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A Kinetic Approach to Investigate the Collective Dynamics of Multi-Agent Systems

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Abstract When the number of interacting agents in a multi-agent system is large, the detailed study of the dynamics of each agent tends to obfuscate the collective, and possibly emergent, dynamics of the multi-agent system as a whole. When the interest is on the collective properties of the multi-agent system, a statistical study of the dynamics of the states of the agents can provide a more effective perspective on the system. In particular, a statistical approach can better focus on the long-time asymptotic properties of the studied multi-agent system. The initial part of this paper outlines a framework to approach the study of the collective properties of multi-agent systems. The framework targets large and decentralized multi-agent systems in which the relevant collective properties emerge from interactions. Then, the paper exemplifies the use of the framework to study the long-time asymptotic properties of multi-agent systems in which agents interact using the symmetric gossip algorithm. The state of each agent is represented as a real number, and the use of the framework shows that all agents exponentially converge to the average of their initial states. The analytic results provided by the framework are confirmed by independent multi-agent simulations. Finally, the paper is concluded with a brief discussion of related works and with an overview of future extensions.

Keywords Collective adaptive systems · Symmetric gossip algorithm · Mathematical kinetic theories.

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1 Introduction

Large and decentralized multi-agent systems (e.g., large fleets of autonomous connected vehicles in an urban area [16,37,44]) are the principal target of relevant research efforts that jointly form the basis of the studies on *collective adaptive systems* (e.g., [23,28]). These systems also find relevant applications in various topics related to *distributed artificial intelligence* (e.g., [30,54]). Usually, the study of the dynamics of these systems assumes that each agent is associated with a state that changes dynamically. The state of an agent changes because of multiple causes and, in particular, it changes because of the interactions that occur when the agent gets in touch with another agent. Accordingly, the descriptions of how interactions change the states of the agents are central to the study of the dynamics of these systems, and they must consider all phenomena that cause agents to change their states. Therefore, these descriptions strongly depend on the considered phenomena and on the specific characteristics of the multi-agent system under investigation.

When the number of agents in the considered multi-agent system is large, the study of the dynamics of the state of each agent is not feasible. For example, consider a large swarm of droids or autonomous taxis in urban areas: on the one hand, tracking the position, velocity, and battery level of each agent might be extremely expensive, if not impossible. On the other hand, what really matters is that the group as a whole achieves its overall objectives, which can include properly shaping the intended choreography for droids or uniformly covering the urban areas for taxis. Actually, when the number of agents in the considered multi-agent system is large, the study of the *collective behavior* [32,53] of the multi-agent system as a whole is preferred, which requires focusing on the characteristics of the states of the agents that jointly form the relevant collective properties of the multi-agent system as a whole. Under the mild assumption that the relevant characteristics of the states of the agents can be represented as real numbers, the collective, and possibly emergent, properties of the considered multi-agent system can be studied using a statistical approach. Although statistical approaches are mainly concerned with the study of aggregate values, which are not normally sufficient to obtain detailed descriptions of the state of each agent, these aggregate values are sufficient to describe the collective dynamics of the multi-agent system as a whole.

This paper reports initial results intended to shed the light on a general-purpose approach to study the asymptotic collective dynamics of large and decentralized multi-agent systems. The discussed approach is based on a specific instantiation of *mathematical kinetic theories* (e.g., [4,5]) that we call *Kinetic Theory of Multi-Agent Systems* (*KTMAS*). Mathematical kinetic theories are not necessarily restricted to the study of physical phenomena, and they are generally intended to investigate the collective properties of groups of interacting peers under the assumption that the relevant characteristics of a group emerge from local interactions among peers and from environmental forces, which is the case for several interesting problems (e.g., [40,41]). Actually, mathematical kinetic theories provide interesting results when the characteristics of studied systems justify a statistical approach and when the interactions among peers are the main causes of the dynamics of studied systems. In this context, KTMAS is instead specifically designed to analytically study the long-time asymptotic properties of large and decentralized multi-agent systems in which agents affect each other's state via direct message passing [6,7].

The major contribution of this paper is to review and extend the results reported in [42] to broaden the discussion on an analytic framework intended to characterize KTMAS. Note that the framework can effectively support both descriptive and prescriptive reasoning on the long-time asymptotic behavior of the collective, and possibly emergent, properties of large and decentralized multi-agent systems. As a descriptive tool, the framework benefits from solid mathematical foundations to represent a valid alternative to multi-agent simulations. As a prescriptive tool, the framework can be effectively used to design the interactions capable of producing the target collective dynamics. These uses of the framework are exemplified in Sect. 3, where the framework is concretely applied to the study of the collective dynamics of multi-agent systems in which agents use the symmetric gossip algorithm [12,13]. The analytic results obtained using the framework are compared with independent multi-agent simulations, and the discussed comparison shows that the analytic results can accurately account for the results of simulations.

This paper is organized as follows. Sect. 2 provides a brief, but detailed, introduction to KTMAS by showing how the adopted analytic framework is obtained from general considerations and appropriate assumptions. Sect. 3 applies the framework to study the dynamics of multi-agent systems in which agents use the symmetric gossip algorithm. Sect. 4 compares the analytic results obtained using the framework with independent multi-agent simulations to confirm the effectiveness of the used approach. Sect. 5 briefly presents relevant related literature. Finally, Sect. 6 concludes the paper and overviews future research.

2 An Overview of the KTMAS Framework

Mathematical kinetic theories are all based on a common general framework (e.g., [4,5]) intended to study the collective dynamics of groups of interacting peers. This general framework assumes that the properties that characterize peers change because of interactions and environmental forces, and that studied groups are so large that their collective, and possibly emergent, properties can be adequately studied using a statistical approach. Note that this general framework is not sufficient to study specific groups of interacting peers because it must be turned into a model by completing it with the details needed to study specific phenomena in specific contexts. A model obtained by completing this general framework with the needed details can then be used to study the collective dynamics of the groups of peers for which the provided details are sufficiently descriptive of interactions and environmental forces.

2.1 The Strong Form of the Boltzmann Equation

The KTMAS framework, as outlined in this section, is a specific instantiation of the general framework of mathematical kinetic theories intended to study large and decentralized multi-agent systems composed of agents that autonomously exchange messages. In particular, the framework studies multi-agent systems composed of a static and large number $n \in \mathbb{N}_+$ of interacting agents, where each agent is uniquely identified by a natural number between 1 and n. Each agent has a state and, without major loss of generality, it is assumed that the state of an agent is represented as a real number $q \in Q$, where $Q \subseteq \mathbb{R}$ is an arbitrary open interval that contains the different states an agent can assume.

Agents experience autonomous interactions in the multi-agent system. Interactions are assumed to take the form of message exchanges, and each interaction involves only two agents. Therefore, the only considered form of interaction regards an agent r (the *receiver*) receiving a message from an agent s (the *sender*). Note that other forms of interaction (e.g., multicast or stigmergic [21]) can be treated by reducing them to pairwise message exchanges or by considering them explicitly in the KTMAS framework. Only the first option is used in this paper, and the second option is left for a future development of the framework.

Interactions are assumed to be instantaneous, which means that if an agent is involved in an interaction, then the interaction is completed before the agent can experience any other interaction. Therefore, interactions are mutually independent, and the effects of an interaction depend only on the states of involved agents. Each agent can interact with any other agent in the multi-agent system. Since interactions are the only interesting events in the current form of the KTMAS framework, time is modeled as a real variable in $\mathbb{R}_{\geq 0}$, and interactions occur at the instants of a Poisson point process with average rate $\nu \in \mathbb{R}_+$.

In its current form, the KTMAS framework does not consider environmental forces. Therefore, the state of an agent can change only because of interactions. Interactions are described in terms of how they change the states of the involved agents. In detail, the current form of the framework assumes that interactions are described in terms of proper *interaction rules* that link the *pre-interaction* states with the respective *post-interaction* states. Interaction rules are specific to the multi-agent system under investigation, and the framework leaves them unspecified until a model of the system is needed to actually study interesting collective, and possibly emergent, properties. Fig. 1 illustrates an example of an interaction in a multi-agent system. In particular, Fig. 1(a) shows that, at time t_i, an agent in state $q_s = 0.3$ interacts with an agent in state $q_r = 0.5$. Instantaneously, the states of the two agents change, and Fig. 1(b) shows the corresponding states $q'_s = 0.36$ and $q'_r = 0.44$ reached after the interaction. Note that previous interactions, which occur at times t_{i-1} and t_{i-2}, are sketched with dashed lines.

In accordance with the general framework of mathematical kinetic theories, $f : Q \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a function, called *density function*, such that $f(q,t)\,\mathrm{d}q$ counts the number of agents whose states are in $(q, q + \mathrm{d}q)$ at time $t \in \mathbb{R}_{\geq 0}$. Note that f is assumed to be sufficiently regular to support the analytic developments discussed in the remaining of this paper. The introduction of the density function f is sufficient to count the number of

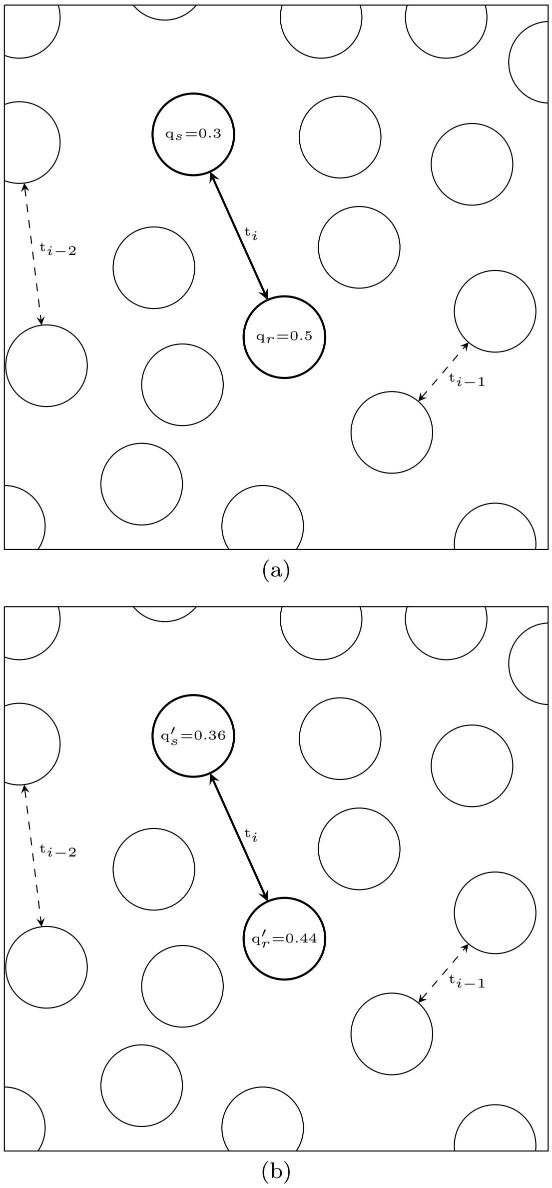

Fig. 1 Illustrative example of an interaction at time t_i with (a) pre-interaction states $q_s = 0.3$ and $q_r = 0.5$, and (b) post-interaction states $q'_s = 0.36$ and $q'_r = 0.44$.

agents $n \in \mathbb{N}_+$ in the multi-agent system

$$n = \int_Q f(q,t)\,\mathrm{d}q, \tag{1}$$

where the dependence of n on t is dropped because the number of agents in the multi-agent system is assumed to be static. Moreover, the introduction of the density function f can be used to characterize two relevant collective properties of the multi-agent system: the average state and the variance of the states. The average state at time $t \in \mathbb{R}_{\geq 0}$ is

$$\bar{q}(t) = \frac{1}{n}\int_Q q\, f(q,t)\,\mathrm{d}q, \tag{2}$$

while the variance of the states at time $t \in \mathbb{R}_{\geq 0}$ is

$$\sigma^2(t) = \frac{1}{n} \int_Q (q - \bar{q}(t))^2 f(q, t)\, \mathrm{d}q. \tag{3}$$

In order to study the evolution of the density function f, and therefore, to study the dynamics of the average state and of the variance of the states, recall that the current form of the KTMAS framework considers multi-agent systems in which interactions involve only two agents. If $t \in \mathbb{R}_{\geq 0}$ is an instant in which an interaction occurs in the multi-agent system, let

$$\mathrm{d}P(s, r, q_s, q_r, t) \tag{4}$$

be the probability that:

1. An arbitrary agent s is involved in the interaction;
2. An arbitrary agent r is involved in the interaction;
3. For an arbitrary $q_s \in Q$, agent s is in a state in $(q_s, q_s + \mathrm{d}q_s)$; and
4. For an arbitrary $q_r \in Q$, agent r is in a state in $(q_r, q_r + \mathrm{d}q_r)$.

One of the major assumptions of mathematical kinetic theories, which is often called *molecular chaos assumption* or *Stosszahlansatz* (German for *assumption on the count of collisions*), regards the possibility of expressing $\mathrm{d}P(s, r, q_s, q_r, t)$ as the product of three factors. These factors correspond to the following three events, which are considered as mutually independent by the molecular chaos assumption:

1. Agent s and agent r are chosen uniformly, and independently of previous interactions, to be the two agents involved in the considered interaction;
2. The state of agent s is in $(q_s, q_s + \mathrm{d}q_s)$; and
3. The state of agent r is in $(q_r, q_r + \mathrm{d}q_r)$.

Note that the molecular chaos assumption oversimplifies the dynamics of the interactions in the studied multi-agent system. However, if the multi-agent system is sufficiently large, and if the interactions can be treated as instantaneous, then the molecular chaos assumption is often sufficiently accurate. If the molecular chaos assumption holds, then $\mathrm{d}P(r, s, q_s, q_r, t)$ can be immediately written as

$$\mathrm{d}P(s, r, q_s, q_r, t) = \frac{2}{n^2} \cdot \frac{f(q_r, t)}{n}\, \mathrm{d}q_r \cdot \frac{f(q_s, t)}{n}\, \mathrm{d}q_s, \tag{5}$$

where the three factors of the product correspond to the probabilities of the three considered events. Actually, the molecular chaos assumption allows treating these events as mutually independent, and therefore their probabilities can be multiplied.

Now, let $\mathrm{d}L(q_r, t)$ be the average *loss* of agents with states in $(q_r, q_r + \mathrm{d}q_r)$ caused by the interaction that occurs at time $t \in \mathbb{R}_{\geq 0}$. Note that $\mathrm{d}L(q_r, t)$ can be computed by properly averaging $\mathrm{d}P(s, r, q_s, q_r, t)$ over the state q_s and over the agents r and s. Therefore,

$$\mathrm{d}L(q_r, t) = \sum_{r=1}^{n} \sum_{s=1}^{n} \int_Q \mathrm{d}P(t, r, s, q_r, q_s). \tag{6}$$

The molecular chaos assumption can be used to explicitate $\mathrm{d}L(q_r, t)$, as follows

$$\mathrm{d}L(q_r, t) = \mathrm{d}q_r \sum_{r=1}^{n} \sum_{s=1}^{n} \int_Q \frac{2}{n^2} \cdot \frac{f(q_r, t) f(q_s, t)}{n^2}\, \mathrm{d}q_s. \tag{7}$$

The previous expression of $\mathrm{d}L(q_r, t)$ can be easily simplified to obtain

$$\mathrm{d}L(q_r, t) = \frac{2}{n^2}\, \mathrm{d}q_r \int_Q f(q_r, t) f(q_s, t)\, \mathrm{d}q_s. \tag{8}$$

Recall that the current form of the KTMAS framework assumes instantaneous interactions, and therefore the considered interaction rule $\mathbf{t} : Q^2 \to Q^2$ is such that

$$(q'_s, q'_r) = \mathbf{t}(q_s, q_r). \tag{9}$$

Note that a loss of agents whose states are in the intervals $(q_s, q_s + \mathrm{d}q_s)$ and $(q_r, q_r + \mathrm{d}q_r)$ equals a gain of agents whose states are in the intervals $(q'_s, q'_s + \mathrm{d}q'_s)$ and $(q'_r, q'_r + \mathrm{d}q'_r)$. Let $\mathrm{d}G(q'_r, t)$ be the average *gain* of agents with states in $(q'_r, q'_r + \mathrm{d}q'_r)$ caused by the interaction that occurs at time $t \in \mathbb{R}_{\geq 0}$. The considerations used to obtain (8) can be used to obtain

$$\mathrm{d}G(q'_r, t) = \frac{2}{n^2}\, \mathrm{d}q_r \int_Q f(q_r, t) f(q_s, t)\, \mathrm{d}q_s. \tag{10}$$

The additional assumption that the interaction rule $\mathbf{t}$ is invertible over Q^2 allows reformulating $\mathrm{d}G(q'_r, t)$ as

$$\mathrm{d}G(q'_r, t) = \frac{2}{n^2}\, \mathrm{d}q'_r \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t)\, \mathrm{d}q'_s, \tag{11}$$

where $J(q'_s, q'_r)$ is the Jacobian of the considered interaction rule, which is defined as the absolute value of the determinant of the Jacobian matrix $\partial\mathbf{t}$ of $\mathbf{t}$, and the inverse of $\mathbf{t}$ is used to compute

$$(q_s, q_r) = \mathbf{t}^{-1}(q'_s, q'_r). \tag{12}$$

Note that the use of words gain and loss to refer to, respectively, increments and decrements of the number of agents in certain states is common to mathematical kinetic theories. Moreover, note that the additional assumption that the considered interaction rule is invertible over Q^2 severely limits the interactions that can be studied using the discussed form of the KTMAS framework. However, the results obtained in this section, in particular the weak form of the Boltzmann equation discussed at the end of this section, can be extended to locally invertible interaction rules at the cost of using a more involved notation.

Given a time interval $[t, t+\mathrm{d}t)$, the assumption that the interactions among agents occur at the instants of a Poisson point process with average rate $\nu \in \mathbb{R}_+$ implies that an average of $\nu\,\mathrm{d}t$ interactions occur in the considered interval. Therefore, the average loss of agents with states in $(q'_r, q'_r + \mathrm{d}q'_r)$ in the considered time interval can be computed using (8) as

$$\beta\,\mathrm{d}t\,\mathrm{d}q'_r \int_Q f(q'_r, t) f(q'_s, t)\,\mathrm{d}q'_s, \tag{13}$$

where $\beta = \frac{2}{n^2}\nu$. Similarly, the average gain of agents with states in $(q'_r, q'_r + \mathrm{d}q'_r)$ in the considered time interval can be computed using (11) as

$$\beta\,\mathrm{d}t\,\mathrm{d}q'_r \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t)\,\mathrm{d}q'_s. \tag{14}$$

Therefore, the balance of the average gain and the average loss can be used to compute the average net increment of agents $f(q'_r, t+\mathrm{d}t)\,\mathrm{d}q'_r - f(q'_r, t)\,\mathrm{d}q'_r$, as follows

$$\beta\,\mathrm{d}q'_r\,\mathrm{d}t \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t) - f(q'_r, t) f(q'_s, t)\,\mathrm{d}q'_s, \tag{15}$$

which immediately leads to the following *(strong form of the) Boltzmann equation*

$$\frac{\partial f}{\partial t}(q'_r, t) = \beta \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t) - f(q'_r, t) f(q'_s, t)\,\mathrm{d}q'_s. \tag{16}$$

Note that the name chosen for this equation is not incidental. Actually, (16) is a generalization to the abstract setting of mathematical kinetic theories of the classic equation devised for the kinetic theory of gases by Ludwig Boltzmann in 1872. The Boltzmann equation for a group of peers is the core of all mathematical kinetic theories because it expresses the dynamics of the group once a model of the considered interactions is chosen and the overall dynamics of interactions is coherent with the molecular chaos assumption.

The explicit expression of (16) requires to carefully select the details to include in the description of the interactions among peers, but these details are normally left unspecified in the general framework of mathematical kinetic theories. Sect. 3 implicitly uses an expression of (16) that details how agents interact to follow the symmetric gossip algorithm. This expression is sufficient to prove the correctness of the algorithm under the assumptions of the KTMAS framework and to study the expected long-time asymptotic dynamics of the studied multi-agent systems.

2.2 The Weak Form of the Boltzmann Equation

The strong form of the Boltzmann equation (16) provides a fine-grained characterization of the dynamics of the states of the agents, which is normally too fine-grained to be feasible for large multi-agent systems. Therefore, the study of the dynamics of relevant collective properties is often preferred. Consider a sufficiently regular *test function* $\phi : Q \to \mathbb{R}$. Then, multiply by $\phi(q'_r)$ and integrate in q'_r over Q both sides of (16) to obtain the following right-hand side

$$\beta \int_Q \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t) - f(q'_r, t) f(q'_s, t) \phi(q_r)\,\mathrm{d}q'_s\,\mathrm{d}q'_r. \tag{17}$$

Now, note that (16) can be rewritten by swapping the roles of agent r and agent s to obtain

$$\frac{\partial f}{\partial t}(q'_s, t) = \beta \int_Q \frac{1}{J(q'_s, q'_r)} f(q_r, t) f(q_s, t) - f(q'_r, t) f(q'_s, t)\,\mathrm{d}q'_r, \tag{18}$$

whose both sides can be multiplied by $\phi(q'_s)$ and integrated in q'_s over Q to obtain the right-hand side

$$\beta \int_Q \int_Q \frac{1}{J(q'_s, q'_r)} f(q_s, t) f(q_r, t) - f(q'_s, t) f(q'_r, t) \phi(q'_s)\,\mathrm{d}q'_r\,\mathrm{d}q'_s. \tag{19}$$

Since $J(q_s, q_r) = J(q_r, q_s)$ and dummy variables in integrals can be freely renamed, previous considerations lead to the following equation

$$\int_Q \frac{\partial f}{\partial t}(q_s, t)\phi(q_s)\,\mathrm{d}q_s = \frac{\beta}{2} \int_Q \int_Q \Delta(q_s, q_r) \cdot f(q_s, t) f(q_r, t)\,\mathrm{d}q_r\,\mathrm{d}q_s, \tag{20}$$

where

$$\Delta(q_s, q_r) = \phi(q'_r) + \phi(q'_s) - \phi(q_r) - \phi(q_s), \tag{21}$$

which depends only on q_s and q_r because of (9). Note that (20) is normally called *weak form of the Boltzmann equation*. Finally, note that if the considered interaction rule is such that $\mathbf{t}(q_r, q_s) = \mathbf{t}(q_s, q_r)$, then (20) can be easily simplified to

$$\int_Q \frac{\partial f}{\partial t}(q_s, t)\phi(q_s)\,\mathrm{d}q_s = \beta \int_Q \int_Q (\phi(q'_r) - \phi(q_r)) \cdot f(q_s, t) f(q_r, t)\,\mathrm{d}q_r\,\mathrm{d}q_s, \tag{22}$$

which is easier to use because it does not depend on q'_s. Also, note that the interaction rule considered in Sect. 3 exhibits this symmetry, and therefore (22) is used in Sect. 3 instead of (20).

The relevance of the weak form of the Boltzmann equation to study the collective properties of multi-agent systems can be clarified as follows. If the details of the interactions that occur in the multi-agent system are made explicit, by stating an explicit expression of the interaction rule, the right-hand sides of (20) and (22) can be also made explicit. Therefore, for an explicit interaction rule and a fixed ϕ, the weak form of the Boltzmann equation becomes an ordinary differential equation that describes the dynamics of the collective property entailed by the chosen ϕ. For example, if $\phi(q) = q$ is chosen, then (22) can be written as

$$n\frac{\mathrm{d}\bar{q}}{\mathrm{d}t}(t) = \beta \int_Q \int_Q (q_r' - q_r)\, f(q_r,t) f(q_s,t)\, \mathrm{d}q_r\, \mathrm{d}q_s, \quad (23)$$

which is the equation used in Sect. 3 to study the average state of the agents. Similarly, if $\phi(q) = (q - \bar{q}(t))^2$ is chosen, then (22) can be written as

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = \beta \int_Q \int_Q \left[(q_r' - \bar{q}(t))^2 - (q_r - \bar{q}(t))^2\right] \cdot f(q_s,t) f(q_r,t) f(q_s,t)\, \mathrm{d}q_r\, \mathrm{d}q_s, \quad (24)$$

and this equation can be used to study the dynamics of the variance of the states of the agents. This choice of ϕ is used in Sect. 3 to prove that the variance of the states of the agents that follow the symmetric gossip algorithm tends to zero as time tends to infinity, which ensures that all agents would eventually tend to have the same asymptotic state.

3 A Study of the Symmetric Gossip Algorithm

The *distributed averaging problem* (e.g., [12,13]) is a well-known problem related to multi-agent systems that finds important applications, for example, in sensor networks and social networks. The motivating application in sensor networks is related to sensors that jointly measure the characteristics of physical phenomena. For example, a toy scenario to motivate the distributed averaging problem regards the sensing of the temperature in a room using a network of sensors [12, 13]. Sensors are deployed to measure the temperature of the region and, to combat minor fluctuations in ambient temperature and noise in sensor readings, sensors need to average their readings. The application in social networks is similar, and it is about *compromise* (e.g., [40]), which is one of the fundamental phenomena that govern opinion formation, and which is considered as the major force that enables decentralized consensus in multi-agent systems. In the assortment of algorithms proposed to solve the distributed averaging problem, the algorithm first introduced in [13], and called *symmetric gossip algorithm* in the nomenclature proposed in [26], can be used to describe a concrete application of the KTMAS framework. The remaining of this section applies the KTMAS framework to study the long-time asymptotic properties of multi-agent systems in which agents follow the symmetric gossip algorithm.

3.1 The model of the multi-agent systems

In the studied multi-agent systems, each agent is characterized by a state $q \in Q$, where $Q \subset \mathbb{R}$ is an open and bounded interval, which is assumed to be $(-1, 1)$ without loss of generality. Each agent is requested to repeatedly exchange messages with other agents to reach consensus on the average of their initial states. Each agent s repeatedly chooses another agent r at random and sends a message to agent r. A message from agent s to agent r contains the current state of agent s, and it is used by agent r to reply and update its state. Given that agents update their states only upon receiving messages, the updates are based on their current states and on the states contained in the received messages.

The symmetric gossip algorithm fixes the function that an agent r uses to update its state upon receiving a message from another agent s. The adopted function is a linear combination of the current state of agent r and of the state contained in the received message. Moreover, the algorithm assumes that, immediately before updating its state, agent r replies with a message containing its state, which is then used by agent s to update its state. Note that the algorithm assumes some form of synchronization because messages and related replies are supposed to contain the current states of interacting agents. This assumption is a characteristic of the symmetric gossip algorithm, and it is considered appropriate for intended applications.

If agent r, in state q_r, interacts with agent s, in state q_s, the symmetric gossip algorithm requires agents to mutually exchange their current states and to update their states using the following interaction rule (adapted from [13])

$$\begin{aligned} q_s' &= t_s(q_s, q_r) = q_s - \gamma(q_s - q_r) \\ q_r' &= t_r(q_s, q_r) = q_r - \gamma(q_r - q_s), \end{aligned} \quad (25)$$

where q_s' and q_r' are the updated states of agent s and of agent r, respectively, and $\gamma \in (0, \frac{1}{2})$ is a parameter of the symmetric gossip algorithm. Following the nomenclature of mathematical kinetic theories, q_s and q_r are the pre-interaction states of agent s and of agent r, respectively, while q_s' and q_r' are the corresponding post-interaction states. Note that the adopted interaction rule $(q'_s, q'_r) = \mathbf{t}(q_s, q_r)$ is invertible because its Jacobian matrix can be written as

$$\partial\mathbf{t}(q_s, q_r) = \begin{pmatrix} 1-\gamma & \gamma \\ \gamma & 1-\gamma \end{pmatrix}, \tag{26}$$

and therefore the Jacobian $J(q_s, q_r) = |1 - 2\gamma|$ is positive for $\gamma \in (0, \frac{1}{2})$. Moreover, note that the adopted interaction rule is such that, for all $q_s \in Q$ and $q_r \in Q$, $t_s(q_s, q_r) = t_r(q_r, q_s)$.

Before using the KTMAS framework to study the long-time asymptotic properties of the multi-agent systems in which the agents follow the symmetric gossip algorithm, some considerations on the adopted interaction rule are needed. First, note that post-interaction states belong to interval $Q = (-1, 1)$ because the following inequalities hold

$$\begin{aligned} |q'_s| &\le (1-\gamma)|q_s| + \gamma|q_r| \le \max\{|q_r|, |q_s|\} < 1 \\ |q'_r| &\le (1-\gamma)|q_r| + \gamma|q_s| \le \max\{|q_r|, |q_s|\} < 1. \end{aligned} \tag{27}$$

Then, note that the following equality can be easily derived from the adopted interaction rule

$$q'_s + q'_r = q_s + q_r, \tag{28}$$

which implies that interactions do not change the average state of the agents. Finally, note that each interaction reduces the distance of the states of the two interacting agents because $\gamma \in (0, \frac{1}{2})$, and therefore the following inequality holds

$$|q'_s - q'_r| = |1 - 2\gamma||q_s - q_r| < |q_s - q_r|. \tag{29}$$

These considerations ensure that, after a sufficiently large number of interactions, all agents would eventually tend to the same state, which is necessarily the average of the initial states. The understanding of how quickly the states of the agents tend to the average of the initial states requires further investigations, as shown in the remaining of this section.

Finally, note that the symmetric gossip algorithm assumes that an underlying interaction protocol [45] is used by agents. In order to ensure that two interacting agents can mutually exchange their states to compute their new states using (25), it is necessary that two messages are exchanged. The KTMAS framework does not provide a specific support to reason on interaction protocols, but in this case the protocol is only intended to implement the adopted interaction rule, and therefore it is supposed to be executed atomically in the scope of a single interaction. Actually, this underlying protocol is not problematic for the KTMAS framework because it is embedded in adopted interaction rule. The presence of this embedded protocol is a characteristic of the studied algorithm, which also justifies the symmetric adjective in its name. Asymmetric variants of the studied algorithm have already been proposed in the literature (e.g., [2,25]). Their study using the KTMAS framework represents an interesting application of the framework reserved for future works.

3.2 Analytic results

The interest now is on using the KTMAS framework to study the dynamics of the states of the agents. The following propositions accurately describe the dynamics of the states of the agents using the KTMAS framework.

Proposition 1 *Given a multi-agent system in which the interactions follow the symmetric gossip algorithm and that satisfies the molecular chaos assumption, the average state of the agents is constant over time.*

Proof The following equation is obtained by setting $\phi(q) = q$ in (22) after ordinary manipulations

$$n\frac{\mathrm{d}\bar{q}}{\mathrm{d}t}(t) = \beta \int_{Q^2} (t_r(q_s, q_r) - q_r) \cdot f(q_s, t) f(q_r, t)\, \mathrm{d}q_s\, \mathrm{d}q_r. \tag{30}$$

Then, using the adopted interaction rule (25) to expand $t_r(q_s, q_r)$, the previous weak form of the Boltzmann equation becomes

$$n\frac{\mathrm{d}\bar{q}}{\mathrm{d}t}(t) = \beta\gamma \int_{Q^2} (q_s - q_r) f(q_s, t) f(q_r, t)\, \mathrm{d}q_r\, \mathrm{d}q_s. \tag{31}$$

Note that the right-hand side of the previous equation can be rewritten as

$$\beta\gamma \left(\int_Q f(q_r, t)\, \mathrm{d}q_r \int_Q q_s f(q_s, t)\, \mathrm{d}q_s - \int_Q f(q_s, t)\, \mathrm{d}q_s \int_Q q_r f(q_r, t)\, \mathrm{d}q_r \right). \tag{32}$$

Therefore, the weak form of the Boltzmann equation for $\phi(q) = q$ becomes

$$\frac{\mathrm{d}\bar{q}}{\mathrm{d}t}(t) = 0, \tag{33}$$

which ensures that the average state of the agents in the multi-agent system is constantly equal to the initial average $q_0 \in Q$. □

Proposition 2 *Given a multi-agent system in which the interactions follow the symmetric gossip algorithm and that satisfies the molecular chaos assumption, the variance of the states of the agents is*

$$\sigma^2(t) = \sigma_0^2\, \mathrm{e}^{-\frac{4}{n}\gamma(1-\gamma)\nu t}, \tag{34}$$

where $n \in \mathbb{N}_+$ is the number of agents in the multi-agent system, $\nu \in \mathbb{R}_+$ is the average interaction rate, $\gamma \in (0, \frac{1}{2})$ is the parameter of the symmetric gossip algorithm, and $\sigma_0^2 = \sigma^2(0)$.

Proof The variance of the states can be studied by setting $\phi(q) = (q - \bar{q})^2$ in the weak form of the Boltzmann equation (22), dropping the dependence of $\bar{q}$ on t because of Proposition 1. Therefore,

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = \beta \int_{Q^2} f(q_s, t) f(q_r, t) \cdot [(t_r(q_s, q_r) - \bar{q})^2 - (q_r - \bar{q})^2] \,\mathrm{d}q_s \,\mathrm{d}q_r. \quad (35)$$

The previous equation can be simplified to

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = \beta \int_{Q^2} f(q_s, t) f(q_r, t) \cdot [t_r^2(q_s, q_r) - q_r^2 - 2\bar{q}(t_r(q_s, q_r) - q_r)] \,\mathrm{d}q_s \,\mathrm{d}q_r. \quad (36)$$

Now, note that the term that contains $(t_r(q_s, q_r) - q_r)$ in the previous equation is proportional to the right-hand side of (30), which Proposition 1 proves to equal zero. Therefore, the previous formulation of the weak form of the Boltzmann equation can be written as

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = \beta \int_{Q^2} f(q_s, t) f(q_r, t) \cdot (t_r^2(q_s, q_r) - q_r^2) \,\mathrm{d}q_s \,\mathrm{d}q_r. \quad (37)$$

The adopted interaction rule (25) can be used to make $t_r(q_s, q_r)$ explicit in the previous equation to obtain

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = \beta \int_{Q^2} f(q_s, t) f(q_r, t) \cdot [\gamma^2(q_r - q_s)^2 - 2\gamma q_r(q_r - q_s)] \,\mathrm{d}q_s \,\mathrm{d}q_r. \quad (38)$$

Simple algebraic manipulations allow obtaining the following formulation of the weak form of the Boltzmann equation for $\phi(q) = (q - \bar{q})^2$

$$n\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = 2\beta\gamma(\gamma - 1)n \cdot \left(\int_Q f(q_r, t) q_r^2 \,\mathrm{d}q_r - n\bar{q}^2 \right). \quad (39)$$

Note that the last factor of the previous equation is nothing but $n\sigma^2(t)$, and therefore, the following ordinary differential equation that describes the dynamics of the variance of the states is obtained

$$\frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t) = 2\beta\gamma(\gamma - 1)n\sigma^2(t). \quad (40)$$

The previous equation can be easily solved to obtain a closed-form expression of the variance of the states

$$\sigma^2(t) = \sigma_0^2 \,\mathrm{e}^{-2\beta\gamma(1-\gamma)nt}, \quad (41)$$

which proves the proposition because $\beta = \frac{2}{n^2}\nu$ and the initial variance of the states is $\sigma_0^2 = \sigma^2(0)$. □

4 A Comparison with Multi-Agent Simulations

Proposition 1 guarantees that the average state of the agents in a multi-agent system in which agents follow the symmetric gossip algorithm is constant over time. Proposition 2 ensures that all agents would eventually tend to the same state, which equals the initial average state for Proposition 1. The following illustrative simulations compare the analytic results derived from previous propositions with the actual behavior of a multi-agent system in which the agents follow the symmetric gossip algorithm. Note that an in-depth comparison between the analytic results derived from previous propositions and the characteristics of the simulated multi-agent system is out of the scope of this paper.

The considered multi-agent system is composed of $n = 100$ agents that follow the symmetric gossip algorithm. For each simulation, the states of the agents are initially set to random values uniformly distributed in $Q = (-1, 1)$, so that the initial variance of the states is $\sigma_0^2 = \frac{1}{3}$. Each simulation comprises $\tau = 10^3$ steps, and at every step, which corresponds to one unit of time, two agents are randomly chosen and their states are updated using (25). Note that in this setting, the average interaction rate ν equals one because only one interaction occurs at each step of the simulation.

Fig. 2 and Fig. 3 show the variances of the states for four simulations obtained using $\gamma \in \{0.1, 0.2, 0.3, 0.4\}$. The corresponding variances computed using (41) are also shown for $\beta = 2 \cdot 10^{-4}$. As expected, the shown variances exponentially decrease toward zero as time increases. A comparison among the plots confirms that the variances obtained using (41) adequately fit the variances obtained by simulations. In particular, for a given $\gamma \in \{0.1, 0.2, 0.3, 0.4\}$, the largest distance between the variance obtained using simulations and the corresponding variance obtained using (41) are: 0.025 for $\gamma = 0.1$, 0.022 for $\gamma = 0.2$, 0.018 for $\gamma = 0.3$, and 0.023 for $\gamma = 0.4$. Also, the plots confirm that the rate of convergence increases as γ increases. This is not surprising because the adopted interaction rule is such that the distance between the post-interaction states of two interacting agents decreases as γ increases in $(0, \frac{1}{2})$. Note that this property of the adopted interaction rule is confirmed by (41) because $\gamma(1 - \gamma) > 0$ if $\gamma \in (0, \frac{1}{2})$.

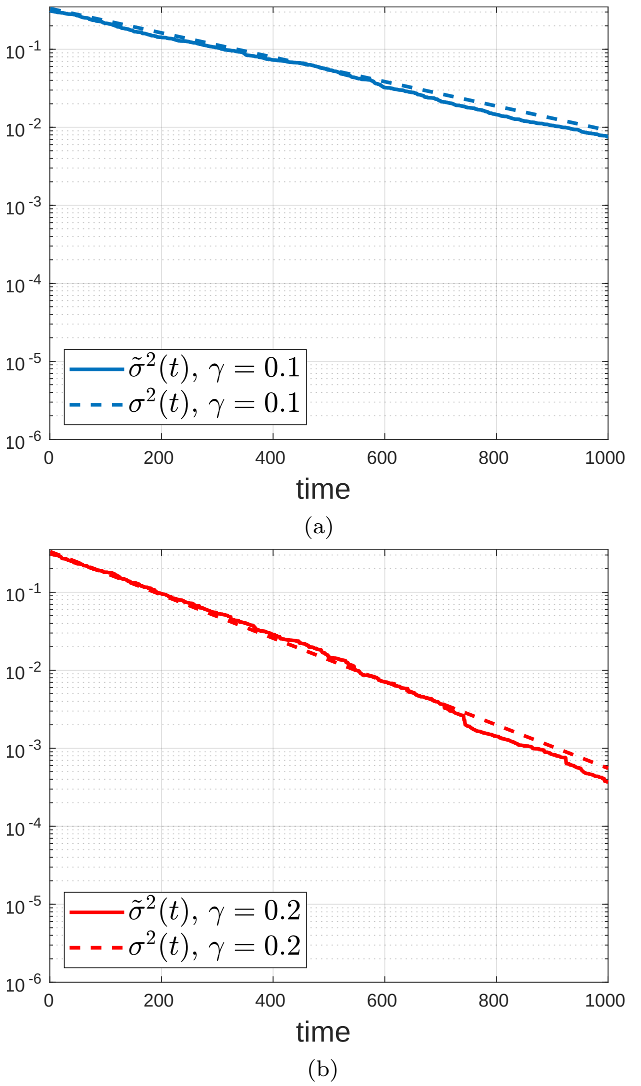

Fig. 2 Plots in logarithmic scale of the variance $\tilde{\sigma}^2(t)$ obtained with multi-agent simulations (solid lines) and the variance $\sigma^2(t)$ computed using (41) with $\beta = 2 \cdot 10^{-4}$ (dashed line) for $n = 100$ agents whose initial states are uniformly distributed in $Q = (-1, 1)$ and that follow the symmetric gossip algorithm with (a) $\gamma = 0.1$ and (b) $\gamma = 0.2$.

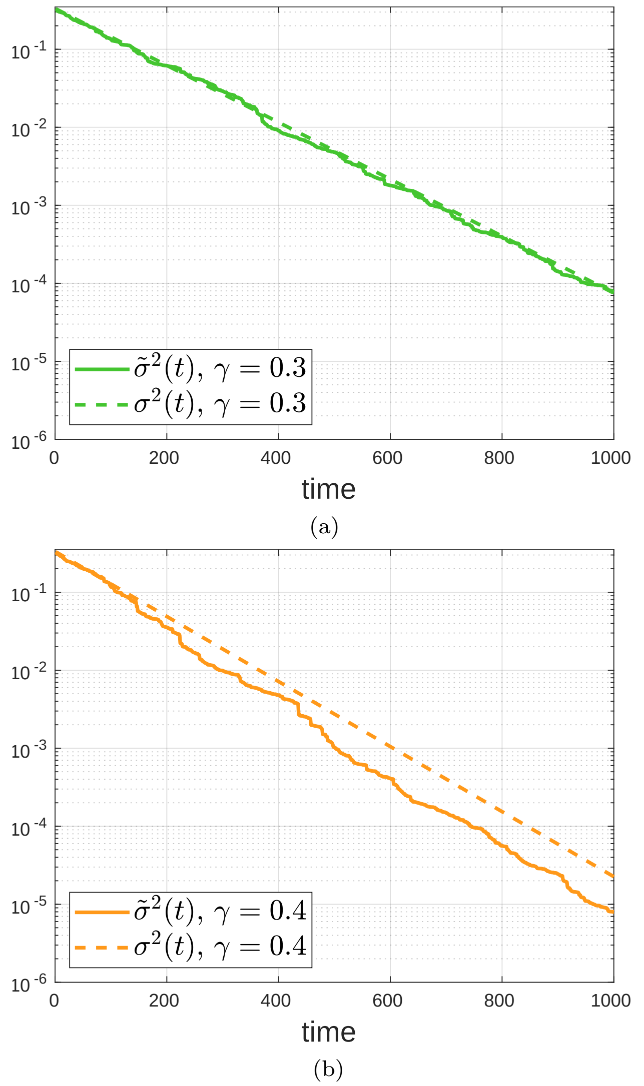

Fig. 3 Plots in logarithmic scale of the variance $\tilde{\sigma}^2(t)$ obtained with multi-agent simulations (solid lines) and the variance $\sigma^2(t)$ computed using (41) with $\beta = 2 \cdot 10^{-4}$ (dashed line) for $n = 100$ agents whose initial states are uniformly distributed in $Q = (-1, 1)$ and that follow the symmetric gossip algorithm with (a) $\gamma = 0.3$ and (b) $\gamma = 0.4$.

5 Related Work

Besides the importance of multi-agent systems for artificial intelligence, as witnessed by the significant body of literature that originated, for example, from [33], large multi-agent systems have been recently attracted considerable attention for their direct link with relevant applications like social networks and sensor networks. It is common opinion that large multi-agent systems have specific peculiarities, and that ordinary methods and tools are not immediately applicable to study them (e.g., [34]). In addition, large multi-agent systems are particularly important in applications that are characterized by decentralized control (e.g., [31]). This is not surprising since decentralized control is assumed to scale for the number of agents better than centralized control, and therefore decentralized control is the most effective, and obvious, choice when the number of agents is large (e.g., [55]).

When multi-agent systems are large and decentralized, the ordinary methods and tools to study their dynamics (e.g., [46]) tend to become unfeasible, and alternative approaches are needed. In these cases, statistical approaches provide preferable ways to investigate the, possibly emergent, properties of multi-agent systems because they move the focus from the dynamics of single agents to the dynamics of the multi-agent system as a whole (e.g., [1,15]).

The KTMAS framework relies on a statistical approach, and it explicitly takes into account that studied multi-agent systems are assumed to be large and decentralized. The framework uses a model-reduction approach to obtain a reduced-order model that preserves the important properties of the original model. As such, the framework is related to relevant works on *chemical reaction networks* (e.g., [18,20,27,52]) and *compartmental models* (e.g., [14]). The analytic nature of the framework ensures that it can be used as a prescriptive tool to answer the major question of collective intelligence: *"How, without any detailed modeling of the overall system, can one set utility functions* [...] *so that the overall dynamics reliably and robustly achieves large values of the provided world utility?"* [53]

Mathematical kinetic theories share relevant similarities with *fluid approximation* (e.g., [10]), which has been recently introduced to study the collective properties of stochastic process algebra models of large populations. Fluid approximation can be applied if the studied model contains many instances of few agent types. It works by treating as continuous the variables that count how many agents of each type are in each state, and by treating the rates of the stochastic transitions as flows, thus obtaining ordinary differential equations that describe the dynamics of the system. For example, [17] uses fluid approximation to relate an approximate majority protocol to chemical reactions in cells, thus obtaining results on the speed of convergence that are strongly related to the results shown in Sect. 3. Note that [19] discusses in detail the relation between fluid approximation and the dynamics of chemical reactions. Similarly to the KTMAS framework, the models discussed in [19] are based on the assumption that molecules react with probabilities proportional to reaction parameters and molecule concentrations.

Similarly to fluid approximation, *mean-field approximation* (e.g., [11]) starts from a stochastic model designed to study systems consisting of a large number of interacting agents, each of which can be in one of few states. Then, mean-field approximation can be used to count how many agents are in a given state, thus obtaining a limit theorem similar to the corresponding limit theorem for fluid approximation. Notably, mean-field approximation was used to study multi-agent systems in which agents follow a variant of the symmetric gossip algorithm discussed in Sect. 3, obtaining analogous results [3]. In-depth comparisons of the KTMAS framework with fluid approximation and mean-field approximation are reserved for future works.

In recent years, models based on the general approach of mathematical kinetic theories have been applied to diverse research domains to study groups of peers that interact within a shared environment under the influence of external forces. The prototypical example of a mathematical kinetic theory is the classic *kinetic theory of gases* (e.g., [35]), which studies the collective properties of gases, like temperature and pressure, starting from the details of the interactions among molecules (or atoms, for noble gases). A rather obvious parallelism between the molecules of a gas and the agents of a multi-agent system can be drawn to adopt generalizations of the kinetic theory of gases to study collective, and possibly emergent, properties of multi-agent systems. For example, [48] studies the similarity between the distribution of wealth in a simple economy and the density of molecules in a gas, and [9] studies the dynamics of wealth taking a similar approach. Similarly, [41,49,51] study models of opinion dynamics using a formalism based on the kinetic theory of gases, while [40] extends previous studies on opinion dynamics using the *kinetic theory of gas mixtures* (e.g., [8]). Unfortunately, besides the general framework of mathematical kinetic theories, few analytic results from the kinetic theory of gases can be used in the KTMAS framework. Actually, the details of the collisions among molecules in gases are significantly different from the details of the interactions among agents in multi-agent systems. Therefore, the KTMAS framework drops the assumption that agents are immersed in the physical world and that they are characterized by mechanical properties, like positions and velocities. The KTMAS framework abstracts this assumption away, thus substantially separating KTMAS from the kinetic theory of gases and its generalizations.

Besides the multiple applications of the general approach of mathematical kinetic theories, the literature proposes several papers that document how models inspired by physics are used to study the collective, and possibly emergent, properties of multi-agent systems. In the early 1990s, the word *econophysics* [22,24,36] was proposed to designate an interdisciplinary research field that applies methods developed by physicists to study economic phenomena. Similarly, the word *sociophysics* [22,29] was introduced to describe an interdisciplinary research field that uses methods inspired by physics to study the behaviors of groups of individuals. Similar points of view have been proposed several times (e.g., [43,47]). All proposals recognize that the collective, and possibly emergent, properties of multi-agent systems can be studied by treating multi-agent systems as *complex systems* [38,50]. The KTMAS framework takes a similar approach, but the framework is not described in terms of adaptations of existing formalisms. Rather, the framework is only based on the characteristics of large and decentralized multi-agent systems.

6 Conclusion

This paper is intended to contribute results to advance the research toward the definition of a complete and coherent KTMAS. The initial part of the paper motivates the urge of effective methods and tools to study the collective, and possibly emergent, properties of multi-agent systems. Then, the paper outlines an analytic framework, based on mathematical kinetic theories, designed to study the collective, and possibly emergent, properties of multi-agent systems taking a statistical perspective. The discussed framework can be considered as the basis of the target KTMAS because it can be effectively used to study the collective, and possibly emergent, properties of multi-agent systems when the studied systems can be considered as large and decentralized. In particular, the discussed KTMAS framework targets the analytic study of long-time asymptotic properties of large and decentralized multi-agent systems in which the collective, and possibly emergent, properties derive from the decentralized interactions among agents. Considered interactions are assumed to be instantaneous and, to work toward analytic results in closed form, sufficiently simple.

It is worth noting that, even if the KTMAS framework is tightly connected to the kinetic theory of gases, it derives from very general considerations regarding the effects of interactions on the states of the agents. Therefore, the equations of the framework are not simple variations of analogous equations of the kinetic theory of gases as obtained, for example, by drawing simplistic analogies between molecules and agents. Rather, the equations obtained in Sect. 2 are genuine derivations from the assumptions taken to describe the effects of interactions on the states of the agents in large and decentralized multi-agent systems.

Finally, the paper describes the application of the KTMAS framework to the study of an illustrative problem intended to show a concrete example of the use of the framework. The problem is to study the collective, and possibly emergent, properties of multi-agent systems in which agents interact following the symmetric gossip algorithm. As expected, the application of the framework allows demonstrating that the states of all agents converge to a single value, which corresponds to the initial average of the states of the agents. Moreover, the framework allows demonstrating that the agents tend to this value exponentially fast. Both these results are confirmed by independent multi-agent simulations. In particular, these simulations show that the collective properties of the simulated multi-agent systems are accurately predicted by the expected exponential convergence to the initial average state.

Methodologically, the major advantages that are expected from the adoption of the KTMAS framework to study the dynamics of large and decentralized multi-agent systems derive from the analytic nature of the framework. The analytic nature of the framework ensures that obtained results can be used both as descriptive tools to explain observations and as prescriptive tools to design the desired dynamics of multi-agent systems. As a descriptive tool, the framework can be used as an alternative to simulations. The validity of the results of simulations depends on how much simulations are representative of the studied multi-agent systems. On the contrary, the validity of analytic results is clearly identified by the assumptions adopted to derive them. As a prescriptive tool, the framework can support the design of multi-agent systems with desired collective, and possibly emergent, properties. The analytic results of the framework can be used to identify the values of the design parameters to ensure that the multi-agent system behaves as intended. Essentially, the KTMAS framework addresses the major challenge of the research on collective intelligence, which regards designing interactions to obtain desired collective, and possibly emergent, properties [53].

The results presented in this paper are not meant to be conclusive. The discussed KTMAS framework targets multi-agent systems under specific assumptions, which could be alleviated in future works. In particular, the future developments of this research intend to provide a solid base for a complete and coherent KTMAS in terms of four relevant extensions. First, stochastic interaction rules are urged to allow introducing randomness in the interactions among agents (e.g. [39]). This is a relevant extension of the discussed framework because stochastic interaction rules can be used to model genuinely random behaviors and behaviors whose details are not sufficiently relevant to deserve exhaustive descriptions. Second, interaction rules that involve more than two agents are needed to extend the reach of the framework to include real-world multi-agent systems based on multicasting. Third, appropriate networks to constraint the exchange of messages among agents are required to study real-world multi-agent systems, which rarely assume that an agent can freely interact with any other agent. Note that, in the long term, the intention is to incorporate dynamic and stochastic networks in the KTMAS framework. Fourth, the treatment of dynamic populations is needed to study multi-agent systems in which agents dynamically join and leave the system. Real-world multi-agent systems are often characterized by dynamic populations, and they often have specific interaction rules that are intended to govern the dynamic inclusion and exclusion of agents.

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References

1. Adiga, A., Kuhlman, C.J., Mortveit, H.S., Vullikanti, A.K.S.: Sensitivity of diffusion dynamics to network uncertainty. Journal of Artificial Intelligence Research **51**, 207–226 (2014)
2. Asensio-Marco, C., Beferull-Lozano, B.: Fast average gossiping under asymmetric links in WSNS. In: Proceedings of the 22nd European Signal Processing Conference (EUSIPCO 2014). pp. 131–135. IEEE (2014)
3. Bakhshi, R., Cloth, L., Fokkink, W., Haverkort, B.: Mean-field analysis for the evaluation of gossip protocols. In: Proceedings of 6th International Conference on the Quantitative Evaluation of Systems (QEST 2009). pp. 247–256. IEEE (2009)
4. Bellomo, N., Burini, D., Dosi, G., Gibelli, L., Knopoff, D., Outada, N., Terna, P., Virgillito, M.E.: What is life? A perspective of the mathematical kinetic theory of active particles. Mathematical Models and Methods in Applied Sciences **31**(9), 1821–1866 (2021)
5. Bellouquid, A., Delitala, M.: Mathematical Modeling of Complex Biological Systems. Modeling and Simulation in Science, Engineering and Technology, Birkhäuser (2006)
6. Bergenti, F., Huhns, M.N.: On the use of agents as components of software systems. In: Methodologies and Software Engineering for Agent Systems: The Agent-Oriented Software Engineering Handbook. pp. 19–32. Kluwer Academic Publishing (2004)
7. Bergenti, F., Ricci, A.: Three approaches to the coordination of multiagent systems. In: Proceedings of the ACM Symposium on Applied Computing (SAC 2002). pp. 367–372. ACM (2002)
8. Bianca, C., Dogbe, C.: On the Boltzmann gas mixture equation: Linking the kinetic and fluid regimes. Communications in Nonlinear Science and Numerical Simulation **29**, 240–256 (2015)
9. Boghosian, B.M.: Kinetics of wealth and the Pareto law. Physical Review E **89**(4) (2014)
10. Bortolussi, L., Hillston, J., Latella, D., Massink, M.: Continuous approximation of collective system behaviour: A tutorial. Performance Evaluation **70**(5), 317–349 (2013)
11. Boudec, J.Y., McDonald, D., Mundinger, J.: A generic mean field convergence result for systems of interacting objects. In: Proceedings of the 4th International Conference on the Quantitative Evaluation of Systems (QEST 2007). IEEE (2007)
12. Boyd, S., Ghosh, A., Prabhakar, B., Shah, D.: Analysis and optimization of randomized gossip algorithms. In: Proceedings of the 2004 IEEE Conference on Decision and Control. pp. 5310–5315. IEEE (2004)
13. Boyd, S., Ghosh, A., Prabhakar, B., Shah, D.: Randomized gossip algorithms. IEEE Transactions on Information Theory **52**(6), 2508–2530 (2006)
14. Brauer, F.: Compartmental models in epidemiology. In: Mathematical Epidemiology. Lecture Notes in Mathematics, vol. 1945, pp. 19–79. Springer (2008)
15. van den Broek, B., Wiegerinck, W., Kappen, B.: Graphical model inference in optimal control of stochastic multi-agent systems. Journal of Artificial Intelligence Research **32**, 95–122 (2008)
16. Bures, T., Nicola, R.D., Gerostathopoulos, I., Hoch, N., Kit, M., Koch, N., Monreale, G.V., Montanari, U., Pugliese, R., Serbedzija, N.B., Wirsing, M., Zambonelli, F.: A life cycle for the development of autonomic systems: The e-mobility showcase. In: Proceedings of the 7th IEEE International Conference on Self-Adaptation and Self-Organizing Systems Workshops (SASOW 2013). pp. 71–76. IEEE (2013)
17. Cardelli, L., Csikász-Nagy, A.: The cell cycle switch computes approximate majority. Scientific Reports **2**(656) (2012)
18. Cardelli, L., Tribastone, M., Tschaikowski, M.: From electric circuits to chemical networks. Natural Computing **19**, 237–248 (2020)
19. Cardelli, L., Tribastone, M., Tschaikowski, M., Vandin, A.: Syntactic Markovian bisimulation for chemical reaction networks. In: Models, Algorithms, Logics and Tools: Essays Dedicated to Kim Guldstrand Larsen on the Occasion of His 60th Birthday. Lecture Notes in Computer Science, vol. 10460, pp. 466–483. Springer (2017)
20. Cardelli, L., Tribastone, M., Tschaikowski, M., Vandin, A.: Symbolic computation of differential equivalences. Theoretical Computer Science **777**, 132–154 (2019)
21. Castelli, G., Mamei, M., Rosi, A., Zambonelli, F.: Engineering pervasive service ecosystems: The SAPERE approach. ACM Transactions on Autonomous and Adaptive Systems **10**(1), 1:1–1:27 (2015)
22. Chakrabarti, B.K., Chakraborti, A., Chatterjee, A.: Econophysics and sociophysics: Trends and perspectives. Wiley (2006)
23. De Nicola, R., Jähnichen, S., Wirsing, M.: Rigorous engineering of collective adaptive systems: Special section. International Journal on Software Tools for Technology Transfer **22**, 389–397 (2020)
24. Dimarco, G., Pareschi, L., Toscani, G., Zanella, M.: Wealth distribution under the spread of infectious diseases. Physical Review E **120**, 022303 (2020)
25. Fagnani, F., Zampieri, S.: Asymmetric randomized gossip algorithms for consensus. IFAC Proceedings Volumes **41**(2), 9052–9056 (2008)
26. Fagnani, F., Zampieri, S.: Randomized consensus algorithms over large scale networks. IEEE Journal on Selected Areas in Communications **26**(4) (2008)
27. Feret, J., Danos, V., Krivine, J., Harmer, R., Fontana, W.: Internal coarse-graining of molecular systems. Proceedings of the National Academy of Sciences **106**(16), 6453–6458 (2009)
28. Ferscha, A.: Collective adaptive systems. In: Adjunct Proceedings of the 2015 ACM International Joint Conference on Pervasive and Ubiquitous Computing and Proceedings of the 2015 ACM International Symposium on Wearable Computers. pp. 893–895. ACM (2015)
29. Galam, S.: Sociophysics: A Physicist's Modeling of Psycho-Political Phenomena. Understanding Complex Systems, Springer (2012)
30. Garcia, A.F., de Lucena, C.J.P., Zambonelli, F., Omicini, A., Castro, J. (eds.): Software Engineering for Large-Scale Multi-Agent Systems, Research Issues and Practical Applications, Lecture Notes in Computer Science, vol. 2603. Springer (2002)

31. Goldman, C.V., Zilberstein, S.: Decentralized control of cooperative systems: Categorization and complexity analysis. Journal of Artificial Intelligence Research **22**, 143–174 (2004)
32. Hillston, J., Pitt, J., Wirsing, M., Zambonelli, F.: Collective adaptive systems: Qualitative and quantitative modelling and analysis. Dagstuhl Reports **4**(12), 68–113 (2014)
33. Huhns, M.N. (ed.): Distributed Artificial Intelligence. Pitman Publishing (1987)
34. Kash, I.A., Friedman, E.J., Halpern, J.Y.: Multiagent learning in large anonymous games. Journal of Artificial Intelligence Research **40**, 571–598 (2011)
35. Liboff, R.L.: Kinetic Theory: Classical, Quantum, and Relativistic Descriptions. Graduate Texts in Contemporary Physics, Springer (2003)
36. Mantegna, R.N., Stanley, H.E.: An Introduction to Econophysics: Correlations and Complexity in Finance. Cambridge University Press (1999)
37. Mariani, S., Cabri, G., Zambonelli, F.: Coordination of autonomous vehicles: Taxonomy and survey. ACM Computing Surveys **54**(1), 19:1–19:33 (2021)
38. Mitchell, M.: Complex systems: Network thinking. Artificial Intelligence **170**, 1194–1212 (2006)
39. Monica, S., Bergenti, F.: An analytic study of opinion dynamics in multi-agent systems with additive random noise. In: AI*IA 2016 Advances in Artificial Intelligence. Lecture Notes in Computer Science, vol. 10037, pp. 105–117. Springer (2016)
40. Monica, S., Bergenti, F.: An analytic study of opinion dynamics in multi-agent systems. Computers & Mathematics with Applications **73**(10), 2272–2284 (2017)
41. Monica, S., Bergenti, F.: Opinion dynamics in multi-agent systems: Selected analytic models and verifying simulations. Computational & Mathematical Organization Theory **23**(3), 423–450 (2017)
42. Monica, S., Bergenti, F., Zambonelli, F.: Toward a kinetic framework to model the collective dynamics of multi-agent systems. In: Proceedings of the 11^{th} International Symposium on Leveraging Applications of Formal Methods, Verification and Validation (ISoLA 2022). Lecture Notes in Computer Science, vol. 13703, pp. 165–180. Springer (2022)
43. Pareschi, L., Toscani, G.: Interacting Multiagent Systems: Kinetic Equations and Montecarlo Methods. Oxford University Press (2013)
44. Piccoli, B., Tosin, A., Zanella, M.: Model-based assessment of the impact of driver-assist vehicles using kinetic theory. Zeitschrift für angewandte Mathematik und Physik **71**, 152 (2020)
45. Poslad, S.: Specifying protocols for multi-agent systems interaction. ACM Transactions on Autonomous and Adaptive Systems **2**(4), 15:1–15:24 (2007)
46. Pynadath, D.V., Tambe, M.: The communicative multiagent team decision problem: Analyzing teamwork theories and models. Journal of Artificial Intelligence Research **16**, 389–423 (2002)
47. Schweitzer, F.: Brownian Agents and Active Particles: Collective Dynamics in the Natural and Social Sciences. Synergetics, Springer (2003)
48. Slanina, F.: Inelastically scattering particles and wealth distribution in an open economy. Physical Review E **69**, 46–102 (2004)
49. Sznajd-Weron, K., Sznajd, J.: Opinion evolution in closed community. International Journal of Modern Physics C **11**, 1157–1166 (2000)
50. Thurner, S., Klimek, P., Hanel, R.: Introduction to the Theory of Complex Systems. Oxford University Press (2018)
51. Weidlich, W.: Sociodynamics: A systematic approach to mathematical modelling in the social sciences. Harwood Academic Publisher (2000)
52. Whitby, M., Cardelli, L., Kwiatkowska, M., Laurenti, L., Tribastone, M., Tschaikowski, M.: PID control of biochemical reaction networks. IEEE Transactions on Automatic Control **67**(2) (2022)
53. Wolpert, D.H., Tumer, K.: Collective intelligence, data routing and Braess' paradox. Journal of Artificial Intelligence Research **16**, 359–387 (2002)
54. Xu, Y., Liao, E., Scerri, P., Yu, B., Lewis, M., Sycara, K.: Towards flexible coordination of large scale multiagent teams. In: Coordination of Large-Scale Multiagent Systems. pp. 287–309. Springer (2006)
55. Ygge, F., Akkermans, H.: Decentralized markets versus central control: A comparative study. Journal of Artificial Intelligence Research **11**, 301–333 (1999)