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Lane-Emden equations and the geometry of Sobolev-Poincaré inequalities

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Abstract

This thesis is addressed to the investigation of the optimal constants in the Sobolev-Poincaré inequalities and in the Hardy inequalities, as well as, to the study of the Lane-Emden equation for the $p-$Laplacian, with subhomogeneous power q in the right-hand side. First, we prove a comparison principle for positive supersolutions and subsolutions of the Lane-Emden equations, then, as applications, we give some results concerning such solutions when defined on convex sets. By exploiting the previous results, we discuss the relation between the embedding of the homogeneous Sobolev space $\mathcal{D}^{1,p}_0$ into L^q and the summability properties of the distance function, thanks to a preliminary study for the sharp constants in Morrey-type and Hardy-type inequalities for general open sets. In turn, this analysis permits to study the asymptotic behaviour of both the optimal constants in the Sobolev-Poincaré inequalities and the positive solutions of the Lane-Emden equation, as the exponent p diverges to ∞, under optimal assumptions on the open set. We also give some geometric lower bounds for sharp Sobolev-Poincaré constants $\lambda_{p,q}$, when $q < p$, on the class of convex bounded open sets: indeed, we prove that $\lambda_{p,q}$ can be bounded from below both in terms of the norm of the distance function in a suitable Lebesgue space, and in terms of a certain power of the inradius of the set. The results so obtained generalize the Makai inequality and the Hersch-Protter inequality, respectively. Finally, in the last part of the thesis, we study the sharp constant in the Hardy inequality in the setting of fractional Sobolev spaces $W^{s,p}$ defined on general open sets, for every $0 < s < 1$. We first list some properties of such a constant and we give a variational characterization, which extends an analogous well-known result for the local case. Then, focusing on the class of convex open sets, we compute the sharp fractional Hardy constant in the regime $s\,p \geq 1$, by constructing suitable supersolutions for the associated equation by means of the distance function. We note that we can avoid the claimed restriction on $s\,p$ when the convex set is an half-space.

Sunto

Questa tesi è dedicata allo studio delle costanti ottime nelle disuguaglianze di Sobolev-Poincaré e nella disuguaglianza di Hardy e allo studio dell'equazione di Lane-Emden per il $p-$Laplaciano, con potenza q sotto-omogenea nel termine a destra. Prima, dimostriamo un principio di confronto tra le soprasoluzioni e le sottosoluzioni positive dell'equazione di Lane-Emden e, come applicazioni, proviamo alcuni risultati relativi a tali soluzioni quando sono definite su insiemi convessi. In seguito, utilizzando i risultati precedenti, discutiamo la relazione tra l'inclusione dello spazio omogeneo di Sobolev $\mathcal{D}_0^{1,p}$ in L^q e le proprietà di sommabilità della funzione distanza, grazie a uno studio preliminare delle costanti ottime in disuguaglianze di tipo Morrey e Hardy per insieme aperti generali. Inoltre questa analisi permette di studiare il comportamento asintotico sia delle costanti ottime nelle disuguaglianze di Sobolev-Poincaré, sia delle soluzioni positive dell'equazione di Lane-Emden, quando l'esponente p diverge a ∞, con ipotesi ottimali sull'insieme aperto. Inoltre, diamo alcuni *lower bound* geometrici ottimali per le costanti ottime di Sobolev-Poincaré $\lambda_{p,q}$, quando $q < p$, nella classe degli insiemi convessi aperti limitati: dimostriamo che $\lambda_{p,q}$ può essere limitato dal basso sia in termini della norma della funzione distanza in un opportuno spazio di Lebesgue, sia in termini di una certa potenza dell'*inradius* dell'insieme. I risultati così ottenuti generalizzano rispettivamente la disuguaglianza di Makai e la disuguaglianza di Hersch-Protter. Nella parte finale della tesi, studiamo la costante ottima della disuguaglianza di Hardy negli spazi di Sobolev frazionari definiti su aperti generali. Prima elenchiamo alcune proprietà di tale costante e ne diamo una caratterizzazione variazionale, che estende un risultato analogo ben noto nel caso locale. Poi, considerando la classe degli aperti convessi, calcoliamo la costante ottima di Hardy, nel regime $s\,p \geq 1$, costruendo opportune soprasoluzioni per l'equazione associata per mezzo della funzione distanza. Osserviamo che è possibile rimuovere la restrizione su $s\,p$ quando l'insieme convesso coincide con un semispazio.

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Contents

Introduction

In the Mathematical Analysis, functional inequalities emerge as powerful tools and they play a crucial role in many questions from Calculus of Variations, Analysis of PDE and Optimization problems. One of the most celebrated functional inequalities in Sobolev spaces is the *Poincaré inequality*, which asserts that, for every $1 \le p < \infty$ and for every $\Omega \subsetneq \mathbb{R}^N$ open set with finite measure, there exists a positive constant $C = C(N, p, \Omega)$ such that

$$\int_\Omega |\nabla u|^p \, dx \ge C \int_\Omega |u|^p \, dx, \qquad \text{for every } u \in C_0^\infty(\Omega), \tag{1}$$

(see [**32**, Corollary 9.19] for a proof). This inequality is widely applied in the Functional Analysis and and a large quantity of extensions has been proved in literature (see for example [**107**, Section 1.1.11] or [**95**, Section 12.2] for some Poincaré-type inequalities).

A very significant generalization is known with the name of *Sobolev-Poincaré's inequality*, given by the following

$$\int_\Omega |\nabla u|^p \, dx \ge C \left(\int_\Omega |u|^q \, dx \right)^{\frac{p}{q}}, \qquad \text{for every } u \in C_0^\infty(\Omega), \tag{2}$$

for every $1 \le q \le p^*$, where

$$p^* = \begin{cases} N\,p/(N-p), & \text{if } p < N, \\ +\infty, & \text{if } p \ge N. \end{cases}$$

(see, for example, [**32**] for a study on Sobolev-Poincaré inequalities when Ω is a bounded open set, as well as, [**107**, Section 1.4]). When the above inequality holds true, we automatically get the existence of the continuous Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, and, when $\Omega \subsetneq \mathbb{R}^N$ has finite measure, such an embedding is also compact with $1 \le q < p^*$ (this is due to Rellich–Kondrachov's Theorem, see for example [**95**, Theorem 11.10]).

For every $1 < p < \infty$, given an open set Ω satisfying inequality (1) or (2) for some positive constant C, we can introduce the *sharp Sobolev-Poincaré constants* defined by

$$\lambda_{p,q}(\Omega) := \inf_{\psi \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla \psi|^p \, dx \; : \; \int_\Omega |\psi|^q \, dx = 1 \right\}, \qquad \text{for every } 1 \le q < \infty. \tag{3}$$

In the *homogenous* case $q = p$, we will write $\lambda_p(\Omega)$ in place of $\lambda_{p,p}(\Omega)$.

Another well-known Poincaré-type inequality is the so-called *Hardy inequality*: it states that, for every bounded open set $\Omega \subset \mathbb{R}^N$ having Lipschitz boundary and for every $1 < p < \infty$, there exists a constant $C = C(N, p, \Omega) > 0$ such that

$$\int_\Omega |\nabla u|^p \, dx \ge C \int_\Omega \frac{|u|^p}{d_\Omega^p} \, dx, \qquad \text{for every } u \in C_0^\infty(\Omega), \tag{4}$$

where d_Ω stands for the *distance function* from the boundary of Ω, which is given by

$$d_\Omega(x) := \inf_{y\in\partial\Omega} |x-y|, \qquad \text{for every } x \in \Omega,$$

and, in this work, it is extended to 0 outside of Ω. Actually, the literature about Hardy's inequalities is quite rich and varied, we refer, for example, to the original paper [**110**], as well as, to the classical monograph [**111**, Theorem 21.3] for a proof of Hardy's inequality. In general, a very interesting problem consists in finding necessary and/or sufficient conditions assuring that an open set admits a Hardy inequality: we refer for example to [**2**, **46**, **58**, **72**, **91**, **92**, **96**] and [**130**] for some classical results in this direction. Another of the most challenging problems is the investigation of the best constant in the Hardy inequality (4), on different classes of open sets, provided it holds true for a fixed $p \in (1,+\infty)$. The latter is defined by

$$\mathfrak{h}_p(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx \,:\, \int_\Omega \frac{|u|^p}{d_\Omega^p}\,dx = 1 \right\}.$$

In the case when $\Omega = \mathbb{R}^N \setminus \{0\}$ the distance function is simply given by

$$d_{\mathbb{R}^N\setminus\{0\}}(x) = |x|, \qquad \text{for every } x \neq 0,$$

and a celebrated result (see [**9**, **96**] and also [**107**, Chapter 1, Section 1.3.1]) states that, for $p \neq N$, it holds

$$\mathfrak{h}_p(\mathbb{R}^N \setminus \{0\}) = \left| \frac{N-p}{p} \right|^p. \tag{5}$$

Moreover, when $\Omega \subsetneq \mathbb{R}^N$ is a convex open set, thanks to the well-known result in [**104**, Theorem 11], for every $1 < p < \infty$, we have

$$\mathfrak{h}_p(\Omega) = \mathfrak{h}_p(\mathbb{H}_+^N) = \left(\frac{p-1}{p}\right)^p,$$

where $\mathbb{H}_+^N = \mathbb{R}^{N-1} \times (0,+\infty)$ is the half-space of $\mathbb{R}^N$ (see also [**106**, Theorem 1]). Actually, some general developments on the problem of determining the sharp Hardy constant can be found in [**104**], as well as, in the paper [**105**].

When we drop the regularity assumption on the boundary of Ω, the Hardy inequality (4) continues to be valid in some particular cases and explicit estimates for $\mathfrak{h}_p(\Omega)$ are also known. Indeed, if $p = N = 2$, thanks to a result due to Ancona ([**5**]), for every simply connected open set $\Omega \subsetneq \mathbb{R}^2$, the Hardy inequality (4) holds with the optimal Hardy constant satisfying

$$\mathfrak{h}_2(\Omega) \geq \frac{1}{16}. \tag{6}$$

Moreover, every general open set $\Omega \subsetneq \mathbb{R}^N$ verifies the Hardy inequality (4) when $p > N$: more precisely, we have that

$$\mathfrak{h}_p(\Omega) \geq \left(\frac{p-N}{p}\right)^p, \tag{7}$$

and the equality is attained when Ω is ball in $\mathbb{R}^N$ from which we remove the center (see [**72**, **96**, **130**] and also [**8**, **41**, **71**] for the proof of the lower bound above); neverthless the determination of the optimal constant $\mathfrak{h}_p(\Omega)$ is still an open problem for the class of general open sets.

In the partcular case $p > N$, another remarkable Poincaré-type inequality is the the classical *Morrey inequality* which states that for every bounded open set $\Omega \subset \mathbb{R}^N$, there exists a constant $M =$

$M(N,p,\Omega) > 0$ such that

$$\int_\Omega |\nabla u|^p\,dx \geq M\,\|u\|^p_{C^{0,\alpha_p}(\Omega)}, \qquad \text{for every } u \in C_0^\infty(\Omega),$$

where $\alpha_p = 1 - N/p$ and

$$\|u\|_{C^{0,\beta}(\Omega)} = \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\beta}(\overline{\Omega})}, \qquad \text{for every } 0 < \beta < 1,$$

with

$$[u]_{C^{0,\beta}(\overline{\Omega})} = \sup_{x\neq y;\, x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\beta},$$

(see, for example, [**32**, Theorem 9.12] for a proof of the Morrey inequality). In light of the last estimate, when $p > N$, we can define the sharp Morrey constant as

$$\mathfrak{m}_p(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx \,:\, [u]_{C^{0,\alpha_p}(\overline{\Omega})} = 1 \right\}. \tag{8}$$

We notice that some remarkable studies on the optimal Morrey constant $\mathfrak{m}_p$ and its extremals have been recently done by Hynd and Seuffert in a series of papers (see [**79**, **80**] and [**81**]).

In this thesis, we will study and generalize the functional inequalities mentioned above, either in the classical Sobolev spaces or in the *non local* setting of Sobolev-Slobodeckiĭ fractional spaces, as well as, we will discuss conditions which guarantee the validity of inequalities (2) and (4) on different classes of open sets and we will provide some geometric estimates for the sharp constants $\lambda_{p,q}(\Omega)$, $\mathfrak{h}_p(\Omega)$ and $\mathfrak{m}_p(\Omega)$.

First, we consider the optimal Morrey constant $\mathfrak{m}_p(\Omega)$ on the class of general open sets and we show that its value is independent on the choice of the open set $\Omega \subset \mathbb{R}^N$. The explicit value for the sharp Morrey constant is still unknown, but we give an interesting lower bound and we determine its asymptotic behaviour as p goes to ∞. Then, as a collateral result, in the case $p > N$ and for every open set $\Omega \subset \mathbb{R}^N$, we find a lower bound for the *generalized* sharp Hardy constant, defined as

$$\mathfrak{h}_{p,q}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx \,:\, \left\| \frac{u}{d_\Omega^{\frac{N}{q}+\frac{p-N}{p}}} \right\|_{L^q(\Omega)} = 1 \right\}, \qquad \text{for } p < q \leq \infty. \tag{9}$$

Moreover, we obtain the following asymptotic result

$$\lim_{p\to\infty} \Big(\mathfrak{h}_p(\Omega)\Big)^{\frac{1}{p}} = \lim_{p\to\infty} \Big(\mathfrak{h}_{p,\infty}(\Omega)\Big)^{\frac{1}{p}} = 1. \tag{10}$$

We point out that the case $q = \infty$ in (9) has been largely investigated by Hynd, Larson and Lindgren in the very recent paper [**77**], where they deeply studied the link between the geometry of the set Ω and the sharp Hardy constant $\mathfrak{h}_{p,\infty}(\Omega)$.

Concerning the standard Poincaré inequality (1), the investigation for its optimal constant $\lambda_p(\Omega)$ proves to be a highly challenging problem, largely studied in the existing literature.

Focusing on the linear case $p = 2$, the minimization problem defining the sharp Poincaré constant

$$\lambda(\Omega) := \inf_{\psi\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla\psi|^2\,dx \,:\, \int_\Omega |\psi|^2\,dx = 1 \right\},$$

is usually referred as the classical *homogeneous eigenvalue problem*: the study of existence and uniqueness of minimizers for $\lambda(\Omega)$ can be found in [**74**] (see Theorem 1.2.5), as well as, in [**69**] (see Theorem 8.38). In this case the Euler-Lagrange equation associated to the minimization problem is given by the following linear equation

$$-\Delta\varphi = \alpha\varphi, \qquad \text{in } \Omega, \tag{11}$$

joined with Dirichlet boundary conditions, where $\alpha = \lambda(\Omega)$ and $\Delta u = \mathrm{div}(\nabla u)$ denotes the classical Laplacian operator. A study on existence and uniqueness of solutions for (11) is provided in [**31**], where the authors deal with a more general continuous functional on the right-hand side. In particular, whenever $\lambda(\Omega)$ is a positive quantity, then we will refer to it as *the first eigenvalue* of the classical Dirichlet Laplacian. Moreover, $\lambda(\Omega)$ can represents also the *principal frequency* of a drum's vibration: rougly speaking, if we envision the set Ω as an elastic membrane anchored along its boundary $\partial\Omega$, then the first eigenvalue $\lambda(\Omega)$ corresponds to the membrane's lowest vibration frequency (see [**116**]).

We notice that there exists a tight connection between the value of $\lambda(\Omega)$ and the geometry of the set Ω and, from this point of view, many *sharp* geometric estimates have been proved for the first eigenvalue. In the class of convex open sets $\Omega \subset \mathbb{R}^N$, the most celebrated estimate is actually the following *Hersch-Protter inequality*

$$\lambda(\Omega) \geq \frac{\pi^2}{4} r_\Omega^{-2}, \tag{12}$$

where $r_\Omega := \|d_\Omega\|_{L^\infty(\Omega)}$ is the *inradius* of Ω. This was first proved by Hersch in [**65**, Théorème 8.1], for every planar convex bounded set Ω, and then generalized to every dimension $N > 2$ by Protter in [**118**, Theorem 2] (see also [**38**]). This inequality is actually optimal, as it holds as an identity, for example, when the convex set $\Omega \subset \mathbb{R}^N$ is a *slab*, such as $\mathbb{R}^{N-1} \times (0,1)$. An upper bound for $\lambda(\Omega)$ in the class of convex sets is given by the *Pólya inequality* proved in [**115**] (see [**83**] for a generalization), which states that $\lambda(\Omega)$ can be bounded from above in a sharp way, by the ratio between the perimeter and the volume of Ω. In a different way, Makai in [**103**] and Hayman in [**73**] showed the existence of a universal constant $C > 0$ such that, in the class of planar simply connected open sets, it holds

$$\lambda(\Omega) \geq C r_\Omega^{-2}. \tag{13}$$

We point out that the exact determination of the sharp constant in (13) is an open problem. Further generalizations to the class of multiply connected sets of order $k > 1$ have been studied by [**44**, **112**, **127**], while for the higher dimensions $N \geq 3$ we refer to [**73**, Theorem 2] and [**127**, Theorem 3]. Another celebrated estimate for $\lambda(\Omega)$, is the so-called *Faber-Krahn inequality*, which consists in a lower bound in terms of the $N-$dimensional measure $|\Omega|$ when it is finite. Finally, in the class of bounded sets, we recall the well-known *Cheeger inequality* (see [**40**, **93**, **94**]) and its reverse formulation shown in [**114**].

In order to establish some geometric estimates for $\lambda_{p,q}(\Omega)$, we devote a part of this thesis to show a comparison principle for positive supersolutions and subsolutions of the following *Lane-Emden equations*

$$-\Delta_p u = \alpha\, |u|^{q-2}\, u, \qquad \text{in } \Omega, \tag{14}$$

with $1 \leq q \leq p$, where the symbol Δ_p stands for the $p-$Laplace operator

$$\Delta_p u = \mathrm{div}\,(|\nabla u|^{p-2}\, \nabla u).$$

Indeed, we observe that if the infimum in the minimization problem (3) is attained on $W_0^{1,p}(\Omega)$, by optimality each minimizer having unitary L^q norm is a solution of (14) with $\alpha = \lambda_{p,q}(\Omega)$. More precisely, in this thesis, we focus on the *sub-homogeneous case*, i.e. on the range $q < p$. In this case, we recall that Lane-Emden equations are connected with another class of well-known equations: the *doubly nonlinear* *slow diffusion equations*, given by

$$\Delta_p u = \partial_t(|u|^{q-2}\,u), \qquad \text{in } (0,+\infty)\times\Omega, \tag{15}$$

coupled with the boundary condition $u(t,x) \equiv 0$ on $\partial\Omega$, for every $t \in (0,+\infty)$. To be more precise, when we look for stationary solutions of (15) of the type $u(t,x) = T(t)\,X(x)$, the spatial part X must solve the Lane-Emden equation (14), while the temporal part T will decay polynomially to 0 with the power $1/(p-q)$, for large time t. Thanks to these first observations, it is possible to understand the long-time behaviour for solutions of (15) and find that this can be predicted by positive solutions of (14), with homogeneous Dirichlet boundary conditions. Indeed, Stan and Vázquez, in [**121**, Theorem 2.1], showed that

$$\lim_{t\to\infty}\left\| t^{\frac{1}{p-q}}u(t,\,\cdot\,) - w^{\Omega,\alpha}_{p,q}(\,\cdot\,)\right\|_{L^\infty(\Omega)} = 0,$$

where $w^{\Omega,\alpha}_{p,q}$ denotes the unique weak positive solution of (14) vanishing at the boundary and $\alpha = (q-1)/(p-q)$. This holds true at least when $\Omega \subsetneq \mathbb{R}^N$ is a bounded open set, with sufficiently smooth boundary, and it generalizes an old result by Aronson and Peletier (see [**7**] and also [**129**] for a simpler proof) for the case $p = 2$ and $1 < q < 2$, when the equation (15) is called the *porous medium equation*.

Our claimed comparison principle states that, when Ω is an open connected set satisfing $\lambda_{p,q}(\Omega) > 0$, given $u, v \in X^{q,p}(\Omega) = \left\{\psi \in L^q(\Omega)\ :\ \nabla\psi \in L^p(\Omega;\mathbb{R}^N)\right\}$, such that v is a positive supersolution and u is a positive subsolution of (14), and $(u-v)$ satisfies a *suitable boundary condition* then $v \geq u$ a.e. in Ω. We note that the proof of this result does not exploit any regularity for the solutions, used on the contrary in other comparison results (see [**119**]). Indeed it relies on the following *hidden convexity* property for the $p-$Dirichlet integral

$$\int_\Omega \left|\nabla\sigma^t\right|^p\,dx \leq (1-t)\int_\Omega |\nabla\psi_0|^p\,dx + t\int_\Omega |\nabla\psi_1|^p\,dx, \qquad \text{for every } t\in[0,1], \tag{16}$$

which we show for every $\psi_0, \psi_1 \in X^{q,p}(\Omega)$, with $\sigma^t = \Big((1-t)\,\psi_0^q + t\,\psi_1^q\Big)^{\frac{1}{q}}$ and $1 \leq q \leq p$. Moreover, we identify the equality cases in (16). Thanks to the hidden convexity property, we have that the *energy functional* naturally associated to (14), given by

$$\mathfrak{F}^\alpha_{p,q}(\psi) := \frac{1}{p}\int_\Omega |\nabla\psi|^p\,dx - \frac{\alpha}{q}\int_\Omega |\psi|^q\,dx, \qquad \text{for every } \psi\in C^\infty_0(\Omega), \tag{17}$$

is actually convex in this suitable sense, despite the presence of the concave lower-order perturbation given by the term $|\psi|^q$. We note that the property (16) generalizes a remarkable result by Benguria originally proved for $p = q = 2$ in his Ph.D thesis [**12**] (see also [**13**, Lemma 4]). The case $p = q \neq 2$ has been treated by Díaz and Saá in [**51**, Lemme 1], and then many extensions have been obtained in order to cover the general case $1 < p < \infty$ with an exponent $1 < q \leq p$ (see, for example, [**11**, **18**, **22**, **50**, **88**, **109**, **123**]). We also refer to the papers [**89**], where a discussion on the equality cases can be found, and [**23**], where the relation between the inequality (16) and the so-called *Picone inequality* is shown.

Our comparison principle can be seen as an extension to the quasilinear case of a result by Kajikiya (see [**87**, Theorem 2.2]), even if Kajikiya's proof is different from ours. In particular, the result in [**87**] concerns only the semilinear case, i.e. $p = 2$, but at the same time it is fairly more general, as it deals with positive solutions to

$$-\Delta u = g(u), \quad \text{in } \Omega, \tag{18}$$

under the assumption that g is sublinear. In the subsequent paper [**86**], such a comparison principle is applied to study geometric properties of positive solutions to (18), hence, in this thesis, we also extend these results to the solutions of Lane-Emden equation (14). Indeed, we apply our comparison principle to get a large quantity of consequences: first, we show a uniqueness result for the minimization of the functional $\mathfrak{F}^{\alpha}_{p,q}$, over functions with given non-negative boundary datum. This also implies the uniqueness of the positive solution of (14), with homogeneous Dirichlet boundary datum, which we denote by $w^{\Omega,\alpha}_{p,q}$. We recall that this retrieves a classical uniqueness result by Díaz and Saá ([**47**]), but with a more general boundary datum and without regularity assumptions on the set, while, when Ω is bounded, a uniqueness theorem has been shown in [**89**, Theorem 1.2], for homogeneous Dirichlet boundary conditions and more general variational integrals with $p-$growth. Moreover, following [**86**, Theorem 2.5], as an immediate application of the comparison principle, we obtain a sharp *pointwise* two-sided estimate for $w^{\Omega,\alpha}_{p,q}$ when Ω is a convex bounded open set. In particular, we get a sharp L^∞ estimate for $w^{\Omega,\alpha}_{p,q}$ which we also extend, thanks to the comparison principle, to every sign-changing solutions in $W^{1,p}_0(\Omega)$ of the equation (14). Finally, as a further application, we obtain a generalized version of the (sharp) Hersch-Protter inequality (12) for the Sobolev-Poincarè constants $\lambda_{p,q}(\Omega)$, when $1 \leq q < p$ and Ω is a convex bounded open set, namely, we prove that

$$\lambda_{p,q}(\Omega)\,|\Omega|^{\frac{p-q}{q}} \geq \left(\frac{\pi_{p,q}}{2}\right)^p \frac{1}{r_\Omega^p}, \tag{19}$$

where $\pi_{p,q} = \big(\lambda_{p,q}((0,1))\big)^{1/p}$. We observe that a first version of the above inequality has been proved for planar convex sets by Makai, in [**102**], when $p = 2$ and $q = 1$. The case $q = p \neq 2$ has been shown by Kajikiya in [**85**] for convex bounded open sets $\Omega \subset \mathbb{R}^N$ (see also [**20**, **48**]). Then, in [**25**, Theorem 1], Brasco and Mazzoleni prove the inequality (19) in the case $p = 2$ and $1 \leq q < 2$, and, in [**49**, Theorem 4.3], Della Pietra, Gavitone and Guarino Lo Bianco extended such a result to cover the case $p \neq 2$ and $q = 1$. We notice that the result provided by Makai follows as corollary of a *stronger* and sharp estimate for the torsional rigidity $T(\Omega) := 1/\lambda_{2,1}(\Omega)$, showed in the aforementioned paper [**102**], namely

$$T(\Omega) \leq \int_\Omega d_\Omega^2\,dx,$$

which holds for planar convex sets Ω and is asymptotically attained by sequences of rectangles which are bounded in one dimension. Inspired by the last result, we show, in the $N-$dimensional case, the following sharp *Makai inequality*

$$\lambda_{p,q}(\Omega) \geq \frac{C_{p,q}}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}},$$

which holds in the class of convex bounded open sets $\Omega \subsetneq \mathbb{R}^N$, when $N \geq 2$ and $1 \leq q < p$, with an optimal explicit constant $C_{p,q}$. The main tool in our proof is a covering argument for polygonal sets, exploited by Makai in the planar case.

In the general case $q \leq p^*$, the sharp Sobolev-Poincaré constants $\lambda_{p,q}(\Omega)$ are strictly connected to the problem of determining conditions on the open set Ω which guarantee the validity of the continuous/compact Sobolev injection $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$. Here the homogeneous Sobolev space $\mathcal{D}^{1,p}_0(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|\nabla\cdot\|_{L^p(\Omega)}$. In particular, we address a part of this thesis to discuss the link between the continuous (and compact) embedding $\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ and the summability of the distance function d_Ω. This question is motivated by a recent result contained in [**28**] by Brasco and Ruffini (see also [**36**]) which states that the existence of the above Sobolev embeddings on a general open set $\Omega \subset \mathbb{R}^N$ is equivalent to a precise summability condition on the so-called *$p-$torsion function*. This is formally defined as the positive solution of (14), with $\alpha = q = 1$, which vanishes on the boundary of Ω (see [**28**, Definitions 2.1 and 2.2] for the precise definition).

In our study, we prove that, when $\Omega \subsetneq \mathbb{R}^N$ is a general open set, $1 \le q < p$ and $p > N$, it holds

$$d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega) \qquad \Longleftrightarrow \qquad \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact},$$

$$d_\Omega \in L^{\infty}(\Omega) \qquad \Longleftrightarrow \qquad \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega),$$

and

$$\lim_{R\to+\infty} \|d_\Omega\|_{L^\infty(\mathbb{R}^N\setminus B_R)} = 0 \qquad \Longleftrightarrow \qquad \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact}.$$

For capacitary reasons, we will see that implications "$\Longrightarrow$" are possible only in the *superconformal case* $p > N$. Indeed, in the case $p \le N$, we will find that the claimed summability conditions on d_Ω are *necessary* for the embeddings to hold, but not sufficient.

The first equivalence stated above follows from the two-sided estimate

$$\mathfrak{h}_p(\Omega) \le \lambda_{p,q}(\Omega) \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}} \le \lambda_p(B_1), \tag{20}$$

which holds for every open set $\Omega \subsetneq \mathbb{R}^N$ when $1 \le q < p$ and $p > N$. Here B_1 denotes the $N-$dimensional ball in $\mathbb{R}^N$, centered in the origin, with radius 1. We prove the lower bound in (20) by using the Hardy inequality (7), while we obtain the upper bound by applying the comparison principle for the Lane-Emden equation. In the case $q = p$, by reasoning as above for the lower bound and by using the scaling property of λ_p for the upper bound, we show the estimates

$$\mathfrak{h}_p(\Omega) \le \lambda_p(\Omega)\, r_\Omega^p \le \lambda_p(B_1), \tag{21}$$

which imply the second equivalence.

Concerning the third equivalence, we remark that this result has been already proved by Adams (see [**2**, Example 6.11] and [**3**, Theorem 2]). Our proof for the implication "$\Longrightarrow$" uses a different argument, whose crucial ingredient is the Hardy inequality (7).

Moreover, by using the asymptotics (10) for $\mathfrak{h}_p(\Omega)$ and thanks to the geometric estimates provided by (20) and (21), we can study the asymptotic behaviour for the Sobolev-Poincaré constants $\lambda_{p,q}(\Omega)$, as $p \to \infty$, within the class of general open sets. We recall some results in the existing literature: for the homogeneous case $q = p$, in [**84**, Lemma 1.5] and in [**68**, Theorem 3.1], the authors show that

$$\lim_{p\to\infty} \Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{r_\Omega},$$

under the restrictive assumption that Ω is a bounded open set of $\mathbb{R}^N$ (see also [**39**, Theorem 5.1]). Actually, we prove that the above limit holds for every general open set Ω. When $q = 1 < p < \infty$, in [**16**, Proposition 2.1] and [**88**, Theorem 1], it is proved that

$$\lim_{p\to\infty} \Big(\lambda_{p,1}(\Omega)\Big)^{\frac{1}{p}} = \left(\int_\Omega d_\Omega\,dx\right)^{-1},$$

for every Ω bounded open set; then such a result has been extended to every set with finite volume in [**33**, Corollary A.4].

In this thesis, we will extend the above asymptotics, by proving that

$$\lim_{p\to\infty} \Big(\lambda_{p,q}(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{\|d_\Omega\|_{L^q(\Omega)}},$$

for every $1 \leq q < \infty$ and for every general open sets Ω, without further regularity assumptions. Moreover, our result covers also the case $q = \infty$. For every $1 < p < \infty$, we can define the supremal Sobolev-Poincaré constants

$$\lambda_{p,\infty}(\Omega) = \min_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx \, : \, \|u\|_{L^\infty(\Omega)} = 1 \right\},$$

recently studied in [**61**] and [**78**]. For every open set $\Omega \subsetneq \mathbb{R}^N$, we prove that

$$\lim_{p \to \infty} \Big(\lambda_{p,\infty}(\Omega) \Big)^{\frac{1}{p}} = \frac{1}{r_\Omega},$$

by generalising the result [**61**, Theorem 3.2] shown for bounded open sets.

The last part of this thesis is devoted to the study of the Hardy inequality in the setting of Sobolev-Slobodeckiĭ fractional spaces. It is formulated as in (4), with the left-hand side replaced by the "global" Gagliardo-Slobodeckiĭ seminorm

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+s\,p}} \, dx \, dy, \qquad \text{with } 0 < s < 1.$$

In this framework, for every $1 < p < \infty$, we define the sharp constant in the fractional Hardy inequality through the following infimum problem

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} \, : \, \int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}} \, dx = 1 \right\}. \tag{22}$$

There exists a large literature about sharp fractional Hardy's inequalities: in [**66**, Theorem 1.1], Frank and Seiringer computed the sharp constant (22), when Ω is the punctured space $\mathbb{R}^N \setminus \{0\}$. In [**21**, Theorem 1.1], Brasco and Cinti proved that the fractional Hardy inequality holds for every $\Omega \subsetneq \mathbb{R}^N$ convex set, every $1 < p < \infty$ and $0 < s < 1$. The claimed result follows by the lower bound

$$\mathfrak{h}_{s,p}(\Omega) \geq \frac{C}{s\,(1-s)},$$

for some explicit constant $C = C(N, p) > 0$. Here, the dependence on the fractional parameter s is optimal, as shown in [**21**, Remark 1.2]. In the setting of convex sets, we recall the remarkable result due to Bogdan and Dyda and concerning the explicit value of the sharp constant $\mathfrak{h}_{s,2}(\mathbb{H}^N_+)$ for every $0 < s < 1$ (see [**17**, Theorem 1.1]). Then, such a result has been extended to the case $p \neq 2$ by Frank and Seiringer (see [**65**, Theorem 1.1]) and, in [**62**, Theorem 5], Filippas, Moschini and Tertikas proved that $\mathfrak{h}_{s,2}(\mathbb{H}^N_+) = \mathfrak{h}_{s,2}(\Omega)$, when $s \geq 1/2$ and $\Omega \subsetneq \mathbb{R}^N$ is a convex open set. To be more precise, the result in [**17**, Theorem 1.1] is given by replacing, in the definition of $\mathfrak{h}_{s,2}(\mathbb{H}^N_+)$, the "global" Gagliardo-Slobodeckiĭ seminorm, defined on the whole set $\mathbb{R}^N \times \mathbb{R}^N$, with the "regional" Gagliardo-Slobodeckiĭ seminorm, which is computed on the set $\mathbb{H}^N_+ \times \mathbb{H}^N_+$. Neverthless, as claimed in [**17**], the value of the "global" sharp Hardy constant can be computed from the "regional" constant. Finally, we recall the paper [**56**], which eventually became a landmark and contains many references on the subject of "regional" fractional Hardy's inequalities; other recent interesting papers are [**54**, **57**, **59**, **122**].

Concerning the study of fractional Hardy's inequalities our initial contribution consists in the following characterization of the fractional sharp Hardy constant: we consider the following equation

$$(-\Delta_p)^s u = \lambda \, \frac{u^{p-1}}{d_\Omega^{s\,p}}, \qquad \text{in } \Omega, \tag{23}$$

where $\lambda \geq 0$ and $(-\Delta_p)^s$ is the *fractional $p-$Laplacian of order s*, formally defined, for $1 < p < \infty$ and $0 < s < 1$, by

$$(-\Delta_p)^s u(x) = 2\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+s\,p}}\,dy,$$

and we establish the identity

$$\mathfrak{h}_{s,p}(\Omega) = \sup\Big\{\lambda \geq 0 \,:\, \text{equation (23) admits a positive local weak supersolution}\Big\}. \tag{24}$$

To obtain this, we employ a variational approach and, inspired by the ideas of [**17**], we show the following equivalence

$$\mathfrak{h}_{s,p}(\Omega) > 0 \qquad \Longleftrightarrow \qquad \begin{array}{c}\text{the equation (23) admits a positive}\\ \text{local weak supersolution for some } \lambda > 0.\end{array}$$

This fact is well-known in the local case: when $p = 2$, it is contained, for example, in the classical paper [**5**, Appendix] by Ancona, while a generalization to $p \neq 2$ has been obtained by Kinnunen and Korte (see [**90**, Theorem 5.1]). In the fractional setting, when $0 < s < 1$ and $p = 2$, Fitzsimmons proved such an equivalence in [**64**, Theorem 1.9], by passing throught a probabilistic approach, which can not be replied in the case $p \neq 2$. The claimed characterization (24) empowers us to derive a lower bound for the fractional sharp Hardy constant on the class of convex sets. To achieve this, we construct suitable supersolutions for the equation (23), by employing a method inspired by [**17**] in the case $p = 2$.

Then, our focus shifts to the computation of the explicit fractional sharp Hardy constants $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$ and $\mathfrak{h}_{s,p}(\Omega)$, when $\Omega \subsetneq \mathbb{R}^N$ is a convex open set. In particular, we establish that they coincide and the proof heavily relies on the characterization (24). However, our argument will force us to restrict to the range $s\,p \geq 1$, while, we do not compute the sharp constant on the class of convex sets, in the case when $s\,p < 1$. We notice that, only when $p = 2$, the problem of determing the sharp constant can be completely solved, by using the *fractional Kelvin transform*, as suggested in [**55**].

Plan of the work

Chapter 1. The first chapter is devoted to the introduction of many definitions and preliminary results, involving the Hölder spaces $C^{0,\alpha}$, the Sobolev spaces $W^{1,p}$ and the Sobolev-Slobodeckiĭ fractional spaces $W^{s,p}$, with $0 < \alpha \leq 1$, $1 < p < \infty$ and $0 < s < 1$ (see Sections 1.1, 1.3 and 1.5, respectively, for standard introductions). In the first section, we consider the space $C_0(\Omega)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the sup norm and the Hölder spaces $C^{0,\alpha}(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^N$ is an open set, and we obtain some interpolation inequalities which hold for functions belonging to these spaces.

Then, in Section 1.2, we discuss some consequences of the integrability of the distance function d_Ω from the boundary $\partial\Omega$ of an open set $\Omega \subsetneq \mathbb{R}^N$. Finally, we prove a version of the Ascoli-Arzelà Theorem for functions defined on *quasibounded* sets, i.e. open sets $\Omega \subset \mathbb{R}^N$ which satisfy

$$\lim_{R\to\infty} \|d_\Omega\|_{L^\infty(\mathbb{R}^N\setminus B_R)} = 0.$$

In the subsequent Sections 1.3 and 1.4, we recall some known results about Sobolev embeddings and interpolation inequalities, and we introduce the homogeneous Sobolev space $\mathcal{D}_0^{1,p}(\Omega)$, defined as the the completion of $C_0^\infty(\Omega)$ functions with respect to the L^p norm of their gradients. We also provide a Poincaré-type inequality for $N-$dimensional balls $B_R(x_0)$ valid for functions which vanish in at least one point of $\partial B_R(x_0)$.

Finally, in Sections 1.5 and 1.6, we introduce the Sobolev-Slobodeckiĭ fractional spaces and a particular fractional weighted space, and we prove some compactness results.

Chapter 2. In this chapter, we focus on two celebrated inequalities in Sobolev spaces: the Morrey inequality and the Hardy inequality. Functional inequalities of Morrey–type and Hardy–type are actually among the most studied estimates in the Calculus of Variations, and the results we are going to introduce about this topic are contained in the work **[Z3]** in collaboration with Lorenzo Brasco and Francesca Prinari.

For every open set $\Omega \subseteq \mathbb{R}^N$ and for every $p > N$, we define the optimal Morrey constant as in (8). The exact value of the sharp constant is still unknown, but, in general, it is independent from the choice of the open set $\Omega \subseteq \mathbb{R}^N$, since $\mathfrak{m}_p(\Omega) = \mathfrak{m}_p(\mathbb{R}^N)$ (see Corollary 2.1.3). With the aim at studying some properties for such a constant, in Lemma 2.1.1, we introduce the following quantity

$$\mu_p(B_1) := \inf_{u\in W^{1,p}(B_1)} \left\{ \int_{B_1} |\nabla u|^p\,dx \,:\, u(0) = 1 \text{ and } u(z) = 0 \right\}, \qquad \text{with } z \in \partial B_1,$$

for every $p > N$. In particular, the definition of $\mu_p(B_1)$ does not depend on the choice of $z \in \partial B_1$ and, by means of it, we can show that for every $p > N$

$$|u(x_0) - u(z)| \leq \frac{R^{1-\frac{N}{p}}}{\left(\mu_p(B_1)\right)^{\frac{1}{p}}} \|\nabla u\|_{L^p(B_R(x_0))}, \qquad \text{for every } u \in W^{1,p}(B_R(x_0)) \text{ and } z \in \partial B_R(x_0).$$

(see Remark 2.1.2). By applying the above inequality, in Corollary 2.1.3, we obtain the following bounds for the sharp Morrey constant

$$\mu_p(B_1) \leq \mathfrak{m}_p(\mathbb{R}^N) \leq N\,\omega_N \left(\frac{p-N}{p-1}\right)^{p-1}, \qquad \text{with } p > N.$$

Moreover, we investigate the asymptotics behaviour for $\mu_p(B_1)$ as $p \to \infty$, which appears to be extremely interesting for itself, as well as for some consequent results, such as the asymptotic behaviour for large p of both the sharp Morrey constant $\mathfrak{m}_p$ and the sharp Hardy–type constant $\mathfrak{h}_{p,q}$ (see below).

In the second part of the chapter, we provide an extension for the classical Hardy inequality, which is possible thanks to interpolation arguments in Lebesgue spaces and to the preliminary study of the Morrey inequality. In particular, in Theorem 2.2.1, we propose a quite interesting generalization of the classical Hardy inequality: for $p > N$, we introduce the sharp Hardy-type constant $\mathfrak{h}_{p,q}(\Omega)$ as in (9) and, without any regularity assumption on the general open set $\Omega \subsetneq \mathbb{R}^N$, we give a lower bound for such a constant by means of the classical sharp Hardy constant $\mathfrak{h}_p(\Omega)$ when $q < \infty$, and the Morrey constant $\mu_p(B_1)$ when $q = \infty$.

Moreover, in Corollary 2.1.3 and in Theorem 2.2.1, we will show that

$$\lim_{p\to\infty} \Big(\mathfrak{h}_{p,q}(\Omega)\Big)^{\frac{1}{p}} = \lim_{p\to\infty} \Big(\mathfrak{m}_p(\Omega)\Big)^{\frac{1}{p}} = 1.$$

The results contained in this chapter will then be crucial in Chapters 5 and 6.

Chapter 3. In this chapter, we introduce one of the main objects of study of this thesis: the *Lane-Emden equation* (14). The results contained in this part of the thesis can be found in the joint paper [**Z4**] with Lorenzo Brasco and Francesca Prinari.

More precisely, we focus on the *sub-homogeneous case* $q < p$, and, after an introduction of the Lane-Emden equation and of the properties of its solutions, we prove a comparison principle theorem for such an equation. With this aim, we consider the *energy functional* $\mathfrak{F}^{\alpha}_{p,q}$ defined as in (17), naturally associated to (14), whose properties we study in Section 3.3. we discuss the so-called *hidden convexity* property for the $p-$Dirichlet integral (see Theorem 3.2.1), i. e. the fact that

$$\psi \mapsto \int_\Omega |\nabla \psi|^p\,dx$$

is convex, on the cone of non-negative functions, along curves of the form

$$t \mapsto \Big((1-t)\,\psi_0^q + t\,\psi_1^q\Big)^{\frac{1}{q}}, \qquad \text{with } t \in [0,1], \qquad \text{for every } 1 \leq q \leq p.$$

Thanks to this property, we obtain that the energy functional $\mathfrak{F}^{\alpha}_{p,q}$ is convex in a certain sense, in spite of the presence, in its definition, of the concave lower-order term, given by the L^q norm.

Then, in the last part of the chapter, we apply the hidden convexity property to prove a comparison principle for positive supersolutions and subsolutions of Lane-Emden equations (see Theorem 3.4.1) on the class of open connected sets Ω such that $\lambda_{p,q}(\Omega) > 0$.
The main ingredients of the proof consist in

- the identification of super/subsolutions of Lane-Emden equation with *super/subminima* of the energy functional $\mathfrak{F}^{\alpha}_{p,q}$ (see Proposition 3.3.6);

- the determination of the equality cases in the hidden convexity property (see Part 3 in Theorem 3.2.1).

Chapter 4. We apply the comparison principle for the Lane-Emden equation to provide a variety of results, contained in the paper [**Z4**], concerning weak positive solutions of (14) when $1 \leq q < p$. More precisely, in this chapter, we present

- a uniqueness result for the minimization of the functional $\mathfrak{F}^{\alpha}_{p,q}$ over functions with given (non-negative) boundary datum U when Ω is a connected open set satisfying $\lambda_{p,q}(\Omega) > 0$ (see Theorem 4.1.2). When we consider homogeneous Dirichlet boundary conditions, we will denote such a minimizer with $w^{\Omega,\alpha}_{p,q}$ and it coincides with the unique positive solution of (14);

- a sharp *pointwise* two-sided estimate for the positive solutions $w^{\Omega,\alpha}_{p,q}$ when Ω is a convex bounded open set, in terms of geometric quantities (see Theorem 4.2.2). More precisely, we prove a lower bound, when Ω is a connected open set satisfying $\lambda_{p,q}(\Omega) > 0$. On the other hand, in the case of convex sets, by exploiting the fact that d_Ω is a weakly $p-$superharmonic function, we show an upper bound for $w^{\Omega,\alpha}_{p,q}$ by constructing a suitable positive supersolution of (14) depending on the distance function d_Ω (see also Lemma 4.2.1), and by applying the comparison principle. By taking the supremum norms of functions involved in the estimates of Theorem 4.2.2, we also obtain a sharp L^∞ estimate on $w^{\Omega,\alpha}_{p,q}$ (see Corollary 4.2.3). Then, this will be quite useful in order to study the asymptotic behaviour, as $p \to \infty$, of $\|w^{\Omega,\alpha}_{p,q}\|_{L^\infty(\Omega)}$, namely, we prove that

$$\|w^{\Omega,\alpha}_{p,q}\|_{L^\infty(\Omega)} \to r_\Omega, \qquad \text{as } p \to \infty,$$

  when Ω is a convex bounded open set (see Corollary 4.2.5);

- a *localization* result for maximum points of $w^{\Omega,\alpha}_{p,q}$ when Ω is a convex bounded open set: we show that, there exists an explicit constant $C = C(N, p, q) \in (0, 1)$ such that, for every maximum point $x_0 \in \Omega$ of $w^{\Omega,\alpha}_{p,q}$ it holds

$$d_\Omega(x_0) \geq C\, r_\Omega,$$

  (see Corollary 4.3.1);

- a "hierarchy" result which asserts that all sign-changing solutions of (14) with homogeneous Dirichlet boundary conditions are "trapped" between the positive solution $w^{\Omega,\alpha}_{p,q}$ and the negative one $-w^{\Omega,\alpha}_{p,q}$ (see Corollary 4.4.1). This result holds on every connected open sets Ω satisfying $\lambda_{p,q}(\Omega) > 0$. As a collateral result, we can extend the L^∞ estimate stated in Corollary 4.2.3 to every sign-changing solution of (14) (see Remark 4.4.2).

Chapter 5. This chapter is devoted to the study of conditions on Ω which assure the validity of the continuous Sobolev embedding

$$\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega), \tag{25}$$

in the range $q \leq p$, and it is based on the first part of a work in collaboration with Lorenzo Brasco and Francesca Prinari ([**Z3**]). This is equivalent to look for necessary and sufficient conditions on Ω such that the Sobolev-Poincaré constants satisfy $\lambda_{p,q}(\Omega) > 0$.

In particular, we aim to investigate the link between the existence of the continuous (and compact) embedding (25) and the summability of the function d_Ω. We notice that, apart for a certain integrability of d_Ω, we will be really careful not to require any regularity conditions on the open set $\Omega \subset \mathbb{R}^N$. More precisely, in Theorem 5.1.1, we show that, for every general open set $\Omega \subset \mathbb{R}^N$ and $p > N$, we have that

$$d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega) \quad \Longleftrightarrow \quad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact,}$$

This equivalence is obtained as a consequence of a two-sided estimate for $\lambda_{p,q}(\Omega)$, in terms of the $L^{\frac{p\,q}{p-q}}$ norm of the distance function.
In a similar way, in Theorem 5.2.1, when $p > N$, we show that

$$d_\Omega \in L^\infty(\Omega) \quad \Longleftrightarrow \quad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega),$$

thanks a two-sided estimate on $\lambda_p(\Omega)$ in term of the inradius r_Ω.
Finally, in Theorem 5.3.1, we give a different proof of the following result first proved by Adams ([**3**]) for $p > N$

$$\lim_{R\to+\infty} \|d_\Omega\|_{L^\infty(\mathbb{R}^N\setminus B_R)} = 0 \quad \Longleftrightarrow \quad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact.}$$

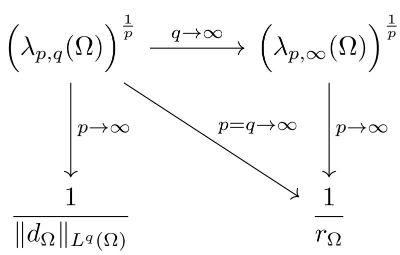
Indeed, for implication "$\Longrightarrow$", we apply the Hardy inequality (7).

In the case $p \le N$, for capacitary reasons, we will find that the above summability conditions on d_Ω are only *necessary* for the embeddings to hold, but not sufficient, hence, the equivalences are possible only in the *superconformal case* $p > N$. Indeed, as a counterexample, in Theorem 5.1.1, we show that, when $1 < p \le N$, there exists an open set $\mathcal{T} \subset \mathbb{R}^N$ such that

$$d_\mathcal{T} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T}) \qquad \text{but} \qquad \mathcal{D}_0^{1,p}(\mathcal{T}) \not\hookrightarrow L^q(\mathcal{T}), \qquad \text{for every } 1 \le q < p.$$

Rougly speaking, the set $\mathcal{T}$ is constructed by removing an infinite sequence of prescribed points from the set $S = (0,1)^{N-1} \times (0,+\infty)$. By using a capacitary result, we have that points have zero $p-$capacity when $p \le N$; hence, $\lambda_{p,q}(\mathcal{T}) = \lambda_{p,q}(S) = 0$ for every $1 \le q < p$. A counterexample is provided also in Theorem 5.2.1 for the homogeneous case $q = p \le N$.

Chapter 6. In this chapter, our main goal is to discuss the asymptotic behaviour for the Sobolev-Poincaré constant $\lambda_{p,q}(\Omega)$ and their extremal functions, as the parameter p goes to ∞. This study is contained in the second part of [**Z3**].
To be more precise, the claimed asymptotic results, contained in Corollaries 6.1.1 and 6.3.1, are consequence of the geometric bounds proved in Theorems 5.1.1 and 5.2.1. Indeed, it is sufficient to raise to the power $1/p$ all sides in (20) and (21) and then send $p \to \infty$, by applying standard convergence theorems.
Hence, we can summarise the asymptotic results with the following diagram



which holds for every $N < p < \infty$, every $1 \le q \le p$ and every general open set $\Omega \subsetneq \mathbb{R}^N$. The elements in the bottom line have to be considered as 0, whenever $d_\Omega \notin L^q(\Omega)$ or $r_\Omega = +\infty$. Moreover, when Ω is such that $d_\Omega \in L^{q_0}(\Omega)$ for some $q_0 < \infty$, then we can close the diagram by

$$\frac{1}{\|d_\Omega\|_{L^q(\Omega)}} \overset{q\to\infty}{\longrightarrow} \frac{1}{r_\Omega},$$

thus making it commutative.

Moreover, we also discuss the asymptotics, as $p \to \infty$, of the extremal functions of $\lambda_{p,q}(\Omega)$. Indeed, when $1 \le q < \infty$, in Theorem 6.2.1, we show that

$$\lim_{p\to\infty} \|w^\Omega_{p,q} - d_\Omega\|_{L^r(\Omega)} = 0, \qquad \text{for every } q \le r \le \infty,$$

under the minimal hypothesis $d_\Omega \in L^q(\Omega)$, when $\Omega \subsetneq \mathbb{R}^N$ is a general open set.

Concerning the homogeneous case $q = p$, when Ω is quasibounded, in Theorem 6.4.1, we find that the extremal functions of $\lambda_p(\Omega)$ converges, as $p \to \infty$, to a solution of the variational problem

$$\min_{u\in W^{1,\infty}(\Omega)} \Big\{ \|\nabla u\|_{L^\infty(\Omega)} : \|u\|_{L^\infty(\Omega)} = 1,\ u \equiv 0 \text{ on } \partial\Omega \Big\} = \frac{1}{r_\Omega}. \tag{26}$$

We note that, when $q = p$, the accumulation points for the extremal functions of $\lambda_p(\Omega)$ does not necessarily coincide with d_Ω, as it happens in the range $q < p$. Finally, in Theorem 6.4.3, we study the asymptotic behaviour of the extremals of $\lambda_{p,\infty}(\Omega)$, and we prove that they converge to solutions of (26).

Chapter 7. In this chapter, we are interested in providing sharp estimates for $\lambda_{p,q}(\Omega)$ in terms of geometric quantities depending on the open set Ω. These results are contained in the two papers [**Z3**] and [**Z4**], in collaboration with Lorenzo Brasco and Francesca Prinari.

First, in Theorem 7.1.1, we prove the following generalization of the Makai inequality in [**102**] to every dimension $N \ge 2$ and every $1 \le q < p < \infty$

$$\lambda_{p,q}(\Omega) \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} \ge C_{p,q}, \tag{27}$$

where $\Omega \subsetneq \mathbb{R}^N$ is a convex open set and $C_{p,q}$ is the optimal constant which comes with an explicit value. The proof consists of two main steps: first, we show the above estimate when Ω coincides with the interior of a polytope in $\mathbb{R}^N$ (see Appendix D for a short introduction of the main properties of polytopes). In particular, we apply the covering argument proposed by Makai by dividing such the polytope into convex subsets with *certain* properties. Then, in the second step, we apply an approximation argument by means of polytopes, to extend the result to general convex sets.

In the subsequent Section 7.2, we compare the explicit sharp constant $C_{p,q}$ with the infimum of the product on the left-hand side of (27) defined on other classes of open subsets $\Omega \subset \mathbb{R}^N$, satisfying $d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega)$. More precisely, we will discuss the cases of general open sets and of planar simply connected open sets.

Then, we devote the last part of the chapter to the proof of another geometric lower bound for $\lambda_{p,q}(\Omega)$ by means of the inradius r_Ω, on the class of convex bounded open sets $\Omega \subset \mathbb{R}^N$, which extends the classical Hersch-Protter inequality. Indeed, by means of two different proofs, we show that inequality (19) holds, for every $1 \le q < p < \infty$. More precisely, in Theorem 7.3.1, this result is obtained as a consequence of the upper bound for the positive solution $w^{\Omega,\alpha}_{p,q}$ of (14), proved in Theorem 4.2.2, while, in Remark 7.3.2, we get the estimate (19) as a corollary of the Makai inequality (27). Finally, by applying the integration formula due to Crasta and Malusa for a set Ω of class C^2 (see [**43**, Theorem 7.1]), in Theorem 7.4.5 we provide an alternative proof for the Hersch-Protter inequality in the homogeneous case $q = p$, first proved by Kajikiya in [**85**] (see also [**20**]).

Chapter 8. In this last chapter, we aim to study the Hardy inequality in the setting of *Sobolev-Slobodeckiĭ fractional spaces*: for every $1 < p < \infty$ and $0 < s < 1$, we define the sharp fractional Hardy constant through the infimum problem (22) and we provide many results concerning the fractional Hardy inequality on the class of convex open sets, obtained in collaboration with Lorenzo Brasco, Francesca Bianchi and Firoj Sk, in [**Z1**] and [**Z2**]; we comment below the most relevant contributions contained in the chapter.

- We discuss the tight relation between the existence of supersolutions for (23) and the variational problem defining $\mathfrak{h}_{s,p}(\Omega)$. In particular, in Theorem 8.3.1, we show that the characterization (24) holds, by proving the equivalence between the fact that $\mathfrak{h}_{s,p}(\Omega) > 0$ and the existence of a positive supersolution for (23), for some constants $\lambda > 0$ (see Lemmas 8.3.2 and 8.3.3). We refer to the application of (24) as the *supersolutions method*, as when we construct a positive supersolution for (23) with an explicit constant λ, then automatically $\mathfrak{h}_{s,p}(\Omega)$ is bounded from below by λ. We also remark that here we do not require any regularity assumption on the set $\Omega \subset \mathbb{R}^N$, hence the supersolutions method can be applied to every general open set;

- when Ω is a convex open set, we construct positive supersolutions to (23) of the form d_Ω^β, for certain values of $\beta \geq 0$. Note that we also include the case when Ω is an half-space $\mathbb{H}^N_+$ (see Sections 8.4 and 8.5). In particular, in Theorem 8.5.2, we obtain that d_Ω^β is a weak supersolutions for (23) with a constant $\lambda = C_{N,s\,p}\,\lambda(\beta)$. Then, in Lemma 8.4.4, we deeply study the properties for the function $\beta \mapsto \lambda(\beta)$, finding that it attains its maximum when $\beta = (s\,p - 1)/p$. At this point, we need to be really carefull, as, by Theorem 8.5.2, this choice of β is feasible only in the regime $s\,p \geq 1$. We notice that the restriction on the product $s\,p$ is not necessary in the case of the half-space.

- we compute the values for the sharp constants $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$ and $\mathfrak{h}_{s,p}(\Omega)$, when Ω is a convex open set (see Section 8.6). For every $0 < s < 1$, in Theorem 8.6.2, we first prove the identity

$$\mathfrak{h}_{s,p}(\mathbb{H}^N_+) = C_{N,s\,p}\,\Lambda_{s,p},$$

  which follows thanks to the supersolutions method and by using an opportune family of trial functions in the definition of $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$, as shown in Theorem 8.6.2 (similar computations can be found in the papers [**65**] and [**66**]). Moreover, in Theorem 8.6.3, under the restriction $s\,p \geq 1$, we obtain that

$$\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}(\mathbb{H}^N_+).$$

  This is proved again thanks to the supersolutions method, as well as, to the fact that

$$\mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{s,p}(\mathbb{H}^N_+), \qquad \text{for every convex open set } \Omega \subset \mathbb{R}^N,$$

  which is an extension, to the non local case, of an argument due to Marcus, Mizel and Pinchover in [**104**] (see Proposition 8.6.1).

  Moreover, inspired by [**55**], in Remark 8.6.5 we point out that the case $p = 2$ is completely solved, that is

$$\mathfrak{h}_{s,2}(\Omega) = \mathfrak{h}_{s,2}(\mathbb{H}^N_+).$$

  without any restriction on $s \in (0, 1)$.

Finally, we observe that the infimum in the definition of $\mathfrak{h}_{s,p}(\Omega)$ is never attained, both in the case when Ω is a convex open set and in the case when $\Omega = \mathbb{H}^N_+$ (see Proposition 8.2.5).

Main notation

In the following we collect some notation we will use in this thesis. If not specified, Ω denotes a general open set contained in $\mathbb{R}^N$.

p^*	critical Sobolev exponent, given by $p\,N/(p-N)$; conventionally, if $p \geq N$, it is $+\infty$
$\|\,\Omega\,\|$	$N-$dimensional Lebesgue measure of the set Ω
$\mathcal{H}^k(\Omega)$	$k-$dimensional Hausdorff measure of the set Ω
1_Ω	characteristic function of Ω, i.e. $1_\Omega(x) = 1$ if $x \in \Omega$, $1_\Omega(x) = 0$ otherwise
$\langle u, v\rangle$	standard Euclidean scalar product between u and v
d_Ω	distance function from the boundary of Ω. i.e. $d_\Omega(x) = \inf_{y\in\partial\Omega} \|x-y\|$ for every $x \in \Omega$
$\mathrm{d}_{\mathcal{H}}(\Omega, \Omega')$	Hausdorff distance between two sets,
	defined as $\min\{\lambda \geq 0 : \Omega \subseteq \Omega' + \lambda B_1, \Omega' \subseteq \Omega + \lambda B_1\}$
r_Ω	inradius of Ω, defined as $\sup\Big\{r > 0\ :\ \text{exists } x_0 \in \Omega \text{ such that } B_r(x_0) \subset \Omega\Big\}$
$B_R(x_0)$	$N-$dimensional open ball centered in x_0 with radius R
B_R	$N-$dimensional open ball centered in the origin with radius R
ω_N	$N-$dimensional Lebesgue measure of $B_1(x_0)$, i.e. $\omega_N = \|B_1(x_0)\|$
$\Omega' \Subset \Omega$	the closure $\overline{\Omega'}$ is a compact set contained in the open set Ω
$\mathbb{H}^1_+$	half-line $(0, +\infty)$
$\mathbb{H}^N_+$	half-space $\mathbb{R}^{N-1} \times (0, +\infty)$
u_+	positive part of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, defined as $u_+ = \max\{u, 0\}$
u_-	negative part of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, defined as $u_- = \max\{-u, 0\}$
$C_0(\Omega)$	completion of $C^\infty_0(\Omega)$ w.r.t. the sup norm
$C^\infty_0(\Omega)$	space of infinitely differentiable functions with compact support in Ω
$C_{\mathrm{bound}}(\overline{\Omega})$	space of continuous and bounded functions on $\overline{\Omega}$
$C^{0,\beta}(\overline{\Omega})$	space of functions in $C_{\mathrm{bound}}(\overline{\Omega})$ such that their seminorm $[u]_{C^{0,\beta}(\overline{\Omega})}$ is finite
$L^p(\Omega)$	space of $p-$integrable Lebesgue measurable functions
$L^1_{\mathrm{loc}}(\mathbb{R}^N)$	space of Lebesgue measurable functions locally integrable in $\mathbb{R}^N$
$W^{1,p}(\Omega)$	space of functions in $L^p(\Omega)$ with first order weak derivatives in $L^p(\Omega)$
$W^{1,p}_0(\Omega)$	closure of $C^\infty_0(\Omega)$ in $W^{1,p}(\Omega)$
$\mathcal{D}^{1,p}_0(\Omega)$	completion of $C^\infty_0(\Omega)$ w.r.t. the norm $\|\nabla \cdot\|_{L^p(\Omega)}$
$\mathrm{Lip}_0(\Omega)$	space of all real-valued Lipschitz functions which vanish at the boundary $\partial\Omega$
$X^{q,p}(\Omega)$	space of functions $u \in L^q(\Omega)$ with first order weak derivatives in $L^p(\Omega)$

$\pi_{p,q}$	one-dimensional Sobolev-Poincaré constant, defined as $\inf\limits_{u\in C_0^\infty((0,1))}\left\{\|u'\|_{L^p([0,1])}\,:\,\|u\|_{L^q([0,1])}=1\right\}$
$[u]_{W^{s,p}(\mathbb{R}^N)}$	*global* Gagliardo seminorm, defined as $\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{\|u(x)-u(y)\|^p}{\|x-y\|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}$
$[u]_{W^{s,p}(\Omega)}$	*regional* Gagliardo seminorm, defined as $\left(\iint_{\Omega\times\Omega}\frac{\|u(x)-u(y)\|^p}{\|x-y\|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}$
$W^{s,p}(\Omega)$	standard fractional Sobolev space defined as $\left\{u\in L^p(\Omega)\,:\,[u]_{W^{s,p}(\mathbb{R}^N)}<+\infty\right\}$
$W^{s,p}_{\rm loc}(\Omega)$	space of functions $u\in L^p_{\rm loc}(\Omega)$ such that $u\in W^{s,p}(\Omega')$ for every $\Omega'\Subset\Omega$
$\widetilde{W}^{s,p}_0(\Omega)$	closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$
$L^\beta_{sp}(\Omega)$	Lebesgue space defined as $\left\{u\in L^\beta_{\rm loc}(\mathbb{R}^N)\,:\,\int_{\mathbb{R}^N}\frac{\|u(x)\|^\beta}{(1+\|x\|)^{N+s\,p}}\,dx<+\infty\right\}$
$\mathcal{X}^{s,p}(\Omega;d_\Omega)$	fractional Sobolev space defined as $\left\{u\in L^p_{s\,p}(\mathbb{R}^N):[u]_{W^{s,p}(\mathbb{R}^N)}<+\infty\ \text{ and }\ \frac{u}{d_\Omega^s}\in L^p(\Omega)\right\}$
$\mathcal{X}^{s,p}(\Omega;d_\Omega)$	closure of $C_0^\infty(\Omega)$ in $\mathcal{X}^{s,p}_0(\Omega;d_\Omega)$
∇	gradient operator, i.e. for every $u\colon\mathbb{R}^N\to\mathbb{R}$, $\nabla u=\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{x_N}\right)$
div	divergence operator, i.e. for every $u\colon\mathbb{R}^N\to\mathbb{R}^N$, $\operatorname{div}u=\sum_{i=1}^N\frac{\partial u}{\partial x_i}$
Δ	standard Laplace operator, i.e. for every $u\colon\mathbb{R}^N\to\mathbb{R}^N$, $\Delta u=\operatorname{div}(\nabla u)=\sum_{i=1}^N\frac{\partial^2 u}{\partial x_i^2}$
Δ_p	$p-$Laplace operator, i.e., for every $u\colon\mathbb{R}^N\to\mathbb{R}^N$, $\Delta_p u=\operatorname{div}\left(\|\nabla u\|^{p-2}\,\nabla u\right)$
$(-\Delta_p)^s$	$p-$Laplace operator of order s, i.e., for every $u\colon\mathbb{R}^N\to\mathbb{R}^N$, it is defined as $(-\Delta_p)^s u(x):=2\,\text{P.V.}\int_{\mathbb{R}^N}\frac{\|u(x)-u(y)\|^{p-2}\,(u(x)-u(y))}{\|x-y\|^{N+s\,p}}\,dy$

CHAPTER 1

PRELIMINARIES

§1.1 The space C_0

We denote by $C_0(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the sup norm. The following simple result will be quite useful.

Lemma 1.1.1. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open set, then it holds that*

$$C_0(\Omega) \subset \Big\{u \in C_{\rm bound}(\overline{\Omega}) \,:\, u = 0 \text{ on } \partial\Omega\Big\}.$$

Here $C_{\rm bound}(\overline{\Omega})$ is the set of continuous and bounded functions on $\overline{\Omega}$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ be a Cauchy sequence, with respect to the sup norm. In particular, it is Cauchy sequence in $C_{\rm bound}(\overline{\Omega})$, which is a Banach space (see for example [**76**, Theorem 7.9]). Thus, there exists $u \in C_{\rm bound}(\overline{\Omega})$ such that u_n converges to u uniformly on $\overline{\Omega}$. Moreover, such a function must vanish at the boundary $\partial\Omega$, as a uniform limit of functions with compact support in Ω. □

For every open set $\Omega \subseteq \mathbb{R}^N$ and $0 < \alpha \le 1$, we introduce the *Holder spaces* $C^{0,\alpha}(\overline{\Omega})$, defined as

$$C^{0,\alpha}(\overline{\Omega}) = \Big\{u \in C_{\rm bound}(\overline{\Omega}) \,:\, [u]_{C^{0,\alpha}(\overline{\Omega})} < +\infty\Big\},$$

where

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x\neq y;\, x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

It is endowed with the natural norm

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{L^\infty(\Omega)} + [u]_{C^{0,\alpha}(\overline{\Omega})}, \qquad \text{for every } u \in C^{0,\alpha}(\overline{\Omega}).$$

We prove that the functions belonging to $C_0(\Omega)$, with finite $C^{0,\alpha}$ seminorm , satisfy the following interpolation–type estimate.

Lemma 1.1.2. *Let $0 < \beta < \alpha \le 1$, let $1 \le \gamma \le \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. For every $u \in C_0(\Omega) \cap L^\gamma(\Omega)$ such that*

$$[u]_{C^{0,\alpha}(\overline{\Omega})} < +\infty,$$

we have

$$[u]_{C^{0,\beta}(\overline{\Omega})} \le C_1\, \|u\|_{L^\gamma(\Omega)}^\theta\, [u]_{C^{0,\alpha}(\overline{\Omega})}^{1-\theta}, \qquad \textit{with } \theta = \frac{\alpha-\beta}{\alpha+\dfrac{N}{\gamma}},$$

for some $C_1 = C_1(N, \alpha, \beta, \gamma) > 0$. *Moreover, if* $1 \leq \gamma < \infty$ *we also have*

$$\|u\|_{L^\infty(\Omega)} \leq C_2 \, \|u\|^{\chi}_{L^\gamma(\Omega)} \, [u]^{1-\chi}_{C^{0,\alpha}(\overline{\Omega})}, \qquad \textit{with } \chi = \frac{\alpha}{\alpha + \dfrac{N}{\gamma}},$$

for some $C_2 = C_2(N, \alpha, \gamma) > 0$.

Proof. By Lemma 1.1.1, we can extend u to a continuous function on the whole $\mathbb{R}^N$, by setting it to be 0 on $\mathbb{R}^N \setminus \overline{\Omega}$. We first observe that for such an extension it holds

$$[u]_{C^{0,\alpha}(\overline{\Omega})} = [u]_{C^{0,\alpha}(\mathbb{R}^N)}. \tag{1.1}$$

Indeed, we may write

$$\begin{aligned} [u]_{C^{0,\alpha}(\mathbb{R}^N)} &= \max\left\{ [u]_{C^{0,\alpha}(\overline{\Omega})}, \sup_{x\in\overline{\Omega}, y\notin\overline{\Omega}} \frac{|u(x)|}{|x-y|^\alpha} \right\} \\ &= \max\left\{ [u]_{C^{0,\alpha}(\overline{\Omega})}, \sup_{x\in\Omega, y\notin\overline{\Omega}} \frac{|u(x)|}{|x-y|^\alpha} \right\}. \end{aligned}$$

If $x \in \Omega$ and $y \notin \overline{\Omega}$ then the segment $\overline{x\,y}$ connecting x and y is such that $\overline{x\,y} \cap \Omega \neq \emptyset$ and $\overline{x\,y} \cap (\mathbb{R}^N \setminus \overline{\Omega}) \neq \emptyset$. Hence, there exists $y_0 \in \overline{x\,y} \cap \partial\Omega$ satisfying

$$|x - y| \geq |x - y_0|.$$

This implies

$$\frac{|u(x)|}{|x-y|^\alpha} \leq \frac{|u(x) - u(y_0)|}{|x - y_0|^\alpha} \leq [u]_{C^{0,\alpha}(\overline{\Omega})},$$

that gives

$$\sup_{x\in\Omega, y\notin\overline{\Omega}} \frac{|u(x)|}{|x-y|^\alpha} \leq [u]_{C^{0,\alpha}(\overline{\Omega})}.$$

This concludes the proof of (1.1).

We now come to the proof of the claimed interpolation inequality. For every $x, y \in \mathbb{R}^N$ such that $x \neq y$, we write

$$\begin{aligned} \frac{|u(x) - u(y)|}{|x-y|^\beta} &= \left(\frac{|u(x) - u(y)|}{|x-y|^\alpha} \right)^{\frac{\beta}{\alpha}} |u(x) - u(y)|^{\frac{\alpha-\beta}{\alpha}} \\ &\leq [u]^{\frac{\beta}{\alpha}}_{C^{0,\alpha}(\overline{\Omega})} \left(|u(x)|^{\frac{\alpha-\beta}{\alpha}} + |u(y)|^{\frac{\alpha-\beta}{\alpha}} \right). \end{aligned} \tag{1.2}$$

Observe that we used the triangle inequality and the sub-additivity of concave powers. In order to estimate the last term, we use that for every $R > 0$ and $z \in B_R(x)$ we have

$$\begin{aligned} |u(x)| \leq |u(x) - u(z)| + |u(z)| &\leq [u]_{C^{0,\alpha}(\mathbb{R}^N)} \, |x - z|^\alpha + |u(z)| \\ &\leq R^\alpha \, [u]_{C^{0,\alpha}(\overline{\Omega})} + |u(z)|, \end{aligned}$$

where we also used (1.1). We now take the integral average of this estimate on $B_R(x)$. This gives

$$\begin{aligned}|u(x)| &\le R^\alpha\,[u]_{C^{0,\alpha}(\overline{\Omega})} + \frac{1}{\omega_N\,R^N}\int_{B_R(x)}|u(z)|\,dz\\ &\le R^\alpha\,[u]_{C^{0,\alpha}(\overline{\Omega})} + (\omega_N\,R^N)^{-\frac{1}{\gamma}}\left(\int_\Omega |u(z)|^\gamma\,dz\right)^{\frac{1}{\gamma}}.\end{aligned}\tag{1.3}$$

With the same argument, we have also

$$|u(y)| \le R^\alpha\,[u]_{C^{0,\alpha}(\overline{\Omega})} + (\omega_N\,R^N)^{-\frac{1}{\gamma}}\left(\int_\Omega |u(z)|^\gamma\,dz\right)^{\frac{1}{\gamma}}.$$

We insert these estimates in (1.2) and use again the subadditivity of concave powers. We get

$$\frac{|u(x)-u(y)|}{|x-y|^\beta} \le 2\,[u]_{C^{0,\alpha}(\overline{\Omega})}^{\frac{\beta}{\alpha}}\left(R^{\alpha-\beta}\,[u]_{C^{0,\alpha}(\overline{\Omega})}^{\frac{\alpha-\beta}{\alpha}} + \left(\frac{1}{\omega_N\,R^N}\right)^{\frac{\alpha-\beta}{\alpha\,\gamma}}\,\|u\|_{L^\gamma(\Omega)}^{\frac{\alpha-\beta}{\alpha}}\right),$$

which is valid for every $x \ne y$ and every $R > 0$. If we now not optimize in $R > 0$, we finally get the desired estimate for the $C^{0,\beta}$ seminorm. The sup norm can be estimated with a similar optimization argument, by using (1.3). □

Remark 1.1.3. We remark that the constants C_1 and C_2 in the previous result are given by

$$C_1 = 2\,\omega_N^{-\frac{\alpha-\beta}{\gamma\,\alpha+N}}\,\left(\frac{\alpha\,\gamma}{N}\right)^{\frac{N}{\alpha\,\gamma+N}}\,\frac{1}{\chi},$$

and

$$C_2 = \omega_N^{-\frac{\chi}{\gamma}}\,\left(\frac{\alpha\,\gamma}{N}\right)^{\frac{N}{\alpha\,\gamma+N}}\,\frac{1}{\chi}.$$

In particular, we notice that C_1 stays uniformly bounded, as α varies in $(\beta, 1]$. Similary, the constant C_2 stays bounded, as α varies in $(0, 1]$. This observation will be useful in the sequel.

§1.2 The distance function

For an open set $\Omega \subsetneq \mathbb{R}^N$ with non-empty boundary, we denote by d_Ω the *distance function* from the boundary $\partial\Omega$, defined by

$$d_\Omega(x) := \inf_{y\in\partial\Omega}|x-y|, \qquad \text{for every } x \in \Omega.$$

We extend d_Ω by 0 in $\mathbb{R}^N \setminus \Omega$. The *inradius* r_Ω of Ω will be the radius of a largest ball contained in Ω. More precisely, this quantity is given by

$$r_\Omega = \sup\Big\{r > 0\ :\ \text{exists } x_0 \in \Omega \text{ such that } B_r(x_0) \subseteq \Omega\Big\}. \tag{1.4}$$

Moreover, a set Ω is said to be *quasibounded* when the following condition on its distance function holds

$$\lim_{R\to+\infty}\|d_\Omega\|_{L^\infty(\mathbb{R}^N\setminus B_R)} = 0,$$

(see for example [**2**, **42**]).

In this section we investigate some consequences of the summability of the distance function. First of all, we prove that when d_Ω^α is summable for some $0<\alpha<\infty$, the set Ω has finite inradius. This comes with an explicit (sharp) bound.

Lemma 1.2.1. *Let $0<\alpha<\infty$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set such that $d_\Omega^\alpha\in L^1(\Omega)$. Then $r_\Omega<+\infty$ and it holds*

$$r_\Omega\le C_{N,\alpha}\left(\int_\Omega d_\Omega^\alpha\,dx\right)^{\frac{1}{N+\alpha}}, \tag{1.5}$$

where the constant $C_{N,\alpha}$ is given by

$$C_{N,\alpha}=\left(N\,\omega_N\int_0^1(1-\varrho)^\alpha\,\varrho^{N-1}\,d\varrho\right)^{-\frac{1}{N+\alpha}}. \tag{1.6}$$

Moreover, inequality (1.5) is sharp, since equality holds for a ball.

To ease of calculation, we note that the integral appearing in the constant $C_{N,\alpha}$ can be expressed through the *Gamma function* Γ, in the following way

$$\int_0^1(1-\varrho)^\alpha\,\varrho^{N-1}\,d\varrho=\frac{\Gamma(N)\,\Gamma(\alpha+1)}{\Gamma(N+\alpha+1)}.$$

Proof. Let $B_r(x_0)\subseteq\Omega$, then we have that

$$(r-|x-x_0|)_+=d_{B_r(x_0)}(x)\le d_\Omega(x),\qquad\text{for every }x\in B_r(x_0).$$

By raising to the power α and integrating, we get

$$\int_{B_r(x_0)}(r-|x-x_0|)_+^\alpha\,dx\le\int_\Omega d_\Omega^\alpha\,dx.$$

By using the change of variable $y=(x-x_0)/r$, from the previous estimate we also get

$$r^{N+\alpha}\le\frac{\displaystyle\int_\Omega d_\Omega^\alpha\,dx}{\displaystyle\int_{B_1}(1-|y|)_+^\alpha\,dy}.$$

If we now take the supremum over the admissible balls, we get the conclusion. □

The next Lemma gives an estimate on the $C^{0,\beta}$ norm of the distance of an open set in terms of a power of the inradius of the set itself.

Lemma 1.2.2. *Let $0<\alpha<\infty$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set such that $r_\Omega<+\infty$. Then for every $0<\beta<1$ we have*

$$[d_\Omega]_{C^{0,\beta}(\overline{\Omega})}\le(2\,r_\Omega)^{1-\beta}.$$

Proof. We extend d_Ω to be 0 outside Ω and consider it as a Lipschitz continuous function defined on the whole $\mathbb{R}^N$, By recalling (1.1), for every $0<\beta<1$ we have

$$[d_\Omega]_{C^{0,\beta}(\overline{\Omega})}=[d_\Omega]_{C^{0,\beta}(\mathbb{R}^N)}.$$

It is now sufficient to write for $t > 0$

$$[d_\Omega]_{C^{0,\beta}(\mathbb{R}^N)} = \max\left\{\sup_{x\neq y;\,|x-y|\le t}\frac{|d_\Omega(x)-d_\Omega(y)|}{|x-y|^\beta},\ \sup_{|x-y|>t}\frac{|d_\Omega(x)-d_\Omega(y)|}{|x-y|^\beta}\right\}.$$

For the first term on the right-hand side, we can just use the $1-$Lipschitz character of d_Ω. For the second one, it is sufficient to use that d_Ω is bounded by r_Ω. This gives

$$[d_\Omega]_{C^{0,\beta}(\mathbb{R}^N)} \le \max\left\{t^{1-\beta},\ \frac{2\,r_\Omega}{t^\beta}\right\},\qquad \text{for every } t>0.$$

By choosing $t = 2\,r_\Omega$, we get the claimed estimate. □

We prove that, when Ω is a general open set, a summability hypothesis on the distance function d_Ω is a sufficient condition to assure that the set Ω is quasibounded.

Lemma 1.2.3. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that $d_\Omega^\alpha \in L^1(\mathbb{R}^N)$, for some $0 < \alpha < \infty$. Then for every $R \ge r_\Omega/2$, we have*

$$d_\Omega(x) \le 2\left(\frac{1}{\omega_N}\int_{\mathbb{R}^N\setminus B_R} d_\Omega^\alpha\,dy\right)^{\frac{1}{N+\alpha}},\qquad \textit{for every } |x| > R + \frac{r_\Omega}{2}.$$

In particular, Ω is quasibounded.

Proof. Let $R \ge r_\Omega/2$ and let $x \in \mathbb{R}^N$ be such that $|x| > R + r_\Omega/2$. If $x \notin \Omega$, then $d_\Omega(x) = 0$ and there is nothing to prove. Let us suppose that $x \in \Omega$, so that $d_\Omega(x) > 0$. We consider the ball

$$B := \left\{y \in \mathbb{R}^N\ :\ |x-y| < \frac{d_\Omega(x)}{2}\right\},$$

and observe that

$$d_\Omega(y) \ge \frac{1}{2}\,d_\Omega(x),\qquad \text{for every } y \in B.$$

We raise to the power α and integrate this inequality over B. This gives

$$2^{-\alpha}\,d_\Omega(x)^\alpha\,|B| \le \int_B d_\Omega(y)^\alpha\,dy,$$

that is

$$\omega_N\,2^{-\alpha-N}\,d_\Omega(x)^{\alpha+N} \le \int_B d_\Omega(y)^\alpha\,dy. \tag{1.7}$$

We then observe that

$$|y| \ge |x| - |y-x| > |x| - \frac{1}{2}\,d_\Omega(x) \ge |x| - \frac{r_\Omega}{2} > R,\qquad \text{for every } y \in B.$$

This gives that $B \subseteq \mathbb{R}^N \setminus B_R$. By using the previous inclusion in (1.7), we get

$$\omega_N\,2^{-\alpha-N}\,d_\Omega(x)^{\alpha+N} \le \int_{\mathbb{R}^N\setminus B_R} d_\Omega(y)^\alpha\,dy.$$

This concludes the proof. □

Remark 1.2.4. The converse implication does not hold true: indeed, there exist quasibounded open sets for which $d_\Omega^\alpha \notin L^1(\mathbb{R}^N)$, for any $0 < \alpha < \infty$ (see Example C.1.1). On the other hand, we easily see that if $\Omega \subseteq \mathbb{R}^N$ is a quasibounded open set, then $d_\Omega \in L^\infty(\mathbb{R}^N)$.

In the next technical lemma, we show that the summability of a *negative* power of the function distance d_Ω implies certain geometric properties of the open set Ω.

Lemma 1.2.5. *Let $N \geq 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that*

$$\int_\Omega \frac{1}{d_\Omega^\alpha}\, dx < +\infty,$$

for some $\alpha > 0$. Then we must have $\alpha < N$. Moreover, we have the estimates

$$r_\Omega \leq \left(\frac{2^\alpha}{\omega_N} \int_\Omega \frac{1}{d_\Omega^\alpha}\, dx\right)^{\frac{1}{N-\alpha}} \qquad \textit{and} \qquad |\Omega| \leq \left(\frac{2^\alpha}{\omega_N}\right)^{\frac{\alpha}{N-\alpha}} \left(\int_\Omega \frac{1}{d_\Omega^\alpha}\, dx\right)^{\frac{N}{N-\alpha}}, \tag{1.8}$$

where r_Ω is defined in (1.4).

Proof. We take $x_0 \in \Omega$ and consider the open ball $B_r(x_0)$ with radius $r = d_\Omega(x_0)$. This implies that

$$B_r(x_0) \subset \Omega \qquad \text{and} \qquad \partial B_r(x_0) \cap \partial\Omega \neq \emptyset.$$

Let us call $\widetilde{x}_0$ a point contained in this intersection. By observing that

$$d_\Omega(x) \leq |x - \widetilde{x}_0|, \qquad \text{for every } x \in B_r(x_0),$$

we get

$$+\infty > \int_\Omega \frac{1}{d_\Omega^\alpha}\, dx \geq \int_{B_r(x_0)} \frac{1}{|x - \widetilde{x}_0|^\alpha}\, dx.$$

By using spherical coordinates, we see that the last integral diverges for $\alpha \geq N$. Thus we get the first statement.

In order to get the claimed estimates, we go on by estimating from below the last integral as follows

$$+\infty > \int_\Omega \frac{1}{d_\Omega^\alpha}\, dx \geq \frac{1}{2^\alpha\, r^\alpha} \int_{B_r(x_0)} dx = \frac{\omega_N}{2^\alpha}\, r^{N-\alpha} = \frac{\omega_N}{2^\alpha}\, d_\Omega(x_0)^{N-\alpha}.$$

Since $\alpha < N$ from the first part of the proof, we can take the supremum on $x_0 \in \Omega$ and get that the distance function is actually bounded. Moreover, we obtain the first estimate in (1.8), thus in particular the inradius is finite. In turn, by using this fact we get

$$\int_\Omega \frac{1}{d_\Omega^\alpha}\, dx \geq \frac{1}{r_\Omega^\alpha} \int_\Omega dx = \frac{|\Omega|}{r_\Omega^\alpha},$$

which shows that the volume is finite, as well, together with the second estimate in (1.8). This concludes the proof. □

We conclude this section with a generalized version of Ascoli-Arzelà Theorem, which is valid when Ω is a quasibounded open set. This generalization is quiet standard, but for completness we include the proof. We then apply this theorem in Chapter 6.

Proposition 1.2.6. *Let $\Omega \subsetneq \mathbb{R}^N$ be a quasibounded open set. Let*

$$\{u_n\}_{n\in\mathbb{N}} \subset \Big\{u \in C_{\mathrm{bound}}(\overline{\Omega})\ :\ u = 0 \text{ on } \partial\Omega\Big\}$$

be a sequence with the following properties:

(a) there exists $M > 0$ such that $\|u_n\|_{L^\infty(\Omega)} \le M$, for every $n \in \mathbb{N}$;

(b) there exist $\delta_0 > 0$ and a function $\omega : (0, \delta_0] \to (0, +\infty)$ such that

$$\lim_{\delta\to 0^+} \omega(\delta) = 0,$$

and for every $n \in \mathbb{N}$

$$\omega_n(\delta) := \sup\Big\{|u_n(x) - u_n(y)|\ :\ x, y \in \overline{\Omega},\ |x - y| \le \delta\Big\} \le \omega(\delta), \qquad \text{for every } 0 < \delta \le \delta_0.$$

Then, there exists $\overline{u} \in C_{\mathrm{bound}}(\Omega)$ vanishing on $\partial\Omega$ such that

$$\lim_{n\to\infty} \|u_n - \overline{u}\|_{L^\infty(\Omega)} = 0,$$

up to a subsequence.

Proof. Let us denote by $k_0 \in \mathbb{N}$ the smallest natural number such that $\overline{\Omega}\cap\overline{B_k}$ is not empty. Thanks to the assumptions, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded and equicontinuous sequence on the compact set $\overline{\Omega}\cap\overline{B_k}$, for every $k \ge k_0$. By applying the classical Ascoli-Arzelá Theorem for compact sets, together with a diagonal argument, we have that there exists a function $\overline{u} \in C(\overline{\Omega})$ such that, up to a subsequence, u_n converges uniformly to $\overline{u}$ on $\overline{\Omega}\cap\overline{B_k}$, for every $k \ge k_0$.

We will show that u_n converges to $\overline{u}$ uniformly on the whole $\overline{\Omega}$. Let $0 < \delta \le \delta_0$, since Ω is quasibounded, there exists $R_\delta > 0$ such that

$$d_\Omega(x) \le \delta, \qquad \text{for } x \in \Omega \setminus B_{R_\delta}.$$

For every $x \in \Omega \setminus B_{R_\delta}$, we take $y \in \partial\Omega$ such that $|x - y| = d_\Omega(x)$. By using property (b), the triangle inequality and the fact that $u_n(y) = 0$, we get that for every $n \in \mathbb{N}$

$$\begin{aligned} |u_n(x) - \overline{u}(x)| &= \lim_{m\to\infty} |u_n(x) - u_m(x)| \\ &\le |u_n(x) - u_n(y)| + \lim_{m\to\infty} |u_m(x) - u_m(y)| \le 2\,\omega(d_\Omega(x)) \le 2\,\omega(\delta). \end{aligned}$$

This implies that

$$\begin{aligned} \|u_n - \overline{u}\|_{L^\infty(\Omega)} &= \max\Big\{\|u_n - \overline{u}\|_{L^\infty(\Omega\setminus B_{R_\delta})}, \|u_n - \overline{u}\|_{L^\infty(\Omega\cap B_{R_\delta})}\Big\} \\ &\le \max\Big\{2\,\omega(\delta), \|u_n - \overline{u}\|_{L^\infty(\Omega\cap B_{R_\delta})}\Big\}. \end{aligned}$$

By taking the limit as n goes to ∞ and exploiting the uniform convergence of u_n to $\overline{u}$ on $\overline{\Omega}\cap\overline{B_{R_\delta}}$, we obtain that

$$\limsup_{n\to\infty} \|u_n - \overline{u}\|_{L^\infty(\Omega)} \le 2\,\omega(\delta), \qquad \text{for } 0 < \delta \le \delta_0.$$

By arbitrariness of δ and using the properties of ω, we get

$$\lim_{n\to\infty} \|u_n - \overline{u}\|_{L^\infty(\Omega)} = 0.$$

This also shows that $\overline{u}$ vanishes on $\partial\Omega$, as a uniform limit of functions with the same property. Moreover, using again that $C_{\rm bound}(\overline{\Omega})$ is a Banach space, we finally get that $\overline{u} \in C_{\rm bound}(\overline{\Omega})$, as well. This concludes the proof. □

§1.3 Sobolev spaces and embeddings

In this first section, we introduce some standard results and definitions concerning the Sobolev spaces. Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set. We indicate by $\mathcal{D}^{1,p}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$, with respect to the norm

$$\psi \mapsto \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}, \qquad \text{for every } \psi \in C^\infty_0(\Omega).$$

We will also indicate by $W^{1,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$, in the usual Sobolev space $W^{1,p}(\Omega)$, endowed with the norm

$$\|\psi\|_{W^{1,p}(\Omega)} = \|\psi\|_{L^p(\Omega)} + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Definition 1.3.1 ($(p,q)-$admissible set)**.** Let $1 < p < \infty$, we will say that an open set $\Omega \subset \mathbb{R}^N$ is $(p,q)-$*admissible* for some $1 \le q < p$, if

$$\lambda_{p,q}(\Omega) := \inf_{\psi\in C^\infty_0(\Omega)} \left\{ \int_\Omega |\nabla\psi|^p\,dx \,:\, \int_\Omega |\psi|^q\,dx = 1 \right\} > 0. \tag{1.9}$$

This is equivalent to require that the homogeneous Sobolev space $\mathcal{D}^{1,p}_0(\Omega)$ is continuously embedded in $L^q(\Omega)$. Hence, in particular, we can refer to $\lambda_{p,q}(\Omega)$ as the sharp Sobolev-Poincaré constant.

Remark 1.3.2. A necessary and sufficient condition for an open set to be $(p,q)-$admissible is given in [**107**, Theorem 15.5.2]. We recall that bounded open sets and, more generally, open sets with finite volume are $(p,q)-$admissible, for every $1 \le q < p$. However, the class of $(p,q)-$admissible sets is larger, since there exist examples of $(p,q)-$admissible sets, which have infinite volume (see [**107**, Section 15.5.3]).

We note that the infimum problem (1.9) can be considered for every $1 \le q \le \infty$. In the following proposition, we show that $\lambda_{p,q}(\Omega)$ can be equivalently defined as an infimum on $W^{1,p}_0(\Omega)$, without requiring any restriction both on q and on the open set Ω.

Proposition 1.3.3. *Let $1 < p < \infty$, $1 \le q \le \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then*

$$\lambda_{p,q}(\Omega) = \inf_{\psi\in C^\infty_0(\Omega)} \left\{ \int_\Omega |\nabla\psi|^p\,dx \,:\, \|\psi\|_{L^q(\Omega)} = 1 \right\} = \inf_{\psi\in W^{1,p}_0(\Omega)} \left\{ \int_\Omega |\nabla\psi|^p\,dx \,:\, \|\psi\|_{L^q(\Omega)} = 1 \right\}.$$

Proof. We first observe that the problem on the right-hand side is actually settled on $W^{1,p}_0(\Omega) \cap L^q(\Omega)$. Since we have $C^\infty_0(\Omega) \subset W^{1,p}_0(\Omega) \cap L^q(\Omega)$, we immediately obtain

$$\inf_{\psi\in C^\infty_0(\Omega)} \left\{ \int_\Omega |\nabla\psi|^p\,dx \,:\, \|\psi\|_{L^q(\Omega)} = 1 \right\} \ge \inf_{\psi\in W^{1,p}_0(\Omega)} \left\{ \int_\Omega |\nabla\psi|^p\,dx \,:\, \|\psi\|_{L^q(\Omega)} = 1 \right\}.$$

In order to prove the reverse inequality, we first observe that if $\lambda_{p,q}(\Omega) = 0$, then from the previous inequality we would get

$$0 = \inf_{\psi \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla \psi|^p \, dx \, : \, \|\psi\|_{L^q(\Omega)} = 1 \right\} \geq \inf_{\psi \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla \psi|^p \, dx \, : \, \|\psi\|_{L^q(\Omega)} = 1 \right\} \geq 0,$$

and thus the desired identity trivially holds true.

Let us now suppose that $\lambda_{p,q}(\Omega) > 0$. For every $u \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ not identically vanishing, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|\nabla u_n - \nabla u\|_{L^p(\Omega)} = \lim_{n\to\infty} \|u_n - u\|_{L^p(\Omega)} = 0.$$

By using the definition of $\lambda_{p,q}(\Omega) > 0$, we have

$$\lambda_{p,q}(\Omega) \, \|u_n - u_m\|_{L^q(\Omega)}^p \leq \|\nabla u_n - \nabla u_m\|_{L^p(\Omega)}^p, \qquad \text{for every } n, m \in \mathbb{N},$$

thus, in particular, $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\Omega)$. This shows that we have

$$\lim_{n\to\infty} \|u_n - u\|_{L^q(\Omega)} = 0,$$

as well. Hence, we get

$$\lambda_{p,q}(\Omega) \leq \lim_{n\to\infty} \frac{\displaystyle\int_\Omega |\nabla u_n|^p \, dx}{\|u_n\|_{L^q(\Omega)}^p} = \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\|u\|_{L^q(\Omega)}^p}.$$

Finally, by taking the infimum on $W_0^{1,p}(\Omega) \cap L^q(\Omega)$ on the right-hand side, we obtain that

$$\lambda_{p,q}(\Omega) \leq \inf_{\psi \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla \psi|^p \, dx \, : \, \|\psi\|_{L^q(\Omega)} = 1 \right\}.$$

□

In the sequel, we prove that $q \mapsto \lambda_{p,q}(\Omega)$ is left-continuous at $q = p$, under the minimal assumption that Ω is $(p,q)-$admissible.

Proposition 1.3.4. *Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set, which is $(p, q_0)-$admissible for some $1 \leq q_0 < p$. Then we have*

$$\lim_{q \nearrow p} \lambda_{p,q}(\Omega) = \lambda_p(\Omega).$$

Proof. Let $\psi \in C_0^\infty(\Omega) \setminus \{0\}$, by definition of $\lambda_{p,q}(\Omega)$ we have for every $1 \leq q < p$

$$\lambda_{p,q}(\Omega) \leq \frac{\displaystyle\int_\Omega |\nabla \psi|^p \, dx}{\left(\displaystyle\int_\Omega |\psi|^q \, dx\right)^{\frac{p}{q}}}.$$

If we now take the limit as q goes to p, we get

$$\limsup_{q\nearrow p} \lambda_{p,q}(\Omega) \le \lim_{q\nearrow p} \frac{\displaystyle\int_\Omega |\nabla\psi|^p\,dx}{\left(\displaystyle\int_\Omega |\psi|^q\,dx\right)^{\frac{p}{q}}} = \frac{\displaystyle\int_\Omega |\nabla\psi|^p\,dx}{\displaystyle\int_\Omega |\psi|^p\,dx}.$$

By arbitrariness of ψ and recalling the definition of $\lambda_p(\Omega)$, we obtain

$$\limsup_{q\nearrow p} \lambda_{p,q}(\Omega) \le \lambda_p(\Omega).$$

In order to complete the proof, we use that for every $q_0 < q < p$ and every $\psi \in C_0^\infty(\Omega)\setminus\{0\}$ we have

$$\|\psi\|_{L^q(\Omega)} \le \|\psi\|_{L^{q_0}(\Omega)}^{1-\vartheta}\,\|\psi\|_{L^p(\Omega)}^{\vartheta}, \qquad \text{with } \vartheta = \frac{p}{q}\,\frac{q-q_0}{p-q_0},$$

by interpolation in Lebesgue spaces. In particular, by using that Ω is $(p, q_0)-$admissible, this entails that

$$\|\psi\|_{L^q(\Omega)} \le \left(\frac{1}{\lambda_{p,q_0}(\Omega)}\right)^{\frac{1-\vartheta}{p}} \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}^{1-\vartheta}\,\|\psi\|_{L^p(\Omega)}^{\vartheta}.$$

We can thus estimate the Rayleigh–type quotient defining $\lambda_{p,q}(\Omega)$ as follows

$$\frac{\displaystyle\int_\Omega |\nabla\psi|^p\,dx}{\left(\displaystyle\int_\Omega |\psi|^q\,dx\right)^{\frac{p}{q}}} \ge \Big(\lambda_{p,q_0}(\Omega)\Big)^{1-\vartheta} \frac{\left(\displaystyle\int_\Omega |\nabla\psi|^p\,dx\right)^{\vartheta}}{\left(\displaystyle\int_\Omega |\psi|^p\,dx\right)^{\vartheta}} \ge \Big(\lambda_{p,q_0}(\Omega)\Big)^{1-\vartheta}\,\Big(\lambda_p(\Omega)\Big)^{\vartheta}.$$

By arbitrariness of ψ, this yields

$$\lambda_{p,q}(\Omega) \ge \Big(\lambda_{p,q_0}(\Omega)\Big)^{1-\vartheta}\,\Big(\lambda_p(\Omega)\Big)^{\vartheta}.$$

If we pass to the limit as q goes to p and observe that

$$\lim_{q\nearrow p} \vartheta = \lim_{q\nearrow p} \frac{p}{q}\,\frac{q-q_0}{p-q_0} = 1,$$

we finally get

$$\liminf_{q\nearrow p} \lambda_{p,q}(\Omega) \ge \lambda_p(\Omega),$$

as desired. □

We need to recall the following family of interpolation–type inequalities

Lemma 1.3.5 (Gagliardo-Nirenberg interpolation inequalities)**.**

$$\|\psi\|_{L^r(\Omega)} \le G_{N,p,q,r}\,\|\nabla\psi\|_{L^p(\Omega)}^{\vartheta}\,\|\psi\|_{L^q(\Omega)}^{1-\vartheta}, \qquad \textit{for every } \psi \in C_0^\infty(\Omega), \tag{1.10}$$

which hold for every $1 < p < \infty$, $1 \le q \le p$ and r satisfying

$$\begin{cases} q < r \le \dfrac{Np}{N-p}, & \text{if } 1 < p < N, \\ q < r < \infty, & \text{if } p = N, \\ q < r \le \infty, & \text{if } p > N, \end{cases}$$

with $G_{N,p,q,r} > 0$ only depending on N, p, q, r and

$$\vartheta = \frac{N\,(r-q)}{N\,r\,(p-q) + p\,q\,r},$$

(for a proof, see for example [**28**, Proposition 2.6]).

By using (1.10), one can prove the following result

Proposition 1.3.6. *Let $1 \le q < p < \infty$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. Then*

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \Longleftrightarrow \qquad W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega).$$

Proof. The implication $\Longrightarrow$ is straightforward. For the converse implication, we suppose that there exists $C > 0$ such that

$$\|\psi\|_{L^q(\Omega)} \le C\,\|\psi\|_{W^{1,p}(\Omega)}, \qquad \text{for every } \psi \in C_0^\infty(\Omega).$$

By applying the Gagliardo-Nirenberg inequality (1.10) with $r = p$, it holds in particular

$$\|\psi\|_{L^q(\Omega)} \le C\,\|\psi\|_{W^{1,p}(\Omega)} \le C\,\left(\|\nabla\psi\|_{L^p(\Omega)} + G_{N,p,q,p}\,\|\nabla\psi\|^{\vartheta}_{L^p(\Omega)}\|\psi\|^{1-\vartheta}_{L^q(\Omega)}\right).$$

With a standard application of Young's inequality with exponents $1/\vartheta$ and $1/(1-\vartheta)$, we can absorb the L^q norm on the right-hand side and get

$$\|\psi\|_{L^q(\Omega)} \le \widetilde{C}\,\|\nabla\psi\|_{L^p(\Omega)},$$

for some $\widetilde{C} > 0$ independent of ψ. Thus, the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ holds, as well. □

The last result of the section deals with *capacity*. Indeed, when $p \le N$, a classical capacity result in Sobolev spaces infers that, since points have zero p-capacity, if one removes points from an open set $\Omega \subseteq \mathbb{R}^N$, the Sobolev space defined on this *new* set does not change, but it remains the same of $W^{1,p}(\Omega)$. This is shown in the following

Lemma 1.3.7 (Removing points in Sobolev spaces)**.** *Let $1 < p \le N$, let $\Omega \subseteq \mathbb{R}^N$ be an open set and $x_0 \in \Omega$, then*

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus \{x_0\}).$$

Proof. The fact that $W_0^{1,p}(\Omega \setminus \{x_0\}) \subseteq W_0^{1,p}(\Omega)$ follows by sets inclusions. In order to prove the reverse inclusion, we first take $\varphi \in C_0^\infty(\Omega)$ and the sequence of cut-off functions $\{u_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip}_0(B_{1/n}(x_0))$, defined by

$$u_n(x) = \begin{cases} 1, & \text{if } |x - x_0| < \dfrac{1}{n^2}, \\[2ex] \dfrac{\log(n\,|x - x_0|)}{-\log n}, & \text{if } \dfrac{1}{n^2} \le |x - x_0| \le \dfrac{1}{n}, \\[2ex] 0, & \text{if } |x - x_0| > \dfrac{1}{n}. \end{cases}$$

Observe that, since Ω is open and $x_0 \in \Omega$, we have that u_n is compactly supported in Ω, for n large enough. We take the sequence $\{\varphi\,(1 - u_n)\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega \setminus \{x_0\})$ and we claim that

$$\lim_{n\to\infty} \|\varphi\,(1 - u_n) - \varphi\|_{W^{1,p}(\Omega)} = 0. \tag{1.11}$$

For the strong convergence in $L^p(\Omega)$ it is sufficient to observe that, thanks to the properties of u_n, we have

$$|\varphi(x)\,(1 - u_n(x)) - \varphi(x)| \le |\varphi(x)|, \qquad \text{for a. e. } x \in \Omega,$$

and

$$\lim_{n\to\infty} \varphi(x)\,(1 - u_n(x)) = \varphi(x), \qquad \text{for a. e. } x \in \Omega.$$

Thus the conclusion follows from the Dominated Convergence Theorem. As for the gradients, we have

$$\begin{aligned} \int_\Omega |\nabla\big(\varphi\,(1 - u_n)\big) - \nabla\varphi|^p\,dx &= \int_\Omega |u_n\,\nabla\varphi + \varphi\,\nabla u_n|^p\,dx \\ &\le 2^{p-1} \int_\Omega |u_n|^p\,|\nabla\varphi|^p\,dx + 2^{p-1} \int_\Omega |\varphi|^p\,|\nabla u_n|^p\,dx. \end{aligned}$$

By using that $\{u_n\}$ converges to 0 almost everywhere in Ω and is equibounded in $L^\infty(\Omega)$, we get

$$\lim_{n\to\infty} \int_\Omega |u_n|^p\,|\nabla\varphi|^p\,dx = 0,$$

again by the Dominated Convergence Theorem. For the second integral, by computing the gradient of u_n and using that φ is bounded, we get

$$\begin{aligned} \int_\Omega |\varphi|^p\,|\nabla u_n|^p\,dx &\le \|\varphi\|^p_{L^\infty(\Omega)}\,\frac{1}{(n\,\log n)^p} \int_{B_{\frac{1}{n}}(x_0)\setminus B_{\frac{1}{n^2}}(x_0)} \frac{1}{|x - x_0|^p}\,dx \\ &= \|\varphi\|^p_{L^\infty(\Omega)}\,\frac{N\,\omega_N}{(n\,\log n)^p} \int_{\frac{1}{n^2}}^{\frac{1}{n}} \varrho^{N-p-1}\,d\varrho. \end{aligned}$$

It is now easy to see that the last integral converges to 0 as n goes to ∞, when $p < N$. If $p = N$, by computing the integral we get that

$$\int_\Omega |\varphi|^p\,|\nabla u_n|^p\,dx \le \|\varphi\|^p_{L^\infty(\Omega)}\,\frac{N\,\omega_N}{(n\,\log n)^p} \int_{\frac{1}{n^2}}^{\frac{1}{n}} \frac{1}{\varrho}\,d\varrho = \|\varphi\|^p_{L^\infty(\Omega)}\,\frac{N\,\omega_N}{n^p\,(\log n)^{p-1}},$$

which converges to 0 as n goes to ∞. Thus we finally get (1.11).

Let us now take $\varphi \in W_0^{1,p}(\Omega)$. By definition, there exists a sequence $\{\varphi_k\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$\lim_{k\to\infty} \|\varphi_k - \varphi\|_{W^{1,p}(\Omega)} = 0.$$

Moreover, from the previous part of the proof, for every $k \in \mathbb{N}$, there exists a sequence $\{\widetilde{\varphi}_{n,k}\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega \setminus \{x_0\})$ such that

$$\lim_{n\to\infty} \|\varphi_k - \widetilde{\varphi}_{n,k}\|_{W^{1,p}(\Omega)} = 0.$$

In particular, for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$\|\varphi_k - \widetilde{\varphi}_{n(k),k}\|_{W^{1,p}(\Omega)} \le \frac{1}{k+1}.$$

By the triangle inequality, we then obtain

$$\begin{aligned}\lim_{k\to\infty} \|\varphi_{n(k),k} - \varphi\|_{W^{1,p}(\Omega\setminus\{x_0\})} &= \lim_{k\to\infty} \|\varphi_{n(k),k} - \varphi\|_{W^{1,p}(\Omega)} \\ &\le \lim_{k\to\infty} \|\varphi_{n(k),k} - \varphi_k\|_{W^{1,p}(\Omega)} + \lim_{k\to\infty} \|\varphi_k - \varphi\|_{W^{1,p}(\Omega)} = 0.\end{aligned}$$

This shows that φ is the limit in the norm $W^{1,p}(\Omega\setminus\{x_0\})$ of elements in $W_0^{1,p}(\Omega\setminus\{x_0\})$. Since the latter is a closed space by construction, we get $\varphi \in W_0^{1,p}(\Omega \setminus \{x_0\})$, as well. This concludes the proof. □

Remark 1.3.8. By iterating the previous result, we immediately get that, if $1 < p \le N$, $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ is an open set, then we have

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus \{x_1, \dots, x_k\}),$$

for every $\{x_1, \dots, x_k\} \subset \Omega$.

In Chapter 2, we will need the following Poincaré–type inequality, for functions which vanish at a point of the boundary. Obviously, this may hold only in the superconformal case $p > N$, i.e. in the case where points have positive $p-$capacity.

Proposition 1.3.9 (A Poincaré inequality). *Let $p > N$ and $u \in W^{1,p}(B_R(x_0))$ be such that $u(z) = 0$, with $z \in \partial B_R(x_0)$. Then there exists a constant $C_{N,p} > 0$ such that*

$$\int_{B_R(x_0)} |u|^p\, dx \le C_{N,p}\, R^p \int_{B_R(x_0)} |\nabla u|^p\, dx. \tag{1.12}$$

Proof. It is sufficient to consider the case $x_0 = 0$ and $R = 1$, then the general case follows by scaling and translating. For the proof of (1.12), we can use a standard contradiction argument, exploiting compact Sobolev embeddings. We assume by contradiction that (1.12) fails to hold with a uniform constant. Thus, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,p}(B_1)$ such that

$$\|u_n\|_{L^p(B_1)} = 1, \quad \lim_{n\to\infty} \|\nabla u_n\|_{L^p(B_1)} = 0, \quad \text{and} \quad u_n(z) = 0 \text{ with } z \in \partial B_1.$$

In particular, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(B_1)$. Hence, thanks to compact Sobolev embeddings for $p > N$, there exists $u \in W^{1,p}(B_1) \cap C(\overline{B_1})$ such that u_n converges to u weakly in $W^{1,p}(B_1)$ and uniformly in $\overline{B_1}$, up to a subsequence. In particular, we get

$$\|\nabla u\|_{L^p(B_1)} \le \liminf_{n\to\infty} \|\nabla u_n\|_{L^p(B_1)} = 0 \qquad \text{and} \qquad u(z) = \lim_{n\to\infty} u_n(z) = 0.$$

Since B_1 is a connected set, the last two facts imply that $u \equiv 0$ in B_1. However, by the uniform convergence we also have

$$\|u\|_{L^p(B_1)} = \lim_{n\to\infty} \|u_n\|_{L^p(B_1)} = 1.$$

This contradicts the fact that u identically vanishes. □

§1.4 The Sobolev space $X^{p,q}$

In this section we introduce the Sobolev space $X^{p,q}$, which is needed in Chapter 3. Let $1 \le q < \infty$ and $1 < p < \infty$, let $\Omega \subset \mathbb{R}^N$ be an open set. We define the Sobolev space

$$X^{q,p}(\Omega) := \Big\{\psi \in L^q(\Omega) \,:\, \nabla\psi \in L^p(\Omega;\mathbb{R}^N)\Big\},$$

endowed with the norm

$$\|\psi\|_{X^{q,p}(\Omega)} := \|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}.$$

Observe that in general we have

$$W^{1,p}(\Omega) \neq X^{q,p}(\Omega),$$

unless $q = p$. For $1 \le q < p$, this can be seen for example by adapting the classical counter-example by Nikodým (see [**107**, Example 1, Section 1.1.4]), to infer existence of an open set $\Omega \subset \mathbb{R}^N$ and a function $\psi_0 \in X^{q,p}(\Omega)$ such that

$$\psi_0 \notin L^p(\Omega).$$

For $q > p$, it is sufficient to take any irregular open set $\Omega \subset \mathbb{R}^N$ such that $W^{1,p}(\Omega)$ does not imbed into $L^q(\Omega)$ (see for example [**70**, page 95]).

We also introduce the space $X^{q,p}_0(\Omega)$, defined as the closure of $C^\infty_0(\Omega)$ in $X^{q,p}(\Omega)$. The following technical results will be useful.

Lemma 1.4.1. *Let $v, U \in X^{q,p}(\Omega)$, then:*

(i) if $v - U \in X^{q,p}_0(\Omega)$, we have $|v| - |U| \in X^{q,p}_0(\Omega)$, as well;

(ii) if $v \in X^{q,p}_0(\Omega)$ and U is non-negative, we have $(v - U)_+ \in X^{q,p}_0(\Omega)$.

Proof. We prove (i) first. By assumption, there exists a sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\Omega)$ such that

$$\lim_{n\to\infty} \|\psi_n - (v - U)\|_{X^{q,p}(\Omega)} = 0.$$

We then define the new sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ by

$$\varphi_n = |\psi_n + U| - |U|, \qquad \text{for every } n \in \mathbb{N}.$$

Observe that each $\varphi_n \in X^{q,p}(\Omega)$ and it has compact support. Thus, with a minor modification of the proof of [**32**, Lemma 9.5], we get $\{\varphi_n\}_{n\in\mathbb{N}} \subset X^{q,p}_0(\Omega)$. By construction and by the triangle inequality, we have

$$\Big|\varphi_n - (|v| - |U|)\Big| = \Big||\psi_n + U| - |v|\Big| \le |\psi_n - (v - U)|.$$

Thus we get

$$\lim_{n\to\infty} \big\|\varphi_n - (|v| - |U|)\big\|_{L^q(\Omega)} = 0.$$

Moreover, it is easily seen that the sequence $\{\nabla\varphi_n\}_{n\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^N)$ is bounded. Thus it is weakly converging to some $\phi \in L^p(\Omega;\mathbb{R}^N)$, up to a subsequence. By using the strong convergence in $L^q(\Omega)$

and the definition of weak gradient, we can identify $\phi = \nabla|v| - \nabla|U|$. By Mazur's Lemma (see [**97**, Theorem 2.13]), we can build a new sequence $\{(\widetilde{\varphi}_n, \widetilde{\phi}_n)\}_{n\in\mathbb{N}} \subset L^q(\Omega) \times L^p(\Omega;\mathbb{R}^N)$, made of convex combinations of $\{(\varphi_n, \nabla\varphi_n)\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} \Big(\|\widetilde{\varphi}_n - (|v| - |U|)\|_{L^q(\Omega)} + \|\widetilde{\phi}_n - \nabla(|v| - |U|)\|_{L^p(\Omega;\mathbb{R}^N)} \Big) = 0.$$

Moreover, by construction we clearly have $\widetilde{\phi}_n = \nabla\widetilde{\varphi}_n$. This permits to show that $|v| - |U|$ is the limit in the $X^{q,p}(\Omega)$ norm of a sequence $\{\widetilde{\varphi}_n\}_{n\in\mathbb{N}} \subset X^{q,p}_0(\Omega)$. Since the latter is a closed subspace, we get the conclusion.

In order to prove (ii), it is sufficient to write

$$(v - U)_+ = \frac{|v - U| + (v - U)}{2} = \frac{(|v - U| - U) + v}{2}.$$

Then we observe that $v \in X^{q,p}_0(\Omega)$ by assumption, while

$$|v - U| - U = |U - v| - |U| \in X^{q,p}_0(\Omega),$$

from point (i), applied to the function $U - v$. Indeed, observe that $(U - v) - U \in X^{q,p}_0(\Omega)$. □

Proposition 1.4.2. *Let $1 \le q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a $(p, q)-$admissible open set. Then we have*

$$X^{q,p}_0(\Omega) = W^{1,p}_0(\Omega) = \mathcal{D}^{1,p}_0(\Omega),$$

and the three spaces are compactly embedded into $L^q(\Omega)$. Consequently, we also have

$$W^{1,p}_0(\Omega) \cap X^{q,p}(\Omega) = W^{1,p}_0(\Omega).$$

Proof. It is sufficient to prove the first fact, the second one being an easy consequence of this. We need to prove that the three norms

$$\|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}, \qquad \|\psi\|_{W^{1,p}(\Omega)} \qquad \text{and} \qquad \|\psi\|_{X^{q,p}(\Omega)},$$

are equivalent on $C^\infty_0(\Omega)$. Observe that under the assumption of $(p, q)-$admissibility of the set Ω, we have

$$\|\psi\|_{X^{q,p}(\Omega)} \le \Big(\lambda_{p,q}(\Omega)^{-\frac{1}{p}} + 1 \Big) \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}, \qquad \text{for every } \psi \in C^\infty_0(\Omega).$$

Moreover, by using the *Gagliardo-Nirenberg interpolation inequality* (1.10) togheter with Young's inequality with conjugate exponents $1/\vartheta$ and $1/(1 - \vartheta)$, we get

$$\|\psi\|_{L^p(\Omega)} \le C'(\|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)}), \qquad \text{for every } \psi \in C^\infty_0(\Omega).$$

This in particular implies that

$$\begin{aligned} \|\psi\|_{W^{1,p}(\Omega)} &= \|\psi\|_{L^p(\Omega)} + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)} \\ &\le C' \left(\|\psi\|_{L^q(\Omega)} + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)} \right) + \|\nabla\psi\|_{L^p(\Omega;\mathbb{R}^N)} \\ &\le (C' + 1) \|\psi\|_{X^{q,p}(\Omega)}, \qquad \text{for every } \psi \in C^\infty_0(\Omega). \end{aligned}$$

The previous estimates show the desired equivalence of the norms on $C^\infty_0(\Omega)$. As for the compactness of the embedding, it is sufficient to recall that for $1 \le q < p$ the embedding of $\mathcal{D}^{1,p}_0(\Omega)$ into $L^q(\Omega)$ is continuous if and only if it is compact (see [**107**, Theorem 15.6.2] or also [**28**, Theorem 1.2]). This fact and the equivalence of the spaces conclude the proof. □

Remark 1.4.3. Under the previous assumptions, by definition of the space $X_0^{q,p}(\Omega)$, it is clear that inequality (1.10) still holds for $\psi \in X_0^{q,p}(\Omega)$.

Remark 1.4.4. With the same argument of Proposition 1.4.2, we also can observe that

$$\lambda_p(\Omega) > 0 \qquad \Longrightarrow \qquad W_0^{1,p}(\Omega) = \mathcal{D}_0^{1,p}(\Omega).$$

Indeed, recall that the positivity of $\lambda_p(\Omega)$ automatically gives that

$$\|\nabla u\|_{L^p(\Omega)} \qquad \text{and} \qquad \|u\|_{W^{1,p}(\Omega)},$$

are equivalent norms on $C_0^\infty(\Omega)$.

§1.5 Sobolev-Slobodeckiĭ fractional spaces

In this section, we introduce some basic properties of *Sobolev-Slobodeckiĭ fractional spaces* (see [**52**] for a friendly introduction to the subject). We then recall these results in Chapter 8, as we will work in the fractional setting.

For $1 < p < \infty$ and $0 < s < 1$, we consider the Sobolev fractional space

$$W^{s,p}(\mathbb{R}^N) = \Big\{ u \in L^p(\mathbb{R}^N) \,:\, [u]_{W^{s,p}(\mathbb{R}^N)} < +\infty \Big\},$$

where

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+s\,p}}\, dx\, dy \right)^{\frac{1}{p}},$$

is usually called *Gagliardo seminorm* or also *Gagliardo-Slobodeckiĭ seminorm*. This is a reflexive Banach space, when endowed with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{W^{s,p}(\mathbb{R}^N)},$$

see for example [**95**, Proposition 17.6 & Theorem 17.41]. For an open set $\Omega \subset \mathbb{R}^N$, we indicate by $\widetilde{W}_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\mathbb{R}^N)$.

The following simple result will be useful: its proof is simply based on standard properties of convolutions and thus it is omitted (see for example [**63**, Lemma 11]).

Lemma 1.5.1. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $\varphi \in W^{s,p}(\mathbb{R}^N)$ has compact support in Ω, then we have $\varphi \in \widetilde{W}_0^{s,p}(\Omega)$.*

In the following two lemmas, we provide some Leibniz–type formulas in fractional Sobolev spaces.

Lemma 1.5.2. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subseteq \mathbb{R}^N$ be an open set, then for every $\eta \in C_0^1(\Omega)$ and $u \in W_{\rm loc}^{s,p}(\Omega)$, the function $\eta\, u$ is compactly supported in Ω and belongs to $W^{s,p}(\mathbb{R}^N)$. In particular, we have*

$$\eta\, u \in \widetilde{W}_0^{s,p}(\Omega).$$

Proof. We consider both η and u to be extended by 0 to $\mathbb{R}^N \setminus \Omega$. In light of Lemma 1.5.1, we only need to show that $\eta\, u \in W^{s,p}(\mathbb{R}^N)$. We take $\Omega'' \Subset \Omega' \Subset \Omega$ such that the support of η is contained in Ω''.

Then we may write

$$[\eta\, u]^p_{W^{s,p}(\mathbb{R}^N)} = [\eta\, u]^p_{W^{s,p}(\Omega')} + 2 \iint_{\Omega'\times(\mathbb{R}^N\setminus\Omega')} \frac{|\eta\,(x)\, u(x)|^p}{|x-y|^{N+s\,p}}\, dx\, dy$$
$$= [\eta\, u]^p_{W^{s,p}(\Omega')} + 2 \iint_{\Omega''\times(\mathbb{R}^N\setminus\Omega')} \frac{|\eta\,(x)\, u(x)|^p}{|x-y|^{N+s\,p}}\, dx\, dy,$$

where we used that η vanishes outside Ω''. For the first term, by using Minkowksi's inequality and Lemma A.3.1, we can estimate it from above by means of the following Leibniz–type rule

$$[\eta\, u]_{W^{s,p}(\Omega')} \leq \left(\int_{\Omega'} |u(x)|^p \left(\int_{\Omega'} \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{N+s\,p}}\, dy\right) dx\right)^{\frac{1}{p}}$$
$$+ \left(\int_{\Omega'} |\eta(y)|^p \left(\int_{\Omega'} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\, dx\right) dy\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{C}{s\,(1-s)}\right)^{\frac{1}{p}} \|u\|_{L^p(\Omega')}\, \|\nabla\eta\|^s_{L^\infty(\mathbb{R}^N)}\, \|\eta\|^{1-s}_{L^\infty(\mathbb{R}^N)} + \|\eta\|_{L^\infty(\mathbb{R}^N)}\, [u]_{W^{s,p}(\Omega')} < +\infty.$$

For the second term, we have

$$2 \iint_{\Omega''\times(\mathbb{R}^N\setminus\Omega')} \frac{|\eta\,(x)\, u(x)|^p}{|x-y|^{N+s\,p}}\, dx\, dy \leq 2\, \|\eta\|^p_{L^\infty(\mathbb{R}^N)} \int_{\Omega''} |u(x)|^p \left(\int_{\mathbb{R}^N\setminus\Omega'} \frac{dy}{|x-y|^{N+s\,p}}\right) dx$$
$$\leq 2\, \|\eta\|^p_{L^\infty(\mathbb{R}^N)} \int_{\Omega''} |u(x)|^p \left(\int_{\mathbb{R}^N\setminus B_{\mathfrak{d}}(x)} \frac{dy}{|x-y|^{N+s\,p}}\right) dx$$
$$= \frac{2\, N\, \omega_N}{s\, p}\, \frac{\|\eta\|^p_{L^\infty(\mathbb{R}^N)}}{\mathfrak{d}^{s\,p}} \int_{\Omega''} |u(x)|^p\, dx < +\infty,$$

where we set $\mathfrak{d} = \mathrm{dist}(\Omega'', \partial\Omega') > 0$. □

Lemma 1.5.3. *Let* $1 < p < \infty$ *and* $0 < s < 1$. *For* $M > 0$, *we take* $u \in W^{s,p}((0, M))$ *and extend it by* 0 *outside* $(0, M)$. *We also suppose that there exist* $C > 0$ *and* $\beta > (s\,p - 1)/p$ *such that*

$$|u(x)| \leq C\, x^\beta, \qquad \text{for a. e. } x \in (0, M).$$

Then, for every $\eta \in C^\infty_0((-\infty, M))$, *we have*

$$u\, \eta \in \widetilde{W}^{s,p}_0(\mathbb{H}^1_+).$$

Moreover, the following estimates hold

$$[u\, \eta]^p_{W^{s,p}(\mathbb{R})} \leq [u\, \eta]^p_{W^{s,p}((0,M))} + \frac{2}{s\,p} \int_0^M \frac{|u(x)\, \eta(x)|^p}{|x|^{s\,p}}\, dx + \frac{2\, M^{p-s\,p}}{s\,p}\, \|\eta'\|^p_{L^\infty(\mathbb{R})}\, \|u\|^p_{L^p((0,M))}, \tag{1.13}$$

and

$$[u\, \eta]_{W^{s,p}((0,M))} \leq \|\eta\|_{L^\infty(\mathbb{R})}\, [u]_{W^{s,p}((0,M))} + \left(\frac{C}{s\,(1-s)}\right)^{\frac{1}{p}} \|u\|_{L^p((0,M))}\, \|\eta'\|^s_{L^\infty(\mathbb{R})}\, \|\eta\|^{1-s}_{L^\infty(\mathbb{R})}, \tag{1.14}$$

for some $C = C(p) > 0$.

Proof. We start by observing that $\widetilde{W}^{s,p}_0(\mathbb{H}^1_+)$ can be identified with the space of functions in $W^{s,p}(\mathbb{R})$ which vanish almost everywhere in $(-\infty, 0]$, thanks to [**63**, Theorem 6]. By construction, it is then sufficient to prove that $\eta\, u \in W^{s,p}(\mathbb{R})$.

It is easy to see that $u\,\eta \in L^p(\mathbb{R})$, hence let us focus on proving that $u\,\eta$ has a finite $W^{s,p}$ seminorm. By construction, this function vanishes almost everywhere outside $(0, M)$. We decompose the seminorm as follows

$$\begin{aligned}[u\,\eta]^p_{W^{s,p}(\mathbb{R})} &= [u\,\eta]^p_{W^{s,p}((0,M))} + 2\int_0^M \int_{-\infty}^0 \frac{|u(x)\,\eta(x)|^p}{|x-y|^{1+s\,p}}\,dy\,dx + 2\int_0^M \int_M^{+\infty} \frac{|u(x)\,\eta(x)|^p}{|x-y|^{1+s\,p}}\,dy\,dx \\ &= [u\,\eta]^p_{W^{s,p}((0,M))} + \frac{2}{s\,p}\int_0^M \frac{|u(x)\,\eta(x)|^p}{|x|^{s\,p}}\,dx + \frac{2}{s\,p}\int_0^M \frac{|u(x)\,\eta(x)|^p}{(M-x)^{s\,p}}\,dx.\end{aligned}$$

In order to estimate the first term on the right-hand side, we proceed similarly as in the proof of Lemma 1.5.2, so to get

$$\begin{aligned}[u\,\eta]_{W^{s,p}((0,M))} &\le \left[\int_0^M |u(x)|^p \left(\int_0^M \frac{|\eta(x)-\eta(y)|^p}{|x-y|^{1+s\,p}}\,dy\right)dx\right]^{\frac{1}{p}} \\ &\quad + \left[\int_0^M |\eta(y)|^p \left(\int_0^M \frac{|u(x)-u(y)|^p}{|x-y|^{1+s\,p}}\,dx\right)dy\right]^{\frac{1}{p}} \\ &\le \left(\frac{C}{s\,(1-s)}\right)^{\frac{1}{p}} \|u\|_{L^p((0,M))}\,\|\eta'\|^s_{L^\infty(\mathbb{R})}\,\|\eta\|^{1-s}_{L^\infty(\mathbb{R})} + \|\eta\|_{L^\infty((0,M))}\,[u]_{W^{s,p}((0,M))}.\end{aligned}$$

In the last inequality, we applied again Lemma A.3.1. As for the other terms, we observe that

$$\int_0^M \frac{|u(x)\,\eta(x)|^p}{|x|^{s\,p}}\,dx,$$

is finite, thanks to the growth assumption on u. Finally, by using that $\eta \in C^\infty_0((-\infty, M))$, we have

$$|u(x)\,\eta(x)|^p = |u(x)|^p\,|\eta(x)-\eta(M)|^p \le \|\eta'\|^p_{L^\infty(\mathbb{R})}\,|u(x)|^p\,(M-x)^p, \qquad \text{for a. e. } x \in (0, M),$$

so that we can infer

$$\int_0^M \frac{|u(x)\,\eta(x)|^p}{(M-x)^{s\,p}}\,dx \le M^{p-s\,p}\,\|\eta'\|^p_{L^\infty(\mathbb{R})}\int_0^M |u(x)|^p\,dx < +\infty.$$

This completes the proof. □

Occasionally, for an open set $\Omega \subset \mathbb{R}^N$, we will need the fractional Sobolev space defined by

$$W^{s,p}(\Omega) = \Big\{u \in L^p(\Omega)\ :\ [u]_{W^{s,p}(\Omega)} < +\infty\Big\},$$

where

$$[u]_{W^{s,p}(\Omega)} := \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}.$$

In this work, we refer to $[u]_{W^{s,p}(\Omega)}$ as the "regional" seminorm on the general open set $\Omega \subset \mathbb{R}^N$. Finally, $W^{s,p}_{\rm loc}(\Omega)$ is the space of functions $u \in L^p_{\rm loc}(\Omega)$ such that $u \in W^{s,p}(\Omega')$ for every $\Omega' \Subset \Omega$.

§1.6 A weighted fractional Sobolev space

For $0 < \beta < \infty$, we also denote by $L^\beta_{s\,p}(\mathbb{R}^N)$ the following weighted Lebesgue space

$$L^\beta_{s\,p}(\mathbb{R}^N) = \left\{ u \in L^\beta_{\rm loc}(\mathbb{R}^N) \,:\, \int_{\mathbb{R}^N} \frac{|u(x)|^\beta}{(1+|x|)^{N+s\,p}}\,dx < +\infty \right\}.$$

We observe that this is a Banach space for $\beta \geq 1$, when endowed with the natural norm. Moreover, it is not difficult to see that

$$L^\beta_{s\,p}(\mathbb{R}^N) \subset L^{\beta'}_{s\,p}(\mathbb{R}^N), \qquad \text{for } 0 < \beta' < \beta < \infty. \tag{1.15}$$

It is sufficient to use that

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{N+s\,p}}\,dx < +\infty, \qquad \text{for every } N \geq 1,\ 1 < p < \infty \ \text{ and } 0 < s < 1,$$

and then apply Jensen's inequality.

Some of the results stated in Chapter 8 crucially rely on certain properties of a fractional Sobolev space, whose definition is inspired by [**5**, Appendix].

Definition 1.6.1. Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set, we define

$$\mathcal{X}^{s,p}(\Omega; d_\Omega) := \left\{ u \in L^p_{s\,p}(\mathbb{R}^N) : [u]_{W^{s,p}(\mathbb{R}^N)} < +\infty \ \text{ and } \ \frac{u}{d_\Omega^s} \in L^p(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{\mathcal{X}^{s,p}(\Omega;d_\Omega)} := [u]_{W^{s,p}(\mathbb{R}^N)} + \left(\int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx \right)^{\frac{1}{p}}, \qquad \text{for every } u \in \mathcal{X}^{s,p}(\Omega; d_\Omega).$$

Then we define $\mathcal{X}^{s,p}_0(\Omega; d_\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $\mathcal{X}^{s,p}(\Omega; d_\Omega)$.

We recall that, for every $1 < p < \infty$ and $0 < s < 1$, the symbol $\mathfrak{h}_{s,p}$ denotes the sharp Hardy constant in the fractional Hardy inequality, defined as in (22).

Remark 1.6.2. We observe that if the open set $\Omega \subsetneq \mathbb{R}^N$ is such that $\mathfrak{h}_{s,p}(\Omega) > 0$, then by a simple density argument we can assure that Hardy's inequality holds in $\mathcal{X}^{s,p}_0(\Omega; d_\Omega)$, as well. That is, we have

$$\mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx \leq [u]^p_{W^{s,p}(\mathbb{R}^N)}, \qquad \text{for every } u \in \mathcal{X}^{s,p}_0(\Omega; d_\Omega).$$

Accordingly, this implies that in this case

$$u \mapsto [u]_{W^{s,p}(\mathbb{R}^N)},$$

defines an equivalent norm on $\mathcal{X}^{s,p}_0(\Omega; d_\Omega)$.

Proposition 1.6.3. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then*

$$\mathcal{X}^{s,p}(\Omega; d_\Omega) \subset W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{s\,p}(\mathbb{R}^N),$$

and we have the estimate

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{(1+|x|)^{N+s\,p}}\,dx\right)^{\frac{1}{p}} \le C_\Omega\,\|u\|_{\mathcal{X}^{s,p}(\Omega;d_\Omega)}, \qquad \text{for every } u\in\mathcal{X}^{s,p}(\Omega;d_\Omega). \tag{1.16}$$

Moreover, $\mathcal{X}^{s,p}(\Omega;d_\Omega)$ *and* $\mathcal{X}^{s,p}_0(\Omega;d_\Omega)$ *are Banach spaces.*

Proof. The first fact is straightforward, by also taking into account (1.15).

We prove the estimate (1.16). We take a ball $B_R(x_0)\Subset\Omega$ such that $B_{2\,R}(x_0)\Subset\Omega$, as well. We then write

$$\begin{aligned}
[u]_{W^{s,p}(\mathbb{R}^N)} &= \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\ge \left(\iint_{B_R(x_0)\times(\mathbb{R}^N\setminus B_{2\,R}(x_0))} \frac{|u(x)-u(y)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}}\\
&\ge \left(\int_{B_R(x_0)}\left(\int_{\mathbb{R}^N\setminus B_{2\,R}(x_0)} \frac{|u(y)|^p}{|x-y|^{N+s\,p}}\,dy\right)dx\right)^{\frac{1}{p}}\\
&\quad - \left(\int_{B_R(x_0)}|u(x)|^p\left(\int_{\mathbb{R}^N\setminus B_{2\,R}(x_0)} \frac{1}{|x-y|^{N+s\,p}}\,dy\right)dx\right)^{\frac{1}{p}},
\end{aligned}$$

thanks to Minkowski's inequality. By observing that

$$|x-y|\ge\frac{1}{2}\,|y-x_0|, \qquad \text{for every } x\in B_R(x_0),\ y\notin B_{2\,R}(x_0),$$

we have

$$\begin{aligned}
\int_{B_R(x_0)}|u(x)|^p\left(\int_{\mathbb{R}^N\setminus B_{2\,R}(x_0)} \frac{1}{|x-y|^{N+s\,p}}\,dy\right)dx &\le \frac{N\,\omega_N\,2^N}{s\,p}\,R^{-s\,p}\int_{B_R(x_0)}|u|^p\,dx\\
&\le \frac{N\,\omega_N\,2^N}{s\,p}\,R^{-s\,p}\int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx\,\|d_\Omega\|^{s\,p}_{L^\infty(B_R(x_0))}.
\end{aligned}$$

This implies that we have

$$\left(\int_{B_R(x_0)}\left(\int_{\mathbb{R}^N\setminus B_{2\,R}(x_0)} \frac{|u(y)|^p}{|x-y|^{N+s\,p}}\,dy\right)dx\right)^{\frac{1}{p}} \le C\,\|u\|_{\mathcal{X}^{s,p}(\Omega;d_\Omega)}, \tag{1.17}$$

for a constant $C=C(N,s,p,\Omega,B_R(x_0))>0$. We now use that

$$|x-y|\le 2\,|x_0-y|, \qquad \text{for every } x\in B_R(x_0),\ y\notin B_{2\,R}(x_0),$$

together with the fact that

$$|x_0-y|\le|x_0|+|y|\le(1+|x_0|)\,(1+|y|).$$

By using these in (1.17), we get

$$\left(\int_{\mathbb{R}^N\setminus B_{2\,R}(x_0)} \frac{|u(y)|^p}{(1+|y|)^{N+s\,p}}\,dy\right)^{\frac{1}{p}} \leq C\,\|u\|_{\mathcal{X}^{s,p}(\Omega;d_\Omega)},$$

possibly with a different constant $C = C(N,s,p,\Omega,B_R(x_0)) > 0$. The proof of estimate (1.16) is almost over: it is now sufficient to add on both sides of the previous estimate the term

$$\left(\int_{B_{2\,R}(x_0)} \frac{|u(y)|^p}{(1+|y|)^{N+s\,p}}\,dy\right)^{\frac{1}{p}}.$$

By using that

$$\begin{aligned}\int_{B_{2\,R}(x_0)} \frac{|u(y)|^p}{(1+|y|)^{N+s\,p}}\,dy \leq \int_{B_{2\,R}(x_0)} |u(y)|^p\,dy &\leq \int_{\Omega} \frac{|u|^p}{d_\Omega^{s\,p}}\,dy\,\|d_\Omega\|^{s\,p}_{L^\infty(B_{2\,R}(x_0))}\\ &\leq C_\Omega\,\|u\|^p_{\mathcal{X}^{s,p}(\Omega;d_\Omega)},\end{aligned}$$

for some constant $C = C(s,p,\Omega,B_{2R}(x_0)) > 0$, we eventually get the desired conclusion.

We prove the second part of the statement. We first observe that it is sufficient to prove that $\mathcal{X}^{s,p}(\Omega;d_\Omega)$ is a Banach space. We take $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}^{s,p}(\Omega;d_\Omega)$ to be a Cauchy sequence. Then we get that this is a Cauchy sequence in the Banach space $L^p(\Omega;d_\Omega^{-s\,p})$ and that

$$\{D^s u_n\}_{n\in\mathbb{N}} \qquad \text{where } D^s\varphi(x,y) := \frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N}{p}+s}},$$

is a Cauchy sequence in $L^p(\mathbb{R}^N\times\mathbb{R}^N)$. This follows from the fact that

$$[u_n]_{W^{s,p}(\mathbb{R}^N)} = \|D^s u_n\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)}.$$

Moreover, according to (1.16), the sequence $\{u_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in the Banach space $L^p_{s\,p}(\mathbb{R}^N)$. The last fact implies that there exists $u \in L^p_{s\,p}(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{|u_n-u|^p}{(1+|x|)^{N+s\,p}}\,dx = 0.$$

In particular, up to a subsequence, we can suppose that u_n converges to u almost everywhere in $\mathbb{R}^N$. By using the completeness of $L^p(\Omega;d_\Omega^{-s\,p})$, we get similarly the existence of $\widetilde{u} \in L^p(\Omega;d_\Omega^{-s\,p})$ such that

$$\lim_{n\to\infty} \int_{\Omega} \frac{|u_n-\widetilde{u}|^p}{d_\Omega^{s\,p}}\,dx = 0.$$

By uniqueness of the limit, we must have $u = \widetilde{u}$ almost everywhere in Ω. Finally, by using that $L^p(\mathbb{R}^N\times\mathbb{R}^N)$ is a Banach space, we get that there exists $\phi \in L^p(\mathbb{R}^N\times\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \|D^s u_n - \phi\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} = 0.$$

This in particular would imply that

$$\lim_{n\to\infty} D^s u_n(x,y) = \phi(x,y), \qquad \text{for a. e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N,$$

up to a subsequence. On the other hand, by using the almost everywhere convergence of u_n previously inferred, we also obtain that

$$\lim_{n\to\infty} D^s u_n(x,y) = D^s u(x,y), \qquad \text{for a. e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N.$$

By using the uniqueness of the limit, we get at the same time that

$$[u]_{W^{s,p}(\mathbb{R}^N)}<+\infty \qquad \text{and} \qquad \lim_{n\to\infty}\|D^s u_n - D^s u\|_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} = \lim_{n\to\infty}[u_n-u]_{W^{s,p}(\mathbb{R}^N)} = 0.$$

This concludes the proof. □

We note that as a straightforward consequence of Lemma 1.2.5, we get the following

Lemma 1.6.4. *Let $1<p<\infty$, $0<s<1$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set. Then for $s\,p\ge N$ the unique constant function contained in $\mathcal{X}^{s,p}(\Omega;d_\Omega)$ is the null one. The same conclusion holds also for $s\,p<N$, if we additionally suppose that $|\Omega|=+\infty$.*

In the next result we compare the two spaces $\widetilde{W}^{s,p}_0(\Omega)$ and $\mathcal{X}^{s,p}_0(\Omega;d_\Omega)$.

Proposition 1.6.5. *Let $1<p<\infty$ and $0<s<1$. Let $\Omega\subsetneq\mathbb{R}^N$ be an open set such that $\mathfrak{h}_{s,p}(\Omega)>0$. Then we have*

$$\widetilde{W}^{s,p}_0(\Omega)\subseteq\mathcal{X}^{s,p}_0(\Omega;d_\Omega), \tag{1.18}$$

and the inclusion is continuous. Moreover, if we assume that $r_\Omega<+\infty$, then

$$\widetilde{W}^{s,p}_0(\Omega)=\mathcal{X}^{s,p}_0(\Omega;d_\Omega),$$

and

$$\varphi\mapsto[\varphi]_{W^{s,p}(\mathbb{R}^N)}, \tag{1.19}$$

is an equivalent norm on this space. Finally, if we further require that $|\Omega|<+\infty$, then we have the continuous embedding

$$\mathcal{X}^{s,p}_0(\Omega;d_\Omega)\hookrightarrow L^p(\Omega),$$

and this is compact, as well.

Proof. By recalling Remark 1.6.2, we know that (1.19) is an equivalent norm on $\mathcal{X}^{s,p}_0(\Omega;d_\Omega)$. Since we trivially have

$$[\varphi]_{W^{s,p}(\mathbb{R}^N)}\le\|\varphi\|_{W^{s,p}(\mathbb{R}^N)}, \qquad \text{for every } \varphi\in C^\infty_0(\Omega),$$

the continuous inclusion (1.18) easily follows.

We now assume that $r_\Omega<+\infty$. In conjuction with Hardy's inequality and recalling (1.4), this yields

$$\int_\Omega|\varphi|^p\,dx\le r_\Omega^{s\,p}\int_\Omega\frac{|\varphi|^p}{d_\Omega^{s\,p}}\,dx\le\frac{r_\Omega^{s\,p}}{\mathfrak{h}_{s,p}(\Omega)}\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)}, \qquad \text{for every } \varphi\in C^\infty_0(\Omega).$$

Thus we get that

$$\varphi\mapsto\|\varphi\|_{\mathcal{X}^{s,p}(\mathbb{R}^N)} \qquad \text{and} \qquad \varphi\mapsto\|\varphi\|_{W^{s,p}(\mathbb{R}^N)},$$

are equivalent norms on $C_0^\infty(\Omega)$, again thanks to Remark 1.6.2. Then the claimed identity of the two closures immediately follows. The last statement is now an easy consequence of the same property for the space $\widetilde{W}_0^{s,p}(\Omega)$, which is well-known. □

Remark 1.6.6. Under the sole assumption that $\mathfrak{h}_{s,p}(\Omega) > 0$, in general we have

$$\widetilde{W}_0^{s,p}(\Omega) \subset \mathcal{X}_0^{s,p}(\Omega; d_\Omega) \qquad \text{and} \qquad \widetilde{W}_0^{s,p}(\Omega) \neq \mathcal{X}_0^{s,p}(\Omega; d_\Omega).$$

As a counter-example, it is sufficient to take any open set $\Omega \subsetneq \mathbb{R}^N$ such that

$$\mathfrak{h}_{s,p}(\Omega) > 0 \qquad \text{and} \qquad \inf_{\varphi \in C_0^\infty(\Omega)} \left\{ [\varphi]^p_{W^{s,p}(\mathbb{R}^N)} \,:\, \int_\Omega |\varphi|^p\,dx = 1 \right\} = 0.$$

For example, we can take Ω to be a half-space. In such a case, we have by construction

$$\widetilde{W}_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega),$$

while

$$\mathcal{X}_0^{s,p}(\Omega; d_\Omega) \not\hookrightarrow L^p(\Omega).$$

We can finally prove a compactness result for the space $\mathcal{X}_0^{s,p}(\Omega)$, under minimal assumptions on the open set Ω. This will be crucially exploited in the proof of Lemma 8.3.3.

Theorem 1.6.7. *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that $\mathfrak{h}_{s,p}(\Omega) > 0$. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ be such that*

$$[u_n]^p_{W^{s,p}(\mathbb{R}^N)} \leq M, \qquad \text{for every } n \in \mathbb{N},$$

for some $M > 0$. Then there exist a function $u \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ and subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} u_{n_k}(x) = u(x), \qquad \text{for a. e. } x \in \Omega.$$

Moreover, for every $\Omega' \Subset \Omega$, we also have

$$\lim_{k\to\infty} \|u_{n_k} - u\|_{L^p(\Omega')} = 0,$$

up to a possible further subsequence.

Proof. We need two distinguish two cases: either $|\Omega| < +\infty$ or $|\Omega| = +\infty$.

Case 1: Ω has finite volume. This is the easiest case: here the result plainly follows from Proposition 1.6.5. We also observe that the last statement actually holds in a stronger form, since we can infer convergence in $L^p(\Omega)$.

Case 2: Ω has infinite volume. We still use the notation $D^s\varphi$ for a measurable function, as in Proposition 1.6.3. Thus, by assumption, we get that $\{D^s u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^p(\mathbb{R}^N \times \mathbb{R}^N)$. This entails that, up to a subsequence, it is weakly converging in $L^p(\mathbb{R}^N \times \mathbb{R}^N)$. For simplicity, we do not relabel the subsequence. Let us call ϕ such a limit. We may apply Mazur's Lemma (see [**97**, Theorem 2.13]) and get that for every $n \in \mathbb{N}$ there exists

$$\{t_\ell(n)\}_{\ell=0}^n \subset [0,1], \qquad \text{such that} \qquad \sum_{\ell=0}^n t_\ell(n) = 1,$$

and such that the new sequence made of convex combinations

$$\widetilde{\phi}_n(x,y)=\sum_{\ell=0}^{n} t_\ell(n)\,D^s u_\ell(x,y),$$

strongly converges in $L^p(\mathbb{R}^N\times\mathbb{R}^N)$ to ϕ, as n goes to ∞. Observe that by construction we have

$$\sum_{\ell=0}^{n} t_\ell(n)\,D^s u_\ell = D^s\left(\sum_{\ell=0}^{n} t_\ell(n)\,u_\ell\right),$$

and

$$\widetilde{u}_n:=\sum_{\ell=0}^{n} t_\ell(n)\,u_\ell\in\mathcal{X}_0^{s,p}(\Omega;d_\Omega),$$

since the latter is a vector space. This proves that $\{D^s\widetilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^N\times\mathbb{R}^N)$ and this, in turn, implies that $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_0^{s,p}(\Omega;d_\Omega)$, thanks to Remark 1.6.2. By using that $\mathcal{X}_0^{s,p}(\Omega;d_\Omega)$ is a Banach space, we get that $\{\widetilde{u}_n\}_{n\in\mathbb{N}}$ converges in this space to a limit function $u\in\mathcal{X}_0^{s,p}(\Omega;d_\Omega)$. In particular, we must have

$$D^s u=\phi.$$

We now want to prove that $\{u_n\}_{n\in\mathbb{N}}$ converges almost everywhere on $\mathbb{R}^N$ to the function u, up to a subsequence. We first observe that all the elements of $\mathcal{X}_0^{s,p}(\Omega;d_\Omega)$ vanish almost everywhere in $\mathbb{R}^N\setminus\Omega$, by construction. Thus we only need to prove convergence almost everywhere in Ω.

We denote by $\{\Omega_k\}_{k\in\mathbb{N}}$ an exhausting sequence for Ω, made of bounded open subsets with smooth boundary: in other words

$$\Omega_k\Subset\Omega,\quad \Omega_k\Subset\Omega_{k+1}\text{ for every }k\in\mathbb{N}\quad\text{and}\quad\bigcup_{k\in\mathbb{N}}\Omega_k=\Omega,$$

see [**45**, Proposition 8.2.1]. We preliminary observe that, thanks to the assumption $\mathfrak{h}_{s,p}(\Omega)>0$, we have for every $k,n\in\mathbb{N}$

$$\int_{\Omega_k}|u_n|^p\,dx\le\|d_\Omega\|_{L^\infty(\Omega_k)}^{s\,p}\int_{\Omega_k}\frac{|u_n|^p}{d_\Omega^{s\,p}}\,dx\le\frac{1}{\mathfrak{h}_{s,p}(\Omega)}\,\|d_\Omega\|_{L^\infty(\Omega_k)}^{s\,p}\,[u_n]_{W^{s,p}(\mathbb{R}^N)}^p\le C_k\,M,$$

which entails that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in each $W^{s,p}(\Omega_k)$. By using the compactness of the embedding $W^{s,p}(\Omega_k)\hookrightarrow L^p(\Omega_k)$ (see for example [**52**, Theorem 7.1]) and a diagonal argument, we can obtain existence of a function $U\in W^{s,p}_{\rm loc}(\Omega)$ and of a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty}u_{n_k}(x)=U(x),\qquad\text{for a. e. }x\in\Omega.$$

We then extend U to be 0 outside Ω: by using Fatou's Lemma and the almost everywhere convergence, we get

$$[U]_{W^{s,p}(\mathbb{R}^N)}^p\le\lim_{k\to\infty}[u_{n_k}]_{W^{s,p}(\mathbb{R}^N)}^p\le M.$$

By further using Hardy's inequality and (1.16), we also get

$$\int_\Omega\frac{|U|^p}{d_\Omega^{s\,p}}\,dx\le\liminf_{k\to\infty}\int_\Omega\frac{|u_{n_k}|^p}{d_\Omega^{s\,p}}\,dx\le\frac{M}{\mathfrak{h}_{s,p}(\Omega)},$$

and

$$\left(\int_{\mathbb{R}^N} \frac{|U|^p}{(1+|x|)^{N+s\,p}}\,dx\right)^{\frac{1}{p}} \le \liminf_{k\to\infty} \left(\int_{\mathbb{R}^N} \frac{|u_{n_k}|^p}{(1+|x|)^{N+s\,p}}\,dx\right)^{\frac{1}{p}}$$
$$\le C_\Omega \liminf_{k\to\infty} \left[[u_{n_k}]_{W^{s,p}(\mathbb{R}^N)} + \left(\int_\Omega \frac{|u_{n_k}|^p}{d_\Omega^{s\,p}}\,dx\right)^{\frac{1}{p}}\right] \le \widetilde{C}.$$

This shows that

$$U \in \mathcal{X}^{s,p}(\Omega; d_\Omega).$$

We now observe that from the first part of the proof, by uniqueness of the weak limit we must have

$$D^s u = D^s U, \qquad \text{a. e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

By recalling the definition of D^s, this in turn implies that there exists a constant $c \in \mathbb{R}$ such that

$$u = U + c, \qquad \text{a. e. in } \mathbb{R}^N.$$

By using that $\mathcal{X}^{s,p}(\Omega; d_\Omega)$ is a vector space, the function constantly equal to c must belong to $\mathcal{X}^{s,p}(\Omega; d_\Omega)$. In light of Lemma 1.6.4, we get that $c = 0$ and thus the desired conclusion holds. □

CHAPTER 2

SOME FUNCTIONAL INEQUALITIES

In this chapter, our goal is to study the sharp constants appearing in the Morrey inequality and in the Hardy inequality. More precisely, in Corollary 2.1.3, we introduce the sharp Morrey constant, defined as

$$\mathfrak{m}_p(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx \,:\, [u]_{C^{0,\alpha_p}(\overline{\Omega})} = 1 \right\}, \qquad \text{where } \alpha_p = 1 - \frac{N}{p}.$$

and we give two geometric bounds for $\mathfrak{m}_p$ which we apply to study its asymptotic behaviour, as p goes to ∞. Then, as a consequence, we can provide a generalization for the classical Hardy inequality, by introducing the sharp Hardy-type constant

$$\mathfrak{h}_{p,q}(\Omega) = \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\,dx \,:\, \left\| \frac{u}{d_\Omega^{\frac{N}{q}+\frac{p-N}{p}}} \right\|_{L^q(\Omega)} = 1 \right\}, \qquad \text{for } p < q \le \infty.$$

In Theorem 2.2.1, we study some geometric bounds for $\mathfrak{h}_{p,q}$, when $p < q \le \infty$, and its asymptotics when $p \to \infty$, in the cases $q = p$ and $q = \infty$.

§2.1 Morrey–type inequality

In order to study the sharp Morrey constant $\mathfrak{m}_p$, we start by investigating some properties for another Morrey–type constant. This result will allow us to prove some geometric estimates for $\mathfrak{m}_p$ and to determine its asymptotic behaviour for large p.

Lemma 2.1.1. *Let $x_0 \in \mathbb{R}^N$ and let $R > 0$. For every $p > N$ and every fixed $z \in \partial B_R(x_0)$, we set*

$$\mu_p(B_R(x_0);\{z\}) := \min_{\varphi\in W^{1,p}(B_R(x_0))} \left\{ \int_{B_R(x_0)} |\nabla \varphi|^p\,dx \,:\, \varphi(x_0) = 1 \text{ and } \varphi(z) = 0 \right\}.$$

Then:

1. *the minimum above is independent of the point $z \in \partial B_R(x_0)$;*
2. *we have the scaling*

$$\mu_p(B_R(x_0);\{z\}) = R^{N-p}\,\mu_p\left(B_1(x_0);\left\{x_0 + \frac{z-x_0}{R}\right\}\right);$$

3. *the family*

$$\left\{ \left(\frac{1}{\omega_N\,R^N}\,\mu_p(B_R(x_0);\{z\}) \right)^{\frac{1}{p}} \right\}_{p>N}$$

is non-decreasing;

4. *we have the following asymptotics*

$$\lim_{p\to\infty}\Big(\mu_p(B_R(x_0);\{z\})\Big)^{\frac{1}{p}}=\frac{1}{R}. \tag{2.1}$$

Proof. We first show that $\mu_p\left(B_R(x_0);\{z\}\right)$ is actually a minimum. To this aim, we consider a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}\subset W^{1,p}(B_R(x_0))$ for the problem defined by $\mu_p\left(B_R(x_0);\{z\}\right)$, i. e.

$$\lim_{n\to\infty}\int_{B_R(x_0)}|\nabla u_n|^p\,dx=\mu_p\left(B_R(x_0);\{z\}\right),\quad u_n(x_0)=1\quad\text{ and }\quad u_n(z)=0.$$

By Proposition 1.3.9, we also have that

$$\int_{B_R(x_0)}|u_n|^p\,dx\le C_{N,p}\,R^p\int_{B_R(x_0)}|\nabla u_n|^p\,dx.$$

This implies that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W^{1,p}(B_R(x_0))$. Hence, thanks to compact Sobolev embeddings for $p>N$, there exists $u\in W^{1,p}(B_R(x_0))\cap C(\overline{B_R(x_0)})$ such that u_n converges to u weakly in $W^{1,p}(B_R(x_0))$ and uniformly in $\overline{B_R(x_0)}$, up to a subsequence. This implies that $u(x_0)=1$ and $u(z)=0$, thus u is an admissible test function for the problem $\mu_p\left(B_R(x_0);\{z\}\right)$. Moreover, by lower semicontinuity we have

$$\int_{B_R(x_0)}|\nabla u|^p\,dx\le\lim_{n\to\infty}\int_{B_R(x_0)}|\nabla u_n|^p\,dx=\mu_p\left(B_R(x_0);\{z\}\right),$$

i. e. u is a minimizer.

The proofs of *part (1) and part (2)* easily follow by standard arguments, while *part (3)* is a consequence of Hölder's inequality.

It remains to prove *part (4)*. We suppose, without loss of generality, that $x_0=0$ and $R=1$. Thus, we have to show that

$$\lim_{p\to\infty}\Big(\mu_p\left(B_1;\{z\}\right)\Big)^{\frac{1}{p}}=1.$$

By *part (3), we already know that such a limit exists*. Since d_{B_1} is admissible for the problem $\mu_p\left(B_1;\{z\}\right)$, we get that

$$\Big(\mu_p\left(B_1;\{z\}\right)\Big)^{\frac{1}{p}}\le\omega_N^{\frac{1}{p}}.$$

This in turn implies that

$$\lim_{p\to\infty}\Big(\mu_p\left(B_1;\{z\}\right)\Big)^{\frac{1}{p}}\le\lim_{p\to\infty}\omega_N^{\frac{1}{p}}=1. \tag{2.2}$$

We now prove that

$$\lim_{p\to\infty}\Big(\mu_p\left(B_1;\{z\}\right)\Big)^{\frac{1}{p}}\ge 1.$$

At this aim, we consider $U_p\in W^{1,p}(B_1)$ a minimizer for $\mu_p\left(B_1;\{z\}\right)$, i. e.

$$\left(\int_{B_1}|\nabla U_p|^p\,dx\right)^{\frac{1}{p}}=\Big(\mu_p\left(B_1;\{z\}\right)\Big)^{\frac{1}{p}},\qquad U_p(0)=1,\qquad\text{and}\qquad U_p(z)=0.$$

We fix $m_0 > N$, by applying Holder's inequality, we have

$$\left(\int_{B_1} |\nabla U_p|^{m_0}\,dx\right)^{\frac{1}{m_0}} \le \omega_N^{\frac{1}{m_0}-\frac{1}{p}} \left(\int_{B_1} |\nabla U_p|^p\,dx\right)^{\frac{1}{p}},$$

for every $p \ge m_0$. Moreover, thanks again to Proposition 1.3.9, we have

$$\int_{B_1} |U_p|^{m_0}\,dx \le C_{N,m_0} \int_{B_1} |\nabla U_p|^{m_0}\,dx.$$

Taking into account (2.2), the above inequality implies that the family $\{U_p\}_{p\ge m_0}$ is bounded in $W^{1,m_0}(B_1)$, for every fixed $m_0 > N$. Then, there exists $U_\infty \in W^{1,m_0}(B_1) \cap C(\overline{B_1})$ and an increasingly diverging sequence $\{p_k^0\}_{k\in\mathbb{N}} \subset [m_0, +\infty)$ such that $\{U_{p_k^0}\}_{k\in\mathbb{N}}$ converges to U_∞ weakly in $W^{1,m_0}(B_1)$ and uniformly in $\overline{B_1}$.

We can recursively repeat the previous argument, by taking an increasingly diverging sequence $\{m_\ell\}_{\ell\in\mathbb{N}} \subset (N, +\infty)$ and each time extracting a subsequence $\{U_{p_k^\ell}\}_{k\in\mathbb{N}}$ from the previous one $\{U_{p_k^{\ell-1}}\}_{k\in\mathbb{N}}$, with $\{p_k^\ell\}_{k\in\mathbb{N}} \subset [m_\ell, +\infty)$. At each step, we get that $U_\infty \in W^{1,m_\ell}(B_1) \cap C(\overline{B_1})$ and that $\{U_{p_k^\ell}\}_{k\in\mathbb{N}}$ converges to U_∞ weakly in $W^{1,m_\ell}(B_1)$ and uniformly in $\overline{B_1}$. Thanks to the claimed convergence, we get for every $\ell \in \mathbb{N}$

$$\begin{aligned}\left(\int_{B_1} |\nabla U_\infty|^{m_\ell}\,dx\right)^{\frac{1}{m_\ell}} &\le \liminf_{k\to\infty} \left(\int_{B_1} |\nabla U_{p_k^\ell}|^{m_\ell}\,dx\right)^{\frac{1}{m_\ell}} \\ &\le \lim_{k\to\infty} \omega_N^{\frac{1}{m_\ell}-\frac{1}{p_k^\ell}} \left(\int_{B_1} |\nabla U_{p_k^\ell}|^{p_k^\ell}\,dx\right)^{\frac{1}{p_k^\ell}} \\ &= \omega_N^{\frac{1}{m_\ell}} \lim_{k\to\infty} \Big(\mu_{p_k^\ell}(B_1;\{z\})\Big)^{\frac{1}{p_k^\ell}} = \omega_N^{\frac{1}{m_\ell}} \lim_{p\to\infty} \Big(\mu_p(B_1;\{z\})\Big)^{\frac{1}{p}}.\end{aligned}$$

By sending ℓ to ∞ and recalling (2.2), it holds that

$$1 \ge \lim_{p\to\infty} \Big(\mu_p(B_1;\{z\})\Big)^{\frac{1}{p}} \ge \|\nabla U_\infty\|_{L^\infty(B_1)}. \tag{2.3}$$

Hence U_∞ is a $1-$Lipschitz continuous function such that $U_\infty(z) = 0$. Accordingly, we get

$$\|\nabla U_\infty\|_{L^\infty(B_1)} \ge \frac{|U_\infty(0)|}{d_{B_1}(0)} = 1,$$

which, combined with (2.3), gives the conclusion. □

Remark 2.1.2. Thanks to Lemma 2.1.1 *part (1)*, fixed a point $z \in \partial B_1$, we can define

$$\mu_p(B_1) := \min_{u\in W^{1,p}(B_1)} \left\{\int_{B_1} |\nabla u|^p\,dx \,:\, u(0) = 1 \text{ and } u(z) = 0\right\}. \tag{2.4}$$

Moreover, as an easy consequence of Lemma 2.1.1, when $p > N$ we get the following estimate

$$|u(x_0) - u(z)| \le \frac{R^{1-\frac{N}{p}}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla u\|_{L^p(B_R(x_0))}, \qquad \text{for } u \in W^{1,p}(B_R(x_0)) \text{ and } z \in \partial B_R(x_0). \tag{2.5}$$

We are now in position to study the sharp Morrey constant.

Corollary 2.1.3 (Sharp Morrey constant)**.** *Let $N < p < \infty$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. We define the sharp Morrey constant*

$$\mathfrak{m}_p(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p\, dx \,:\, [u]_{C^{0,\alpha_p}(\overline{\Omega})} = 1 \right\}, \qquad \textit{where } \alpha_p := 1 - \frac{N}{p}.$$

Then the constant $\mathfrak{m}_p(\Omega)$ is independent of Ω, i.e. we have

$$\mathfrak{m}_p(\Omega) = \mathfrak{m}_p(\mathbb{R}^N).$$

Moreover, we have

$$\mu_p(B_1) \le \mathfrak{m}_p(\mathbb{R}^N) \le N\,\omega_N \left(\frac{p-N}{p-1}\right)^{p-1}, \tag{2.6}$$

and

$$\lim_{p\to\infty} \Big(\mathfrak{m}_p(\mathbb{R}^N)\Big)^{\frac{1}{p}} = 1.$$

Proof. We first show that $\mathfrak{m}_p(\Omega)$ is independent of Ω. The fact that $\mathfrak{m}_p(\mathbb{R}^N) \le \mathfrak{m}_p(\Omega)$ follows by monotonicity with respect to sets inclusion and the fact that

$$[u]_{C^{0,\alpha_p}(\overline{\Omega})} = [u]_{C^{0,\alpha_p}(\mathbb{R}^N)}, \qquad \text{for every } u \in C_0^\infty(\Omega),$$

see (1.1).

In order to show that $\mathfrak{m}_p(\Omega) \le \mathfrak{m}_p(\mathbb{R}^N)$, let $u \in C_0^\infty(\mathbb{R}^N)$ and let

$$u_r(x) = u\left(\frac{x-x_0}{r}\right), \qquad \text{with } x_0 \in \mathbb{R}^N \text{ and } r > 0.$$

Since u has compact support, we have $u_r \in C_0^\infty(\Omega)$ for some suitable x_0 and r small enough. Then, by scaling and thanks to (1.1), it holds

$$\mathfrak{m}_p(\Omega) \le \frac{\displaystyle\int_\Omega |\nabla u_r|^p\, dx}{[u_r]^p_{C^{0,\alpha_p}(\overline{\Omega})}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_r|^p\, dx}{[u_r]^p_{C^{0,\alpha_p}(\mathbb{R}^N)}} = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^p\, dx}{[u]^p_{C^{0,\alpha_p}(\mathbb{R}^N)}}.$$

By taking the infimum on $C_0^\infty(\mathbb{R}^N)$ on the right-hand side, we get the claimed inequality.

We now come to the proof of (2.6). Let $u \in C_0^\infty(\mathbb{R}^N)$, for the lower bound it is sufficient to prove that

$$|u(x) - u(y)| \le \frac{1}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla u\|_{L^p(\Omega)}\, |x-y|^{\alpha_p}, \qquad \text{for every } x, y \in \mathbb{R}^N. \tag{2.7}$$

If $x = y$, then (2.7) trivially holds, thus let us assume that $x \ne y$. Without loss of generality, we assume $u(x) > u(y)$ and we define

$$v(z) := \frac{u(z) - u(y)}{u(x) - u(y)}, \qquad \text{for } z \in \mathbb{R}^N,$$

which satisfies $v(x) = 1$ and $v(y) = 0$. Since $v \in W^{1,p}(B_R(x))$ with $R = |x-y|$, we have from (2.5) that

$$1 = |v(x)| \leq \frac{|x-y|^{\alpha_p}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla v\|_{L^p(B_R(x))} = \frac{|x-y|^{\alpha_p}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \frac{1}{|u(x)-u(y)|} \|\nabla u\|_{L^p(B_R(x))}.$$

From this estimate, we get

$$|u(x)-u(y)| \leq \frac{|x-y|^{\alpha_p}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla u\|_{L^p(B_R(x))} \leq \frac{|x-y|^{\alpha_p}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla u\|_{L^p(\mathbb{R}^N)},$$

which is the claimed inequality (2.7).

As for the upper bound, for every $u \in C_0^\infty(B_1) \setminus \{0\}$, by the first part of the proof and the very definition of $\mathfrak{m}_p$, we have

$$\mathfrak{m}_p(\mathbb{R}^N) = \mathfrak{m}_p(B_1) \leq \frac{\|\nabla u\|^p_{L^p(B_1)}}{[u]^p_{C^{0,\alpha_p}(B_1)}}.$$

Moreover, by using that u is compactly supported in B_1, we have

$$[u]_{C^{0,\alpha_p}(B_1)} \geq \sup_{x\in B_1, y\in\partial B_1} \frac{|u(x)|}{|x-y|^{\alpha_p}} \geq |u(0)|.$$

Thus, for every $u \in C_0^\infty(B_1)$ such that $|u(0)| \neq 0$, we get

$$\mathfrak{m}_p(\mathbb{R}^N) \leq \frac{\|\nabla u\|^p_{L^p(B_1)}}{|u(0)|^p}.$$

By density, the last estimate is still true for functions $u \in W_0^{1,p}(B_1)$. Now we consider the function

$$u(x) = (1 - |x|^{\frac{p-N}{p-1}}) \in W_0^{1,p}(B_1),$$

hence there exists $u_n \in C_0^\infty(B_1)$ such that u_n converges to u in $W_0^{1,p}(B_1)$. Since

$$[u]_{C^{0,\alpha_p}(\overline{B_1})} = \sup_{x\neq y, x,y\in\overline{B_1}} \frac{\Big||x|^{\frac{p-N}{p-1}} - |y|^{\frac{p-N}{p-1}}\Big|}{|x-y|^{\alpha_p}} \geq 1.$$

Thus, it holds that

$$\Big(\mathfrak{m}_p(\mathbb{R}^N)\Big)^{\frac{1}{p}} \leq \frac{\|\nabla u_n\|_{L^p(B_1)}}{|u(0)|} \leq \|\nabla u\|_{L^p(\mathbb{R}^N)} = (N\,\omega_N)^{\frac{1}{p}} \left(\frac{p-N}{p-1}\right)^{\frac{p-1}{p}}.$$

This shows the claimed upper bound.

Finally, by taking the $p-$rooth in (2.6) and using (2.1), we get the desired asymptotics for $\mathfrak{m}_p$. □

§2.2 Hardy–type inequality

We now apply the results of the previous section to study a generalization of the sharp Hardy constant. The following Hardy inequality for general open sets has been originally proved for $q = p$ in [**96**] (see also [**72**] and [**130**]), without determining an explicit constant. The latter can be found in [**8**, **41**, **71**]. We generalize these results to cover the case $p \leq q \leq \infty$. In particular, we will focus on the asymptotic behaviour of the sharp constant, as p goes to ∞.

Theorem 2.2.1 (Hardy's inequality). *Let $N < p < \infty$ and $p \leq q \leq \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. We define*

$$\mathfrak{h}_{p,q}(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx \, : \, \left\| \frac{u}{d_\Omega^{\frac{N}{q}+\frac{p-N}{p}}} \right\|_{L^q(\Omega)} = 1 \right\}, \qquad \textit{for } p < q \leq \infty,$$

and

$$\mathfrak{h}_p(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx \, : \, \left\| \frac{u}{d_\Omega} \right\|_{L^p(\Omega)} = 1 \right\}.$$

We have that

$$\mathfrak{h}_{p,q}(\Omega) \geq \Big(\mathfrak{h}_p(\Omega) \Big)^{\frac{p}{q}} \Big(\mathfrak{h}_{p,\infty}(\Omega) \Big)^{\frac{q-p}{q}}, \qquad \textit{for } p < q < \infty, \tag{2.8}$$

and

$$\mathfrak{h}_p(\Omega) \geq \left(\frac{p-N}{p} \right)^p, \qquad \mathfrak{h}_{p,\infty}(\Omega) \geq \mu_p(B_1), \tag{2.9}$$

where $\mu_p(B_1)$ is the same constant as in (2.4). Moreover, it holds

$$\lim_{p \to \infty} \Big(\mathfrak{h}_p(\Omega) \Big)^{\frac{1}{p}} = \lim_{p \to \infty} \Big(\mathfrak{h}_{p,\infty}(\Omega) \Big)^{\frac{1}{p}} = 1. \tag{2.10}$$

Proof. We first prove the lower bound in the extremal case, i.e. for $q = \infty$. Let $x \in \Omega$ and let $\overline{x} \in \partial\Omega$ be such that

$$|x - \overline{x}| = d_\Omega(x).$$

For every $u \in C_0^\infty(\Omega)$ and every $p > N$, we thus get from (2.5)

$$|u(x)| \leq \frac{d_\Omega(x)^{1-\frac{N}{p}}}{(\mu_p(B_1))^{\frac{1}{p}}} \left(\int_{B_{d_\Omega(x)}(x)} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

By taking the supremum over Ω, we get

$$\left\| \frac{u}{d_\Omega^{1-\frac{N}{p}}} \right\|_{L^\infty(\Omega)} \leq \frac{1}{(\mu_p(B_1))^{\frac{1}{p}}} \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

This gives the desired Hardy inequality result for $q = \infty$, together with the claimed lower bound in (2.9). In the case $p = q$, the estimate in (2.9) comes from [**8**, **41**], as already recalled.

The case $p < q < \infty$ now simply follows from interpolation of the two endpoints. Indeed, for every $u \in C_0^\infty(\Omega)$, we have

$$\left(\int_\Omega \frac{|u|^q}{d_\Omega^{\gamma\,q}}\,dx\right)^{\frac{p}{q}} \leq \left(\int_\Omega \frac{|u|^p}{d_\Omega^p}\,dx\right)^{\frac{p}{q}} \left\| \frac{u}{d_\Omega^{\frac{\gamma\,q-p}{q-p}}} \right\|_{L^\infty(\Omega)}^{(q-p)\,\frac{p}{q}}$$

where we set for simplicity

$$\gamma = \frac{N}{q} + \frac{p-N}{p}.$$

We observe that

$$\frac{\gamma\,q-p}{q-p} = 1 - \frac{N}{p},$$

thus by using the definitions of $\mathfrak{h}_p(\Omega)$ and $\mathfrak{h}_{p,\infty}(\Omega)$, we obtain

$$\left(\int_\Omega \frac{|u|^q}{d_\Omega^{\gamma\,q}}\,dx\right)^{\frac{p}{q}} \leq \left(\frac{1}{\mathfrak{h}_p(\Omega)}\right)^{\frac{p}{q}} \left(\frac{1}{\mathfrak{h}_{p,\infty}(\Omega)}\right)^{\frac{q-p}{q}} \int_\Omega |\nabla u|^p\,dx.$$

By taking the infimum over $u \in C_0^\infty(\Omega)$, we get the lower bound (2.8).

In order to prove the last statement, for the case $q = \infty$, from (2.9) and Lemma 2.1.1 we have

$$\liminf_{p\to\infty} \Big(\mathfrak{h}_{p,\infty}(\Omega)\Big)^{\frac{1}{p}} \geq \lim_{p\to\infty} \Big(\mu_p(B_1)\Big)^{\frac{1}{p}} = 1.$$

In the case $p = q$, we directly have

$$\liminf_{p\to\infty} \Big(\mathfrak{h}_p(\Omega)\Big)^{\frac{1}{p}} \geq \lim_{p\to\infty} \frac{p-N}{p} = 1.$$

In order to prove that the $\limsup$ is smaller than or equal to 1, it is sufficient to use a suitable test function. We first observe that, by a standard density argument, for every $p > N$ and $p \leq q \leq \infty$, the Hardy inequality

$$\mathfrak{h}_{p,q}(\Omega) \left\| \frac{\varphi}{d_\Omega^{\frac{N}{q}+\frac{p-N}{p}}} \right\|_{L^q(\Omega)}^p \leq \int_\Omega |\nabla\varphi|^p\,dx,$$

still holds in both spaces $\mathcal{D}_0^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$, for every open set $\Omega \subsetneq \mathbb{R}^N$. Then, for every $x_0 \in \Omega$, we take

$$\varphi(x) = \big(r - |x-x_0|\big)_+ \in W_0^{1,p}(\Omega), \qquad \text{for } r = d_\Omega(x_0).$$

Thus we get

$$\Big(\mathfrak{h}_{p,\infty}(\Omega)\Big)^{\frac{1}{p}} \leq \frac{(\omega_N\,r^N)^{\frac{1}{p}}}{\left\| \dfrac{\varphi}{d_\Omega^{\frac{p-N}{p}}} \right\|_{L^\infty(\Omega)}}, \qquad \text{and} \qquad \Big(\mathfrak{h}_p(\Omega)\Big)^{\frac{1}{p}} \leq \frac{(\omega_N\,r^N)^{\frac{1}{p}}}{\left\| \dfrac{\varphi}{d_\Omega} \right\|_{L^p(\Omega)}}.$$

By using that

$$\lim_{p\to\infty} \frac{(\omega_N\, r^N)^{\frac{1}{p}}}{\left\| \dfrac{\varphi}{d_\Omega^{\frac{p-N}{p}}} \right\|_{L^\infty(\Omega)}} = \lim_{p\to\infty} \frac{(\omega_N\, r^N)^{\frac{1}{p}}}{\left\| \dfrac{\varphi}{d_\Omega} \right\|_{L^p(\Omega)}} = \inf_{x\in B_r(x_0)} \frac{d_\Omega(x)}{(r-|x-x_0|)_+} \le \frac{d_\Omega(x_0)}{r} = 1,$$

we then obtain the desired conclusion. □

Remark 2.2.2. By a standard density argument, for every $p > N$ and $p \le q \le \infty$ the Hardy inequality

$$\mathfrak{h}_{p,q}(\Omega) \left\| \frac{u}{d_\Omega^{\frac{N}{q}+\frac{p-N}{p}}} \right\|_{L^q(\Omega)}^p \le \int_\Omega |\nabla u|^p\, dx,$$

still holds in both spaces $\mathcal{D}^{1,p}_0(\Omega)$ and $W^{1,p}_0(\Omega)$, for every open set $\Omega \subsetneq \mathbb{R}^N$.

CHAPTER 3

COMPARISON PRINCIPLE FOR THE LANE-EMDEN EQUATION

§3.1 Weak solutions to the Lane-Emden equation

Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set, we introduce the *Lane-Emden equation*

$$-\Delta_p u = \alpha\, |u|^{q-2}\, u, \qquad \text{in } \Omega, \tag{3.1}$$

where $\alpha > 0$ is a given constant and $1 \leq q < p^*$, with

$$p^* = \begin{cases} N\, p/(N-p), & \text{if } p < N, \\ +\infty, & \text{if } p \geq N. \end{cases}$$

In this chapter, we will focus on the *sub-homogeneous case*, i.e. we will always consider the case

$$1 \leq q < p.$$

By using the same notation of Section 1.4, we recall the following definition

Definition 3.1.1. Let $1 < q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set. We say that a function $v \in X^{q,p}(\Omega)$ is a:

- *weak supersolution of* (3.1) if

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \psi \rangle\, dx \geq \alpha \int_\Omega |v|^{q-2}\, v\, \psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega),\ \psi \geq 0;$$

- *weak subsolution of* (3.1) if

$$\int_\Omega \langle |\nabla v|^{p-2}\, \nabla v, \nabla \psi \rangle\, dx \leq \alpha \int_\Omega |v|^{q-2}\, v\, \psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega),\ \psi \geq 0;$$

- *weak solution of* (3.1) if

$$\int_\Omega \langle |\nabla v|^{p-2}\, \nabla v, \nabla \psi \rangle\, dx = \alpha \int_\Omega |v|^{q-2}\, v\, \psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega). \tag{3.2}$$

In the above inequalities, we can also admit test functions in $X_0^{q,p}(\Omega)$, by a standard density argument. In the case $q = 1$, for a non-negative function v we extend the previous definitions, by using the convention

$$|v|^{q-2}\, v = v^{q-1} = 1.$$

In the next Lemma, we collect some standard properties of solutions of Lane-Emden equation.

Proposition 3.1.2 (Properties of solutions). *Let $\Omega \subset \mathbb{R}^N$ be an open set, $u \in X^{q,p}(\Omega)$ be a weak solution of (3.1), then*

- *if $t > 0$, the rescaled function*
$$u_t(x) = t^{\frac{p}{p-q}} u\Big(\frac{x - x_0}{t}\Big)$$
is a weak solution of (3.1) in $x_0 + t\,\Omega$;
- *if $\beta > 0$, the function*
$$v = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-q}} u,$$
is a weak solution of
$$-\Delta_p v = \beta\, |v|^{q-2}\, v, \qquad in\ \Omega.$$

The same claims apply to subsolutions and supersolutions.

Then, the main tool we will prove in this chapter is a comparison principle for the Lane-Emden equation, whose proof is based on the *hidden convexity* property for the $p-$Dirichlet integral. This will imply that the *energy functional* naturally associated to (3.1), given by

$$\mathfrak{F}^{\alpha}_{p,q}(\psi) := \frac{1}{p}\int_\Omega |\nabla \psi|^p\, dx - \frac{\alpha}{q}\int_\Omega |\psi|^q\, dx, \qquad \text{for every } \psi \in C^\infty_0(\Omega),$$

would be convex in a suitable sense, hence, we can adapt to our equation the standard proofs for the classical comparison principles. Finally, we remark that in this chapter, we will not make any regularity assumption on the boundary of the open set Ω and this will allow us to work on a large class of sets, possibly unbounded or even with infinite volume. More precisely, we usually ask to the open set Ω to be $(p, q)-$*admissible*, which implies that the sharp Sobolev-Poincaré constants $\lambda_{p,q}(\Omega)$ are positive (see Definition 1.3.1).

§3.2 Hidden convexity

We will prove the following *hidden convexity property* of the $p-$Dirichlet integral. This is nowadays well-known among experts, we give here a general statement under minimal assumptions both on the open set and on the functions. We also identify the equality cases, by means of an enhanced version of the inequality (see equation (3.9) below).

Theorem 3.2.1 (Hidden convexity). *Let $1 \le q < \infty$, $1 < p < \infty$ and $1 \le r \le \min\{p, q\}$. Let $\Omega \subset \mathbb{R}^N$ be an open set, for every pair of non-negative functions $v, w \in X^{q,p}(\Omega)$, we set*

$$\sigma^t := ((1-t)\, v^r + t\, w^r)^{\frac{1}{r}}, \qquad \textit{for every } t \in [0, 1]. \tag{3.3}$$

Then $\sigma^t \in X^{q,p}(\Omega)$ and

$$\int_\Omega |\nabla \sigma^t|^p\, dx \le (1-t)\int_\Omega |\nabla v|^p\, dx + t\int_\Omega |\nabla w|^p\, dx \quad \textit{for every } t \in [0, 1]. \tag{3.4}$$

Moreover, when Ω is connected:

- *if $\boxed{1 = r < p}$, the equality for some $t \in (0, 1)$ holds in (3.4) if and only if $v = w + C$, with $C \in \mathbb{R}$;*

- *if* $\boxed{1 < r < p}$, *the equality for some* $t \in (0,1)$ *holds in* (3.4) *if and only if*

$$\textit{either} \qquad v = w \qquad \textit{or} \qquad w \textit{ and } v \textit{ are both constant};$$

- *if* $\boxed{1 < r = p}$ *and in addition we have*

$$\frac{1}{v} \in L^\infty_{\rm loc}(\Omega) \qquad \textit{and} \qquad \frac{1}{w} \in L^\infty_{\rm loc}(\Omega), \tag{3.5}$$

  the equality for some $t \in (0,1)$ *holds in* (3.4) *if and only if* $v = C\,w$*, with* $C > 0$.

Proof. For $r = 1$ there is nothing to prove, in this case σ^t is just the usual convex combination of v and w. Then (3.4) is just the usual convexity of the $p-$Dirichlet integral. As for the equality cases, from the strict convexity of the map $z \mapsto |z|^p$, we get that if (3.4) holds as an identity for some $t \in (0,1)$, then we must have

$$\nabla v = \nabla w \qquad \text{a. e. in } \Omega.$$

If Ω is connected, this implies that $v - w$ must be constant in Ω.

We now focus on the case $1 < r \le \min\{p,q\}$. We will first show that σ^t belongs to $X^{q,p}(\Omega)$ and satisfies (3.4). Then we will focus on the equality cases in the latter, under the claimed assumptions.

Part 1: properties of σ^t. It is easy to prove that $\sigma^t \in L^q(\Omega)$. Indeed, observe that

$$\int_\Omega |\sigma^t|^q\,dx = \int_\Omega \left((1-t)\,v^r + t\,w^r\right)^{\frac{q}{r}}\,dx \le (1-t)\int_\Omega v^q\,dx + t\int_\Omega w^q\,dx < +\infty,$$

thanks to the convexity of the map $\tau \mapsto \tau^{q/r}$. For $\varepsilon > 0$ and $t \in (0,1)$, we consider the C^1 function $G_{\varepsilon,t}$ defined on $[0,+\infty) \times [0,+\infty)$ by

$$G_{\varepsilon,t}(s_1,s_2) := \left((1-t)\,(s_1+\varepsilon)^r + t\,(s_2+\varepsilon)^r\right)^{\frac{1}{r}} - \varepsilon.$$

Then, we take the vector field

$$\Phi(x) = (v(x), w(x)) \in X^{q,p}(\Omega;\mathbb{R}^2).$$

Since

$$\nabla G_{\varepsilon,t}(s_1,s_2) = \left(\frac{(1-t)\,(s_1+\varepsilon)^{r-1}}{\left((1-t)\,(s_1+\varepsilon)^r + t\,(s_2+\varepsilon)^r\right)^{\frac{r-1}{r}}}, \frac{t\,(s_2+\varepsilon)^{r-1}}{\left((1-t)\,(s_1+\varepsilon)^r + t\,(s_2+\varepsilon)^r\right)^{\frac{r-1}{r}}}\right),$$

it is easily seen that $G_{\varepsilon,t}$ has a bounded gradient. Thanks to the Chain Rule in Sobolev spaces (see for example [**97**, Theorem 6.16]), we have that

$$\sigma^t_\varepsilon = G_{\varepsilon,t} \circ \Phi \in X^{q,p}(\Omega).$$

Observe that we also used that $G_{\varepsilon,t}(0,0) = 0$, to guarantee that σ^t_ε has the required summability, when Ω has infinite volume.

We remark that for every $s_1, s_2 \ge 0$ and $t \in (0,1)$, the quantity $\varepsilon \mapsto G_{\varepsilon,t}(s_1,s_2)$ is monotone decreasing. Indeed, observe that

$$\frac{d}{d\varepsilon}G_{\varepsilon,t}(s_1,s_2) = \frac{(1-t)\,(s_1+\varepsilon)^{r-1} + t\,(s_2+\varepsilon)^{r-1}}{\left((1-t)\,(s_1+\varepsilon)^r + t\,(s_2+\varepsilon)^r\right)^{\frac{r-1}{r}}} - 1 \le 0,$$

thanks to the concavity of the map $\tau \mapsto \tau^{(r-1)/r}$. Thus in particular we have

$$\sigma^t_{\varepsilon_2} \leq \sigma^t_{\varepsilon_1} \leq \sigma^t, \qquad \text{for } 0 < \varepsilon_1 < \varepsilon_2.$$

By using the Monotone Convergence Theorem and the fact that $\sigma^t \in L^q(\Omega)$, we thus obtain

$$\lim_{\varepsilon\to 0} \int_\Omega |\sigma^t_\varepsilon - \sigma^t|^q\, dx = \lim_{\varepsilon\to 0} \int_\Omega (\sigma^t - \sigma^t_\varepsilon)^q\, dx = 0.$$

We now compute the gradient of σ^t_ε

$$\begin{aligned} \nabla \sigma^t_\varepsilon &= \frac{(1-t)\,(v+\varepsilon)^{r-1}}{((1-t)\,(v+\varepsilon)^r + t\,(w+\varepsilon)^r)^{\frac{r-1}{r}}}\,\nabla v + \frac{t\,(w+\varepsilon)^{r-1}}{((1-t)\,(v+\varepsilon)^r + t\,(w+\varepsilon)^r)^{\frac{r-1}{r}}}\,\nabla w \\ &= (G_{\varepsilon,t}(v,w)+\varepsilon)\left(\frac{(1-t)\,(v+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\,\frac{\nabla v}{v+\varepsilon} + \frac{t\,(w+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\,\frac{\nabla w}{w+\varepsilon}\right). \end{aligned} \tag{3.6}$$

We now observe that

$$\frac{(1-t)\,(v+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r} \qquad \text{and} \qquad \frac{t\,(w+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r},$$

are both non-negative, less than or equal to 1 and their sum gives 1, thanks to the definition of $G_{\varepsilon,t}$. Thus they can be regarded as the coefficients of a convex combination. By using the convexity of the map $z \mapsto |z|^r$, we thus obtain

$$|\nabla \sigma^t_\varepsilon|^r \leq (G_{\varepsilon,t}(v,w)+\varepsilon)^r \left(\frac{(1-t)\,(v+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\left|\frac{\nabla v}{v+\varepsilon}\right|^r + \frac{t\,(w+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\left|\frac{\nabla w}{w+\varepsilon}\right|^r\right).$$

After some simplifications the previous estimate can be rewritten as

$$|\nabla \sigma^t_\varepsilon|^r \leq (1-t)\,|\nabla v|^r + t\,|\nabla w|^r. \tag{3.7}$$

We now raise to the power p/r, integrate over Ω and use the convexity of the map $\tau \mapsto \tau^{p/r}$. We finally obtain

$$\int_\Omega |\nabla \sigma^t_\varepsilon|^p\, dx \leq (1-t) \int_\Omega |\nabla v|^p\, dx + t \int_\Omega |\nabla w|^p\, dx. \tag{3.8}$$

This shows that $\{\nabla \sigma^t_\varepsilon\}_{\varepsilon>0}$ is a bounded subset of $L^p(\Omega;\mathbb{R}^N)$, for every $t \in (0,1)$. Thus, there exists an infinitesimal sequence $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ such that $\nabla \sigma^t_{\varepsilon_n}$ weakly converges to a vector field $\phi^t \in L^p(\Omega;\mathbb{R}^N)$. By using the definition of weak gradient and the strong convergence in L^q previously inferred, it is standard to show that we must have $\phi^t = \nabla \sigma^t$. This proves that

$$\sigma^t \in X^{q,p}(\Omega),$$

as claimed. Moreover, by taking the limit in (3.8) and using the lower semicontinuity of the L^p norm with respect to the weak convergence, we finally establish (3.4), at once.

Part 2: enhanced hidden convexity. We go back for a moment to (3.6). Then, rather than simply using the convexity of $z \mapsto |z|^r$, we use a "quantified" version of such a property. This is expressed by the inequalities of Lemma A.1.1 and Lemma A.1.3. Thus we now obtain:

- if $r \geq 2$, in place of (3.7) we get

$$|\nabla \sigma_\varepsilon^t|^r + C\, \frac{t\,(1-t)\,(w+\varepsilon)^r\,(v+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\,\left|\frac{\nabla v}{v+\varepsilon} - \frac{\nabla w}{w+\varepsilon}\right|^r \leq (1-t)\,|\nabla v|^r + t\,|\nabla w|^r,$$

which can be further simplified into

$$|\nabla \sigma_\varepsilon^t|^r + C\,t\,(1-t)\,\frac{|(w+\varepsilon)\,\nabla v - (v+\varepsilon)\,\nabla w|^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r} \leq (1-t)\,|\nabla v|^r + t\,|\nabla w|^r;$$

- if $1 < r < 2$, in place of (3.7) we get

$$\begin{aligned}|\nabla \sigma_\varepsilon^t|^r &+ C\, \frac{t\,(1-t)\,(w+\varepsilon)^r\,(v+\varepsilon)^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\,\left(\left|\frac{\nabla v}{v+\varepsilon}\right|^2 + \left|\frac{\nabla w}{w+\varepsilon}\right|^2\right)^{\frac{r-2}{2}}\,\left|\frac{\nabla v}{v+\varepsilon} - \frac{\nabla w}{w+\varepsilon}\right|^2\\ &\leq (1-t)\,|\nabla v|^r + t\,|\nabla w|^r,\end{aligned}$$

which can be further simplified into

$$\begin{aligned}|\nabla \sigma_\varepsilon^t|^r &+ C\,t\,(1-t)\,\frac{\left(|(w+\varepsilon)\,\nabla v|^2 + |(v+\varepsilon)\,\nabla w|^2\right)^{\frac{r-2}{2}}}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\,|(w+\varepsilon)\,\nabla v - (v+\varepsilon)\,\nabla w|^2\\ &\leq (1-t)\,|\nabla v|^r + t\,|\nabla w|^r.\end{aligned}$$

By recalling the definition of $G_{\varepsilon,t}$, it is not difficult to see that for almost every $x \in \Omega$, we have [1]

- if $r \geq 2$

$$\lim_{\varepsilon\to 0^+} \frac{|(w+\varepsilon)\,\nabla v - (v+\varepsilon)\,\nabla w|^r}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r} = \mathcal{R}_r(v,w) := \begin{cases} \dfrac{|w\,\nabla v - v\,\nabla w|^r}{(1-t)\,v^r + t\,w^r}, & \text{if } w(x)+v(x) > 0,\\[2ex] 0, & \text{if } w(x)=v(x)=0;\end{cases}$$

- while for $1 < r < 2$

$$\begin{aligned}\lim_{\varepsilon\to 0^+} &\frac{\left(|(w+\varepsilon)\,\nabla v|^2 + |(v+\varepsilon)\,\nabla w|^2\right)^{\frac{r-2}{2}}\,|(w+\varepsilon)\,\nabla v - (v+\varepsilon)\,\nabla w|^2}{(G_{\varepsilon,t}(v,w)+\varepsilon)^r}\\ &= \mathcal{R}_r(v,w) := \begin{cases} \dfrac{\left(|w\,\nabla v|^2 + |v\,\nabla w|^2\right)^{\frac{r-2}{2}}\,|w\,\nabla v - v\,\nabla w|^2}{(1-t)\,v^r + t\,w^r}, & \text{if } w(x)+v(x) > 0,\\[2ex] 0, & \text{if } w(x)=v(x)=0.\end{cases}\end{aligned}$$

As before, we now raise to the power p/r the pointwise estimates above, integrate over Ω and use the convexity and super-additivity of the map $\tau \mapsto \tau^{p/r}$. A further limit as ε goes to 0, in conjuction with

[1] We use that for a function $v \in W^{1,1}_{\rm loc}(\Omega)$ we have

$$\nabla v = 0, \qquad \text{a.e. in } \{x\in\Omega\,:\, v(x)=0\},$$

see for example [**97**, Theorem 6.19].

Fatou's Lemma, finally gives

$$\begin{aligned}\int_\Omega |\nabla \sigma^t|^p\,dx + C\,\left(t\,(1-t)\right)^{\frac{p}{r}} \int_\Omega \left(\mathcal{R}_r(v,w)\right)^{\frac{p}{r}}\,dx \\ \le \int_\Omega \Big((1-t)\,|\nabla v|^r + t\,|\nabla w|^r\Big)^{\frac{p}{r}}\,dx \\ \le (1-t)\int_\Omega |\nabla v|^p\,dx + t\int_\Omega |\nabla w|^p\,dx.\end{aligned} \tag{3.9}$$

Part 3: equality cases for $r < p$. We now suppose that Ω is connected. If equality holds in (3.4) for some $t \in (0,1)$, in particular we must have equality in (3.9), as well. Thus we have that

$$\int_\Omega \Big((1-t)\,|\nabla v|^r + t\,|\nabla w|^r\Big)^{\frac{p}{r}}\,dx = (1-t)\int_\Omega |\nabla v|^p\,dx + t\int_\Omega |\nabla w|^p\,dx,$$

and

$$\int_\Omega \left(\mathcal{R}_r(v,w)\right)^{\frac{p}{r}}\,dx = 0.$$

The first fact implies that

$$|\nabla v| = |\nabla w|, \qquad \text{a.e. in } \Omega, \tag{3.10}$$

thanks to the strict convexity of $\tau \mapsto \tau^{p/r}$. The second fact implies that

$$w\,\nabla v = v\,\nabla w, \qquad \text{a.e. in } \Omega, \tag{3.11}$$

thanks to the definition of $\mathcal{R}_r(v,w)$. We claim that (3.10) and (3.11) imply that $\nabla w = \nabla v$ almost everywhere in Ω. Indeed, let us call

$$E = \Big\{x \in \Omega\,:\, \nabla w(x) \neq \nabla v(x)\Big\},$$

and let us suppose that $|E| > 0$. By recalling that w and v are non-negative, from (3.11) we get in particular

$$w\,|\nabla v| = v\,|\nabla w|, \qquad \text{a.e. in } E.$$

By recalling (3.10) and the definition of E, the last identity in turn implies that

$$w = v, \qquad \text{a.e. in } E.$$

On the other hand, we know that (see again [**97**, Theorem 6.19])

$$\nabla w = \nabla v, \qquad \text{a.e. in } \Big\{x \in \Omega\,:\, w(x) = v(x)\Big\}.$$

The last properties in particular give that

$$\nabla w = \nabla v, \qquad \text{a.e. in } E.$$

By recalling the definition of E, this is a contradiction. This gives that

$$\nabla(w - v) = 0, \qquad \text{a.e. in } \Omega,$$

and thus $w - v = C$ in Ω for some constant C, since Ω is connected. We can spend this information in (3.11), so to get

$$(v + C)\,\nabla v = v\,\nabla v, \qquad \text{a. e. in } \Omega,$$

that is $C\,\nabla v$ vanishes almost everywhere in Ω. This implies that either $C = 0$ (and thus $w = v$) or v is constant in Ω (and the same is true for w).

Part 4: equality cases for $r = p$. We suppose again that Ω is connected and assume the stronger condition (3.5) on v and w. If equality holds in (3.4), from (3.9) we get in particular that

$$\int_\Omega \mathcal{R}_p(v, w)\,dx = 0.$$

This gives again (3.11), as above. The assumption on v and w entails that for every open set Ω' compactly contained in Ω, there exists $c_{\Omega'} > 0$ such that $v, w \geq c_{\Omega'}$ almost everywhere in Ω'. This permits to infer that

$$\log v \in W^{1,1}_{\rm loc}(\Omega) \qquad \text{and} \qquad \log w \in W^{1,1}_{\rm loc}(\Omega),$$

and the chain rule formula holds for their distributional gradients. We then obtain

$$\nabla(\log v - \log w) = \frac{\nabla v}{v} - \frac{\nabla w}{w} = 0, \qquad \text{a. e. in } \Omega,$$

thanks to (3.11). Since Ω is an open connected set, the difference $\log v - \log w$ must be constant almost everywhere in Ω. This in turn permits to conclude that $v = C\,w$ almost everywhere in Ω, for some $C > 0$. The proof is now concluded. □

Actually, if the curve $t \mapsto \sigma^t$ of the previous result is built from two Sobolev functions sharing the same boundary datum, the same remains true for the curve itself. This is the content of the next result.

Proposition 3.2.2. *Let $1 \leq q < \infty$, $1 < p < \infty$ and $1 \leq r \leq \min\{p, q\}$. Let $\Omega \subset \mathbb{R}^N$ be an open set, for every pair of non-negative functions $v, w \in X^{q,p}(\Omega)$, we still denote by σ^t the curve of functions defined by* (3.3)*. If there exists $U \in X^{q,p}(\Omega)$ such that*

$$v - U \in X^{q,p}_0(\Omega) \qquad \textit{and} \qquad w - U \in X^{q,p}_0(\Omega),$$

then

$$\sigma^t - U \in X^{q,p}_0(\Omega).$$

Proof. By assumption, there exist two sequences $\{\psi_n\}_{n\in\mathbb{N}}, \{\varphi_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\Omega)$ such that

$$\lim_{n\to\infty} \|\psi_n - (v - U)\|_{X^{q,p}(\Omega)} = 0,$$

and

$$\lim_{n\to\infty} \|\varphi_n - (w - U)\|_{X^{q,p}(\Omega)} = 0.$$

Then we set

$$v_n := \psi_n + U \in X^{q,p}(\Omega) \qquad \text{and} \qquad w_n := \varphi_n + U \in X^{q,p}(\Omega),$$

and observe that the first one converges to v, while the second one to w. For every $t \in [0, 1]$, thanks to Theorem 3.2.1, we know that

$$\sigma^t_n := ((1 - t)\,v_n^r + t\,w_n^r)^{\frac{1}{r}} \in X^{q,p}(\Omega).$$

Moreover, since by construction $\sigma_n^t - U$ has compact support in Ω, we have that $\sigma_n^t - U \in X_0^{q,p}(\Omega)$ (it is sufficient to adapt the proof of [[**32**], Lemma 9.5] to our Sobolev space). In order to conclude the proof, it is sufficient to show that

$$\lim_{n\to\infty} \|\sigma_n^t - \sigma^t\|_{L^q(\Omega)} = 0, \qquad \text{for every } t \in (0,1), \tag{3.12}$$

and

$$\sup_{n\in\mathbb{N}} \|\nabla\sigma_n^t\|_{L^p(\Omega;\mathbb{R}^N)} < +\infty, \qquad \text{for every } t \in (0,1). \tag{3.13}$$

Indeed, thanks to the reflexivity of $L^p(\Omega;\mathbb{R}^N)$, from (3.13) we would get that $\nabla\sigma_n^t - \nabla U$ weakly converges to some $\phi^t \in L^p(\Omega;\mathbb{R}^N)$, up to a subsequence. Then (3.12) and the definition of weak gradient would permit to show that $\phi^t = \nabla\sigma^t - \nabla U$. Thus $\sigma^t - U$ would coincide with the weak limit of a sequence in $X_0^{q,p}(\Omega)$. As in the proof of Lemma 1.4.1, we can appeal to Mazur's Lemma to show that $\sigma^t - U$ is also a strong limit of a sequence in $X_0^{q,p}(\Omega)$. We finally conclude that $\sigma^t - U \in X_0^{q,p}(\Omega)$, as claimed.

The strong convergence in L^q can be obtained by observing that [2]

$$\sigma^t = \left\| \Big((1-t)^{1/r}\, v, t^{1/r}\, w\Big) \right\|_{\ell^r} \qquad \text{and} \qquad \sigma_n^t = \left\| \Big((1-t)^{1/r}\, v_n, t^{1/r}\, w_n\Big) \right\|_{\ell^r}.$$

Then, by using the triangle inequality for the ℓ^r norm

$$\Big| \|z\|_{\ell^r} - \|\xi\|_{\ell^r} \Big| \le \|z-\xi\|_{\ell^r} \qquad \text{for } z,\xi \in \mathbb{R}^2,$$

it follows

$$|\sigma_n^t - \sigma^t| \le \left((1-t)\,|v_n - v|^r + t\,|w_n - w|^r\right)^{\frac{1}{r}}.$$

By raising to the power q, integrating over Ω and using the convexity of the map $\tau \mapsto \tau^{q/r}$, we get

$$\int_\Omega |\sigma_n^t - \sigma^t|^q\, dx \le (1-t) \int_\Omega |v_n - v|^q\, dx + t \int_\Omega |w_n - w|^q\, dx.$$

By recalling that $\{v_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ strongly converge in $L^q(\Omega)$ to v and w, respectively, we get (3.12).

As for the estimate (3.13), by applying the hidden convexity inequality (3.4), we have

$$\int_\Omega |\nabla\sigma_n^t|^p\, dx \le (1-t) \int_\Omega |\nabla v_n|^p\, dx + t \int_\Omega |\nabla w_n|^p\, dx,$$

and the right-hand side is bounded, uniformly with respect to n. The proof is over. □

§3.3 The energy functional

We recall the notion of *superminimum* and *subminimum* for the energy functional

$$\mathfrak{F}_{p,q}^{\alpha}(\psi) := \frac{1}{p}\int_\Omega |\nabla\psi|^p\, dx - \frac{\alpha}{q}\int_\Omega |\psi|^q\, dx, \qquad \text{for every } \psi \in X^{q,p}(\Omega),$$

[2] We denote by $\|z\|_{\ell^q}$ the norm

$$\|z\|_{\ell^q} = (|z_1|^q + |z_2|^q)^{\frac{1}{q}}, \qquad \text{for every } z = (z_1, z_2) \in \mathbb{R}^2$$

which is naturally attached to our Lane-Emden equation (3.1). When $\alpha = 1$, we will write $\mathfrak{F}_{p,q}$ in place of $\mathfrak{F}^1_{p,q}$.

Definition 3.3.1. Let $\alpha > 0$, $1 < p < \infty$ and $1 \le q < p$. For every $v \in X^{q,p}(\Omega)$, we introduce the classes

$$\mathcal{A}^+(v) = \Big\{\psi \in X^{q,p}(\Omega)\,:\,\psi \ge v \text{ on } \Omega,\, \psi - v \in X^{q,p}_0(\Omega)\Big\},$$

and

$$\mathcal{A}^-(v) = \Big\{\psi \in X^{q,p}(\Omega)\,:\,\psi \le v \text{ on } \Omega,\, \psi - v \in X^{q,p}_0(\Omega)\Big\}.$$

Then we say that v is a:

- *superminimum for* $\mathfrak{F}^\alpha_{p,q}$ if

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \mathfrak{F}^\alpha_{p,q}(v), \qquad \text{for every } \psi \in \mathcal{A}^+(v);$$

- *subminimum for* $\mathfrak{F}^\alpha_{p,q}$ if

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \mathfrak{F}^\alpha_{p,q}(v), \qquad \text{for every } \psi \in \mathcal{A}^-(v);$$

- *minimum for* $\mathfrak{F}^\alpha_{p,q}$ if

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \mathfrak{F}^\alpha_{p,q}(v), \qquad \text{for every } \psi \in X^{q,p}(\Omega) \text{ such that } \psi - v \in X^{q,p}_0(\Omega).$$

Remark 3.3.2. It is a routine fact to show that a function v is a minimum for $\mathfrak{F}^\alpha_{p,q}$ if and only if it is both a superminimum and a subminimum.

We start with the following existence result, which holds under minimal assumptions.

Theorem 3.3.3 (Existence)**.** *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open set which is $(p, q)-$admissible. Then for every $U \in X^{q,p}(\Omega)$ the following problem*

$$\inf_{\psi \in X^{q,p}(\Omega)} \Big\{\mathfrak{F}^\alpha_{p,q}(\psi)\,:\,\psi - U \in W^{1,p}_0(\Omega)\Big\},$$

admits a solution. Moreover, we have:

- *for $1 < q < p$, each minimizer is a solution of the Lane-Emden equation* (3.1)*;*
- *for $q = 1$, each minimizer u satisfies*

$$-\alpha \le -\Delta_p u \le \alpha, \qquad \textit{in } \Omega,$$

  in weak sense;
- *for $q = 1$, each non-negative minimizer u (provided it exists) satisfies*

$$-\Delta_p u = \alpha, \qquad \textit{in } \Omega,$$

  in weak sense.

Proof. We first observe that the class of admissible functions is not empty, since the function U itself is admissible. We will use the Direct Method in the Calculus of Variations. At this aim, let us first prove that the infimum above is finite. For every admissible ψ, we have

$$\begin{aligned}\int_\Omega |\psi|^q\,dx &\le 2^{q-1}\int_\Omega |\psi-U|^q\,dx + 2^{q-1}\int_\Omega |U|^q\,dx\\ &\le \frac{2^{q-1}}{(\lambda_{p,q}(\Omega))^{\frac{q}{p}}}\,\|\nabla\psi-\nabla U\|^q_{L^p(\Omega;\mathbb{R}^N)} + 2^{q-1}\int_\Omega |U|^q\,dx,\end{aligned}\tag{3.14}$$

where we used the assumptions that $\psi - U \in W^{1,p}_0(\Omega)$ and that Ω is $(p,q)-$admissible. We now observe that

$$\|\nabla\psi-\nabla U\|^q_{L^p(\Omega;\mathbb{R}^N)} \le 2^{q-1}\,\|\nabla\psi\|^q_{L^p(\Omega;\mathbb{R}^N)} + 2^{q-1}\,\|\nabla U\|^q_{L^p(\Omega;\mathbb{R}^N)}.$$

By inserting this in (3.14), we finally get

$$\int_\Omega |\psi|^q\,dx \le C_1\,\|\nabla\psi\|^q_{L^p(\Omega;\mathbb{R}^N)} + C_2\,\|U\|^q_{X^{q,p}(\Omega)},\tag{3.15}$$

for some $C_1 = C_1(N,p,q,\Omega) > 0$ and $C_2 = C_2(N,p,q,\Omega) > 0$. This yields

$$\begin{aligned}\mathfrak{F}^\alpha_{p,q}(\psi) &= \frac{1}{p}\int_\Omega |\nabla\psi|^p\,dx - \frac{\alpha}{q}\int_\Omega |\psi|^q\,dx\\ &\ge \frac{1}{p}\int_\Omega |\nabla\psi|^p\,dx - C_1\,\frac{\alpha}{q}\left(\int_\Omega |\nabla\psi|^p\,dx\right)^{\frac{q}{p}} - C_2\,\frac{\alpha}{q}\,\|U\|^q_{X^{q,p}(\Omega)}.\end{aligned}$$

By a suitable application of the generalized Young's inequality

$$a\,b \le \delta\,\frac{q}{p}\,a^{\frac{p}{q}} + \delta^{-\frac{q}{p-q}}\,\frac{p-q}{p}\,b^{\frac{p}{p-q}}, \qquad \text{for } a,b\ge 0 \text{ and } \delta>0,$$

we then obtain

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \frac{1}{p}\,(1-\delta)\int_\Omega |\nabla\psi|^p\,dx - \delta^{-\frac{q}{p-q}}\,\frac{p-q}{p\,q}\,(C_1\,\alpha)^{\frac{p}{p-q}} - C_2\,\frac{\alpha}{q}\,\|U\|^q_{X^{q,p}(\Omega)}.$$

If we now choose $\delta = 1/2$, we finally end up with

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \frac{1}{2\,p}\int_\Omega |\nabla\psi|^p\,dx - M,\tag{3.16}$$

where $M = M(N,p,q,\alpha,\Omega,U) > 0$. This in particular shows that the infimum of $\mathfrak{F}^\alpha_{p,q}$ over the claimed set is finite.

Let us call m this infimum and consider a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ of admissible functions, such that

$$\mathfrak{F}^\alpha_{p,q}(\psi_n) \le m + \frac{1}{n+1}, \qquad \text{for every } n\in\mathbb{N}.$$

By using the estimates (3.15) and (3.16), we can then infer that

$$\sup_{n\in\mathbb{N}} \|\psi_n\|_{X^{q,p}(\Omega)} < +\infty.$$

Thus, we get that ψ_n weakly converges in $L^q(\Omega)$ to ψ and $\nabla\psi_n$ weakly converges in $L^p(\Omega;\mathbb{R}^N)$ to a vector field ϕ, up to a subsequence. It is easily seen, by the definition of weak gradient, that it must result $\phi = \nabla\psi$. This shows that $\psi \in X^{q,p}(\Omega)$.

We still need to show that $\psi - U \in W_0^{1,p}(\Omega)$. In order to prove this, we observe that the sequence $\{\psi_n - U\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ is bounded in the norm $X^{q,p}(\Omega)$. However, thanks to Proposition 1.4.2, we know that $W_0^{1,p}(\Omega) = X_0^{q,p}(\Omega)$. This in turn permits to show that $\{\psi_n - U\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$. Thus $\psi_n - U$ weakly converges in $W_0^{1,p}(\Omega)$, up to a subsequence. By uniqueness, such a limit must coincide with $\psi - U$ and belongs to $W_0^{1,p}(\Omega)$, since the latter is weakly closed. Moreover, by using again that Ω is $(p,q)-$admissible, we can infer

$$\lim_{n\to\infty} \|\psi_n - U\|_{L^q(\Omega)} = \|\psi - U\|_{L^q(\Omega)},$$

up to a subsequence, thanks to Proposition 1.4.2. Thus $\{\psi_n\}_{n\in\mathbb{N}}$ is actually strongly converging in $L^q(\Omega)$. This fact and the weak lower semicontinuity of the $p-$Dirichlet integral show that

$$\mathfrak{F}^{\alpha}_{p,q}(\psi) \le \liminf_{n\to\infty} \mathfrak{F}^{\alpha}_{p,q}(\psi_n).$$

By construction of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, the function ψ is the desired minimizer.

The last part of the statement is now standard for $1 < q < p$, it is sufficient to observe that the functional is Gateaux differentiable and compute its first variation.

In the limit case $q = 1$ we must be more careful. Let us take u a minimizer. For every $t > 0$ and $\psi \in C_0^\infty(\Omega)$ non-negative, we have

$$\mathfrak{F}^{\alpha}_{p,1}(u + t\,\psi) \ge \mathfrak{F}^{\alpha}_{p,1}(u).$$

By using the definition of $\mathfrak{F}_{1,\alpha}$ and the triangle inequality for the absolute value, this implies

$$\frac{1}{p}\int_\Omega \frac{|\nabla u + t\,\nabla\psi|^p - |\nabla u|^p}{t}\,dx + \alpha\int_\Omega \psi\,dx \ge 0,$$

where we further divided by $t > 0$. By taking the limit as t goes to 0, we get

$$\int_\Omega \langle |\nabla u|^{p-2}\,\nabla u, \nabla\psi\rangle\,dx \ge -\alpha\int_\Omega \psi\,dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega),\ \psi \ge 0.$$

This exactly means that $-\Delta_p u \ge -\alpha$, in weak sense. The other differential inequality is obtained in the same way, by taking this time $t < 0$. This is left to the reader.

Finally, still in the case $q = 1$, let us suppose that u is a non-negative minimizer. Then, if we define the *convex* functional

$$\widetilde{\mathfrak{F}}^{\alpha}_{p,1}(\psi) := \frac{1}{p}\int_\Omega |\nabla\psi|^p\,dx - \alpha\int_\Omega \psi\,dx,$$

we get that u minimizes this functional, as well. It is sufficient to observe that

$$\widetilde{\mathfrak{F}}^{\alpha}_{p,1}(\psi) \ge \mathfrak{F}^{\alpha}_{p,1}(\psi) \ge \mathfrak{F}^{\alpha}_{p,1}(u) = \widetilde{\mathfrak{F}}^{\alpha}_{p,1}(u).$$

for every admissible ψ. This new functional is Gateaux differentiable (because the lower order term is now linear) and its critical points exactly satisfy the weak formulation of the equation $-\Delta_p u = \alpha$. This concludes the proof. □

Remark 3.3.4. Recall that, thanks to Proposition 1.4.2, in the previous result one could equivalently write the condition $\psi - U \in W^{1,p}_0(\Omega)$ as

$$\psi - U \in X^{q,p}_0(\Omega).$$

Moreover, in the particular case $U \in W^{1,p}_0(\Omega)$, under the previous assumptions we get

$$\inf_{\psi \in X^{q,p}(\Omega)} \left\{ \mathfrak{F}^\alpha_{p,q}(\psi) \,:\, \psi - U \in W^{1,p}_0(\Omega) \right\} = \inf_{\psi \in W^{1,p}_0(\Omega)} \mathfrak{F}^\alpha_{p,q}(\psi),$$

still thanks to Proposition 1.4.2.

Remark 3.3.5 (Existence of a positive minimizer)**.** We notice that thanks to Theorem 3.3.3, we can infer the existence of a positive minimizer for

$$\inf_{\psi \in X^{q,p}(\Omega)} \left\{ \mathfrak{F}^\alpha_{p,q}(\psi) \,:\, \psi - U \in W^{1,p}_0(\Omega) \right\},$$

when $\Omega \subset \mathbb{R}^N$ is a $(p, q)-$admissible open set. In particular, it is sufficient to observe that if u is a minimizer then $|u|$ is a minimizer, as well. Indeed, $|u| - U \in W^{1,p}_0(\Omega)$ thanks to Lemma 1.4.1 and to the fact that U is non-negative, and, since $\mathfrak{F}^\alpha_{p,q}$ is even, we have that $|u|$ is still admissible and it is a positive minimizer.

The hidden convex structure of the functional $\mathfrak{F}^\alpha_{p,q}$ permits to establish an equivalence between supersolutions/subsolutions of (3.1) and superminima/subminima of the functional $\mathfrak{F}^\alpha_{p,q}$. More precisely, we have the following

Proposition 3.3.6. *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $v \in X^{q,p}(\Omega)$ be a positive function. The following facts hold:*

1. *v is a weak supersolution of (3.1) if and only if it is a superminimum for $\mathfrak{F}^\alpha_{p,q}$;*
2. *v is a weak subsolution of (3.1) if and only if it is a subminimum for $\mathfrak{F}^\alpha_{p,q}$.*

Proof. We focus on proving the first statement. The proof of the second one runs similarly and it is thus left to the reader. Moreover, the fact that a superminimum is a weak supersolution, it follows in a standard way, by computing the first variation of the functional. We can thus reduce to prove the converse implication.

We suppose that v is a positive supersolution of (3.1). Let $\psi \in \mathcal{A}^+(v)$, our aim is to show that

$$\mathfrak{F}^\alpha_{p,q}(\psi) \ge \mathfrak{F}^\alpha_{p,q}(v). \tag{3.17}$$

We first observe that if we set

$$\widetilde{v} = \alpha^{-\frac{1}{p-q}}\, v,$$

then by Proposition 3.1.2 this is a supersolution of the Lane-Emden equation with $\alpha = 1$. Moreover, we have

$$\mathfrak{F}_{p,q}(\widetilde{v}) = \alpha^{-\frac{p}{p-q}}\, \mathfrak{F}^\alpha_{p,q}(v).$$

Analogously, if $\psi \in \mathcal{A}^+(v)$ and we set

$$\widetilde{\psi} = \alpha^{-\frac{1}{p-q}}\, \psi,$$

then $\widetilde{\psi} \in \mathcal{A}^+(\widetilde{v})$ and

$$\mathfrak{F}_{p,q}(\widetilde{\psi}) = \alpha^{-\frac{p}{p-q}}\, \mathfrak{F}^\alpha_{p,q}(\psi).$$

Thus, in order to prove (3.17), we can reduce to prove that

$$\mathfrak{F}_{p,q}(\widetilde{\psi}) \geq \mathfrak{F}_{p,q}(\widetilde{v}). \tag{3.18}$$

We now build the curve $\sigma^t = ((1-t)\,\widetilde{v}^q + t\,\widetilde{\psi}^q)^{1/q}$ and observe that

$$\frac{\sigma^t - \widetilde{v}}{t} \in X_0^{q,p}(\Omega),$$

thanks to Proposition 3.2.2. Furthermore, by monotonicity, we observe that

$$(\sigma^t)^q = (1-t)\,\widetilde{v}^q + t\,\widetilde{\psi}^q = \widetilde{v}^q + t\,(\widetilde{\psi}^q - \widetilde{v}^q) \geq \widetilde{v}^q,$$

since $\widetilde{\psi} \geq \widetilde{v}$. This shows that

$$\sigma^t \in \mathcal{A}^+(\widetilde{v}).$$

From (3.4), we get

$$\int_\Omega |\nabla \sigma^t|^p\,dx \leq (1-t)\int_\Omega |\nabla \widetilde{v}|^p\,dx + t\int_\Omega |\nabla \widetilde{\psi}|^p\,dx, \qquad \text{for every } t \in [0,1],$$

and also, by construction, we have

$$\int_\Omega (\sigma^t)^q\,dx = (1-t)\int_\Omega \widetilde{v}^q + t\int_\Omega \widetilde{\psi}^q\,dx, \qquad \text{for every } t \in [0,1].$$

Thus in particular we obtain

$$\frac{\mathfrak{F}_{p,q}(\sigma^t) - \mathfrak{F}_{p,q}(\widetilde{v})}{t} \leq \mathfrak{F}_{p,q}(\widetilde{\psi}) - \mathfrak{F}_{p,q}(\widetilde{v}), \qquad \text{for every } t \in (0,1]. \tag{3.19}$$

We focus on the term on the left-hand side. We use the “above tangent” inequality

$$\frac{1}{p}\,|z|^p \geq \frac{1}{p}\,|z_0|^p + \langle |z_0|^{p-2} z_0, z - z_0\rangle, \qquad \text{for every } z_0, z \in \mathbb{R}^N,$$

with the choices $z = \nabla \sigma^t$ and $z_0 = \nabla \widetilde{v}$. Thus we get

$$\begin{aligned}
\frac{\mathfrak{F}_{p,q}(\sigma^t) - \mathfrak{F}_{p,q}(\widetilde{v})}{t} &= \frac{1}{p}\int_\Omega \frac{|\nabla \sigma^t|^p - |\nabla \widetilde{v}|^p}{t}\,dx - \frac{1}{q}\int_\Omega \frac{(\sigma^t)^q - \widetilde{v}^q}{t}\,dx \\
&\geq \int_\Omega \left\langle |\nabla \widetilde{v}|^{p-2}\,\nabla \widetilde{v}, \frac{\nabla \sigma^t - \nabla \widetilde{v}}{t}\right\rangle dx - \frac{1}{q}\int_\Omega \frac{\widetilde{v}^q + t\,(\widetilde{\psi}^q - \widetilde{v}^q) - \widetilde{v}^q}{t}\,dx \\
&\geq \int_\Omega \widetilde{v}^{q-1}\,\frac{\sigma^t - \widetilde{v}}{t}\,dx - \frac{1}{q}\int_\Omega (\widetilde{\psi}^q - \widetilde{v}^q)\,dx,
\end{aligned}$$

where in the last inequality we used that $\widetilde{v}$ is a supersolution and the fact that $\sigma^t - \widetilde{v}$ is a feasible test function. Moreover, we observe that almost everywhere in Ω it holds

$$\lim_{t\to 0^+} \frac{\sigma^t - \widetilde{v}}{t} = \frac{d}{dt}\sigma^t|_{t=0} = \frac{1}{q}\,\widetilde{v}^{1-q}\,(\widetilde{\psi}^q - \widetilde{v}^q).$$

Then we can pass to the limit as t goes to 0: from (3.19) and the estimate above, we obtain

$$\mathfrak{F}_{p,q}(\widetilde{\psi}) - \mathfrak{F}_{p,q}(\widetilde{v}) \geq \frac{1}{q}\int_\Omega \widetilde{v}^{q-1}\,\widetilde{v}^{1-q}\,(\widetilde{\psi}^q - \widetilde{v}^q)\,dx - \frac{1}{q}\int_\Omega (\widetilde{\psi}^q - \widetilde{v}^q)\,dx = 0,$$

thanks to Fatou's Lemma and to the fact that $v > 0$ almost everywhere in Ω, by assumption. This finally established (3.18), as desired. □

Remark 3.3.7. Of course, the previous result implies that a positive function $v \in X^{q,p}(\Omega)$ is a weak solution of (3.1) if and only if it is a solution of

$$\min_{\psi\in X^{q,p}(\Omega)} \left\{\mathfrak{F}^\alpha_{p,q}(\psi)\,:\,\psi - v \in W^{1,p}_0(\Omega)\right\},$$

i.e. if it minimizes $\mathfrak{F}^\alpha_{p,q}$ with respect to its own boundary datum.

The standard result in the next Proposition is proved when $\alpha = 1$ by simplicity, but it can be generalized for every $\alpha > 0$.

Proposition 3.3.8. *Let $1 < p < \infty$, $1 \leq q < p$ and let $\Omega \subset \mathbb{R}^N$ be a (p,q)-admissible connected open set. Then*

$$\min_{\psi\in W^{1,p}_0(\Omega)} \mathfrak{F}_{p,q}(\psi) = \frac{q-p}{p\,q}\left(\frac{1}{\lambda_{p,q}(\Omega)}\right)^{\frac{q}{p-q}}. \tag{3.20}$$

Proof. Since the minimum problem can be equivalently settled on $W^{1,p}_0(\Omega)\setminus\{0\}$, as a positive minimizer exists thanks to Remark 3.3.5, we can write that

$$\min_{\psi\in W^{1,p}_0(\Omega)} \mathfrak{F}_{p,q}(\psi) = -\max_{\psi\in W^{1,p}_0(\Omega)\setminus\{0\},t>0}\left\{\frac{t^q}{q}\int_\Omega |\psi|^q\,dx - \frac{t^p}{p}\int_\Omega |\nabla\psi|^p\,dx\right\}.$$

It is easily seen that, for every $\psi \in W^{1,p}_0(\Omega)\setminus\{0\}$, the function

$$t \mapsto \frac{t^q}{q}\int_\Omega |\psi|^q\,dx - \frac{t^p}{p}\int_\Omega |\nabla\psi|^p\,dx$$

is maximal for

$$t_0 = \left(\frac{\displaystyle\int_\Omega |\psi|^q\,dx}{\displaystyle\int_\Omega |\nabla\psi|^p\,dx}\right)^{\frac{1}{p-q}}.$$

With such a choice of t, we get

$$\frac{t_0^q}{q}\int_\Omega |\psi|^q\,dx - \frac{t_0^p}{p}\int_\Omega |\nabla\psi|^p\,dx = \frac{p-q}{p\,q}\,\frac{\left(\displaystyle\int_\Omega |\psi|^q\,dx\right)^{\frac{p}{p-q}}}{\left(\displaystyle\int_\Omega |\nabla\psi|^p\,dx\right)^{\frac{q}{p-q}}}.$$

Then, by recalling the definition of $\lambda_{p,q}(\Omega)$, we get (3.20).

□

§3.4 Comparison principle for solutions of the Lane-Emden equation

The main tool of this chapter is the following comparison principle for positive supersolutions and subsolutions of the Lane-Emden equation (3.1). This is proved under minimal assumptions, both on the set and on the functions.

Theorem 3.4.1 (Comparison principle)**.** *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set. Assume that $u, v \in X^{q,p}(\Omega)$ are two positive functions, such that u is a subsolution and v is a supersolution of* (3.1)*. If $(u-v)_+ \in X_0^{q,p}(\Omega)$, then*

$$v \ge u, \qquad \textit{a. e. in } \Omega.$$

Proof. We first observe that for $q = 1$, the result is well-known. It can be obtained with exactly the same proof of the comparison principle for $p-$harmonic functions (see for example [**98**, Theorem 2.15]). For completeness, we sketch the argument: it is sufficient to take the test function $\psi = (u-v)_+ \in X_0^{q,p}(\Omega)$ in the weak formulations for u and v. This gives

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla(u-v)_+\rangle\, dx \le \alpha \int_\Omega (u-v)_+\, dx,$$

and

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla(u-v)_+\rangle\, dx \ge \alpha \int_\Omega (u-v)_+\, dx.$$

By subtracting them, we obtain

$$\int_{\{u>v\}} \langle |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\,\nabla v, \nabla u - \nabla v\rangle\, dx \le 0.$$

If we now use that the vector field $z \mapsto |z|^{p-2}\, z$ is strictly monotone[3], we get the desired conclusion with standard arguments. We leave the details to the reader.

Let us now focus on the case $1 < q < p$. With Proposition 3.3.6 and the hidden convexity property at hand, we can essentially reproduce the proof of the classical comparison principle for *strictly convex* integral functionals of the Calculus of Variations, see for example [**70**, Lemma 1.1]. The identification of equality cases in the hidden convexity property will play a crucial role.

We define $\varphi = \min\{v, u\}$ and observe that it has the following properties

$$\varphi \in X^{q,p}(\Omega), \qquad u - \varphi = (u-v)_+ \in X_0^{q,p}(\Omega) \qquad \text{and} \qquad \varphi \le u \text{ in } \Omega,$$

so that it belongs to $\mathcal{A}^-(u)$. Observe that, since u is a positive subsolution of (3.1), from Proposition 3.3.6 it is a subminimum for $\mathfrak{F}^\alpha_{p,q}$, as well. By using this and the properties of φ, we get

$$\mathfrak{F}^\alpha_{p,q}(\varphi) \ge \mathfrak{F}^\alpha_{p,q}(u). \tag{3.21}$$

[3] In other words, we have

$$\langle |z|^{p-2}\, z - |w|^{p-2}\, w, z - w\rangle \ge 0,$$

and the equality holds if and only if $z = w$. This property is a consequence of the strict convexity of the map $z \mapsto |z|^p/p$, whose gradient is precisely $z \mapsto |z|^{p-2}\, z$.

By recalling that the weak gradient of φ is given by (see [**97**, Corollary 6.18])

$$\nabla\varphi = \begin{cases} \nabla v, & \text{a. e. on } \{v < u\}, \\ \nabla u, & \text{a. e. on } \{u < v\}, \\ \nabla v = \nabla u, & \text{a. e. on } \{v = u\}, \end{cases}$$

inequality (3.21) entails that

$$\frac{1}{p}\int_{\{v<u\}} |\nabla v|^p\,dx - \frac{\alpha}{q}\int_{\{v<u\}} |v|^q\,dx \ge \frac{1}{p}\int_{\{v<u\}} |\nabla u|^p\,dx - \frac{\alpha}{q}\int_{\{v<u\}} |u|^q\,dx.$$

If we add the quantity

$$\frac{1}{p}\int_{\{u\le v\}} |\nabla v|^p\,dx - \frac{\alpha}{q}\int_{\{u\le v\}} |v|^q\,dx,$$

on both sides, we end up with

$$\mathfrak{F}^{\alpha}_{p,q}(v) \ge \mathfrak{F}^{\alpha}_{p,q}(\psi), \qquad \text{where } \psi = \max\{u, v\} \in X^{q,p}(\Omega). \tag{3.22}$$

Observe that ψ has the following properties

$$\psi - v = (u - v)_+ \in X^{q,p}_0(\Omega) \qquad \text{and} \qquad v \le \psi \text{ in } \Omega,$$

i. e. it belongs to $\mathcal{A}^+(v)$. Since v is a positive supersolution of (3.1), it is a superminimum of $\mathfrak{F}^{\alpha}_{p,q}$ (again thanks to Proposition 3.3.6). This remark and the fact that $\psi \in \mathcal{A}^+(v)$ imply that we must have

$$\mathfrak{F}^{\alpha}_{p,q}(v) \le \mathfrak{F}^{\alpha}_{p,q}(\psi),$$

as well. Thus equation (3.22) must be an identity.

We now set

$$\sigma = \left(\frac{v^q + \psi^q}{2}\right)^{\frac{1}{q}} \in X^{q,p}(\Omega),$$

and use the hidden convexity property (3.4) with $t = 1/2$, so to obtain

$$\mathfrak{F}^{\alpha}_{p,q}(\sigma) \le \frac{1}{2}\,\mathfrak{F}^{\alpha}_{p,q}(v) + \frac{1}{2}\,\mathfrak{F}^{\alpha}_{p,q}(\psi) = \mathfrak{F}^{\alpha}_{p,q}(v). \tag{3.23}$$

Since $\psi - v \in X^{q,p}_0(\Omega)$, by Proposition 3.2.2 we have $\sigma - v \in X^{q,p}_0(\Omega)$, as well. Moreover, since by construction $\psi \ge v$, we also have $\sigma \ge v$. We can thus test the superminimality of v against σ and get that actually also (3.23) must hold as an equality. In particular, we have

$$\int_\Omega |\nabla\sigma|^p\,dx = \frac{1}{2}\int_\Omega |\nabla v|^p\,dx + \frac{1}{2}\int_\Omega |\nabla\psi|^p\,dx.$$

We use the equality cases of Theorem 3.2.1 with $r = q < p$ and $t = 1/2$, so to get that

either $\quad \psi = v \quad$ or $\quad \psi$ and v are both constant.

In the first case we directly get the desired conclusion, since $\psi = \max\{u, v\}$. The second case can not occur, since v is positive and from the equation we easily see that a positive constant can not be a supersolution. This concludes the proof. □

Remark 3.4.2. The assumption $(u - v)_+ \in X_0^{q,p}(\Omega)$ is a weak surrogate of the usual condition

$$v \geq u \qquad \text{on } \partial\Omega,$$

appearing in comparison principles. Whenever a trace theory is available (for example, if Ω is smooth enough), the two conditions coincide.

CHAPTER 4

SOME CONSEQUENCES OF COMPARISON PRINCIPLE

In the present chapter, our aim is to apply the comparison principle, proved in Theorem 3.4.1, to obtain some interesting consequences for weak solutions of Lane-Emden equation (3.1). To be more precise, we will show the following results:

- a uniqueness theorem for solutions of the minimization problem for the functional $\mathfrak{F}_{p,q}^{\alpha}$ with a general non-negative boundary datum, from which it also follows the uniqueness for the weak positive solution of the Lane-Emden equation (3.1);
- a sharp *pointwise* two-sided estimate on the unique positive solution $w_{p,q}^{\Omega,\alpha}$ with homogeneous Dirichlet boundary datum and, as a direct consequence, a sharp L^{∞} geometric estimate on $w_{p,q}^{\Omega,\alpha}$;
- a study on the asymptotics for L^{∞} norms of $w_{p,q}^{\Omega,\alpha}$, as $p \to \infty$, which follows thanks to the previous two-sided estimate;
- a *localization result* for maximum points of $w_{p,q}^{\Omega,\alpha}$;
- a "hierarchy" for solutions of (3.1) and an L^{∞} two-sided estimate for all sign changing solutions of (3.1).

§4.1 Uniqueness of positive solutions of the Lane-Emden equation

We start by applying the comparison principle to prove the uniqueness theorem for positive solutions of Lane-Emden equation (3.1).
We first need to recall the following

Definition 4.1.1 (Weakly $p-$superharmonic function)**.** Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be an open set. We say that $u \in L^1_{\rm loc}(\Omega)$ is *weakly $p-$superharmonic in* Ω if $\nabla u \in L^p(\Omega;\mathbb{R}^N)$ and

$$\int_\Omega \langle |\nabla u|^{p-2}\,\nabla u, \nabla \psi\rangle\,dx \geq 0, \qquad \text{for every } \psi \in C_0^\infty(\Omega),\ \psi \geq 0. \tag{4.1}$$

For the next result, we will use that, if u_+ and u_- are the two functions

$$u_+ = \max\{u,0\} \qquad \text{and} \qquad u_- = \max\{-u,0\},$$

then we can write $u = u_+ - u_-$.

Theorem 4.1.2 (Uniqueness of minimizers)**.** *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set, which is $(p,q)-$admissible . For every function $U \in X^{q,p}(\Omega)$, the minimization problem*

$$\inf_{\psi \in X^{q,p}(\Omega)} \Big\{ \mathfrak{F}^{\alpha}_{p,q}(\psi) \,:\, \psi - U \in W^{1,p}_0(\Omega) \Big\},$$

admits:

(i) exactly one solution, when $U \in X^{q,p}(\Omega) \setminus W^{1,p}_0(\Omega)$ is non-negative. Moreover, such a solution is positive;

(ii) exactly two solutions, when $U \in W^{1,p}_0(\Omega)$. In this case, both solutions have constant sign and they coincide, up to the choice of the sign.

Proof. Observe that we already know that the minimization problem does admit a solution, by virtue of Theorem 3.3.3. We will also rely on the optimality conditions, contained in the same result.

First of all we note that a minimizer u can not identically vanish. Furthermore, in both cases (i) and (ii), if u is a minimizer then $|u|$ is a minimizer, as well, as we observed in Remark 3.3.5. Hence, by minimality for both u and $|u|$, we deduce that u_+ is weakly $p-$superharmonic. Indeed this is trivial in the case (i) since the null function is not admissible for the minimization problem. In the case (ii) it is sufficient to take a function $\psi \in W^{1,p}_0(\Omega) \setminus \{0\}$ and $0 < t \ll 1$ to get

$$\mathfrak{F}^{\alpha}_{p,q}(t\,\psi) = \frac{t^p}{p} \int_\Omega |\nabla \psi|^p \,dx - \frac{\alpha\, t^q}{q} \int_\Omega |\psi|^q \,dx < 0 = \mathfrak{F}^{\alpha}_{p,q}(0),$$

since $q < p$.

Indeed, when $1 < q < p$ we have

$$\int_\Omega \langle |\nabla u|^{p-2}\, \nabla u, \nabla \psi \rangle \,dx = \alpha \int_\Omega |u|^{q-2}\, u\, \psi \,dx, \qquad \text{for every } \psi \in W^{1,p}_0(\Omega),$$

and

$$\int_\Omega \langle |\nabla |u||^{p-2}\, \nabla |u|, \nabla \psi \rangle \,dx = \alpha \int_\Omega |u|^{q-1}\, \psi \,dx, \qquad \text{for every } \psi \in W^{1,p}_0(\Omega).$$

We can sum up these two integral identities: by observing that

$$\frac{|u|^{q-2}\, u + |u|^{q-1}}{2} = (u_+)^{q-1}, \qquad \text{a. e. in } \Omega,$$

and[1]

$$\frac{|\nabla u|^{p-2}\, \nabla u + |\nabla |u||^{p-2}\, \nabla |u|}{2} = |\nabla u_+|^{p-2}\, \nabla u_+, \qquad \text{a. e. in } \Omega,$$

we then obtain

$$\int_\Omega \langle |\nabla u_+|^{p-2}\, \nabla u_+, \nabla \psi \rangle \,dx = \alpha \int_\Omega (u_+)^{q-1}\, \psi \,dx, \qquad \text{for every } \psi \in W^{1,p}_0(\Omega).$$

[1] It is sufficient to use the following classical fact from the theory of Sobolev spaces

$$\nabla |u| = \begin{cases} \nabla u, & \text{a. e on } \{u > 0\}, \\ -\nabla u, & \text{a. e on } \{u < 0\}, \\ 0, & \text{a. e on } \{u = 0\}, \end{cases}$$

see for example [**97**, Theorem 6.17].

In particular, u_+ is a weakly $p-$superharmonic function on Ω. In the case $q = 1$, a little additional care is needed. From Theorem 3.3.3, we know that

$$\int_\Omega \langle |\nabla u|^{p-2}\,\nabla u, \nabla\psi\rangle\,dx \geq -\alpha\int_\Omega \psi\,dx, \qquad \text{for every } \psi \in W_0^{1,p}(\Omega),\ \psi \geq 0,$$

and

$$\int_\Omega \langle |\nabla |u||^{p-2}\,\nabla |u|, \nabla\psi\rangle\,dx = \alpha\int_\Omega \psi\,dx, \qquad \text{for every } \psi \in W_0^{1,p}(\Omega).$$

We can sum up the two relations as above, when testing with a non-negative ψ: the terms containing α cancel and we now directly get that u_+ is weakly $p-$superharmonic.

We can now prove uniqueness of the minimizer. Let us start with case (i). In this case any minimizer must be positive. Indeed, let u be a minimizer. The assumption $u - U \in W_0^{1,p}(\Omega)$ and the fact that $U \in X^{q,p}(\Omega) \setminus W_0^{1,p}(\Omega)$ is non-negative entail that we must have

$$u_+ \not\equiv 0.$$

Indeed, if u_+ would identically vanish, we would have $u = -u_-$ and $|u| = u_-$. By Lemma 1.4.1 we know that $|u| - U \in W_0^{1,p}(\Omega)$. Thus we would obtain

$$U = \frac{(U-u) + (U - |u|)}{2} \in W_0^{1,p}(\Omega),$$

which contradicts the assumption on U.

Since u_+ is weakly $p-$superharmonic on the connected set Ω, the minimum principle implies that we must have $u_+ > 0$ almost everywhere in Ω. In particular, we get

$$u = u_+ > 0 \qquad \text{in } \Omega,$$

as desired.

Now we assume by contradiction that the minimization problem above admits two distinct minimizers $v, u \in X^{q,p}(\Omega)$. From the discussion above, v and u are positive functions. Of course, we have

$$(u-v)_+ \in W_0^{1,p}(\Omega) \qquad \text{and} \qquad (v-u)_+ \in W_0^{1,p}(\Omega).$$

We can then apply Theorem 3.4.1, first by considering u as a subsolution of (3.1) and v as a supersolution, then the other way round. In conclusion we get

$$v \geq u \quad \text{a.e. in } \Omega \qquad \text{and} \qquad u \geq v \quad \text{a.e. in } \Omega.$$

This implies that v and u must coincide in Ω, thus obtaining a contradiction.

We now focus on case (ii), which is slightly subtler. In this case, we first prove that each minimizer is either positive or negative. Indeed, let u be a minimizer, we write

$$u = u_+ - u_- \qquad \text{and observe that} \qquad u_+, u_- \in W_0^{1,p}(\Omega).$$

If $u_+ \not\equiv 0$ then the same argument as above shows that $u = u_+ > 0$ almost everywhere in Ω. If on the contrary $u_+ \equiv 0$, we get $u = -u_-$. Since the functional $\mathfrak{F}_{p,q}^\alpha$ is even, we get that $-u = u_-$ is still a minimizer. It is actually a non-negative minimizer, thus it solves the relevant Euler-Lagrange equation, by Theorem 3.3.3. In particular, u_- is a non-negative weakly $p-$superharmonic function, which is not identically vanishing. Again by the minimum principle, we get that u_- must be positive on Ω and thus the desired conclusion follows.

Finally, we assume to have two distinct minimizers $v, u \in W_0^{1,p}(\Omega)$. The previous considerations and Theorem 3.3.3 show that v and u are both constant sign solutions of (3.1), not identically vanishing. If we assume that they are both positive, by using Theorem 3.4.1 as in the first part of the proof, we get again that $u = v$ in Ω, which is not possible. Similarly, if both are negative, then $-u$ and $-v$ are positive solutions and again we get that they must coincide. The only possibility left is thus that u is positive and v is negative: in this case, we can apply the comparison principle as before, to the pair u and $-v$. This finally gives that we must have

$$u = -v, \qquad \text{a. e. in } \Omega.$$

The proof is now over. □

By joining the previous result and Remark 3.3.7, we get the following uniqueness result for the Lane-Emden equation (3.1): when Ω is a $(p, q)-$admissible open connected set, it is proved that there exists a unique weak positive solution of (3.1) with general non-negative boundary datum.

Corollary 4.1.3. *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set, which is $(p, q)-$admissible. For every $U \in X^{q,p}(\Omega)$ non-negative, the boundary value problem*

$$\begin{cases} -\Delta_p u = \alpha\, |u|^{q-2}\, u, \ \text{in } \Omega, \\ u - U \in W_0^{1,p}(\Omega), \\ u > 0, \ \text{in } \Omega, \end{cases}$$

admits a unique solution.

Definition 4.1.4. Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set, which is $(p, q)-$admissible. We will indicate by $w_{p,q}^{\Omega,\alpha} \in W_0^{1,p}(\Omega)$ the unique positive solution of

$$\min_{\psi \in W_0^{1,p}(\Omega)} \mathfrak{F}_{p,q}^{\alpha}(\psi).$$

In light of Theorem 4.1.2, such a definition is well-posed. We also observe that, by Corollary 4.1.3, such a function is also the unique positive solution of (3.1) with homogeneous Dirichlet boundary conditions. In the case $\alpha = 1$, we will simply indicate this function by $w_{p,q}^{\Omega}$. By recalling Remark 3.1.2, we have the relation

$$w_{p,q}^{\Omega,\alpha} = \alpha^{\frac{1}{p-q}}\, w_{p,q}^{\Omega}. \tag{4.2}$$

Remark 4.1.5. Note that, when $1 < p < \infty$, $1 \le q < p$ and $\Omega \subset \mathbb{R}^N$ is a $(p, q)-$admissible connected open set, then

$$\int_\Omega |\nabla w_{p,q}^{\Omega}|^p\, dx = \int_\Omega |w_{p,q}^{\Omega}|^q\, dx = \left(\frac{1}{\lambda_{p,q}(\Omega)}\right)^{\frac{q}{p-q}}. \tag{4.3}$$

Indeed, by using that $w_{p,q}^{\Omega}$ satisfies the weak formulation (3.2) (with $\alpha = 1$) and by taking $w_{p,q}^{\Omega}$ as a test function, we get that

$$\int_\Omega |\nabla w_{p,q}^{\Omega}|^p\, dx = \int_\Omega |w_{p,q}^{\Omega}|^q\, dx.$$

By recalling the definition of $\mathfrak{F}_{p,q}$, this implies

$$\mathfrak{F}_{p,q}(w_{p,q}^{\Omega}) = \frac{q-p}{p\,q} \int_\Omega |w_{p,q}^{\Omega}|^q\, dx,$$

and thanks to (3.20), equality (4.3) easily follows.

Remark 4.1.6 (The case $q = 1$ in a ball)**.** In the case when Ω is ball in $\mathbb{R}^N$, the function $w_{p,1}^{\Omega;\alpha}$ can be explicitly computed, in particular, for every $N \geq 1$, we have that

$$w_{p,1}^{B_1}(x) = \frac{p-1}{p} N^{-\frac{1}{p-1}} \left(1 - |x|^{\frac{p}{p-1}}\right), \qquad \text{for } x \in B_1.$$

Then the explicit general form of $w_{p,1}^{\Omega;\alpha}$ can be obtained by using the properties for solutions of (3.1) stated in Proposition 3.1.2.

We conclude this section, with the following result, which collects some basic properties of the unique positive solution in the case of an interval. Then, this preliminary Lemma will be applied to prove some geometric bounds for positive solutions in the next section.

Lemma 4.1.7 (One-dimensional case)**.** *Let $1 \leq q < p < \infty$. If we denote by $I = (-1, 1)$, the function $w_I \in W_0^{1,p}(I)$ has the following properties:*

1. *it is even, monotone increasing on $(-1, 0)$ and monotone decreasing on $(0, 1)$;*
2. *both w_I and $|w_I'|^{p-2}\, w_I'$ belongs to $C^1(\overline{I})$;*
3. *it is the unique solution of*
$$\begin{cases} -(|w_I'|^{p-2}\, w_I')' &= w_I^{q-1}, \quad \text{in } (-1,0), \\ w_I(-1) &= 0, \\ w_I'(0) &= 0; \end{cases} \tag{4.4}$$
4. *$w_I'(t) > 0$ for every $t \in (-1, 0)$;*
5. *it holds*
$$\int_{-1}^{0} |w_I|^q \, dt = \left(\frac{2}{\pi_{p,q}}\right)^{\frac{p\,q}{p-q}}, \tag{4.5}$$
where $\pi_{p,q}$ is defined in (4.11).

Proof. We proceed point by point.

1. The fact that w_I is even follows from its uniqueness. Indeed, if this were not true, the new function $\widetilde{w}_I(t) = w_I(-t)$ would be another positive minimizer of $\mathfrak{F}_{q,1}$. As for the claimed monotonicity, we observe that the new function
$$\widehat{w}_I(t) = \begin{cases} \displaystyle\int_{-1}^{t} |w_I'(\tau)|\, d\tau, & \text{for } t \in (-1,0), \\ \displaystyle\int_{t}^{1} |w_I'(\tau)|\, d\tau, & \text{for } t \in (0,1), \end{cases}$$
is still admissible, monotone on both subintervals and such that
$$|\widehat{w}_I'(t)| = |w_I'(t)| \qquad \text{and} \qquad \widehat{w}_I(t) \geq w_I(t), \qquad \text{for a.e. } t \in I.$$
Thus $\widehat{w}_I$ is still a minimizer and thus, by uniqueness, it must coincide with w_I;

2. by minimality, we know that w_I is a weak solution of

$$\begin{cases} -(|w_I'|^{p-2}\, w_I')' = w_I^{q-1} & \text{in } I, \\ w_I(-1) = w_I(1) = 0. \end{cases}$$

By [**53**, Theorem 3.1], we know that such a problem admits a unique positive solution $u \in C^1(\overline{I})$ such that $|u'|^{p-2}\, u' \in C^1(\overline{I})$, as well. Such a solution is also a weak solution and thus, by uniqueness, it must coincide with w_I. This proves the claimed regularity properties of w_I;

3. this simply follows from the previous point and the symmetry of w_I;

4. since $w_I > 0$ and $w_I' \geq 0$ in $(-1,0)$, by using (4.4), we get that

$$-((w_I')^{p-1})' = w_I^{q-1} > 0, \qquad \text{in } (-1,0).$$

This implies that $(w_I')^{p-1}$ is strictly decreasing and the same is true for w_I', as well. In particular,

$$w_I'(t) > w_I'(0) = 0, \qquad \text{for every } t \in (-1,0),$$

as claimed;

5. finally, in order to prove (4.5), it is sufficient to recall that, by scalings, it holds

$$\lambda_{p,q}(I) = 2^{\frac{q-p}{q}} \left(\frac{\pi_{p,q}}{2}\right)^p.$$

Bu putting this in identity (4.3), we get the desired conclusion.

The proof is over. □

§4.2 Geometric estimates for positive solutions of the Lane-Emden equation

This section is devoted to the study of some geometric bounds for both positive solutions of Lane-Emden equation and their L^∞ norm, as a consequence of the application of the comparison principle. The following expedient lemma will be useful, in order to construct positive supersolution of (3.1). We point out that here the restriction on $q < p$ is not needed.

Lemma 4.2.1. *Let $1 < p < \infty$ and $1 \leq q < \infty$. Let $f \in C^1([a,b])$ be a non-negative and non-decreasing function, such that*

$$|f'|^{p-2} f' \in C^1([a,b]),$$

and which satisfies

$$-(|f'|^{p-2} f')' = C\, f^{q-1}, \qquad \textit{in } [a,b],$$

for some $C > 0$. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a weakly $p-$superharmonic function. Moreover we assume that $a \leq u \leq b$ almost everywhere in Ω. Then the composition $\phi = f \circ u$ satisfies

$$-\Delta_p \phi \geq C\, |\nabla u|^p\, \phi^{q-1}, \qquad \textit{in } \Omega,$$

in weak sense.

Proof. We insert in (4.1) the test function

$$\psi = (|f'(u)|^{p-2} f'(u))\, \eta,$$

with $\eta \in C_0^\infty(\Omega)$ a non-negative function. Thanks to the assumptions on f and u, the Chain Rule formula ensures that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and

$$\nabla \psi = (|f'|^{p-2}\, f')'(u)\, \nabla u\, \eta + |f'(u)|^{p-2}\, f'(u)\, \nabla \eta.$$

We thus get

$$0 \le -C \int_\Omega |\nabla u|^p\, f(u)^{q-1} \eta\, dx + \int_\Omega |f'(u)|^{p-2}\, f'(u)\, \langle |\nabla u|^{p-2}\, \nabla u, \nabla \eta\rangle\, dx,$$

where we also used the equation satisfied by f. This gives in particular that

$$\int_\Omega \langle |\nabla f(u)|^{p-2}\, \nabla f(u), \nabla \eta\rangle\, dx \ge C \int_\Omega |\nabla u|^p f(u)^{q-1} \eta\, dx.$$

By recalling the definition $\phi = f \circ u$, we can conclude. □

We will use the previous result to construct a special supersolution to the Lane-Emden equation with some geometric contents in the class of convex sets. We recall that we use the notation $I = (-1, 1)$.

Theorem 4.2.2 (Two-sided pointwise estimate)**.** *Let $\alpha > 0$ and $1 \le q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set, which is $(p, q)-$admissible. Then if $B_r(x_0) \subset \Omega$, it holds*

$$w_{p,q}^{B_1,\alpha}\left(\frac{x - x_0}{r}\right) \le r^{-\frac{p}{p-q}}\; w_{p,q}^{\Omega,\alpha}(x), \qquad \textit{for a. e. } x \in \Omega, \tag{4.6}$$

where the function on the left-hand side is extended by zero to the whole Ω. Moreover, if Ω is bounded and convex, it also holds

$$r_\Omega^{-\frac{p}{p-q}}\; w_{p,q}^{\Omega,\alpha}(x) \le w_{I,\alpha}\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right), \qquad \textit{for a. e. } x \in \Omega. \tag{4.7}$$

Finally, both estimates are sharp.

Proof. We prove separately the lower and upper bounds. In both cases, we heavily rely on the comparison principle of Theorem 3.4.1. By recalling (4.2), it is sufficient to prove the result for $\alpha = 1$.

Lower bound. Let $w_{B_r(x_0)}$ be the positive solution of (3.1) in $B_r(x_0) \subset \Omega$, with $\alpha = 1$. Then, by Remark 3.1.2 and the uniqueness of the positive solution, we know that

$$w_{B_r(x_0)}(x) = r^{\frac{p}{p-q}}\; w_{p,q}^{B_1}\left(\frac{x - x_0}{r}\right).$$

Since $w_{B_r(x_0)} \in W_0^{1,p}(B_r(x_0))$ and $w_\Omega \ge 0$ on $B_r(x_0)$, by Lemma 1.4.1 part (ii), we have that

$$(w_{B_r(x_0)} - w_\Omega)_+ \in X_0^{q,p}(B_r(x_0)) = W_0^{1,p}(B_r(x_0)).$$

Moreover both $w_{B_r(x_0)}$ and w_Ω are positive solutions to (3.1) in $B_r(x_0)$. Hence, thanks to Theorem 3.4.1, we obtain

$$r^{\frac{p}{p-q}}\, w_{p,q}^{B_1}\left(\frac{x-x_0}{r}\right) = w_{B_r(x_0)}(x) \le w_\Omega(x), \qquad \text{for a. e. } x \in B_r(x_0). \tag{4.8}$$

Moreover, we can extend $w_{B_r(x_0)}$ to the whole Ω by setting it to be zero in $\Omega \setminus B_r(x_0)$. Then (4.8) holds almost everywhere in Ω.

Upper bound. We define

$$u = \frac{d_\Omega}{r_\Omega} - 1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and observe that this is a weakly $p-$superharmonic function. Indeed, since Ω is convex, the distance function d_Ω is concave and thus weakly superharmonic (see [**6**]). By further observing that $|\nabla d_\Omega| = 1$ almost everywhere in Ω, we get that it is actually weakly $p-$superharmonic, for every $1 < p < \infty$. Also observe that by construction, we have $-1 \le u \le 0$.

If we consider the composition $\phi = w_I \circ u$, in light of Lemma 4.2.1 and of the properties of w_I contained in Lemma 4.1.7, we know that $\phi \in W_0^{1,p}(\Omega)$ is a weak positive solution of

$$-\Delta_p \phi \ge \frac{1}{r_\Omega^p}\, \phi^{q-1} \qquad \text{in } \Omega.$$

By recalling Remark 3.1.2, if we define

$$\widetilde{\phi} = r_\Omega^{\frac{p}{p-q}}\, \phi,$$

then this satisfies

$$-\Delta_p \widetilde{\phi} \ge \widetilde{\phi}^{q-1}, \qquad \text{in } \Omega.$$

Moreover, we have that both w_Ω and $\widetilde{\phi}$ belongs to $W_0^{1,p}(\Omega)$. Thus $(w_\Omega - \widetilde{\phi})_+ \in W_0^{1,p}(\Omega)$ and by the comparison principle it holds

$$w_\Omega \le \widetilde{\phi} = r_\Omega^{\frac{p}{p-q}}\, w_I\left(\frac{d_\Omega}{r_\Omega} - 1\right), \qquad \text{a. e. in } \Omega,$$

as desired.

Sharpness. It is straightfoward to see that the lower bound in (4.6) is sharp. It is sufficient to take Ω to be any $N-$dimensional open ball and $B_r(x_0) = \Omega$, to get equality in the lower bound.

The upper bound (4.7) is slightly more complicated: indeed, the function

$$w_I\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right),$$

"virtually" coincides with the function $w_{p,q}^\Omega$ for the slab $\Omega = \mathbb{R}^{N-1} \times I$. However, this choice is not feasible, since w_Ω is not well-defined in our framework. Indeed, the set $\Omega = \mathbb{R}^{N-1} \times I$ is not $(p,q)-$admissible for any $1 \le q < p$ and the minimization problem in Definition 4.1.4 is not well-posed.

We go through an approximation argument. For every $n \in \mathbb{N} \setminus \{0\}$, we take

$$\Omega_n = \left(-\frac{n}{2}, \frac{n}{2}\right)^{N-1} \times I,$$

then from Lemma B.0.1 we have that

$$\lim_{n\to\infty} w_{\Omega_n}(x',x_N) = w_I(x_N), \qquad \text{for a. e. } (x',x_N)\in\mathbb{R}^{N-1}\times I. \tag{4.9}$$

On the other hand, by using that $r_{\Omega_n} = 1$ for $n \geq 2$ and that

$$\lim_{n\to\infty} d_{\Omega_n}(x',x_N) = 1-|x_N|, \qquad \text{for } (x',x_N)\in\mathbb{R}^{N-1}\times I$$

we obtain that

$$\lim_{n\to\infty} w_I\left(\frac{d_{\Omega_n}(x)}{r_{\Omega_n}}-1\right) = w_I(-|x_N|) = w_I(x_N). \tag{4.10}$$

By comparing (4.9) and (4.10), we get the claimed sharpness. □

For the next result, we need to introduce the following notation for the one-dimensional Sobolev-Poincaré constant

$$\pi_{p,q} := \inf_{u\in C_0^\infty((0,1))}\left\{\|u'\|_{L^p([0,1])}\ :\ \|u\|_{L^q([0,1])}=1\right\}, \tag{4.11}$$

for every $1<p\leq\infty$ and $1\leq q\leq\infty$.

As a straightforward application of Theorem 4.2.2, we get the following

Corollary 4.2.3 (Two-sided L^∞ estimate)**.** *Let $\alpha>0$ and $1\leq q<p<\infty$. Let $\Omega\subset\mathbb{R}^N$ be an open bounded convex set. Then we have*

$$w_{p,q}^{B_1,\alpha}(0) \leq r_\Omega^{-\frac{p}{p-q}}\,\|w_{p,q}^{\Omega,\alpha}\|_{L^\infty(\Omega)} \leq \left(\alpha\left(\frac{2}{\pi_{p,q}}\right)^p\right)^{\frac{1}{p-q}}\left(\frac{q\,p-q+p}{p}\right)^{\frac{1}{q}}, \tag{4.12}$$

and both estimates are sharp.

Proof. By still recalling (4.2), we can take $\alpha=1$, without loss of generality. Let $B_r(x_0)\subset\Omega$, thus in particular $r\leq r_\Omega$, by definition of inradius. By passing to the supremum in (4.6), we obtain

$$w_{p,q}^{B_1}(0) = \left\|w_{p,q}^{B_1}\left(\frac{\cdot - x_0}{r}\right)\right\|_{L^\infty(\Omega)} \leq r^{-\frac{p}{p-q}}\,\|w_{p,q}^{\Omega,\alpha}\|_{L^\infty(\Omega)}.$$

The first equality follows from the fact that $w_{p,q}^{B_1}$ is a radially symmetric decreasing function. This well-known property of $w_{p,q}^{B_1}$ can be proved by applying a radially symmetric decreasing rearrangement and then appealing to the so-called *Pólya-Szegő principle*, see for example [**88**, Theorem 3]. By arbitrariness of the radius r in the estimate above, we get the lower bound in (4.12).

As for the upper bound in (4.12), we use that w_I is increasing on $(-1,0)$, thus to finish we just need to prove that

$$w_I(0) = \left(\frac{q\,p-q+p}{p}\right)^{\frac{1}{q}}\left(\frac{2}{\pi_{p,q}}\right)^{\frac{p}{p-q}}. \tag{4.13}$$

We now use the identity (4.5) and the equation (4.4) solved by w_I, in order to determine $w_I(0)$. Since w_I' does not vanish in $(-1,0)$, by multiplying equation (4.4) by w_I', we get

$$-(|w_I'|^{p-2}\,w_I')'\,w_I' = w_I^{q-1}\,w_I',$$

which can be rewritten as

$$-(p-1)\,\frac{d}{dt}\frac{|w_I'|^p}{p} = \frac{d}{dt}\frac{w_I^q}{q}.$$

Upon integrating this on $[t, 0]$ and using that $w_I'(0) = 0$, we obtain

$$(p-1)\left(\frac{|w_I'(t)|^p}{p}\right) = \frac{(w_I(0))^q}{q} - \frac{(w_I(t))^q}{q}.$$

which implies

$$(w_I'(t))^p = \frac{p}{q\,(p-1)}((w_I(0))^q - (w_I(t))^q). \tag{4.14}$$

Finally, by using again the equation, the evenness of w_I and (4.14), we obtain

$$\int_{-1}^{0} |w_I(t)|^q\,dt = \int_{-1}^{0} |w_I'(t)|^p\,dt = \frac{p}{q\,(p-1)}\,(w_I(0))^q - \frac{p}{q\,(p-1)}\,\int_{-1}^{0} |w_I(t)|^q\,dt,$$

hence

$$w_I(0) = \left(\frac{q\,p - q + p}{p}\right)^{\frac{1}{q}} \left(\int_{-1}^{0} |w_I(t)|^q\,dt\right)^{\frac{1}{q}}.$$

From this and (4.5), we finally get (4.13). The sharpness of our L^∞ estimate is now a straightforward consequence of the sharpness of (4.6). □

Remark 4.2.4 (More general sets)**.** Apart for the simple lower bound (4.6), all the results of Section 4.2 have been proved under the assumption that Ω is convex. Actually, in the proof of Theorem 4.2.2, convexity was used in the upper bound (4.7) only to assure that the distance function d_Ω was weakly superharmonic. Thus the upper bound (4.7) (and, thus, all its consequences prove in the nexts sections) continues to hold for all sets such that

$$-\Delta d_\Omega \geq 0, \qquad \text{in } \Omega, \tag{4.15}$$

in weak sense. For completeness, we recall that condition (4.15) is equivalent to require that Ω is convex in dimension $N = 2$, but it is otherwise a weaker condition for $N \geq 3$, see [**6**].

As a consequence of Corollary 4.2.3, we can study the asymptotic behaviour for the L^∞ norms of positive solutions $w^{\Omega,\alpha}_{p,q}$ of Lane-Emden equation, as $p \to \infty$, when Ω is a convex bounded open set.

Corollary 4.2.5. *Let $1 < p < \infty$, $1 \leq q < p$ and $\Omega \subset \mathbb{R}^N$ be a convex bounded open set. Then*

$$\lim_{p\to\infty} \|w^{\Omega}_{p,q}\|_{L^\infty(\Omega)} = r_\Omega.$$

In particular, for the ball B_1 it holds

$$\lim_{p\to\infty} w^{B_1}_{p,q}(0) = 1. \tag{4.16}$$

Proof. For every $\eta \in C_0^\infty(\Omega)$ and $\varepsilon > 0$, the function

$$\psi = \frac{|\eta|^p}{(w^{\Omega}_{p,q} + \varepsilon)^{p-1}} \in W_0^{1,p}(\Omega),$$

is a feasible test function in the equation (3.2) for $w^\Omega_{p,q}$. Hence, by applying Picone's inequality for the $p-$Laplacian (see [4]), we get that

$$\int_\Omega (w^\Omega_{p,q})^{\,q-1}\,\frac{|\eta|^p}{(w^\Omega_{p,q}+\varepsilon)^{p-1}}\,dx = \int_\Omega \left\langle |\nabla w^\Omega_{p,q}|^{p-2}\,\nabla w^\Omega_{p,q}, \nabla\left(\frac{|\eta|^p}{(w^\Omega_{p,q}+\varepsilon)^{p-1}}\right)\right\rangle dx$$
$$\leq \int_\Omega |\nabla \eta|^p\,dx.$$

Since $w^\Omega_{p,q}\in W^{1,p}_0(\Omega)$ is positive by the minimum principle, by sending $\varepsilon\to 0$, we get that

$$\int_\Omega \frac{|\eta|^p}{(w^\Omega_{p,q})^{p-q}}\,dx \leq \int_\Omega |\nabla\eta|^p\,dx. \tag{4.17}$$

By recalling the definition of $\lambda_p(\Omega)$ and that $w^\Omega_{p,q}$ is bounded, the latter estimate implies that

$$1\leq \|w^\Omega_{p,q}\|^{p-q}_{L^\infty(\Omega)}\,\lambda_p(\Omega). \tag{4.18}$$

By raising to the power $1/p$ and using that

$$\lim_{p\to\infty}\Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}}=\frac{1}{r_\Omega},$$

(see [**84**, Lemma 1.5]), from (4.18) we obtain

$$r_\Omega\leq \liminf_{p\to\infty}\|w^\Omega_{p,q}\|_{L^\infty(\Omega)}.$$

On the other hand, by applying the geometric upper bound (4.12), we have that

$$\|w^\Omega_{p,q}\|_{L^\infty(\Omega)}\leq \left(\frac{2}{\pi_{p,q}}\right)^{\frac{p}{p-q}}\left(\frac{q\,p-q+p}{p}\right)^{\frac{1}{q}}\,r_\Omega^{\frac{p-q}{p}}. \tag{4.19}$$

We claim that

$$\pi_{p,q}\geq 2^{\left(1-\frac{1}{p}\right)}\left(q\left(1-\frac{1}{p}\right)+1\right)^{\frac{1}{q}}, \tag{4.20}$$

for every $1<p<\infty$ and $1\leq q<\infty$. Postponing the proof of this fact for a moment, we see that (4.19) and (4.20) would give

$$\limsup_{p\to\infty}\|w^\Omega_{p,q}\|_{L^\infty(\Omega)}\leq r_\Omega,$$

as desired. We are left with establishing (4.20). Let $\varphi\in C^\infty_0((0,1))$, for every $t\in(0,1)$, we have that

$$|\varphi(t)|=|\varphi(t)-\varphi(0)|\leq\int_0^t|\varphi'|\,dt\leq t^{1-\frac{1}{p}}\,\|\varphi'\|_{L^p([0,1])},$$

and

$$|\varphi(t)|=|\varphi(t)-\varphi(1)|\leq\int_t^1|\varphi'|\,dt\leq (1-t)^{1-\frac{1}{p}}\,\|\varphi'\|_{L^p([0,1])}.$$

By raising to the power q and integrating the first estimate on $(0,1/2)$ and the second one on $(1/2,1)$, we get

$$\int_0^{1/2} |\varphi(t)|^q\,dt \leq \frac{1}{q\left(1-\dfrac{1}{p}\right)+1}\left(\frac{1}{2}\right)^{q\left(1-\frac{1}{p}\right)+1} \|\varphi'\|^q_{L^p([0,1])},$$

and

$$\int_{1/2}^{1} |\varphi(t)|^q\,dt \leq \frac{1}{q\left(1-\dfrac{1}{p}\right)+1}\left(\frac{1}{2}\right)^{q\left(1-\frac{1}{p}\right)+1} \|\varphi'\|^q_{L^p([0,1])}.$$

Then, by summing up, we obtain

$$\int_0^{1} |\varphi(t)|^q\,dt \leq \frac{1}{q\left(1-\dfrac{1}{p}\right)+1}\left(\frac{1}{2}\right)^{q\left(1-\frac{1}{p}\right)} \|\varphi'\|^q_{L^p([0,1])}.$$

By raising to the power $1/q$ on both sides and using the arbitrariness of φ, we get (4.20). □

Remark 4.2.6. We point out that inequality (4.17) holds for general open sets: in this case, if the open set is not $(p,q)-$admissible, the function $w^{\Omega}_{p,q}$ has to be carefully defined. We refer to [**28**, Proposition 4.5] for the case of the $p-$torsion function, i.e. the case $q=1$ and $1<p<\infty$. The general case is contained in [**24**, Lemma 4.1] and [**128**, Corollary 3.3].

§4.3 Localization of maximum points

The two-sided estimates proved in Theorem 4.2.2 give quite a precise description of $w^{\Omega,\alpha}_{p,q}$, in terms of geometric quantities. It is thus possible to use this description to give a simple localization estimate for the maximum points of $w^{\Omega,\alpha}_{p,q}$. This is the content of the following

Corollary 4.3.1. *Let $\alpha>0$ and $1\leq q<p<\infty$. Let $\Omega\subset\mathbb{R}^N$ be an open bounded convex set. For every maximum point $x_0\in\Omega$ of $w^{\Omega,\alpha}_{p,q}$, we have*

$$d_\Omega(x_0)\geq C_{N,p,q}\,r_\Omega,$$

where the constant $0<C_{N,p,q}<1$ is defined by

$$C_{N,p,q} := \left(\frac{q\,(p-1)}{p}\right)^{\frac{1}{p}} \int_0^{w^{B_1}_{p,q}(0)} \frac{1}{((w_I(0))^q-\tau^q)^{\frac{1}{p}}}\,d\tau,$$

and $w_I(0)$ has been evaluated in (4.13).

Proof. Again by (4.2), we see that the location of maximum points is independent of $\alpha>0$. Thus we can take $\alpha=1$. We also observe that by standard results from Elliptic Regularity, we know that $w^{\Omega}_{p,q}$ is continuous on $\overline{\Omega}$ (see for example [**70**, Theorem 7.8]). Thus, it does admit maximum points on Ω. If $x_0\in\Omega$ is such a maximum point, we get from (4.6) and (4.7)

$$w^{B_1}_{p,q}(0)\leq r_\Omega^{-\frac{p}{p-q}}\,w^{\Omega}_{p,q}(x_0)\leq w_I\left(\frac{d_\Omega(x_0)}{r_\Omega}-1\right).$$

This in particular entails that

$$w_{p,q}^{B_1}(0) \leq w_I \left(\frac{d_\Omega(x_0)}{r_\Omega} - 1 \right). \tag{4.21}$$

We recall that $d_\Omega/r_\Omega - 1 \leq 0$ and observe that

$$w_I : (-1, 0] \to (0, w_I(0)]$$

is increasing, thus invertible. By taking the inverse function of w_I, we get

$$w_I^{-1}(w_{p,q}^{B_1}(0)) + 1 \leq \frac{d_\Omega(x_0)}{r_\Omega}. \tag{4.22}$$

We now observe that for every $y \in (0, w_I(0)]$ we can write

$$w_I^{-1}(y) = w_I^{-1}(0) + \int_0^y \frac{1}{w_I'(w_I^{-1}(\tau))}\, d\tau = -1 + \int_0^y \frac{1}{w_I'(w_I^{-1}(\tau))}\, d\tau.$$

We can use (4.14) to evalute the derivative inside the integral. This gives

$$w_I^{-1}(y) + 1 = \left(\frac{q\,(p-1)}{p} \right)^{\frac{1}{p}} \int_0^y \frac{1}{(w_I(0)^q - \tau^q)^{\frac{1}{p}}}\, d\tau.$$

By recalling the definition of $C_{N,p,q}$, from (4.22) we get the desired conclusion. □

Remark 4.3.2. A result similar to the previous one has been obtained by Magnanini and Poggesi in [**101**]. We refer in particular to [**101**, Remark 4.8], even if the estimate is not explicitly stated for our equation. Their proof is different from ours, it is based on obtaining a refined gradient bound for the solution w_Ω. Their method of proof is inspired by the so-called $P-$*functions method*, introduced by Payne in [**113**].

For the case $q = 1$, by using the explicit expression of $w_{B_1(0)}$ and w_I (see Remark 4.1.6), directly from (4.21) it is not difficult to get the expression

$$C_{N,p,1} = 1 - (1 - N^{-\frac{1}{p-1}})^{\frac{p-1}{p}}.$$

This is worse than the constant $N^{-1/p}$ obtained by Magnanini and Poggesi, see [**101**, Corollary 4.3]. In both cases, we observe that such a constant tends to 1, as p goes to ∞. Accordingly, the maximum points of w_Ω get closer and closer to the maximum points of the distance function d_Ω.

§4.4 Hierarchy of solutions of the Lane-Emden equation

By combining the comparison principle with standard properties of solutions to elliptic PDEs, we get the following "hierarchy" of solutions for the Lane-Emden equation (3.1) with homogeneous Dirichlet boundary conditions. This asserts that all solutions must be "trapped" between $w_{p,q}^{\Omega,\alpha}$ and $-w_{p,q}^{\Omega,\alpha}$.

Corollary 4.4.1 (Hierarchy of solutions)**.** *Let $\alpha > 0$ and $1 < q < p < \infty$. Let $\Omega \subset \mathbb{R}^N$ be an open connected set, which is $(p, q)-$admissible. Then for every sign-changing weak solution $v \in W_0^{1,p}(\Omega)$ of* (3.1) *with homogeneous Dirichlet boundary conditions, we have*

$$|v| \leq w_{p,q}^{\Omega,\alpha}, \qquad a.\,e.\ in\ \Omega.$$

Proof. Let $v \in W_0^{1,p}(\Omega)$ be a solution of (3.1). We claim that $V := \max\{v, w_{p,q}^{\Omega,\alpha}\} \in W_0^{1,p}(\Omega)$ is a positive weak subsolution of the same equation. If this were true, then we would get from Theorem 3.4.1 that

$$v \le V \le w_{p,q}^{\Omega,\alpha}, \qquad \text{a. e. in } \Omega.$$

By repeating the argument with $-v$ (which is still a weak solution of the same equation), we would finally get the desired conclusion.

We are left with proving that $\max\{v, w_{p,q}^{\Omega,\alpha}\}$ is a weak subsolution. This is quite classical, we briefly sketch the argument: for every $n \in \mathbb{N} \setminus \{0\}$, we take

$$H_n(t) = \begin{cases} 0, & \text{if } t \le 0, \\ n\,t, & \text{if } 0 \le t \le 1/n, \\ 1, & \text{if } t \ge 1/n, \end{cases}$$

i.e. this is a Lipschitz approximation of the Heaviside step function. For every $\psi \in C_0^\infty(\Omega)$ non-negative, we then insert in the weak formulations of the equations for v and $w_{p,q}^{\Omega,\alpha}$, the test functions

$$\varphi = H_n(v - w_{p,q}^{\Omega,\alpha})\,\psi \qquad \text{and} \qquad \varphi = (1 - H_n(v - w_{p,q}^{\Omega,\alpha}))\,\psi,$$

respectively. We thus get

$$\begin{aligned}\int_\Omega \langle |\nabla v|^{p-2}\,\nabla v, \nabla v - \nabla w_{p,q}^{\Omega,\alpha}\rangle\, H_n'(v - w_{p,q}^{\Omega,\alpha})\,\psi\,dx + \int_\Omega \langle |\nabla v|^{p-2}\,\nabla v, \nabla \psi\rangle\, H_n(v - w_{p,q}^{\Omega,\alpha})\,dx \\ = \alpha \int_\Omega |v|^{q-2}\,v\,H_n(v - w_{p,q}^{\Omega,\alpha})\,\psi\,dx,\end{aligned}$$

and

$$\begin{aligned}-\int_\Omega \langle |\nabla w_{p,q}^{\Omega,\alpha}|^{p-2}\,\nabla w_{p,q}^{\Omega,\alpha}, \nabla v - \nabla w_{p,q}^{\Omega,\alpha}\rangle\, H_n'(v - w_{p,q}^{\Omega,\alpha})\,\psi\,dx \\ + \int_\Omega \langle |\nabla w_{p,q}^{\Omega,\alpha}|^{p-2}\,\nabla w_{p,q}^{\Omega,\alpha}, \nabla\psi\rangle\,(1 - H_n(v - w_{p,q}^{\Omega,\alpha}))\,dx \\ = \alpha \int_\Omega (w_{p,q}^{\Omega,\alpha})^{q-1}\,(1 - H_n(v - w_{p,q}^{\Omega,\alpha}))\,\psi\,dx.\end{aligned}$$

We now sum up these two identities, use that the vector field $z \mapsto |z|^{p-2}\,z$ is monotone and that H_n is non-decreasing. We can thus obtain

$$\begin{aligned}\int_\Omega \langle |\nabla v|^{p-2}\,\nabla v, \nabla\psi\rangle\, H_n(v - w_{p,q}^{\Omega,\alpha})\,dx + \int_\Omega \langle |\nabla w_{p,q}^{\Omega,\alpha}|^{p-2}\,\nabla w_{p,q}^{\Omega,\alpha}, \nabla\psi\rangle\,(1 - H_n(v - w_{p,q}^{\Omega,\alpha}))\,dx \\ \le \alpha \int_\Omega |v|^{q-2}\,v\,H_n(v - w_{p,q}^{\Omega,\alpha})\,\psi\,dx \\ + \alpha \int_\Omega (w_{p,q}^{\Omega,\alpha})^{q-1}\,(1 - H_n(v - w_{p,q}^{\Omega,\alpha}))\,\psi\,dx.\end{aligned}$$

We can now pass to the limit as n goes to ∞, with a straightforward application of the Lebesgue Dominated Convergence Theorem. By observing that for almost every $x \in \Omega$ we have

$$\lim_{n\to\infty} H_n(v(x) - w_{p,q}^{\Omega,\alpha}(x)) = \begin{cases} 1, & \text{if } v(x) \ge w_{p,q}^{\Omega,\alpha}(x), \\ 0, & \text{otherwise,} \end{cases}$$

and recalling that (see again [**97**, Corollary 6.18])

$$\nabla V = \nabla \max\{v, w_{p,q}^{\Omega,\alpha}\} = \begin{cases} \nabla w_{p,q}^{\Omega,\alpha}, & \text{a. e. on } \{v < w_{p,q}^{\Omega,\alpha}\}, \\ \nabla v, & \text{a. e. on } \{w_{p,q}^{\Omega,\alpha} < v\}, \\ \nabla v = \nabla w_{p,q}^{\Omega,\alpha}, & \text{a. e. on } \{v = w_{p,q}^{\Omega,\alpha}\}, \end{cases}$$

we finally obtain

$$\int_\Omega \langle |\nabla V|^{p-2}\, \nabla V, \nabla \psi\rangle\, dx \leq \alpha \int_\Omega V^{q-1}\, \psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega),\ \psi \geq 0.$$

Thus V is a positive weak subsolution of (3.1) and the proof is over. □

Remark 4.4.2 (Universal L^∞ estimate)**.** By combining Corollaries 4.4.1 and 4.2.3, we get in particular that, for every convex bounded open set $\Omega \subset \mathbb{R}^N$, it holds

$$r_\Omega^{-\frac{p}{p-q}}\, \|v\|_{L^\infty(\Omega)} \leq \left(\alpha \left(\frac{2}{\pi_{p,q}}\right)^p\right)^{\frac{1}{p-q}} \left(\frac{q\,p - q + p}{p}\right)^{\frac{1}{q}},$$

for every solution $v \in W_0^{1,p}(\Omega)$ of (3.1).

CHAPTER 5

COMPACT SOBOLEV EMBEDDINGS

The main goal of this chapter is to discuss conditions on Ω assuring the validity of the continuous embedding

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \tag{5.1}$$

in the case when $1 \leq q \leq p$.

First, we recall some known facts concerning Sobolev embeddings. We start with the following remarkable equivalence

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is continuous} \qquad \Longleftrightarrow \qquad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact},$$

which holds in the *sub-homogeneous case* $q < p$. This result is shown in [**107**, Theorem 15.6.2] (see also [**28**, Theorem 1.2], for a different proof). Note that in the homogenous case $q = p$, it ceases to be valid. Moreover, for the *super-homogeneous case* $q > p$, we have

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is continuous} \qquad \Longleftrightarrow \qquad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is continuous},$$

and

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact} \qquad \Longleftrightarrow \qquad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \text{ is compact},$$

where q is such that

$$p < q \begin{cases} < \dfrac{N\,p}{N-p}, & \text{if } 1 < p < N, \\ < \infty, & \text{if } p = N, \\ \leq \infty, & \text{if } p \geq N. \end{cases}$$

We refer to [**107**, Theorem 15.4.1] and [**107**, Theorem 15.6.1] for these equivalences. In light of these facts, in this chapter, we will essentially limit ourselves to discuss the cases $q < p$ and $q = p$ and we only consider the case $q > p$ as a consequence.

To be more precise, with this results at hand, we aim to characterize the existence of a continuous/compact Sobolev embedding (5.1) in terms of a certain integrability of the distance function d_Ω. To this aim, we need to recall some standard properties for the sharp Sobolev-Poincaré constants $\lambda_{p,q}(\Omega)$, defined in (1.9), which will be useful in the sequel. Let $\Omega \subset \mathbb{R}^N$ be an open set, then, for every $1 < p < \infty$ and $1 \leq q \leq \infty$, it holds that

- $\lambda_{p,q}(\Omega) = \lambda_{p,q}(\Omega + x_0)$, for every x_0, i.e. $\lambda_{p,q}(\Omega)$ is invariant with respect to translations;
- if $\Omega' \subset \Omega$, then $\lambda_{p,q}(\Omega) \leq \lambda_{p,q}(\Omega')$, i.e. $\lambda_{p,q}(\Omega)$ is monotone non-increasing with respect to set inclusion;
- $\lambda_{p,q}(t\,\Omega) = t^{\,N-p-N\frac{p}{q}}\,\lambda_{p,q}(\Omega)$ for every $t > 0$ (*scaling property*).

For the ease of presentation of our main embeddings results, we distinguish between the three cases $q < p$, $q = p$ and $q > p$.

§5.1 Embedding theorem for the case $q < p$

Theorem 5.1.1. *Let $1 \le q < p < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. The following facts hold:*

(i) we have that
$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \qquad \Longrightarrow \qquad d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega),$$
and the following upper bound holds
$$\lambda_{p,q}(\Omega) \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} \le \lambda_p(B_1); \tag{5.2}$$

(ii) moreover, if $N < p < \infty$, then we also have
$$d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega) \qquad \Longrightarrow \qquad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega),$$
and the following lower bound holds
$$\mathfrak{h}_p(\Omega) \le \lambda_{p,q}(\Omega) \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}}, \tag{5.3}$$
where $\mathfrak{h}_p(\Omega)$ is the sharp Hardy constant (see Theorem 2.2.1);

(iii) finally, if $p \le N$, there exists an open set $\mathcal{T} \subsetneq \mathbb{R}^N$ such that
$$d_{\mathcal{T}} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T}) \qquad \textit{but} \qquad \mathcal{D}_0^{1,p}(\mathcal{T}) \not\hookrightarrow L^q(\mathcal{T}).$$

Proof. We prove each point separately.

(i) Let $x_0 \in \Omega$. Since both $w_{p,q}^{B_1}$ and $w_{p,q}^{\Omega}$ are continuous functions, evaluating the lower bound in Theorem 4.2.2 at $x = x_0$ and $r = d_\Omega(x_0)$, we get
$$d_\Omega(x_0)^{\frac{p}{p-q}}\, w_{p,q}^{B_1}(0) \le w_{p,q}^{\Omega}(x_0). \tag{5.4}$$
Then, by raising to the power q both sides of (5.4), integrating on Ω and exploiting (4.3), we get
$$\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx \le \left(w_{p,q}^{B_1}(0)\right)^{-q} \int_\Omega (w_{p,q}^{\Omega})^q(x)\, dx = \left(w_{p,q}^{B_1}(0)\right)^{-q} \left(\frac{1}{\lambda_{p,q}(\Omega)} \right)^{\frac{q}{p-q}}.$$
By using (4.18) for the ball B_1
$$\left(w_{p,q}^{B_1}(0)\right)^{-q} \le \Big(\lambda_p(B_1)\Big)^{\frac{q}{p-q}},$$
we get the claimed summability of d_Ω, together with the upper bound in (5.2);

(ii) let us suppose that $d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega)$ and $p > N$. For every $u \in C_0^\infty(\Omega)$, a joint application of Hölder's and Hardy's inequalities (see Theorem 2.2.1) leads to

$$\begin{aligned}\int_\Omega |u|^q\,dx &\le \left(\int_\Omega \frac{|u|^p}{d_\Omega^p}\,dx\right)^{\frac{q}{p}} \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{p}}\\ &\le \Big(\mathfrak{h}_p(\Omega)\Big)^{-\frac{q}{p}} \left(\int_\Omega |\nabla u|^p\,dx\right)^{\frac{q}{p}} \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{p}}.\end{aligned}$$

This in turn implies that

$$\frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\left(\displaystyle\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}}} \ge \frac{\mathfrak{h}_p(\Omega)}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}}.$$

By taking the infimum on $C_0^\infty(\Omega)$ on the left-hand side, we get the lower bound in (5.3). This in particular shows that $\lambda_{p,q}(\Omega) > 0$, i.e. we have the embedding

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega),$$

as desired;

(iii) we construct an open set $\mathcal{T} \subset \mathbb{R}^N$ such that, under the assumption $1 < p \le N$

- $d_\mathcal{T} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T})$, hence $d_\mathcal{T} \in L^\alpha(\mathcal{T})$ for every $\alpha \in [1, +\infty]$;
- $\mathcal{D}_0^{1,p}(\mathcal{T})$ is not compactly embedded in $L^q(\mathcal{T})$, for every $1 \le q < p$.

We consider the $(N-1)-$dimensional open hypercube $Q = (0,1)^{N-1} \subset \mathbb{R}^{N-1}$ and we define

$$C_k = Q \times (k, k+1], \qquad \text{for every } k \in \mathbb{N}.$$

Then, for every $k \in \mathbb{N}$, we take a dyadic partition of C_k, made of $2^{k\,N}$ cubes with side length 2^{-k}. We indicate by $C_k(j)$ each of these cubes, with $j = 1, \dots, 2^{k\,N}$. We also denote by $x_k(j)$ the center of the cube $C_k(j)$ and by

$$S_k := \Big\{x_k(i) : 1 \le i \le 2^{k\,N}\Big\},$$

the collection of all these centers, at a given $k \in \mathbb{N}$. Finally, we call *infinite fragile tower* the open set given by

$$\mathcal{T} = \bigcup_{k\in\mathbb{N}} (C_k \setminus S_k)\,.$$

We first show that the condition $d_\mathcal{T} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T})$ is satisfied. Indeed, we first observe that

$$r_\mathcal{T} = \frac{5}{12},$$

which implies that $d_\mathcal{T} \in L^\infty(\mathcal{T})$. Moreover, we have

$$d_\mathcal{T}(x) \le 2^{-k-1}\sqrt{N}, \qquad \text{for } x \in C_k(j) \setminus \{x_k(j)\},\ \ j = 1, \dots, 2^{k\,N} \text{ and } k \in \mathbb{N},$$

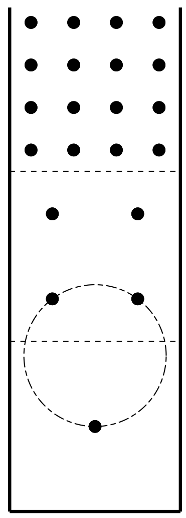


FIGURE 5.1: The construction of the set $\mathcal{T}$. The horizontal dashed lines denote the separation lines between the cubes C_k. The dashed circular line highlights the ball with maximal radius $r_{\mathcal{T}}$.

by construction. Then

$$\begin{aligned}\int_{\mathcal{T}} d_{\mathcal{T}}\,dx &= \sum_{k\in\mathbb{N}}\sum_{j=1}^{2^{kN}}\int_{C_k(j)\setminus\{x_k(j)\}} d_{\mathcal{T}}\,dx \\ &\le \frac{\sqrt{N}}{2}\sum_{k\in\mathbb{N}}\left(\frac{1}{2^k}\right)|C_k\setminus S_k| = \frac{\sqrt{N}}{2}\sum_{k\in\mathbb{N}}\left(\frac{1}{2^k}\right) = \sqrt{N}.\end{aligned}$$

We now show that for every $1 \le q < p \le N$, we have

$$\lambda_{p,q}(\mathcal{T}) = 0.$$

This would imply that $\mathcal{D}_0^{1,p}(\mathcal{T})$ is not continuously embedded in $L^q(\mathcal{T})$. At this aim, for every $m \in \mathbb{N}$, we introduce the *truncated tower*

$$\mathcal{T}_m = \left(\bigcup_{k=0}^{m}(C_k\setminus S_k)\right)\setminus(Q\times\{m+1\}).$$

This is a bounded open set contained in $\mathcal{T}$, thus, by monotonicity with respect to set inclusion, we have

$$\lambda_{p,q}(\mathcal{T}) \le \lambda_{p,q}(\mathcal{T}_m).$$

Therefore, in order to get the desired conclusion, it is sufficient to show that

$$\lim_{m\to\infty}\lambda_{p,q}(\mathcal{T}_m) = 0.$$

Since $p \leq N$, we know that points have zero $p-$capacity and thus we have (see [**126**, Chapter 17])

$$\lambda_{p,q}(\mathcal{T}_m) = \lambda_{p,q}(Q \times (0, m+1)).$$

By appealing to [**19**, Main Theorem], the last quantity can be estimated from above by

$$\begin{aligned}\lambda_{p,q}(Q \times (0, m+1)) &\leq \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{\mathcal{H}^{N-1}(Q \times (0, m+1))}{|Q \times (0, m+1)|^{1-\frac{1}{p}+\frac{1}{q}}}\right)^p \\ &\leq \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{2\,(N-1)\,(m+1)+2}{(m+1)^{1-\frac{1}{p}+\frac{1}{q}}}\right)^p.\end{aligned}$$

By using that $q < p$, it is easily seen that the last term converges to 0, as m goes to ∞. This gives the desired conclusion.

The proof is now over. $\square$

Before proceeding further, a couple of comments are in order on the geometric estimates obtained in the previous result.

Remark 5.1.2. For $p = 2$, the lower bound (5.2) has been obtained in [**14**, Theorem 3]. The proof there is simpler: up to some technical issues, it is simply based on using the trial function d_Ω in the definition of $\lambda_{2,q}(\Omega)$. However, this produces a poorer estimate: observe that the constant appearing in [**14**, equation (9)] blows-up as $q \nearrow p = 2$. This is not the case for our estimate (5.2).

§5.2 Embedding theorem for the case $q = p$: continuity

Theorem 5.2.1. *Let $1 < p < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. The following facts hold:*

(i) we have that

$$\mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega) \qquad \Longrightarrow \qquad d_\Omega \in L^\infty(\Omega),$$

and the following upper bound holds

$$\lambda_p(\Omega)\, r^p_\Omega \leq \lambda_p(B_1); \tag{5.5}$$

(ii) moreover, if $N < p < \infty$, then we also have

$$d_\Omega \in L^\infty(\Omega) \qquad \Longrightarrow \qquad \mathcal{D}^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega),$$

and the following lower bound holds

$$\mathfrak{h}_p(\Omega) \leq \lambda_p(\Omega)\, r^p_\Omega; \tag{5.6}$$

(iii) finally, if $p \leq N$, then for the open set $\mathcal{P} := \mathbb{R}^N \setminus \mathbb{Z}^N$ we have

$$d_\mathcal{P} \in L^\infty(\mathcal{P}) \qquad \textit{but} \qquad \mathcal{D}^{1,p}_0(\mathcal{P}) \not\hookrightarrow L^p(\mathcal{P}).$$

Proof. (i) Let $\lambda_p(\Omega) > 0$ and let $\{B_{r_n}(x_n)\}_{n\in\mathbb{N}} \subset \Omega$ be a sequence of balls such that r_n converges to r_Ω as n goes to ∞. Thanks to the monotonicity with respect to sets inclusion of λ_p, we get that

$$\lambda_p(\Omega) \leq \lambda_p(B_{r_n}(x_n)).$$

In particular, using the scaling properties of λ_p, we obtain that

$$r_n^p \leq \frac{\lambda_p(B_1)}{\lambda_p(\Omega)},$$

and, by sending n to ∞, we get $r_\Omega < +\infty$ and the upper bound in (5.5);

(ii) let us suppose that $r_\Omega < +\infty$ and $N < p < \infty$. By applying the Hardy inequality of Theorem 2.2.1, we have that

$$\int_\Omega |u|^p\,dx \leq r_\Omega^p \int_\Omega \frac{|u|^p}{d_\Omega^p}\,dx \leq \frac{1}{\mathfrak{h}_p(\Omega)}\, r_\Omega^p \int_\Omega |\nabla u|^p\,dx, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

By taking the infimum on $C_0^\infty(\Omega)$, we get the lower bound in (5.6). In particular, if $r_\Omega < +\infty$, then $\lambda_p(\Omega) > 0$ and thus the continuous embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ holds true;

(iii) it is sufficient to note that

$$\lambda_p(\mathcal{P}) \leq \lambda_p(B_m \setminus \mathbb{Z}^N), \qquad \text{for every } m \in \mathbb{N}.$$

Thanks to the assumption $p \leq N$, again by [**126**, Chapter 17] it holds

$$\lambda_p(B_m \setminus \mathbb{Z}^N) = \lambda_p(B_m).$$

By using the scale property of λ_p, we get that

$$\lambda_p(\mathcal{P}) \leq \lim_{m\to\infty} \lambda_p(B_m) = \lim_{m\to\infty} \frac{\lambda_p(B_1)}{m^p} = 0.$$

This gives the desired conclusion.

The proof is concluded. □

§5.3 Embedding theorem for the case $q = p$: compactness

Theorem 5.3.1. *Let $1 < p < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. The following facts hold:*

(i) we have that

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact} \qquad \Longrightarrow \qquad \Omega \text{ is quasibounded};$$

(ii) moreover, if $N < p < \infty$, then we also have

$$\Omega \text{ is quasibounded} \qquad \Longrightarrow \qquad \mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \text{ is compact};$$

(iii) finally, if $p \leq N$ and $\mathcal{T} \subsetneq \mathbb{R}^N$ is the same open set of Theorem 5.1.1, then $\mathcal{T}$ is quasibounded and the embedding $\mathcal{D}_0^{1,p}(\mathcal{T}) \hookrightarrow L^p(\mathcal{T})$ is continuous, but not compact.

Proof. (i) This follows from [**2**, Example 6.11]. For completeness, we sketch the idea of the proof: let us suppose that Ω is not quasibounded. Then there exists a sequence of balls $\{B_r(x_n)\}_{n\in\mathbb{N}} \subset \Omega$, with $r > 0$ fixed and

$$\lim_{n\to\infty} |x_n| = +\infty.$$

We consider $\psi \in C_0^\infty(B_1) \setminus \{0\}$ and then we simply set

$$\psi_n(x) = \psi\left(\frac{x - x_n}{r}\right), \qquad \text{for } x \in B_r(x_n),\ n \in \mathbb{N}.$$

It is easily seen that $\{\psi_n\}_{n\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{1,p}(\Omega)$, but it can not converge in $L^p(\Omega)$;

(ii) this result can be found in [**3**, Theorem 2], but here we give an alternative proof, which relies on the Hardy inequality of Theorem 2.2.1. Let $p > N$ and assume that Ω is quasibounded. Thus, the distance d_Ω is a bounded function. By Theorem 5.2.1, we already know that $\mathcal{D}_0^{1,p}(\Omega)$ is a functional space continuously embedded in $L^p(\Omega)$. Moreover, Remark 1.4.4 guarantees that we have

$$\mathcal{D}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega).$$

Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega)$ be a bounded sequence. We can extend these functions by 0 outside Ω and consider them as elements of $W^{1,p}(\mathbb{R}^N)$. In order to apply the classical Riesz–Fréchet–Kolmogorov Theorem, we first observe that by Theorem 5.2.1 we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^p(\mathbb{R}^N)$, as well.

Moreover, the bound on the L^p norm of the gradients guarantees that translations converge to 0 in $L^p(\Omega)$ uniformly in n, i. e.

$$\lim_{|h|\to 0} \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^N} |u_n(x+h) - u_n(x)|^p \, dx = 0.$$

The crucial point is to exclude the “loss of mass at infinity”. For this, we exploit the assumption that Ω is quasibounded. The latter entails that for every $\varepsilon > 0$, there exists $R > 0$ such that

$$\|d_\Omega\|_{L^\infty(\Omega\setminus B_R)} < \varepsilon.$$

Let $\eta_R \in C^\infty(\mathbb{R}^N)$ be such that

$$0 \le \eta_R \le 1, \quad \eta_R = 1 \text{ in } \mathbb{R}^N \setminus B_{R+1}, \quad \eta_R \equiv 0 \text{ in } B_R, \quad |\nabla \eta_R| \le C,$$

for some universal constant $C > 0$. Then

$$\sup_{n\in\mathbb{N}} \|\nabla(u_n \eta_R)\|_{L^p(\Omega)} \le \sup_{n\in\mathbb{N}} \|\nabla u_n\|_{L^p(\Omega)} + C \sup_{n\in\mathbb{N}} \|u_n\|_{L^p(\Omega)} =: M < +\infty.$$

Since the functions $u_n\,\eta_R$ belong to $\mathcal{D}_0^{1,p}(\Omega)$, by applying Hölder's and Hardy's inequalities (see Theorem 2.2.1 and Remark 2.2.2), for every $n \in \mathbb{N}$ we have that

$$\begin{aligned}\left(\int_{\Omega\setminus B_{R+1}} |u_n|^p \, dx\right)^{\frac{1}{p}} &\le \|d_\Omega\|_{L^\infty(\Omega\setminus B_R)} \left(\int_\Omega \frac{|u_n\,\eta_R|^p}{d_\Omega^p} \, dx\right)^{\frac{1}{p}} \\ &\le \varepsilon\, \mathfrak{h}_p(\Omega)^{-\frac{1}{p}} \left(\int_\Omega |\nabla(u_n\,\eta_R)|^p \, dx\right)^{\frac{1}{p}} \le \varepsilon\, \mathfrak{h}_p(\Omega)^{-\frac{1}{p}}\, M.\end{aligned}$$

We can thus appeal to the Riesz–Fréchet–Kolmogorov Theorem and get that, up to a subsequence, $\{u_n\}_{n\in\mathbb{N}}$ strongly converges in $L^p(\Omega)$;

(iii) we consider the set $\mathcal{T}$ defined as in the proof of Theorem 5.1.1 part (iii). Since $d_\mathcal{T} \in L^1(\mathcal{T})$, by applying Lemma 1.2.3, we have that $\mathcal{T}$ is quasibounded. Moreover, the embedding $\mathcal{D}_0^{1,p}(\mathcal{T}) \hookrightarrow$

$L^p(\mathcal{T})$ holds. Indeed, it is sufficient to notice that $\mathcal{T}$ is bounded in one direction. Thus, again by Remark 1.4.4, we can infer

$$W_0^{1,p}(\mathcal{T}) = \mathcal{D}_0^{1,p}(\mathcal{T}).$$

However, the embedding $\mathcal{D}_0^{1,p}(\mathcal{T}) \hookrightarrow L^p(\mathcal{T})$ cannot be compact. Indeed, we take $v \in C_0^\infty(Q \times (0,1))$ not identically zero and we build a bounded sequence $\{v_k\}_{k\in\mathbb{N}}$ by simply translating v in the vertical direction, i.e. for every $k \in \mathbb{N}$ we set

$$v_k(x', x_N) = v(x', x_N - k), \qquad \text{for every } (x', x_N) \in Q \times (k, k+1).$$

By appealing again to [**126**, Chapter 17], we have that

$$\begin{aligned} v_k \in C_0^\infty(Q \times (k, k+1)) \subset W_0^{1,p}(Q \times (k, k+1)) \\ = W_0^{1,p}((Q \times (k, k+1)) \setminus S_k) \subset W_0^{1,p}(\mathcal{T}) = \mathcal{D}_0^{1,p}(\mathcal{T}), \end{aligned}$$

for every $k \in \mathbb{N}$. Hence, the sequence $\{v_k\}_{k\in\mathbb{N}}$ is bounded in $\mathcal{D}_0^{1,p}(\mathcal{T})$ and $\|v_k\|_{L^p(\mathcal{T})} > 0$ is constant. This shows that $\{v_k\}_{k\in\mathbb{N}}$ cannot admit a converging subsequence in $L^p(\mathcal{T})$.

This concludes the proof. □

§5.4 The super-homogeneous case $q > p$ and beyond

In what follows, for an open set $\Omega \subset \mathbb{R}^N$ and for $0 < \beta \le 1$, we consider the space $C^{0,\beta}(\overline{\Omega})$ defined as in Section 1.1.

As a consequence of the previous embedding theorems, it easily follows the next result for the super-homogeneous case $N < p < q$.

Corollary 5.4.1. *Let $p > N$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. The following facts hold:*

(i) if $d_\Omega \in L^\infty(\Omega)$, then we have

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \text{for every } p \le q \le \infty,$$

and

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow C_0(\Omega) \cap C^{0,\beta}(\overline{\Omega}), \qquad \text{for every } 0 < \beta \le \alpha_p;$$

(ii) if Ω is quasibounded, then the above embeddings are compact, for

$$p \le q \le \infty \qquad \text{and} \qquad 0 < \beta < \alpha_p;$$

(iii) if $d_\Omega \in L^\gamma(\Omega)$, for some $p/(p-1) \le \gamma < \infty$, then we have

$$\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \text{for every } \frac{p\,\gamma}{p+\gamma} \le q \le \infty,$$

and such an embedding is compact.

Proof. (i) Let $d_\Omega \in L^\infty(\Omega)$. The existence of the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is a consequence of Theorem 5.2.1 part (ii). By using the Gagliardo-Nirenberg interpolation inequality (1.10) with $q = p$, it follows that $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded in every $L^q(\Omega)$ with $p \le q \le \infty$.

As for the embedding in Hölder spaces: we observe at first that from the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain that each $\{u_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ which is a Cauchy sequence in the norm of

$\mathcal{D}_0^{1,p}(\Omega)$, is a Cauchy sequence in the sup norm, as well. Thus, by recalling the definition of the completion space $C_0(\Omega)$, we get that $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow C_0(\Omega)$. By using this fact and Corollary 2.1.3, we thus get that $\mathcal{D}_0^{1,p}(\Omega)$ is continuously embedded in $C_0(\Omega)\cap C^{0,\alpha_p}(\overline{\Omega})$. Then Lemma 1.1.2 gives the desired conclusion;

(ii) we now suppose that Ω is quasibounded. In order to prove the first statement, it is sufficient to observe that the embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact for $q = p$ thanks to Theorem 5.3.1 part (ii). By applying again the Gagliardo-Nirenberg inequality (1.10) with $q = p$, we conclude.

The case of $C_0(\Omega) \cap C^{0,\beta}(\overline{\Omega})$ follows as above, by combining Morrey's inequality and Lemma 1.1.2;

(iii) we first recall that the assumption $d_\Omega \in L^\gamma(\Omega)$, for some $p/(p-1) \le \gamma < \infty$, implies that Ω is a quasibounded set (see Lemma 1.2.3). The compact embedding $\mathcal{D}_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ easily follows by Theorem 5.1.1 part (ii), when $q = p\,\gamma/(p+\gamma)$, while the case $q = \infty$ has just been proved in the part (ii) above. We conclude, by interpolation, that the embedding is compact for every $p\,\gamma/(p+\gamma) \le q \le \infty$.

The proof is now complete □

Remark 5.4.2. It is not difficult to see that the compact embedding of Corollary 5.4.1 part (ii) *does not* extend up to the borderline case $\beta = \alpha_p = 1 - N/p$. This can be seen by means of a standard scaling argument: take $\Omega = B_1$ and $\psi \in C_0^\infty(B_1) \setminus \{0\}$. We define the sequence

$$\psi_n(x) = n^{\frac{N-p}{p}}\, \psi(n\,x), \qquad \text{for } n \in \mathbb{N}.$$

It is easily seen that

$$\|\nabla \psi_n\|_{L^p(B_1)} = \|\nabla \psi\|_{L^p(B_1)} \qquad \text{and} \qquad [\psi_n]_{C^{0,\alpha_p}(B_1)} = [\psi]_{C^{0,\alpha_p}(B_1)}.$$

On the other hand, by construction, we have that ψ_n converges uniformly to 0 as n goes to ∞, since $N - p < 0$. Thus, for this sequence we cannot have convergence in the norm of $C^{0,\alpha_p}(\overline{B_1})$.

We complete the previous result by studying some geometric estimates for the Sobolev-Poincaré constants $\lambda_{p,q}$ in the case $N < p < q$, as well.

Corollary 5.4.3. *Let $N < p < \infty$, $p \le q \le \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. We have that*

$$\lambda_{p,q}(\Omega) > 0 \qquad \Longleftrightarrow \qquad r_\Omega < +\infty,$$

and

$$\frac{\mathfrak{h}_{p,q}(\Omega)}{r_\Omega^{p-N+N\,\frac{p}{q}}} \le \lambda_{p,q}(\Omega) \le \frac{\lambda_{p,q}(B_1)}{r_\Omega^{p-N+N\,\frac{p}{q}}}, \tag{5.7}$$

with $\mathfrak{h}_{p,q}(\Omega)$ defined in Theorem 2.2.1. Moreover, if Ω is quasibounded, then there exists a minimizer $u_{p,q} \in W_0^{1,p}(\Omega)$ for the problem defining $\lambda_{p,q}(\Omega)$.

Proof. Let us assume $\lambda_{p,q}(\Omega) > 0$ and let $\{B_{r_n}(x_n)\}_{n\in\mathbb{N}} \subset \Omega$ be a sequence of balls such that r_n goes to r_Ω, as n goes to ∞. As in the proof of Theorem 5.2.1 part (i), it follows that

$$r_n^{p-N+N\frac{p}{q}} \le \frac{\lambda_{p,q}(B_1)}{\lambda_{p,q}(\Omega)},$$

and, by sending n to ∞, we get $r_\Omega < +\infty$ and the upper bound in (5.7).

In order to prove the reverse implication, we first observe that this has already been proved in Theorem 5.2.1 for the case $q = p$ part (ii). For the case $p < q \leq \infty$, it is sufficient to use the same argument, in conjuction with the general Hardy inequality of Theorem 2.2.1. This comes with the lower bound in (5.7).

We now discuss the existence result for the minimizer $u_{p,q} \in W_0^{1,p}(\Omega)$, under the stronger assumption that Ω is quasibounded. We first note that Theorem 5.2.1 and Remark 1.4.4 guarantee the identity

$$\mathcal{D}_0^{1,p}(\Omega) = W_0^{1,p}(\Omega),$$

thanks to Remark 1.4.4. Then, the existence of a minimizer is now an easy consequence of the Direct Method in the Calculus of Variations, once observed that $W_0^{1,p}(\Omega)$ is weakly closed and that we have the compact embeddings of Corollary 5.4.1 at our disposal. □

Remark 5.4.4. We notice that the value of $\lambda_{p,\infty}(B_1)$ can be made explicit: according to [**125**, Theorem 2E] we have

$$\lambda_{p,\infty}(B_1) = \left(\frac{p-N}{p-1}\right)^{p-1} N\,\omega_N, \qquad \text{for } p > N.$$

This implies that the upper bound for the sharp Morrey constant in (2.6) can be rewritten as

$$\mathfrak{m}_p(\mathbb{R}^N) \leq \lambda_{p,\infty}(B_1).$$

Moreover, such a value is uniquely attained by the functions

$$u(x) = \pm\left(1 - |x|^{\frac{p-N}{p-1}}\right)_+.$$

We refer to [**61**, **78**] for a detailed study of the variational problem associated to $\lambda_{p,\infty}$, in the case of bounded open sets.

CHAPTER 6

ASYMPTOTICS FOR SOBOLEV-POINCARÉ CONSTANTS AND ITS EXTREMALS

This chapter is addressed to the study of the asymptotic behaviour for $\lambda_{p,q}$ and their relevant extremals (provided they exist), as the parameter p goes to ∞.

In particular, we aim to extend some known convergence results to the Sobolev-Poincaré constants $\lambda_{p,q}(\Omega)$ for any $q \in [1, +\infty]$ and to the general case of open sets, without further regularity assumptions on Ω.

Then, as an application of such asymptotic results, we also study the convergence of the unique positive solution $w^{\Omega}_{p,q}$ of (3.1), of the extremal functions of $\lambda_p(\Omega)$ and of $\lambda_{p,\infty}(\Omega)$, as $p \to \infty$.

§6.1 Asymptotics for $\lambda_{p,q}$

In this section, we study the asymptotic behaviour of $\lambda_{p,q}(\Omega)$, when $\Omega \subset \mathbb{R}^N$ is a general open set and for every $1 \leq q \leq \infty$.

Corollary 6.1.1. *Let $1 \leq q < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then*

$$\lim_{p\to\infty} \Big(\lambda_{p,q}(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{\|d_\Omega\|_{L^q(\Omega)}},$$

and

$$\lim_{p\to\infty} \Big(\lambda_{p,\infty}(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{r_\Omega}.$$

In the previous equations, the right-hand sides have to be considered 0, if $d_\Omega \notin L^q(\Omega)$ or $r_\Omega = +\infty$, respectively.

Proof. We start with the case $q = \infty$. If $r_\Omega = +\infty$, thanks to Corollary 5.4.3, there is nothing to prove. Let us assume $r_\Omega < +\infty$, it is sufficient to take the $p-$rooth in (5.7) and use that

$$\lim_{p\to\infty} \Big(\lambda_{p,\infty}(B_1)\Big)^{\frac{1}{p}} = 1,$$

(see Remark 5.4.4) and (2.10). This gives the desired conclusion as p goes to ∞.

We now consider the case $q < \infty$. We first suppose that $d_\Omega \in L^q(\Omega)$. Observe that for every $p > 2\,q$, we have

$$d_\Omega(x)^{\frac{p\,q}{p-q}} \leq r_\Omega^{\frac{q^2}{p-q}}\, d_\Omega(x)^q \leq \Big(\max\{1, r_\Omega\}\Big)^q\, d_\Omega(x)^q, \qquad \text{for every } x \in \Omega,$$

thus we can apply the Dominated Convergence Theorem to get that

$$\lim_{p\to\infty}\left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q\,p}}=\left(\int_\Omega d_\Omega^q\,dx\right)^{\frac{1}{q}}. \tag{6.1}$$

Moreover, by Theorem 5.1.1, for every $p>q$ and $p>N$, we have the two-sided estimate

$$\mathfrak{h}_p(\Omega)\le\lambda_{p,q}(\Omega)\left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}\le\lambda_p(B_1).$$

By raising this estimate to the power $1/p$, using (2.10), (6.1) and the following fact

$$\lim_{p\to\infty}\Big(\lambda_p(B_1)\Big)^{\frac{1}{p}}=1,$$

(see [**84**, Lemma 1.5]), we get the desired conclusion.

We now suppose that $d_\Omega\notin L^q(\Omega)$. Let $n_0\in\mathbb{N}$ such that $\Omega_n:=\Omega\cap B_n\neq\emptyset$ for every $n\ge n_0$. By applying the first part of this proof to the set Ω_n with $n\ge n_0$, we have that

$$\lim_{p\to\infty}\Big(\lambda_{p,q}(\Omega_n)\Big)^{\frac{1}{p}}=\frac{1}{\|d_{\Omega_n}\|_{L^q(\Omega_n)}}.$$

Hence, by using the monotonicity of $\lambda_{p,q}$ with respect to the set inclusion, we get that

$$\limsup_{p\to\infty}\Big(\lambda_{p,q}(\Omega)\Big)^{\frac{1}{p}}\le\frac{1}{\|d_{\Omega_n}\|_{L^q(\Omega_n)}}, \tag{6.2}$$

for every $n\ge n_0$. We extend each distance function d_{Ω_n} equal to 0 in $\mathbb{R}^N\setminus\Omega_n$. We note that the family $\{d_{\Omega_n}\}_{n\ge n_0}$ is nondecreasing with respect to n. Thus, in order to conclude, it is sufficient to prove that

$$\lim_{n\to\infty}d_{\Omega_n}(x)=d_\Omega(x),\qquad\text{for every } x\in\Omega. \tag{6.3}$$

Indeed, by passing to the limit in (6.2) as n goes to ∞ and by using Monotone Convergence Theorem, we get that

$$\limsup_{p\to\infty}\Big(\lambda_{p,q}(\Omega)\Big)^{\frac{1}{p}}=0.$$

In order to show (6.3), we note that, for every $x\in\Omega$, there exists $n_x\ge n_0$ such that $B_{d_\Omega(x)}(x)\subset\Omega_n$, for every $n\ge n_x$. This implies that

$$d_\Omega(x)=d_{\Omega_n}(x),\qquad\text{for every } n\ge n_x.$$

This concludes the proof. □

§6.2 Asymptotics for the solution of the Lane-Emden equation

In this section, we study the asymptotic behavior of $w_{p,q}^\Omega$, as p goes to ∞, under the asumption that $d_\Omega\in L^q(\Omega)$, for some $1\le q<\infty$. We first observe that, Lemma 1.2.1 implies $d_\Omega\in L^\infty(\Omega)$, as well, and, by interpolation, we have that $d_\Omega\in L^{\frac{p\,q}{p-q}}(\Omega)$ for every $q<p<\infty$. Hence, by Theorem 5.1.1, Ω is

$(p,q)-$admissible for every $p > \max\{N,q\}$ and, the uniqueness Theorem 4.1.2 assures the existence of a unique positive solution $w^\Omega_{p,q} \in W^{1,p}_0(\Omega)$ to (3.1) for every $q < p$.

Theorem 6.2.1. *Let $1 \le q < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open connected set such that $d_\Omega \in L^q(\Omega)$. Then*

$$\lim_{p\to\infty} \|w^\Omega_{p,q} - d_\Omega\|_{L^r(\Omega)} = 0 \qquad \textit{and} \qquad \lim_{p\to\infty} \|w^\Omega_{p,q} - d_\Omega\|_{C^{0,\beta}(\overline{\Omega})} = 0 \tag{6.4}$$

for every $q \le r \le \infty$ and every $0 < \beta < 1$.

Proof. We will first show that (6.4) holds to $r = q$. Then, by interpolation, we will obtain all the other claimed convergences.

Part 1: convergence in $L^q(\Omega)$. We extend each function $w^\Omega_{p,q}$ to $\mathbb{R}^N$ by setting it to be zero in $\mathbb{R}^N \setminus \Omega$. First of all, we note that, by using (4.3) and Corollary 6.1.1, we have

$$\lim_{p\to\infty} \int_\Omega |\nabla w^\Omega_{p,q}|^p \, dx = \lim_{p\to\infty} \int_\Omega |w^\Omega_{p,q}|^q \, dx = \lim_{p\to\infty} \left(\frac{1}{\Big(\lambda_{p,q}(\Omega)\Big)^{\frac{1}{p}}} \right)^{\frac{p\,q}{p-q}} = \int_\Omega d^q_\Omega \, dx, \tag{6.5}$$

which implies

$$\lim_{p\to\infty} \|\nabla w^\Omega_{p,q}\|_{L^p(\Omega)} = 1. \tag{6.6}$$

Moreover, by applying (2.5), we find the upper bound

$$0 < w^\Omega_{p,q}(x) \le \frac{d_\Omega(x)^{\alpha_p}}{\Big(\mu_p(B_1)\Big)^{\frac{1}{p}}} \|\nabla w^\Omega_{p,q}\|_{L^p(\Omega)}, \qquad \text{for every } x \in \Omega, \tag{6.7}$$

where $\alpha_p = 1 - N/p$. On the other hand, thanks to (5.4), we obtain the lower bound

$$(d_\Omega(x))^{\frac{p}{p-q}} \, w^{B_1}_{p,q}(0) \le w^\Omega_{p,q}(x), \qquad \text{for every } x \in \Omega. \tag{6.8}$$

By sending p to ∞ in (6.7) and (6.8) and taking into account (2.1), (6.6) and (4.16), we get that

$$\lim_{p\to\infty} w^\Omega_{p,q}(x) = d_\Omega(x), \qquad \text{for every } x \in \Omega.$$

Thanks to [**30**, Theorem 1], the pointwise convergence, combined with the convergence of the L^q norm given by (6.5), implies that

$$\lim_{p\to\infty} \|w^\Omega_{p,q} - d_\Omega\|_{L^q(\Omega)} = 0.$$

Part 2: convergence in $L^\infty(\Omega)$. By Corollary 5.4.1, we have that $w^\Omega_{p,q} \in C_0(\Omega) \cap C^{0,\alpha_p}(\overline{\Omega})$. Moreover, by applying the estimate on the sharp Morrey constant of Corollary 2.1.3, we have that $w^\Omega_{p,q}$ satisfies

$$[w^\Omega_{p,q}]_{C^{0,\alpha_p}(\overline{\Omega})} \le \left(\frac{1}{\mathfrak{m}_p(\Omega)} \right)^{\frac{1}{p}} \|\nabla w^\Omega_{p,q}\|_{L^p(\Omega)}.$$

By using (6.6) and Corollary 2.1.3, we have

$$\limsup_{p\to\infty} [w^\Omega_{p,q}]_{C^{0,\alpha_p}(\overline{\Omega})} \le 1,$$

and thus in particular the seminorms $[w^\Omega_{p,q}]_{C^{0,\alpha_p}(\overline{\Omega})}$ are uniformly bounded, for p large enough. We also observe that by Lemma 1.2.2, we have

$$[d_\Omega]_{C^{0,\alpha_p}(\overline{\Omega})} \leq (2\, r_\Omega)^{1-\alpha_p}.$$

We now apply Lemma 1.1.2 to $w^\Omega_{p,q} - d_\Omega$, with $\alpha = \alpha_p$ and $\gamma = q$. Thus, for every $0 < \beta < 1$ and every p such that $\alpha_p > \beta$, we have

$$[w^\Omega_{p,q} - d_\Omega]_{C^{0,\beta}(\overline{\Omega})} \leq C_1\, \|w^\Omega_{p,q} - d_\Omega\|^{\theta_p}_{L^q(\Omega)}\, [w^\Omega_{p,q} - d_\Omega]^{1-\theta_p}_{C^{0,\alpha_p}(\overline{\Omega})}, \qquad \text{with } \theta_p = \frac{\alpha_p - \beta}{\alpha_p + \dfrac{N}{q}},$$

and

$$\|w^\Omega_{p,q} - d_\Omega\|_{L^\infty(\Omega)} \leq C_2\, \|w^\Omega_{p,q} - d_\Omega\|^{\chi_p}_{L^q(\Omega)}\, [w^\Omega_{p,q} - d_\Omega]^{1-\chi_p}_{C^{0,\alpha_p}(\overline{\Omega})}, \qquad \text{with } \chi_p = \frac{\alpha_p}{\alpha_p + \dfrac{N}{q}}.$$

We observe that, for p diverging to ∞, the exponent α_p goes to 1. Thus, the constants C_1 and C_2, which depend on p through α_p, stay uniformly bounded as p goes to ∞ (see Lemma 1.1.2 and Remark 1.1.3). By using this fact, the bound on the C^{0,α_p} seminorms inferred above and the convergence in L^q proved in Part (1) of the proof, the previous interpolation estimates give

$$\lim_{p\to\infty} \Big(\|w^\Omega_{p,q} - d_\Omega\|_{L^\infty(\overline{\Omega})} + [w^\Omega_{p,q} - d_\Omega]_{C^{0,\beta}(\overline{\Omega})} \Big) = 0.$$

Finally, the convergence in $L^r(\Omega)$ for $q < r < \infty$ can be obtained by interpolation in Lebesgue spaces. □

§6.3 Asymptotics for λ_p

The following corollary generalizes the result shown, independently, in [**68**, Theorem 3.1] and [**84**, Lemma 1.2]. While these treat the case of bounded open sets, we enlarge the result to cover every open set, without further restrictions.

Corollary 6.3.1. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open set, then*

$$\lim_{p\to\infty} \Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{r_\Omega}, \tag{6.9}$$

where the right-hand side has to be considered 0*, if $r_\Omega = +\infty$.*

Proof. First of all, we note that for every $0 < r < r_\Omega$ there exists a ball $B_r(x_r) \subset \Omega$. Hence, by applying [**84**, Lemma 1.5], it holds

$$\limsup_{p\to\infty} \Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}} \leq \limsup_{p\to\infty} \Big(\lambda_p(B_r(x_r))\Big)^{\frac{1}{p}} = \frac{1}{r}.$$

By sending $r \to r_\Omega$, we get

$$\limsup_{p\to\infty} \Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}} \leq \frac{1}{r_\Omega},$$

where the right-hand side is 0 when $r_\Omega = +\infty$. In order to obtain the reverse inequality when $r_\Omega < +\infty$, it is sufficient to apply (5.6) and (2.10), to get that

$$\liminf_{p\to\infty} \Big(\lambda_p(\Omega)\Big)^{\frac{1}{p}} \geq \frac{1}{r_\Omega} \lim_{p\to\infty} \Big(\mathfrak{h}_p(\Omega)\Big)^{\frac{1}{p}} = \frac{1}{r_\Omega}.$$

This concludes the proof. □

§6.4 Asymptotics for the first $p-$eigenfunction

We first recall that for every function $u \in W^{1,\infty}(\Omega)$ vanishing on the boundary $\partial\Omega$, we have

$$|u(x)| \leq d_\Omega(x)\,\|\nabla u\|_{L^\infty(\Omega)}, \qquad \text{for every } x \in \Omega.$$

This may be seen as a limit case of Hardy's inequality. In particular $\pm\, d_\Omega/r_\Omega$ is a solution of the following minimization problem

$$\min_{u\in W^{1,\infty}(\Omega)} \Big\{\|\nabla u\|_{L^\infty(\Omega)} \,:\, \|u\|_{L^\infty(\Omega)} = 1,\, u \equiv 0 \text{ on } \partial\Omega\Big\} = \frac{1}{r_\Omega}, \tag{6.10}$$

provided Ω has finite inradius.

In the following theorem we aim to show that the extremals for $\lambda_p(\Omega)$ converge to solutions of the variational problem (6.10), as p goes to ∞.

Theorem 6.4.1. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open connected quasibounded set. Then, for every $N < p < \infty$, there exists a unique positive solution $u_p \in W^{1,p}_0(\Omega)$ of the problem $\lambda_p(\Omega)$. Moreover, the family $\{u_p\}_{p>N}$ is precompact in $C^{0,\beta}(\overline{\Omega})$ for every $0 < \beta < 1$ and every accumulation point u_∞ is a solution of (6.10), possibly different from d_Ω/r_Ω.*

Proof. We first observe that $\lambda_p(\Omega) > 0$, thanks to the assumption on Ω and Theorem 5.2.1. Thus, by Remark 1.4.4, we have

$$W^{1,p}_0(\Omega) = \mathcal{D}^{1,p}_0(\Omega).$$

By Theorem 5.3.1, for every $p > N$ the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Hence, by using also Lemma 1.3.3, it follows that, for every $p > N$, there exists a positive solution $u_p \in W^{1,p}_0(\Omega)$ of the minimization problem defining $\lambda_p(\Omega)$. Uniqueness can now be inferred by using the *hidden convexity* principle stated in Theorem 3.2.1. See also **[4]** and **[99]** for other proofs of the uniqueness.

Without loss of generality, let $\{p_n\}_{n\in\mathbb{N}}$ be an increasing sequence diverging at ∞. In particular, there exists $n_0 \in \mathbb{N}$ such that $p_n > N$ for every $n \geq n_0$. We denote by u_{p_n} the unique positive solution of the problem defining $\lambda_{p_n}(\Omega)$. By applying (2.5) and (2.7), we find that

$$|u_{p_n}(x)| \leq \frac{1}{\Big(\mu_{p_n}(B_1)\Big)^{\frac{1}{p_n}}} \Big(\lambda_{p_n}(\Omega)\Big)^{\frac{1}{p_n}} d_\Omega(x)^{\alpha_{p_n}}, \qquad \text{for every } x \in \overline{\Omega}, \tag{6.11}$$

and

$$|u_{p_n}(x) - u_{p_n}(y)| \leq \frac{1}{\Big(\mu_{p_n}(B_1)\Big)^{\frac{1}{p_n}}} \Big(\lambda_{p_n}(\Omega)\Big)^{\frac{1}{p_n}} |x-y|^{\alpha_{p_n}}, \qquad \text{for every } x, y \in \overline{\Omega}. \tag{6.12}$$

Since Ω is quasibounded, the previous estimates assures that we can apply Proposition 1.2.6. Thus, we get that $\{u_{p_n}\}_{n\geq n_0}$ converges uniformly to a function $u_\infty \in C_0(\Omega)$, up to a subsequence. By passing to the limit in (6.11) and in (6.12) as n goes to ∞, we obtain that

$$|u_\infty(x)| \leq \frac{1}{r_\Omega}\, d_\Omega(x), \qquad \text{for every } x \in \overline{\Omega},$$

and

$$|u_\infty(x) - u_\infty(y)| \leq \frac{1}{r_\Omega}\, |x-y|, \qquad \text{for every } x, y \in \overline{\Omega}.$$

Observe that we also used (2.1) and (6.9). In particular $u_\infty \in W^{1,\infty}(\Omega)$ and it satisfies

$$\|u_\infty\|_{L^\infty(\Omega)} \leq 1 \qquad \text{and} \qquad \|\nabla u_\infty\|_{L^\infty(\Omega)} \leq \frac{1}{r_\Omega}.$$

If we also prove that $\|u_\infty\|_{L^\infty(\Omega)} \geq 1$, we can conclude that u_∞ is a minimizer for the problem (6.10).

In order to show this, for every $R > 0$, we take $\eta_R \in C^\infty(\mathbb{R}^N)$ such that

$$0 \leq \eta_R \leq 1, \quad \eta_R = 1 \text{ in } \mathbb{R}^N \setminus B_{R+1}, \quad \eta_R = 0 \text{ in } B_R, \quad |\nabla \eta_R| \leq C,$$

for some universal constant $C > 0$. Then

$$\begin{aligned}\sup_{n\geq n_0} \|\nabla(u_{p_n}\eta_R)\|_{L^{p_n}(\Omega)} &\leq \sup_{n\geq n_0} \|\nabla u_{p_n}\|_{L^{p_n}(\Omega)} + C \sup_{n>N} \|u_{p_n}\|_{L^{p_n}(\Omega)} \\ &= \sup_{n\geq n_0} \Big(\lambda_{p_n}(\Omega)\Big)^{\frac{1}{p_n}} + C \\ &\leq \frac{1}{r_\Omega} \sup_{n\geq n_0} \Big(\lambda_{p_n}(B_1)\Big)^{\frac{1}{p_n}} + C =: M < +\infty.\end{aligned}$$

We notice that $M > 0$ only depends on N and r_Ω. Now, thanks to (2.10), there exists $\overline{p} > N$ such that

$$\Big(\mathfrak{h}_p(\Omega)\Big)^{\frac{1}{p}} \geq \frac{1}{2}, \qquad \text{for every } p \geq \overline{p}.$$

Since Ω is quasibounded, for every $0 < \varepsilon \leq 1/2$, there exists $R_\varepsilon > 0$ such that

$$\|d_\Omega\|_{L^\infty(\Omega\setminus B_{R_\epsilon})} < \frac{\varepsilon}{2\,M}.$$

By using the properties of η and applying Hölder's and Hardy's inequalities to $u_{p_n}\eta_{R_\varepsilon} \in W^{1,p}_0(\Omega)$, for every $p_n \geq \max\{\overline{p}, p_{n_0}\}$, we have that

$$\begin{aligned}\int_{\Omega\setminus B_{R_\varepsilon+1}} |u_{p_n}|^{p_n}\,dx = \int_{\Omega\setminus B_{R_\varepsilon+1}} |u_{p_n}\eta_{R_\varepsilon}|^{p_n}\,dx &\leq \|d_\Omega\|^{p_n}_{L^\infty(\Omega\setminus B_{R_\varepsilon})} \int_\Omega \frac{|u_{p_n}\,\eta_{R_\epsilon}|^{p_n}}{d_\Omega^{p_n}}\,dx \\ &\leq \|d_\Omega\|^{p_n}_{L^\infty(\Omega\setminus B_{R_\varepsilon})}\, \frac{1}{\mathfrak{h}_{p_n}(\Omega)} \int_\Omega |\nabla(u_{p_n}\,\eta_{R_\varepsilon})|^{p_n}\,dx \\ &\leq \varepsilon^{p_n}.\end{aligned}$$

Hence

$$1 = \int_{\Omega\setminus B_{R_\varepsilon+1}} |u_{p_n}|^{p_n}\,dx + \int_{B_{R_\varepsilon+1}} |u_{p_n}|^{p_n}\,dx \leq \varepsilon^{p_n} + \int_{B_{R_\varepsilon+1}} |u_{p_n}|^{p_n}\,dx,$$

that is

$$\left(1-\varepsilon^{p_n}\right)^{\frac{1}{p_n}} \leq \left(\omega_N\left(R_\epsilon+1\right)^N\right)^{\frac{1}{p_n}} \sup_{B_{R_\epsilon+1}} |u_{p_n}|, \qquad \text{for every } p_n \geq \max\{\overline{p}, p_{n_0}\}.$$

By exploiting the uniform convergence of the sequence $\{u_{p_n}\}_{n\geq n_0}$ to u_∞ on compact sets, if we take the limit as n goes to ∞, we get

$$1 \leq \sup_{B_{R_\varepsilon+1}} |u_\infty| \leq \sup_{\Omega} |u_\infty|,$$

which proves the claim for every Ω quasibounded set.

In order to get the convergence in $C^{0,\beta}(\overline{\Omega})$ for $0<\beta<1$, we can use the same interpolation argument as in *Part 2* of the proof of Theorem 6.2.1. It is sufficient to observe that it holds

$$[u_\infty]_{C^{0,\beta}(\overline{\Omega})} \leq 2^{1-\beta} \|u_\infty\|^{1-\beta}_{L^\infty(\Omega)} \|\nabla u_\infty\|^{\beta}_{L^\infty(\Omega)},$$

an estimate that can be proved by repeating the proof of Lemma 1.2.2. We leave the details to the reader. ☐

Remark 6.4.2. We underline that, differently from the case $1\leq q<p$, when $p=q$ it may happen that the accumulation points of the family $\{u_p\}_{p>N}$ do not coincide with d_Ω (see [**68**, Corollary 4.7]). We refer to [**61**, Theorem 3.14] for a study of the multiplicity of extremals for problem (6.10), in the case of open bounded sets.

Let us define the supremal problem

$$\lambda_{p,\infty}(\Omega) = \min_{u\in W_0^{1,p}(\Omega)} \left\{\int_\Omega |\nabla u|^p\,dx \,:\, \|u\|_{L^\infty(\Omega)}=1\right\}.$$

We notice that in [**61**] and [**78**], a deeper study on $\lambda_{p,\infty}$ and its extremals can be found.

Hence, with the same argument applied in the proof of Theorem 6.4.1, we can show a similar result for minimizers of $\lambda_{p,\infty}(\Omega)$, whose existence is given by Corollary 5.4.3 when Ω is quasibounded. This generalizes [**61**, Theorem 3.3]. We omit the proof.

Theorem 6.4.3. *Let $N<p$ and let $\Omega\subsetneq\mathbb{R}^N$ be a quasibounded open set. Let $u_{p,\infty}\in W_0^{1,p}(\Omega)$ be a positive solution of $\lambda_{p,\infty}(\Omega)$. Then the family $\{u_{p,\infty}\}_{p>N}$ is precompact in $C^{0,\beta}(\overline{\Omega})$ for every $0<\beta<1$ and every accumulation point u_∞ is a solution of* (6.10).

CHAPTER 7

GEOMETRIC ESTIMATES FOR SOBOLEV-POINCARÉ CONSTANTS

Several estimates have been studied in literature for the Sobolev-Poincaré constants $\lambda_{p,q}$ and for shape functionals involving such quantities (for example, see [**15**, **34**]); in particular, in this chapter, we provide some generalizations of two well-known geometric inequalities: the Makai inequality and the Hersch-Protter inequality.

In the whole chapter we will find the symbol $\pi_{p,q}$, which we recall to denote the one-dimensional Sobolev-Poincaré constant, defined as

$$\pi_{p,q} := \inf_{u \in C_0^\infty((0,1))} \left\{ \|u'\|_{L^p([0,1])} \ : \ \|u\|_{L^q([0,1])} = 1 \right\}.$$

§7.1 The Makai inequality for $\lambda_{p,q}$

In this section, we prove the generalization for the Makai inequality. The proof is inspired by the covering argument for polygonal sets exploited by Makai in the planar case. In the N-dimensional case, thanks to a standard approximation argument, we can restrict ourselves to consider the case when $\Omega \subsetneq \mathbb{R}^N$ is the interior of a polytope K (see Appendix D.1 for an introduction to polytopes). In this case, in order to prove the claimed inequality, our key tool is given by Lemma D.2.1 where we construct a suitable covering of Ω by means of convex sets Ω_i, every one satisfying the property that $\partial\Omega_i \cap \Omega$ is the graph of a continuous function defined on a facet S_i of the polytope K.

Theorem 7.1.1 (Makai's inequality)**.** *Let $1 \leq q < p < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be a convex bounded open set. Then, the following lower bound holds*

$$\lambda_{p,q}(\Omega) \geq \frac{C_{p,q}}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}}}, \tag{7.1}$$

where $C_{p,q}$ is the positive constant given by

$$C_{p,q} = \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}}.$$

Moreover, the estimate (7.1) is sharp.

Proof. By following [**102**], we divide the proof in three parts: first, we prove the lower bound (7.1) when Ω is the interior of a polytope K, then, by applying an approximation argument, we show that such a lower bound holds when $\Omega \subsetneq \mathbb{R}^N$ is a general convex set. Finally, we show that (7.1) is asymptotically sharp for slab-type sequences.

Part 1: Makai's inequality for a polytope. Without loss of generality, let us suppose that $0 \in \Omega$. Moreover, in this step we assume that $\Omega = \mathrm{int}(K)$, where $K \subset \mathbb{R}^N$ is a polytope. According to the notation in Lemma D.2.1, we consider the subsets Ω_i given by (D.1), with $i \in \{1, \dots, h\}$.

Now, we will show that, for every $i \in \{1, \dots, h\}$ and for every $u \in C_0^\infty(\Omega)$, it holds

$$\int_{\Omega_i} |u|^q\, dx \le \left(\frac{2}{\pi_{p,q}}\right)^q \left(\frac{pq+p-q}{p-q}\right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} d_\Omega^{\frac{p\,q}{p-q}}\, dx\right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} |\nabla u|^p\, dx\right)^{\frac{q}{p}}. \tag{7.2}$$

Indeed, let $i \in \{1, \dots, h\}$ and, up to translations and rotations, we can assume that

$$H_i = \{(y,t) \in \mathbb{R}^{N-1} \times \mathbb{R} :\, t = 0\}$$

is the affine hyperplane containing S_i. Then

$$t = d_{S_i}(y,t) = d_\Omega(y,t), \qquad \text{for every } (y,t) \in \Omega_i. \tag{7.3}$$

Thanks to Lemma D.2.1, we have that $\Pi_i : \overline{\partial\Omega_i \cap \Omega} \to S_i$ is a bijective and continuous function between two compact sets. Hence, it has a continuous inverse functions $\Pi_i^{-1} : S_i \to \overline{\partial\Omega_i \cap \Omega}$. Defining $S_i' = \{y \in \mathbb{R}^{N-1} :\, (y,0) \in S_i\}$, we obtain that there exists a continuous function $f_i : S_i' \to [0, +\infty)$ such that

$$f_i(y) = |\Pi_i^{-1}(y,0) - (y,0)|$$

is a continuous function such that, for every $y \in S_i'$, it holds

$$f_i(y) = t \qquad \Longleftrightarrow \qquad (y,t) \in \overline{\partial\Omega_i \cap \Omega}. \tag{7.4}$$

It is easy to show that

$$\Omega_i = \Big\{(y,t) \in S_i' \times \mathbb{R} :\, 0 < t \le f_i(y)\Big\}. \tag{7.5}$$

Indeed, the inclusion "$\subseteq$" follows by using (7.3) and (7.4), while the converse one "$\supseteq$" is an application of Parts (3) and (6) of Lemma D.2.1, taking into account that

$$f_i(y) > 0 \qquad \Longleftrightarrow \qquad (y,0) \in \mathrm{relint}(S_i).$$

Now, we recall that

$$\left(\frac{\pi_{p,q}}{2}\right)^p = \min_{\varphi \in W^{1,p}((0,1))\setminus\{0\}} \left\{ \frac{\displaystyle\int_0^1 |\varphi'|^p\, dt}{\left(\displaystyle\int_0^1 |\varphi|^q\, dt\right)^{\frac{p}{q}}} : \varphi(0) = 0 \right\},$$

(see [**20**, Lemma A.1]), which implies that, for every $s > 0$

$$\left(\int_0^s |\varphi|^q\, dt\right)^{\frac{p}{q}} \le \left(\frac{2}{\pi_{p,q}}\right)^p s^{\frac{pq+p-q}{q}} \int_0^s |\varphi'|^p\, dt, \qquad \text{for every } \varphi \in C_0^\infty((0,s]). \tag{7.6}$$

Hence, for every $u \in C_0^\infty(\Omega)$ and for every $i \in \{1, \dots, h\}$, thanks to formula (7.5), by using Fubini's Theorem and (7.6) with $s = f_i(y)$, we get

$$\begin{aligned} \int_{\Omega_i} |u(x)|^q \, dx &= \int_{S_i'} \int_0^{f_i(y)} |u(y,t)|^q \, dt \, dy \\ &\le \left(\frac{2}{\pi_{p,q}} \right)^q \int_{S_i'} f_i(y)^{\frac{pq+p-q}{p}} \left(\int_0^{f_i(y)} \left| \frac{\partial u}{\partial t}(y,t) \right|^p \, dt \right)^{\frac{q}{p}} dy, \end{aligned} \tag{7.7}$$

Then, by applying the Hölder inequality, it follows

$$\begin{aligned} \int_{\Omega_i} |u(x)|^q \, dx &\le \left(\frac{2}{\pi_{p,q}} \right)^q \left(\int_{S_i'} f_i(y)^{\frac{pq+p-q}{p-q}} \, dt \right)^{\frac{p-q}{p}} \left(\int_{S_i'} \int_0^{f_i(y)} \left| \frac{\partial u}{\partial t}(y,t) \right|^p \, dy \, dt \right)^{\frac{q}{p}} \\ &\le \left(\frac{2}{\pi_{p,q}} \right)^q \left(\int_{S_i'} f_i(y)^{\frac{pq+p-q}{p-q}} \, dt \right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} |\nabla u|^p \, dx \right)^{\frac{q}{p}}. \end{aligned} \tag{7.8}$$

Taking into account (7.3), we have that

$$\begin{aligned} \int_{S_i'} f_i(y)^{\frac{pq+p-q}{p-q}} \, dy &= \left(\frac{pq+p-q}{p-q} \right) \int_{S_i'} \left(\int_0^{f_i(y)} t^{\frac{p\,q}{p-q}} \, dt \right) dy \\ &= \left(\frac{pq+p-q}{p-q} \right) \int_{S_i'} \left(\int_0^{f_i(y)} (d_\Omega(y,t))^{\frac{p\,q}{p-q}} \, dt \right) dy \\ &= \left(\frac{pq+p-q}{p-q} \right) \int_{\Omega_i} d_\Omega^{\frac{p\,q}{p-q}} \, dx. \end{aligned} \tag{7.9}$$

By combining (7.8) and (7.9), for every $i \in \{1, \dots, h\}$, we obtain (7.2).

Finally, since

$$\Omega = \bigcup_{i=1}^h \Omega_i,$$

by summing with respect to the index $i \in \{1, \dots, h\}$ in (7.2), it follows that

$$\int_\Omega |u|^q \, dx \le \left(\frac{2}{\pi_{p,q}} \right)^q \left(\frac{pq+p-q}{p-q} \right)^{\frac{p-q}{p}} \sum_{i=1}^h \left(\int_{\Omega_i} d_\Omega^{\frac{p\,q}{p-q}} \, dx \right)^{\frac{p-q}{p}} \left(\int_{\Omega_i} |\nabla u|^p \, dx \right)^{\frac{q}{p}}.$$

By applying Hölder's inequality

$$\|a\, b\|_{\ell^1} \le \|a\|_{\ell^r} \, \|b\|_{\ell^{r'}},$$

with $r = p/q$, we get

$$\int_\Omega |u|^q \, dx \le \left(\frac{2}{\pi_{p,q}} \right)^q \left(\frac{pq+p-q}{p-q} \right)^{\frac{p-q}{p}} \left(\int_\Omega d_\Omega^{\frac{p\,q}{p-q}} \, dx \right)^{\frac{p-q}{p}} \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{q}{p}}.$$

and by raising to the power p/q on both sides, this in turns implies

$$\frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\left(\displaystyle\int_\Omega |u|^q\,dx\right)^{\frac{p}{q}}} \geq \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}} \frac{1}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}}, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

Taking the infimum on $C_0^\infty(\Omega)$ on the left-hand side, we get that for every polytope $K \subset \mathbb{R}^N$, the set $\Omega = \mathrm{int}(K)$ satisfies the lower bound (7.1), as desired.

Part 2: approximation argument. If $\Omega \subsetneq \mathbb{R}^N$ is a general convex bounded open set, thanks to [**120**, Theorem 1.8.19], for every $0 < \varepsilon \ll 1$, there exists a polytope K_ε such that

$$K_\varepsilon \subset \Omega \subset \overline{\Omega} \subset \frac{1}{1-\varepsilon} K_\varepsilon.$$

In particular, we have that

$$(1-\varepsilon)\,\Omega \subset \mathrm{int}(K_\varepsilon).$$

By applying the scaling property of $\lambda_{p,q}$ and its monotonicity with respect to the set inclusion, we have that

$$\frac{\lambda_{p,q}(\Omega)}{(1-\varepsilon)^{p-N+N\frac{p}{q}}} = \lambda_{p,q}((1-\varepsilon)\,\Omega) \geq \lambda_{p,q}(\mathrm{int}(K_\varepsilon)),$$

and, thanks to Part (1) of the proof, we obtain

$$\frac{\lambda_{p,q}(\Omega)}{(1-\varepsilon)^{p-N+N\frac{p}{q}}} \geq \frac{C_{p,q}}{\left(\displaystyle\int_{\mathrm{int}(K_\varepsilon)} d_{\mathrm{int}(K_\varepsilon)}^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}} \geq \frac{C_{p,q}}{\left(\displaystyle\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}},$$

where the last inequality follows from the fact that $d_{\mathrm{int}(K_\varepsilon)}(x) \leq d_\Omega(x)$, for every $x \in K_\epsilon$. Then, by sending $\varepsilon \to 0^+$, we get that Ω satisfies (7.1).

Part 3: sharpness. Now we will show that estimate (7.1) is asymptotically sharp for slab-type sequences

$$\Omega_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^{N-1} \times (0,1), \qquad \text{with } L \geq 1.$$

With this aim, we denote by $y = (x_1, \dots, x_{N-1})$ a point of $\mathbb{R}^{N-1}$. Let S_0 and S_1 be the facets of Ω_L contained, respectively, in the hyperplanes $\{(y,t) \in \mathbb{R}^N : t = 0\}$ and $\{(y,t) \in \mathbb{R}^N : t = 1\}$, and we define the *lateral surface* $\mathcal{S}_L$ of Ω_L as the following union:

$$\mathcal{S}_L := \bigcup_{i=1}^{N-1} \left\{(y,t) \in \Omega_L : x_i = \pm\frac{L}{2},\, 0 < t < 1\right\}.$$

Then, we define the set Ω_1 as

$$\Omega_1 = \left\{x \in \Omega_L : d_{\mathcal{S}_L}(x) \geq 1\right\}.$$

Since $|\Omega_1| = (L-2)^{N-1}$, we obtain that

$$|\Omega_L \setminus \Omega_1| = L^{N-1} - (L-2)^{N-1} \sim L^{N-2}\,C_N, \qquad \text{as } L \to \infty,$$

which implies

$$\int_{\Omega_L\setminus\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx \le r_{\Omega_L}^{\frac{p\,q}{p-q}}|\Omega_L\setminus\Omega_1| \sim L^{N-2}\,C_N\,\left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}, \qquad \text{as } L\to\infty.$$

Moreover, since, for every $(y,t)\in\Omega_1$, it holds that

$$d_{\Omega_L}(y,t)=d_{S_0\cup S_1}(y,t)=\min\{t,1-t\},$$

we obtain

$$\int_{\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dt = \left(\int_{\left(-\frac{L}{2}+1,\frac{L}{2}-1\right)^{N-1}} dy\right)\left(2\int_0^{1/2} t^{\frac{p\,q}{p-q}}\,dt\right) = (L-2)^{N-1}\,\left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}\,\frac{p-q}{pq+p-q}.$$

In particular, since

$$\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx = \int_{\Omega_L\setminus\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx + \int_{\Omega_1} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx,$$

we have the following asymptotic behavior

$$\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx \sim L^{N-1}\,\left(\frac{1}{2}\right)^{\frac{p\,q}{p-q}}\,\frac{p-q}{pq+p-q}, \qquad \text{as } L\to\infty.$$

Hence, we finally obtain, as $L\to\infty$

$$\frac{C_{p,q}}{\left(\displaystyle\int_{\Omega_L} d_{\Omega_L}^{\frac{pq}{p-q}}\,dx\right)^{\frac{p-q}{q}}} = \left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}}\,\frac{1}{\left(\displaystyle\int_{\Omega_L} d_{\Omega_L}^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}}} \sim (\pi_{p,q})^p\,\frac{1}{L^{\frac{(p-q)(N-1)}{q}}}.$$

On the other hand, by [**19**, Main Theorem], the following upper bound holds

$$\lambda_{p,q}(\Omega) < \left(\frac{\pi_{p,q}}{2}\right)^p\left(\frac{P(\Omega)}{|\Omega|^{1-\frac{1}{p}+\frac{1}{q}}}\right)^p$$

for every $\Omega\subset\mathbb{R}^N$ convex bounded open set and it is asimptotically sharp for slab–type sequences Ω_L, as $L\to\infty$. Finally, since

$$P(\Omega_L)\sim 2\,L^{N-1} \qquad \text{and} \qquad |\Omega_L|\sim L^{N-1}, \qquad \text{as } L\to\infty,$$

we obtain

$$\lambda_{p,q}(\Omega_L)\sim(\pi_{p,q})^p\frac{1}{L^{\frac{(p-q)(N-1)}{q}}}, \qquad \text{as } L\to\infty.$$

The proof is over. □

§7.2 Makai's costants for non-convex sets

We address this section to the study of optimal constants in the Makai-type inequality for other classes of open sets. To this aim, we define

$$\widetilde{C}_{p,q} = \inf\left\{\lambda_{p,q}(\Omega)\,\|d_\Omega\|^p_{L^{\frac{p\,q}{p-q}}(\Omega)} : \Omega \subsetneq \mathbb{R}^N \text{ is an open set, } d_\Omega \in L^{\frac{p\,q}{p-q}}(\Omega)\right\},$$

and

$$\widehat{C}_{2,q} = \inf\left\{\lambda_{2,q}(\Omega)\,\|d_\Omega\|^2_{L^{\frac{2\,q}{2-q}}(\Omega)} : \Omega \subsetneq \mathbb{R}^2 \text{ is a simply connected open set, } d_\Omega \in L^{\frac{2\,q}{2-q}}(\Omega)\right\}.$$

Thanks to the implications (i) in Theorems 5.1.1 and 5.2.1, here the assumptions on the integrability of d_Ω are necessary conditions in order to have $\lambda_{p,q}(\Omega) > 0$. In the sequel, every time we consider the case $q = p$, we assume that q has to be larger than 1. We first list some known facts:

- when $1 < p \le N$, it holds that

  $$\widetilde{C}_{p,q} = 0, \qquad \text{for every } q \le p.$$

  Indeed, we consider a bounded open set $\Omega \subsetneq \mathbb{R}^N$ and we remove from it a periodic array of n points, by denoting as Ω_n the open set so constructed. As $p \le N$, points in $\mathbb{R}^N$ have zero $p-$capacity (see [**126**, Chapter 17]), then, for every $q \le p$, it holds

  $$\lambda_{p,q}(\Omega_n) = \lambda_{p,q}(\Omega), \qquad \text{for every } n \in \mathbb{N}.$$

  Since r_{Ω_n} tends to 0, as $n \to \infty$, the above equality implies that

  $$\widetilde{C}_{p,q} \le \limsup_{n\to\infty} \lambda_{p,q}(\Omega_n)\,\|d_{\Omega_n}\|^p_{L^{\frac{p\,q}{p-q}}(\Omega_n)} \le \lambda_{p,q}(\Omega)\,|\Omega|^{\frac{p-q}{q}} \limsup_{n\to\infty} r^p_{\Omega_n} = 0,$$

  for every $q \le p$. In particular, in this range, it follows that

  $$\widetilde{C}_{p,q} < C_{p,q};$$

- when $p > N$,

  $$\widetilde{C}_{p,q} \ge \left(\frac{p-N}{p}\right)^p > 0, \qquad \text{for every } 1 \le q < p, \tag{7.10}$$

  thanks to (5.3) and (2.9);

- when $1 \le q \le 2$,

  $$\widehat{C}_{2,q} \ge \frac{1}{16} > 0 = \widetilde{C}_{2,q},$$

  thanks to the joint application of (5.3) with $p = 2$ and an old result due to Ancona ([**5**]), which states that

  $$\mathfrak{h}_2(\Omega) \ge \frac{1}{16}, \qquad \text{for every } \Omega \subset \mathbb{R}^2 \text{ simply connected set.}$$

  Moreover, by using a different argument, in [**103**] Makai showed that

  $$\frac{1}{4} \le \widehat{C}_{2,2} < \frac{\pi^2}{4},$$

where, in order to prove the upper bound, he exhibited a simply connected open set $\Omega \subsetneq \mathbb{R}^2$ satisfying

$$\lambda_2(\Omega)\, r_\Omega^2 < \frac{\pi^2}{4}. \tag{7.11}$$

We notice that, thanks to the optimality for the Hersch-Protter inequality (12), we have that

$$C_{2,2} = \frac{\pi^2}{4},$$

hence, from (7.11), it also follows that $\widehat{C}_{2,2} < C_{2,2}$.

Hence, with these results at hand, some natural questions that arise are whether

- $\widetilde{C}_{p,q} < C_{p,q}$, when $p > N$ and $1 \le q < N$;
- $\widetilde{C}_{p,p} < C_{p,p}$, when $p > N$;
- $\widehat{C}_{2,q} < C_{2,q}$, when $N = 2$ and $1 \le q < 2$.

In this section, we give some partial answers to the above list of questions.

We first focus on the class of general open sets when $q < N < p$.

Proposition 7.2.1 (Case of general open sets in $\mathbb{R}^N$). *Let $1 \le q < N$, then there exists $\overline{p} = \overline{p}(q) > N$ such that*

$$\widetilde{C}_{p,q} < C_{p,q}, \qquad \text{for every } p \in (q, \overline{p}]. \tag{7.12}$$

Proof. We consider the *infinite fragile tower* set $\mathcal{T} \subset \mathbb{R}^N$ defined as in Theorem 5.1.1. By contruction, it satisfies the following properties:

- $d_{\mathcal{T}} \in L^1(\mathcal{T}) \cap L^\infty(\mathcal{T})$;
- $\lambda_{p,q}(\mathcal{T}) = 0$, for every $1 \le q < p \le N$,

Moreover, thanks to (7.10), $\lambda_{p,q}(\mathcal{T}) > 0$, for every $p > N$ and $1 \le q \le p$. In order to show (7.12), it is sufficient to prove that

$$\lim_{p \searrow N} \lambda_{p,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} < \lim_{p \searrow N} C_{p,q}.$$

With this aim, we observe that, since

$$d_{\mathcal{T}}^{\frac{p\,q}{p-q}} \le r_{\mathcal{T}}^{\frac{p\,q}{p-q} - \frac{N\,q}{N-q}}\, d_{\mathcal{T}}^{\frac{N\,q}{N-q}} \in L^1(\mathcal{T}),$$

by using the Dominated Convergence theorem, the following limit holds

$$\lim_{p \searrow N} \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\, dx \right)^{\frac{p-q}{q}} = \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{N\,q}{N-q}}\, dx \right)^{\frac{N-q}{q}}. \tag{7.13}$$

Moreover, we have that

$$\lim_{p \searrow N} C_{p,q} = \lim_{p \searrow N} \left(\frac{\pi_{p,q}}{2} \right)^p \left(\frac{p-q}{pq + p - q} \right)^{\frac{p-q}{q}} = C_{N,q}. \tag{7.14}$$

The last limit follows by taking into account that, as computed in [**124**, equation (7)],

$$\pi_{p,q} = \frac{2}{q}\left(1+\frac{q}{p'}\right)^{\frac{1}{q}}\left(1+\frac{p'}{q}\right)^{-\frac{1}{p}} B\left(\frac{1}{q},\frac{1}{p'}\right),$$

where $p' = p/(p-1)$ and B is the *Euler Beta function*, which is continuous on $(0,+\infty)$. If we show that

$$\limsup_{p\searrow N} \lambda_{p,q}(\mathcal{T}) \leq \lambda_{N,q}(\mathcal{T}), \tag{7.15}$$

by using (7.13), (7.14) and (7.15), we obtain that

$$\lim_{p\searrow N} \lambda_{p,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{p\,q}{p-q}}\,dx\right)^{\frac{p-q}{q}} \leq \lambda_{N,q}(\mathcal{T}) \left(\int_{\mathcal{T}} d_{\mathcal{T}}^{\frac{N\,q}{N-q}}\,dx\right)^{\frac{N-q}{q}} = 0 < C_{N,q} = \lim_{p\searrow N} C_{p,q},$$

which gives the desired conclusion.
In order to prove the claim (7.15), we note that, for every $1 \leq r < \infty$ and for every open set $\Omega \subset \mathbb{R}^N$, it holds

$$\lim_{p\searrow r} \|\nabla\varphi\|^p_{L^p(\Omega)} = \|\nabla\varphi\|^r_{L^r(\Omega)}, \qquad \text{for every } \varphi \in C_0^\infty(\Omega). \tag{7.16}$$

Indeed, if $\varphi \in C_0^\infty(\Omega)$, then, for every $p > r$, it follows that

$$\int_\Omega |\nabla\varphi|^p\,dx = \int_\Omega |\nabla\varphi|^{p-r}\,|\nabla\varphi|^r\,dx \leq \|\nabla\varphi\|^{p-r}_{L^\infty(\Omega)} \int_\Omega |\nabla\varphi|^r\,dx,$$

which implies

$$\limsup_{p\searrow r} \|\nabla\varphi\|^p_{L^p(\Omega)} \leq \|\nabla\varphi\|^r_{L^r(\Omega)}.$$

On the other hand, by Fatou's Lemma, we also have that

$$\int_\Omega |\nabla\varphi|^r\,dx \leq \liminf_{p\searrow r} \int_\Omega |\nabla\varphi|^p\,dx.$$

By applying (7.16) with $r = N$, we obtain

$$\limsup_{p\searrow N} \lambda_{p,q}(\mathcal{T}) \leq \limsup_{p\searrow N} \frac{\displaystyle\int_{\mathcal{T}} |\nabla\varphi|^p\,dx}{\left(\displaystyle\int_{\mathcal{T}} |\varphi|^q\,dx\right)^{p/q}} = \frac{\displaystyle\int_{\mathcal{T}} |\nabla\varphi|^N\,dx}{\left(\displaystyle\int_{\mathcal{T}} |\varphi|^q\,dx\right)^{N/q}}, \qquad \text{for every } \varphi \in C_0^\infty(\mathcal{T}),$$

and by taking the infimum on $C_0^\infty(\mathcal{T})$, this easily implies (7.15). □

Remark 7.2.2. Let $1 \leq q < \infty$. We notice that

$$\lim_{p\to\infty} \left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} = \lim_{p\to\infty} (C_{p,q})^{\frac{1}{p}} = 1.$$

Indeed, by Corollary 6.1.1, it holds that

$$\lim_{p\to\infty} \frac{\pi_{p,q}}{2} = \frac{1}{2}\frac{1}{\left(\displaystyle\int_0^1 (\min\{t,1-t\})^q\,dx\right)^{\frac{1}{q}}} = \frac{1}{(q+1)^{\frac{1}{q}}},$$

which implies

$$\lim_{p\to\infty} (C_{p,q})^{\frac{1}{p}} = \lim_{p\to\infty} \left(\frac{\pi_{p,q}}{2}\right)^{p} \left(\frac{p-q}{pq+p-q}\right)^{\frac{p-q}{q}} = 1.$$

Then, taking into account also (7.10), it easily follows that

$$1 \le \liminf_{p\to\infty} \left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} \le \limsup_{p\to\infty} \left(\widetilde{C}_{p,q}\right)^{\frac{1}{p}} \le \lim_{p\to\infty} (C_{p,q})^{\frac{1}{p}} = 1.$$

In the next result, we restrict to consider $N = 2$, and we will discuss the cases of the simply connected open sets when $p = 2$ and $q < 2$ and the general open sets when $q = p > 2$.

Proposition 7.2.3 (Planar case). *Let $N = 2$, then there exist $1 \le \overline{q} < 2$ and $\overline{p} > 2$ such that*

$$\widehat{C}_{2,q} < C_{2,q}, \qquad \textit{for every } q \in [\overline{q}, 2], \tag{7.17}$$

and

$$\widetilde{C}_{p,p} < C_{p,p}, \qquad \textit{for every } p \in [2, \overline{p}]. \tag{7.18}$$

Proof. We first construct an explicit example of a simply connected open set $\widetilde{\Omega} \subset \mathbb{R}^2$ such that

$$\lambda(\widetilde{\Omega})\, r^2_{\widetilde{\Omega}} < C_{2,2} = \frac{\pi^2}{4}. \tag{7.19}$$

Let $A \subset \mathbb{R}^2$ be an annulus from which we remove a segment, that is

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 : 1 < \sqrt{x_1^2 + x_2^2} < 2\right\} \setminus \left\{(x_1, 0) \in \mathbb{R}^2 : 1 < x_1 < 2\right\}.$$

Then it holds that $\lambda(A) = \pi^2$ (see [**112**], page 551). Now, for a fixed $0 < \varepsilon < 1$, we consider the simply connected open set

$$\widetilde{\Omega} = A \cup \left\{(x_1, x_2) \in \mathbb{R}^2 : \sqrt{4 - x_2^2} \le x_1 < 2 + \varepsilon, -\varepsilon < x_2 < \varepsilon\right\} \setminus \left\{(x_1, 0) \in \mathbb{R}^2 : 1 < x_1 < 2 + \epsilon\right\}.$$

Since $A \subset \widetilde{\Omega}$ and $|\widetilde{\Omega} \setminus A| \neq \emptyset$, we get that

$$\lambda(\widetilde{\Omega}) < \lambda(A) = \pi^2.$$

Being $r_A = r_{\widetilde{\Omega}} = 1/2$, the above inequality implies (7.19). Now, we recall that, by Proposition 1.3.4, it holds

$$\lim_{q\nearrow 2} \lambda_{2,q}(\widetilde{\Omega}) = \lambda(\widetilde{\Omega}) \qquad \text{and} \qquad \lim_{q\nearrow 2} \pi_{2,q} = \pi,$$

hence, by combining the above limits with (7.19), we get

$$\lim_{q\nearrow 2} \lambda_{2,q}(\widetilde{\Omega}) \left(\int_{\widetilde{\Omega}} d_{\widetilde{\Omega}}^{\frac{2q}{2-q}}\, dx\right)^{\frac{2-q}{q}} = \lambda(\widetilde{\Omega})\, r^2_{\widetilde{\Omega}} < \frac{\pi^2}{4} = \lim_{q\nearrow 2} C_{2,q}.$$

This gives the desired conclusion (7.17).

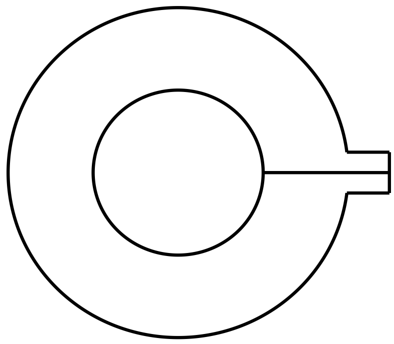


FIGURE 7.1: The set $\widetilde{\Omega}$ obtained from an annulus adding a *small tooth* and removing a segment.

Finally, we note that, by applying (7.16) with $r = 2$, for every $\varphi \in C_0^\infty(\widetilde{\Omega})$, it holds that

$$\limsup_{p\searrow 2} \lambda_p(\widetilde{\Omega}) \le \limsup_{p\searrow 2} \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla\varphi|^p\,dx}{\displaystyle\int_{\widetilde{\Omega}} |\varphi|^p\,dx} \le \limsup_{p\searrow 2} \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla\varphi|^p\,dx}{|\widetilde{\Omega}|^{\frac{2-p}{2}} \left(\displaystyle\int_{\widetilde{\Omega}} |\varphi|^2\,dx\right)^{\frac{p}{2}}} = \frac{\displaystyle\int_{\widetilde{\Omega}} |\nabla\varphi|^2\,dx}{\displaystyle\int_{\widetilde{\Omega}} |\varphi|^2\,dx},$$

and, by taking the infimum on $\varphi \in C_0^\infty(\widetilde{\Omega})$, we get that

$$\limsup_{p\searrow 2} \lambda_p(\widetilde{\Omega}) \le \lambda(\widetilde{\Omega}).$$

Hence,

$$\limsup_{p\searrow 2} \lambda_p(\widetilde{\Omega})\, r_{\widetilde{\Omega}}^{p} \le \lambda(\widetilde{\Omega})\, r_{\widetilde{\Omega}}^{2} < \frac{\pi^2}{4} = \lim_{p\searrow 2} C_{p,p},$$

which implies the desired conclusion (7.18). □

Remark 7.2.4 (Makai-type inequality in the homogenous case)**.** Let $\Omega \subsetneq \mathbb{R}^2$ be a simply connected open set, by combining the Makai inequality ([**103**])

$$\lambda(\Omega) \ge \frac{1}{4}\, r_\Omega^{-2},$$

with estimate (1.5) when $N = 2$, we have that

$$\lambda(\Omega) \ge \frac{1}{4}\left(\frac{2\pi}{(\alpha+1)(\alpha+2)}\right)^{\frac{2}{2+\alpha}} \frac{1}{\left(\displaystyle\int_\Omega d_\Omega^\alpha\,dx\right)^{\frac{2}{2+\alpha}}}, \qquad \text{for every } \alpha > 0.$$

Reasoning in the same way, when $\Omega \subset \mathbb{R}^N$ is an open set and $p > N$, by joining together (5.6) and (1.5), we get that

$$\lambda_p(\Omega) \geq \frac{\mathfrak{h}_p(\Omega)}{C_{N,\alpha}^p} \frac{1}{\left(\displaystyle\int_\Omega d_\Omega^\alpha\,dx\right)^{\frac{p}{N+\alpha}}},$$

where the constant $C_{N,\alpha}$ is defined as in (1.6).

§7.3 The Hersch-Protter inequality for $\lambda_{p,q}$

In this section, we prove an extension for the Hersch-Protter-Kajikiya inequality to $\lambda_{p,q}(\Omega)$ on the class of convex sets. The first proof we provide relies on Theorem 4.2.2, which is turn follows from the comparison principle for Lane-Emden equations.

Theorem 7.3.1 (Hersch-Protter–type inequality for $\lambda_{p,q}$)**.** *Let $1 \leq q < p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a convex bounded open set. Then the following lower bound holds*

$$\lambda_{p,q}(\Omega)\,|\Omega|^{\frac{p-q}{q}} \geq \left(\frac{\pi_{p,q}}{2}\right)^p \frac{1}{r_\Omega^p}. \tag{7.20}$$

Moreover, the estimate is sharp.

Proof. We first prove that the claimed inequality holds for every convex bounded open set. We take $w_{p,q}^{\Omega,\alpha} \in W_0^{1,p}(\Omega)$ the unique positive solution of (3.1) with homogeneous Dirichlet boundary conditions, corresponding to the choice $\alpha = \lambda_{p,q}(\Omega)$. By uniqueness, this must coincide with the positive minimizer of

$$\lambda_{p,q}(\Omega) = \min_{\psi\in W_0^{1,p}(\Omega)} \left\{\int_\Omega |\nabla\psi|^p\,dx \,:\, \int_\Omega |\psi|^q\,dx = 1\right\},$$

which is a positive solution of the same boundary value problem, by optimality. Thus in particular we have

$$\int_\Omega |w_{p,q}^{\Omega,\alpha}|^q\,dx = 1. \tag{7.21}$$

We also observe that from Theorem 4.2.2, we get

$$w_{p,q}^{\Omega,\alpha}(x) \leq r_\Omega^{\frac{p}{p-q}}\, w_{p,q}^{I,\alpha}\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right), \qquad \text{in } \Omega.$$

We raise to the power q and integrate over Ω. By taking (7.21) into account and using (4.2) with $\alpha = \lambda_{p,q}(\Omega)$, this yields

$$1 \leq (\lambda_{p,q}(\Omega)\, r_\Omega^p)^{\frac{q}{p-q}} \int_\Omega \left[w_{p,q}^I\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right)\right]^q dx. \tag{7.22}$$

In order to conclude the proof, we need to extract the geometrical content from the last integral.

By using the *Coarea Formula* and the fact that $|\nabla d_\Omega| = 1$ almost everywhere in Ω, we can write

$$\int_\Omega \left[w_{p,q}^I\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right)\right]^q dx = \int_0^{r_\Omega} \left[w_{p,q}^I\left(\frac{t}{r_\Omega} - 1\right)\right]^q P(t)\,dt,$$

where we set

$$P(t) = \mathcal{H}^{N-1}(\partial\Omega_t) \qquad \text{and} \qquad \Omega_t = \Big\{x \in \Omega \,:\, d_\Omega(x) > t\Big\}.$$

We introduce the function

$$\xi(t) = \int_0^t \left[w^I_{p,q}\left(\frac{\tau}{r_\Omega} - 1\right)\right]^q d\tau.$$

It is not difficult to see that $t \mapsto \xi(t)/t$ is monotone increasing, while $t \mapsto P(t)$ is decreasing by convexity of Ω (see for example [**37**, Lemma 2.2.2]). By appealing to [**25**, Lemma A.1], we get

$$\int_0^{r_\Omega} \left[w^I_{p,q}\left(\frac{t}{r_\Omega} - 1\right)\right]^q P(t)\,dt \le \frac{\xi(r_\Omega)}{r_\Omega} \int_0^{r_\Omega} P(t)\,dt.$$

By recalling the definition of ξ and using again Coarea Formula, this is the same as

$$\int_0^{r_\Omega} \left[w^I_{p,q}\left(\frac{t}{r_\Omega} - 1\right)\right]^q P(t)\,dt \le \frac{|\Omega|}{r_\Omega} \int_0^{r_\Omega} \left[w^I_{p,q}\left(\frac{t}{r_\Omega} - 1\right)\right]^q dt.$$

Finally, by making the change of variable $s = t/r_\Omega - 1$, we obtain

$$\int_\Omega \left[w^I_{p,q}\left(\frac{d_\Omega(x)}{r_\Omega} - 1\right)\right]^q dx \le |\Omega| \int_{-1}^0 w^I_{p,q}(s)^q\,ds.$$

By inserting this estimate into (7.22) and recalling (4.5), we eventually conclude the proof of the inequality.

We are left to show the *sharpness*. As in the case $p = 2$, we show that inequality (7.20) is asymptotically sharp for the slab-type sequence

$$\Omega_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^{N-1} \times I \subset \mathbb{R}^N.$$

Indeed, for $L > 1$ we have that

$$r_{\Omega_L} = 1 \qquad \text{hence} \qquad \left(\frac{\pi_{p,q}}{2}\right)^p \frac{1}{r^p_{\Omega_L}} = \left(\frac{\pi_{p,q}}{2}\right)^p. \tag{7.23}$$

In order to estimate $\lambda_{p,q}(\Omega_L)$, we use that (see [**19**, Main Theorem])

$$\lambda_{p,q}(\Omega_L) \le \left(\frac{\pi_{p,q}}{2}\right)^p \left(\frac{P(\Omega_L)}{|\Omega_L|^{1-\frac{1}{p}+\frac{1}{q}}}\right)^p.$$

By joining this estimate and (7.20), we get

$$1 \le \left(\frac{2}{\pi_{p,q}}\right)^p r^p_{\Omega_L}\, |\Omega_L|^{\frac{p-q}{q}}\, \lambda_{p,q}(\Omega_L) \le r^p_{\Omega_L} \left(\frac{P(\Omega_L)}{|\Omega_L|}\right)^p.$$

If we now recall (7.23) and use that

$$P(\Omega_L) \sim 2\,L^{N-1}, \qquad |\Omega_L| = 2\,L^{N-1}, \qquad \text{as } L \to +\infty,$$

we get

$$\lim_{L\to+\infty} \left[\left(\frac{2}{\pi_{p,q}}\right)^p r^p_{\Omega_L}\, |\Omega_L|^{\frac{p-q}{q}}\, \lambda_{p,q}(\Omega_L)\right] = 1,$$

which proves the claimed sharpness of the estimate. □

In the next remark, we observe that an alternative proof can be obtained as an easy consequence of the Makai inequality.

Remark 7.3.2 (Alternative proof for (7.20))**.** We observe that, thanks to Theorem 7.1.1, we can give an alternative proof for the general case $1 \le q < p < \infty$ of the Hersch-Protter-Kajikiya–type inequality (7.20). Indeed, by using the same notation as in Theorem 7.1.1, let $\Omega = \mathrm{int}(K)$, where K is a polytope, then thanks to Part (3) of Lemma D.2.1, it holds that

$$f_i(y) \le r_\Omega, \qquad \text{for every } y \in S_i'. \tag{7.24}$$

By combining (7.9) and (7.24), we obtain that

$$\begin{aligned}
\int_\Omega d_\Omega^{\frac{p\,q}{p-q}}\,dx &= \sum_{i=1}^h \int_{S_i'} \int_0^{f_i(y)} (d_\Omega(y,t))^{\frac{p\,q}{p-q}}\,dt\,dy \\
&= \frac{p-q}{pq+p-q} \sum_{i=1}^h \int_{S_i'} (f_i(y))^{\frac{p\,q}{p-q}+1}\,dy \\
&\le \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} \left(\sum_{i=1}^h \int_{S_i'} f_i(y)\,dy \right) \\
&= \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} \left(\sum_{i=1}^h \int_{S_i'} \int_0^{f_i(y)} dt\,dy \right) \\
&= \frac{p-q}{pq+p-q} r_\Omega^{\frac{p\,q}{p-q}} |\Omega|.
\end{aligned} \tag{7.25}$$

and, by applying the above estimate in (7.1), we get the desired lower bound (7.20). In order to obtain the desired lower bound when $\Omega \subset \mathbb{R}^N$ is a general convex open set, it is sufficient to follow the same approximation argument as in the Part (2) of Theorem 7.1.1.

Remark 7.3.3. Let $\Omega \subsetneq \mathbb{R}^N$ be a convex bounded open set, then, from Remark 7.3.2, it follows that

$$\int_\Omega d_\Omega^\alpha\,dx \le |\Omega|\,\frac{r_\Omega^\alpha}{\alpha+1}, \qquad \text{for every } \alpha \ge 1. \tag{7.26}$$

Indeed, if $\alpha > 1$, taking $p = \frac{\alpha}{\alpha-1}$ and $q = 1$, we have that $\frac{p\,q}{p-q} = \alpha$ and, reasoning as in Remark 7.3.2, we get (7.25). The case $\alpha = 1$ follows by sending $\alpha \to 1^+$ in (7.25). Moreover, such an estimate is sharp and the equality is asymptotically attained by slab–type sequences Ω_L.

Remark 7.3.4 (Upper bound by means of the distance function in the homogeneous case)**.** Let $\Omega \subset \mathbb{R}^N$ be a convex bounded open set, then by (7.26), we have that

$$r_\Omega \ge \left(\frac{\alpha+1}{|\Omega|}\right)^{\frac{1}{\alpha}} \left(\int_\Omega d_\Omega^\alpha\,dx\right)^{\frac{1}{\alpha}}.$$

Then, thanks to the scaling property of $\lambda_p(\Omega)$, we get the following upper bound in terms of the integral of a positive power of d_Ω and the measure of Ω

$$\lambda_p(\Omega) \le \lambda_p(B_1) \left(\frac{|\Omega|}{\alpha+1}\right)^{\frac{p}{\alpha}} \frac{1}{\left(\int_\Omega d_\Omega^\alpha\,dx\right)^{\frac{p}{\alpha}}}.$$

However, such an estimate is not sharp.

Remark 7.3.5 (Extension to general open sets)**.** When $N < p < \infty$, we can provide an Hersch-Protter-Kajikiya–type inequality for general open sets Ω: indeed, when $1 \le q < p$ and $|\Omega| < \infty$, then Theorem 5.1.1 implies

$$\lambda_{p,q}(\Omega)\, |\Omega|^{\frac{p-q}{q}} \ge \frac{\mathfrak{h}_p(\Omega)}{r_\Omega^p},$$

while, when $q = p$ and $r_\Omega < \infty$, from Theorem 5.2.1, it follows that

$$\lambda_p(\Omega) \ge \frac{\mathfrak{h}_p(\Omega)}{r_\Omega^p}.$$

Such an extension can be also found in [**117**, Theorem 1.4.1], with a different proof and a poorer constant: the result in [**117**] is stated for *bounded* open sets, however a closer inspection of the proof reveals that it still works for open sets with finite inradius.

Nevertheless, in the above inequalities, the constant $\mathfrak{h}_p(\Omega)$ is very likely not to be sharp, hence, in the case $N < p < \infty$, it would be interesting to determine

- the sharp constant $C_1(p, N) > 0$ such that
$$\lambda_{p,q}(\Omega)\, |\Omega|^{\frac{p-q}{q}} \ge \frac{C_1(N, p)}{r_\Omega^p},$$
for every $\Omega \subset \mathbb{R}^N$ open set with $|\Omega| < \infty$;
- the sharp constant $C_2(p, N) > 0$ such that
$$\lambda_p(\Omega) \ge \frac{C_2(N, p)}{r_\Omega^p},$$
for every $\Omega \subset \mathbb{R}^N$ open set with $r_\Omega < \infty$.

Moreover, we can observe that, thanks to (2.9), both sharp constants are bounded from below by $\left(\frac{p-N}{p}\right)^p$.

§7.4 Alternative proof for the Hersch-Protter inequality for λ_p

We now focus on the homogenous case $q = p$. A result which immediatly follows from (7.20) by simply taking the limit as $q \nearrow p$[1], is the sharp Hersch-Protter-Kajikiya inequality for the $p-$Laplacian

$$\lambda_p(\Omega) \ge \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_\Omega^p}, \tag{7.27}$$

where $\Omega \subsetneq \mathbb{R}^N$ is a convex bounded open set.

In this section, we will give a further alternative proof of (7.27) by exploiting a change of variables formula proved by Crasta and Malusa in [**43**, Theorem 7.1], when the domain of integration is a connected open set of class C^2, and then using a suitable approximation result for convex sets.

With this aim, we need some preliminary results. First of all, we recall the change of variables formula which follows from [**43**, Theorem 7.1] when $K = \overline{B_1}$ (see also [**43**, Example 5.6])

[1]We use here that $q \mapsto \lambda_{p,q}(\Omega)$ is left-continuous at $q = p$ when $\Omega \subset \mathbb{R}^N$ is a bounded open set (see Proposition 1.3.4).

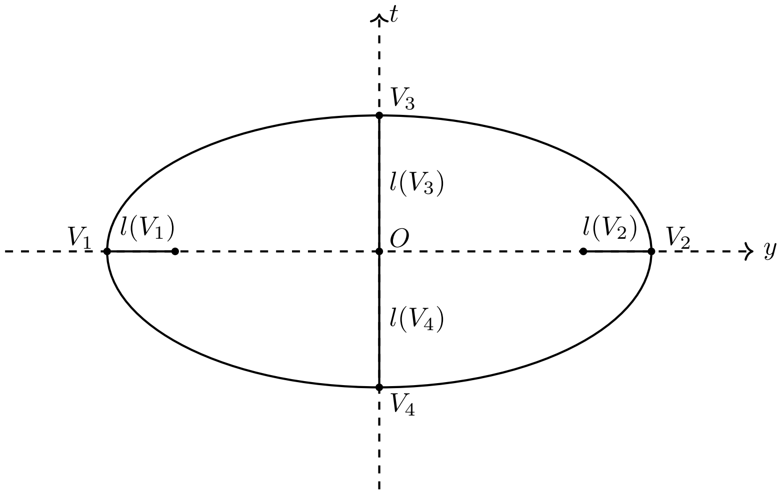


FIGURE 7.2: The values $l(V_i)$ where V_i are the vertices of the ellipse with equation $\frac{y^2}{a^2} + \frac{t^2}{b^2} = 1$, with $a > b > 0$,and $\Omega = \left\{(y,t) \in \mathbb{R}^2 : \frac{y^2}{a^2} + \frac{t^2}{b^2} < 1\right\}$. We have that

$$l(V_1) = l(V_2) = \frac{b^2}{a} < a \text{ and } l(V_3) = l(V_4) = b.$$

Theorem 7.4.1 (Crasta and Malusa's integration formula)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded open connected set of class C^2. Let $l : \partial\Omega \to \mathbb{R}$ be the function given by*

$$l(x) = \sup\left\{|x - z| : z \in \overline{\Omega} \text{ and } x \in \Pi(z)\right\},$$

where

$$\Pi(z) = \{x \in \partial\Omega : d_\Omega(z) = |x - z|\},$$

and let $\Phi = \partial\Omega \times \mathbb{R} \to \mathbb{R}^N$ be the map defined by

$$\Phi(x,t) = x + t\nu(x), \qquad \text{for every } (x,t) \in \partial\Omega \times \mathbb{R},$$

where, for every $x \in \partial\Omega$, $\nu(x)$ is the inward normal unit vector to $\partial\Omega$ at x. Then, for every $h \in L^1(\Omega)$, it holds

$$\int_\Omega h(x)\,dx = \int_{\partial\Omega}\left(\int_0^{l(x)} h(\Phi(x,t))\prod_{i=1}^{N-1}(1 - tk_i(x))\,dt\right)d\mathcal{H}^{N-1}(x). \tag{7.28}$$

Here $k_1(x), \ldots, k_{n-1}(x)$ are the principal curvatures of $\partial\Omega$ at x, i. e. the eigenvalues of the Weingarten map $W(x) : T_x \to T_x$, where T_x denotes the tangent space to $\partial\Omega$ at x. Thanks to the C^2 regularity assumption on Ω, it follows that k_i is continuous on $\partial\Omega$ for every $i \in \{1, \ldots, N-1\}$. Moreover, by [**43**, Proposition 3.10, Lemma 4.1, Theorem 6.7], l is a positive and continuous function on $\partial\Omega$.

Remark 7.4.2. We note that, when Ω is the interior of a polytope, with the notation in the proof of Lemma D.2.1, if $S_i \subset H_i = \{(y,t) \in \mathbb{R}^{N-1} \times \mathbb{R} : t = 0\}$, then we have that

$$l(x) = |x - \Pi_i^{-1}(x)| = f_i(y), \qquad \text{for every } x = (y,0) \in \operatorname{relint}(S_i).$$

Now, we investigate some properties of weighted Rayleigh quotients for every $1 < p < \infty$, defined as

$$\mu_p(w,(0,L)) := \inf_{\psi \in C_0^\infty((0,L])\setminus\{0\}} \left\{ \frac{\displaystyle\int_0^L |\psi'(t)|^p\, w(t)\, dt}{\displaystyle\int_0^L |\psi(t)|^p\, w(t)\, dt} \right\},$$

where $w : (0,L) \to \mathbb{R}$ is a monotone non-increasing positive function, $w \not\equiv 0$. When $w \equiv 1$ on $(0,L)$, we will write $\mu_p(w,(0,L)) = \mu_p(1,(0,L))$.
With the aim to show that $\mu_p(1,(0,L)) \le \mu_p(w,(0,L))$ for every monotone non-decreasing positive weight $w : (0,L) \to \mathbb{R}$, first we prove that there exists a monotone non-decreasing positive minimizer for $\mu_p(1,(0,L))$.

Lemma 7.4.3. *Let $1 < p < \infty$, then*

$$\mu_p(1,(0,L)) = L^{-p}\left(\frac{\pi_p}{2}\right)^p.$$

In particular, there exists a positive and monotone non-decreasing solution of the minimization problem

$$\mu_p(1,(0,L)) = \inf_{\psi \in W^{1,p}((0,L))\setminus\{0\},\psi(0)=0} \left\{ \frac{\displaystyle\int_0^L |\psi'(t)|^p\, dt}{\displaystyle\int_0^L |\psi(t)|^p\, dt} \right\}. \tag{7.29}$$

Proof. First we note that, thanks to the density of $C_0^\infty((0,L])$ in the subspace $\{\varphi \in W^{1,p}(0,L) : \varphi(0) = 0\}$, we have that (7.29) holds. Moreover, following the proof of [**20**, Lemma A.1], we have that

$$\mu_p(1,(0,L)) = L^{-p}\mu_p(1,(0,1)) = L^{-p}\left(\frac{\pi_p}{2}\right)^p,$$

and there exists a positive symmetric function $v \in W_0^{1,p}\left(\left(-\frac{1}{2},\frac{1}{2}\right)\right)$ which is monotone non-increasing on $\left(0,\frac{1}{2}\right)$, such that

$$\pi_p^p = \frac{\displaystyle\int_0^{\frac{1}{2}} |v'(t)|^p\, dt}{\displaystyle\int_0^{\frac{1}{2}} |v(t)|^p\, dt}.$$

Then, it is easy to show that the function $\tilde{v} \in W^{1,p}((0,L))$ defined by

$$\tilde{v}(t) := v\left(\frac{L-t}{2L}\right)$$

satisfies $\tilde{v}(0) = 0$ and it is a positive monotone non-decreasing minimizer of the problem (7.29). $\square$

Now we are in position to show the following minimization result.

Theorem 7.4.4. *Let $1 < p < \infty$ and let w be a positive and monotone non-increasing function on $(0,L)$, then*

$$\mu_p(w,(0,L)) \ge \mu_p(1,(0,L)). \tag{7.30}$$

Proof. First, we show that, for every positive and monotone non-increasing weight $w \in L^\infty((0,L))$, it holds

$$\mu_p(1,(0,L)) \int_0^L |\varphi|^p w\, dt \le \int_0^L |\varphi'|^p\, w\, dt, \qquad \text{for every } \varphi \in C_0^\infty((0,L]). \tag{7.31}$$

Indeed, let $v \in W^{1,p}((0,L))$ be a positive and monotone non-decreasing eigenfunction for $\mu_p(1,(0,L))$ whose existence is ensured by Lemma 7.4.3. Then v is a weak solution of

$$\begin{cases} -(|v'|^{p-2}v')' = \mu_p(1,(0,L))\, v^{p-1}, & \text{in } (0,L), \\ v(0) = 0. \end{cases}$$

Moreover, fixed a mollifier $\delta \in C_0^\infty(\mathbb{R})$, given by

$$\delta(x) := \begin{cases} e^{\frac{1}{|x|^2-1}}, & \text{if } |x|<1, \\ 0, & \text{if } |x| \ge 1, \end{cases}$$

for $0<\varepsilon<1$, we define

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\,\delta\left(\frac{x}{\varepsilon}\right) \in C^\infty(\mathbb{R}).$$

Then $w_\varepsilon = w * \delta_\varepsilon \in C^\infty((\varepsilon, L-\varepsilon))$ and, as $\varepsilon \to 0$, w_ε pointwise converges to w a.e. on $(0,L)$. Moreover, $w_\varepsilon' \le 0$. Indeed, for every $t, t' \in (\varepsilon, L-\varepsilon)$ such that $t > t'$, we have that

$$\begin{aligned} w_\varepsilon(t) = (w*\delta_\varepsilon)(t) &= \int_0^L \delta_\varepsilon(t-y)\, w(y)\, dy = \frac{1}{\varepsilon}\int_0^L \delta\left(\frac{t-y}{\varepsilon}\right) w(y)\, dy \\ &= \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \delta\left(\frac{t-y}{\varepsilon}\right) w(y)\, dy = \int_{-1}^1 \delta(z)\, w(t-\varepsilon z)\, dz \\ &\le \int_{-1}^1 \delta(z)\, w(t'-\varepsilon z)\, dz = w_\varepsilon(t'), \end{aligned}$$

where in the last inequality we use that $\delta(z) > 0$, for every $z \in (-1,1)$ and w is a monotone non-increasing function. Now, by using that $w_\varepsilon' \le 0$ and $v' \ge 0$ a.e. on $(0,L)$ (thanks to Lemma 7.4.3), for every $\varphi \in C_0^\infty((0,L])$, we have that

$$\begin{aligned} \mu_p(1,(0,L)) \int_0^L |\varphi|^p w_\varepsilon\, dt &= \mu_p(1,(0,L)) \int_0^L v^{p-1} \frac{|\varphi|^p}{v^{p-1}} w_\varepsilon\, dt \\ &= \int_0^L |v'|^{p-2}|v'| \left(\frac{|\varphi|^p}{v^{p-1}}\, w_\varepsilon\right)' dt \\ &= \int_0^L |v'|^{p-2}v' \left(\frac{|\varphi|^p}{v^{p-1}}\right)' w_\varepsilon\, dt + \int_0^L |v'|^{p-2}v' \frac{|\varphi|^p}{v^{p-1}} w_\varepsilon'\, dt \\ &\le \int_0^L |v'|^{p-2}v' \left(\frac{|\varphi|^p}{v^{p-1}}\right)' w_\varepsilon\, dt. \end{aligned}$$

By applying Picone's inequality on the last integral (see [4]), the above inequality implies

$$\mu_p(1,(0,L)) \int_0^L |\varphi|^p w_\varepsilon\, dt \le \int_0^L |\varphi'|^p\, w_\varepsilon\, dt.$$

Since $\|w_\varepsilon\|_{L^\infty((0,L))} \le \|w\|_{L^\infty((0,L))}$, as $\varepsilon \to 0$, by using the Dominated Convergence Theorem, we obtain that w satisfies (7.31).

Now, we remove the assumption that w is bounded. For every $M > 0$, we define $w_M := \min\{w, M\} \in L^\infty((0,L))$. By applying (7.31), we have that

$$\mu_p(1,(0,L)) \int_0^L |\varphi|^p w_M \, dt \le \int_0^L |\varphi'|^p \, w_M \, dt \le \int_0^L |\varphi'|^p \, w \, dt, \qquad \text{for every } \varphi \in C_0^\infty((0,L])$$

and, sending $M \to \infty$, we get that also w satisfies (7.31).

Finally, passing to the infimum on functions $\varphi \in C_0^\infty((0,L])$ in (7.31), we obtain the desired estimate (7.30). □

Now, by applying the preliminary results shown in this section, we can prove the following

Theorem 7.4.5 (Hersch-Protter-Kajikiya inequality)**.** *Let $1 < p < \infty$ and let $\Omega \subsetneq \mathbb{R}^N$ be a convex bounded open set. Then, the following lower bound holds*

$$\lambda_p(\Omega) \ge \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_\Omega^p}. \tag{7.32}$$

Moreover, the estimate is sharp.

Proof. We divide the proof in two parts.

Part 1: inequality for C^2 convex bounded sets. We first suppose that Ω is a convex bounded open set of class C^2. Let $u \in C_0^\infty(\Omega)$ then, by using formula (7.28), we have that

$$\int_\Omega |u(x)|^p \, dx = \int_{\partial\Omega} \left(\int_0^{l(x)} |u(x + t\nu(x))|^p \prod_{i=1}^{N-1} (1 - t k_i(x)) \, dt \right) d\mathcal{H}^{N-1}(x) \tag{7.33}$$

and

$$\int_\Omega |\nabla u(x)|^p \, dx = \int_{\partial\Omega} \left(\int_0^{l(x)} |\nabla u(x + t\nu(x))|^p \prod_{i=1}^{N-1} (1 - t k_i(x)) \, dt \right) d\mathcal{H}^{N-1}(x). \tag{7.34}$$

Now we fix $x \in \partial\Omega$ and let $v \in C_0^\infty((0, l(x)])$ be defined by $v(t) := u(x + t\nu(x))$ for every $t \in (0, l(x)]$. Furthermore we introduce the weight $w_x : (0, l(x)) \to \mathbb{R}$ given by

$$w_x(t) = \prod_{i=1}^{N-1} (1 - t k_i(x)) \in L^\infty(0, l(x)).$$

It is easy to verify that the weight w_x is monotone non-increasing. Being $l(x) > 0$, there exists $z = z(x) \in \Omega$, such that $x \in \Pi(z)$ and $l(x) = d_\Omega(z)$. Hence

$$1 - t k_i(x) > 1 - d_\Omega(z) \, k_i(x) \ge 0, \qquad \text{for every } t \in (0, l(x)),$$

where the last inequality follows from [**43**, Lemma 5.4]. In particular, this implies that $w_x > 0$ on $(0, l(x))$. Being satisfied all the hypotheses of Theorem 7.4.4, we have that

$$\begin{aligned}\int_0^{l(x)} |u(x+t\nu(x))|^p \, w_x(t)\, dt &= \int_0^{l(x)} |v(t)|^p \, w_x(t)\, dt \\ &\le \frac{1}{\mu_p(w_x,(0,l(x)))} \int_0^{l(x)} |v'(t)|^p \, w_x(t)\, dt \\ &\le \frac{1}{\mu_p(1,(0,l(x)))} \int_0^{l(x)} |v'(t)|^p \, w_x(t)\, dt.\end{aligned}$$

Then, applying Lemma 7.4.3 and taking into account that $l(x) \le r_\Omega$ for every $x \in \partial\Omega$, we obtain that

$$\begin{aligned}\int_0^{l(x)} |u(x+t\nu(x))|^p \, w_x(t)\, dt &\le \left(\frac{2}{\pi_p}\right)^p l(x)^p \int_0^{l(x)} |v'(t)|^p \, w_x(t)\, dt \\ &\le \left(\frac{2}{\pi_p}\right)^p r_\Omega^p \int_0^{l(x)} |\nabla u(x+t\nu(x))|^p \, w_x(t)\, dt, \qquad \text{for every } x \in \partial\Omega.\end{aligned}$$

By exploiting the above estimate in (7.33) and then using (7.34), we get

$$\int_\Omega |u|^p\, dx \le \left(\frac{2}{\pi_p}\right)^p r_\Omega^p \left(\int_{\partial\Omega} \left(\int_0^{l(x)} |\nabla u(x+t\nu(x))|^p \, w_x(t)\, dt\right) d\mathcal{H}^{N-1}(x)\right) = \left(\frac{2}{\pi_p}\right)^p r_\Omega^p \int_\Omega |\nabla u|^p\, dx,$$

that is

$$\frac{\displaystyle\int_\Omega |\nabla u|^p\, dx}{\displaystyle\int_\Omega |u|^p\, dx} \ge \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_\Omega^p}, \qquad \text{for every } u \in C_0^\infty(\Omega).$$

Taking the infimum on $C_0^\infty(\Omega)$, we obtain that (7.32) holds when Ω is a convex bounded open set of class C^2.

Part 2: inequality for convex bounded sets. We will apply an approximation argument to show the validity of (7.32) for every convex bounded open set. Let $\Omega \subset \mathbb{R}^N$ be a convex bounded open set, then, thanks to [**60**, Section 4.3], there exists a sequence $\{C_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^N$ of convex bounded closed sets of class C^2 such that, for every $k \in \mathbb{N}$, the following properties hold:

- $C_{k+1} \subset C_k \subset C_1$, for every $k \in \mathbb{N}$;
- $\overline{\Omega} \subset C_k$ and

$$d_{\mathcal{H}}(C_k, \overline{\Omega}) = \min\left\{\lambda \ge 0 : \overline{\Omega} \subset C_k + \lambda B_1, C_k \subset \overline{\Omega} + \lambda B_1\right\} \le \frac{1}{k},$$

  i. e. $\{C_k\}_{k\in\mathbb{N}}$ converges to $\overline{\Omega}$, as $k \to \infty$, in the sense of Hausdorff.

This implies that, for every $\varepsilon > 0$, there exists $k_1 = k_1(\varepsilon) \in \mathbb{N}$, such that

$$(1-\varepsilon)\,\overline{\Omega} \subset C_k, \qquad \text{for every } k \ge k_1,$$

which leads to

$$(1-\varepsilon)\,\Omega \subset \operatorname{int}(C_k), \qquad \text{for every } k \ge k_1.$$

Let r_k be the inradius of $\Omega_k := \mathrm{int}(C_k)$, for every $k \in \mathbb{N}$. Then, thanks to the monotonicity of λ_p with respect to the inclusion of sets and by using Part (1) on Ω_k, we obtain that

$$\frac{\lambda_p(\Omega)}{(1-\varepsilon)^p} \geq \lambda_p(\Omega_k) \geq \left(\frac{\pi_p}{2}\right)^p \frac{1}{r_k^p}. \tag{7.35}$$

Since

$$\Omega \subset \Omega_k \subset C_1, \qquad \text{for every } k \in \mathbb{N},$$

we can consider the co-Hausdorff distance between Ω and Ω_k, given by

$$\mathrm{d}^{\mathcal{H}}(\Omega_k, \Omega) = \mathrm{d}_{\mathcal{H}}(C_1 \setminus \Omega_k, C_1 \setminus \Omega),$$

and, being Ω and Ω_k convex open sets, we also have that

$$\mathrm{d}^{\mathcal{H}}(\Omega_k, \Omega) = \mathrm{d}_{\mathcal{H}}(\partial\Omega_k, \partial\Omega) = \mathrm{d}_{\mathcal{H}}(C_k, \overline{\Omega}) \leq \frac{1}{k}.$$

Hence, as $k \to \infty$, the sequence $\{\Omega_k\}_{k\in\mathbb{N}}$ converges to Ω in the sense of co-Hausdorff. By using [**35**, Lemma 4.4], we have that

$$r_k \to r_\Omega, \qquad \text{as } k \to \infty.$$

Hence, by sending $k \to \infty$ and then $\varepsilon \to 0$ in (7.35), we finally get (7.32).

Finally, we notice that the inequality is sharp. Indeed, the equality can be attained by different class of sets, for example by infinite slabs, as $\mathbb{R}^{N-1} \times (0,1)$, or asymptotically by the family of *collapsing pyramids* $C_\alpha = \text{convex hull}\left((-1,1)^{N-1} \cup \{(0,\ldots,0,\alpha)\}\right)$, as proved in [**20**, Theorem 1.2]. The proof is over. □

CHAPTER 8

THE SHARP HARDY INEQUALITY ON SOBOLEV-SLOBODECKIĬ FRACTIONAL SPACES

§8.1 Some preliminaries

In this chapter, we use the same notation of Sections 1.5 and 1.6.
Our aim is to study the Hardy inequality in the setting of *Sobolev-Slobodeckiĭ fractional spaces*. More precisely, when $1 < p < \infty$ and $0 < s < 1$, we study the sharp constant in the *fractional* $(s,p)-$*Hardy inequality*

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} \geq C \int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx, \qquad \text{for every } u \in C_0^\infty(\Omega), \tag{8.1}$$

which is defined by the following variational problem

$$\mathfrak{h}_{s,p}(\Omega) := \inf_{u\in C_0^\infty(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} \, : \, \int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx = 1 \right\}. \tag{8.2}$$

Observe that both integral quantities appearing in (8.1) have the same scaling, thus it is easily seen that $\mathfrak{h}_{s,p}(\Omega)$ can not depend on the size of Ω, i.e. we have that

$$\mathfrak{h}_{s,p}(\mu\,\Omega) = \mathfrak{h}_{s,p}(\Omega), \qquad \text{for every } \mu > 0.$$

Moreover, it can be proved that

$$\mathfrak{h}_{s,p}(\Omega) = \mathfrak{h}_{s,p}(t\,\Omega + x_0), \qquad \text{for every } t > 0,\ x_0 \in \Omega. \tag{8.3}$$

The first result of this chapter consists in a characterization for the fractional sharp Hardy constant: we prove that there is a tight link between the infimum problem $\mathfrak{h}_{s,p}(\Omega)$ and the existence of solutions for the equation

$$(-\Delta_p)^s u = \lambda\,\frac{u^{p-1}}{d_\Omega^{s\,p}}, \qquad \text{in } \Omega, \tag{8.4}$$

where $\lambda \geq 0$. The operator $(-\Delta_p)^s$ is the *fractional* $p-$*Laplacian of order* s, formally defined for $1 < p < \infty$ and $0 < s < 1$ by

$$(-\Delta_p)^s u(x) = 2\,\text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}\,(u(x)-u(y))}{|x-y|^{N+s\,p}}\,dy.$$

Usually, it can be also defined in weak form by the first variation of the convex functional

$$u \mapsto \frac{1}{p}\,[u]^p_{W^{s,p}(\mathbb{R}^N)}.$$

We first give the definition of solutions and supersolutions for (8.4). To this aim, for every $1 < p < \infty$, we denote by $J_p : \mathbb{R} \to \mathbb{R}$ the monotone increasing continuous function defined by

$$J_p(t) = |t|^{p-2}t, \qquad \text{for every } t \in \mathbb{R}.$$

Definition 8.1.1. We will say that $u \in W^{s,p}_{\rm loc}(\Omega) \cap L^{p-1}_{s\,p}(\mathbb{R}^N)$ is a

- *local weak supersolution* of (8.4) if

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy \geq \lambda \int_\Omega \frac{|u(x)|^{p-2}\,u(x)}{d_\Omega(x)^{s\,p}}\,\varphi(x)\,dx, \tag{8.5}$$

  for every non-negative $\varphi \in W^{s,p}(\mathbb{R}^N)$ with compact support in Ω;

- *local weak solution* of (8.4) if (8.5) holds as an equality, for every $\varphi \in W^{s,p}(\mathbb{R}^N)$ with compact support in Ω.

Note that, under the assumptions taken on u and the test functions, the previous definition is well-posed, i.e.

$$\frac{J_p(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}} \in L^1(\mathbb{R}^N\times\mathbb{R}^N).$$

We also observe that by a density argument, we can always consider test functions belonging to $\widetilde{W}^{s,p}_0(\Omega)$.

Then, we devote the second part of the chapter to the computation of the sharp fractional Hardy constants $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$ and $\mathfrak{h}_{s,p}(\Omega)$, when $\Omega \subsetneq \mathbb{R}^N$ is a convex open set.

§8.2 Minimizers for the sharp fractional Hardy constant

Let $1 < p < \infty$ and $0 < s < 1$. For an open set $\Omega \subsetneq \mathbb{R}^N$ we define the sharp fractional $(s,p)-$Hardy constant as in (8.2). In this section we aim to discuss some properties for the minimizers of $\mathfrak{h}_{s,p}(\Omega)$; in particular we discuss their uniqueness on the class of general open sets and then we give conditions on Ω to not have existence.

Proposition 8.2.1 (Equivalent definition on $\widetilde{W}^{s,p}_0(\Omega)$)**.** *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set, then*

$$\mathfrak{h}_{s,p}(\Omega) = \inf_{u\in\widetilde{W}^{s,p}_0(\Omega)} \left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} \,:\, \int_\Omega \frac{|u|^p}{d^{s\,p}_\Omega}\,dx = 1 \right\}. \tag{8.6}$$

Proof. We first note that, by definition of the space $\widetilde{W}^{s,p}_0(\Omega)$, the fact that the infimum over $\widetilde{W}^{s,p}_0(\Omega)$ is less than or equal to $\mathfrak{h}_{s,p}(\Omega)$ simply follows from the fact that we enlarged the class of admissible functions.

To prove the converse inequality, we first observe that if $\mathfrak{h}_{s,p}(\Omega) = 0$ then there is nothing to prove. If on the contrary $\mathfrak{h}_{s,p}(\Omega) > 0$, then for every $u \in \widetilde{W}^{s,p}_0(\Omega)\setminus\{0\}$ we know that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\Omega)$ converging to u in $W^{s,p}(\mathbb{R}^N)$. Without loss of generality, we can assume that u_n

converges almost everywhere to u, as well. We then have

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} = \lim_{n\to\infty} [u_n]^p_{W^{s,p}(\mathbb{R}^N)} \geq \mathfrak{h}_{s,p}(\Omega) \liminf_{n\to\infty} \int_\Omega \frac{|u_n|^p}{d_\Omega^{s\,p}}\,dx \geq \mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx,$$

where we used Hardy's inequality in the first inequality and Fatou's Lemma in the second one. □

In the next Lemma we show the existence and the uniqueness for the minimizers of $\mathfrak{h}_{s,p}(\Omega)$ when Ω is a general open set.

Lemma 8.2.2. *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. If $\mathfrak{h}_{s,p}(\Omega)$ admits a non-trivial minimizer $u \in \widetilde{W}^{s,p}_0(\Omega)$, then this has constant sign in Ω and $u \neq 0$ almost everywhere in Ω. Moreover, the minimizer is unique, up to the choice of the sign and it is a weak solution of (8.4), with $\lambda = \mathfrak{h}_{s,p}(\Omega)$.*

Proof. Let us suppose that (8.6) admits a minimizer $u \in \widetilde{W}^{s,p}_0(\Omega)$, in particular this implies that $\mathfrak{h}_{s,p}(\Omega) > 0$. We observe that

$$\Big||a| - |b|\Big| \leq |a - b|, \qquad \text{for every } a, b \in \mathbb{R},$$

and the inequality is strict, whenever $a\,b < 0$. This yields

$$\mathfrak{h}_{s,p}(\Omega) \leq \big[|u|\big]^p_{W^{s,p}(\mathbb{R})} \leq [u]^p_{W^{s,p}(\mathbb{R})} = \mathfrak{h}_{s,p}(\Omega),$$

and thus it must result

$$u(x)\,u(y) \geq 0, \qquad \text{for a. e. } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

This shows that u has constant sign almost everywhere in Ω. Without loss of generality, we can suppose that u is non-negative.

We then observe that u must be a minimizer of the following functional

$$\mathcal{F}(\varphi) := \frac{1}{p}\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\mathfrak{h}_{s,p}(\Omega)}{p} \int_\Omega \frac{|\varphi|^p}{d_\Omega^{s\,p}}\,dx, \qquad \text{for every } \varphi \in \widetilde{W}^{s,p}_0(\Omega),$$

as well. Indeed, by definition of $\mathfrak{h}_{s,p}(\Omega)$, we have $\mathcal{F}(\varphi) \geq 0$ for every admissible function and $\mathcal{F}(u) = 0$. Moreover, u is non-trivial, due to the normalization on the weighted L^p norm.

By minimality, we get that u must be a non-trivial non-negative weak solution of the Euler-Lagrange equation, which is given by (8.4) with $\lambda = \mathfrak{h}_{s,p}(\Omega)$. By the minimum principle (see [**23**, Theorem A.1]), we directly obtain that $u > 0$ almost everywhere in Ω, if the latter is connected. If Ω has more than one connected component, the same conclusion can be drawn by proceeding as in [**26**, Proposition 2.6], thanks to the nonlocality of the operator.

We now show the uniqueness for the positive minimizer of $\mathfrak{h}_{s,p}(\Omega)$. For this, it is sufficient to exploit Lemma A.3.2. Let us take $u, v \in \widetilde{W}^{s,p}_0(\Omega)$ two positive minimizers of $\mathfrak{h}_{s,p}(\Omega)$ and set

$$\sigma = \left(\frac{1}{2}\,u^p + \frac{1}{2}\,v^p\right)^{\frac{1}{p}},$$

Thanks to (A.4), we get that $\sigma \in \widetilde{W}^{s,p}_0(\Omega)$ is still a minimizer for $\mathfrak{h}_{s,p}(\Omega)$. Thus (A.4) holds as an identity. By Lemma A.3.2, this means that there exists a constant C such that

$$u = C\,v, \qquad \text{a. e. in } \Omega.$$

Finally, the normalization on the weighted norm implies that $C = 1$. This concludes the proof. □

Remark 8.2.3. In the local case, the uniqueness of an extremal for $\mathfrak{h}_{s,p}$ (provided it exists) can be found for example in [**105**, Proposition 3.2]. Differently from [**105**], here we found useful to rely on a hidden convexity argument, rather than on Picone's inequality.

We devote the last part of this section to show that under some hypothesis both on the open set Ω and on the supersolutions of (8.4), the infimum in $\mathfrak{h}_{s,p}(\Omega)$ is not attained, namely, we have not existence of minimizers. We first need the following definition.

Definition 8.2.4. Let $\Omega \subsetneq \mathbb{R}^N$ be an open set. We say that $\partial\Omega$ is *locally continuous at* $x_0 \in \partial\Omega$ if there exist:

- an open $N-$dimensional hyper-rectangle Q_{δ_0,δ_1} centered at the origin, defined by

$$Q_{\delta_0,\delta_1} = (-\delta_0, \delta_0)^{N-1} \times (-\delta_1, \delta_1), \qquad \text{with } \delta_0, \delta_1 > 0;$$

- a linear isometry $\mathcal{O} : \mathbb{R}^N \to \mathbb{R}^N$ such that $\mathcal{O}(x_0) = 0$;

- a continuous function $\Psi : (-\delta_0, \delta_0)^{N-1} \to (-\delta_1, \delta_1)$;

such that

$$Q_{\delta_0,\delta_1} \cap \mathcal{O}(\Omega) = \Big\{x = (x', x_N) \in Q_{\delta_0,\delta_1} \, : \, \Psi(x') < x_N < \delta_1\Big\},$$

and

$$Q_{\delta_0,\delta_1} \cap \mathcal{O}(\partial\Omega) = \Big\{x = (x', x_N) \in Q_{\delta_0,\delta_1} \, : \, x_N = \Psi(x')\Big\}.$$

Roughly speaking, this means that $\partial\Omega$ coincides with the graph of a continuous function, in a small rectangular neighborhood of x_0.

Proposition 8.2.5. *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set, which is locally continuous at a point $x_0 \in \partial\Omega$. Let us suppose that there exists a positive local weak supersolution u of* (8.4) *with $\lambda = \mathfrak{h}_{s,p}(\Omega)$, such that*

$$u \geq \frac{1}{C}\, d_\Omega^{\frac{s\,p-1}{p}}, \qquad \text{in } \Omega. \tag{8.7}$$

Then the infimum $\mathfrak{h}_{s,p}(\Omega)$ is not attained.

Proof. We first show that for such a set, we have

$$1/d_\Omega \notin L^1(\Omega). \tag{8.8}$$

At this aim, we can assume without loss of generality that

$$x_0 = (0, \dots, 0) \qquad \text{and} \qquad \mathcal{O} = \mathrm{Id},$$

so that

$$Q_{\delta_0,\delta_1}(x_0) \cap \Omega = \Big\{x = (x', x_N) \in Q_{\delta_0,\delta_1}(x_0) \, : \, \Psi(x') < x_N < \delta_1\Big\}.$$

We then observe that (see Figure 8.1)

$$d_\Omega(x) \leq |x_N - \Psi(x')| = (x_N - \Psi(x')), \qquad \text{for every } x = (x', x_N) \in Q_{\delta_0,\delta_1}(x_0) \cap \Omega.$$

This implies that

$$\int_\Omega \frac{1}{d_\Omega}\, dx \geq \int_{Q_{\delta_0,\delta_1}(x_0)\cap\Omega} \frac{1}{d_\Omega}\, dx \geq \int_{(-\delta_0,\delta_0)^{N-1}} \left(\int_{\Psi(x')}^{\delta_1} \frac{1}{x_N - \Psi(x')}\, dx_N\right) dx'.$$

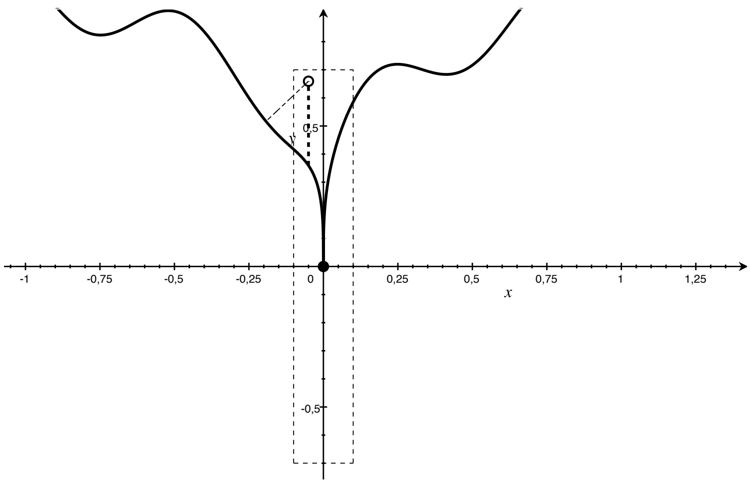


FIGURE 8.1: For (x', x_N) around a continuity point for the boundary, the "vertical" distance $x_N - \Psi(x')$ (in bold dashed line) is always larger than its distance from the boundary.

By observing that the last integral is diverging, we get (8.8).

We now argue by contradiction and suppose that $v \in \widetilde{W}_0^{s,p}(\Omega)$ is a minimizer for $\mathfrak{h}_{s,p}(\Omega)$. This in particular implies that $\mathfrak{h}_{s,p}(\Omega) > 0$. By Lemma 8.2.2, we can suppose that v is positive. We then take a sequence $\{v_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega)$ approximating v in $W^{s,p}(\mathbb{R}^N)$. Without loss of generality, we can take each v_n to be non-negative and suppose that they converge to v almost everywhere, as well. We then insert in the weak formulation of the equation for u the test function

$$\varphi = \frac{v_n^p}{u^{p-1}},$$

which is admissible thanks to Lemma 1.5.2 and (8.7). This leads to

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))}{|x-y|^{N+s\,p}} \left(\frac{v_n(x)^p}{u(x)^{p-1}} - \frac{v_n(y)^p}{u(y)^{p-1}}\right) dx\,dy \geq \mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{v_n^p}{d_\Omega^{s\,p}}\,dx. \tag{8.9}$$

We now set

$$\mathcal{R}(v_n, u) := |v_n(x) - v_n(y)|^p - J_p(u(x)-u(y)) \left(\frac{v_n(x)^p}{u(x)^{p-1}} - \frac{v_n(y)^p}{u(y)^{p-1}}\right),$$

and observe that by Lemma A.2.1 this is always a non-negative quantity. With the previous notation, from equation (8.9) we get that

$$\mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{v_n^p}{d_\Omega^{s\,p}}\,dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\mathcal{R}(v_n,u)}{|x-y|^{N+s\,p}}\,dx\,dy \leq [v_n]^p_{W^{s,p}(\mathbb{R}^N)}.$$

We now pass to the limit in the previous estimate and use Fatou's Lemma on the second term on the left-hand side: this yields

$$\mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{v^p}{d_\Omega^{s\,p}}\,dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\mathcal{R}(v,u)}{|x-y|^{N+s\,p}}\,dx\,dy \leq [v]^p_{W^{s,p}(\mathbb{R}^N)}.$$

By recalling that v solves (8.6), the previous inequality gives

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\mathcal{R}(v,u)}{|x-y|^{N+s\,p}}\,dx\,dy = 0.$$

Since by Lemma A.2.1 we have $\mathcal{R}(v,u)\geq 0$ almost everywhere, this in turn implies that

$$0=\mathcal{R}(v,u)=|v(x)-v(y)|^p - J_p(u(x)-u(y))\left(\frac{v(x)^p}{u(x)^{p-1}}-\frac{v(y)^p}{u(y)^{p-1}}\right),\qquad \text{for a. e. } (x,y)\in\Omega\times\Omega.$$

By using the equality cases in the discrete Picone inequality, it follows that there exists a constant $C>0$ such that

$$u = C\,v,\qquad \text{a. e. in } \Omega.$$

This fact and the assumption (8.7) imply in particular that

$$v\geq \frac{1}{C}\,d_\Omega^{\frac{sp-1}{p}},\qquad \text{in } \Omega,$$

possibly for a different constant $C>0$. By minimality of v, it follows

$$+\infty > [v]^p_{W^{s,p}(\mathbb{R}^N)} = \mathfrak{h}_{s,p}(\Omega)\int_\Omega \frac{|v|^p}{d_\Omega^{s\,p}}\,dx \geq \frac{\mathfrak{h}_{s,p}(\Omega)}{C^p}\int_\Omega \frac{1}{d_\Omega}\,dx.$$

This finally gives a contradiction with (8.8). □

§8.3 The supersolutions method

In this section, we prove a characterization for the sharp Hardy constant in Sobolev-Slobodeckiĭ spaces in terms of positive local weak supersolutions of the relevant Euler-Lagrange equation. Our interest in the existence of positive supersolutions for (8.4) is explained in the following result.

Theorem 8.3.1. *Let $1<p<\infty$, $0<s<1$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set. Then we have*

$$\mathfrak{h}_{s,p}(\Omega)=\sup\Big\{\lambda\geq 0\ :\ \text{equation (8.4) admits a positive local weak supersolution}\Big\}. \tag{8.10}$$

The proof of this result relies on the next Lemmas 8.3.2 and 8.3.3.

Lemma 8.3.2. *Let $1<p<\infty$, $0<s<1$ and let $\Omega\subsetneq\mathbb{R}^N$ be an open set. Then:*

(i) if there exists $\lambda\geq 0$ such that the equation (8.4) admits a positive local weak supersolution u, then $\lambda\leq\mathfrak{h}_{s,p}(\Omega)$;

(ii) in particular, if u is a positive weak solution in $\widetilde{W}^{s,p}_0(\Omega)$, then $\lambda=\mathfrak{h}_{s,p}(\Omega)$ and u is a minimizer for $\mathfrak{h}_{s,p}(\Omega)$.

Proof. In order to prove *(i)*, for every $\eta\in C_0^\infty(\Omega)$, we test the weak formulation with

$$\varphi=\frac{|\eta|^p}{(\varepsilon+u)^{p-1}},$$

where $\varepsilon > 0$. We observe that this is a feasible test function, thanks to Lemma 1.5.2. By using the *discrete Picone inequality* (see [**66**, Lemma 2.6] or [**23**, Proposition 4.2]), we obtain

$$\lambda \int_\Omega \frac{u^{p-1}}{d_\Omega^{s\,p}} \frac{|\eta|^p}{(\varepsilon+u)^{p-1}}\,dx \le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{J_p(u(x)-u(y))\left(\dfrac{|\eta|^p}{(\varepsilon+u)^{p-1}}(x) - \dfrac{|\eta|^p}{(\varepsilon+u)^{p-1}}(y)\right)}{|x-y|^{N+s\,p}}\,dx\,dy$$
$$\le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\Big||\eta(x)|-|\eta(y)|\Big|^p}{|x-y|^{N+s\,p}}\,dx\,dy \le [\eta]^p_{W^{s,p}(\mathbb{R}^N)}.$$

In the last inequality we used that

$$\big[|\eta|\big]^p_{W^{s,p}(\mathbb{R}^N)} \le [\eta]^p_{W^{s,p}(\mathbb{R}^N)}, \tag{8.11}$$

and the inequality is strict, unless η has constant sign almost everywhere (see the proof of Lemma 8.2.2). By taking the limit as ε goes to 0 on the left-hand side, using that u is positive on Ω and the arbitrariness of $\eta \in C_0^\infty(\Omega)$, this finally gives that $\lambda \le \mathfrak{h}_{s,p}(\Omega)$, as desired.

In order to prove point *(ii)*, we observe that if $u \in \widetilde{W}_0^{s,p}(\Omega)$, we can test the weak formulation of the equation with the solution itself. This yields

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} = \lambda \int_\Omega \frac{u^p}{d_\Omega^{s\,p}}\,dx.$$

On the other hand, by definition of $\mathfrak{h}_{s,p}(\Omega)$, we know that

$$\mathfrak{h}_{s,p}(\Omega) \int_\Omega \frac{u^p}{d_\Omega^{s\,p}}\,dx \le [u]^p_{W^{s,p}(\mathbb{R}^N)}.$$

This shows that $\mathfrak{h}_{s,p}(\Omega) \le \lambda$. Since the reverse inequality holds from *(i)*, we conclude that it must result $\lambda = \mathfrak{h}_{s,p}(\Omega)$. □

In the next Lemma, we will use the weighted space $\mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ studied in Section 1.6.

Lemma 8.3.3. *Let $1 < p < \infty$, $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that*

$$\mathfrak{h}_{s,p}(\Omega) > 0.$$

Then for every $0 \le \lambda < \mathfrak{h}_{s,p}(\Omega)$ there exists a positive local weak supersolution $u_\lambda \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ of the equation (8.4). More precisely, the function u_λ is a weak solution of the equation

$$(-\Delta_p)^s u = \lambda\,\frac{u^{p-1}}{d_\Omega^{s\,p}} + 1_B, \qquad \textit{in } \Omega, \tag{8.12}$$

where $B \Subset \Omega$ is a fixed ball.

Proof. We first observe that, for every $\varphi \in \mathcal{X}_0^{s,p}(\Omega, d_\Omega)$, we have

$$\int_B |\varphi|\,dx \le |B|^{\frac{p-1}{p}}\,\|d_\Omega\|^s_{L^\infty(B)} \left(\int_\Omega \frac{|\varphi|^p}{d_\Omega^{s\,p}}\,dx\right)^{\frac{1}{p}} \le |B|^{\frac{p-1}{p}}\,\|d_\Omega\|^s_{L^\infty(B)} \left(\frac{1}{\mathfrak{h}_{s,p}(\Omega)}\right)^{\frac{1}{p}} [\varphi]_{W^{s,p}(\mathbb{R}^N)},$$

thanks to Hölder's inequality, the definition of $\mathfrak{h}_{s,p}(\Omega)$ and the fact that Hardy's inequality holds in $\mathcal{X}_0^{s,p}(\Omega; d_\Omega)$, as well (see Remark 1.6.2). This shows that we have the continuous embedding $\mathcal{X}_0^{s,p}(\Omega; d_\Omega) \hookrightarrow L^1(B)$, for every $B \Subset \Omega$ as in the statement.

Let $0 \le \lambda < \mathfrak{h}_{s,p}(\Omega)$, we consider the functional

$$\mathfrak{F}_\lambda(\varphi) = \frac{1}{p}\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda}{p}\int_\Omega \frac{|\varphi|^p}{d_\Omega^{s\,p}}\,dx - \int_B \varphi\,dx, \qquad \text{for every } \varphi \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega).$$

We will construct the desired supersolution as a minimizer of the following problem

$$m(\lambda) := \inf_{\varphi \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)} \mathfrak{F}_\lambda(\varphi).$$

We first notice that by Hardy's inequality we have, for every $\varphi \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$

$$\begin{aligned}
\mathfrak{F}_\lambda(\varphi) &\ge \frac{1}{p}\left(1 - \frac{\lambda}{\mathfrak{h}_{s,p}(\Omega)}\right)[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \int_B \varphi\,dx \\
&\ge \frac{1}{p}\left(1 - \frac{\lambda}{\mathfrak{h}_{s,p}(\Omega)}\right)[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{p-1}{p}\,\varepsilon^{\frac{1}{1-p}}\int_B d_\Omega^{\frac{s\,p}{p-1}}\,dx - \frac{\varepsilon}{p}\int_B \frac{|\varphi|^p}{d_\Omega^{s\,p}}\,dx \\
&\ge \frac{1}{p}\left(1 - \frac{\lambda}{\mathfrak{h}_{s,p}(\Omega)}\right)[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{p-1}{p}\,\varepsilon^{\frac{1}{1-p}}\int_B d_\Omega^{\frac{s\,p}{p-1}}\,dx - \frac{\varepsilon}{p}\,\frac{1}{\mathfrak{h}_{s,p}(\Omega)}\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)},
\end{aligned}$$

with $\varepsilon > 0$, where we also used Young's inequality. In particular, by choosing

$$\varepsilon = \frac{\mathfrak{h}_{s,p}(\Omega) - \lambda}{2},$$

we can infer that

$$\mathfrak{F}_\lambda(\varphi) \ge c_1\,[\varphi]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{1}{C_1}, \qquad \text{for every } \varphi \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega), \tag{8.13}$$

where $c_1 > 0$ and $C_1 > 0$ do not depend on φ. This in particular shows that $m(\lambda) > -\infty$.

Let us now take a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ such that

$$\mathfrak{F}_\lambda(u_n) \le m(\lambda) + \frac{1}{n+1}, \qquad \text{for every } n \in \mathbb{N}.$$

By appealing to (8.13), we get in particular that there exists a constant $M > 0$ such that

$$[u_n]^p_{W^{s,p}(\mathbb{R}^N)} \le M, \qquad \text{for every } n \in \mathbb{N}.$$

By applying Theorem 1.6.7, we can infer existence of $u \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$ such that the sequence converges almost everywhere in $\mathbb{R}^N$ and such that

$$\int_B u_n\,dx = \int_B u\,dx + o(1), \qquad \text{as } n \to \infty,$$

up to a subsequence. Observe that by construction we have

$$m(\lambda) + \frac{1}{n+1} \ge \frac{1}{p}\,[u_n]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda}{p}\int_\Omega \frac{|u_n|^p}{d_\Omega^{s\,p}}\,dx - \int_B u_n\,dx \ge m(\lambda),$$

which in particular implies that

$$\frac{1}{p}\,[u_n]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda}{p}\int_\Omega \frac{|u_n|^p}{d_\Omega^{s\,p}}\,dx - \int_B u_n\,dx = m(\lambda) + o(1), \qquad \text{as } n\to\infty. \tag{8.14}$$

By applying the Brézis-Lieb Lemma (see [**30**, Theorem 1] and also [**29**, Lemma 2.2]), we get

$$\frac{\lambda}{p}\int_\Omega \frac{|u_n|^p}{d_\Omega^{s\,p}}\,dx = \frac{\lambda}{p}\int_\Omega \frac{|u|^p}{d_\Omega^{s\,p}}\,dx + \frac{\lambda}{p}\int_\Omega \frac{|u_n-u|^p}{d_\Omega^{s\,p}}\,dx + o(1), \qquad \text{as } n\to\infty,$$

and

$$[u_n]^p_{W^{s,p}(\mathbb{R}^N)} = [u]^p_{W^{s,p}(\mathbb{R}^N)} + [u_n-u]^p_{W^{s,p}(\mathbb{R}^N)} + o(1), \qquad \text{as } n\to\infty.$$

By inserting these informations in (8.14), we obtain

$$\mathfrak{F}_\lambda(u) + \frac{1}{p}\,[u_n-u]^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda}{p}\int_\Omega \frac{|u_n-u|^p}{d_\Omega^{s\,p}}\,dx = m(\lambda) + o(1), \qquad \text{as } n\to\infty.$$

We can now use Hardy's inequality for the function $u_n - u \in \mathcal{X}_0^{s,p}(\Omega; d_\Omega)$. Thanks to the choice of λ, it holds that

$$\mathfrak{F}_\lambda(u) \le m(\lambda) + o(1), \qquad \text{as } n\to\infty,$$

and by taking the limit as n goes to ∞, we finally get that u is a minimizer.

By minimality, we get that u must be non-negative. Indeed, by using (8.11) and observing that

$$-\int_B u\,dx \ge -\int_B |u|\,dx,$$

we have

$$\mathfrak{F}_\lambda(u) \ge \mathfrak{F}_\lambda(|u|).$$

Moreover, the inequality sign in the latter is strict, unless u has constant sign almost everywhere. By virtue of the inequality for the integral on B, we get that such a sign must be non-negative, i.e. we must have $u \ge 0$ almost everywhere in Ω. Moreover, by minimality u is a weak solution of the Euler-Lagrange equation (8.12), as claimed. This in particular proves that $u \not\equiv 0$, thanks to the presence of the term 1_B. Observe that (see Proposition 1.6.3)

$$\mathcal{X}_0^{s,p}(\Omega) \subset W^{s,p}_{\text{loc}}(\Omega) \cap L^{p-1}_{s\,p}(\mathbb{R}^N),$$

thus u is a local weak supersolution, in the sense of Definition 8.1.1. Finally, by using the minimum principle, we get that u is positive on Ω (we can proceed as in the proof of Lemma 8.2.2, for example). □

By joining the previous two technical results, we finally get the characterization of the sharp fractional $(s,p)-$Hardy constant stated in Theorem 8.3.1.

Proof of Theorem 8.3.1. We first observe that the set of admissible λ is non-empty: indeed, it always contains $\lambda = 0$. To see this, it is sufficient to observe that any positive constant function is a local weak solution of

$$(-\Delta_p)^s u = 0, \qquad \text{in } \Omega,$$

which is (8.4) for $\lambda = 0$.

In order to prove the claimed identity, we first consider the case $\mathfrak{h}_{s,p}(\Omega) = 0$. Then, the previous discussion and Lemma 8.3.2 imply that the set of admissible λ is actually given by the singleton $\{0\}$. Thus the conclusion holds.

In the case $\mathfrak{h}_{s,p}(\Omega) > 0$, again by Lemma 8.3.2, we have that $\mathfrak{h}_{s,p}(\Omega) \geq \lambda$ for every λ such that (8.4) admits a positive local weak supersolution. On the other hand, from Lemma 8.3.3 we have that for every $\varepsilon > 0$ if we take

$$\mathfrak{h}_{s,p}(\Omega) - \varepsilon < \lambda < \mathfrak{h}_{s,p}(\Omega),$$

then (8.4) admits a positive local weak supersolution. This concludes the proof. □

In the next two sections we aim to construct supersolutions for $\mathbb{H}^1_+$, $\mathbb{H}^N_+$ and convex sets $\Omega \subset \mathbb{R}^N$, in order to apply Theorem 8.3.1. This will imply a lower bound for $\mathfrak{h}_{s,p}$, which we will prove to be optimal in Section 8.6.

§8.4 Construction of supersolutions for the half-line

In what follows, for $t > 0$, we use the notation

$$I_\varepsilon(t) := \Big(\frac{t}{1+\varepsilon}, (1+\varepsilon)t\Big), \qquad \text{for } 0 < \varepsilon \ll 1. \tag{8.15}$$

We still use the notation $\mathbb{H}^1_+ = (0, +\infty)$ for the half-line. Let $\beta \in \mathbb{R}$, we set

$$U_\beta(t) := t^\beta, \qquad \text{for } t \in \mathbb{H}^1_+,$$

and extend it by 0 to the complement of $\mathbb{H}^1_+$. In particular, in the borderline case $\beta = 0$, this has to be intended as the characteristic function of $\mathbb{H}^1_+$.

The next result collects some properties of U_β which will be useful in the sequel.

Lemma 8.4.1. *Let $1 < p < \infty$ and $0 < s < 1$. For every $\beta \in \mathbb{R}$ we have $U_\beta \in W^{s,p}_{\rm loc}(\mathbb{H}^1_+)$. Moreover, U_β has the following further properties:*

- *for*
$$\frac{s\,p - 1}{p} < \beta,$$
we have $U_\beta \in W^{s,p}((0, M))$, for every $M > 0$;
- *for*
$$-\frac{1}{p-1} < \beta < \frac{s\,p}{p-1},$$
we have $U_\beta \in L^{p-1}_{s\,p}(\mathbb{R})$.

Proof. We observe that U_β is locally Lipschitz on $\mathbb{H}^1_+$, for every $\beta \in \mathbb{R}$. This easily implies that $U_\beta \in W^{s,p}_{\rm loc}(\mathbb{H}^1_+)$.

Let us now suppose that $\beta > (s\,p - 1)/p$. From the fact that $U_\beta \in W^{s,p}_{\rm loc}(\mathbb{H}^1_+)$, we get that for every $0 < \varepsilon < M$ we have

$$\int_\varepsilon^M \int_\varepsilon^M \frac{|U_\beta(t) - U_\beta(y)|^p}{|t-y|^{1+s\,p}}\, dt\, dy < +\infty.$$

We show that this is uniformly bounded with respect to ε. For $\beta > s$ this is straightforward, it is sufficient to use that U_β is either $\beta-$Hölder continuous (for $s < \beta < 1$) or even Lipschitz continuous (for $\beta \geq 1$) on $[0, M]$.

We thus assume $(s\,p-1)/p<\beta\le s$. By using the definition of U_β, Fubini's Theorem and the change of variable $y=\tau\,t$, we get

$$\begin{aligned}\int_\varepsilon^M\int_\varepsilon^M \frac{|U_\beta(t)-U_\beta(y)|^p}{|t-y|^{1+s\,p}}\,dt\,dy &= \int_\varepsilon^M\left(\int_{\frac{\varepsilon}{t}}^{\frac{M}{t}} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right) t^{\beta\,p-s\,p}\,dt\\ &=\int_\varepsilon^M\left(\int_{\frac{\varepsilon}{t}}^{1} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right) t^{\beta\,p-s\,p}\,dt\\ &+\int_\varepsilon^M\left(\int_{1}^{\frac{M}{t}} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right) t^{\beta\,p-s\,p}\,dt.\end{aligned} \tag{8.16}$$

We now observe that

$$\int_{\frac{\varepsilon}{t}}^1 \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau\le\int_0^1 \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau<+\infty.$$

For second integral, we observe that

$$\frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\sim\frac{1}{\tau^{1+s\,p-\beta\,p}},\qquad \text{for } \tau\to+\infty,$$

and the last function is integrable on $[1,+\infty)$, for $\beta<s$. Thus we get

$$\int_1^{\frac{M}{t}} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau\le\int_1^{+\infty} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau<+\infty.$$

This discussion entails that

$$\int_\varepsilon^M\int_\varepsilon^M \frac{|U_\beta(t)-U_\beta(y)|^p}{|t-y|^{1+s\,p}}\,dt\,dy\le C\int_\varepsilon^M t^{\beta\,p-s\,p}\,dt=C\,\frac{M^{\beta\,p-s\,p+1}-\varepsilon^{\beta\,p-s\,p+1}}{\beta\,p-s\,p+1},$$

and the last quantity is bounded as ε goes to 0, thanks to the fact that $\beta>(s\,p-1)/p$. We thus proved the claimed property of U_β, for $(s\,p-1)/p<\beta<s$.

We still miss the borderline case $\beta=s$. From (8.16), we can infer

$$\int_\varepsilon^M\int_\varepsilon^M \frac{|U_s(t)-U_s(y)|^p}{|t-y|^{1+s\,p}}\,dt\,dy\le\int_\varepsilon^M\left(\int_0^1 \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right)dt+\int_\varepsilon^M\left(\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right)dt.$$

The first integral on the right-hand side is uniformly bounded in ε, but now we have to pay attention to the fact that

$$\lim_{t\to0^+}\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau=+\infty.$$

We can proceed as follows: we write

$$\int_\varepsilon^M\left(\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right)dt=\int_\varepsilon^{\frac{M}{2}}\left(\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right)dt+\int_{\frac{M}{2}}^M\left(\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau\right)dt$$

and observe that for $0 < t < M/2$, we have

$$\int_\varepsilon^{\frac{M}{2}} \left(\int_1^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau \right) dt \le \frac{M}{2} \int_1^2 \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau + \int_\varepsilon^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau \right) dt$$

and, at last

$$\int_\varepsilon^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \frac{|1-\tau^s|^p}{|1-\tau|^{1+s\,p}}\,d\tau \right) dt \le 2^{1+s\,p} \int_\varepsilon^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \tau^{p\,s-1-s\,p}\,d\tau \right) dt = 2^{1+s\,p} \int_\varepsilon^{M} \log\left(\frac{M}{2\,t}\right) dt.$$

The last integral is uniformly bounded, as ε goes to 0. This finally proves that $U_s \in W^{s,p}((0, M))$.

Finally, we observe that

$$U_\beta \in L^{p-1}_{\rm loc}(\mathbb{R}) \qquad \Longleftrightarrow \qquad \beta\,(p-1) > -1,$$

and

$$\int_{\mathbb{R}} \frac{U_\beta^{p-1}}{(1+|t|)^{1+s\,p}}\,dt = \int_0^{+\infty} \frac{t^{\beta\,(p-1)}}{(1+t)^{1+s\,p}}\,dt < +\infty \qquad \Longleftrightarrow \qquad -1 < \beta\,(p-1) < s\,p.$$

This concludes the proof. □

Remark 8.4.2. For later reference, we observe that in the previous proof for

$$\frac{s\,p-1}{p} < \beta < s,$$

we proved the following upper bound

$$[U_\beta]^p_{W^{s,p}((0,M))} \le \left(\int_0^1 \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau + \int_1^{+\infty} \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}}\,d\tau \right) \frac{M^{\beta\,p-s\,p+1}}{\beta\,p-s\,p+1}.$$

By making the change of variable $\tau = 1/\xi$ in the second integral, this can also be rewritten as

$$[U_\beta]^p_{W^{s,p}((0,M))} \le \left(\int_0^1 \frac{|1-\tau^\beta|^p}{|1-\tau|^{1+s\,p}} \left(1+\tau^{s\,p-\beta\,p-1}\right) d\tau \right) \frac{M^{\beta\,p-s\,p+1}}{\beta\,p-s\,p+1}. \tag{8.17}$$

In the next result, we compute the fractional $p-$Laplacian of order s for U_β. This generalizes [**82**, Lemma 3.1] to the case $\beta \neq s$.

Proposition 8.4.3. *Let $1 < p < \infty$ and $0 < s < 1$. For every*

$$-\frac{1}{p-1} < \beta < \frac{s\,p}{p-1},$$

the function U_β is a local weak solution of (8.4) *in $\mathbb{H}^1_+$, with*

$$\lambda = \lambda(\beta) = 2\int_0^1 \frac{J_p(1-t^\beta)}{(1-t)^{1+s\,p}} \left(1-t^{s\,p-1-\beta\,(p-1)}\right) dt + \frac{2}{s\,p}. \tag{8.18}$$

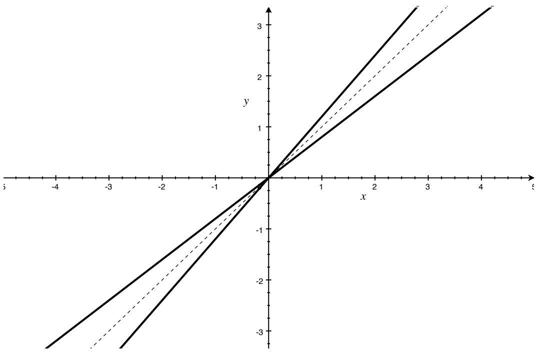


FIGURE 8.2: The set $\mathcal{O}_\varepsilon$ is the conical region "centered" around the line $y = t$.

Moreover, if we define the family of functions on $\mathbb{H}^1_+$ *by*

$$F_\varepsilon(t) = 2 \int_{\mathbb{R}\setminus I_\varepsilon(t)} \frac{J_p(U_\beta(t) - U_\beta(y))}{|t-y|^{1+s\,p}}\, dy, \qquad \textit{for } 0 < \varepsilon < 1, \tag{8.19}$$

where $I_\varepsilon(t)$ *is defined by* (8.15)*, we get that this converges to*

$$F_0(t) = \lambda(\beta)\, \frac{U_\beta(t)^{p-1}}{t^{s\,p}},$$

uniformly on compact subsets of $\mathbb{H}^1_+$*, as* ε *goes to* 0.

Proof. Let us take $\varphi \in C^\infty_0(\mathbb{H}^1_+)$, we observe that by the Dominated Convergence Theorem we have

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{J_p(U_\beta(t) - U_\beta(y))\,(\varphi(t) - \varphi(y))}{|t-y|^{1+s\,p}}\, dt\, dy = \lim_{\varepsilon\to 0^+} \iint_{(\mathbb{R}\times\mathbb{R})\setminus\mathcal{O}_\varepsilon} \frac{J_p(U_\beta(t) - U_\beta(y))\,(\varphi(t) - \varphi(y))}{|t-y|^{1+s\,p}}\, dt\, dy,$$

where

$$\mathcal{O}_\varepsilon = \left\{(t,y) \in \mathbb{R}\times\mathbb{R} \,:\, \min\left\{\frac{t}{1+\varepsilon}, (1+\varepsilon)t\right\} \le y \le \max\left\{\frac{t}{1+\varepsilon}, (1+\varepsilon)t\right\}\right\},$$

see Figure 8.2. For every $0 < \varepsilon < 1$, by proceeding as in [**21**, Lemma 2.3], we have

$$\frac{J_p(U_\beta(t) - U_\beta(y))}{|t-y|^{1+s\,p}}\, \varphi(t) \in L^1((\mathbb{R}\times\mathbb{R})\setminus\mathcal{O}_\varepsilon).$$

Thus we can use Fubini's Theorem and a change of variable, to write

$$\iint_{(\mathbb{R}\times\mathbb{R})\setminus\mathcal{O}_\varepsilon} \frac{J_p(U_\beta(t) - U_\beta(y))\,(\varphi(t) - \varphi(y))}{|t-y|^{1+s\,p}}\, dt\, dy = 2\int_0^{+\infty} \left(\int_{\mathbb{R}\setminus I_\varepsilon(t)} \frac{J_p(U_\beta(t) - U_\beta(y))}{|t-y|^{1+s\,p}}\, dy\right) \varphi(t)\, dt.$$

Observe that we used that φ is compactly supported on $\mathbb{H}^1_+$. By recalling the definition (8.19), up to now we have obtained

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{J_p(U_\beta(t) - U_\beta(y))\,(\varphi(t) - \varphi(y))}{|t-y|^{1+s\,p}}\, dt\, dy = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} F_\varepsilon(t)\,\varphi(t)\, dt, \tag{8.20}$$

for every $\varphi \in C_0^\infty(\mathbb{H}^1_+)$. We now manipulate this quantity, for a fixed $0<\varepsilon<1$: by recalling that U_β identically vanishes in $(-\infty,0]$, for $t>0$ we have

$$\begin{aligned}F_\varepsilon(t) &= 2\int_{\mathbb{R}\setminus I_\varepsilon(t)} \frac{J_p(U_\beta(t)-U_\beta(y))}{|t-y|^{1+s\,p}}\,dy\\ &= 2\int_{\mathbb{H}^1_+\setminus I_\varepsilon(t)} \frac{J_p(t^\beta-y^\beta)}{|t-y|^{1+s\,p}}\,dy + 2\int_{-\infty}^0 \frac{t^{\beta\,(p-1)}}{|t-y|^{1+s\,p}}\,dy.\end{aligned}$$

The second integral can be directly computed: this gives

$$\int_{-\infty}^0 \frac{t^{\beta\,(p-1)}}{|t-y|^{1+s\,p}}\,dy = \frac{1}{s\,p}\,\frac{U_\beta(t)^{p-1}}{t^{s\,p}},$$

where we used the definition of $U_\beta(t)$. For the first integral in the definition of F_ε, by performing the change of variable $y=\tau\,t$, we obtain

$$\begin{aligned}\int_{\mathbb{H}^1_+\setminus I_\varepsilon(t)} \frac{J_p(t^\beta-y^\beta)}{|t-y|^{1+s\,p}}\,dy &= \frac{t^{\beta\,(p-1)}}{t^{s\,p}}\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + \frac{t^{\beta\,(p-1)}}{t^{s\,p}}\int_{1+\varepsilon}^{+\infty} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau\\ &= \frac{U_\beta(t)^{p-1}}{t^{s\,p}}\left(\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + \int_{1+\varepsilon}^{+\infty} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau\right),\end{aligned}$$

again thanks to the definition of U_β. Thus we have obtained

$$F_\varepsilon(t) = \lambda_\varepsilon(\beta)\,\frac{U_\beta(t)^{p-1}}{t^{s\,p}}, \qquad \text{for every } t\in\mathbb{H}^1_+,\ 0<\varepsilon<1, \tag{8.21}$$

where

$$\lambda_\varepsilon(\beta) = 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + 2\int_{1+\varepsilon}^{+\infty} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + \frac{2}{s\,p}.$$

By inserting this in (8.20), we have

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{J_p(U_\beta(t)-U_\beta(y))\,(\varphi(t)-\varphi(y))}{|t-y|^{1+s\,p}}\,dt\,dy = \left(\lim_{\varepsilon\to 0^+}\lambda_\varepsilon(\beta)\right)\int_{\mathbb{R}} \frac{U_\beta(t)^{p-1}}{t^{s\,p}}\,\varphi(t)\,dt. \tag{8.22}$$

To conclude the proof, we only need to show that for $\lambda(\beta)$ defined by (8.18), we have

$$\lambda(\beta) = \lim_{\varepsilon\to 0^+}\lambda_\varepsilon(\beta), \qquad \text{for every } -\frac{1}{p-1}<\beta<\frac{s\,p}{p-1}.$$

We first observe that the case $\beta=0$ is simple: in this case we have

$$J_p(t^\beta-1)=0, \qquad \text{for } t\in(0,1),$$

and thus we directly get

$$\lambda(0)=\lambda_\varepsilon(0)=\frac{2}{s\,p}.$$

We can thus suppose that $\beta \neq 0$. By recalling the definition of $\lambda_\varepsilon(\beta)$ above and performing the change of variable $\tau = 1/\zeta$ in the second integral, we get for $0 < \varepsilon < 1$

$$\begin{aligned}\lambda_\varepsilon(\beta) &= 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\zeta^{-\beta})}{|1-\zeta^{-1}|^{1+s\,p}}\,\frac{d\zeta}{\zeta^2} + \frac{2}{s\,p}\\ &= 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,d\tau + 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(\zeta^\beta-1)}{|\zeta-1|^{1+s\,p}}\,\zeta^{s\,p-1-\beta\,(p-1)}d\zeta + \frac{2}{s\,p}\\ &= 2\int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,\left(1-\tau^{s\,p-1-\beta\,(p-1)}\right)\,d\tau + \frac{2}{s\,p}\end{aligned}$$

By observing that

$$1-\varepsilon < \frac{1}{1+\varepsilon}, \qquad \text{for } 0<\varepsilon<1,$$

we can write

$$\lambda_\varepsilon(\beta) = 2\int_0^{1-\varepsilon} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,(1-\tau^{s\,p-1-\beta\,(p-1)})\,d\tau + 2\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{J_p(\tau^\beta-1)}{|\tau-1|^{1+s\,p}}\,\tau^{s\,p-1-\beta\,(p-1)}\,d\tau + \frac{2}{s\,p}.$$

We claim that

$$\lim_{\varepsilon\to 0^+}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{J_p(\tau^\beta-1)}{|\tau-1|^{1+s\,p}}\,\tau^{s\,p-1-\beta\,(p-1)}\,d\tau = 0. \tag{8.23}$$

Observe at first that for $0<\varepsilon<1/2$, we have

$$\begin{aligned}\tau^{s\,p-1-\beta\,(p-1)} &\le \max\left\{(1-\varepsilon)^{s\,p-1-\beta\,(p-1)}, \left(\frac{1}{1+\varepsilon}\right)^{s\,p-1-\beta\,(p-1)}\right\}\\ &\le \max\left\{2^{-s\,p+1+\beta\,(p-1)}, 1\right\} = C, \qquad \text{for } 1-\varepsilon<\tau<\frac{1}{1+\varepsilon}.\end{aligned}$$

Thus we get

$$\left|\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{J_p(\tau^\beta-1)}{|\tau-1|^{1+s\,p}}\,\tau^{s\,p-1-\beta\,(p-1)}\,d\tau\right| \le C\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{|\tau^\beta-1|^{p-1}}{|\tau-1|^{1+s\,p}}\,d\tau.$$

By using Lemma A.2.2, we can further estimate for $0<\varepsilon<1/2$

$$\begin{aligned}\left|\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{J_p(\tau^\beta-1)}{|\tau-1|^{1+s\,p}}\,\tau^{s\,p-1-\beta\,(p-1)}\,d\tau\right| &\le C\,|\beta|^{p-1}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \max\left\{\tau^{\beta-1},1\right\}^{p-1}(1-\tau)^{p-2-s\,p}\,d\tau\\ &\le C\,|\beta|^{p-1}\max\left\{2^{1-\beta},1\right\}^{p-1}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}}(1-\tau)^{p-2-s\,p}\,d\tau.\end{aligned}$$

By a direct computation, we now get

$$\lim_{\varepsilon\to 0^+}\int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}}(1-\tau)^{p-2-s\,p}\,d\tau = 0,$$

which in turn implies (8.23). On the other hand, by a Taylor expansion, we have

$$\frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,(1-\tau^{s\,p-1-\beta\,(p-1)}) \sim \beta^{p-1}\left(s\,p-1-\beta\,(p-1)\right)(1-\tau)^{p\,(1-s)-1}, \qquad \text{for } \tau \nearrow 1^-,$$

which shows that

$$\frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}}\,(1-\tau^{s\,p-1-\beta\,(p-1)}) \in L^1((0,1)).$$

These facts permit to establish that

$$\lambda(\beta) = \lim_{\varepsilon\to 0^+} \lambda_\varepsilon(\beta), \qquad \text{for every } -\frac{1}{p-1} < \beta < \frac{s\,p}{p-1},$$

thus from (8.22) we get that U_β is a local weak solution of the claimed equation.

The last statement about the convergence of F_ε is an easy consequence of formula (8.21). □

The next result discusses some properties for the function $\lambda(\beta)$ defined in (8.18). This in particular permits to select a special solution among all the functions U_β: this corresponds to the choice

$$\beta = \frac{s\,p-1}{p}.$$

Indeed, for this function, the constant λ is *the largest possible*. This extends to $1 < p < \infty$ a similar discussion contained in the proof of [**17**, Theorem 1].

Proposition 8.4.4. *Let $1 < p < \infty$ and $0 < s < 1$. Let us consider the function*

$$\beta \mapsto \lambda(\beta), \qquad \textit{defined by (8.18) on the interval } \left(-\frac{1}{p-1}, \frac{s\,p}{p-1}\right).$$

Then this has the following properties:

1. *it is monotonically decreasing for $\beta > (s\,p-1)/p$ and monotonically increasing for $\beta < (s\,p-1)/p$. In particular, we have*

$$\lambda(\beta) \le \lambda\left(\frac{s\,p-1}{p}\right) = 2\int_0^1 \frac{\left|1-t^{\frac{s\,p-1}{p}}\right|^p}{(1-t)^{1+s\,p}}\,dt + \frac{2}{s\,p};$$

2. *there exists $\beta^* = \beta^*(s,p)$ such that*

$$-\frac{1}{p-1} < \beta^* < \frac{s\,p-1}{p} \qquad \textit{and} \qquad \lambda(\beta^*) = \lambda(s) = 0.$$

In particular, we have

$$\lambda(\beta) \ge 0 \qquad \Longleftrightarrow \qquad \beta^* \le \beta \le s.$$

Proof. We proceed similarly as in [**23**, Lemma B.1], but making a more complete study. For every $0 < t < 1$, we consider the function defined by

$$g(\beta) = J_p(1-t^\beta)\left(1-t^{s\,p-1-\beta\,(p-1)}\right), \qquad \text{for } -\frac{1}{p-1} \le \beta < \frac{s\,p}{p-1}.$$

We now discuss separately the cases $\beta < 0$ and $\beta > 0$. We start with the latter. By observing that $\log t < 0$ and that

$$J_p'(1-t^\beta) = (p-1)\,(1-t^\beta)^{p-2} \qquad \text{and} \qquad J_p(1-t^\beta) = (1-t^\beta)^{p-1},$$

we get

$$\begin{aligned} g'(\beta) &= (p-1)\,t^{s\,p-1-\beta\,(p-1)}\,\log t\,J_p(1-t^\beta) \\ &\quad - (p-1)\,t^\beta\,\log t\,\left(1-t^{s\,p-1-\beta\,(p-1)}\right)\,J_p'(1-t^\beta), \end{aligned} \tag{8.24}$$

By observing that $\log t < 0$, $J_p'(1-t^\beta) > 0$ and that $J_p(1-t^\beta) > 0$ for $\beta > 0$, we get

$$\begin{aligned} g'(\beta) \ge 0 \quad &\Longleftrightarrow \quad t^{s\,p-1-\beta\,(p-1)}\,(1-t^\beta) - t^\beta\left(1-t^{s\,p-1-\beta\,(p-1)}\right) \le 0 \\ &\Longleftrightarrow \quad t^{s\,p-1-\beta\,(p-1)} \le t^\beta \\ &\Longleftrightarrow \quad t^{s\,p-1-\beta\,p} \le 1. \end{aligned}$$

Since $0 < t < 1$, the last requirement is equivalent to

$$\beta \le \frac{s\,p-1}{p}.$$

This implies that:

- if $s\,p < 1$, then g is monotone decreasing on the whole interval $(0, (s\,p)/(p-1))$ and thus

$$g(\beta) \le g(0) = 0, \qquad \text{for every } 0 \le \beta < \frac{s\,p}{p-1};$$

- if $s\,p \ge 1$, then g is monotone increasing on $(0, (s\,p-1)/p)$ and monotone decreasing on $((s\,p-1)/p, s\,p/(p-1))$. In particular, it is maximal at $\beta = (s\,p-1)/p$ and thus

$$g(\beta) \le g\left(\frac{s\,p-1}{p}\right) = \left(1-t^{\frac{s\,p-1}{p}}\right)^p, \qquad \text{for every } 0 \le \beta < \frac{s\,p}{p-1}.$$

  We also observe that
$$\left(1-t^{\frac{s\,p-1}{p}}\right)^p \ge 0.$$

We now perform a similar discussion for $\beta < 0$: from (8.24), by noticing that this time

$$J_p(1-t^\beta) = -(t^\beta-1)^{p-1} \qquad \text{and} \qquad J_p'(1-t^\beta) = J_p'(t^\beta-1) = (p-1)\,(t^\beta-1)^{p-2},$$

we get again

$$g'(\beta) \ge 0 \quad \Longleftrightarrow \quad \beta \le \frac{s\,p-1}{p}.$$

As above, this implies that:

- if $s\,p < 1$, then g is monotone increasing on $(-1/(p-1), (s\,p-1)/p)$ and monotone decreasing on $((s\,p-1)/p, 0)$. In particular, it is maximal at $\beta = (s\,p-1)/p$ and thus

$$g(\beta) \le g\left(\frac{s\,p-1}{p}\right) = \left(t^{\frac{s\,p-1}{p}}-1\right)^p, \qquad \text{for every } -\frac{1}{p-1} < \beta < 0.$$

We also observe that

$$\left(t^{\frac{s\,p-1}{p}}-1\right)^p \geq 0;$$

- if $s\,p \geq 1$, then g is monotone increasing on the whole interval $(-1/(p-1), 0)$ and thus

$$g(\beta) \leq \lim_{\tau\to 0^-} g(\tau) = 0, \qquad \text{for every } -\frac{1}{p-1} < \beta < 0.$$

In particular, this finally permits to infer that

$$\max_{-\frac{1}{p-1}<\beta<\frac{s\,p}{p-1}} g(\beta) \leq \left|1 - t^{\frac{s\,p-1}{p}}\right|^p.$$

and such a maximal value is uniquely attained at $\beta = (s\,p-1)/p$. By recalling that by definition

$$\lambda(\beta) = 2\int_0^1 \frac{g(\beta)}{(1-t)^{1+s\,p}}\,dt + \frac{2}{s\,p},$$

the properties of λ claimed in (1) follow from the above detailed discussion on g.

Finally, the fact that $\lambda(s) = 0$ has been proved in [**82**, Lemma 3.1]. The existence of the exponent β^* now follows by using the monotonicity and continuity of λ, together with the fact that

$$\lambda\left(\frac{s\,p-1}{p}\right) > 0 \qquad \text{and} \qquad \lim_{\beta\to\left(-\frac{1}{p-1}\right)^+} \lambda(\beta) = -\infty.$$

This concludes the proof. □

Remark 8.4.5 (The exponent β^*)**.** For $p = 2$, the function $\lambda(\beta)$ is given by

$$\lambda(\beta) = \int_0^1 \frac{1-t^\beta}{(1-t)^{1+2\,s}}\,\left(1 - t^{2\,s-1-\beta}\right)\,dt + \frac{1}{s}.$$

It is not difficult to see that such a function is symmetric with respect to the maximum point $(2\,s-1)/2$, i.e. we have

$$\lambda(2\,s-1-\beta) = \lambda(\beta), \qquad \text{for every } -1 < \beta < 2\,s. \tag{8.25}$$

Accordingly, the exponent β^* in this case is simply given by

$$\beta^* = 2\,s - 1 - s = s - 1.$$

With such a choice, in view of (8.25), we have

$$\lambda(s-1) = \lambda(s) = 0.$$

Another case where β^* can be explicitly determined is when $s\,p = 1$. In this case, we have

$$\lambda(\beta) = \int_0^1 \frac{J_p(1-t^\beta)}{(1-t)^2}\,\left(1 - t^{-\beta\,(p-1)}\right)\,dt + 2,$$

and we observe that

$$J_p(1-t^\beta)\,\left(1 - t^{-\beta\,(p-1)}\right) = J_p(1-t^{-\beta})\,\left(1 - t^{\beta\,(p-1)}\right),$$

thanks to the oddness and the homogeneity of J_p. This shows that $\beta \mapsto \lambda(\beta)$ is an even function, i.e.

$$\lambda(\beta) = \lambda(-\beta), \qquad \text{for every } -\frac{1}{p-1} < \beta < \frac{1}{p-1}.$$

Thus, by recalling that $\lambda(s) = 0$, we get in this case that $\beta^* = -s = -1/p$.

§8.5 Construction of supersolutions for convex sets

In what follows, for an open set $\Omega \subsetneq \mathbb{R}^N$ we will use the notation

$$U_\beta := d_\Omega^\beta,$$

where the function is extended by 0 to the complement $\mathbb{R}^N \setminus \Omega$. In the borderline case $\beta = 0$, this coincides with the characteristic function of Ω. In this section, we will show that, for some values of β, the function U_β is a local weak supersolution when Ω is a convex open set and, in the particular case when Ω is an half-space, U_β is exactly a solution.

Lemma 8.5.1. *Let $1 < p < \infty$ and $0 < s < 1$. Let $\Omega \subsetneq \mathbb{R}^N$ be an open convex set. For every*

$$0 \leq \beta < \frac{s\,p}{p-1},$$

we have

$$U_\beta \in W^{s,p}_{\mathrm{loc}}(\Omega) \cap L^{p-1}_{s\,p}(\mathbb{R}^N).$$

If Ω is a half-space, then this property is still true for

$$-\frac{1}{p-1} < \beta < \frac{s\,p}{p-1}.$$

Proof. The fact that $U_\beta \in W^{s,p}_{\mathrm{loc}}(\Omega)$ easily follows from its local Lipschitz character. In order to show that $U_\beta \in L^{p-1}_{s\,p}(\mathbb{R}^N)$, if Ω is bounded it is enough to show that $U_\beta \in L^{p-1}(\Omega)$. To prove this, we can confine ourselves to consider $\beta < 0$ (otherwise there is nothing to prove). By using the Coarea Formula and indicating by r_Ω the supremum of d_Ω over Ω, we have

$$\int_\Omega U_\beta^{p-1}\,dx = \int_0^{r_\Omega} t^{\beta\,(p-1)}\,\mathcal{H}^{N-1}(\{x \in \Omega\,:\, d_\Omega = t\})\,dt \leq \mathcal{H}^{N-1}(\partial\Omega) \int_0^{r_\Omega} t^{\beta\,(p-1)}\,dt.$$

We then observe that the last integral is finite, provided $\beta > -1/(p-1)$. Observe that we used the monotonicity of the surface area of *convex sets* with respect to set inclusion (see [**37**, Lemma 2.2.2]), in the last estimate.

If Ω is an unbounded convex set, not coinciding with $\mathbb{R}^N$, the proof above still shows that $U_\beta \in W^{s,p}_{\mathrm{loc}} \cap L^{p-1}_{\mathrm{loc}}(\Omega)$. In order to conclude, we need to prove that

$$\int_{\mathbb{R}^N} \frac{U_\beta^{p-1}}{(1+|x|)^{N+s\,p}}\,dx < +\infty, \qquad \text{if } \beta < \frac{s\,p}{p-1}.$$

For $\beta \leq 0$ such a property is straightforward. For $\beta > 0$, it is sufficient to fix $x_0 \in \partial\Omega$ and observe that (recall that U_β vanishes outside Ω)

$$U_\beta(x) \leq |x - x_0|^\beta, \qquad \text{for every } x \in \mathbb{R}^N.$$

We then obtain

$$\int_{\mathbb{R}^N} \frac{U_\beta^{p-1}}{(1+|x|)^{N+s\,p}}\,dx \le \int_{\mathbb{R}^N} \frac{|x-x_0|^{\beta\,(p-1)}}{(1+|x|)^{N+s\,p}}\,dx.$$

It is easily seen that the last integral converges if $\beta\,(p-1) < s\,p$.

Finally, if Ω is a half-space, we can suppose without loss of generality that

$$\Omega = \mathbb{H}^+_N = \mathbb{R}^{N-1}\times(0,+\infty).$$

The case $N=1$ is already contained in Lemma 8.4.1, thus we consider $N \ge 2$. We take $-1/(p-1) < \beta < 0$ and we decompose

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{U_\beta^{p-1}}{(1+|x|)^{N+s\,p}}\,dx &= \int_{\mathbb{H}^N_+} \frac{x_N^{\beta\,(p-1)}}{(1+|x|)^{N+s\,p}}\,dx \\
&= \int_{\{x\in\mathbb{H}^N_+\,:\,x_N\ge 1\}} \frac{x_N^{\beta\,(p-1)}}{(1+|x|)^{N+s\,p}}\,dx \\
&+ \int_{\mathbb{R}^{N-1}} \left(\int_0^1 \frac{x_N^{\beta\,(p-1)}}{\left(1+\sqrt{x_N^2+|x'|^2}\right)^{N+s\,p}}\,dx_N \right) dx' \\
&\le \int_{\{x\in\mathbb{H}^N_+\,:\,x_N\ge 1\}} \frac{1}{(1+|x|)^{N+s\,p}}\,dx \\
&+ \int_{\mathbb{R}^{N-1}} \frac{dx'}{(1+|x'|)^{N+s\,p}} \left(\int_0^1 x_N^{\beta\,(p-1)}\,dx_N \right).
\end{aligned}$$

Thanks to the choice of β, the last integral is finite. □

For every $k\in\mathbb{N}$ and $\alpha>0$, we recall that we set

$$\mathcal{I}(k;\alpha) = \int_0^{+\infty} t^k\,(1+t^2)^{-\frac{k+2+\alpha}{2}}\,dt.$$

Then we observe that for $N \ge 2$ and every $m>0$, by using the $(N-1)-$dimensional spherical coordinates and a change of variable, we have

$$\int_{\mathbb{R}^{N-1}} \frac{dy'}{(m^2+|x'-y'|^2)^{\frac{N+s\,p}{2}}} = \frac{(N-1)\,\omega_{N-1}}{m^{1+s\,p}}\,\mathcal{I}(N-2;s\,p). \tag{8.26}$$

In what follows, we still denote by $\lambda(\beta)$ the constant given by (8.18), while $C_{N,s\,p}$ is defined as

$$C_{N,s\,p} := \begin{cases} (N-1)\,\omega_{N-1}\,\mathcal{I}(N-2;s\,p), & \text{for } N\ge 2,\\[2ex] 1, & \text{for } N=1. \end{cases} \tag{8.27}$$

We refer to Remark 8.5.3 below, for a comment about the sharpness of the restriction $\beta \ge 0$.

Theorem 8.5.2. *Let $1<p<\infty$ and $0<s<1$. Let $\Omega \subsetneq \mathbb{R}^N$ be an open convex set. Then:*

1. *if*

$$0 \le \beta < \frac{s\,p}{p-1},$$

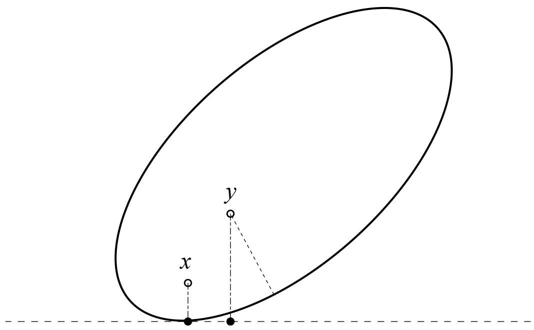


FIGURE 8.3: The supporting hyperplane for Ω at $\overline{x}$.

the function U_β is a local weak supersolution of (8.4)*, with $\lambda = C_{N,s\,p}\,\lambda(\beta)$;*

2. *if Ω is a half-space and*
$$-\frac{1}{p-1} < \beta < \frac{s\,p}{p-1},$$
the function U_β is a local weak solution of (8.4)*, still with $\lambda = C_{N,s\,p}\,\lambda(\beta)$.*

Proof. We will use a simple geometric construction, already exploited in the proof of [**21**, Proposition 3.2], in conjunction with the formula (8.26). We take $x \in \Omega$ and let $\overline{x} \in \partial\Omega$ be a point such that
$$d_\Omega(x) = |x - \overline{x}|.$$

Since Ω is convex, there exists a supporting hyperplane for it at the point $\overline{x}$. Without loss of generality, we can suppose that such a supporting hyperplane coincides with
$$\partial\mathbb{H}^N_+ = \mathbb{R}^{N-1} \times \{0\},$$

and thus
$$x = (x', x_N) \text{ with } x_N > 0, \qquad \overline{x} = (x', 0) \qquad \text{and} \qquad d_\Omega(x) = x_N.$$
Moreover, we have $\Omega \subset \mathbb{H}^N_+$. We now observe that for every other $y = (y', y_N) \in \Omega$, by convexity it results
$$d_\Omega(y) \le y_N,$$

see Figure 8.3. By using this fact and recalling that U_β vanishes in the complement of Ω, we actually have
$$d_\Omega(y) \le (y_N)_+, \qquad \text{for every } y = (y', y_N) \in \mathbb{R}^N.$$
By recalling the definition of U_β, we thus get that for every $y \in \mathbb{R}^N$ and $\beta \ge 0$
$$U_\beta(x) - U_\beta(y) = d_\Omega(x)^\beta - d_\Omega(y)^\beta \ge (x_N)^\beta_+ - (y_N)^\beta_+. \tag{8.28}$$

For every $0 < \varepsilon \ll 1$ and for every $x \in \mathbb{R}^N$, we introduce the following slab
$$\mathcal{K}_\varepsilon(x) = \left\{ y \in \mathbb{R}^N \,:\, \min\left\{\frac{x_N}{1+\varepsilon}, (1+\varepsilon)\,x_N\right\} \le y_N \le \max\left\{\frac{x_N}{1+\varepsilon}, (1+\varepsilon)\,x_N\right\}\right\}.$$

Recalling that $x_N > 0$, we now use (8.28) and the monotonicity of $\tau \mapsto J_p(\tau)$: we obtain for $\beta \geq 0$ and $0 < \varepsilon \ll 1$

$$\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{J_p(U_\beta(x) - U_\beta(y))}{|x-y|^{N+s\,p}}\, dy \geq \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right)}{|x-y|^{N+s\,p}}\, dy. \tag{8.29}$$

If $N \geq 2$, the last integral can be written as

$$\begin{aligned}\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{J_p(U_\beta(x) - U_\beta(y))}{|x-y|^{N+s\,p}}\, dy &\geq \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right)}{|x-y|^{N+s\,p}}\, dy \\ &= \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right) \\ &\quad \times \left(\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x_N - y_N|^2 + |x'-y'|^2)^{\frac{N+s\,p}{2}}}\right) dy_N,\end{aligned}$$

where $I_\varepsilon(x_N)$ is the same interval as in (8.15). If we now use (8.26) with $m = |x_N - y_N|$, we get

$$\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x_N - y_N|^2 + |x'-y'|^2)^{\frac{N+s\,p}{2}}} = \frac{C_{N,s\,p}}{|x_N - y_N|^{1+s\,p}}.$$

Thus, we obtain from (8.29)

$$\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{J_p(U_\beta(x) - U_\beta(y))}{|x-y|^{N+s\,p}}\, dy \geq C_{N,s\,p} \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \frac{J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right)}{|x_N - y_N|^{1+s\,p}}\, dy_N.$$

By recalling that we set $C_{1,s\,p} = 1$, the above formula obviously holds for $N = 1$, as well: actually, it coincides with (8.29).

By the definition (8.19) and the identity (8.21), we have for $x_N > 0$

$$2 \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \frac{J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right)}{|x_N - y_N|^{1+s\,p}}\, dy_N = F_\varepsilon(x_N) = \lambda_\varepsilon(\beta)\, \frac{x_N^{\beta(p-1)}}{x_N^{s\,p}} = \lambda_\varepsilon(\beta)\, \frac{U_\beta(x)^{p-1}}{d_\Omega(x)^{s\,p}}.$$

Moreover, we recall that (see the proof of Proposition 8.4.3)

$$\begin{aligned}\lambda_\varepsilon(\beta) &= 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{J_p(1-\tau^\beta)}{|1-\tau|^{1+s\,p}} \left(1 - \tau^{s\,p - \beta\,(p-1)}\right) d\tau \\ &\quad + 2 \int_{1-\varepsilon}^{\frac{1}{1+\varepsilon}} \frac{J_p(\tau^\beta - 1)}{|\tau - 1|^{1+s\,p}}\, \tau^{s\,p-1-\beta\,(p-1)} d\tau + \frac{2}{s\,p},\end{aligned}$$

and

$$\lim_{\varepsilon \to 0^+} \lambda_\varepsilon(\beta) = \lambda(\beta). \tag{8.30}$$

Thus we have

$$2\, C_{N,s\,p} \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \frac{J_p\left((x_N)_+^\beta - (y_N)_+^\beta\right)}{|x_N - y_N|^{1+s\,p}}\, dy_N = C_{N,s\,p}\, \lambda_\varepsilon(\beta)\, \frac{U_\beta(x)^{p-1}}{d_\Omega(x)^{s\,p}}.$$

This in turn leads to

$$2\int_{\mathbb{R}^N\setminus\mathcal{K}_\varepsilon(x)} \frac{J_p(U_\beta(x)-U_\beta(y))}{|x-y|^{N+s\,p}}\,dy \ge C_{N,s\,p}\,\lambda_\varepsilon(\beta)\,\frac{U_\beta(x)^{p-1}}{d_\Omega(x)^{s\,p}}.$$

We take $\varphi\in C_0^\infty(\Omega)$ non-negative, multiply the previous inequality by $\varphi(x)$ and integrate over Ω. We get

$$2\int_\Omega\left(\int_{\mathbb{R}^N\setminus\mathcal{K}_\varepsilon(x)} \frac{J_p(U_\beta(x)-U_\beta(y))}{|x-y|^{N+s\,p}}\,dy\right)\varphi(x)\,dx \ge C_{N,s\,p}\,\lambda_\varepsilon(\beta)\int_\Omega \frac{U_\beta(x)^{p-1}}{d_\Omega(x)^{s\,p}}\,\varphi(x)\,dx. \tag{8.31}$$

On the other hand, we have

$$\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\in L^1(\mathbb{R}^N\times\mathbb{R}^N),$$

thanks to Lemma 8.5.1. Thus by the Dominated Convergence Theorem, we get

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\qquad=\lim_{\varepsilon\to0^+}\iint_{(\mathbb{R}^N\times\mathbb{R}^N)\setminus\mathcal{C}_\varepsilon}\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy,\end{aligned}$$

where

$$\mathcal{C}_\varepsilon=\Big\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N\,:\,y\in\mathcal{K}_\varepsilon(x)\Big\}.$$

Moreover, for every $0<\varepsilon\ll1$ we have

$$\frac{J_p(U_\beta(x)-U_\beta(y))}{|x-y|^{N+s\,p}}\,\varphi(x)\in L^1((\mathbb{R}^N\times\mathbb{R}^N)\setminus\mathcal{C}_\varepsilon).$$

Thus with a simple change of variables, we get

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\qquad=\lim_{\varepsilon\to0^+}\iint_{(\mathbb{R}^N\times\mathbb{R}^N)\setminus\mathcal{C}_\varepsilon}\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy.\\ &\qquad=2\lim_{\varepsilon\to0^+}\int_\Omega\left(\int_{\mathbb{R}^N\setminus\mathcal{K}_\varepsilon(x)}\frac{J_p(U_\beta(x)-U_\beta(y))}{|x-y|^{N+s\,p}}\,dy\right)\varphi(x)\,dx.\end{aligned} \tag{8.32}$$

Observe that we also used Fubini's Theorem for every fixed $0<\varepsilon\ll1$, in order to arrive at the last integral. By joining (8.31), (8.32) and (8.30), we finally get

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{J_p(U_\beta(x)-U_\beta(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{N+s\,p}}\,dx\,dy\ge C_{N,s\,p}\,\lambda(\beta)\int_\Omega\frac{U_\beta(x)^{p-1}}{d_\Omega(x)^{s\,p}}\,\varphi(x)\,dx,$$

which is the desired conclusion for $\beta\ge0$.

In order to prove the second statement, we first observe that if Ω is a half-space, we can assume for simplicity that

$$\Omega=\mathbb{H}^N_+.$$

Then, in the case $N = 1$, the statement has been proved in Proposition 8.4.3. For $N \geq 2$, it suffices to observe that we have equalities everywhere in the previous argument, even for $\beta < 0$, provided it is an admissible exponent. □

Remark 8.5.3 (Optimality of Theorem 8.5.2)**.** As a consequence of Proposition 8.4.4, we have that

$$C_{N,s\,p}\,\lambda(\beta) \leq C_{N,s\,p}\,\lambda\left(\frac{s\,p-1}{p}\right), \qquad \text{for every } -\frac{1}{p-1} < \beta < \frac{s\,p}{p-1}.$$

Thus, even in the more general case of a convex subset Ω, the choice

$$\beta = \frac{s\,p-1}{p},$$

still produces a supersolution of (8.4), which has the largest possible λ, among supersolutions of this type. However, it should be noticed that, in light of Theorem 8.5.2, such a choice is now feasible *only for*

$$\frac{s\,p-1}{p} \geq 0 \qquad \text{i. e.} \qquad s\,p \geq 1,$$

unless Ω is a half-space. Moreover, if the convex set Ω is not a half-space, *such a result is optimal* in the following sense: already in the borderline case $s\,p = 1$, the function U_β with $\beta < 0$ *is not* a supersolution of (8.4). See Lemma C.2.1 in the Appendix.
Hence, for every $1 < p < \infty$ and $0 < s < 1$, we can define the constant

$$\Lambda_{s,p} := 2\int_0^1 \frac{\left|1 - t^{\frac{s\,p-1}{p}}\right|^p}{(1-t)^{1+s\,p}}\,dt + \frac{2}{s\,p}, \tag{8.33}$$

such that $\Lambda_{s,p} = \lambda\left(\frac{s\,p-1}{p}\right)$.

§8.6 The sharp fractional Hardy inequality for convex sets

In this last section, we determine the sharp constant $\mathfrak{h}_{s,p}$ on the half-space $\mathbb{H}^N_+$ and on the class convex sets. We notice that, when $\Omega \subset \mathbb{R}^N$ is a convex set, we need to restrict to the case $s\,p \geq 1$, in order to compute the sharp constant $\mathfrak{h}_{s,p}(\Omega)$. The key point in the proof is the application of the supersolutions method: in Section 8.5, we found that the function $d_\Omega^{(s\,p-1)/p}$ is a local weak supersolution of (8.4) with constant $\lambda = C_{N,s\,p}\,\Lambda_{s,p}$, only when the power $(s\,p-1)/p$ is non negative, i.e. when $s\,p \geq 1$. Hence, thanks to this fact, from Theorem 8.3.1, we get that $\mathfrak{h}_{s,p}(\Omega) \geq C_{N,s\,p}\,\Lambda_{s,p}$. When $s\,p < 1$, then the supersolutions method can not be applied with the same supersolution $d_\Omega^{(s\,p-1)/p}$, as, in general, negative powers of the distance function do not give supersolutions of (8.4). As an example, in Section C.2 in the Appendix, we show that d_I^β is not even locally weakly $(s,p)-$superharmonic on $I = (0,1)$ when $p = 2$, $s = 1/2$ and $\beta < 0$. We can avoid this limitation on the product $s\,p$, when Ω is an half-space.

We start with the following result, which compares the two sharp constants $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$ and $\mathfrak{h}_{s,p}(\Omega)$, when Ω is a convex set.

Proposition 8.6.1. *Let $1 < p < \infty$ and $0 < s < 1$. For every $\Omega \subsetneq \mathbb{R}^N$ convex open set, we have*

$$\mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{s,p}(\mathbb{H}^N_+).$$

Proof. In dimension $N = 1$, we suppose $\Omega \subsetneq \mathbb{R}$ to be a bounded interval. Thanks to (8.3), we can assume that $\Omega = I = (0, 1)$. We take $\psi \in C_0^\infty(\mathbb{H}_+^1)$ and define the rescaled function

$$\psi_\varepsilon(t) = \psi\left(\frac{t}{\varepsilon}\right).$$

We observe that for $\varepsilon > 0$ sufficiently small, we have $\psi_\varepsilon \in C_0^\infty(I)$. We compute

$$[\psi_\varepsilon]^p_{W^{s,p}(\mathbb{R})} = \varepsilon^{1-s\,p}\,[\psi]^p_{W^{s,p}(\mathbb{R})},$$

and

$$\int_I \frac{|\psi_\varepsilon|^p}{d_I^{s\,p}}\,dt = \int_0^1 \frac{\left|\psi\left(\dfrac{t}{\varepsilon}\right)\right|^p}{(\min\{t, 1-t\})^{s\,p}}\,dt = \varepsilon^{1-s\,p}\int_0^{\frac{1}{\varepsilon}} \frac{|\psi(\tau)|^p}{\left(\min\left\{\tau, \varepsilon^{-1}-\tau\right\}\right)^{s\,p}}\,d\tau.$$

By recalling that ψ is compactly supported, we get that for $0 < \varepsilon \ll 1$ we have

$$\min\left\{\tau, \varepsilon^{-1} - \tau\right\} = \tau, \qquad \text{for } \tau \text{ in the support ot } \psi.$$

In conclusion, we get for every $\psi \in C_0^\infty(\mathbb{H}_+^1)$

$$\mathfrak{h}_{s,p}(I) \le \lim_{\varepsilon\to 0^+} \frac{[\psi_\varepsilon]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_I \frac{|\psi_\varepsilon|^p}{d_I^{s\,p}}\,dt} = \lim_{\varepsilon\to 0^+} \frac{[\psi]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_0^{\frac{1}{\varepsilon}} \frac{|\psi(\tau)|^p}{\tau^{s\,p}}\,d\tau} = \frac{[\psi]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_0^{+\infty} \frac{|\psi(\tau)|^p}{\tau^{s\,p}}\,d\tau}.$$

By arbitrariness of ψ, this gives the claimed inequality.

For the case $N \ge 2$, we can repeat the same proof of [**104**, Theorem 5], which deals with the local case, up to some very minor modifications. We just recall that the proof in [**104**] is based on a scaling argument as in the one-dimensional case exposed above, together with the fact that a convex set admits a tangent hyperplane at almost every boundary point. □

For an half-space, we can determine the sharp Hardy constant without restrictions on the product $s\,p$.

Theorem 8.6.2. *Let $1 < p < \infty$ and $0 < s < 1$. For every $N \ge 1$ we have*

$$\mathfrak{h}_{s,p}(\mathbb{H}_+^N) = C_{N,s\,p}\,\Lambda_{s,p},$$

where $\Lambda_{s,p}$ and $C_{N,s\,p}$ are defined by (8.33) and (8.27), respectively. Moreover, such a constant is not attained.

Proof. By combining (8.10) and Theorem 8.5.2 for $\Omega = \mathbb{H}_+^N$, we immediately obtain

$$\mathfrak{h}_{s,p}(\mathbb{H}_+^N) \ge C_{N,s\,p}\,\lambda(\beta), \qquad \text{for every } -\frac{1}{p-1} < \beta < \frac{s\,p}{p-1}.$$

Moreover, by Proposition 8.4.4, we know that the right-hand side is maximal for $\beta = (s\,p-1)/p$ and thus

$$\mathfrak{h}_{s,p}(\mathbb{H}_+^N) \ge C_{N,s\,p}\,\lambda\left(\frac{s\,p-1}{p}\right) = C_{N,s\,p}\,\Lambda_{s,p}.$$

In order to prove that the right-hand side actually gives the sharp constant, we distinguish two cases: $N = 1$ and $N \ge 2$. We will show that the latter reduces to the former: this is quite a standard fact for the Hardy inequality, but we prefer to give the details, since some non-trivial computations are needed.

For the case $N = 1$, we will use a slightly different family of trial functions with respect to [**65**, **66**]: this permits to treat the cases $s\,p < 1$ and $s\,p \geq 1$ at the same time.

Sharpness: case $N = 1$. We need to prove that

$$\mathfrak{h}_{s,p}(\mathbb{H}^1_+) \leq \Lambda_{s,p}.$$

We take a cut-off function $\psi \in C_0^\infty\left((-\infty, 2)\right)$ such that

$$0 \leq \psi \leq 1, \qquad \psi \equiv 1, \text{ on } [0,1], \qquad |\psi'| \leq \widetilde{C},$$

and we use the trial function

$$\phi_\beta = U_\beta\,\psi, \qquad \text{with } \frac{s\,p-1}{p} < \beta < s.$$

According to Lemma 8.4.1 and Lemma 1.5.3, this function belongs to $\widetilde{W}_0^{s,p}(\mathbb{H}^1_+)$. In light of the estimate (1.13) and the properties of the cut-off, we get

$$\mathfrak{h}_{s,p}(\mathbb{H}^1_+) \leq \frac{[\phi_\beta]^p_{W^{s,p}(\mathbb{R})}}{\displaystyle\int_0^{+\infty} \frac{|\phi_\beta(x)|^p}{x^{s\,p}}\,dx} \leq \frac{[U_\beta\,\psi]^p_{W^{s,p}((0,2))}}{\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx} + \frac{2}{s\,p} + \frac{2^{1+p-s\,p}}{s\,p}\,\widetilde{C}^p\,\frac{\|U_\beta\|^p_{L^p((0,2))}}{\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx}. \tag{8.34}$$

We evaluate separately the two quotients on the right-hand side. For the first one, by using the estimate (1.14), we have

$$\frac{[U_\beta\,\psi]_{W^{s,p}((0,2))}}{\left(\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx\right)^{\frac{1}{p}}} \leq \frac{[U_\beta]_{W^{s,p}((0,2))}}{\left(\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx\right)^{\frac{1}{p}}} + \left(\frac{C}{s\,(1-s)}\right)^{\frac{1}{p}}\widetilde{C}^s\,\frac{\|U_\beta\|_{L^p((0,2))}}{\left(\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx\right)^{\frac{1}{p}}}.$$

Moreover, by recalling the definition of U_β, we note that

$$\lim_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} \int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx = \lim_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} \int_0^2 \frac{\psi^p}{x^{s\,p-\beta\,p}}\,dx = +\infty,$$

thus the denominator diverges as β goes to $(s\,p-1)/p$. Coming back to (8.34), this entails that

$$\begin{aligned}\mathfrak{h}_{s,p}(\mathbb{H}^1_+) &\leq \limsup_{\beta\to\left(\frac{s\,p-1}{p}\right)^+}\left[\frac{[U_\beta\,\psi]^p_{W^{s,p}((0,2))}}{\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx} + \frac{2}{s\,p} + \frac{2^{p+1-s\,p}}{s\,p}\,\widetilde{C}^p\,\frac{\|U_\beta\|^p_{L^p((0,2))}}{\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx}\right]\\ &\leq \limsup_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} \frac{[U_\beta]^p_{W^{s,p}((0,2))}}{\displaystyle\int_0^2 \frac{(U_\beta\,\psi)^p}{x^{s\,p}}\,dx} + \frac{2}{s\,p} \leq \limsup_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} \frac{[U_\beta]^p_{W^{s,p}((0,2))}}{\displaystyle\int_0^1 \frac{(U_\beta)^p}{x^{s\,p}}\,dx} + \frac{2}{s\,p}.\end{aligned}$$

We claim that

$$\limsup_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} \frac{[U_\beta]^p_{W^{s,p}((0,2))}}{\displaystyle\int_0^1 \frac{(U_\beta)^p}{x^{s\,p}}\,dx} \leq 2\int_0^1 \frac{\left|1-t^{\frac{s\,p-1}{p}}\right|^p}{(1-t)^{1+s\,p}}\,dt, \tag{8.35}$$

this would conclude the proof, by recalling the definition (8.33) of $\Lambda_{s,p}$. By using the form of U_β we have

$$\int_0^1 \frac{(U_\beta)^p}{x^{s\,p}}\,dx = \int_0^1 x^{\beta\,p-s\,p}\,dt = \frac{1}{\beta\,p - s\,p + 1}.$$

Thus in order to prove (8.35), we just need to show that

$$\limsup_{\beta\to\left(\frac{s\,p-1}{p}\right)^+} (\beta\,p - s\,p+1)\,[U_\beta]^p_{W^{s,p}(0,2)} \le 2\int_0^1 \frac{\left|1-t^{\frac{s\,p-1}{p}}\right|^p}{(1-t)^{1+s\,p}}\,dt.$$

By recalling the estimate (8.17) from Remark 8.4.2, we have

$$[U_\beta]^p_{W^{s,p}((0,2))} \le \left(\int_0^1 \frac{|1-t^\beta|^p}{|1-t|^{1+s\,p}}\,\left(1+t^{s\,p-p\,\beta-1}\right)\,dt\right)\,\frac{2^{\beta\,p-s\,p+1}}{\beta\,p-s\,p+1}.$$

Hence, by taking the limit as β goes to $(s\,p-1)/p$ and using the Dominated Convergence Theorem, we get (8.35), as desired. This proves the sharpness for $N=1$.

Sharpness: case $N \ge 2$. We will show that this can be reduced to the previous case, by proceeding as in [**65**, Theorem 1.1] and [**108**, Proposition 3.2]. Let $\eta \in C_0^\infty((0,+\infty))$ and $\chi \in C_0^\infty((-1,1)^{N-1})$, we use the test function

$$\varphi = \chi_M(x')\,\eta(x_N), \qquad \text{where } \chi_M(x') := \frac{M^{\frac{1-N}{p}}}{\|\chi\|_{L^p(\mathbb{R}^{N-1})}}\,\chi\left(\frac{x'}{M}\right),$$

for some $M>0$. Observe that by construction the function χ_M has compact support on $(-M,M)^{N-1}$ and unit L^p norm. We thus obtain

$$\mathfrak{h}_{s,p}(\mathbb{H}^N_+) \le \frac{[\eta\,\chi_M]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_{\mathbb{H}^N_+} \frac{(\eta\,\chi_M)^p}{x_N^{s\,p}}\,dx} = \frac{[\eta\,\chi_M]^p_{W^{s,p}(\mathbb{R}^N)}}{\displaystyle\int_0^{+\infty} \frac{\eta^p}{x_N^{s\,p}}\,dx_N},$$

where in the last identity we used Fubini's Theorem and the properties of χ_M. In order to estimate the seminorm, we first use Minkowski's inequality

$$\begin{aligned}[\eta\,\chi_M]_{W^{s,p}(\mathbb{R}^N)} &\le \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\chi_M(x')|^p\,|\eta(x_N)-\eta(y_N)|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}} \\ &\quad + \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(y_N)|^p\,|\chi_M(x')-\chi_M(y')|^p}{|x-y|^{N+s\,p}}\,dx\,dy\right)^{\frac{1}{p}},\end{aligned}$$

and we focus separately on the two integrals on the right-hand side. For the first integral, we use Fubini's Theorem and the identity (8.26), so to get

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\chi_M(x')|^p\,|\eta(x_N)-\eta(y_N)|^p}{|x-y|^{N+s\,p}}\,dx\,dy \\ &\quad = C_{N,s\,p}\left(\int_{\mathbb{R}^{N-1}} |\chi_M(x')|^p\,dx'\right)\left(\iint_{\mathbb{R}\times\mathbb{R}} \frac{|\eta(x_N)-\eta(y_N)|^p}{|x_N-y_N|^{1+s\,p}}\,dx_N\,dy_N\right) = C_{N,s\,p}\,[\eta]^p_{W^{s,p}(\mathbb{R})}.\end{aligned}$$

On the other hand, by using a computation similar to (8.26), we have

$$\int_{\mathbb{R}} \frac{1}{(|x_N-y_N|^2+|x'-y'|^2)^{\frac{N+s\,p}{2}}}\,dx_N = \frac{2\displaystyle\int_0^{+\infty}(1+t^2)^{-\frac{N+s\,p}{2}}\,dt}{|x'-y'|^{N-1+s\,p}} = \frac{C}{|x'-y'|^{N-1+s\,p}}.$$

Thus, it holds

$$\begin{aligned}&\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(y_N)|^p\,|\chi_M(x')-\chi_M(y')|^p}{|x-y|^{N+s\,p}}\,dx\,dy\\ &\qquad = C\int_{\mathbb{R}}|\eta(y_N)|^p\,dy_N\left(\iint_{\mathbb{R}^{N-1}\times\mathbb{R}^{N-1}}\frac{|\chi_M(x')-\chi_M(y')|^p}{|x'-y'|^{N-1+s\,p}}\,dx'\,dy'\right)\\ &\qquad = C\,\frac{\|\eta\|^p_{L^p(\mathbb{R})}}{\|\chi\|^p_{L^p(\mathbb{R}^{N-1})}}\,\frac{[\chi]^p_{W^{s,p}(\mathbb{R}^{N-1})}}{M^{s\,p}}.\end{aligned}$$

In the last identity we used the definition of χ_M and a change of variable. Then, it follows that

$$\Big(\mathfrak{h}_{s,p}(\mathbb{H}^N_+)\Big)^{\frac{1}{p}} \le (C_{N,s\,p})^{\frac{1}{p}}\,\frac{[\eta]_{W^{s,p}(\mathbb{R})}}{\left(\displaystyle\int_0^{+\infty}\frac{\eta^p}{x_N^{s\,p}}\,dx\right)^{\frac{1}{p}}} + C^{\frac{1}{p}}\,\frac{\|\eta\|_{L^p(\mathbb{R})}}{M^s\,\|\chi\|_{L^p(\mathbb{R}^{N-1})}}\,\frac{[\chi]_{W^{s,p}(\mathbb{R}^{N-1})}}{\left(\displaystyle\int_0^{+\infty}\frac{\eta^p}{x_N^{s\,p}}\,dx\right)^{\frac{1}{p}}}$$

By letting M go to $+\infty$ and thanks to the arbitrariness of $\eta\in C_0^\infty((0,+\infty))$, we obtain

$$\mathfrak{h}_{s,p}(\mathbb{H}^N_+) \le C_{N,s\,p}\,\mathfrak{h}_{s,p}(\mathbb{H}^1_+) = C_{N,s\,p}\,\Lambda_{s,p},$$

as desired. The last identity follows from the sharpness for $N=1$.

The fact that $\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$ is not attained follows directly from Proposition 8.2.5, since by Theorem 8.5.2 we found a local weak solution of (8.4) with $\lambda=\mathfrak{h}_{s,p}(\mathbb{H}^N_+)$, of the form

$$u = d_{\mathbb{H}^N_+}^{\frac{s\,p-1}{p}}.$$

The proof is over.

□

Theorem 8.6.3. *Let $1<p<\infty$ and $0<s<1$ be such that $s\,p\ge 1$. Then for every $\Omega\subsetneq\mathbb{R}^N$ convex open set, we have*

$$\mathfrak{h}_{s,p}(\Omega) = C_{N,s\,p}\,\Lambda_{s,p},$$

where $\Lambda_{s,p}$ and $C_{N,s\,p}$ are defined by (8.33) and (8.27), respectively. Moreover, such a constant is not attained.

Proof. We suppose that Ω is not a half-space, otherwise there is nothing to prove. By appealing to (8.10) and Theorem 8.5.2, we get

$$\mathfrak{h}_{s,p}(\Omega) \ge C_{N,s\,p}\,\lambda(\beta), \qquad \text{for every } 0\le\beta<\frac{s\,p}{p-1}. \tag{8.36}$$

Again by Proposition 8.4.4, we know that the right-hand side is maximal for $\beta = (s\,p-1)/p$: such a choice is feasible, thanks to the assumption $s\,p \geq 1$. We thus get

$$\mathfrak{h}_{s,p}(\Omega) \geq C_{N,s\,p}\,\lambda\left(\frac{s\,p-1}{p}\right) = C_{N,s\,p}\,\Lambda_{s,p}.$$

On the other hand, by Proposition 8.6.1 we know that

$$\mathfrak{h}_{s,p}(\Omega) \leq \mathfrak{h}_{s,p}(\mathbb{H}^N_+).$$

By combining the latter with Theorem 8.6.2, we finally get

$$\mathfrak{h}_{s,p}(\Omega) \leq C_{N,s\,p}\,\Lambda_{s,p},$$

as well. This proves that $\mathfrak{h}_{s,p}(\Omega)$ has the claimed expression.

Finally, the fact that $\mathfrak{h}_{s,p}(\Omega)$ is not attained follows directly from Proposition 8.2.5, since by Theorem 8.5.2 we found a local weak supersolution of (8.4) with $\lambda = \mathfrak{h}_{s,p}(\Omega)$, having the form

$$u = d_\Omega^{\frac{s\,p-1}{p}}.$$

This concludes the proof. □

Remark 8.6.4 (A lower bound in the case $s\,p < 1$)**.** As already said, in the case $s\,p < 1$ the maximal choice for β is not feasible. In this case we can choose in (8.36) the exponent $\beta = 0$ and get at least a lower bound, i.e.

$$\mathfrak{h}_{s,p}(\Omega) \geq C_{N,s\,p}\,\lambda(0) = C_{N,s\,p}\,\frac{2}{s\,p}.$$

In this last remark, we would notice that, when $p = 2$, the case $2\,s < 1$ is completely solved.

Remark 8.6.5 (The case $0 < s < 1$ and $p = 2$)**.** In the case $p = 2$ we can compute the sharp constant in the whole range $0 < s < 1$. Indeed, in [**Z2**, Section 6.3], we showed that, for every $\Omega \subsetneq \mathbb{R}^N$ open convex set, we have

$$\mathfrak{h}_{s,2}(\Omega) = C_{N,s,2}\,\Lambda_{s,2},$$

where $\Lambda_{s,2}$ and $C_{N,s,2}$ are defined by (8.33) and (8.27), respectively. Also in this case such a constant is not attained. In order to prove the claimed identity, we follow the same argument as in [**55**]. For the one-dimensional Hardy inequality, it is sufficient to consider

$$f(t) := t^{\frac{2\,s-1}{2}}\,(1-t)^{\frac{2\,s-1}{2}} = d_{(0,+\infty)}^{\frac{2\,s-1}{2}}\left(\frac{1}{t}-1\right),$$

which is a positive local weak solution of

$$(-\Delta)^s u = \Lambda_{s,2}\,\frac{u}{\big(t\,(1-t)\big)^{2\,s}}, \qquad \text{in } I := (0,1),$$

and, by observing that

$$\frac{1}{t-a} + \frac{1}{b-t} \geq \frac{1}{d_I(t)}, \qquad \text{for } t \in I,$$

we get that

$$\mathfrak{h}_{s,2}((a,b)) \geq \Lambda_{s,2}. \tag{8.37}$$

Here, the function f coincides with the *fractional Kelvin transform* of the "shifted" function

$$x \mapsto d_{(0,+\infty)}^{\frac{2s-1}{2}}(x-1), \qquad \text{for every } x \in (1,+\infty);$$

In dimension $N \geq 2$, we reproduce the proof of [**100**, Theorem 1.1] for convex sets, and by applying a *reduction formula* (see [**100**, Lemma 2.4]) and (8.37), we have that

$$\mathfrak{h}_{s,2}(\Omega) \leq \mathfrak{h}_{s,2}(\mathbb{H}^N_+) = C_{N,2\,s}\,\Lambda_{s,2}.$$

Finally, the reverse inequality follows thanks to Proposition 8.6.1 and Theorem 8.6.2.

APPENDIX A

A COLLECTION OF USEFUL INEQUALITIES

§A.1 Quantified convexity of power functions

Lemma A.1.1. *Let $r \geq 2$. For every $z, w \in \mathbb{R}^N$ and every $t \in [0, 1]$ we have*

$$t\,|z|^r + (1-t)\,|w|^r \geq |t\,z + (1-t)\,w|^r + C\,t\,(1-t)\,|z-w|^r,$$

where $C = C(r) > 0$.

Proof. For simplicity, we set $F(z) = |z|^r$. For $t = 0$ or $t = 1$ there is nothing to prove, so let us assume $0 < t < 1$. From [**99**, Lemma 4.2, equation (4.3)], we know that there exists $C_r > 0$ such that

$$F(\xi) \geq F(\zeta) + \langle \nabla F(\zeta), \xi - \zeta\rangle + C_r\,|\xi - \zeta|^r, \qquad \text{for every } \xi, \zeta \in \mathbb{R}^N. \tag{A.1}$$

We use (A.1) with

$$\xi = z \qquad \text{and} \qquad \zeta = t\,z + (1-t)\,w.$$

We obtain

$$F(z) \geq F(t\,z + (1-t)\,w) + (1-t)\,\langle \nabla F(t\,z + (1-t)\,w), z - w\rangle + C_r\,(1-t)^r\,|z-w|^r. \tag{A.2}$$

Similary, we use (A.1) with

$$\xi = w \qquad \text{and} \qquad \zeta = t\,z + (1-t)\,w$$

This now yields

$$F(w) \geq F(t\,z + (1-t)\,w) + t\,\langle \nabla F(t\,z + (1-t)\,w), w - z\rangle + C_r\,t^r\,|z-w|^r. \tag{A.3}$$

We multiply (A.2) by t, then multiply (A.3) by $1-t$ and sum up. The outcome is the following

$$\begin{aligned}(1-t)\,F(w) + t\,F(z) \geq &\; F(t\,z + (1-t)\,w)\\ &+ C_r\,\left[(1-t)^{r-1} + t^{r-1}\right]\,t\,(1-t)\,|z-w|^r.\end{aligned}$$

By using convexity of the function $\tau \mapsto \tau^{r-1}$, we get

$$(1-t)^{r-1} + t^{r-1} \geq 2^{2-r},$$

and thus the conclusion. □

Remark A.1.2. We observe that the extra term $C_r\,|\xi-\zeta|^r$ in (A.1) permits to prove an improved version of the classical Jensen inequality for the convex function $F(z)=|z|^r$, containing a suitable remainder term. A general class of functions which satisfy this kind of stronger Jensen's inequality is widely studied in [**1**].

Lemma A.1.3. *Let $1<r<2$. For every $z,w\in\mathbb{R}^N$ and every $t\in[0,1]$ we have*

$$t\,|z|^r+(1-t)\,|w|^r\geq|t\,z+(1-t)\,w|^r+C\,t\,(1-t)\,\left(|z|^2+|w|^2\right)^{\frac{r-2}{2}}\,|z-w|^2,$$

where $C=C(r)>0$.

Proof. The proof is the same as that of Lemma A.1.1. It is sufficient to use this time [**99**, Lemma 4.2, equation (4.4)]. We leave the details to the reader. □

§A.2 Pointwise inequalities

We recall the following discrete version of Picone's inequality, taken from [**23**, Proposition 4.2] (see also [**66**, Lemma 2.6]). We explicitly state the equality cases.

Lemma A.2.1 (Discrete Picone's inequality)**.** *Let $1<p<\infty$, for every $a,b>0$ and $c,d\geq0$ we have*

$$J_p(a-b)\,\left(\frac{c^p}{a^{p-1}}-\frac{d^p}{b^{p-1}}\right)\leq|c-d|^p.$$

Moreover, equality holds if and only if

$$\frac{c}{a}=\frac{d}{b}.$$

Proof. We first observe that if $c=0$, the inequality is equivalent to

$$J_p(b-a)\,\frac{d^p}{b^{p-1}}\leq d^p.$$

If we also have $d=0$, then this is trivially true. If $d>0$, then this is equivalent to

$$J_p(b-a)\,\frac{1}{b^{p-1}}\leq1\qquad\text{that is}\qquad\left|1-\frac{a}{b}\right|^{p-2}\left(1-\frac{a}{b}\right)\leq1.$$

Since a and b are both positive, it is easily seen that the last inequality is true, actually with the strict inequality sign.

We then suppose $c\neq0$: we first observe that the left-hand side can be rewritten as

$$\begin{aligned}J_p(a-b)\,\left(\frac{c^p}{a^{p-1}}-\frac{d^p}{b^{p-1}}\right)&=J_p\left(1-\frac{b}{a}\right)\,\left(c^p-d^p\,\left(\frac{a}{b}\right)^{p-1}\right)\\&=c^p\,J_p\left(1-\frac{b}{a}\right)\,\left(1-\left(\frac{d}{c}\right)^p\,\left(\frac{a}{b}\right)^{p-1}\right),\end{aligned}$$

thanks to the homogeneity of J_p. If we introduce the shortcut notation

$$t=\frac{b}{a}\qquad\text{and}\qquad A=\frac{d}{c},$$

we then get that the claimed inequality is equivalent to

$$J_p(1-t)\left(1-\frac{A^p}{t^{p-1}}\right) \leq |1-A|^p, \qquad \text{for every } t>0,\ A\geq 0.$$

It is not difficult to see that the function

$$\Phi(t) = J_p(1-t)\left(1-\frac{A^p}{t^{p-1}}\right),$$

is monotone increasing for $t < A$ and monotone decreasing for $t > A$. The choice $t = A$ thus corresponds to the *unique* maximum point, for which we have

$$\Phi(t) \leq \Phi(A) = |1-A|^p.$$

This concludes the proof. □

The following simple inequality will be useful somewhere in Section 3.4.

Lemma A.2.2. *Let $\beta \neq 0$, then we have*

$$|1-\tau^\beta| \leq |\beta|\, \max\left\{\tau^{\beta-1}, 1\right\}(1-\tau), \qquad \textit{for every } \tau\in(0,1),$$

and

$$|1-\tau^\beta| \leq |\beta|\, \max\left\{\tau^{\beta}, \frac{1}{\tau}\right\}(\tau-1), \qquad \textit{for every } \tau>1.$$

Proof. We start with the case $\beta > 0$. By basic Calculus, we have for $\tau \in (0,1)$

$$|1-\tau^\beta| = 1-\tau^\beta = \beta\, \xi^{\beta-1}\,(1-\tau),$$

for some $\tau < \xi < 1$. We observe that the quantity $\xi \mapsto \beta\, \xi^{\beta-1}$ is increasing for $\beta > 1$ and decreasing for $0 < \beta < 1$. This gives the desired conclusion.

The case $\beta < 0$ is treated similarly. We have this time for $\tau \in (0,1)$

$$|1-\tau^\beta| = \tau^\beta - 1 = \beta\, \xi^{\beta-1}\,(\tau-1) = (-\beta)\, \xi^{\beta-1}\,(1-\tau),$$

for some $\tau < \xi < 1$. By using that $\xi^{\beta-1} < \tau^{\beta-1}$, we get the conclusion, here as well.

Finally, for $\tau > 1$ it is sufficient to write

$$|1-\tau^\beta| = \left|1-\left(\frac{1}{\tau}\right)^{-\beta}\right|,$$

and then use the previous inequality, with $-\beta$ in place of β and $1/\tau$ in place of τ. □

§A.3 Inequalities in the fractional setting

In this section we provide two inequalities in the setting of fractional Sobolev spaces: a fractional interpolation–type inequality for smooth functions and a fractional hidden convexity inequality.

Lemma A.3.1 (Fractional interpolation–type inequality for smooth functions)**.** *Let* $1 < p < \infty$ *and* $0 < s < 1$*, then for every* $\varphi \in C^1_0(\mathbb{R}^N)$ *we have*

$$\sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dy \le \frac{C}{s\,(1-s)}\,\|\nabla\varphi\|^{s\,p}_{L^\infty(\mathbb{R}^N)}\,\|\varphi\|^{(1-s)\,p}_{L^\infty(\mathbb{R}^N)},$$

for some $C = C(N,p) > 0$.

Proof. We pick $\delta > 0$, then for $x \in \mathbb{R}^N$ we split the integral in two parts

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dy &= \int_{B_\delta(x)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dy + \int_{\mathbb{R}^N\setminus B_\delta(x)} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+s\,p}}\,dy\\ &\le \|\nabla\varphi\|^p_{L^\infty(\mathbb{R}^N)} \int_{B_\delta(x)} |x-y|^{p\,(1-s)-N}\,dy\\ &+ 2^p\,\|\varphi\|^p_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N\setminus B_\delta(x)} |x-y|^{-N-s\,p}\,dy\\ &= \frac{N\,\omega_N}{p}\left(\frac{\|\nabla\varphi\|^p_{L^\infty(\mathbb{R}^N)}}{1-s}\,\delta^{p\,(1-s)} + 2^p\,\frac{\|\varphi\|^p_{L^\infty(\mathbb{R}^N)}}{s}\,\delta^{-s\,p}\right).\end{aligned}$$

By optimizing in $\delta > 0$, we get the desired result. □

Lemma A.3.2 (Fractional hidden convexity)**.** *Let* $1 < p < \infty$ *and* $0 < s < 1$*. Let* $\Omega \subset \mathbb{R}^N$ *be an open set, for every two non-negative functions* $u, v \in \widetilde{W}^{s,p}_0(\Omega)$*, we set*

$$\sigma = \left(\frac{1}{2}\,u^p + \frac{1}{2}\,v^p\right)^{\frac{1}{p}}.$$

Then $\sigma \in \widetilde{W}^{s,p}_0(\Omega)$ *and there holds*

$$[\sigma]^p_{W^{s,p}(\mathbb{R}^N)} \le \frac{1}{2}\,[u]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{2}\,[v]^p_{W^{s,p}(\mathbb{R}^N)}. \tag{A.4}$$

Moreover, if equality holds in (A.4)*, and* u*,* v *are both positive almost everywhere in* Ω*, then there exists a constant* C *such that*

$$u = C\,v, \qquad \text{a. e. in } \Omega.$$

Proof. The proof of (A.4) and the identification of equality cases are contained in [**67**, Lemma 4.1 & Theorem 4.2]. We just show here that σ belongs to the relevant fractional Sobolev space, a fact that seems to have been overlooked in the literature. We first notice that

$$\int_\Omega |\sigma|^p\,dx = \frac{1}{2}\int_\Omega u^p\,dx + \frac{1}{2}\int_\Omega v^p\,dx < +\infty,$$

and by (A.4) we have in particular

$$[\sigma]_{W^{s,p}(\mathbb{R}^N)} < +\infty.$$

This shows that $\sigma \in W^{s,p}(\mathbb{R}^N)$. We now consider $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}} \subset C^\infty_0(\Omega)$ two sequences which converge respectively to u and v in $W^{s,p}(\mathbb{R}^N)$. Since u and v are positive, without loss of generality we can take u_n and v_n to be non-negative. Moreover, up to pass to a subsequence, we can suppose to have almost everywhere convergence.

We set

$$\sigma_n = \left(\frac{1}{2}\left(u_n + \frac{1}{n}\right)^p + \frac{1}{2}\left(v_n + \frac{1}{n}\right)^p\right)^{\frac{1}{p}} - \frac{1}{n}, \qquad \text{for every } n \in \mathbb{N} \setminus \{0\},$$

and observe that $\{\sigma_n\}_{n\in\mathbb{N}} \subset C_0^\infty(\Omega) \subset \widetilde{W}_0^{s,p}(\Omega)$. Moreover, σ_n converges to σ almost everywhere, as n goes to ∞. We claim that

$$\lim_{n\to\infty} \|\sigma_n - \sigma\|_{L^p(\Omega)} = 0. \tag{A.5}$$

Indeed, thanks to Fatou's Lemma, it holds that

$$\liminf_{n\to\infty} \int_\Omega |\sigma_n|^p \, dx \geq \int_\Omega |\sigma|^p \, dx.$$

Conversely, we observe that[1]

$$\sigma_n \leq \left(\frac{1}{2}\, u_n^p + \frac{1}{2}\, v_n^p\right)^{\frac{1}{p}}, \qquad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

By raising to the power p and taking the limit, we get

$$\limsup_{n\to\infty} \int_\Omega |\sigma_n|^p \, dx \leq \limsup_{n\to\infty} \left[\frac{1}{2} \int_\Omega u_n^p \, dx + \frac{1}{2} \int_\Omega v_n^p \, dx\right] = \int_\Omega |\sigma|^p \, dx.$$

These facts entail that we have convergence of the L^p norms. By joining this with the almost everywhere convergence, we get (A.5) from the so-called *Brézis-Lieb Lemma* (see [**30**, Theorem 1]).

We also observe that $[\sigma_n]_{W^{s,p}(\mathbb{R}^N)}$ is bounded. Indeed, we can apply the convexity inequality (A.4) as follows

$$\begin{aligned}
[\sigma_n]^p_{W^{s,p}(\mathbb{R}^N)} = \left[\sigma_n + \frac{1}{n}\right]^p_{W^{s,p}(\mathbb{R}^N)} &\leq \frac{1}{2}\left[u_n + \frac{1}{n}\right]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{2}\left[v_n + \frac{1}{n}\right]^p_{W^{s,p}(\mathbb{R}^N)} \\
&= \frac{1}{2}\,[u_n]^p_{W^{s,p}(\mathbb{R}^N)} + \frac{1}{2}\,[v_n]^p_{W^{s,p}(\mathbb{R}^N)},
\end{aligned}$$

and observe that the last terms are uniformly bounded, by construction. The uniform bound on $\|\sigma_n\|_{W^{s,p}(\mathbb{R}^N)}$ and the reflexivity of the space $W^{s,p}(\mathbb{R}^N)$ entail that σ_n weakly converges, up to subsequences, to a function in $\widetilde{W}_0^{s,p}(\Omega)$, the latter being a weakly closed subspace of $W^{s,p}(\mathbb{R}^N)$. By the uniqueness of the limit, such a function must coincide with σ, which then belongs to $\widetilde{W}_0^{s,p}(\Omega)$. □

[1] This follows by noticing that the function

$$h(\varepsilon) = \left(\frac{1}{2}\,(a+\varepsilon)^p + \frac{1}{2}\,(b+\varepsilon)^p\right)^{\frac{1}{p}} - \varepsilon, \qquad \text{for every } a, b \geq 0,$$

is monotone decreasing with respect to $\varepsilon \geq 0$, thus $h(\varepsilon) \leq h(0)$.

APPENDIX B

ASYMPTOTICS OF THE POSITIVE SOLUTION IN A SLAB-TYPE SEQUENCE

For every $L > 0$, we indicate by

$$\Omega_L = \left(-\frac{L}{2}, \frac{L}{2}\right)^{N-1} \times I, \tag{B.1}$$

where we recall that $I = (-1, 1)$. We then have the following convergence result.

Lemma B.0.1. *Let $1 \leq q < p < \infty$, we define the function*

$$U_\infty(x', x_N) = w^I_{p,q}(x_N), \qquad \text{for } x' \in \mathbb{R}^{N-1}, x_N \in I.$$

Then for every $L_0 > 0$ we have

$$\lim_{n\to\infty} \|w^{\Omega_n}_{p,q} - U_\infty\|_{L^p(\Omega_{L_0})} = 0.$$

Proof. For simplicity we will write w_Ω in place of $w^\Omega_{p,q}$, which is defined as in Definition 4.1.4. We will adapt to our nonlinear situation a related argument from [**BFR**, Lemma 7.2], for the case $p = 2$ and $q = 1$. We extend all functions w_{Ω_L} to the whole slab $\mathbb{R}^{N-1} \times I$, by putting them constantly equal to 0 outside Ω_L. Observe that, by Elliptic Regularity (see for example [**70**, Theorem 7.8]), we know that w_{Ω_L} is Hölder continuous on $\overline{\Omega_L}$ and thus it takes the homogeneous Dirichlet boundary condition in classical pointwise sense. Accordingly, the extended functions are Hölder continuous on $\mathbb{R}^{N-1} \times I$.

We first observe that

$$w_{\Omega_{L_2}} \geq w_{\Omega_{L_1}}, \qquad \text{for } L_2 \geq L_1,$$

by the comparison principle of Theorem 3.4.1, thanks to the fact that $\Omega_{L_2} \supset \Omega_{L_1}$. Thus, we get that $\{w_{\Omega_L}\}_{L>0}$ is a family of monotone increasing continuous functions. Moreover, we have the uniform upper bound

$$w_{\Omega_L} \leq w_I(0), \qquad \text{in } \mathbb{R}^{N-1} \times I, \tag{B.2}$$

thanks to (4.6). Then, the pointwise limit

$$U(x) = \lim_{L\to+\infty} w_{\Omega_L}(x), \qquad \text{for } x \in \mathbb{R}^{N-1} \times I, \tag{B.3}$$

is well-defined. Observe that this is a bounded function, which still satisfies (B.2). We also notice that, if we fix $L_0 > 0$ as in the statement, we have

$$\lim_{L\to+\infty} \int_{\Omega_{4L_0}} |w_{\Omega_L} - U|^p \, dx = \lim_{n\to\infty} \int_{\Omega_{4L_0}} (U - w_{\Omega_L})^p \, dx = 0, \tag{B.4}$$

thanks to the Monotone Convergence Theorem. We devote the rest of the proof to show that

$$U = U_\infty, \qquad \text{in } \mathbb{R}^{N-1} \times I. \tag{B.5}$$

Let us now work with the sequence $\{w_{\Omega_n}\}_{n\geq 4L_0}$, where Ω_n is defined by (B.1), with the choice $L = n$. We take $\eta \in C^\infty(\overline{\Omega_{4L_0}})$ a cut-off function such that

$$0 \leq \eta \leq 1, \qquad \eta \equiv 1 \quad \text{on } \overline{\Omega_{2L_0}}, \qquad \|\nabla \eta\|_{L^\infty} \leq \frac{C}{L_0},$$

and

$$\eta \equiv 0 \quad \text{on } \Omega_{4L_0} \setminus \Omega_{3L_0}.$$

Then we insert the test function $\psi = \eta^p\, w_{\Omega_n}$ in the weak formulation of (3.1) for w_{Ω_n}. We get

$$\int_{\Omega_n} |\nabla w_{\Omega_n}|^p\, \eta^p\, dx + p \int_{\Omega_n} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla \eta\rangle\, \eta^{p-1}\, w_{\Omega_n}\, dx = \int_{\Omega_n} w_{\Omega_n}^q\, \eta^p\, dx.$$

By Young's inequality, for every $\delta > 0$ we have

$$\begin{aligned} p\,\langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla \eta\rangle\, \eta^{p-1}\, w_{\Omega_n} &\geq -\delta\,(p-1)\,|\nabla w_{\Omega_n}|^p\, \eta^p \\ &\quad - \delta^{-(p-1)}\, w_{\Omega_n}^p\, |\nabla \eta|^p. \end{aligned}$$

By choosing $\delta = 1/(2\,p-2)$, we then obtain the Caccioppoli-type inequality

$$\frac{1}{2} \int_{\Omega_n} |\nabla w_{\Omega_n}|^p\, \eta^p\, dx \leq (2\,p-2)^{p-1} \int_{\Omega_n} |\nabla \eta|^p\, w_{\Omega_n}^p\, dx + \int_{\Omega_n} w_{\Omega_n}^q\, \eta^p\, dx.$$

By using this estimate, the properties of η and the upper bound (B.2), we then obtain in particular

$$\int_{\Omega_{2L_0}} |\nabla w_{\Omega_n}|^p\, dx \leq C, \qquad \text{for every } n \geq 4\,L_0. \tag{B.6}$$

for some uniform constant $C > 0$. This implies that the sequence $\{w_{\Omega_n}\}_{n\geq 4L_0}$ is bounded in $W^{1,p}(\Omega_{2L_0})$. Thus it weakly converges to a function $u \in W^{1,p}(\Omega_{2L_0})$: by uniqueness of the limit, we must have $u = U$. This in particular implies that U belongs to $W^{1,p}(\Omega_{2L_0})$. By also using the compactness of the trace embedding (see [**95**, Corollary 18.4])

$$W^{1,p}(\Omega_{2L_0}) \hookrightarrow L^p(\partial\Omega_{2L_0}),$$

and the boundary condition

$$w_{\Omega_n} = 0, \qquad \text{on } (-L_0, L_0)^{N-1} \times \{-1,\, 1\},$$

we get that the trace of U must have the same property.

We claim that U weakly solves the Lane-Emden equation (3.1) in Ω_{L_0}. We take $\zeta \in C^\infty(\overline{\Omega_{3L_0}})$ a cut-off function such that

$$0 \leq \zeta \leq 1, \qquad \zeta \equiv 1 \quad \text{on } \overline{\Omega_{L_0}}, \qquad \|\nabla \zeta\|_{L^\infty} \leq \frac{C}{L_0},$$

and

$$\zeta \equiv 0 \quad \text{on } \Omega_{3L_0} \setminus \Omega_{2L_0}.$$

Then we have

$$\begin{aligned}\int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla w_{\Omega_n} - \nabla U\rangle\, \zeta\, dx &= \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla((w_{\Omega_n} - U)\,\zeta)\rangle\, dx \\ &\quad - \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla \zeta\rangle\, (w_{\Omega_n} - U)\, dx \\ &= \int_{\Omega_{2L_0}} w_{\Omega_n}^{q-1}\, (w_{\Omega_n} - U)\, \zeta\, dx \\ &\quad - \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla \zeta\rangle\, (w_{\Omega_n} - U)\, dx,\end{aligned} \tag{B.7}$$

where we used the equation for w_{Ω_n}, tested against the function $\psi = \zeta\,(w_{\Omega_n} - U)$. Indeed, this is a feasible test function, thanks to the condition on the trace of $w_{\Omega_n} - U$. We now observe that

$$\lim_{n\to\infty} \int_{\Omega_{2L_0}} w_{\Omega_n}^{q-1}\, (w_{\Omega_n} - U)\, \zeta\, dx = 0,$$

thanks to the uniform bound (B.2) and to the strong convergence (B.4). Moreover, by using (B.6) and again (B.4), we also get

$$\lim_{n\to\infty} \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla \zeta\rangle\, (w_{\Omega_n} - U)\, dx = 0.$$

On account of (B.7), these yield

$$\lim_{n\to\infty} \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n}, \nabla w_{\Omega_n} - \nabla U\rangle\, \zeta\, dx = 0.$$

Moreover, we also have

$$\lim_{n\to\infty} \int_{\Omega_{2L_0}} \langle |\nabla U|^{p-2}\, \nabla U, \nabla w_{\Omega_n} - \nabla U\rangle\, \zeta\, dx = 0,$$

thanks to the weak convergence in $W^{1,p}(\Omega_{2L_0})$ of w_{Ω_n}. By subtracting the last two equations in display, we get

$$\lim_{n\to\infty} \int_{\Omega_{2L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n} - |\nabla U|^{p-2}\, \nabla U, \nabla w_{\Omega_n} - \nabla U\rangle\, \zeta\, dx = 0,$$

as well. By recalling that ζ is constantly equal to 1 on $\overline{\Omega_{L_0}}$ and that the integrand is non-negative, we get in particular

$$\lim_{n\to\infty} \int_{\Omega_{L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\, \nabla w_{\Omega_n} - |\nabla U|^{p-2}\, \nabla U, \nabla w_{\Omega_n} - \nabla U\rangle\, dx = 0.$$

For $p \geq 2$, by recalling the inequality (see [**98**, Section 10, equation (I)])

$$\langle |z|^{p-2}\, z - |w|^{p-2}\, w, z - w\rangle \geq 2^{2-p}\, |z - w|^p, \qquad \text{for every } z, w \in \mathbb{R}^N,$$

we immediately obtain

$$\lim_{n\to\infty} \|\nabla w_{\Omega_n} - \nabla U\|_{L^p(\Omega_{L_0};\mathbb{R}^N)} = 0. \tag{B.8}$$

For $1 < p < 2$, it is slightly more complicate: we need to use the inequality (see [**98**, Section 10, equation (VII)])

$$\langle |z|^{p-2}\, z - |w|^{p-2}\, w, z - w\rangle \geq (p-1)\,|z-w|^2\,(1+|z|^2+|w|^2)^{\frac{p-2}{2}}, \qquad \text{for every } z, w \in \mathbb{R}^N.$$

This permits to infer, thanks to Hölder's inequality, that we have

$$\begin{aligned}\int_{\Omega_{L_0}} |\nabla w_{\Omega_n} - \nabla U|^p\, dx &\leq \left(\int_{\Omega_{L_0}} |\nabla w_{\Omega_n} - \nabla U|^2\,(1+|\nabla w_{\Omega_n}|^2+|\nabla U|^2)^{\frac{p-2}{2}}\, dx\right)^{\frac{p}{2}} \\ &\times \left(\int_{\Omega_{L_0}} (1+|\nabla w_{\Omega_n}|^2+|\nabla U|^2)^{\frac{p}{2}}\, dx\right)^{\frac{2-p}{2}} \\ &\leq \left(\frac{1}{p-1}\int_{\Omega_{L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\,\nabla w_{\Omega_n} - |\nabla U|^{p-2}\,\nabla U, \nabla w_{\Omega_n} - \nabla U\rangle\, dx\right)^{\frac{p}{2}} \\ &\times \left(\int_{\Omega_{L_0}} (1+|\nabla w_{\Omega_n}|^2+|\nabla U|^2)^{\frac{p}{2}}\, dx\right)^{\frac{2-p}{2}}\end{aligned}$$

By using that the last integral is uniformly bounded thanks to (B.6), while the other one converges to 0, we get (B.8) for $1 < p < 2$, as well.

Thanks to the strong convergence (B.8), we can now pass to the limit in

$$\int_{\Omega_{L_0}} \langle |\nabla w_{\Omega_n}|^{p-2}\,\nabla w_{\Omega_n}, \nabla\psi\rangle\, dx = \int_{\Omega_{L_0}} w_{\Omega_n}^{q-1}\,\psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega_{L_0}),$$

and obtain

$$\int_{\Omega_{L_0}} \langle |\nabla U|^{p-2}\,\nabla U, \nabla\psi\rangle\, dx = \int_{\Omega_{L_0}} U^{q-1}\,\psi\, dx, \qquad \text{for every } \psi \in C_0^\infty(\Omega_{L_0}),$$

i. e. U is a solution of the Lane-Emden equation in Ω_{L_0}, as claimed.

Next, we claim that U actually does not depend on the variable x', but only on x_N. To prove this, let us fix two points

$$X = (x'_0, x_N),\ Y = (x'_1, x_N) \in \mathbb{R}^{N-1}\times I, \qquad \text{with } x'_0 \neq x'_1.$$

By definition, we have

$$U(X) = \lim_{n\to\infty} w_{\Omega_n}(X) \qquad \text{and} \qquad U(Y) = \lim_{n\to\infty} w_{\Omega_n}(Y).$$

We introduce the "horizontally" translated set $\widetilde{\Omega}_n = \Omega_n - X + Y$ and observe that we clearly have

$$w_{\widetilde{\Omega}_n}(x) = w_{\Omega_n}(x + X - Y), \qquad \text{for } x \in \widetilde{\Omega}_n.$$

We notice that by construction, we have that there exists $n_0 = n_0(|X-Y|) \in \mathbb{N}$ such that

$$\Omega_{\frac{n}{4}} \subset \widetilde{\Omega}_n \subset \Omega_{4\,n}, \qquad \text{for every } n \geq n_0.$$

Again by the comparison principle of Theorem 3.4.1, we know that

$$w_{\Omega_{\frac{n}{4}}} \leq w_{\widetilde{\Omega}_n} \leq w_{\Omega_{4\,n}},$$

and moreover, we have

$$\lim_{n\to\infty} w_{\Omega_{\frac{n}{4}}}(x) = \lim_{n\to\infty} w_{\Omega_{4\,n}}(x) = U(x),$$

thanks to (B.3). This in turn implies that

$$\lim_{n\to\infty} w_{\widetilde{\Omega}_n}(x) = U(x),$$

as well. We then obtain

$$U(X) = \lim_{n\to\infty} w_{\Omega_n}(X) = \lim_{n\to\infty} w_{\widetilde{\Omega}_n}(Y) = U(Y),$$

as desired.

By resuming, we get that U is a positive weak solution of

$$\begin{cases} -\Delta_p U &= U^{q-1}, \quad \text{in } \Omega_{L_0}, \\ U &= 0, \qquad \text{on } \left(-\dfrac{L_0}{2}, \dfrac{L_0}{2}\right)^{N-1} \times \{-1, 1\}, \end{cases}$$

which does not depend on the variable x'. In particular, we have that $x_N \mapsto U(x', x_N)$ is a weak solution of the same one-dimensional problem solved by w_I, as well. By uniqueness of the solution and arbitrariness of L_0, we finally get (B.5). This concludes the proof. □

Remark B.0.2. The L^p convergence in the previous result can actually be upgraded to a uniform convergence. It is sufficient to observe that U_∞ is continuous, that $\{w_{\Omega_n}\}_{n\in\mathbb{N}}$ is a sequence of monotone increasing continuous functions and then use Dini's Theorem. We leave the details to the reader.

APPENDIX C

TWO PATHOLOGICAL EXAMPLES

§C.1 An infinite strip with slowly shrinking ends

In the next example, we consider a quasibounded open set for which $d_\Omega^\gamma \notin L^1(\mathbb{R}^N)$, for any $0 < \gamma < \infty$. Sets of this type have been considered for example in [**10**] and [**47**, Section 7].

Example C.1.1. *For every $\alpha > 0$ and $x_1 \in \mathbb{R}$, we set*

$$f_1(x_1) = \frac{1}{\log(2 + x_1^2)} \qquad \textit{and} \qquad f_\alpha(x_1) = f_1\left(\frac{x_1}{\alpha}\right) = \frac{1}{\log\left(2 + \left(\dfrac{x_1}{\alpha}\right)^2\right)}.$$

Then we consider the quasibounded open set

$$\Omega_\alpha = \Big\{x = (x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 \in \mathbb{R},\, |x_2| < f_\alpha(x_1)\Big\}, \qquad \textit{with } \alpha^2 > (\log 2)^{-3}.$$

Observe that for this set we have $d_{\Omega_\alpha}^\gamma \notin L^1(\Omega_\alpha)$, for any $0 < \gamma < \infty$. Thus, by Theorem 5.1.1 part (i), we have

$$\mathcal{D}_0^{1,2}(\Omega_\alpha) \not\hookrightarrow L^q(\Omega_\alpha), \qquad \textit{for every } 1 \leq q < 2.$$

On the other hand, since Ω_α is bounded in the x_2 direction, we easily get that $\lambda_2(\Omega_\alpha) > 0$, that is

$$\mathcal{D}_0^{1,2}(\Omega_\alpha) \hookrightarrow L^2(\Omega_\alpha).$$

As for the compactness of this embedding, we observe that this cannot be directly inferred from Theorem 5.2.1, since we are in the critical situation $p = 2 = N$. Nevertheless, we are going to show that actually such an embedding is compact, if α is large enough, thanks to the peculiar geometry of the set Ω_α. In particular, the Dirichlet-Laplacian on Ω_α has a discrete spectrum.

We define

$$\Omega_{\alpha,R} := \Omega_\alpha \cap \Big((-R, R) \times (-R, R)\Big), \qquad \textit{for } R \geq R_0 = \frac{1}{\log 2}.$$

We denote by w_{Ω_α} the torsion function of Ω_α, defined as

$$w_{\Omega_\alpha} := \lim_{R \to \infty} w_{\Omega_{\alpha,R}},$$

where $w_{\Omega_{\alpha,R}} \in W_0^{1,2}(\Omega_{\alpha,R})$ is the torsion function of the bounded set $\Omega_{\alpha,R}$, i. e. it solves

$$-\Delta u = 1, \qquad \textit{in } \Omega_{\alpha,R},$$

*(see [**28**, Definition 2.2]). We observe that w_{Ω_α} is a bounded function, by [**28**, Theorem 1.3]. In order to prove the compactness of the embedding of $\mathcal{D}^{1,2}_0(\Omega_\alpha) \hookrightarrow L^2(\Omega_\alpha)$, it is sufficient to prove that*

$$\lim_{R\to\infty} \|w_{\Omega_\alpha}\|_{L^\infty(\Omega_\alpha\setminus B_R)} = 0, \tag{C.1}$$

*thanks to [**28**, Theorem 1.3]. We will achieve (C.1) by exploiting the geometry of Ω_α in order to construct a suitable upper barrier. For every $\alpha > 0$ and $x_1 \in \mathbb{R}$, we set*

$$F_1(x_1) = (f_1(x_1))^2 \qquad \textit{and} \qquad F_\alpha(x_1) := F_1\left(\frac{x_1}{\alpha}\right) = (f_\alpha(x_1))^2.$$

Observe that F_1 has a bounded second order derivative, i.e. there exists $L > 0$ such that

$$|F_1''(t)| \le L, \qquad \textit{for every } t \in \mathbb{R}.$$

Thus, if we take $\alpha > 0$ such that $\alpha^2 > (\log 2)^{-3}$, we obtain

$$|F_\alpha''(x_1)| = \frac{1}{\alpha^2}\left|F_1''\left(\frac{x_1}{\alpha}\right)\right| \le 2\,(1 - C_\alpha), \qquad \textit{for every } x_1 \in \mathbb{R}, \tag{C.2}$$

where

$$C_\alpha := 1 - \frac{L}{2\alpha^2}.$$

With such a choice of α, we consider the function

$$U_\alpha(x_1, x_2) = \frac{F_\alpha(x_1) - x_2^2}{2\,C}, \qquad \textit{for every } (x_1, x_2) \in \Omega_\alpha.$$

We claim that this is the desired upper barrier. Indeed, by construction we have $U_\alpha \ge 0$ and, thanks to (C.2), it holds

$$-\Delta U_\alpha(x_1, x_2) = \frac{1}{C} - \frac{F_\alpha''(x_1)}{2\,C} \ge 1, \qquad \textit{for every } (x_1, x_2) \in \Omega_\alpha.$$

By applying the Comparison Principle in every $\Omega_{\alpha,R}$ we get that

$$w_{\Omega_{\alpha,R}}(x) \le U_\alpha(x) \le \frac{F_\alpha(x_1)}{2\,C}, \qquad \textit{for every } x = (x_1, x_2) \in \Omega_{\alpha,R},$$

and such an estimate does not depend on R. Hence, by sending R to ∞, we have that

$$w_{\Omega_\alpha}(x) \le \frac{F_\alpha(x_1)}{2\,C}, \qquad \textit{for every } x = (x_1, x_2) \in \Omega_\alpha.$$

By using the properties of $F_\alpha = (f_\alpha)^2$ and the previous estimate, we eventually get (C.1).

§C.2 Negative powers of the distance in the borderline case $s = 1/2$

We consider for $-1 < \beta < 0$

$$U_\beta(t) = d_I(t)^\beta = \left(\min\{t, 1-t\}\right)^\beta, \qquad \text{for } t \in I = (0, 1).$$

We extend this function by 0 outside the interval I. We want to estimate its fractional Laplacian of order $1/2$.

Lemma C.2.1. *Under the assumptions above, we have*

$$(-\Delta)^{\frac{1}{2}} U_\beta \le \beta\, H(t)\, U_\beta(t) + \frac{2}{t\,(1-t)}\, U_\beta(t), \qquad \text{in } I, \tag{C.3}$$

in weak sense, where H is the continuous function on $I \setminus \{1/2\}$ defined by

$$H(t) = -\frac{2}{t\,(1-t)} + \frac{2}{d_I(t)}\, \log\left(\frac{4\,t\,(1-t)}{(1-2\,t)^2}\right).$$

This has the following properties:

- *it is symmetric with respect to $1/2$, i.e.*

$$H(t) = H(1-t), \qquad \text{for } t \in I \setminus \left\{\frac{1}{2}\right\};$$

- *it belongs to $L^q_{\mathrm{loc}}(I)$ for every $1 \le q < \infty$;*
- *it satisfies*

$$H(t) \sim -4\, \log\left(\frac{1}{2} - t\right)^2, \qquad \text{for } t \to \frac{1}{2},$$

  thus the right-hand side of (C.3) diverges to $-\infty$ as t approaches $1/2$.

In particular, in this case the function U_β is not even locally weakly $(1/2)-$superharmonic on I.

Proof. We first show that U_β satisfies (C.3) in $I \setminus \{1/2\}$. Let us take $\varphi \in C^\infty_0(I \setminus \{1/2\})$ non-negative, then there exists $0 < \delta_0 < 1/4$ such that its support is contained in the set

$$\mathcal{I}_{\delta_0} = \left[\delta_0, \frac{1}{2} - \delta_0\right] \cup \left[\frac{1}{2} + \delta_0, 1 - \delta_0\right].$$

For every $\varepsilon > 0$ and $t \in I$, we set

$$\mathfrak{I}_\varepsilon(t) = (t-\varepsilon, t+\varepsilon) \qquad \text{and} \qquad \mathcal{D}_\varepsilon = \Big\{(t,y) \in \mathbb{R}\times\mathbb{R} \,:\, t-\varepsilon \le y \le t+\varepsilon\Big\}.$$

By using Fubini's Theorem and a change of variable, we can write as usual

$$\begin{aligned}
&\iint_{\mathbb{R}\times\mathbb{R}} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\,(\varphi(t) - \varphi(y))}{|t-y|^2}\, dt\, dy \\
&= \lim_{\varepsilon\to 0} \iint_{(\mathbb{R}\times\mathbb{R})\setminus\mathcal{D}_\varepsilon} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\,(\varphi(t) - \varphi(y))}{|t-y|^2}\, dt\, dy \\
&= 2\, \lim_{\varepsilon\to 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t-y|^2}\, dy\right) \varphi(t)\, dt \\
&= 2\, \lim_{\varepsilon\to 0} \int_{\mathcal{I}_{\delta_0}} \left(\int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t-y|^2}\, dy\right) \varphi(t)\, dt.
\end{aligned} \tag{C.4}$$

We first observe that, by using that $U_\beta(t) = U_\beta\,(1-t)$, we have for $t \in \mathcal{I}_{\delta_0}$

$$\int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(t)-U_\beta(y)}{|t-y|^2}\,dy = \int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(1-t)-U_\beta(1-y)}{|t-y|^2}\,dy$$
$$= \int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(1-t)-U_\beta(1-y)}{|(1-t)-(1-y)|^2}\,dy = \int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(1-t)} \frac{U_\beta(1-t)-U_\beta(\tau)}{|(1-t)-\tau|^2}\,d\tau.$$

This shows that it is sufficient to consider $t \in [\delta_0, 1/2-\delta_0]$. For almost every $t \in [\delta_0, 1/2-\delta_0]$ and every $0 < \varepsilon < \delta_0$, we have[1]

$$\int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(t)-U_\beta(y)}{|t-y|^2}\,dy = \int_0^{t-\varepsilon} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{\frac{1}{2}}^1 \frac{t^\beta-(1-y)^\beta}{(y-t)^2}\,dy$$
$$+ \int_1^{+\infty} \frac{t^\beta}{(y-t)^2}\,dy + \int_{-\infty}^0 \frac{t^\beta}{(t-y)^2}\,dy$$
$$= \int_0^{t-\varepsilon} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{\frac{1}{2}}^1 \frac{t^\beta-(1-y)^\beta}{(y-t)^2}\,dy$$
$$+ \frac{U_\beta(t)}{(1-t)} + \frac{U_\beta(t)}{t}.$$

To estimate the remaining integrals, we use the “above tangent” property for the convex function $\tau \mapsto \tau^\beta$, to infer that

$$t^\beta - y^\beta \le \beta\, t^{\beta-1}\,(t-y), \qquad \text{for } y \in (0, t-\varepsilon) \cup \left(t+\varepsilon, \frac{1}{2}\right),$$

and

$$t^\beta - (1-y)^\beta \le \beta\, t^{\beta-1}\,(t+y-1), \qquad \text{for } y \in \left(\frac{1}{2}, 1\right).$$

These yield

$$\int_0^{t-\varepsilon} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta-y^\beta}{|t-y|^2}\,dy + \int_{\frac{1}{2}}^1 \frac{t^\beta-(1-y)^\beta}{(y-t)^2}\,dy$$
$$\le \beta\, t^{\beta-1} \int_0^{t-\varepsilon} \frac{t-y}{|t-y|^2}\,dy + \beta\, t^{\beta-1} \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t-y}{|t-y|^2}\,dy + \beta\, t^{\beta-1} \int_{\frac{1}{2}}^1 \frac{(t+y-1)}{(y-t)^2}\,dy.$$

The last integrals can be explicitly computed. We have

$$\int_0^{t-\varepsilon} \frac{t-y}{|t-y|^2}\,dy + \int_{t+\varepsilon}^{1/2} \frac{t-y}{|t-y|^2}\,dy = \log t - \log\left(\frac{1}{2}-t\right),$$

[1] Observe that by construction $t-\varepsilon > 0$ and $t+\varepsilon < 1/2$.

and

$$\begin{aligned}\int_{1/2}^{1} \frac{(t+y-1)}{(y-t)^2}\,dy &= t\int_{1/2}^{1} \frac{1}{(y-t)^2}\,dy + \int_{1/2}^{1} \frac{y-1}{(y-t)^2}\,dy\\ &= t\left[\left(\frac{1}{2}-t\right)^{-1} - (1-t)^{-1}\right]\\ &\quad -\frac{1}{2}\left(\frac{1}{2}-t\right)^{-1} + \left[\log(1-t) - \log\left(\frac{1}{2}-t\right)\right].\end{aligned}$$

This finally gives that for $t \in [\delta_0, 1/2-\delta_0]$, we have

$$\int_{\mathbb{R}\setminus\mathfrak{I}_\varepsilon(t)} \frac{U_\beta(t)-U_\beta(y)}{|t-y|^{1+2\,s}}\,dy \le \beta\, t^{\beta-1}\, G(t) + \frac{U_\beta(t)}{t\,(1-t)}, \tag{C.5}$$

where we set for simplicity

$$\begin{aligned}G(t) = \left[\log t - \log\left(\frac{1}{2}-t\right)\right] + t\left[\left(\frac{1}{2}-t\right)^{-1} - (1-t)^{-1}\right]\\ -\frac{1}{2}\left(\frac{1}{2}-t\right)^{-1} + \left[\log(1-t) - \log\left(\frac{1}{2}-t\right)\right].\end{aligned}$$

With simple manipulations, we see that this can be also written as

$$G(t) = -\frac{1}{1-t} + \log\left(\frac{4\,t\,(1-t)}{(1-2\,t)^2}\right),$$

and thus this is a continuous function on $(0, 1/2)$ such that

$$G \in L^q_{\mathrm{loc}}((0,1/2]),\ \text{ for every } 1 \le q < \infty \qquad \text{and} \qquad \lim_{t\to\left(\frac{1}{2}\right)^-} G(t) = +\infty,$$

because of the logarithmic term. If we now define

$$H(t) := 2\,\frac{G(t)}{t}, \quad \text{for } t \in \left(0, \frac{1}{2}\right) \qquad \text{and} \qquad H(t) := H(1-t), \quad \text{for } t \in \left(\frac{1}{2}, 1\right),$$

from (C.4) and (C.5), by recalling that φ is non-negative we finally get

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{\Big(U_\beta(t)-U_\beta(y)\Big)\,(\varphi(t)-\varphi(y))}{|t-y|^2}\,dt\,dy \le \int_I \left[\beta\, H(t) + \frac{2}{t\,(1-t)}\right] U_\beta(t)\,\varphi(t)\,dt.$$

This shows that U_β is a local weak subsolution of the the claimed equation (C.3), at least in the open set $I \setminus \{1/2\}$.

In order to show that U_β is a local weak subsolution on the whole interval I, it is sufficient to use a standard trick to "fill the hole": we take $\varphi \in C^\infty_0(I)$ non-negative and for every natural number $n \ge 5$

we take $\psi_n \in C^\infty(\overline{I})$ such that

$$\psi_n \equiv 1 \text{ on } \overline{I} \setminus \left[\frac{1}{2} - \frac{2}{n}, \frac{1}{2} + \frac{2}{n}\right], \qquad \psi_n \equiv 0 \text{ on } \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right],$$

and

$$0 \le \psi_n \le 1, \qquad |\psi_n'| \le C\, n.$$

The seminorm of ψ_n can be estimated by using its properties and an interpolation inequality (see [**27**, Corollary 2.2]), i. e.

$$\begin{aligned} [\psi_n]^2_{W^{\frac{1}{2},2}(I)} = [1-\psi_n]^2_{W^{\frac{1}{2},2}(I)} &\le C \left(\int_I |\psi_n'|^2\, dt\right)^{\frac{1}{2}} \left(\int_I |1-\psi_n|^2\, dt\right)^{\frac{1}{2}} \\ &\le C \left(\int_{\frac{1}{2}-\frac{2}{n}}^{\frac{1}{2}+\frac{2}{n}} |\psi_n'|^2\, dt\right)^{\frac{1}{2}} \left(\int_{\frac{1}{2}-\frac{2}{n}}^{\frac{1}{2}+\frac{2}{n}} |1-\psi_n|^2\, dt\right)^{\frac{1}{2}} \le \widetilde{C}, \end{aligned}$$

where $\widetilde{C}$ does not depend on n. This in particular implies that the sequence $\{\Psi_n\}_{n\ge 5} \subset L^2(I \times I)$ defined by

$$\Psi_n(t,y) := \frac{\psi_n(t) - \psi_n(y)}{|t-y|}, \qquad \text{for a. e. } (t,y) \in I \times I, \tag{C.6}$$

is bounded in $L^2(I \times I)$, since by construction

$$\|\Psi_n\|_{L^2(I\times I)} = [\psi_n]_{W^{\frac{1}{2},s}(I)}.$$

Thus, up to a subsequence, it converges weakly in $L^2(I \times I)$. Thanks to the properties of ψ_n, such a limit function must coincide with the null one.

The test function $\varphi\, \psi_n$ belongs to $C_0^\infty(I \setminus \{1/2\})$ and is non-negative. From the first part we get

$$\begin{aligned} &\iint_{\mathbb{R}\times\mathbb{R}} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t)\,\psi_n(t) - \varphi(y)\,\psi_n(y)\right)}{|t-y|^2}\, dt\, dy \\ &\qquad \le \int_I \left[H(t)\, U_\beta(t) + \frac{2}{t\,(1-t)}\right] U_\beta(t)\, \varphi(t)\, \psi_n(t)\, dt. \end{aligned} \tag{C.7}$$

We wish to pass to the limit in (C.7), as n goes to ∞: for the right-hand side, it is easily seen that

$$\lim_{n\to\infty} \int_I \left[H(t)\, U_\beta(t) + \frac{2}{t\,(1-t)}\right] U_\beta(t)\, \varphi(t)\, \psi_n(t)\, dt = \int_I \left[H(t)\, U_\beta(t) + \frac{2}{t\,(1-t)}\right] U_\beta(t)\, \varphi(t)\, dt,$$

by the Dominated Convergence Theorem. As for the left-hand side, we split the integral as follows:

$$\begin{aligned} &\iint_{\mathbb{R}\times\mathbb{R}} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t)\,\psi_n(t) - \varphi(y)\,\psi_n(y)\right)}{|t-y|^2}\, dt\, dy \\ &\quad = \iint_{I''\times I''} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t)\,\psi_n(t) - \varphi(y)\,\psi_n(y)\right)}{|t-y|^2}\, dt\, dy \\ &\quad + 2 \iint_{I'\times(\mathbb{R}\setminus I'')} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\,\varphi(t)\,\psi_n(t)}{|t-y|^2}\, dt\, dy, \end{aligned}$$

where $I' \Subset I'' \Subset I$ and I' contains the support of φ. For the last integral we can easily pass to the limit as n goes to ∞, for the first one we proceed as follows

$$\begin{aligned}\iint_{I''\times I''} & \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t)\,\psi_n(t) - \varphi(y)\,\psi_n(y)\right)}{|t-y|^2}\,dt\,dy \\ &= \iint_{I''\times I''} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t) - \varphi(y)\right)}{|t-y|^2}\,\frac{\psi_n(t) + \psi_n(y)}{2}\,dt\,dy \\ &+ \iint_{I''\times I''} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\psi_n(t) - \psi_n(y)\right)}{|t-y|^2}\,\frac{\varphi(t) + \varphi(y)}{2}\,dt\,dy.\end{aligned}$$

By using that

$$\frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t) - \varphi(y)\right)}{|t-y|^2} \in L^1(I''\times I''),$$

and the properties of ψ_n, it is easily seen that

$$\begin{aligned}\lim_{n\to\infty} \iint_{I''\times I''} & \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t) - \varphi(y)\right)}{|t-y|^2}\,\frac{\psi_n(t) + \psi_n(y)}{2}\,dt\,dy \\ &= \iint_{I''\times I''} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t) - \varphi(y)\right)}{|t-y|^2}\,dt\,dy,\end{aligned}$$

again thanks to the Dominated Convergence Theorem. Finally, the last integral is the most delicate one: with the notation (C.6), we can write

$$\frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\psi_n(t) - \psi_n(y)\right)}{|t-y|^2}\,\frac{\varphi(t) + \varphi(y)}{2} = \Phi(t,y)\,\Psi_n(t,y),$$

where

$$\Phi(t,y) = \frac{\Big(U_\beta(t) - U_\beta(y)\Big)}{|t-y|}\,\frac{\varphi(t) + \varphi(y)}{2} \in L^2(I''\times I'').$$

Thus, by using the weak convergence of $\{\Psi_n\}_{n\geq 5}$ previously discussed, we get

$$\lim_{n\to\infty} \iint_{I''\times I''} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\psi_n(t) - \psi_n(y)\right)}{|t-y|^2}\,\frac{\varphi(t) + \varphi(y)}{2}\,dt\,dy = 0.$$

Finally, we obtain that we can pass to the limit in (C.7) as n goes to ∞ and obtain

$$\iint_{\mathbb{R}\times\mathbb{R}} \frac{\Big(U_\beta(t) - U_\beta(y)\Big)\left(\varphi(t) - \varphi(y)\right)}{|t-y|^2}\,dt\,dy \leq \int_I \left[H(t)\,U_\beta(t) + 2\,\frac{U_\beta(t)}{(1-t)} + 2\,\frac{U_\beta(t)}{t}\right]\varphi(t)\,dt,$$

for every $\varphi \in C_0^\infty(I)$ non-negative, as desired. □

APPENDIX D

SOME FACTS ABOUT POLYTOPES

§D.1 Properties of polytopes

This appendix is devoted to the study of some properties of polytopes in $\mathbb{R}^N$, which are largely used in Section 7.1.

For every convex set $C \subset \mathbb{R}^N$, we will denote by $\mathrm{relint}(C)$ its *relative interior* and by $\mathrm{relbd}(C)$ its *relative boundary*, once we regard C as a subset of its convex hull. We define the dimension of C as the dimension of its convex hull and we indicate it by $\dim(C)$. Conventionally, the empty set has dimension -1.

Definition D.1.1. Let $C \subset \mathbb{R}^N$ be a not empty convex closed set. A convex subset $S \subset C$ is a face of C if each segment $[x, y] \subset C$ satisfying $S \cap \mathrm{relint}([x, y]) \neq \emptyset$ is contained in S.
We denote by $\mathcal{F}(C)$ the set of all faces of C and by $\mathcal{F}_i(C)$ the set of all faces of C having dimension i, for every $0 \leq i \leq \dim(C) - 1$. The empty set and C itself are faces of C; the other faces are called *proper*. Every $(\dim(C) - 1)$-dimensional face of C is called a *facet* of C.

We can summarise the main properties of faces in the following theorem (see [**120**, Section 2.1]).

Theorem D.1.2. *Let $C \subset \mathbb{R}^N$ be a not empty convex bounded closed set. Then*

1. *the faces of C are closed;*
2. *if $F \neq C$ is a face of C, then $F \cap \mathrm{relint}(C) = \emptyset$;*
3. *if G and F are faces of C, then $G \cap F$ is a face of C;*
4. *if G is a face of F and F is a face of C, then G is a face of C;*
5. *each point $x \in C$ is contained in $\mathrm{relint}(F)$ for a unique face $F \in \mathcal{F}(C)$.*

Definition D.1.3. We say that $K \subset \mathbb{R}^N$ is a polytope if it is the convex hull of finitely many points of $\mathbb{R}^N$.

We recall that, thanks to [**120**, Theorem 1.1.11 and Theorem 2.4.7], a polytope is a compact convex set. Moreover, each proper face of K is itself a polytope and is contained in some facet of K.

Furthermore, we recall that if K is a polytope and $0 \in \mathrm{int}(K)$, then the following facts hold:

(i) the *polar set* $K^\circ = \{x \in \mathbb{R}^N : \langle x, y\rangle \leq 1, \text{ for every } y \in K\}$ is itself a polytope;

(ii) if F is a face of K then the conjugate set $\widehat{F} = \{x \in K^\circ : \langle x, y\rangle = 1, \text{ for every } y \in F\}$ is itself a polytope such that
$$\dim(\widehat{F}) = N - \dim(F) - 1;$$

(iii) if $F, G \in \mathcal{F}(K)$ are such that $F \subset G$, then $\widehat{F} \supset \widehat{G}$;

(iv) the application $F \mapsto \widehat{F}$ is a bijection from $\mathcal{F}(K)$ to $\mathcal{F}(K^\circ)$.

In the sequel we need the following result.

Lemma D.1.4. *Let K be a polytope and assume that $0 \in \operatorname{int}(K)$. Then,*

1. *if S, S' are facets of K such that $S \neq S'$ then $S \cap \operatorname{relint}(S') = \emptyset$;*
2. *for every facet S of K, if $z \in \operatorname{relbd}(S)$, then there exists another facet $\widetilde{S} \neq S$ of K such that $z \in \widetilde{S}$. In particular, $z \in \operatorname{relbd}(S) \cap \operatorname{relbd}(\widetilde{S})$.*

Proof. 1. Without loss of generality, we assume that $F = S \cap S' \neq \emptyset$. Since $S \neq S'$, we have that F is a proper face of S'. Then, by Part (2) of Theorem D.1.2, we have

$$S \cap \operatorname{relint}(S') \subset S \cap S' \cap \operatorname{relint}(S') = F \cap \operatorname{relint}(S') = \emptyset;$$

2. let S be a facet of K and let $z \in \operatorname{relbd}(S)$. Then, there exists a facet F of S such that $z \in F$. Then, by Part (4) of Theorem D.1.2, we have that $F \in \mathcal{F}(K)$ and $\dim(F) = N - 2$. This implies $\dim(\widehat{F}) = 1$. Therefore, there exist exactly two points x', x'', such that $x' \neq x''$ and

$$\widehat{F} = [x', x''].$$

Then, thanks to properties (iv) and (ii) above, there exist two facets $S', S'' \in \mathcal{F}(K)$, with $S' \neq S''$, such that $\widehat{S'} = \{x'\}$ and $\widehat{S''} = \{x''\}$, and, by property (iii), it holds

$$z \in F \subset \widehat{\{x'\}} \cap \widehat{\{x''\}} = S' \cap S'',$$

namely, there exists at least another facet $\widetilde{S} \neq S$ containing z. Since $z \in \operatorname{relbd}(S) \cap \widetilde{S} \subset S \cap \widetilde{S}$ and, thanks to Part (1) of this lemma, $S \cap \operatorname{relint}(\widetilde{S}) = \emptyset$, we can conclude that $z \in \operatorname{relbd}(S) \cap \operatorname{relbd}(\widetilde{S})$. □

§D.2 A generalization for Makai's covering argument

In the next lemma, we extend to every dimension the argument applied by Makai in [**102**] in order to divide the interior of a polytope K, when $\operatorname{int}(K) \neq \emptyset$, in a finite number of suitable convex open subsets with a particular property.

Lemma D.2.1. *Let $N \geq 2$ and $K \subset \mathbb{R}^N$ be a polytope such that $0 \in \operatorname{int}(K)$. Let $\Omega = \operatorname{int}(K)$ and let $S_1, S_2, \ldots, S_h$ be the facets of K. For every $i \in \{1, \ldots, h\}$, let $\Pi_i : \mathbb{R}^N \to H_i$ be the orthogonal projection on the affine hyperplane H_i containing S_i. Define*

$$\Omega_i = \Big\{ x \in \Omega : d_{S_i}(x) = d_\Omega(x) \Big\}, \tag{D.1}$$

where

$$d_{S_i}(x) = \min_{y \in S_i} |x - y|.$$

Then, for every $i \in \{1, \ldots, h\}$, the following facts hold:

1. *if $x \in \Omega_i$ then $\Pi_i(x) \in S_i$. In particular $\Pi_i(x)$ is the unique minimizer of the problem defining $d_{S_i}(x)$;*

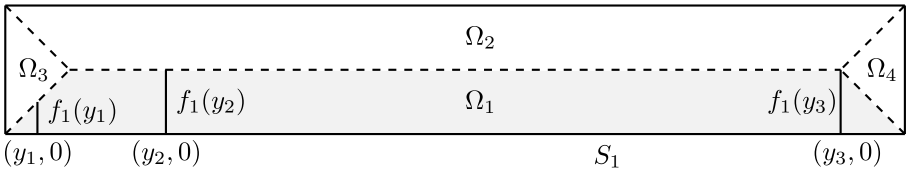


FIGURE D.1: The partition of $\Omega = \left(-\frac{L}{2}, \frac{L}{2}\right) \times (0,1)$, when $L > 1$, by means of the sets Ω_i, and the function $f_1 = f_1(y)$ with $(y, 0) \in S_1$. We note that $f_1(y_3) = 1/2 = L/2 - y_3$.

2. $\Pi_i(x) \in \operatorname{relint}(S_i)$, *for every* $x \in \Omega_i$*;*
3. Ω_i *is a convex set;*
4. $\operatorname{int}(\Omega_i) \neq \emptyset$*;*
5. Ω_i *can be included in a rectangle with base* S_i *and height* r_Ω*;*
6. *for every* $x_0 \in \operatorname{relint}(S_i)$*, there exists a unique* $y_0 \in \partial\Omega_i \cap \Omega$ *such that* $\Pi_i(y_0) = x_0$.

In particular, the restriction $\Pi_i : \partial\Omega_i \cap \Omega \to \operatorname{relint}(S_i)$ *is a continuous bijection.*

Proof. First of all, we notice that, since $\operatorname{int}(K) \neq \emptyset$, we have that $\overline{\Omega} = K$. Now, we prove every part separately. Let us fix $i \in \{1, \dots, h\}$.

1. Let $x \in \Omega_i$. By contradiction, if $\Pi_i(x) \notin K$, then the segment $[x, \Pi_i(x)]$ would intersect $\partial\Omega$ in a point z. Since $z \neq \Pi_i(x)$, then $z \in S_j$, with $S_j \neq S_i$. Hence, as $x \in \Omega_i$, we would have that
$$d_\Omega(x) \leq d_{S_j}(x) \leq |x - z| < |x - \Pi_i(x)| = \min_{y \in H_i} |x - y| \leq d_{S_i}(x) = d_\Omega(x),$$
which is a contradiction. In particular, using the fact that $\Pi_i(x) \in S_i$, we get that
$$d_{S_i}(x) = \min_{y \in S_i} |x - y| \leq |x - \Pi_i(x)| = \min_{y \in H_i} |x - y| \leq d_{S_i}(x)$$
that is, $\Pi_i(x)$ is the unique minimizer of the problem defining $d_{S_i}(x)$;

2. by contradiction, let us suppose that $\Pi_i(x) \in S_i \setminus \operatorname{relint}(S_i) = \operatorname{relbd}(S_i)$. By Lemma D.1.4, Part (2), there exists $S_j \neq S_i$ such that $\Pi_i(x) \in S_j$. Since
$$d_{S_j}(x) \leq |x - \Pi_i(x)| = d_{S_i}(x) = d_\Omega(x) \leq d_{S_j}(x),$$
we get that
$$d_{S_j}(x) = |x - \Pi_i(x)|.$$
Being $\Pi_i(x) \in S_j$, by uniqueness, we would obtain $\Pi_j(x) = \Pi_i(x) \in H_i \cap H_j$. Then $H_i \equiv H_j$, giving the contradiction $S_i = S_j$;

3. by contradiction, assume that there exist $x, y \in \Omega_i$ and $\lambda \in (0,1)$ such that $z = \lambda\, x + (1 - \lambda)\, y \notin \Omega_i$. Hence $z \in \Omega_j$ for some $j \neq i$, and
$$d_\Omega(z) = d_{S_j}(z) < d_{S_i}(z). \tag{D.2}$$

Since Ω is a convex set, then the distance function d_Ω is a concave function (see [**6**]), hence, it follows that

$$d_\Omega(z) = d_\Omega(\lambda\, x + (1-\lambda)\, y) \geq \lambda\, d_\Omega(x) + (1-\lambda)\, d_\Omega(y). \tag{D.3}$$

On the other hand, by Part (1), we have that $\Pi_i(x), \Pi_i(y) \in S_i$. By linearity, we get that

$$\Pi_i(z) = \lambda\, \Pi_i(x) + (1-\lambda)\, \Pi_i(y) \in S_i,$$

which implies that

$$d_{S_i}(z) = |z - \Pi_i(z)| \leq \lambda |x - \Pi_i(x)| + (1-\lambda)|y - \Pi_i(y)| = \lambda\, d_\Omega(x) + (1-\lambda)\, d_\Omega(y). \tag{D.4}$$

By combining (D.3), (D.4) and (D.2), we obtain a contradiction;

4. let $x_0 \in \mathrm{relint}(S_i)$, then, by Part(1) of Lemma D.1.4, we have that $x_0 \notin S_j$ for every $S_j \neq S_i$. We set $g_j(\,\cdot\,) = d_{S_j}(\,\cdot\,) - d_{S_i}(\,\cdot\,)$, then, it holds that

$$g_j(x_0) = d_{S_j}(x_0) > 0.$$

Being g_j a continuous function on $\mathbb{R}^N$, there exists $B_\rho(x_0)$ such that $g_j > 0$ on $B_\rho(x_0)$, for every $j \neq i$. Hence $d_\Omega = d_{S_i}$ on the open set $B_\rho(x_0) \cap \Omega \neq \emptyset$, which implies that $B_\rho(x_0) \cap \Omega \subset \Omega_i$, giving the desired conclusion;

5. without loss of generality, suppose that

$$H_i = \{x = (y,t) \in \mathbb{R}^{N-1} \times \mathbb{R} :\, t = 0\}.$$

Hence, thanks to Part (3), one of the following inclusion holds

$$\Omega_i \subset \{(y,t) \in \mathbb{R}^{N-1} \times \mathbb{R} :\, t \geq 0\} \qquad \text{or} \qquad \Omega_i \subset \{(y,t) \in \mathbb{R}^{N-1} \times \mathbb{R} :\, t \leq 0\}.$$

In both cases, since

$$|t| = d_{S_i}(x) = d_\Omega(x) \leq r_\Omega, \qquad \text{for every } x \in \Omega_i,$$

we obtain the claimed conclusion;

6. let $x_0 \in \mathrm{relint}(S_i)$. Since $\mathrm{int}(\Omega_i) \neq \emptyset$, we can take an open half-line r_{x_0} with origin x_0 such that it is perpendicular to S_i in x_0 and $r_{x_0} \cap \mathrm{int}(\Omega_i) \neq \emptyset$. Being Ω_i bounded set and r_{x_0} a connected set, we obtain that $r_{x_0} \cap \partial\Omega_i \neq \emptyset$. Moreover, as $\partial\Omega_i = (\partial\Omega_i \cap \Omega) \cup S_i$ and $r_{x_0} \cap S_i = \emptyset$, we also have that

$$\Sigma_i(x_0) := r_{x_0} \cap \partial\Omega_i \cap \Omega \neq \emptyset.$$

Now, we will show that $\Sigma_i(x_0)$ consists in exactly a point. Indeed, we note that if $y' \in \Sigma_i(x_0)$, then

$$]x_0, y'[=]x_0, x] \cup [x, y'[\subset \mathrm{int}(\Omega_i), \qquad \text{for every } x \in r_{x_0} \cap \mathrm{int}(\Omega_i).$$

In particular, if $y', y'' \in \Sigma_i(x_0)$, with $y' \neq y''$, then we would obtain the contradiction

$$y'' \in]x_0, y'[\subset \mathrm{int}(\Omega_i) \qquad \text{or} \qquad y' \in]x_0, y''[\subset \mathrm{int}(\Omega_i).$$

□

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This thesis is based on the following papers:

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