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An analysis of precautionary behavior in retirement decision making with an application to pension system reform

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ABSTRACT

We analyze how precautionary motives affect the decisions of a risk-averse agent on saving, labor supply and retirement. In a setting where there is a random shock which affects agent disutility from work, we show that uncertainty directly affects retirement age and saving, but leaves labor supply during working age unchanged. In particular, a precautionary motive for retirement always arises, which pushes the agent to bring forward retirement in the presence of a risk on the cost of work effort. Moreover, prudence and a sufficiently high level of absolute temperance are sufficient conditions for precautionary saving. In this setting, we also study the effects of two common reforms of the pension system: an increase in pension contributions and a cut in pension benefits. The conditions for the agent to postpone retirement and increase labor supply are studied. This makes it possible to characterize the circumstances when the financial soundness of the pension system improves after these reforms.

1. Introduction

The retirement decision is complex and multifaceted, and is influenced by various characteristics of the decision maker, including health, family, individual pension incentives and preferences for leisure.[2] A trait which is common to some of these dimensions is the role of retirement as a means to insure against risks of different nature. In particular, retirement reduces the uncertainty which arises in the labor market. This uncertainty may be due to unemployment, causing household income to become risky (Magnani, 2020), or to a health risk which arises as a consequence of working activity (Pestieau and Racionero, 2016a, 2016b). Precautionary motives thus affect the retirement decision as well as those decisions which are closely intertwined with the timing of retirement, i.e. saving decisions and labor supply decisions. Once retired, in fact, an agent stops earning a labor income and must finance consumption with pension payments and savings, which implies that labor supply, the level of saving and the timing of retirement must be jointly planned.[3]

Our analysis studies the effects of precautionary motives on these decisions when agent disutility from work becomes uncertain. This risk on the cost of work effort (or work effort risk) summarizes the effects of the different shocks which can affect working activity in old age. One example is health issues, which increase the actual cost of effort. Another example is shocks on preferences which cause the opportunity cost of working to increase; for instance, the value of spare time for elderly workers might increase because it can be spent taking care of grandchildren and/or parents, or simply traveling around the world.

Following a long tradition in decision theory which dates back to Pratt (1964), we focus on the case of small risks, and show that uncertainty caused by a random shock on the disutility from work directly affects the choice of the retirement age and the level of saving, but leaves unchanged labor supply during working age. In this context, a

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[2] The effects of these characteristics on the retirement decision are the focus of many studies. See Coile (2015) for a survey.

[3] In this context, the points in time when the agent stops working and when pension benefits are claimed may not be the same, but available evidence shows that most of the time they coincide. Data reported by OECD (2017) and Goll (2020) for the E.U., and by Waldron (2020) for the U.S., show that very few people receive pensions while working. Gustman and Steinmeier (2015) further claim that the spike in retirements at age 62, which is the most prominent feature in retirement behavior in the U.S., is accompanied by a spike in benefit claiming at that age.

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precautionary motive for retirement which pushes the agent to bring forward retirement always arises. However, this is not the case for saving. Prudence and a sufficiently high level of absolute temperance are required, as they are sufficient conditions for precautionary saving.

The results of this positive analysis allow us to further address an important normative question, i.e. what type of reform of the pension system can achieve the purpose of financial stabilization.

Instability is a common problem in pension systems in different countries, regardless of their nature (Castañeda et al., 2021). Lower birth rates and increased longevity, resulting in aging populations, in fact, have made the financing of pension payments increasingly difficult. Solutions put forward to restore the financial soundness of the system often involve an increase in pension contributions and/or a cut in pension benefits.

Such measures are intuitively beneficial because they increase revenues and decrease payments. However, an actual improvement is achieved only if these reforms do not backfire by leading workers to reduce the labor supply, causing a drop in revenues of the system, and/or by leading them to bring forward retirement, possibly increasing pension payments.[4] We investigate this issue, and focus on how an increase in pension contributions and a cut in pension benefits affect labor supply during working age and during old age, in order to identify specific conditions where the financial soundness of the pension system improves.

A reform that increases the pension contribution rate achieves the goal of improving the pension system balance sheet if two sufficient conditions are satisfied. The first condition is that interactions between retirement and saving cause saving to foster late retirement. The second condition requires the real return paid to workers on pension contributions to exceed the real return of a portfolio including liquidity and a riskless asset. When these conditions occur, total pension contributions increase together with labor supply, and financial soundness improves if the pension system obtains a real return on these funds higher than the real return paid to workers.

When a cut in pension benefits is implemented, no clear cut predictions are obtained. This reform, which causes a decline in pension payments, typically triggers a decrease in lifetime labor supply and thus also a decrease in pension contributions. As a consequence, the net effect on the financial soundness of the system is ambiguous.

These results are obtained in a setting where agent preferences are described by a general utility function, and this is the main element of novelty of the present analysis. Their scope thus is broader than for results obtained in most of previous literature on retirement decision and pension system reforms.

The paper has the following structure. Section 2 reviews the related literature, and Section 3 presents the baseline model. Section 4 analyzes the interactions between the precautionary motives for retirement and for saving, in the presence of a health risk. Section 5 discusses the effects of an increase in pension contributions and a cut in pension payments. Section 6 concludes.

2. Related literature

We contribute to the literature which studies the interactions between the social security system and the labor market in the presence of risk.

The paper builds on the work of Magnani (2020) which studies retirement and saving decisions in the presence of a labor-income risk, and analyzes the precautionary motive for retirement. In particular, Magnani (2020), focusing on the case of small risks, derives a set of conditions on decision maker preferences which allow precautionary retirement and precautionary saving to occur, and considers the mutual interactions between them.

Our paper differs from Magnani (2020) in that we do not study the effects of a random shock on labor income, but focus on a small risk which affects the disutility from work (as in Kuhn et al., 2015). Moreover, we introduce labor supply in the young age among the choice variables, and explicitly model the pension system in order to analyze the outcomes of different reforms.

We contribute to the precautionary saving literature, surveyed in Baiardi et al. (2020), by considering, for the first time, the interactions between saving, retirement and labor supply, in the presence of uncertainty, and their effects on the financial soundness of the pension system. In this field of research, other authors consider the joint decision on labor supply and saving in the presence of uncertainty, but ignore the choice over the timing of retirement. In particular, Flodén (2006) studies this issue in a setting where, unlike the present analysis, the decision maker has a bivariate utility function, which depends on consumption and leisure. Nocetti and Smith (2011) further enrich this framework introducing a utility function which makes it possible to disentangle risk preferences from preferences toward intertemporal substitution.

A different body of literature focuses on the interactions between the design of the pension system and the labor market, in the presence of different risks (typically longevity and health risks). There are three main strands of literature, which make use of calibrated life cycle models, on this issue.

The first strand considers how the setup and proposed reforms to the social security system affect retirement and labor supply. Rust and Phelan (1997) show that labor supply is positively affected by pension eligibility age, while other studies find that changes in pension benefits strongly affect labor supply (Deng et al., 2021; French, 2005; French and Jones, 2011; Laun et al., 2019; Malkova, 2020; Gustman and Steinmeier, 2009; Maurer et al., 2021). Moreover, Haan and Prowse (2014) and Gustman and Steinmeier (2005) focus on an increase in the pension age, and show that this reform generates sizable labor supply responses.

The second strand of the literature studies the links between retirement, longevity and health. Different authors analyze these links in settings where health is an exogenous variable (Bloom et al., 2007, 2014; d'Albis et al., 2012), and in settings where investments in health are possible (Dalgaard and Strulik, 2012; Galama et al., 2013; Kuhn et al., 2015).

The third strand of the literature focuses on the interactions between the precautionary motive for saving and retirement decisions. In particular, Jappelli et al. (2021), and Van Santen (2019) focus on the risk deriving from uncertainty in pension income, while Caliendo et al. (2016) focus on the risk deriving from uncertainty about the timing of retirement.

None these strands of the literature however, explicitly consider how different risk preferences affect worker behavior. Life cycle models in fact typically adopt specific univariate expected utility functions which constrain the possible results of the analysis when uncertainty is introduced.

[4] Early retirement is possible in many pension systems which allow for flexibility in the timing of retirement. In these systems, workers are permitted to retire within a certain age range, for example between 62 and 67 years old (OECD, 2017 and Van Vuuren, 2014). The effect of early retirement is twofold. If workers bring their retirement forward, there is a negative impact on pension contributions because the labor supply decreases. The amount of pension payments made may also increase, if early retirement is accompanied by early benefit claiming, and if the resulting reduction in life-long earnings is not actuarially sound. It is in fact the case that early retirement is accompanied by early benefit claiming, as noted above (Footnote 3, page 1), but also that the resulting reduction in life-long earnings is not actuarially sound. Most pension systems in fact do not provide for automatic adjustment mechanisms of the type described by Bravo et al. (2023). Mechanisms linking retirement age and pension benefits to increased longevity are present in only half of OECD countries, and, in some cases, they do not cover all the components of a pension system (OECD, 2019).

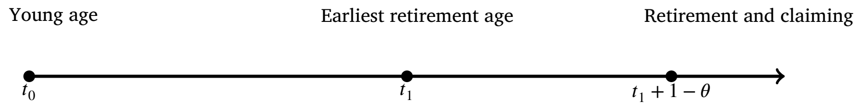

Fig. 1. Timing of the model.

The present paper adopts a different approach, and studies the retirement decision, and the decisions closely intertwined with it, from a more general perspective. In fact no further restrictions on preferences are introduced beyond the adoption of a univariate, time separable, expected utility function. This makes it possible to analyze how precautionary motives affect decisions on retirement, saving and labor supply, and also to characterize the conditions on risk preferences which allow for different reforms to be effective at enhancing the financial soundness of the pension system.

Lastly, it is worth pointing out the complementarity between our paper and the work by Baltas et al. (2022). These authors extend the reach of uncertainty and analyze model uncertainty in a dynamic setting where, as in the present study, a defined contribution pension fund operates. In this context, the manager of the fund does not fully trust the model adopted to handle the portfolio of investments, because the exact probability law of the stochastic risk factors involved is not known. This is because of the long-term nature of pension fund investments and connected risks.

From an analytical point of view, model uncertainty is handled by resorting to techniques of robust optimization and robust control which are also widely used in estimation theory.[5] The results supply new insights into optimal investment decisions and are complementary to the results of our analysis, which takes the point of view of fund members rather than the fund manager.

This suggests a fruitful avenue for future research: introducing model uncertainty in the present setting could lead to deeper understanding on worker retirement decision in the presence of long-term risks.

3. The baseline model

The model considers a representative agent who lives for two periods, and is young in the first period and old in the second period. While young, the agent supplies an amount l_1 of labor on the market and earns a post-tax labor income $l_1 \cdot w(1-\tau)$, where w is the wage rate and τ is the pension contribution rate.

Income is allocated to consumption and saving, denoted by $s \in \Re^+$. Saving is invested in financial markets, and in the second period, yields the return $s \cdot R$ where $R \geq 1$ is the riskless gross return on saving in real terms.

In the second period, an old agent is entitled to retire, implying that the beginning of the period corresponds to the earliest possible regular retirement age. Following available data (Footnote 3, page 1), we further assume that retirement and pension benefit claiming happen at the same time.

Workers can adjust the labor supply in old age, and the timing of retirement by choosing the fraction $\theta \in \left[\bar{\theta}, 1\right]$ of the second period when they do not work, implying that $1-\theta$ is in fact second period labor supply. This fraction is adjusted, in the first place, by choosing the retirement age, i.e. by means of an adjustment of labor supply at the extensive margin, and, in the second place, through a reduction in hours worked, i.e. by means of an adjustment of labor supply at the intensive margin.

Since adjustments at the extensive margin are, in fact, the main driver of second period labor supply, we refer to changes in θ as changes in retirement behavior, for the sake of simplicity.[6] The parameter $\bar{\theta} > 0$ thus defines the upper bound for the labor supply of an old worker, determined by the mandatory retirement age.

In this setting, old age is split in two subperiods. As shown in the diagram (see Fig. 1), the first subperiod is the last part of a worker's working life, and the second subperiod approximately corresponds to retirement.

Second period labor supply is thus $l_2 = 1-\theta$, and post-tax labor income amounts to $(1-\theta)\, w(1-\tau)$.

During retirement, the agent receives a pension payment $P(\theta) = k + \alpha \cdot \beta \cdot w \cdot \tau \cdot l_1 + \beta \cdot w \cdot \tau \cdot l_2$, where $k \geq 0$ is a fixed benefit not proportional to labor income, $\alpha > 0$ is the real return on pension contributions in the first period, and $\beta > 0$ is the real rate of return on pension contributions in the second period.

These parameters characterize the generosity of the pension system, and define its actuarial fairness and actuarial neutrality. The benchmark case where the system is actuarially fair and neutral requires, in fact, beyond $k = 0$, that $\alpha = R$ and $\beta = 1$ hold, so that the real return on pension contributions in the first period is equal to the riskless real return on saving, and the real return on pension contributions in the second period is equal to the real return on liquidity.

The agent has time-separable preferences described by the utility function

$$V\left(y_1, y_2\right) = u\left(y_1\right) + v\left(y_2\right),$$

where y_t denotes income in period t $(t = 1, 2)$, and u and v are Von Neumann-Morgenstern utility functions defining utility in the first period and in the second period respectively. For the sake of simplicity, we assume that the intertemporal discount rate is embedded in the utility function v.

Denote by u_1, u_{11}, u_{111} and u_{1111}(v_1, v_{11}, v_{111} and v_{1111}) the derivatives of u (and v), from the first to the fourth. Functions u and v are assumed to be strictly increasing and strictly concave ($u_1 > 0$, $v_1 > 0$, $u_{11} < 0$, $v_{11} < 0$), and four times continuously differentiable.

We further assume that working implies a disutility in monetary terms. When the agent is young, disutility from work, i.e. the cost of work effort, is described by the function $f(l)$, such that $f'(l) \geq 0$, $f'(0) = 0$, and $f''(l) > 0$. When the agent is old, the cost of work effort increases by a factor $(1+\phi)$. This increase is meant to capture the effects of health shocks and/or of increased preferences for leisure by the elderly.[7]

From previous assumptions it follows that first period income is

$$y_1 = w(1-\tau)l_1 - s - f\left(l_1\right),$$

[5] Consider in particular, the robustification of non-parametric regression techniques such as multivariate adaptive regression splines (MARS, see Friedman, 1991) and conic MARS (CMARS, see Weber et al., 2012), with respective outcomes robust MARS (RMARS, see Özmen et al., 2023) and robust CMARS (RCMARS, see Özmen et al., 2011).

[6] This is coherent with the empirical findings by Blundell et al. (2016) and OECD (2017) which show that, on average, employed workers aged 60-64 work only slightly fewer hours per week than those aged 50-54.

[7] This setting is analogous to a setting where the agent has a time-separable bivariate utility function which depends on wealth and leisure. These variables are substitutes in the sense of Edgeworth-Pareto, a condition that, according to the definition by Samuelson (1974, p.1270), requires the cross-derivative of the utility function to have a negative sign. The absolute value of this cross derivative defines the rate at which the two variables can be substituted. In the present setting, the cost of work effort $f(l)$ sets this rate, which increases to $f(l)(1+\phi)$ for elderly workers, because the cost of work effort increases in old age.

while second period income is

$$y_2 = s \cdot R + l_2 \cdot w(1-\tau) + P(\theta) - (1+\phi) f\left(l_2\right) =$$
$$= s \cdot R + (1-\theta) w(1-\tau+\beta\cdot\tau) + k + \alpha\cdot\beta\cdot w\cdot\tau\cdot l_1 +$$
$$-(1+\phi) f(1-\theta).$$

In this setting, workers consider total utility in the second period, and do not distinguish between utility in the subperiod when they are working and utility in the subperiod when they retire. We adopt this assumption to capture relevant features of the retirement decision. The first feature is the fact that by adjusting their retirement age workers also adjust the ratio between total life length and retirement length, or, equivalently, between total life length and the length of working activity. The second feature is the fact that changing the retirement age also changes the composition of income in terms of labor income and pension payments, and the degree of exposure to work effort risk. In planning the path of lifetime consumption and utility, the choice on the timing of retirement is therefore mostly about achieving an optimal balance between work and retirement.

Consider initially a setting where there is no uncertainty. The agent solves the following maximization problem:

$$\max_{\{s,\theta,l_1\}} V\left(s,\theta,l_1\right) = u\left(w(1-\tau)l_1 - s - f\left(l_1\right)\right) + \tag{1}$$
$$+v(s\cdot R + (1-\theta) w(1-\tau+\beta\cdot\tau) + k + \alpha\cdot\beta\cdot w\cdot\tau\cdot l_1 +$$
$$-(1+\phi) f(1-\theta)).$$

As in Magnani (2020), we assume that all choice variables, i.e. optimal saving, s^c, the optimal timing of retirement, θ^c, and optimal labor supply in the young age l_1^c are simultaneously chosen at the beginning of the first period. Retirement in fact sets the moment in time when workers stop earning a labor income and start relying on pension payments and the return on saving in order to finance consumption. A sensible planning of life-time utility requires labor supply during working age and saving choices to be coherent with the timing of retirement. Households thus consider these issues as parts of the same decision process.[8]

In the absence of uncertainty, optimal saving, s^c, is defined by the following first-order condition:

$$\frac{\partial V\left(s^c,\theta^c,l_1^c\right)}{\partial s} = -u_1\left(y_1\right) + R v_1\left(y_2\right) = 0,$$

which implies

$$u_1\left(y_1\right) = R v_1\left(y_2\right), \tag{2}$$

i.e. requires the marginal disutility of reducing first period income by the amount of saving to be equal to the marginal benefit of increasing second period income by the return on saving.

The optimal timing of retirement, θ^c, is defined by the following first-order condition:

$$\frac{\partial V\left(s^c,\theta^c,l_1^c\right)}{\partial \theta} = v_1\left(y_2\right)\left[(1+\phi) f'\left(1-\theta^c\right) - w\left(1-\tau+\beta\tau\right)\right] = 0,$$

which implies

$$w\left(1-\tau+\beta\tau\right) = (1+\phi) f'\left(1-\theta^c\right), \tag{3}$$

so that the marginal cost of early retirement (and of reducing second period labor supply) is equal to its marginal benefit.

Lastly, consider optimal labor supply in the first period, l_1^c:

$$\frac{\partial V\left(s^c,\theta^c,l_1^c\right)}{\partial l_1} = u_1\left(y_1\right)\left[w(1-\tau) - f'\left(l_1^c\right)\right] + v_1\left(y_2\right)\alpha\beta\tau w = 0. \tag{4}$$

By substituting Equation (2), simplifying and reordering the terms, we can rewrite this equation as

$$f'\left(l_1^c\right) = w\cdot\frac{(1-\tau)R + \alpha\beta\tau}{R}, \tag{5}$$

so that the marginal cost of increasing first period labor supply equals its marginal benefit.

In the Appendix (Subsection A.1) we show that the second order conditions (SOCs) for Problem (1) are satisfied. Hence s^c, θ^c and l_1^c maximize agent utility.

Note now that in the absence of uncertainty, there are no interactions between optimal retirement and optimal first period labor supply and saving, because $\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial s\partial\theta} = \frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial l_1\partial\theta} = 0$ hold. In this setting thus, optimal retirement is part of a decision process which is independent from saving and first period labor supply choices.

Interactions between retirement and saving only occur when a work effort risk is considered. In the presence of uncertainty, a precautionary motive arises which affects agent decisions, and causes the choices on saving and retirement to be intertwined.

4. The effects of a random shock on the disutility from work

We now introduce uncertainty into the baseline model, in the form of a random shock on the disutility from work in the second period. This generic shock captures the effect of different push and pull factors which affect the retirement decision. For instance, it can represent a deterioration in health status which pushes a worker into retirement because the cost of work effort increases; or it can represent an increase in the opportunity cost of working due to spouse retirement or to child/elderly care needs, which pulls the worker into retirement due to a higher value of leisure.[9]

4.1. The model with uncertainty

Define then the stochastic second period income as

$$\tilde{y}_2 = s\cdot R + (1-\theta) w(1-\tau+\beta\cdot\tau) + k + \alpha\cdot\beta\cdot w\cdot\tau\cdot l_1 +$$
$$-\left(1+\tilde{\phi}\right) f(1-\theta),$$

where $\tilde{\phi}$ is a random variable distributed on $\left[0,\bar{\phi}\right]$ with an expected value $E[\tilde{\phi}] = \phi$, so that $E\left[\tilde{y}_2\right] = y_2$.

The maximization problem of the agent becomes:

$$\max_{\{s,\theta,l_1\}} E\left[V\left(s,\theta,l_1\right)\right] = u\left(w(1-\tau)l_1 - s - f\left(l_1\right)\right) + \tag{6}$$
$$+E[v(s\cdot R + (1-\theta) w(1-\tau+\beta\cdot\tau) +$$
$$+k + \alpha\cdot\beta\cdot w\cdot\tau\cdot l_1 - \left(1+\tilde{\phi}\right) f(1-\theta)].$$

Optimal saving, s^u, the optimal timing of retirement, θ^u, and labor supply in the first period l_1^u are defined by the following first-order conditions:

$$\frac{\partial E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s} = -u_1\left(y_1\right) + RE\left[v_1\left(\tilde{y}_2\right)\right] = 0,$$

which requires

$$u_1\left(y_1\right) = RE\left[v_1\left(\tilde{y}_2\right)\right], \tag{7}$$

[8] It is worth noting that this assumption is not crucial for the results of the analysis. The same results are obtained in a model where saving is chosen in the first period, and the timing of retirement is chosen in the second period. But in the presence of uncertainty on the disutility from work, the decision on the timing of retirement must be taken before observing the realization of the random variable which describes the risk on the cost of work effort.

[9] On the empirical relevance of these push and pull factors see De Preter et al. (2013) and Scharn et al. (2018).

$$\frac{\partial E\left[V\left(s^{u},\theta^{u},l_{1}^{u}\right)\right]}{\partial\theta} = E\left[v_{1}\left(\tilde{y}_{2}\right)\left(f'\left(1-\theta^{u}\right)\left(1+\tilde{\phi}\right)-w(1-\tau+\beta\tau)\right)\right]=0, \tag{8}$$

and

$$\frac{\partial E\left[V\left(s^{u},\theta^{u},l_{1}^{u}\right)\right]}{\partial l_{1}} = u_{1}\left(y_{1}\right)\left[w(1-\tau)-f'\left(l_{1}^{u}\right)\right]+E\left[v_{1}\left(\tilde{y}_{2}\right)\right]\alpha\beta\tau w=0. \tag{9}$$

Following previous literature on similar problems, such as Dionne and Eeckhoudt (1984) and Brianti et al. (2018), we assume that SOCs for Problem (7) are satisfied in a neighborhood of s^u, θ^u and l_1^u. More details on this are available in the Appendix (Subsection A.2).

4.2. Saving, retirement and labor supply decisions

In this section, we investigate how a risk on the cost of work effort affects agent decisions and study the impact of precautionary motives on the timing of retirement, on the levels of saving and on first period labor supply. Unlike the case with certainty, in this setting it is possible to show that the choice variables in the maximization problem of the agent are interdependent.[10] As a consequence, we also characterize the interactions among these variables when workers have different risk preferences.

4.2.1. The precautionary motive for retirement, saving and labor supply

We compare the setting with uncertainty to the setting without uncertainty, to identify the conditions for precautionary saving, precautionary retirement and precautionary first period labor supply to arise. Following a long tradition in decision theory which starts from Pratt (1964), we focus on the case of small risks.

Consider initially the optimal timing of retirement.

Proposition 1. *In the presence of a small risk on the cost of work effort, a precautionary motive for retirement always arises, such that $\theta^u > \theta^c$ holds.*

Proof. See the Appendix (Subsection A.3). □

An increase in θ, i.e. an earlier retirement, has three different effects. The first effect is the reduction in the disutility due to uncertainty, which follows from shorter working activity and from a reduction in the exposure to work effort risk. The second effect is the decrease in disutility from work due to lower labor effort. The third effect is a reduction in labor and pension income due to lower labor supply in old age. The last two effects have the same magnitude, at $\theta = \theta^c$, implying that the first effect causes precautionary retirement, in accordance with the role played by retirement as an insurance device against work effort risk.

Consider now optimal saving.

Proposition 2. *In the presence of a small risk on the cost of work effort, a precautionary motive for saving arises and $s^u > s^c$ holds, if $v_{111} \geq 0$ and*

$$-\frac{v_{1111}\left(y_{2}\right)}{v_{111}\left(y_{2}\right)} \geq -\frac{2f'\left(1-\theta^{u}\right)}{f\left(1-\theta^{u}\right)\left(f'\left(1-\theta^{u}\right)\left(1+\phi\right)-w(1-\tau+\beta\tau)\right)}. \tag{10}$$

Proof. See the Appendix (Subsection A.4). □

The precautionary motive for saving can arise under the usual condition that the agent is prudent. This condition has been widely studied since the seminal papers by Leland (1968), Sandmo (1970) and Dréze and Modigliani (1972).

If the third derivative of the utility function is positive, the marginal utility of income is larger in the presence of uncertainty than in the presence of certainty. Following Chiu and Eeckhoudt (2010), we can interpret the condition $v_{111}(y) \geq 0$ as the "precautionary effect" or "apportionment effect" which captures agent preference to bear an (additive) risk when income is higher, or equivalently, to disaggregate the harm of risk and that of lower income.

But in the present setting, where the decision maker also chooses the optimal timing of retirement, prudence is not the only condition required for precautionary saving. Additional conditions must be satisfied: the agent must be temperate and the level of absolute temperance must be sufficiently high.[11]

The reason is that retirement, on one hand, provides insurance against work effort risk, and, on the other hand, decreases the level of income during old age. As a consequence, there is interaction between the precautionary motives for retirement and for saving, which depends on the interplay between risk coverage and the change in the marginal utility of second period income. In the next subsection we analyze this issue in more detail.

Focus lastly on optimal labor supply in the first period, and notice that precautionary motives play no role in this decision.

Proposition 3. *In the presence of a small risk on the cost of work effort, no precautionary motive for first period labor supply arises and $l_1^u = l_1^c = \bar{l}_1$ holds.*

Proof. By substituting Equation (7) into Equation (9), and by simplifying and reordering the terms, Equation (5) is obtained. □

4.2.2. Separation result

Uncertainty affects specific aspects of retirement decision, namely, the optimal timing of retirement and the optimal level of saving. Both these variables in fact, are used to deal with risk. Retirement provides insurance against work effort risk, and saving affects marginal utility of a risk-averse agent.

However, the decision on first period labor supply is not affected by uncertainty, and the same labor supply is chosen both in the setting with certainty and in the setting with uncertainty.

This shows that the optimizing behavior of the agent entails separating the decisions concerning the management of risk from the decision concerning income maximization. Saving and the timing of retirement, which respond to precautionary issues, serve the former purpose while first period labor supply serves the latter purpose.

Labor supply during young age in fact, depends only on the parameters of the pension system and on the real return on saving. These elements jointly define the marginal benefit of working in the first period and the relative benefit of present labor income with respect to future labor income. By adjusting labor supply during young age the agent maximizes thus total income, which provides more opportunities for intertemporal income allocation and reduces the marginal cost of early retirement.

The separation result presented above result can be further extended to the case studied by Dionne and Eeckhoudt (1984) which concerns the substitution between insurance and saving in a two-period model. These authors find that, in the presence of a fair insurance premium, a "separation" occurs between insurance and saving decisions. A risk-averse agent, in fact, buys full insurance to minimize the effects of uncertainty on utility, and relies on saving for the purpose of consumption smoothing.

A similar result is obtained in the present setting. When the marginal cost of increasing insurance against work effort risk by means of retirement equals the marginal decrease in expected loss, the agent chooses

[10] See Equations (19), (20) and (21) in the Appendix (Subsection A.2).

[11] It is worth noting that, in the case of widely adopted utility functions like CRRA or CARA, Inequality (10) is satisfied for $f'(1-\theta^u)$ sufficiently low.

to retire as early as possible. Since the “premium” paid is actuarially fair, full insurance is optimal.

Proposition 4. *When the condition*

$$w(1-\tau+\beta\tau)=(1+\phi)f'\left(1-\theta^u\right)$$

holds, the optimal timing of retirement entails $\theta^u=1$.

Proof. Rewrite Equation (8) as

$$\frac{\partial E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial\theta}=f'\left(1-\theta^u\right)cov\left[v_1\left(\tilde{y}_2\right),\tilde{\phi}\right]+$$
$$+E\left[v_1\left(\tilde{y}_2\right)\right]\left[(1+\phi)f'\left(1-\theta^u\right)-w(1-\tau+\beta\tau)\right].$$

If $w(1-\tau+\beta\tau)=(1+\phi)f'\left(1-\theta^u\right)$, previous equation becomes

$$\frac{\partial E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial\theta}=f'\left(1-\theta^u\right)cov\left[v_1\left(\tilde{y}_2\right),\tilde{\phi}\right]\geq 0,$$

so that $\theta^u=1$ is optimal for the agent. □

Under the previous hypothesis, retirement is used to insure against work effort risk, while saving is used for consumption smoothing. This completes the separation result, showing that each tool available to the agent has a specific purpose. First period labor supply realizes income maximization, savings realizes consumption smoothing, and retirement realizes risk management.

4.3. Risk preferences and retirement decision making

The joint use of retirement and saving to deal with risk and intertemporal wealth allocation makes it necessary to analyze their interactions in more depth. This allows us to shed light on how the precautionary motives for retirement and for saving affect each other. We thus compare different solutions for the maximization problem of the agent, and consider the relationship between the optimal timing of retirement and the optimal amount of saving.

Proposition 5. *In the presence of a small risk on the cost of work effort, early retirement fosters saving and* $\frac{\partial s^u}{\partial\theta^u}\geq 0$ *holds, if:*

- $v_{111}\geq 0$*, and Inequality (10) holds;*
- $v_{111}<0$*, and Inequality (10) holds in the reverse direction.*

Proof. See the Appendix (Subsection A.5). □

The conditions for early retirement to increase saving crucially depend on whether the agent is prudent or not. Consider first a prudent agent.

If $v_{111}\geq 0$ holds, a crowding in effect only occurs if the agent is temperate and absolute temperance is sufficiently high. Note that these conditions are the same as those required for precautionary saving to occur. If the agent is prudent then, $\frac{\partial s^u}{\partial\theta^u}\geq 0$ captures the interplay between the precautionary motives for retirement and for saving.

Consider how these variables interact.

An increase in θ^u, i.e. earlier retirement, causes second period income and the exposure to work effort risk to decrease. This has different effects on the consumption smoothing motive and on the precautionary motive for saving.

Early retirement increases the incentive to save for consumption smoothing reasons, because it causes second period income to decrease. The effects of early retirement on precautionary saving, however, are ambiguous.

On the one hand, the precautionary motive for saving is weakened because risk exposure decreases. On the other hand, earlier retirement causes v_{111} to increase, because second period income decreases and the agent is temperate, as required for Inequality (10) to hold. This implies that the impact of a lower second period income on the disutility caused by uncertainty is now larger. As a consequence, the precautionary motive for saving is strengthened.

Adopting a method analogous to that used in Magnani (2017), we can clarify this point by rewriting Inequality (10) as follows:

$$\frac{v_{1111}\left(y_2\right)}{v_{111}\left(y_2\right)}\cdot\theta^u\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)$$
$$\geq\frac{2\theta^u}{f\left(1-\theta^u\right)}. \tag{11}$$

The left hand side of the above inequality is the elasticity of prudence, with respect to θ, i.e. with respect to a decrease in the retirement age, or, in analytical terms, $\frac{\partial v_{111}(y_2)}{\partial\theta^u}\cdot\frac{\theta^u}{v_{111}(y_2)}$. This quantity measures responsiveness to early retirement of the decrease in the disutility caused by risk, which follows from an increase in the level of saving (or, in general, from an increase in second period income). In other words, it is the responsiveness of the precautionary motive for saving to the reduction in second period income due to early retirement.

The right hand side of the inequality, on the other hand, is the absolute value of the elasticity of the variance of second period income with respect to a decrease in the retirement age. Note in fact that a decrease in retirement age causes a reduction in the variance of second period income by lowering the exposure to work effort risk. In analytical terms it is $\left|\frac{\partial var\left[\tilde{y}_2\right]}{\partial\theta^u}\right|\cdot\frac{\theta^u}{var\left[\tilde{y}_2\right]}$, and measures the responsiveness of the precautionary motive for saving to the increase in the insurance against work effort risk provided by early retirement.

So, if the degree of absolute temperance is sufficiently high and Inequality (10) holds, the elasticity of prudence with respect to early retirement is larger than the elasticity of the variance of second period income with respect to early retirement. This implies that the positive effect on the precautionary motive for saving of a decrease in income in old age is larger than the negative effect due to lower exposure to work effort risk. As a consequence, the precautionary motive for saving is strengthened by early retirement.

These results are in line with the findings by Magnani (2020), who shows that precautionary retirement and precautionary saving can coexist if the agent is prudent, and that a crowding out effect between precautionary saving and precautionary retirement more easily occurs if the agent is intemperate. Further analogies are found with the findings of other authors who consider the interaction between labor supply and precautionary saving.

In particular, Flodén (2006) finds that, in the presence of risk in the form of uncertainty on future wages, labor supply flexibility increases the precautionary saving of a prudent agent. Similarly in our setting, where retirement allows for this flexibility, a crowding in effect exists between precautionary saving and precautionary retirement.

More recently Nocetti and Smith (2011), in a model with endogenous labor supply, find that, in the presence of labor or non-labor income risk, under plausible assumptions, a precautionary motive for saving arises, current supply of labor increases and expected future supply of labor decreases. Since in the present setting, early retirement is in fact a reduction in future labor supply, our results are line with those obtained by these authors.[12]

Consider now the case where the agent is imprudent. If $v_{111}<0$, no precautionary motive for saving emerges, and the direction of Inequality (10) must be reversed for the crowding in effect to occur. This

[12] This analogy is particularly relevant, because Nocetti and Smith (2011) adopt a richer preference structure than the structure adopted here, which makes it possible to disentangle risk preferences from preferences toward intertemporal substitution. This is evidence that our results are not driven by specific assumptions on the utility function.

condition is looser than that obtained under the assumption that the agent is prudent. Note in fact that

$$-\frac{v_{1111}\left(y_2\right)}{v_{111}\left(y_2\right)}<-\frac{2f'\left(1-\theta^u\right)}{f\left(1-\theta^u\right)\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)}$$

always holds if the agent is temperate and $v_{1111} \le 0$. However, if the agent is intemperate and $v_{1111} > 0$, a sufficiently low level of absolute temperance is required.

The fact that an imprudent agent wants to decrease income in the period where uncertainty is present drives this result. Early retirement in fact becomes increasingly beneficial as saving increases, because a higher second period income increases the marginal disutility caused by work effort risk. This in turn, encourages the purchase of insurance in the form of early retirement.

The incentive to bring forward retirement is especially strong if the agent is also temperate. As second period income increases due to higher saving, the agent in fact becomes more imprudent, and the push to escape work effort risk through retirement becomes stronger.

On the other hand, if the agent is intemperate, the increase in second period income reduces imprudence and the marginal benefit of insurance. A sufficiently low level of absolute temperance is thus required for the crowding in effect to emerge.

Note lastly that a crowding out effect between early retirement and saving can only occur when the sufficient conditions stated in Proposition 5 are not satisfied.

Corollary 6. *In the presence of a small risk on the cost of work effort, early retirement reduces the incentive to save, and $\frac{\partial s^u}{\partial \theta^u} < 0$ holds, only if:*

- $v_{111} \ge 0$*, and Inequality (10) holds in the reverse direction;*
- $v_{111} < 0$*, and Inequality (10) holds.*

Proof. Straightforwardly follows from Proposition 5. □

5. An analysis of different pension system reforms

In this section we investigate the effects on lifetime labor supply and on the financial soundness of the pension system of two different reforms: an increase in the pension contribution rate and a cut in pension payments. These reforms have been proposed to improve the financial performances of pension schemes in many countries faced with an aging population. However, their benefits crucially depend on how people respond to them, because expenditures and revenues of the pension system directly depend on worker choices concerning labor supply and the optimal timing of retirement.

5.1. Effects of an increase in the pension contribution rate

Consider initially the effects of an increase in the pension contribution rate on retirement, i.e. on second period labor supply.

Proposition 7. *When the contribution rate τ increases, the optimal timing of retirement does not change and $\frac{\delta\theta^u}{\delta\tau} = 0$, if $\alpha = R$ and $\beta = 1$.*

The agent postpones retirement and $\frac{\delta\theta^u}{\delta\tau} < 0$ holds, if:

- $\frac{\partial s^u}{\partial \theta^u} < 0$, $\beta \ge 1$, *and* $R\cdot\bar{l}_1+1-\theta^u < \alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$;
- $\frac{\partial s^u}{\partial \theta^u} > 0$, $\beta \ge 1$, *and* $R\cdot\bar{l}_1+1-\theta^u > \alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

The agent brings forward retirement and $\frac{\delta\theta^u}{\delta\tau} > 0$, if:

- $\frac{\partial s^u}{\partial \theta^u} < 0$, $\beta \le 1$, *and* $R\cdot\bar{l}_1+1-\theta^u > \alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$;
- $\frac{\partial s^u}{\partial \theta^u} > 0$, $\beta \le 1$ *and* $R\cdot\bar{l}_1+1-\theta^u < \alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

Proof. See the Appendix (Subsection A.6). □

The choice of the agent to either postpone or bring forward retirement crucially depends on the marginal effect on second period income of an increase in contribution. Note in fact that the increase in τ causes pension contribution to marginally increase by $\bar{l}_1+1-\theta^u$, implying that pension payments and second period income also marginally increase by $\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

This increase in pension payments comes at a cost, since the additional pension contributions due to a higher τ cannot be invested in financial markets. As a consequence, second period income decreases at the margin by $R\cdot\bar{l}_1+1-\theta^u$.

In this context, an increase in the contribution rate has no effect, if the pension system is actuarially fair. This happens if the real return on pension contributions in the first period, α, is equal to the real return on saving, R, and if the real return on pension contributions in the second period, β, is 1. These benchmark values can be interpreted as the real return on a riskless asset and the real return on liquidity respectively.

Under these circumstances $\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta = R\cdot\bar{l}_1+1-\theta^u$ holds, and second period income does not vary in response to an increase in the contribution rate. As a consequence, no adjustment is required in second period labor supply, and the retirement decision is unaffected.

However, if $\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta > R\cdot\bar{l}_1+1-\theta^u$, the pension system pays a higher real rate of return on pension contributions than the real rate of return paid by financial markets. Hence, the increase in the contribution rate causes second period income to increase, implying that the increase in the contribution rate has the same effects as an increase in saving.

In this case, if agent preferences are such that $\frac{\partial s^u}{\partial \theta^u} < 0$ or, equivalently, $\frac{\partial \theta^u}{\partial s^u} < 0$, and saving fosters late retirement, the second period labor supply increases when postponing retirement is not disincentivized. This requires the real rate of return on pension contributions in the second period to be actuarially fair or advantageous ($\beta \ge 1$). If this is the case, the agent delays retirement.

On the other hand, when $\frac{\partial s^u}{\partial \theta^u} > 0$ or, equivalently, $\frac{\partial \theta^u}{\partial s^u} > 0$ hold, and saving fosters early retirement, second period labor supply decreases when postponing retirement is disincentivized and $\beta \le 1$. The agent thus brings forward retirement.

Consider now the circumstance where $\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta < R\cdot\bar{l}_1+1-\theta^u$ holds, and the pension system pays a lower real rate of return on pension contributions than the real rate of return paid by financial markets. In this case, the increase in the contribution rate causes second period income to decrease, and has the same effects as a decrease in saving.

As a consequence, if agent preferences are such that saving fosters late retirement ($\frac{\partial \theta^u}{\partial s^u} < 0$), when postponing retirement is disincentivized ($\beta \le 1$), the agent brings forward retirement and second period labor supply decreases. Symmetrically, if agent preferences are such that saving fosters early retirement ($\frac{\partial \theta^u}{\partial s^u} > 0$), and postponing retirement is not disincentivized ($\beta \ge 1$), second period labor supply increases and the agent delays retirement.

Focus now on first period labor supply.

Proposition 8. *When the contribution rate τ increases, first period labor supply does not change, and $\frac{\delta\bar{l}_1}{\delta\tau} = 0$ holds, if $R = \alpha\cdot\beta$.*

The agent increases first period labor supply, and $\frac{\delta\bar{l}_1}{\delta\tau} > 0$ holds, if $R < \alpha\cdot\beta$.

The agent decreases first period labor supply, and $\frac{\delta\bar{l}_1}{\delta\tau} < 0$ holds, if $R > \alpha\cdot\beta$.

Proof. See the Appendix (Subsection A.7). □

In this context, if $\alpha = R$ and $\beta = 1$ hold, the pension system is actuarially neutral, and the present value of accrued pension payments generated by increasing labor supply either in the first period or in the second period is the same. An increase in the contribution rate has the same effect as an increase in saving, and does not change the relative benefits of working in the first period or in the second period. As a consequence, the first period labor supply does not change either. For the same reason this outcome is also obtained under the more general condition that $\alpha \cdot \beta = R$.

A sufficient condition for first period labor supply to increase in response to an increase in τ, is for the real return on pension contributions paid in the first period to exceed the real return on saving ($\alpha \cdot \beta > R$). When this is the case, the pension system is more efficient than financial markets at the intertemporal transfer of income.

A higher contribution rate thus causes an increase in the marginal benefit of working in the first period, and labor supply increases. Symmetrically, when the real return on pension contributions paid in the first period is lower than the real return on saving ($\alpha \cdot \beta < R$), first period labor supply decreases.

Lastly, we analyze lifetime labor supply.

Corollary 9. *When the contribution rate τ increases there are no effects on labor supply in the first period and in the second period if $\beta = 1$ and $\alpha = R$; Labor supply increases both in the first period and in the second period if $\frac{\partial s^u}{\partial \theta^u} < 0$, $\alpha \cdot \beta > R$ and $\beta \geq 1$.*

Labor supply increases in the first period and decreases in the second period if

- $\frac{\partial s^u}{\partial \theta^u} < 0$, $\alpha \cdot \beta > R$, $\beta \leq 1$, *and* $R \cdot \bar{l}_1 + 1 - \theta^u > \alpha \cdot \beta \cdot \bar{l}_1 + (1 - \theta^u)\beta$;
- $\frac{\partial s^u}{\partial \theta^u} > 0$, $\alpha \cdot \beta > R$, $\beta \leq 1$ *and* $R \cdot \bar{l}_1 + 1 - \theta^u < \alpha \cdot \beta \cdot \bar{l}_1 + (1 - \theta^u)\beta$.

Labor supply decreases in the first period and increases in the second period if

- $\frac{\partial s^u}{\partial \theta^u} < 0$, $\alpha \cdot \beta < R$, $\beta \geq 1$ *and* $R \cdot \bar{l}_1 + 1 - \theta^u < \alpha \cdot \beta \cdot \bar{l}_1 + (1 - \theta^u)\beta$;
- $\frac{\partial s^u}{\partial \theta^u} > 0$, $\alpha \cdot \beta < R$, $\beta \geq 1$ *and* $R \cdot \bar{l}_1 + 1 - \theta^u > \alpha \cdot \beta \cdot \bar{l}_1 + (1 - \theta^u)\beta$.

Labor supply decreases both in the first period and in the second period if $\frac{\partial s^u}{\partial \theta^u} < 0$, $\alpha \cdot \beta < R$, $\beta \leq 1$.

Proof. Straightforwardly follows from Proposition 7 and Proposition 8. □

We now discuss the implications of these results for the pension system, and focus on the case where an increase in the contribution rate obtains a sure improvement in financial soundness. This requires two specific circumstances to occur.

The first circumstance is that the agent increases first period labor supply and postpones retirement so that overall contributions paid to the system increase, jointly with lifetime labor supply.

The sufficient conditions for this to happen entail that agent preferences are such that saving encourages late retirement, and that $\alpha \cdot \beta > R$ and $\beta \geq 1$ hold. The former inequality implies that the real return paid to workers on first period contributions exceeds the real return on a riskless asset. The latter inequality implies that the real return paid to workers on second period contributions exceeds the real return on liquidity.

The second circumstance which permits the financial soundness of the pension system to improve is that the real return obtained on pension contributions exceeds the real return paid back to workers, so that, after the reform, additional financial resources are generated by the increase in overall contributions.[13]

5.2. Effects of a cut in pension payments

Consider now the effects of a reform that reduces the generosity of the pension system. A cut in pension payments can be implemented by decreasing either the real return on contributions in the first period, α, or the real return on contributions in the second period, β.

We first analyze the case of a reduction in α, and study its effects on the timing of retirement.

Proposition 10. *When the real return on pension contributions in the first period, α, decreases:*

- *the agent postpones retirement, and $\frac{\delta \theta^u}{\delta \alpha} \geq 0$ holds, if $\frac{\partial s^u}{\partial \theta^u} \geq 0$;*
- *the agent brings forward retirement, and $\frac{\delta \theta^u}{\delta \alpha} < 0$ holds, if $\frac{\partial s^u}{\partial \theta^u} < 0$.*

Proof. See the Appendix (Subsection A.8). □

Note now that, differently from the case of an increase in τ, a reduction in α always triggers a decrease in first period labor supply.

Proposition 11. *When the real return on pension contributions in the first period, α, decreases, the agent decreases first period labor supply, and $\frac{\partial \bar{l}_1}{\partial \alpha} \geq 0$ holds.*

Proof. Analogous to the proof of Proposition 8. □

The overall effects on labor supply are summarized below.

Corollary 12. *When the real return on pension contributions in the first period, α, decreases:*

- *labor supply decreases in the first and in the second period, if $\frac{\partial s^u}{\partial \theta^u} > 0$;*
- *labor supply decreases in the first period and increases in the second period, if $\frac{\partial s^u}{\partial \theta^u} \leq 0$.*

Proof. Straightforwardly follows from Propositions 10 and 11. □

The main effect of a reduction in the real return paid to workers on pension contributions in the first period is a reduction in the marginal benefits of working in the first period, which triggers a decrease in labor supply and in labor income. As a consequence, saving also decreases for consumption smoothing reasons.

The effects of lower savings on retirement depend on agent preferences.

If $\frac{\partial \theta^u}{\partial s^u} \geq 0$ or, equivalently, $\frac{\partial s^u}{\partial \theta^u} \geq 0$ holds, saving fosters early retirement, and the agent postpones retirement. Second period labor supply thus increases.

If $\frac{\partial \theta^u}{\partial s^u} < 0$ or, equivalently, $\frac{\partial s^u}{\partial \theta^u} < 0$ holds, saving discourages early retirement, and the agent brings forward retirement. In this case, both second period labor supply and first period labor supply decrease in response to the reform.

The above results show that no clear cut conclusions on the effects of a reduction in α on the financial soundness of the pension system can be drawn. The reform in fact, causes a sure reduction in first period labor supply, and ambiguously impacts the timing of retirement and second period labor supply.

The net variation in total contributions paid by workers is uncertain, except when saving discourages early retirement. In this case, a negative variation in pension contributions occurs with certainty.

[13] The real returns on pension contributions, obtained or paid by the pension system, in the first period and in the second period, can be different from the real return on saving, R, and from the real return on liquidity, 1, respectively. For instance, in a pay-as-you-go system, these rates of return depend on demographic trends and are not correlated to the real interest rate prevailing on financial markets. In a fully funded system, investments can be made in risky assets, which yield higher returns than riskless assets.

On the one hand, the cut in pension payments thus produces savings, but, on the other hand, it may result in a decrease in total revenues of the system, due to a lower lifetime labor supply.

A different reform which brings about a cut in pension payments is a reduction in the real return on pension contribution in the second period, i.e. a reduction in β. Its effects on the timing of retirement and on second period labor supply are summarized below.

Proposition 13. *When the real return on pension contributions in the second period, β, decreases, the agent brings forward retirement and $\frac{\partial \theta^u}{\partial \beta} \leq 0$ holds, if $\frac{\partial s^u}{\partial \theta^u} \leq 0$.*

Proof. See the Appendix (Subsection A.9). □

Another significant result follows from this proposition.

Corollary 14. *A necessary condition for $\frac{\partial \theta^u}{\partial \beta} > 0$ to hold, and for the retirement age to increase when the real return on pension contributions in the second period, β, decreases, is $\frac{\partial s^u}{\partial \theta^u} > 0$.*

Proof. Straightforwardly follows from Proposition 13. □

Consider now first period labor supply.

Proposition 15. *When the real return on pension contributions in the second period, β, decreases, the agent decreases first period labor supply, and $\frac{\partial \bar{l}_1}{\partial \beta} \geq 0$ holds.*

Proof. Analogous to the proof of Proposition 8. □

The effects of this reform on labor supply are analogous to the effects of decrease in the real return on pension contributions in the first period α, and are easily understood. Since a reduction in β reduces the benefits from working in the first period, labor supply during young age decreases. Moreover, the incentives to postpone retirement and to work longer decline together with the price of insuring against work effort risk. As a consequence, optimal retirement age, second period labor supply, and lifetime labor supply all decrease.

In this context, the only circumstance where a reduction in the real return on pension contributions in the second period can trigger an increase in the retirement age is when agent preferences are such that saving fosters early retirement. This condition also implies that a decrease in second period income reduces the marginal benefit of early retirement. As a consequence, a lower β, causing second period income to decrease, may push the agent to postpone retirement.

As in the case of a reduction in α, a reduction in the real return on pension contributions in the second period does not supply sure benefits to the pension system either. Indeed, lower pension payments result in a reduction in expenditures. However, a decrease in lifetime labor supply is likely to occur, which causes pension system revenues to shrink, and the net impact on financial soundness to become ambiguous.

6. Final remarks

The present analysis considers the effects of uncertainty, in the form of a random shock which affects the disutility from work effort, on the retirement decision of a risk-averse agent. In this setting, the retirement decision includes not only the definition of the optimal timing of retirement, but also the choices on saving and labor supply during working age, which are relevant for a sensible planning of the consumption path along the life cycle.

Adopting a simple two period model which includes a stylized pension system, we show that a risk on the cost of work effort affects the timing of retirement and the level of saving, but does not affect labor supply in the first period.

In this context, a precautionary motive for retirement always arises, and provides an incentive to bring forward retirement in order to insure against risk. However, specific conditions need to be satisfied for the agent to engage in precautionary saving: he/she must be prudent and have a sufficiently high level of absolute temperance.

The irrelevance of precautionary motives in the optimal choice of labor supply in the first period highlights that a separation exists between this decision, and the decisions on the timing of retirement and on the level of saving. The agent in fact, adjusts first period labor supply for the purpose of maximizing income, and allocates retirement and saving to the purposes of risk management and consumption smoothing.

A more precise distinction between these variables emerges when the price paid to insure against work effort risk by means of retirement is fair. Under this hypothesis, first period labor supply realizes income maximization, saving realizes consumption smoothing, and retirement realizes optimal risk management, so that each tool is used by the agent for a specific aim.

In the absence of a fair insurance price however, retirement and saving interact. When the agent engages in precautionary retirement and precautionary saving, a crowding in effect may emerge, such that early retirement increases the level of saving, if specific conditions on agent preferences are satisfied.

Interactions between these variables are very important because they drive the outcomes of specific reforms of the pension system: an increase in pension contribution rate and a cut in pension payments. We study their effects on worker labor supply and retirement decisions, and discuss the impact on revenues and expenditures of the pension system. This yields insight on the potential benefits for the financial soundness of pension schemes.

The case of an increase in the pension contribution rate supplies two important results.

The first result concerns the ineffectiveness of this reform when the pension system is both actuarially fair and actuarially neutral. In this case, worker optimal choices on first period labor supply and on the timing of retirement do not vary, and no change occurs in revenues and expenditures of the pension system either.

The second result concerns the conditions for the reform to improve the financial soundness of the system. This outcome requires labor supply and total pension contributions to jointly increase, and only occurs if workers preferences are such that saving discourages early retirement, and if the real return on pension contributions paid to workers exceeds the real return of a portfolio including a riskless asset and liquidity. Lastly, it must be the case that the pension system obtains a real return on worker contributions higher than the real return paid to workers.

The second reform being analyzed is a cut in pension payments. In this case, it is not possible to characterize the conditions when a sure improvement in the financial soundness of the pension system occurs. This reform in fact has no clear cut effects on first period labor supply or on second period labor supply, implying that the net variation in total contributions paid to the system is ambiguous. As a consequence, savings are generated with certainty, but the revenues of the system can either decrease or increase.

Since we adopt a simplified setting, our results are subject to several limitations, concerning, for instance, liquidity constraints which may result in inefficiently low saving, and the adoption of a stylized model of the pension system.

Most importantly though, we do not account for two main issues.

The first issue is the fact that the length of worker life is uncertain. This source of risk is not considered, but an extension of the model in this direction would be a fruitful avenue for future research.

The second issue concerns our simplified description of financial markets where only a riskless asset exists. Including assets with safe and risky returns in the present model would be an interesting extension which would make it possible to jointly study decisions concerning the optimal portfolio problem and the timing of retirement.[14] This would also make it possible to extend the analysis to the study of the effects on retirement of regime switches of the type that occurred in financial markets after the year 2008: the sub-prime crisis, the sovereign debt crisis, the COVID-19 pandemics and recently the Russian invasion of Ukraine.[15]

CRediT authorship contribution statement

Marco Magnani: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Resources, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The author did not receive support from any organization for the submitted work. The author has no relevant financial or non-financial interests to disclose.

Data availability

No data was used for the research described in the article.

Appendix A

A.1. Second order conditions for the maximization problem in the baseline model

SOCs require the Hessian of maximization Problem (1) to be a semidefinite negative matrix when $s = s^c$, $\theta = \theta^c$ and $l = l_1^c$ hold. Focus now on the elements of this matrix:

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 s} = u_{11}\left(y_1\right) + R^2 v_{11}(y_2) < 0, \tag{12}$$

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 \theta} = v_{11}(y_2)\left[(1+\phi)\cdot f'\left(1-\theta^c\right) - w\left(1-\tau+\beta\tau\right)\right]^2 + -v_1(y_2)(1+\phi)\cdot f''\left(1-\theta^c\right),$$

which, by Equation (3) simplifies to

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 \theta} = -v_1(y_2)(1+\phi) f''\left(1-\theta^c\right) < 0, \tag{13}$$

and

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 l_1} = u_{11}\left(y_1\right)\left[w(1-\tau) - f'\left(l_1^c\right)\right]^2 - u_1\left(y_1\right) f''\left(l_1^c\right) + +v_{11}\left(y_2\right)(\alpha\beta\tau w)^2 < 0. \tag{14}$$

Use now Equation (2) to rewrite Equation (4) as

$$\frac{\partial V\left(s^c,\theta^c,l_1^c\right)}{\partial l_1} = v_1\left(y_2\right)\left\{R\left[w(1-\tau) - f'\left(l_1^c\right)\right] + \alpha\beta\tau w\right\} = 0 \tag{15}$$

and take the first derivative of previous equation with respect to s to obtain

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial l_1 \partial s} = R v_{11}\left(y_2\right)\left\{R\left[w(1-\tau) - f'\left(l_1^c\right)\right] + \alpha\beta\tau w\right\} = 0,$$

by Equation (15).

Note further that, by Equation (3),

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial s \partial \theta} = v_{11}(y_2) R\left[(1+\phi) f'\left(1-\theta^c\right) - w\left(1-\tau+\beta\tau\right)\right] = 0,$$

and

$$\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial l_1 \partial \theta} = v_{11}(y_2)\alpha\beta\tau w\left[(1+\phi) f'\left(1-\theta^c\right) - w\left(1-\tau+\beta\tau\right)\right] = 0,$$

hold.

As a consequence the Hessian is a diagonal matrix

$$H = \begin{pmatrix} \frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 s} & 0 & 0 \\ 0 & \frac{\partial^2 V\left(s^u,\theta^u,l_1^u\right)}{\partial^2 \theta} & 0 \\ 0 & 0 & \frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 l_1} \end{pmatrix},$$

which is easily proven to be semidefinite negative. Hence SOCs are satisfied.

A.2. Second order conditions for the maximization problem in the model where a random shock on the disutility from work is present

SOCs require the Hessian of maximization Problem (7), defined as

$$H = \begin{pmatrix} \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s \partial \theta} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s \partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial \theta \partial s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 \theta} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial \theta \partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1 \partial s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1 \partial \theta} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} \end{pmatrix}$$

to be a semidefinite negative matrix. This condition depends on the following equations:

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} = u_{11}\left(y_1\right) + R^2 E\left[v_{11}\left(\tilde{y}_2\right)\right] \le 0, \tag{16}$$

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 \theta} = E\left[v_{11}\left(\tilde{y}_2\right)\left(f'\left(1-\theta^u\right)\left(1+\tilde{\phi}\right) - w(1-\tau+\beta\tau)\right)^2\right] + -E\left[v_1\left(\tilde{y}_2\right)\left(1+\tilde{\phi}\right) f''\left(1-\theta^u\right)\right] \le 0, \tag{17}$$

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} = u_{11}\left(y_1\right)\left[w(1-\tau) - f'\left(\bar{l}_1\right)\right]^2 - u_1\left(y_1\right) f''\left(\bar{l}_1\right) + +E\left[v_{11}\left(\tilde{y}_2\right)\right](\alpha\beta\tau w)^2,$$

which, by substituting Equation (5) and Equation (16), can be rewritten as

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} = \left(\frac{w\alpha\beta\tau}{R}\right)^2 \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} - f''\left(\bar{l}_1\right) u_1\left(y_1\right) \le 0, \tag{18}$$

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s \partial \theta} = RE\left[v_{11}\left(\tilde{y}_2\right)\left(f'\left(1-\theta^u\right)\left(1+\tilde{\phi}\right) - w(1-\tau+\beta\tau)\right)\right], \tag{19}$$

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s \partial l_1} = -u_{11}\left(y_1\right)\left[w(1-\tau) - f'\left(\bar{l}_1\right)\right] + + E\left[v_{11}\left(\tilde{y}_2\right)\right] R\alpha\beta\tau w,$$

which, by substituting Equation (5) and Equation (16), can be rewritten as

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[14] The optimal portfolio problem in a dynamic model which includes a defined contribution pension system, is studied by Temocin et al. (2018).

[15] In this regard, the paper by Savku and Weber (2022) provides a useful setting for the analysis of the effects on investors behavior of regime switches.

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s \partial l_1}=\frac{w\alpha\beta\tau}{R}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\leq 0, \tag{20}$$

and

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}$$
$$=w\alpha\beta\tau E\left[v_{11}\left(\tilde{y}_2\right)\left(f'\left(1-\theta^u\right)\left(1+\tilde{\phi}\right)-w(1-\tau+\beta\tau)\right)\right],$$

which, by substituting Equation (19), can be rewritten as

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}=\frac{w\alpha\beta\tau}{R}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}. \tag{21}$$

Consider now the determinant of the Hessian, $\det(H)$:

$$\det(H)=\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\left[\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1}+-\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}\right)^2\right]-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\cdot$$
$$\cdot\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1}-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s}\right)+\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1}\cdot$$
$$\cdot\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\theta}-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s}\right).$$

Substitute now Equations (18), (21), and (20) to obtain:

$$\det(H)=\left(\frac{w\alpha\beta\tau}{R}\right)^2\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}+$$
$$-u_1\left(y_1\right)f''\left(\bar{l}_1\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}+$$
$$-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\left(\frac{w\alpha\beta\tau}{R}\right)^2\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s\partial\theta}\right)^2+$$
$$+u_1\left(y_1\right)f''\left(\bar{l}_1\right)\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\right)^2+$$
$$+\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s\partial\theta}\right)^2\left(\frac{w\alpha\beta\tau}{R}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial^2 s}+$$
$$-\left(\frac{w\alpha\beta\tau}{R}\right)^2\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial^2 s}\right)^2\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}\right)\right]}{\partial^2\theta},$$

or, simplifying and reordering the terms,

$$\det(H)=-u_1\left(y_1\right)f''\left(\bar{l}_1\right)\cdot\left[\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}+-\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\right)^2\right]. \tag{22}$$

In order for the SOCs of the maximization problem to be satisfied $\frac{\partial^2 V\left(s^c,\theta^c,l_1^c\right)}{\partial^2 s}\leq 0$ must hold, and this is the case by Equation (16). Moreover the conditions $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}-\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\right)^2\geq 0$ and $\det(H)\leq 0$ must be fulfilled. By Equation (22), both these inequalities hold if the first inequality is satisfied. Following previous literature on similar problems, such as Dionne and Eeckhoudt (1984) and Brianti et al. (2018), we thus assume that $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}-\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\right)^2\geq 0$ holds in a neighborhood of s^u, θ^u and $\bar{l}_1$, so that SOCs are satisfied.

A.3. Proof of Proposition 1

Consider a second-order Taylor expansion of Equation (8) in point ϕ:

$$\frac{\partial E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta} \tag{23}$$
$$\simeq v_1\left(y_2\right)\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)+$$
$$+\frac{1}{2}\cdot f\left(1-\theta^u\right)var\left[\tilde{\phi}\right]\cdot\left(v_{111}\left(y_2\right)f\left(1-\theta^u\right)\cdot\right.$$
$$\cdot\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)+$$
$$\left.-2v_{11}\left(y_2\right)f'\left(1-\theta^u\right)\right)=0.$$

Evaluate previous equation in $\theta=\theta^c$, and note that, by Equation (3),

$$\frac{\partial E\left[V\left(s^u,\theta^c,\bar{l}_1\right)\right]}{\partial\theta}\simeq -f'\left(1-\theta^c\right)\cdot$$
$$\cdot f\left(1-\theta^c\right)var\left[\tilde{\phi}\right]v_{11}\left(y_2\right)\geq 0,$$

implying further that $\theta^u>\theta^c$ holds, since it is $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2\theta}\leq 0$.

A.4. Proof of Proposition 2

Focus initially on the conditions for $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\geq 0$ to hold, and consider a second-order Taylor expansion of Equation (19) around point ϕ

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \tag{24}$$
$$\simeq Rv_{11}\left(y_2\right)\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)+$$
$$+R\cdot\frac{1}{2}\cdot f\left(1-\theta^u\right)var\left[\tilde{\phi}\right]\left(v_{1111}\left(y_2\right)f\left(1-\theta^u\right)\cdot\right.$$
$$\cdot\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)+$$
$$\left.-2\cdot v_{111}\left(y_2\right)f'\left(1-\theta^u\right)\right).$$

Proposition 1 shows that $\theta^u>\theta^c$ holds, implying that

$$f'\left(1-\theta^u\right)(1+\phi)<w(1-\tau+\beta\tau),$$

by Equation (3), and the first term in Equation (24) is positive. Assume now that the agent is prudent so that $v_{111}\geq 0$. For the second term in Equation (24) and $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}$ to be positive it must be the case that

$$v_{1111}f\left(y_2\right)\left(1-\theta^u\right)\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)$$
$$\geq 2v_{111}\left(y_2\right)f'\left(1-\theta^u\right), \tag{25}$$

because $R\cdot\frac{1}{2}\cdot f\left(1-\theta^u\right)var\left[\tilde{\phi}\right]$ is positive. Reordering the terms of previous inequality gives

$$-\frac{v_{1111}\left(y_2\right)}{v_{111}\left(y_2\right)}\geq -\frac{2f'\left(1-\theta^u\right)}{f\left(1-\theta^u\right)\left(f'\left(1-\theta^u\right)(1+\phi)-w(1-\tau+\beta\tau)\right)}.$$

Note now that if $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}\right)\right]}{\partial s\partial\theta}\geq 0$, a sufficient condition for $s^u>s^c$ to hold, is $\frac{\partial E\left[V\left(s^c,\theta^c,\bar{l}_1\right)\right]}{\partial s}\geq 0$, because $\theta^u>\theta^c$ and also $\frac{\partial E\left[V\left(s^c,\theta^u,\bar{l}_1\right)\right]}{\partial s}\geq$

$\frac{\partial E\left[V\left(s^c,\theta^c,\bar{l}_1\right)\right]}{\partial s}$.

The condition $\frac{\partial E\left[V\left(s^c,\theta^c,\bar{l}_1\right)\right]}{\partial s} \geq 0$ requires

$$-u_1\left(y_1\left(s^c,\theta^c,\bar{l}_1\right)\right) + RE\left[v_1\left(\tilde{y}_2\left(s^c,\theta^c,\bar{l}_1\right)\right)\right] \geq 0,$$

or, by Equation (2),

$$-Rv_1\left(y_2\left(s^c,\theta^c,\bar{l}_1\right)\right) + RE\left[v_1\left(\tilde{y}_2\left(s^c,\theta^c,\bar{l}_1\right)\right)\right] \geq 0.$$

This inequality holds if and only if $v_{111} \geq 0$. So in order to have $s^u > s^c$, it must be the case that $v_{111} \geq 0$ and Inequality (10) hold.

A.5. Proof of Proposition 5

By the Implicit Function Theorem we have that

$$\frac{\partial s^u}{\partial \theta^u} = -\frac{\det\begin{pmatrix} \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\theta} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} \end{pmatrix}}{\det\begin{pmatrix} \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} \end{pmatrix}} =$$

$$-\frac{\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\theta}}{\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} - \left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1}\right)^2}.$$

Note that by SOCs, the denominator of the above fraction is positive, implying that $\frac{\partial s^u}{\partial \theta^u} \geq 0$ requires

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\theta} +$$

$$-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} \geq 0,$$

or, substituting Equations (18), (20) and (21),

$$\left(\frac{w\alpha\beta\tau}{R}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} +$$

$$-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\left(\frac{w\alpha\beta\tau}{R}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} +$$

$$+f''\left(\bar{l}_1\right)u_1\left(y_1\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}$$

$$= f''\left(\bar{l}_1\right)u_1\left(y_1\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \geq 0.$$

For the inequality above to be satisfied, it is required that $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \geq 0$, a condition which holds under different circumstances when the agent is prudent and when the agent is imprudent.

Following the same steps as in the proof of Proposition 2, it is clear that if the agent is prudent and $v_{111} \geq 0$, Inequality (10) is sufficient to have $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \geq 0$.

Analyze now the case where $v_{111} < 0$ and the agent is imprudent. In this case, following the same steps as in the proof of Proposition 2, it is possible to show that, for Inequality (25) to be satisfied, and for $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \geq 0$ to hold, it is required that Inequality (10) holds in the reverse direction.

A.6. Proof of Proposition 7

By the implicit function theorem, we have that

$$\frac{\partial \theta^u}{\partial \tau} = -\frac{\det\begin{pmatrix} \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1} \\ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\tau} & \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} \end{pmatrix}}{\det\left(H\right)}.$$

Since $\det\left(H\right) \leq 0$ holds by SOCs, the sign of $\frac{\partial\theta^u}{\partial\tau}$ is the same as the sign of the determinant of the matrix at the numerator of the fraction in the right hand side of previous equation, which we denote by H^τ.

Consider then $\det\left(H^\tau\right)$:

$$\det\left(H^\tau\right) = \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot$$

$$\cdot\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}\cdot\right.$$

$$\left.\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\tau}\right) - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot$$

$$\cdot\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 l_1} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial l_1}\cdot\right.$$

$$\left.\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s}\right) + \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial l_1}\cdot$$

$$\cdot\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\tau} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau}\cdot\right.$$

$$\left.\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial s}\right).$$

Substitute Equations (18), (21), and (20) to obtain:

$$\det\left(H^\tau\right) = \left(\frac{w\alpha\beta\tau}{R}\right)^2\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau} +$$

$$-u_1\left(y_1\right)f''\left(\bar{l}_1\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau} +$$

$$-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\left(\frac{w\alpha\beta\tau}{R}\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s\partial\theta}\cdot$$

$$\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\tau} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot$$

$$\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s\partial\theta}\left(\frac{w\alpha\beta\tau}{R}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} +$$

$$+u_1\left(y_1\right)f''\left(\bar{l}_1\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot\frac{\partial^2 V\left(s^u,\theta^u,\bar{l}_1\right)}{\partial\theta\partial s} +$$

$$+\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot$$

$$\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial s\partial\theta}\left(\frac{w\alpha\beta\tau}{R}\right)^2\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial^2 s} +$$

$$+\left(\frac{w\alpha\beta\tau}{R}\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot$$

$$\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial l_1\partial\tau} - \left(\frac{w\alpha\beta\tau}{R}\right)^2\left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,l_1^u\right)\right]}{\partial^2 s}\right)^2\cdot$$

$$\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}\right)\right]}{\partial\theta\partial\tau},$$

or, simplifying and reordering the terms,

$$\det\left(H^{\tau}\right)=u_1\left(y_1\right)f''\left(\bar{l}_1\right)\cdot \left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau}\right), \tag{26}$$

where

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau} = u_{11}\left(y_1\right)w\cdot\bar{l}_1+RE\left[v_{11}\left(\tilde{y}_2\right)\right]w\left(\alpha\beta\bar{l}_1+(1-\theta^u)(\beta-1)\right)\le 0,$$

and

$$\begin{aligned}\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau} &= E\left[v_{11}\left(\tilde{y}_2\right)\left(f'(1-\theta^u)\left(1+\tilde{\phi}\right)-w(1-\tau+\beta\tau)\right)\right]\cdot \\ &\cdot w\left(\alpha\bar{l}_1+(1-\theta^u)(\beta-1)\right)+ \\ &-E\left[v_1\left(\tilde{y}_2\right)\right]w(\beta-1) \\ &=\frac{1}{R}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot w\left(\alpha\beta\bar{l}_1+(1-\theta^u)(\beta-1)\right)+ \\ &-E\left[v_1\left(\tilde{y}_2\right)\right]w(\beta-1).\end{aligned}$$

Since $u_1\left(y_1\right)f''\left(\bar{l}_1\right)\ge 0$ holds, the sign of Equation (26) and of $\det\left(H^{\tau}\right)$ depends on the term in brackets which, using previous equations and Equation (16), can be rewritten as:

$$\begin{aligned}&\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}+ \\ &-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau} \\ &=u_{11}\left(y_1\right)w\bar{l}_1\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}+ \\ &+RE\left[v_{11}\left(\tilde{y}_2\right)\right]w\left(\alpha\beta\bar{l}_1+(1-\theta^u)(\beta-1)\right)\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}+ \\ &-u_{11}\left(y_1\right)\frac{1}{R}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot w\left(\alpha\beta\bar{l}_1+(1-\theta^u)(\beta-1)\right)+ \\ &-RE\left[v_{11}\left(\tilde{y}_2\right)\right]\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\cdot w\left(\alpha\beta\bar{l}_1+(1-\theta^u)(\beta-1)\right)+ \\ &+E\left[v_1\left(\tilde{y}_2\right)\right]w(\beta-1)(u_{11}\left(y_1\right)+R^2E\left[v_{11}\left(\tilde{y}_2\right)\right]),\end{aligned}$$

or, simplifying and reordering the terms,

$$\begin{aligned}&\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\tau}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}+ \\ &-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\tau}\end{aligned} \tag{27}$$

$$\begin{aligned}&=u_{11}\left(y_1\right)w\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}\left(\bar{l}_1-\frac{\alpha\beta}{R}\cdot\bar{l}_1-\frac{(1-\theta^u)}{R}(\beta-1)\right)+ \\ &+E\left[v_1\left(\tilde{y}_2\right)\right]w(\beta-1)(u_{11}\left(y_1\right)+R^2\cdot E\left[v_{11}\left(\tilde{y}_2\right)\right]).\end{aligned} \tag{28}$$

It is easy to see that this equation is equal to 0, so that $\frac{\partial\theta^u}{\partial\tau}=0$ holds, if $\alpha=R$ and $\beta=1$.

Focus now on the sufficient conditions for $\frac{\partial\theta^u}{\partial\tau}<0$ to hold. The second term in Equation (27) is negative or nil for $\beta\ge 1$. There are two sets of sufficient conditions for the first term to be negative as well, which depend on the sign of $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}$. If $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}>0$, it must be the case that

$$\bar{l}_1-\frac{\alpha\beta}{R}\cdot\bar{l}_1-\frac{1-\theta^u}{R}(\beta-1)>0,$$

or, reordering the terms

$$R\cdot\bar{l}_1+1-\theta^u>\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta. \tag{29}$$

Hence, sufficient conditions for $\frac{\partial\theta^u}{\partial\tau}<0$ to hold, are $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}>0$, or equivalently $\frac{\partial s^u}{\partial\theta^u}>0$, $\beta\ge 1$ and $R\cdot\bar{l}_1+1-\theta^u>\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

If $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}<0$, Inequality (29) must be reversed, implying that sufficient conditions for $\frac{\partial\theta^u}{\partial\tau}\le 0$ to hold are $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}<0$, or equivalently $\frac{\partial s^u}{\partial\theta^u}<0$, $\beta\ge 1$ and $R\cdot\bar{l}_1+1-\theta^u<\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

Lastly, consider the sufficient conditions for $\frac{\partial\theta^u}{\partial\tau}>0$ to hold. The second term in Equation (27) is positive or nil for $\beta\le 1$. There are two sets of sufficient conditions for the first term to be positive as well, which depend on the sign of $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}$. These are $R\cdot\bar{l}_1+1-\theta^u<\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$, if $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}>0$, and $R\cdot\bar{l}_1+1-\theta^u>\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$, if $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}<0$. As a consequence, $\frac{\partial\theta^u}{\partial\tau}>0$ holds, if $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}>0$, $\beta\le 1$ and $R\cdot\bar{l}_1+1-\theta^u<\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$, or if $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}<0$ and $\beta\le 1$ and $R\cdot\bar{l}_1+1-\theta^u>\alpha\cdot\beta\cdot\bar{l}_1+(1-\theta^u)\beta$.

A.7. Proof of Proposition 8

Consider Equation (5). Since the function $f'\left(l_1\right)$ is strictly increasing, it is also invertible and

$$\bar{l}_1=f'^{-1}\left(w\cdot\frac{(1-\tau)R+\alpha\beta\tau}{R}\right)$$

holds. It follows that:

$$\frac{\partial\bar{l}_1}{\partial\tau}=\frac{w}{f''\left(\bar{l}_1\right)}\left(\frac{\alpha\beta}{R}-1\right),$$

implying that $\frac{\partial\bar{l}_1}{\partial\tau}\ge 0$ holds if $R\le\alpha\cdot\beta$.

A.8. Proof of Proposition 10

Following the same steps as those presented in the proof of Proposition 7, it is possible to show that the sign of $\frac{\partial\theta^u}{\partial\alpha}$ is the same as the sign of the equation below:

$$\det\left(H^{\alpha}\right)=u_1\left(y_1\right)f''\left(\bar{l}_1\right)\cdot \left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\alpha}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\alpha}\right),$$

where $\det\left(H^{\alpha}\right)$ is the determinant of the matrix H^{α}, whose definition is analogous to the definition of H^{τ} in the proof of Proposition 7. Moreover,

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\alpha}=R\cdot E\left[v_{11}\left(\tilde{y}_2\right)\right]\beta w\tau\bar{l}_1\le 0,$$

and

$$\begin{aligned}\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\alpha} &= E\left[v_{11}\left(\tilde{y}_2\right)\left(f'(1-\theta^u)\left(1+\tilde{\phi}\right)-w(1-\tau+\beta\tau)\right)\right]\cdot \\ &\cdot\beta w\tau\bar{l}_1 \\ &=\frac{1}{R}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}\cdot\beta w\tau\bar{l}_1,\end{aligned}$$

hold, so that

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\alpha}\cdot\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s}-\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s}\cdot$$

$$\cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\alpha}$$

$$= R \cdot E\left[v_{11}\left(\tilde{y}_2\right)\right] \beta w \tau \bar{l}_1 \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} +$$

$$-\frac{\beta w \tau \bar{l}_1}{R} \cdot u_{11}\left(y_1\right) \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} - R \cdot E\left[v_{11}\left(\tilde{y}_2\right)\right] \beta w \tau \bar{l}_1 \cdot$$

$$\cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}$$

$$= -\frac{\beta w \tau \bar{l}_1}{R} \cdot u_{11}\left(y_1\right) \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}.$$

As a consequence, it is

$$\det\left(H^\alpha\right) = -u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot \frac{\beta w \tau \bar{l}_1}{R} \cdot u_{11}\left(y_1\right) \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} \geq 0.$$

Since $-u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot \frac{w\tau\bar{l}_1}{R} \cdot u_{11}\left(y_1\right) \geq 0$ holds, the sign of $\frac{\partial\theta^u}{\partial\alpha}$ is the same as the sign of $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta}$, or equivalently of $\frac{\partial s^u}{\partial\theta^u}$.

A.9. Proof of Proposition 13

Following the same steps as those presented in the proof of Proposition 7, it is possible to show that the sign of $\frac{\partial\theta^u}{\partial\beta}$ is the same as the sign of the equation below:

$$\det\left(H^\beta\right) = u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot \left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\beta} \right),$$

where $\det\left(H^\beta\right)$ is the determinant of the matrix H^β, whose definition is analogous to the definition of H^τ in the proof of Proposition 7. Moreover,

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta} = R \cdot E\left[v_{11}\left(\tilde{y}_2\right)\right] w\tau\left(1-\theta^u+\alpha\bar{l}_1\right) \leq 0,$$

and

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\beta}$$

$$= E\left[v_{11}\left(\tilde{y}_2\right)\left(f'\left(1-\theta^u\right)\left(1+\tilde{\phi}\right) - w(1-\tau+\beta\tau)\right)\right] \cdot$$

$$\cdot w\tau\left(1-\theta^u+\alpha\bar{l}_1\right) - E\left[v_1\left(\tilde{y}_2\right)\right] w\tau$$

$$= \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\theta} + - E\left[v_1\left(\tilde{y}_2\right)\right] w\tau,$$

hold, so that, making use of the equation above makes it possible to obtain:

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} + - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\beta}$$

$$= \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot \left(\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta} - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} \cdot \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R} \right) + \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} E\left[v_1\left(\tilde{y}_2\right)\right] w\tau.$$

Substituting Equation (16) and $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta}$ in the previous equation gives

$$\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial s\partial\beta} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} + - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial\beta}$$

$$= \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot R \cdot E\left[v_{11}\left(\tilde{y}_2\right)\right] w\tau\left(1-\theta^u+\alpha\bar{l}_1\right) +$$

$$- \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot u_{11}\left(y_1\right) \cdot \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R} +$$

$$- \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot R \cdot E\left[v_{11}\left(\tilde{y}_2\right)\right] w\tau\left(1-\theta^u+\alpha\bar{l}_1\right) +$$

$$+ \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} E\left[v_1\left(\tilde{y}_2\right)\right] w\tau +$$

$$= - \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot u_{11}\left(y_1\right) \cdot \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R} +$$

$$+ \frac{\partial^2 V\left(s^u,\theta^u,\bar{l}_1\right)}{\partial^2 s} E\left[v_1\left(\tilde{y}_2\right)\right] w\tau.$$

As a consequence

$$\det\left(H^\beta\right) = u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial^2 s} E\left[v_1\left(\tilde{y}_2\right)\right] w\tau +$$

$$- u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot \frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \cdot u_{11}\left(y_1\right) \cdot \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R},$$

implying that a sufficient condition for $\frac{\partial\theta^u}{\partial\beta} \leq 0$ to hold, is $\frac{\partial^2 E\left[V\left(s^u,\theta^u,\bar{l}_1\right)\right]}{\partial\theta\partial s} \leq 0$ (or equivalently $\frac{\partial s^u}{\partial\theta^u} \leq 0$). Note in fact, that the first term in the equation above is always negative and that $-u_1\left(y_1\right) f''\left(\bar{l}_1\right) \cdot u_{11}\left(y_1\right) \cdot \frac{w\tau\left(1-\theta^u+\alpha\bar{l}_1\right)}{R} \geq 0$ holds.

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