,v) =& f_2^{(v)}(u,v)vu^2 + f_2^{(u)}(u,v)\left(u^2 + vu - u\right)u - f_4(u,v)u,\\
h_{6,1}(u,v) =& f_3^{(u)}(u,v)u^2 - f_3^{(u)}(u,v)u + f_3^{(v)}(u,v)vu,\\
h_{6,2}(u,v) =& -\frac{(-u^4 + 2u^3 - u^2 + vu^2)f_3^{(u)}(u,v) + (-vu^3 + vu^2)f_3^{(v)}(u,v) + vf_5(u,v)(u,v)u}{v^2},\\
k_{1,1}(u,v) =& -f_1^{(v)}(u,v)v,\\
k_{1,2}(u,v) =& -\frac{-vf_1^{(v)}(u,v) + f_1^{(v)}(u,v) - f_1^{(u)}(u,v)u}{u^2},
\end{aligned}$$

$$
\begin{aligned}
k_{2,1}(u,v) =& f_2^{(u)}(u,v)vu^2 + f_2^{(v)}(u,v)vu^2,\\
k_{2,2}(u,v) =& f_2^{(v)}(u,v),\\
k_{3,1}(u,v) =& -\frac{f_3^{(u)}(u,v)u^2}{v},\\
k_{3,2}(u,v) =& -\frac{f_3^{(u)}(u,v)u + f_3^{(v)}(u,v)v}{v^2},\\
k_{4,1}(u,v) =& f_2^{(v)}(u,v)u^2v,\\
k_{4,2}(u,v) =& -vf_2^{(v)}(u,v) + f_2^{(v)}(u,v) - f_2^{(u)}(u,v)u - 2f_2(u,v),\\
k_{5,1}(u,v) =& -\frac{f_3^{(u)}(u,v)u^3 + f_3^{(v)}(u,v)vu^2}{v},\\
k_{5,2}(u,v) =& -\frac{f_3^{(u)}(u,v)u^2 - f_3^{(u)}(u,v)vu + f_3^{(v)}(u,v)vu - f_3^{(v)}(u,v)v^2 + f_3^{(v)}(u,v)v + vf_6(u,v)}{v^2},\\
k_{6,1}(u,v) =& f_1^{(v)}(u,v)v^2 - f_1^{(v)}(u,v)v + f_1^{(u)}(u,v)uv,\\
k_{6,2}(u,v) =& -\frac{(-u^2-u+vu)f_1^{(u)}(u,v) + (-vu+v^2-2v+1)f_1^{(v)}(u,v) + f_4(u,v)u}{u^2},\\
l_{1,1}(u,v) =& -f_2^{(u)}(u,v)uv,\\
l_{1,2}(u,v) =& -f_2^{(u)}(u,v)v^3 - f_2^{(v)}(u,v)v^3,\\
l_{2,1}(u,v) =& -\frac{f_3^{(u)}(u,v)u^2 - f_3^{(u)}(u,v)u + f_3^{(v)}(u,v)vu}{v},\\
l_{2,2}(u,v) =& f_3^{(u)}(u,v)v,\\
l_{3,1}(u,v) =& -\frac{-f_1^{(v)}(u,v)v^2 - f_1^{(u)}(u,v)uv}{u},\\
l_{3,2}(u,v) =& \frac{f_1^{(v)}(u,v)v^3}{u^2},\\
l_{4,1}(u,v) =& \frac{f_3^{(u)}(u,v)u}{v},\\
l_{4,2}(u,v) =& f_3^{(u)}(u,v)v + f_3^{(v)}(u,v)v - 2f_3(u,v),\\
l_{5,1}(u,v) =& \frac{f_1^{(v)}(u,v)v}{u},\\
l_{5,2}(u,v) =& -\frac{-f_1^{(v)}(u,v)v^2 + f_1^{(u)}(u,v)uv^2 + f_1^{(v)}(u,v)uv^2 + uf_5(u,v)v}{u^2},\\
l_{6,1}(u,v) =& -f_2^{(u)}(u,v)uv - f_2^{(v)}(u,v)uv,\\
l_{6,2}(u,v) =& -f_2^{(u)}(u,v)v^3 - f_2^{(v)}(u,v)v^3 - f_2^{(v)}(u,v)v^2 + f_2^{(u)}(u,v)uv^2 + f_2^{(v)}(u,v)uv^2 - f_6(u,v)v,\\
g_{1,1}(u,v) =& \frac{v}{2}\left(uf_4^{(u)}(u,v) + vf_4^{(v)}(u,v) + f_4(u,v) + 2uf_2^{(u)}(u,v)\right)
\end{aligned}
$$

$$
\begin{aligned}
g_{1,2}(u,v) =& -uv^2 f_2^{(u)}(u,v) - uv^2 f_2^{(v)}(u,v) - \frac{v^2}{2} f_4^{(v)}(u,v) \\
g_{2,1}(u,v) =& \frac{1}{2}\left(2\frac{v}{u}(f_1^{(v)}(u,v) - v f_1^{(v)}(u,v) - u f_1^{(u)}(u,v)) - u f_5^{(u)}(u,v) - v f_5^{(v)}(u,v)\right) \\
g_{2,2}(u,v) =& \frac{1}{2}\left(v f_5^{(v)}(u,v) + f_5(u,v) + 2(\frac{v_2}{u} - \frac{v}{u} + v) f_1^{(v)}(u,v) + 2v f_1^{(u)}(u,v)\right) \\
g_{3,1}(u,v) =& \frac{u}{2}\left(-f_6(u,v) + u f_6^{(u)}(u,v) + v f_6^{(v)}(u,v) + 2(u - \frac{u}{v} - \frac{u^2}{v}) f_3^{(u)}(u,v) + 2(v-u) f_3^{(v)}(u,v)\right) \\
g_{3,2}(u,v) =& \frac{u}{2}\left(f_6(u,v) - v f_6^{(v)}(u,v) - 2u f_3^{(u)}(u,v) - 2v f_3^{(v)}(u,v)\right)
\end{aligned}
$$

Similar results can be obtained for all the other possible descendent correlators. This is reproducing the fact that the full four-point supercorrelator of stress tensor multiplets is determined by the superprimary.

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