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PHILOSOPHICAL TRANSACTIONS A

rsta.royalsocietypublishing.org

Research

Article submitted to journal

Subject Areas:
Mathematical Physics, Kinetic Theory

Keywords:
wealth distribution, non–conservative transitions, continuous trading limit, Fokker-Planck equations

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Kinetic model for international trade allowing transfer of individuals

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We propose a kinetic model to describe trade among different populations, living in different countries. The interaction rules are assumed depending on the trading propensity of each population and also on non deterministic (random) effects. Moreover, the possible transfers of individuals from one country to another are also taken into account, by means of suitable Boltzmann–type operators. Consistent macroscopic equations for number density and mean wealth of each country are derived from the kinetic equations, and the effects of transfers on their equilibrium values are commented on. Finally, a suitable continuous trading limit is considered, leading to a simpler system of Fokker–Planck–type kinetic equations, with specific contributions accounting for transfers.

1. Introduction

The kinetic theory proposed by Boltzmann for the description of rarefied gases has been then generalized to various kinds of interacting systems, providing reasonable models for the evolution of wealth distribution [16], for the opinion formation [28], for the spread of an epidemics [14] and so on. For the evolution of wealth, the pioneering models based on binary exchanges, with individuals saving a fraction of their wealth in each trade, may be found in [8,9]. A good kinetic model should be able to reproduce the fact, noted by the Italian economist V. Pareto in [25], that for high values of wealth the distribution function may be approximated by a suitable inverse power of the wealth itself; this means that a great part of the total wealth of the considered country is owned by a small fraction of the population [10,11]. A model able to justify in a rigorous way the formation (for long times) of Pareto tails has been proposed by Cordier, Pareschi and Toscani in [12]; the basic idea consists in taking into account not only the deterministic wealth exchanges between agents, but also random effects allowing to amplify gains or losses in the market. This CPT model has been extensively investigated, also from the numerical point of view [24], and generalized introducing for instance wealth–dependent transaction parameters [4], the influence of personal knowledge [23], possible debts [27], taxation and redistribution of the collected wealth [5], and so on. The major part of kinetic models for socio–economic sciences concern the investigation of exchanges (of goods, wealth, opinion, ...) among a single population, characterized thus by a unique distribution function. However, in several econophysics problems it is natural to study the time–evolution of different interacting populations. This leads to a system of kinetic equations, one for each population, similarly to the Boltzmann description of gas mixtures, widely investigated in the pertinent literature [7,18,20]. In [15,17] the authors propose the generalization of the CPT model [12] to a set of different populations: for the i–th population, the interaction operator is provided by a sum of binary Boltzmann operators, each one describing the effects due to trades with individuals of only another population j; of course, the case $i = j$ reproduces the classical CPT operator for the "domestic" trade, while options $j \neq i$ describe "international" trade between two different countries. In this paper we aim at following this line, but in addition we allow transfers of agents from a country to another. The passage of individuals from a category to another has already been taken into account in models for the spread of an epidemics (essentially of SIR type), where a susceptible individual could become infected and then removed for healing or death [14]. In such models the number of transfers is essentially measured by the number of probable contacts among individuals, and no exchanges of goods occur during the shift. In the present work we adopt a different approach: we assume that one of the two agents decides to change country during a binary trading (in order to join his trading partner, or to invest his money in a different market). Such phenomenon, combining classical trading with transfers of individuals, may be modelled by scattering operators of Boltzmann type similar to the ones used for gas mixtures undergoing chemical reactions (where particles change their nature, i.e. pass from one species to another) [13,20,26]. Unlike previous CPT–type models for wealth distribution, in the present frame the number of individuals of each country is no more constant, and this will cause significant modifications in the evolution of macroscopic moments and in the trend to the equilibrium configuration.

Specifically, the paper is organized as follows. In Section 2 we introduce our frame and assumptions, as well as kinetic equations with interaction rules and all required Boltzmann operators for domestic and international trade. Particular attention is given to the construction of transfer operators, at first to their "strong form" and then also to their "weak form", that is useful to derive macroscopic equations even if it turns out to be more complicated than the one for the classical CPT operator because of the lack of symmetry between the gain and the loss terms in the transfers. Then, Section 3 is devoted to the evolution equations for number density and mean wealth of single countries, and to the discussion of their equilibrium values for varying parameters. In Section 4 we perform a suitable asymptotic analysis (continuous trading limit) that, as usual in this kind of models [12], allows to pass from integro–differential Boltzmann kinetic equations to a set of PDEs of Fokker–Planck type, with additional terms due to the transfer operators. Section 5 finally contains some conclusions and perspectives.

2. Boltzmann equations for international trade and individual transfers

In this section we build up a kinetic model for the international trade among various populations (typically, of different countries), taking also into account the possibility of individual transfers from one country to another. The domestic and international trade will be modelled by means of binary interaction rules similar to the ones proposed in [12,17], while transfers will be described by operators of Boltzmann type similar to the ones modelling bimolecular and reversible chemical reactions [13,20,21]. In order to simplify a bit the presentation (especially of the transfer operators), we show the present model for a set of two populations $i = 1, 2$, but it could be easily extended to N populations.

We aim at studying the evolution of the wealth distribution function of two countries, $f_i(v,t)$ (with $i = 1, 2$) depending on the individual wealth $v \in \mathbb{R}_+$ and on time $t \in \mathbb{R}_+$. As usual in such kinetic models, debts are not allowed, namely we assume that there are no individuals with negative wealth. More precisely, for the sake of continuity of wealth distributions as $v \to 0$, we assume also that $f_i(0,t) = 0$, so that all agents have a positive wealth. The moments of distribution functions

$$\mathcal{M}_s^i = \int_0^{+\infty} v^s \, f_i(v,t) \, dv \,, \qquad s \geq 0,$$

provide information on the wealth distribution. The option $s = 0$ yields the number of individuals of each population

$$\rho_i = \int_0^{+\infty} f_i(v,t) \, dv \,, \tag{2.1}$$

that will vary in time due to individual transfers. The moments $\mathcal{M}_1^i =: M_i$ correspond to the total wealth of the i–th country; the mean wealth is consequently provided by $m_i = M_i/\rho_i$. The second moment $\mathcal{M}_2^i$ is related to the variance of the i–th distribution with respect to its mean value. It is well known that, in realistic economies, for $v \to +\infty$ the wealth distribution is asymptotic to an inverse power law of v, therefore not all moments are convergent; specifically, one defines

$$\alpha_i = \sup \left\{ s \in \mathbb{R}_+ : \mathcal{M}_s^i < +\infty \right\},$$

which is called Pareto index of the i–th distribution, in honour of the Italian economist V. Pareto that in [25] pointed out this phenomenon.

The Boltzmann equations for distributions $f_i(v,t)$ may be cast as

$$\frac{\partial f_i(v,t)}{\partial t} = \sum_{j=1}^{2} \mathcal{Q}_i(f_i, f_j)(v) + \mathcal{Q}_i^T(f_1, f_2)(v) \,, \qquad i = 1, 2. \tag{2.2}$$

Here, $\mathcal{Q}_i(f_i, f_j)$ is the binary trading operator, describing the effects on the i–th distribution due to trade with individuals of j–th population (including of course the case $j = i$); then, $\mathcal{Q}_i^T(f_1, f_2)$ takes into account transfers from the first to the second population, or vice versa.

The construction of $\mathcal{Q}_i(f_i, f_j)$ is well known [12,24], but we summarize the basic steps in order to be able to generalize them to the new operator $\mathcal{Q}_i^T(f_1, f_2)$. When a pair of individuals of countries (i,j), with wealths (v,w) respectively, interacts through a binary trade, the post–collision wealths (v^*, w^*) are defined by a proper trading rule

$$(v^*, w^*) = h_{ij}(v, w)$$

which characterizes the model. If the trades are considered as independent events, namely the number of encounters in a time unit between individuals with wealths (v,w) is simply provided by $\chi_{ij} \, f_i(v,t) \, dv \, f_j(w,t) \, dw$ (where χ_{ij} is the interaction probability, assumed independent of wealths for simplicity, and such that $\chi_{ji} = \chi_{ij}$), then the binary Boltzmann operator $\mathcal{Q}_i(f_i, f_j)$ is given by the difference between a gain and a loss term as

$$\mathcal{Q}_i(f_i, f_j)(v) = \chi_{ij} \int_0^{+\infty} \left(\frac{1}{J_{ij}} \, f_i(v_*) \, f_j(w_*) - f_i(v) \, f_j(w) \right) dw \,; \tag{2.3}$$

in the gain term, (v_*, w_*) denote the pre–interaction wealths producing (v,w) as post–interaction wealths, and J_{ij} is the Jacobian of the transformation h_{ij}. Indeed, in the form (2.3) of the interaction operator it is implicitly assumed the trading rule to be invertible. In order to be able to relax this assumption, which might be somehow restrictive since trade is governed also by non deterministic parameters, it turns out to be useful to write the operator in weak form. Indeed, by multiplying by a smooth function $\varphi(v)$ and integrating over $v \in \mathbb{R}_+$, one gets

$$\int_0^{+\infty} \varphi(v)\, \mathcal{Q}_i(f_i, f_j)(v)\, dv = \chi_{ij} \iint_0^{+\infty} \varphi(v) \left(\frac{1}{J_{ij}}\, f_i(v_*)\, f_j(w_*) - f_i(v)\, f_j(w) \right) dv\, dw$$

$$= \chi_{ij} \left\{ \iint_0^{+\infty} \varphi(v)\, f_i(v_*)\, f_j(w_*) dv_*\, dw_* - \iint_0^{+\infty} \varphi(v)\, f_i(v)\, f_j(w)\, dv\, dw \right\} \tag{2.4}$$

$$= \chi_{ij} \iint_0^{+\infty} \Big[\varphi(v^*) - \varphi(v) \Big]\, f_i(v)\, f_j(w)\, dv\, dw\,,$$

where last equality takes into account that $(v, w) = h_{ij}(v_*, w_*)$ as well as $(v^*, w^*) = h_{ij}(v, w)$. The weak form (2.4) involves thus only the post–trade wealth and not the pre–trade one, therefore there is no need to invert the interaction rule.

In this paper we assume the interaction rule of the CPT model [12,17], since it is known to be able to reproduce a realistic long time behaviour, with the formation of Pareto tails. Specifically, we take

$$\left\{ \begin{array}{lcl} v^* & = & (1-\gamma_i)\, v + \gamma_j\, w + \eta_{ij}\, v \\ w^* & = & \gamma_i\, v + (1-\gamma_j)\, w + \eta_{ji}\, w \end{array} \right. \tag{2.5}$$

The parameter γ_i represents the exchange propensity of the i–th population, namely the fraction of the owned wealth that one individual gives to the trading partner. The quantities η_{ij} (with $i, j = 1, 2$) denote random variables, taking into account the non deterministic effects of the market, and that, as shown for the first time in [12], turn out to be crucial in order to recover the wealth diffusion and the formation of Pareto tails. Random variables η_{ij} are assumed identically distributed, with zero mean (random gain and loss effects compensate) and with variance σ_{ij}^2, depending on the interacting species in order to reflect the different amount of risks in the various countries. In order to ensure that the post–trade wealths are non–negative, we assume $\eta_{ij} \geq -1 + \gamma_i$, $\forall i, j = 1, 2$. The presence of random variables in the trading rule (2.5) implies that the weak form of the trading Boltzmann operator has to be cast as

$$\int_0^{+\infty} \varphi(v)\, \mathcal{Q}_i(f_i, f_j)(v)\, dv = \chi_{ij} \left\langle \iint_0^{+\infty} \Big[\varphi(v^*) - \varphi(v) \Big]\, f_i(v)\, f_j(w)\, dv\, dw \right\rangle, \tag{2.6}$$

where $\langle \cdot \rangle$ denotes the operation of mean with respect to all random variables.

We build up now the operators $\mathcal{Q}_i^T$, describing the transfers of individuals. We assume that transfers occur after a binary interaction (with also some small change of goods). We consider both the situation in which an interaction between a pair of individuals of the same country gives rise to the transfer of one of them to the other country, i.e.

$$(a) \quad 1 + 1 \to 1 + 2\,, \qquad\qquad (b) \quad 2 + 2 \to 1 + 2\,, \tag{2.7}$$

and also the reverse transfers, where a trade between individuals of different populations makes them to decide to live in the same country, i.e.

$$(c) \quad 1 + 2 \to 1 + 1\,, \qquad\qquad (d) \quad 1 + 2 \to 2 + 2\,. \tag{2.8}$$

Boltzmann–type operators $\mathcal{Q}_i^T$ should thus take into account the effects due to all four types of transitions (a), (b), (c), (d), hence

$$\mathcal{Q}_i^T(v) = \sum_{j \in \{a,b,c,d\}} \mathcal{Q}_i^{T(j)}(v)\,. \tag{2.9}$$

We detail the construction of operators relevant to transfer (a), the others may be built up analogously. In the interaction (a), we denote as before by (v^*, w^*) the post–trade wealths of individuals of populations $(1, 2)$, respectively, corresponding to pre–trade wealths (v, w). The interaction rule k_{11}^{12} such that $(v^*, w^*) = k_{11}^{12}(v, w)$ will be explicitly defined later, and it will fulfill the obvious symmetry property $k_{11}^{12}(w, v) = (w^*, v^*)$. Concerning population 1, let us bear in mind that $f_1(v, t)\, dv$ represents the number of individuals with wealth in $(v, v + dv)$, and notice that in each time unit the transfer (a) causes both a loss and a gain of such agents. The gain term is provided by the encounters such that the post–interaction wealth of individual 1 belongs to $(v, v + dv)$. For any fixed $w \in \mathbb{R}_+$, the number of trades providing as output a pair (v, w) is

$$\beta_{11}^{12}\, f_1(v_*)\, f_1(w_*)\, dv_*\, dw_* = \beta_{11}^{12}\, f_1(v_*)\, f_1(w_*)\, \frac{1}{J_{11}^{12}}\, dv\, dw\,,$$

where $(v_*, w_*) = \left(k_{11}^{12}\right)^{-1}(v, w)$, and β_{11}^{12} is the (constant) probability that the transfer (a) occurs. Integrating over all possible w we get the gain term

$$\left(\mathcal{Q}_1^{T(a)}\right)_+(v)\, dv = \beta_{11}^{12}\, dv \int_0^{+\infty} \frac{1}{J_{11}^{12}}\, f_1(v_*) f_1(w_*)\, dw\,.$$

On the other hand, the loss term takes into account all interactions involving individuals with wealth v:

$$\left(\mathcal{Q}_1^{T(a)}\right)_-(v)\, dv = -2\, \beta_{11}^{12}\, f_1(v)\, dv \int_0^{+\infty} f_1(w)\, dw\,,$$

hence the total Boltzmann operator may be cast as

$$\mathcal{Q}_1^{T(a)}(v) = \beta_{11}^{12} \int_0^{+\infty} \left(\frac{1}{J_{11}^{12}}\, f_1(v_*) f_1(w_*) - 2\, f_1(v)\, f_1(w) \right) dw\,. \tag{2.10}$$

For population 2, the transition (a) produces only a gain of individuals,

$$\left(\mathcal{Q}_2^{T(a)}\right)_+(w)\, dw = \beta_{11}^{12}\, dw \int_0^{+\infty} \frac{1}{J_{11}^{12}}\, f_1(v_*) f_1(w_*)\, dv\,,$$

hence, owing to the symmetry of the interaction rule,

$$\mathcal{Q}_2^{T(a)}(v) = \beta_{11}^{12} \int_0^{+\infty} \frac{1}{J_{11}^{12}}\, f_1(v_*) f_1(w_*)\, dw\,. \tag{2.11}$$

In order to get the evolution of macroscopic fields, use will be made of the weak form of the interaction operators

$$\int_0^{+\infty} \varphi(v)\, \mathcal{Q}_1^{T(a)}(v)\, dv = \beta_{11}^{12} \iint_0^{+\infty} \varphi(v)\, f_1(v_*) f_1(w_*) dv_*\, dw_* - 2\, \beta_{11}^{12} \iint_0^{+\infty} \varphi(v)\, f_1(v)\, f_1(w)\, dv\, dw\,, \tag{2.12}$$

$$\int_0^{+\infty} \varphi(v)\, \mathcal{Q}_2^{T(a)}(v)\, dv = \beta_{11}^{12} \iint_0^{+\infty} \varphi(v)\, f_1(v_*)\, f_1(w_*)\, dv_*\, dw_*\,. \tag{2.13}$$

The operators for the interaction (b) in (2.7) are very similar, one has formally to exchange the role of indices 1 and 2. Of course the transition probabilities and the trading rules could be different, i.e. $\beta_{22}^{12} \neq \beta_{11}^{12}$ and $k_{22}^{12}(v, w) \neq k_{11}^{12}(v, w)$. Skipping all details, Boltzmann operators in strong form read as

$$\mathcal{Q}_1^{T(b)}(v) = \beta_{22}^{12} \int_0^{+\infty} \frac{1}{J_{22}^{12}}\, f_2(v_*) f_2(w_*)\, dw\,, \tag{2.14}$$

$$\mathcal{Q}_2^{T(b)}(v) = \beta_{22}^{12} \int_0^{+\infty} \left(\frac{1}{J_{22}^{12}}\, f_2(v_*) f_2(w_*) - 2\, f_2(v)\, f_2(w) \right) dw\,. \tag{2.15}$$

In the interaction (c), population 1 involves one individual in the trading pair, and two individuals after the trade, thus this country has a net gain of one individual; on the other hand, population 2 loses one individual and its operator will be made only by a loss term. We denote by (v, w) the pre–trade wealths of the pair (1,2), and we define the interaction rule $(v^*, w^*) = k_{12}^{11}(v, w)$. Since both agents could have post–trade wealth in $(v, v + dv)$, one has to pay attention to the gain term of population 1; indeed, we have

$$\mathcal{Q}_1^{T(c)}(v) = \beta_{12}^{11} \int_0^{+\infty} \left(\frac{1}{J_{12}^{11}} \, f_1(v_*) \, f_2(w_*) + \frac{1}{J_{12}^{11}} \, f_1(\bar{v}_*) \, f_2(\bar{w}_*) - f_1(v) \, f_2(w) \right) dw \,, \tag{2.16}$$

where $(v_*, w_*) = \left(k_{12}^{11}\right)^{-1}(v, w)$, and $(\bar{v}_*, \bar{w}_*) = \left(k_{12}^{11}\right)^{-1}(w, v)$; notice that it could be $(\bar{v}_*, \bar{w}_*) \neq (w_*, v_*)$, since the interacting individuals are of different countries, therefore we cannot exchange their indices inside the operator and in the interaction rule. For country 2, the operator is much simpler and is provided by

$$\mathcal{Q}_2^{T(c)}(v) = -\,\beta_{12}^{11} \, f_2(v) \int_0^{+\infty} f_1(w) \, dw \,. \tag{2.17}$$

Analogous results are in order for the transfer (d), by exchanging the role of populations 1 and 2; specifically, we have

$$\mathcal{Q}_1^{T(d)}(v) = -\,\beta_{12}^{22} \, f_1(v) \int_0^{+\infty} f_2(w) \, dw \,, \tag{2.18}$$

$$\mathcal{Q}_2^{T(d)}(v) = \beta_{12}^{22} \int_0^{+\infty} \left(\frac{1}{J_{12}^{22}} \, f_1(v_*) \, f_2(w_*) + \frac{1}{J_{12}^{22}} \, f_1(\bar{v}_*) \, f_2(\bar{w}_*) - f_1(w) \, f_2(v) \right) dw \,, \tag{2.19}$$

where $(v_*, w_*) = \left(k_{12}^{22}\right)^{-1}(v, w)$, and $(\bar{v}_*, \bar{w}_*) = \left(k_{12}^{22}\right)^{-1}(w, v)$.

Notice that, since transfer Boltzmann operators may be non symmetric, it is not convenient to follow a procedure analogous to (2.4) in order to look for a more manageable version of the weak form of such operators. As already shown in (2.12)–(2.13) for the operators relevant to transfer (a), in the weak form use will be made only of the property $(1/J_{ij}^{hk}) \, dv \, dw = dv_* \, dw_*$. For the transfers (b), (c), (d) we get

$$\int_0^{+\infty} \varphi(v) \, \mathcal{Q}_1^{T(b)}(v) \, dv = \beta_{22}^{12} \iint_0^{+\infty} \varphi(v) \, f_2(v_*) \, f_2(w_*) \, dv_* \, dw_* \,, \tag{2.20}$$

$$\int_0^{+\infty} \varphi(v) \, \mathcal{Q}_2^{T(b)}(v) \, dv = \beta_{22}^{12} \iint_0^{+\infty} \varphi(v) \, f_2(v_*) \, f_2(w_*) \, dv_* \, dw_* - 2\,\beta_{22}^{12} \iint_0^{+\infty} \varphi(v) \, f_2(v) \, f_2(w) \, dv \, dw \,, \tag{2.21}$$

$$\begin{aligned} \int_0^{+\infty} \varphi(v) \, \mathcal{Q}_1^{T(c)}(v) \, dv &= \beta_{12}^{11} \iint_0^{+\infty} \varphi(v) \, f_1(v_*) \, f_2(w_*) \, dv_* \, dw_* \\ + \beta_{12}^{11} \iint_0^{+\infty} \varphi(v) \, f_1(\bar{v}_*) \, f_2(\bar{w}_*) \, d\bar{v}_* \, d\bar{w}_* &- \beta_{12}^{11} \iint_0^{+\infty} \varphi(v) \, f_1(v) \, f_2(w) \, dv \, dw \,, \end{aligned} \tag{2.22}$$

$$\int_0^{+\infty} \varphi(v) \, \mathcal{Q}_2^{T(c)}(v) \, dv = -\,\beta_{12}^{11} \iint_0^{+\infty} \varphi(v) \, f_2(v) \, f_1(w) \, dv \, dw \,, \tag{2.23}$$

$$\int_0^{+\infty} \varphi(v) \, \mathcal{Q}_1^{T(d)}(v) \, dv = -\,\beta_{12}^{22} \iint_0^{+\infty} \varphi(v) \, f_1(v) \, f_2(w) \, dv \, dw \,, \tag{2.24}$$

$$\begin{aligned} \int_0^{+\infty} \varphi(v) \, \mathcal{Q}_2^{T(d)}(v) \, dv &= \beta_{12}^{22} \iint_0^{+\infty} \varphi(v) \, f_1(v_*) \, f_2(w_*) \, dv_* \, dw_* \\ + \beta_{12}^{22} \iint_0^{+\infty} \varphi(v) \, f_1(\bar{v}_*) \, f_2(\bar{w}_*) \, d\bar{v}_* \, d\bar{w}_* &- \beta_{12}^{22} \iint_0^{+\infty} \varphi(v) \, f_2(v) \, f_1(w) \, dv \, dw \,. \end{aligned} \tag{2.25}$$

In each transfer (a), (b), (c), (d), the most important event is the passage of one individual from one country to the other, and the wealth exchange could also be small. For this reason we take interaction rules very simple and completely deterministic, where each population is characterized by an exchange parameter ω_i; specifically, for $(i,j,h,k) \in \{(1,1,1,2),\ (2,2,1,2),\ (1,2,1,1),\ (1,2,2,2)\}$, the trading rule $(v^*, w^*) = k_{ij}^{hk}(v,w)$ is defined as

$$\begin{cases} v^* = (1-\omega_i)\, v + \omega_j\, w \\ w^* = \omega_i\, v + (1-\omega_j)\, w \end{cases} \tag{2.26}$$

More complicated effects, as the non–deterministic phenomena in the wealth evolution, are already taken into account in the trading rules (2.5), which involve any possible pair of agents $(i,j) \in \{(1,1),\ (1,2),\ (2,1),\ (2,2)\}$. Main notations introduced in this section are summarized in the following table.

Classical trades (without transfer)	
(v^*, w^*)	post–interaction wealths corresponding to (v,w)
(v_*, w_*)	pre–interaction wealths corresponding to (v,w)
χ_{ij}	interaction probabilities
γ_i	trading propensity of the i–th population
η_{ij}	random variables with zero mean and variance σ_{ij}^2
Trades with transfers	
(v^*, w^*)	post–interaction wealths corresponding to (v,w)
(v_*, w_*)	pre–interaction wealths corresponding to (v,w)
$(\bar v_*, \bar w_*)$	pre–interaction wealths corresponding to (w,v)
β_{ij}^{ik}	probability of the transfer $i+j \to i+k$
ω_i	exchange parameter of the i–th population in a transfer

3. Macroscopic equations for individual numbers and mean wealths

In the kinetic model presented in the previous section, the bi–population binary trades cause a passage of wealth from a population to the other, and some of these interactions give also rise to transfer of individuals. In this section we derive evolution equations for the number of individuals and the mean wealth of both populations.

The equations for densities ρ_i $(i = 1,2)$ are provided by the weak forms of the Boltzmann equations (2.2) corresponding to the test function $\varphi(v) = 1$. As expected, from (2.6) one has that contribution of binary trades vanishes, and the variations of the individual numbers are due only to the transfers described by the operators $\mathcal{Q}_i^{T(j)}$, with $i = 1,2$ and $j = a,b,c,d$. Putting $\varphi(v) = 1$ in the corresponding weak forms we get

$$\begin{aligned} \frac{d\rho_1}{dt} &= -\beta_{11}^{12}\,(\rho_1)^2 + \beta_{22}^{12}\,(\rho_2)^2 + \left(\beta_{12}^{11} - \beta_{12}^{22}\right)\rho_1\,\rho_2\,, \\ \frac{d\rho_2}{dt} &= \beta_{11}^{12}\,(\rho_1)^2 - \beta_{22}^{12}\,(\rho_2)^2 - \left(\beta_{12}^{11} - \beta_{12}^{22}\right)\rho_1\,\rho_2\,. \end{aligned} \tag{3.1}$$

The stationary state of such equations is achieved when

$$\rho_2 = \alpha\,\rho_1\,, \qquad \text{with} \qquad \alpha = \frac{-\left(\beta_{12}^{11} - \beta_{12}^{22}\right) + \sqrt{\left(\beta_{12}^{11} - \beta_{12}^{22}\right)^2 + 4\,\beta_{11}^{12}\,\beta_{22}^{12}}}{2\,\beta_{22}^{12}}\,,$$

thus, taking into account that the sum of individuals is constant, i.e. $\rho_1 + \rho_2 = \bar\rho$, the unique equilibrium state is provided by

$$(\rho_1)_\infty = \frac{\bar\rho}{1+\alpha}\,, \qquad (\rho_2)_\infty = \frac{\alpha\,\bar\rho}{1+\alpha}\,. \tag{3.2}$$

If the probabilities of transfer towards the i–th country take a common value, namely

$$\beta_1^2 := \beta_{11}^{12} = \beta_{12}^{22}\,, \qquad \beta_2^1 := \beta_{22}^{12} = \beta_{12}^{11}\,, \tag{3.3}$$

and this assumption will be adopted from now on, then $\alpha = \beta_1^2/\beta_2^1$ and the equilibrium values become

$$(\rho_1)_\infty = \frac{\beta_2^1}{\beta_1^2 + \beta_2^1}\,\bar{\rho}\,, \qquad (\rho_2)_\infty = \frac{\beta_1^2}{\beta_1^2 + \beta_2^1}\,\bar{\rho}\,. \tag{3.4}$$

In the particular case $\beta_1^2 = \beta_2^1$, we get $(\rho_1)_\infty = (\rho_2)_\infty = \bar{\rho}/2$.

We focus the attention now on the evolution of the total wealth M_i of each country, owing to the weak forms of Boltzmann equations with $\varphi(v) = v$. From (2.6), since in the trading rule (2.5) one has $v^* - v = (\eta_{ij} - \gamma_i)\,v + \gamma_j\,w$ and the random variables have zero mean wealth, the contributions due to operators $\mathcal{Q}_i(f_i, f_j)$ are

$$\int_0^{+\infty} v\,\mathcal{Q}_i(f_i, f_j)(v)\,dv = \chi_{ij}\Big(\gamma_j\,\rho_i\,M_j - \gamma_i\,\rho_j\,M_i\Big)\,. \tag{3.5}$$

For the transfers among two different countries, resorting to the rule (2.26) and bearing in mind the relations between pre- and post–trade wealths inside each weak operator, we get, under the assumptions (3.3),

$$\begin{aligned}
\int_0^{+\infty} v\,\mathcal{Q}_1^{T(a)}(v)\,dv &= -\int_0^{+\infty} v\,\mathcal{Q}_2^{T(a)}(v)\,dv = -\,\beta_1^2\,\rho_1\,M_1\,,\\
\int_0^{+\infty} v\,\mathcal{Q}_1^{T(b)}(v)\,dv &= -\int_0^{+\infty} v\,\mathcal{Q}_2^{T(b)}(v)\,dv = \beta_2^1\,\rho_2\,M_2\,,\\
\int_0^{+\infty} v\,\mathcal{Q}_1^{T(c)}(v)\,dv &= -\int_0^{+\infty} v\,\mathcal{Q}_2^{T(c)}(v)\,dv = \beta_2^1\,\rho_1\,M_2\,,\\
\int_0^{+\infty} v\,\mathcal{Q}_1^{T(d)}(v)\,dv &= -\int_0^{+\infty} v\,\mathcal{Q}_2^{T(c)}(v)\,dv = -\,\beta_1^2\,\rho_2\,M_1\,.
\end{aligned} \tag{3.6}$$

Therefore, equations for total wealths M_i read as

$$\begin{aligned}
\frac{dM_1}{dt} &= \chi_{12}\Big(\gamma_2\,\rho_1\,M_2 - \gamma_1\,\rho_2\,M_1\Big) - \beta_1^2\,\bar{\rho}\,M_1 + \beta_2^1\,\bar{\rho}\,M_2\,,\\
\frac{dM_2}{dt} &= \chi_{12}\Big(\gamma_1\,\rho_2\,M_1 - \gamma_2\,\rho_1\,M_2\Big) + \beta_1^2\,\bar{\rho}\,M_1 - \beta_2^1\,\bar{\rho}\,M_2\,.
\end{aligned} \tag{3.7}$$

Notice that, since random effects have been taken with zero mean and bounded from below in order to have non–negative post–trade wealths, we obtain that total wealth is preserved, i.e. $M_1 + M_2 = \bar{M}$. In other more complicated formulations, with unbounded random variables and suitable Heaviside functions in the kernels of Boltzmann equations in order to remove interactions giving non admissible (negative) wealths, it has been proved that total wealth can increase exponentially in time [12].

By combining results (3.7) and (3.1), we get the following equations for the mean wealth $m_i = M_i/\rho_i$ of each country:

$$\begin{aligned}
\frac{dm_1}{dt} &= \chi_{12}\,\rho_2\Big(\gamma_2\,m_2 - \gamma_1\,m_1\Big) + \beta_2^1\,\rho_2\left(1 + \frac{\rho_2}{\rho_1}\right)(m_2 - m_1)\,,\\
\frac{dm_2}{dt} &= \chi_{12}\,\rho_1\Big(\gamma_1\,m_1 - \gamma_2\,m_2\Big) + \beta_1^2\,\rho_1\left(1 + \frac{\rho_1}{\rho_2}\right)(m_1 - m_2)\,.
\end{aligned} \tag{3.8}$$

The stationary points of this system, taking into account the values of equilibrium number densities (3.4), are provided by the manifold

$$m_2 = \xi\,m_1\,, \qquad \text{with} \qquad \xi = \frac{\chi_{12}\,\gamma_1 + \beta_1^2 + \beta_2^1}{\chi_{12}\,\gamma_2 + \beta_1^2 + \beta_2^1}\,.$$

Bearing in mind that $\rho_1\, m_1 + \rho_2\, m_2 = \bar{M}$, the unique equilibrium state is given by

$$
\begin{aligned}
(m_1)_\infty &= \frac{\bar{M}}{(\rho_1)_\infty + \xi\,(\rho_2)_\infty} &= \frac{\bar{M}}{\bar{\rho}}\,\frac{\beta_1^2 + \beta_2^1}{\xi\,\beta_1^2 + \beta_2^1}\,,\\
(m_2)_\infty &= \frac{\xi\,\bar{M}}{(\rho_1)_\infty + \xi\,(\rho_2)_\infty} &= \frac{\bar{M}}{\bar{\rho}}\,\frac{\xi\,\beta_1^2 + \xi\,\beta_2^1}{\xi\,\beta_1^2 + \beta_2^1}\,.
\end{aligned}
\tag{3.9}
$$

Notice that if the two populations have the same trading propensity, i.e. $\gamma_1 = \gamma_2$, then the parameter $\xi = 1$ and $(m_1)_\infty = (m_2)_\infty$, thus at the equilibrium configuration the mean wealth of the two countries is the same. On the other hand, if for instance $\gamma_1 > \gamma_2$, namely the individuals of the first country are more favourable to exchanging their wealth, then the mean wealth of the second population will increase and $(m_2)_\infty > (m_1)_\infty$. The total wealths $(M_1)_\infty = (\rho_1)_\infty\,(m_1)_\infty$ and $(M_2)_\infty = (\rho_2)_\infty\,(m_2)_\infty$ of course depend also on the numbers of individuals that, as discussed above, are influenced by the transfer rates from one country to the other.

4. Continuous trading limit and Fokker-Planck approximation

As known in several kinetic models in socio–economic sciences [1–3,24], it is very difficult to deduce analytical properties on the behaviour of distribution functions starting from the Boltzmann equations (2.2). Even in our frame, more complicated due to transfers of individuals, it could be convenient to study the asymptotic regime usually called “continuous trading limit”, since it could lead to simpler kinetic equations of Fokker–Planck type, as in [12]. This regime is based on the fact that, since the trading rules (2.5) and (2.26) concern each single binary interaction, it is highly probable that the amount of exchanged wealth is very small so that, in other words, the differences between post– and pre–interaction wealths are almost negligible. The market is thus described as a huge amount of small binary trades, and it can be seen as a continuum.

We measure all interaction coefficients in terms of a small parameter ε. In the trading rules (2.5) we set the trading propensities γ_i and the variance of random variables η_{ij} as

$$
\gamma_i = \tilde{\gamma}_i\,\varepsilon\,, \qquad \sigma_{ij}^2 = \lambda_{ij}\,\varepsilon\,, \qquad i = 1, 2, \tag{4.1}
$$

and, analogously, in the interaction rules causing transfers we set

$$
\omega_i = \tilde{\omega}_i\,\varepsilon\,, \qquad i = 1, 2. \tag{4.2}
$$

Moreover, since transfers of individuals are usually much rarer than wealth exchanges (2.5), we take the following assumption for the interaction probabilities

$$
\max\left\{\beta_1^2, \beta_2^1\right\} \ll \min\left\{\chi_{ij}, \quad i,j = 1,2\right\}, \tag{4.3}
$$

and more specifically we assume

$$
\beta_1^2 = \tilde{\beta}_1^2\,\varepsilon\,, \qquad \beta_2^1 = \tilde{\beta}_2^1\,\varepsilon\,. \tag{4.4}
$$

Since we are interested in the long time behaviour of distributions, taking into account all phenomena involved in the model, we measure time in the same unit setting $\tau = \varepsilon\, t$. The weak form of the Boltzmann equations for the scaled distributions $g_i(v,\tau) = f_i(v,t)$, $i = 1, 2$, may thus be cast as

$$
\frac{d}{d\tau}\int_0^{+\infty} \varphi(v)\, g_i(v,\tau)\, dv = \frac{1}{\varepsilon}\sum_{j=1}^{2}\int_0^{+\infty} \varphi(v)\, Q_i(g_i, g_j)(v)\, dv + \frac{1}{\varepsilon}\int_0^{+\infty} \varphi(v)\, Q_i^T(g_1, g_2)\, dv\,. \tag{4.5}
$$

We investigate the asymptotic limit $\varepsilon \to 0$. In the weak form of trading operators (2.6), we can resort to a second order Taylor expansion of the test function $\varphi(v^*)$ around v, obtaining

$$\varphi(v^*) - \varphi(v) = \varphi'(v)\,(v^* - v) + \frac{1}{2}\,\varphi''(\tilde{v})\,(v^* - v)^2 =$$

$$= \varphi'(v)\,\left[\varepsilon\,(\tilde{\gamma}_j\,w - \tilde{\gamma}_i v) + \eta_{ij}\,v\right] + \frac{1}{2}\,\varphi''(\tilde{v})\,\left[\varepsilon^2\,(\tilde{\gamma}_j\,w - \tilde{\gamma}_i v)^2 + \eta_{ij}^2\,v^2 + 2\,\varepsilon\,\eta_{ij}\,v\,(\tilde{\gamma}_j\,w - \tilde{\gamma}_i v)\right],$$

where $\tilde{v} = \theta\,v + (1-\theta)\,v^*$ for some suitable $\theta \in [0,1]$. Bearing in mind that random variables η_{ij} have zero mean and variance $\lambda_{ij}\,\varepsilon$, we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{j=1}^{2}\int_0^{+\infty}\varphi(v)\,\mathcal{Q}_i(g_i,g_j)(v)\,dv =$$

$$= \sum_{j=1}^{2}\chi_{ij}\iint_0^{+\infty}\left[\varphi'(v)\,(\tilde{\gamma}_j\,w - \tilde{\gamma}_i v) + \frac{\lambda_{ij}}{2}\,\varphi''(\tilde{v})\,v^2\right] g_i(v)\,g_j(w)\,dv\,dw$$

$$= \sum_{j=1}^{2}\chi_{ij}\int_0^{+\infty}\left[\varphi'(v)\,\rho_j(\tilde{\gamma}_j\,m_j - \tilde{\gamma}_i v) + \frac{\lambda_{ij}}{2}\,\rho_j\varphi''(\tilde{v})\,v^2\right] g_i(v)\,dv\,.$$

The fact that all higher order terms of the Taylor expansion, including the ones involving moments of the random variables, are $O(\varepsilon)$ and thus vanish in this limit may be rigorously proved [12]. By integration by parts, we can pass the derivatives from the test function to the distribution $g_i(v)$ and, under the assumptions that $g_i(0,t) = 0$ (no individuals with zero wealth) and $g_i(v,0)$ has moments $\mathcal{M}_s$ bounded for $s = 2+\delta$ for some $\delta > 0$ (see [12,17] for further details), we get

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{j=1}^{2}\int_0^{+\infty}\varphi(v)\,\mathcal{Q}_i(g_i,g_j)(v)\,dv =$$

$$= \sum_{j=1}^{2}\chi_{ij}\int_0^{+\infty}\varphi(v)\left\{-\,\rho_j\,\frac{\partial}{\partial v}\left[(\tilde{\gamma}_j\,m_j - \tilde{\gamma}_i v)\,g_i(v)\right] + \frac{\lambda_{ij}}{2}\,\rho_j\,\frac{\partial^2}{\partial v^2}\left(v^2\,g_i(v)\right)\right\} dv\,. \tag{4.6}$$

Concerning the transfer operators, notice that the interaction rule (2.26) in the present scaling implies $\varphi(v^*) = \varphi(v) + O(\varepsilon) = \varphi(v_*) + O(\varepsilon)$. Consequently, taking into account that β_1^2 and β_2^1 are $O(\varepsilon)$, for the operators relevant to transfer (a) we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{+\infty}\varphi(v)\,\mathcal{Q}_1^{T(a)}(v)\,dv = \tilde{\beta}_1^2\iint_0^{+\infty}\varphi(v_*)\,g_1(v_*)g_1(w_*)dv_*\,dw_* - 2\,\tilde{\beta}_1^2\iint_0^{+\infty}\varphi(v)\,g_1(v)\,g_1(w)\,dv\,dw$$

$$= -\,\tilde{\beta}_1^2\,\rho_1\int_0^{+\infty}\varphi(v)\,g_1(v)\,dv\,,$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{+\infty}\varphi(v)\,\mathcal{Q}_2^{T(a)}(v)\,dv = \tilde{\beta}_1^2\iint_0^{+\infty}\varphi(v_*)\,g_1(v_*)\,g_1(w_*)\,dv_*\,dw_* = \tilde{\beta}_1^2\,\rho_1\int_0^{+\infty}\varphi(v)\,g_1(v)\,dv\,. \tag{4.7}$$

The contributions due to operators relevant to transfers (b), (c) and (d) may be analogously computed. In conclusion, skipping further details, the scaled distributions $g_1(v)$ and $g_2(v)$ are weak solutions to the following equations

$$\begin{aligned}
\frac{\partial g_1}{\partial t} &= \frac{1}{2}\Big(\chi_{11}\,\lambda_{11}\,\rho_1 + \chi_{12}\,\lambda_{12}\,\rho_2\Big)\frac{\partial^2}{\partial v^2}\Big(v^2\,g_1(v)\Big) - \frac{\partial}{\partial v}\Big\{\big[\chi_{11}\,\rho_1\,\tilde{\gamma}_1\,m_1 + \chi_{12}\,\rho_2\,\tilde{\gamma}_2\,m_2 \\
&\quad - (\chi_{11}\,\rho_1 + \chi_{12}\,\rho_2)\,\tilde{\gamma}_1\,v\big]g_1(v)\Big\} - \tilde{\beta}_1^2\,\bar{\rho}\,g_1(v) + \tilde{\beta}_2^1\,\bar{\rho}\,g_2(v)\,, \\
\frac{\partial g_2}{\partial t} &= \frac{1}{2}\Big(\chi_{12}\,\lambda_{21}\,\rho_1 + \chi_{22}\,\lambda_{22}\,\rho_2\Big)\frac{\partial^2}{\partial v^2}\Big(v^2\,g_2(v)\Big) - \frac{\partial}{\partial v}\Big\{\big[\chi_{12}\,\rho_1\,\tilde{\gamma}_1\,m_1 + \chi_{22}\,\rho_2\,\tilde{\gamma}_2\,m_2 \\
&\quad - (\chi_{12}\,\rho_1 + \chi_{22}\,\rho_2)\,\tilde{\gamma}_2\,v\big]g_2(v)\Big\} + \tilde{\beta}_1^2\,\bar{\rho}\,g_1(v) - \tilde{\beta}_2^1\,\bar{\rho}\,g_2(v)\,.
\end{aligned} \tag{4.8}$$

These equations are, as expected, of Fokker–Planck type; in the derivation use has been made of the assumption $g_i(v,t)=0$ for $v \le 0$, but in [27] it has been shown that Fokker–Planck descriptions are well suited even in presence of some debts. The first term on the right hand sides is a diffusion operator, measuring the spread of wealth due to non–deterministic events; indeed, diffusion coefficients depend on the variances λ_{ij} of rescaled random variables, and vanish in absence of random effects. The second addend on the right hand sides of (4.8) is a drift term, forcing the agents wealth towards a suitable weighted average of the mean wealths of the two countries. Finally, unlike classical CPT model [12] and its extension to international trade [17], here there appear contributions accounting for the transfers of individuals from one country to the other, and these make (4.8) a system of coupled PDEs for the unknown kinetic distributions $g_1(v)$ and $g_2(v)$. A similar coupling has been found in [14], where a generalized SIR model has been recovered from kinetic equations, allowing passages of susceptible individuals to the infected class and then to the removed one.

It is in general not possible to compute analytically the steady distributions of system (4.8). It is well known that steady solutions of pairs of Fokker–Planck equations could show a bimodal shape, already discussed analytically and numerically in various socio–economic problems, without ([17]) or with ([14]) transfers. This behaviour is expected also in the present model for suitable choices of parameters, since the sum of the two equations (4.8) provides an equation for total distribution analogous to that in [17]. The appearance of two peaks is due to the fact that single country distributions (that are both vanishing for $v \to 0^+$ and for $v \to +\infty$) assume their maximum value in correspondence to different wealths. Anyway, as extensively discussed in [19], a bimodal distribution may appear when the saving propensity for one population is really higher than for the other (namely if the behaviour of the two countries is very different). In our model, the mobility of individuals could compensate the differences between the two countries or else favour the growth of one country (in number of individuals and in total wealth); in both these scenarios the formation of bimodal distributions would become less probable. Just as illustrative example, explicit results may be obtained in the particular case in which transfer from a selected country to the other is much more probable than the reverse one. Let us consider the scaling $\beta_2^1 = \tilde\beta_2^1\,\varepsilon$ and $\beta_1^2 = O(\varepsilon^2)$. This assumption could model for instance a situation in which the passage from country 1 to country 2 is allowed only if the amount of wealth of the individual in transfer is very high; indeed, the costs related to transfers should be taken into account in the interaction rules or in the kernels of collision operators. In this regime, the steady configuration is provided to the leading order by all individuals living in country 1 thus, neglecting $O(\varepsilon)$ corrections that would vanish in the considered limit $\varepsilon \to 0$, one has $(\rho_1)_\infty = \bar\rho$, $(m_1)_\infty = \bar M/\bar\rho$, and $(\rho_2)_\infty = 0$, $(m_2)_\infty = 0$. Consequently, the steady distribution $(g_2)_\infty$ is also vanishing, while $(g_1)_\infty$ is provided by the solution to the equation

$$\frac{\partial}{\partial v}\left\{\frac{\lambda_{11}}{2}\,\frac{\partial}{\partial v}\Big(v^2\,(g_1)_\infty(v)\Big) - \tilde\gamma_1\left(\frac{\bar M}{\bar\rho} - v\right)(g_1)_\infty(v)\right\} = 0\,. \tag{4.9}$$

This is essentially the same equation obtained from CPT–like trading models for a single population [4,12], and by standard techniques we get that the steady state may be cast as

$$(g_1)_\infty(v) = C\,v^{-\left(\frac{2\,\tilde\gamma_1}{\lambda_{11}}+2\right)}\,\exp\left(-\,\frac{2\,\tilde\gamma_1\,\bar M}{\lambda_{11}\,\bar\rho}\,\frac{1}{v}\right). \tag{4.10}$$

The kinetic model is thus able to recover a realistic long time behaviour, with formation of Pareto tails; specifically, the Pareto index of distribution $(g_1)_\infty$ for country 1 is $\alpha_1 = 1 + 2\,\tilde\gamma_1/\lambda_{11}$, and it depends both on the trading propensity and on the random effects of such population.

5. Conclusion and perspectives

In this paper we have proposed a kinetic model to describe wealth distribution of a set of countries, taking into account also possible transfers of individuals from one country to another. The interaction rules are of the type proposed in [12], since they have already been proved to reproduce realistic long–time behaviours, as the formation of Pareto tails and, for suitable options for the random variables not treated in this paper, also a time exponential increasing of the total wealth. The transfers are assumed to occur as consequence of a trading interaction, and their effects are described through Boltzmann–type operators, provided by the difference between a gain and a loss term. The lack of symmetry between a transfer and the reverse one (the probability of the reverse transfer is different, and also the interaction rule may vary) does not allow to simplify the weak form of Boltzmann operators, and the inverse trading rule, determining the pre–trade wealths corresponding to given post–trade ones, is explicitly required. Unlike previous models for a simple market economy (even with international trade), in the present model the number of individuals of each population is not constant (because of transfers), and this has implications on the macroscopic equations and on the equilibrium configuration. The equilibrium states for the number of agents and the mean wealth of each population are explicitly determined and discussed in the case of two populations. Then, a continuous trading asymptotic limit is investigated, corresponding to a small amount of exchange in each interaction and to a low probability of transfers; in this regime, Boltzmann kinetic equations may be approximated by simpler PDEs of Fokker–Planck type, but with additional contributions with respect to the original CPT model [12,17] due to transfers of individuals. The equilibrium distributions may be explicitly recovered in a special case, showing the desired Pareto tails, with Pareto index depending on the variance of random variables. Unlike many papers on the matter, in this work we have kept in the Boltzmann kernels different trading probabilities for any pair of interacting populations, and also specific transfer probabilities in the additional transfer operators. These (constant) parameters affect of course the evolution of the model and the equilibrium values of macroscopic fields.

In this paper the model has been described in detail for a set of two interacting populations, but it could be obviously extended to $N > 2$ different countries. The construction of Boltzmann–type operators (and their weak form) would remain the same, only with indices ranging from 1 to N. Even the macroscopic equations for population densities and total or mean wealth could be derived analogously to the procedure described in this paper. However, the steady values of these fields (provided essentially by the solutions of linear systems) could not be determined explicitly, since the constraints corresponding to preservations of total number of individuals and of total wealth are not enough to provide a unique equilibrium configuration; a manifold of admissible equilibria would appear, and their stability properties should be investigated by classical tools of qualitative analysis of dynamical systems. The asymptotic reduction of the Boltzmann equations to a set of Fokker–Planck–type PDEs could also be performed as in the case $N = 2$.

As future work, the kinetic description of transfers could be generalized in several ways. For instance, since mobility has an intrinsic cost, it could be allowed only if the individual wealth overcomes a suitable threshold V; this would imply the presence of unit step functions in the kernel of transfer Boltzmann operators, analogously to kinetic models for reacting gas mixtures, where a chemical reaction may occur only if the impinging kinetic energy is enough [20,21]. The construction of macroscopic equations in this case is not trivial at all, and probably further approximations should be adopted, since Boltzmann integrals with thresholds may be cast in explicit form essentially when distributions take a Maxwellian shape (and this does not usually occur in socio–economic frames). Of course, when the threshold $V \to 0^+$ one should recover the model studied in the present paper, while the limit $V \to +\infty$ should provide the classical CPT model (with no mobility), but all intermediate options are completely open problems. A simpler way to take into account the costs due to the mobility could be the subtraction of a small fraction of individuals wealth in each interaction giving rise to a transfer. In order to preserve the conservation of total wealth in the considered economic system, the subtracted amount should be suitably redistributed among the populations; suggestions for the construction of a proper redistribution operator may be found in [5]. A possible simplification of the mathematical investigation of such models could occur if the transfers of individuals are described by simpler kinetic operators of BGK–type, similarly to BGK models for inert or reacting gas mixtures [6,22], that could also allow the analytical and numerical investigation of quite complicated asymptotic regimes for N populations, with for instance some transfers much more probable than the others, as it happens in realistic societies.

Data Accessibility. All data are provided in the paper.

Competing Interests. The author declares that she has no competing interests.

Funding. This research has been funded by Ministero dell'Università e della Ricerca of the Italian Government, through the PRIN project Multiscale Phenomena in Continuum Mechanics: Singular Limits, Off-Equilibrium and Transitions (grant number 2017YBKNCE).

Acknowledgements. The author thanks the support by the University of Parma, and by the Italian National Group of Mathematical Physics (GNFM-INdAM).

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