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*Some geometric estimates
for fractional Poincaré inequalities*

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Abstract

This thesis is devoted to study the following two problems: giving geometric lower bounds for the first eigenvalue of the fractional Dirichlet-Laplacian of order s ; determining the sharp constant for the fractional Hardy inequality. The two problems are tightly connected: indeed, for an open set having finite inradius, every lower bound on the sharp Hardy constant immediately translates into a lower bound for the first eigenvalue.

For the first problem, we prove a geometric lower bound in terms of the *inradius* of the set, in the case of open planar sets having nontrivial topology. This is valid for $1/2 < s < 1$ only, and we show that this condition is sharp. Moreover, by constructing suitable counter-examples, we prove that our lower bound is optimal, in many respects. The result is obtained through some non-trivial adaptations to the fractional case of techniques employed by Hayman and Taylor, to handle the case of the usual Laplacian operator. In adapting these techniques, we will develop some technical tools, which are interesting in themselves, in the context of fractional Sobolev spaces.

For the second problem, we determine the sharp constant in the fractional Hardy inequality for open convex sets in every dimension. This is done by constructing suitable local weak supersolutions with geometric content, to the relevant Euler-Lagrange equation. The latter is a weighted eigenvalue-type equation, containing a negative power of the distance from the boundary. For $1/2 \leq s < 1$, such supersolution is given by a suitable power of the distance function. The case $0 < s < 1/2$ is much more difficult, since such a construction fails. In this case, we employ a directional decomposition method, which permits to reduce the problem to dimension 1. This technique is due to Loss and Sloane, in the fractional setting. In particular, we can compute the sharp constant in the whole range $0 < s < 1$. This completes a result which was left open in the literature.

Sunto

In questa tesi studiamo i due seguenti problemi: dare stime geometriche dal basso per il primo autovalore del Laplaciano frazionario di ordine s con condizioni di Dirichlet; determinare la costante ottima per la disuguaglianza di Hardy frazionaria. Tali problemi sono strettamente connessi: infatti, per un insieme aperto con inradius finito, ogni stima dal basso per la costante ottima di Hardy si può convertire in modo immediato in una stima dal basso per il primo autovalore.

Per il primo problema, proviamo una stima geometrica dal basso in termini dell'*inradius* dell'insieme, nel caso di aperti del piano con topologia non banale. Questo vale solo per $1/2 < s < 1$, e mostriamo che tale condizione è ottima. Inoltre, costruendo controesempi appropriati, proviamo che questa stima è ottimale in molti aspetti. Tale risultato è ottenuto adattando al caso frazionario in modo non banale le tecniche utilizzate da Hayman e Taylor, per il caso del Laplaciano classico. A tal proposito, elaboreremo alcuni strumenti tecnici che sono interessanti di per sé nel contesto degli spazi di Sobolev frazionari.

Per il secondo problema, determiniamo la costante ottima della disuguaglianza di Hardy frazionaria per aperti convessi in ogni dimensione. Per fare ciò, costruiamo una soprasoluzione debole locale, avente un contenuto geometrico, alla rispettiva equazione di Eulero-Lagrange. Quest'ultima è un'equazione agli autovalori pesata, contenente una potenza negativa della funzione che indica la distanza dal bordo dell'insieme. Per $1/2 \leq s < 1$, tale soprasoluzione è data da una potenza appropriata della funzione distanza. Il caso $0 < s < 1/2$ è più complesso, in

quanto tale costruzione non è valida. In questo caso, applichiamo un metodo di decomposizione delle direzioni, sviluppato da Loss e Sloane nel contesto frazionario, che ci permette di ridurre il problema alla dimensione 1. Più precisamente, possiamo calcolare la costante ottima in tutto l'intervallo $0 < s < 1$ e questo completa un problema che era stato lasciato aperto in letteratura.

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Introduction

I Fractional Poincaré inequality

I.1 Background

The first part of this thesis is devoted to provide geometric estimates for the sharp *fractional* Poincaré constant of planar open sets, possibly satisfying some additional topological and geometric restrictions. As we will explain below, this will be connected with giving geometric bounds for the first eigenvalue of the *fractional Laplacian of order s* , a nonlocal operator that in the last years attracted a lot of interest, both for its theoretical aspects and its possible applications (see for example [30, 55, 57] for more details). We recall that, for a smooth function u , this is the operator defined by

$$(-\Delta)^s u(x) := 2 \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad \text{for } x \in \mathbb{R}^N,$$

up to some normalization constant (see [37]).

However, in order to gently introduce the reader to the problems we want to address in this thesis, it is certainly better to start with their classical counterparts. Thus, for an open set $\Omega \subseteq \mathbb{R}^N$, we introduce the quantity

$$\lambda_1(\Omega) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}. \quad (\text{I.1})$$

By its very definition, this is nothing but the sharp constant for the classical Poincaré inequality

$$C \int_{\Omega} |u|^2 \leq \int_{\Omega} |\nabla u|^2 dx, \quad \text{for every } u \in C_0^\infty(\Omega),$$

whose proof, possibly under some suitable assumptions on Ω , can be found in all the textbooks on Sobolev spaces. This is one of the first examples of *functional inequality*, whose importance in many fields of Mathematics can not be underestimated. On this subject, it is mandatory to refer to the fundamental book [86].

In particular, an open set Ω supports the previous Poincaré inequality if and only if $\lambda_1(\Omega) > 0$. By a density argument, the infimum in (I.1) is unchanged if the class of admissible functions is extended to the whole $W_0^{1,2}(\Omega) \setminus \{0\}$. Here $W_0^{1,2}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the usual Sobolev space $W^{1,2}(\Omega)$.

We recall that, whenever the infimum in (I.1) turns to be a minimum on $W_0^{1,2}(\Omega)$, then λ_1 is called the *first eigenvalue of the Dirichlet-Laplacian on Ω* . The name is justified by the fact that this is the smallest $\lambda \in \mathbb{R}$ such that the equation

$$-\Delta u = \lambda u,$$

is weakly solved in $W_0^{1,2}(\Omega) \setminus \{0\}$. In this case, a corresponding weak solution takes the name of¹ *first eigenfunction of the Dirichlet-Laplacian on Ω* .

On the other hand, it may happen that $\lambda_1(\Omega) > 0$, but the infimum above is not attained in $W_0^{1,2}(\Omega)$. In this case, the continuous embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ fails to be compact and the spectrum of the Dirichlet-Laplacian on Ω is not discrete (see [11, Chapter 10, Section 1.1, Theorem 5]). Nevertheless, $\lambda_1(\Omega)$ still plays an important role, since it coincides in any case with the infimum of the spectrum of the Dirichlet-Laplacian on Ω (see [11, Chapter 10, Section 1.1]). This situation occurs for example in the case of the “slab”

$$\mathcal{S} = \mathbb{R}^{N-1} \times (-1, 1),$$

where $\lambda_1(\mathcal{S}) = \pi^2/4$ but there is no first eigenfunction in $W_0^{1,2}(\Omega)$ associated to it.

The quantity $\lambda_1(\Omega)$ plays an important role in both Classical and Quantum Mechanics, where usually it can be regarded as the minimal energy of a physical system. An interpretation in Classical Mechanics is certainly well-known to the reader: if we consider $\Omega \subseteq \mathbb{R}^2$ as the vibrating membrane of a drum fixed along its boundary, then $\sqrt{\lambda_1(\Omega)}$ turns out to be proportional to the least possible frequency of stationary vibration. This simply follows from a space-time separation of variables in the wave equation.

In Quantum Mechanics, we can think of a quantum particle confined to move in a region $\Omega \subseteq \mathbb{R}^3$ by an ideal confining potential (i.e. a potential vanishing in Ω and identically equal to $+\infty$ in its complement). Then its stationary state of lowest energy coincides with $\lambda_1(\Omega)$, up to some normalization constant which has no bearing for our mathematical treatment (see for example [79, Chapter 11] for more details).

In spite of its importance, in general the quantity $\lambda_1(\Omega)$ can be explicitly computed only for some particular sets. Thus, it is quite natural to inquire the possibility of retrieving informations on $\lambda_1(\Omega)$, from some (possibly simple) geometric features of the set Ω . In a nutshell, one aims at proving some upper and lower bounds on $\lambda_1(\Omega)$ in terms of geometric quantities of Ω . Of course, the same problem can be tackled for other eigenvalues of the Laplacian, possibly with different boundary conditions and even for spectral functionals of more general differential operators (or even not necessarily differential operators, like the fractional Laplacian). This is a very well-studied field, extremely active still nowadays: we refer to the classical book [93] and to the modern presentation given in [2, 65], for an introduction into this realm.

In particular, in this spirit, in the last century many authors undertook to give estimates for $\lambda_1(\Omega)$ by using for example:

- the *volume* $|\Omega|$ of the open set $|\Omega|$ (i.e. its N -dimensional Lebesgue measure);
- the *diameter* $\text{diam}(\Omega)$;
- the *perimeter* $P(\Omega)$ (possibly defined in distributional sense, see [81]);
- the *inradius* r_Ω , that is

$$r_\Omega = \sup \{ r > 0 : \exists x_0 \in \Omega \text{ s.t. } B_r(x_0) \subseteq \Omega \}; \quad (\text{I.2})$$

or even combinations of them. Here, we recall the most prominent example: the celebrated *Faber-Krahn inequality*. This asserts that N -dimensional balls minimize λ_1 , among open subsets

¹Observe that $\lambda_1(\Omega)$ could be a minimum, even if the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is not compact. We refer to our paper [B3] for a variational point of view on the spectral analysis of a particular class of open sets having this property.

of \mathbb{R}^N having fixed volume. By keeping into account the scaling properties of both λ_1 and the volume, this can be recast into the following geometric lower bound

$$\lambda_1(\Omega) \geq \mathcal{C}_{\text{FK}} |\Omega|^{-\frac{2}{N}}, \quad \text{with } \mathcal{C}_{\text{FK}} := \lambda_1(B_1) |B_1|^{\frac{2}{N}},$$

see for example [65, Theorem 3.2.1]. Here we indicated by B_1 the N -dimensional ball of radius 1, centered at the origin. Observe that $-2/N$ is the unique power on the volume which makes the previous inequality scale invariant: indeed, from its definition we see that λ_1 scales like a length to the power of -2 , that is

$$\lambda_1(t\Omega) = \frac{1}{t^2} \lambda_1(\Omega), \quad \text{for every } t > 0.$$

Thus, from the Faber-Krahn inequality, we can bound λ_1 from below, in terms of the volume. However, even if this is a sharp lower bound, we notice that such an estimate becomes *useless* whenever we encounter situations where

$$|\Omega| = +\infty \quad \text{and} \quad \lambda_1(\Omega) > 0.$$

A very simple example is given by the slab $\mathcal{S} = \mathbb{R}^{N-1} \times (-1, 1)$ previously mentioned. More generally, an important class of open sets falling into this case is given by the *curved waveguides*, i.e. the tubular neighborhoods of infinite smooth curves, without self-intersections. Without any attempt of completeness, we refer to [3, 45, 46, 59] and [75] for some thorough studies on the spectral properties of these sets.

Observe that for sets of this type, seeking for estimates in terms of quantities like the perimeter or the diameter would be equally useless. On the contrary, we observe that in these examples the inradius is finite: one could bravely guess that some estimates in terms of r_Ω are possible. To start with, let us observe that the following upper bound

$$\lambda_1(\Omega) \leq \frac{\lambda_1(B_1)}{r_\Omega^2},$$

holds for *every open set* $\Omega \subseteq \mathbb{R}^N$, in *every dimension* N . In the case $r_\Omega = +\infty$, the right-hand side has to be intended as 0. The proof of the previous estimate is extremely simple: it is sufficient to observe that if $B_r(x_0)$ is a ball contained in Ω , then by monotonicity of λ_1 with respect to the set inclusion, we get

$$\lambda_1(\Omega) \leq \lambda_1(B_r(x_0)) = \frac{\lambda_1(B_1)}{r^2}.$$

By taking the supremum over the radius of balls contained in Ω , we get the desired conclusion. The estimate above is a quantitative version of the following statement: if Ω contains arbitrarily large balls, then the set fails in supporting Poincaré's inequality. More generally, if Ω contains very large balls, then λ_1 must be small.

A rough intuition could suggest that actually the converse statement should be true, as well: if one thinks to the two-dimensional physical interpretation of Ω as the vibrating membrane of a drum fixed along its boundary, then one could expect that if the fundamental frequency of vibration $\lambda_1(\Omega)$ is very low, the drum must be “very large”, in some sense. Then, one could guess that it is possible to revert the previous estimate and get a lower bound on $\lambda_1(\Omega)$ in terms of r_Ω . Unfortunately, in general this is not the case, as shown by the following

Counter-example. Let $\Omega := \mathbb{R}^2 \setminus \mathbb{Z}^2$. Then we have

$$r_\Omega = \frac{\sqrt{2}}{2}, \quad \text{while} \quad \lambda_1(\Omega) = 0.$$

Indeed, for every $n \in \mathbb{N}$, we get

$$\lambda_1(\Omega) \leq \lambda_1(Q_n \setminus \mathbb{Z}^2) = \lambda_1(Q_n) = n^{-2} \lambda_1(Q_1),$$

where Q_n denotes the square $(-n, n) \times (-n, n)$. The first inequality simply follows from the monotonicity of λ_1 with respect to the set inclusion, while the last identity is due to the scaling properties of λ_1 . Finally, the identity $\lambda_1(Q_n \setminus \mathbb{Z}^2) = \lambda_1(Q_n)$ is given by the fact that points are removable, since we are in dimension 2 (see for example [86, Chapter 2, Section 2.2]).

The key-point of the previous construction is that, in dimension $N \geq 2$, for every open set $\Omega \subseteq \mathbb{R}^N$ we have

$$W_0^{1,2}(\Omega \setminus \{x_0\}) = W_0^{1,2}(\Omega),$$

for every $x_0 \in \Omega$. There is nothing special with points: actually, the same equality stays true whenever we remove a compact set with zero *capacity*. For a compact set Σ contained to an open bounded set $E \subseteq \mathbb{R}^N$, this is defined by the following variational quantity

$$\text{cap}(\Sigma; E) := \inf_{u \in C_0^\infty(E)} \left\{ \|\nabla u\|_{L^2(E)}^2 : u \geq 1 \text{ on } \Sigma \right\}.$$

This is the correct way to measure sets which are “negligible” in the sense of the relevant Sobolev space $W_0^{1,2}$. In light of this fact, the previous **Counter-example** shows that it is possible to make r_Ω finite and at the same violating the validity of the Poincaré inequality, by simply constructing sets with a complement which is too “thin”, in the sense of capacity.

This intuitive idea can be translated into a rigorous result, as shown in [86, Chapter 15, Section 4], which proves a necessary and sufficient condition on Ω which guarantees $\lambda_1(\Omega) > 0$. Let us denote by $Q_r(x_0)$ a cube with sides parallel to the axis, centered at a point $x_0 \in \mathbb{R}^N$ and having sides of length $2r$. Then [86, Theorem 15.4.2.1] shows that $\lambda_1(\Omega) > 0$ if and only if there exists a radius $\delta = \delta(\Omega) > 0$ such that

$$\inf_{x_0 \in \mathbb{R}^N} \text{cap} \left(\overline{Q_\delta(x_0)} \setminus \Omega; Q_{2\delta}(x_0) \right) > 0.$$

This last condition essentially means that Ω has a complement which is locally “fat” in the sense of capacity and this property holds uniformly in every region of the space. However, this kind of “analytic” condition is quite difficult to translate at a “geometric” level: in other words, it is difficult to extrapolate from the previous fatness condition of the complement of Ω , a geometric lower bound on $\lambda_1(\Omega)$.

In any case, the previous discussion clarifies that the simple geometric condition $r_\Omega < +\infty$ is not sufficient to ensure the validity of the Poincaré inequality. Thus, some further assumptions on the open set are needed. In order to clarify the scopes of the first part of the thesis, we will now list the most remarkable classical results in this direction, listing situations where the lower bound

$$\lambda_1(\Omega) \geq \frac{C}{r_\Omega^2}, \tag{I.3}$$

is known to be feasible.

• **Convex sets.** The first author to obtain a (sharp) lower bound of the type (I.3) was J. Hersch in 1960. Namely, in [67, Théorème 8.1] he showed that, if $\Omega \subseteq \mathbb{R}^2$ is an open *convex* set, then (I.3) holds with

$$C = \frac{\pi^2}{4}.$$

This constant is sharp and equality is attained for the strip $\mathcal{S} = \mathbb{R} \times (-1, 1)$, for example. The proof by Hersch is based on a clever geometric argument, which permits to reduce the problem to proving the estimate for triangles.

Some years later, M. H. Protter in [96, Theorem 2] generalized Hersch's proof to any dimension, by obtaining the estimate above with the same constant $\pi^2/4$, which is sharp in any dimension. Equality is attained for example by slabs, i.e. sets which coincide with $\mathbb{R}^{N-1} \times (-1, 1)$, up to a rigid movement.

We also point out that in more recent years, R. Kajikiya in [72] has proved the Hersch-Protter inequality, by means of a completely different proof, based on the fact that in a convex set the distance function d_Ω is weakly superharmonic. Such a proof extends immediately to the case of the p -Laplacian.

• **Simply connected sets in \mathbb{R}^2 .** In 1965, E. Makai in [82, Equation (5)] showed that in the class of open simply connected sets in the plane, it is still possible to get a lower bound on λ_1 in terms of the inradius. Namely, he proved that it holds with $C = 1/4$ for this class of sets. For the scopes of this thesis, it is useful to recall that Makai's proof is based in particular on the *Coarea Formula*, in one of its prototypical versions. Thus, his proof is not suitable for an extension to the fractional case we are interested in, because of the lack of a genuine Coarea Formula for fractional Sobolev spaces.

In 1978 W. K. Hayman, unaware of Makai's result, showed the same kind of lower bound for simply connected subsets of \mathbb{R}^2 , by means of a completely different proof with respect to Makai. For this reason, we call this kind of lower bound the *Makai-Hayman inequality*.

It is interesting to remark that the result by Makai is quantitatively better than the one by Hayman: indeed, the latter obtains the same lower bound, but with a factor $1/900$ (!) in place of the $1/4$ found by Makai.

This could suggest that the attribution of this result to both authors is maybe too generous. On the contrary, as we have shown in [B2] and as we will report in Chapter 3 of this thesis, despite providing a poorer constant, the method of proof by Hayman is elementary, flexible and robust enough to be generalized to other situations, where Makai's and other approaches become too complicate or do not seem feasible.

In the same paper, Hayman also observed that in dimensions $N \geq 3$ an estimate like (I.3) is not possible, not even under severe topological restrictions: he observed that by taking a ball B , from which we remove an increasing number of inward pointing spikes, one can construct a sequence of open *starshaped* sets $\Omega_n \subseteq \mathbb{R}^N$ such that

$$\lim_{n \rightarrow \infty} r_{\Omega_n} = 0 \quad \text{and} \quad \lambda_1(\Omega_n) = \lambda_1(B),$$

see [63, Section 4]. Again, the construction is possible due to capacity effects: segments in dimension $N \geq 3$ have zero capacity, i.e. λ_1 is unaltered by the removal of segments. For more details on the construction of this counter-example, one can also have a look at [86, Chapter 4, Section 3, Example].

Even if this falls outside the scopes of the present research, we can not resist the temptation of pointing out that the determination of the sharp constant in the Makai-Hayman inequality, i.e.

$$\mathcal{C}_{\text{MH}} := \inf \left\{ \lambda_1(\Omega) r_\Omega^2 : \Omega \subseteq \mathbb{R}^2 \text{ simply connected with } r_\Omega < +\infty \right\},$$

is still an open problem. An important step towards the direction of solving this problem has been undertaken in [7], by R. Bañuelos and T. Carroll. There, the two authors provided yet another proof of the Makai-Hayman inequality, which permits to infer in particular that

$$\mathcal{C}_{\text{MH}} > 0.6197.$$

This “considerably” improves the lower bound $1/4$ found by Makai. The proof by Bañuelos and Carroll is quite sophisticated: it is based on obtaining the following L^∞ estimate on the *torsion function* w_Ω of Ω

$$\|w_\Omega\|_{L^\infty(\Omega)} \lesssim \left(\sup_{x \in \Omega} \varrho_\Omega(x) \right)^2.$$

The rightmost quantity $\varrho_\Omega(x)$ is the so-called *conformal radius of Ω at the point x* (see for example [6]), while we recall that w_Ω is characterized as the unique solution of

$$-\Delta u = 1, \quad \text{in } \Omega,$$

with homogeneous Dirichlet boundary conditions. One then obtains the Makai-Hayman inequality by joining the previous key estimate with the following ones

$$\lambda_1(\Omega) \|w_\Omega\|_{L^\infty(\Omega)} \geq 1 \quad \text{and} \quad \sup_{x \in \Omega} \varrho_\Omega(x) \lesssim r_\Omega.$$

The latter is a consequence of the *Koebe one quarter Theorem* (see [42, Chapter 2]), while the first one just follows by an integration by parts (see for example [66, Section 3]).

In the same paper [19], the authors also obtained the upper bound

$$\mathcal{C}_{\text{MH}} < 2.13,$$

by estimating from above λ_1 for a suitable family of simply connected domains, called *Goodman domains* (see [60] for such domains). For these sets it is possible to give quite precise estimates on λ_1 , by using conformal mappings. This upper bound was then slightly improved by P. R. Brown in [28], where he refined the same technique.

Remark. In his review of Makai’s paper (see [68]), J. Hersch formulated a conjecture on the sharp constant \mathcal{C}_{MH} above. Indeed, he suggested that an optimal set providing the sharp value \mathcal{C}_{MH} could be the following

$$\mathbf{H} := \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1, x < 0 \right\} \cup \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < |x_2| < 1, x_1 \geq 0 \right\},$$

that we call Hersch’s pipe. This is a slit disk, to which two infinite tubes of constant width are attached. Observe that this set could also be seen as an infinite curved wave guide: for example, in [46, Example 4.3] this specific example is called “bookcover”.

In [B3, Theorem 7.6], we disproved this conjecture, by means of a singular perturbation technique. Namely, we have shown that adding a suitable (finite) number of thin tubes on the flat part of $\partial\mathbf{H}$ eventually leads to an increase of the product $\lambda_1(\mathbf{H}) r_{\mathbf{H}}^2$. In this part, we greatly rely on the technique recently studied (in greater generality) in [1] by L. Abatangelo and R. Ognibene (see also [47]).

• **Multiply connected sets in \mathbb{R}^2 .** Actually, there is nothing special about simply connected sets in the plane: one can still prove a lower bound of the form (I.3) for planar sets having, roughly speaking, a fixed numbers of “holes”. In light of the **Counter-example** above, it must be expected that the constant C in (I.3) should now depend on the number of “holes”. Let us first give the precise definition of sets we want to consider.

Definition. Let us indicate by $(\mathbb{R}^2)^*$ the one-point compactification of \mathbb{R}^2 , i.e. the compact space obtained by adding to \mathbb{R}^2 the point at infinity. We say that an open connected set $\Omega \subseteq \mathbb{R}^2$ is multiply connected of order k if its complement in $(\mathbb{R}^2)^*$ has k connected components. When $k = 1$, we will simply say that Ω is simply connected.

For this class of sets, R. Osserman, M. E. Taylor and C. Croke independently showed that (I.3) still holds, with a constant $C \sim k^{-1}$. More precisely, the pioneer of this type of result was R. Osserman: in [92, Theorem p. 546] he showed that

$$\lambda_1(\Omega) \geq \min \left\{ \frac{1}{4}, \frac{1}{k^2} \right\} \frac{1}{r_\Omega^2},$$

for every $\Omega \subseteq \mathbb{R}^2$ multiply connected sets of order k . This estimate follows by combining a refinement of the so-called Cheeger inequality and a Bonnesen–type inequality. Two years later, in [104] M. E. Taylor improved this result, by showing that

$$\lambda_1(\Omega) \geq \frac{C}{k} \frac{1}{r_\Omega^2}, \tag{I.4}$$

still for planar multiply connected sets of order k . The main novelty with respect to [92] is the improvement of the estimate in its dependence on the “topological” index k : indeed, it can be shown that the factor k^{-1} is optimal, as k diverges to $+\infty$.

We will not give details about Taylor’s proof here, for a very simple reason: despite being the one producing the worst constant, it is the most flexible and robust one. In particular, we will show in the first part of the thesis that his method of proof can be adapted (with some non-trivial efforts) to cover the fractional case we are interested in. Thus, we will give an insight into Taylor’s proof when presenting our results in the next section. Here, we only anticipate that the concept of *capacity* plays a pivot role in this proof: it goes straight to the core of the problem, in a sense.

Finally, in [33] C. B. Croke further improved Taylor’s result, by making explicit the universal constant C appearing in (I.4). However, even his constant is not sharp. His argument is based on a refinement of Osserman’s one.

Remark. *We also mention the paper [61] of S. E. Graversen and M. Rao, which contains the same type of result. Their proof is based on the theory of Brownian motion and the dependence on k is (slightly) worse when compared with the ones by Croke and Taylor.*

For completeness, we cite the recent preprint [17], where F. Bozzola and L. Brasco extended the Croke-Osserman-Taylor inequality to cover the more general case of sharp Sobolev-Poincaré constants (see [17, Theorem 3.4] for the explicit statement).

I.2 Main goal

As we anticipated at the beginning, our aim is to extend the previous results to the setting of fractional Sobolev spaces. We refer to the recent books [43] and [77] for a modern introduction to these spaces, even if the reader will find all the needed definitions and basic results in Chapter 1. More specifically, our primary target in this part of the thesis is the quest for a fractional counterpart of the Croke-Osserman-Taylor inequality, i.e. an estimate like (I.4) in the class of planar multiply connected sets.

In order to neatly present the problem we want to tackle, let us now try to enter more into the business: we fix a parameter $0 < s < 1$, which stands for a fractional order of differentiation. For every open set $\Omega \subseteq \mathbb{R}^N$ we define the following quantity

$$\lambda_1^s(\Omega) := \inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{[u]_{W^{s,2}(\mathbb{R}^N)}^2}{\|u\|_{L^2(\Omega)}^2}, \tag{I.5}$$

where

$$[u]_{W^{s,2}(\mathbb{R}^N)} := \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}},$$

is the so-called *Gagliardo-Slobodeckii seminorm*. As usual, by a density argument, we can enlarge the class of admissible function to $\widetilde{W}_0^{s,2}(\Omega) \setminus \{0\}$. The latter is the closure of $C_0^\infty(\Omega)$ into the Sobolev-Slobodeckii space $W^{s,2}(\mathbb{R}^N)$.

The comments made after the definition of $\lambda_1(\Omega)$ can be repeated almost verbatim: by definition, $\lambda_1^s(\Omega)$ coincides with the sharp constant for the following fractional version of Poincaré inequality

$$C \int_{\Omega} |u|^2 \leq [u]_{W^{s,2}(\mathbb{R}^N)}^2, \quad \text{for every } u \in C_0^\infty(\Omega).$$

Observe that in this context, functions are compactly supported in Ω , but they have to be thought as functions identically vanishing on the complement of Ω . In other words, the previous Poincaré inequality holds for functions which enjoys a “nonlocal” homogeneous Dirichlet conditions, i.e. we have to think to the complement $\mathbb{R}^N \setminus \Omega$ as the “boundary” for this inequality. Of course, for an open set Ω this Poincaré inequality holds if and only if $\lambda_1^s(\Omega) > 0$.

As in the classical case, also the quantity (I.5) is connected to an eigenvalue problem. Indeed, if the infimum in (I.5) is a minimum on $\widetilde{W}_0^{s,2}(\Omega)$, then λ_1^s is called *first eigenvalue of the Dirichlet fractional Laplacian of order s of Ω* . In this case $\lambda_1^s(\Omega)$ is the smallest real number λ such that the following boundary value problem

$$(-\Delta)^s u = \lambda u,$$

admits non trivial solutions in $\widetilde{W}_0^{s,2}(\Omega)$, as well. In weak formulation, this means that

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy = \lambda \int_{\Omega} u \varphi dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

Remark I.1. *We can build a bridge between λ_1^s and λ_1 by recalling the following celebrated convergence result due to J. Bourgain, H. Brezis and P. Mironescu*

$$\lim_{s \nearrow 1} (1 - s) [u]_{W^{s,2}(\mathbb{R}^N)}^2 = C_N \|\nabla u\|_{L^2(\Omega)}^2, \quad \text{for every } u \in C_0^\infty(\Omega), \quad (\text{I.6})$$

see [15] or also [43, Chapter 3]. This shows in particular that

$$\limsup_{s \nearrow 1} (1 - s) \lambda_1^s(\Omega) \leq C_N \lambda_1(\Omega).$$

Under some mild regularity assumptions on Ω , one can actually prove that

$$\lim_{s \nearrow 1} (1 - s) \lambda_1^s(\Omega) = C_N \lambda_1(\Omega),$$

but for general open sets this may fail to be true (see [18, Lemma A.1 & Remark A.2] for more details).

With the quest for geometric estimates on λ_s^1 in mind, we can observe that, with exactly the same proof as in the classical case, one can immediately get the following (sharp!) upper bound

$$\lambda_1^s(\Omega) \leq \frac{\lambda_1^s(B_1)}{r_\Omega^{2s}},$$

in terms of r_Ω only. This estimate is obviously in scale invariant form, just notice that now λ_1^s scales like a length to the power of $-2s$. Of course, such an estimate can not be reverted: we can essentially copy the same counter-example of the local case.

Counter-example. Let $0 < s < 1$ and $\Omega := \mathbb{R}^2 \setminus \mathbb{Z}^2$. Then we have

$$r_\Omega = \frac{\sqrt{2}}{2}, \quad \text{while} \quad \lambda_1^s(\Omega) = 0.$$

Indeed, similarly to what we did with $s = 1$, for every $n \in \mathbb{N}$ we get

$$\lambda_1^s(\Omega) \leq \lambda_1^s(Q_n \setminus \mathbb{Z}^2) = n^{-2s} \lambda_1^s(Q_1).$$

The first inequality follows by the monotonicity of λ_1^s with respect to the set inclusion and the fact that points have zero s -capacity (see Lemma 4.2.1 below).

As before, one could expect that the previous counter-example is “optimal”, in the sense that preventing the sets to have an infinite number of “holes” could lead to a lower bound on λ_s^1 in terms of r_Ω only. We then arrive at the main goal of the first part of the thesis: proving a lower bound of the type

$$\lambda_1^s(\Omega) \geq \frac{C_{k,s}}{r_\Omega^{2s}}, \tag{I.7}$$

for open multiply connected sets of order k , in the plane. This would be the fractional version of the Croke-Osserman-Taylor inequality, evoked at the beginning of this section. Moreover, we are interested in getting (I.7) with a “good” constant $C_{k,s}$. In other words, in light of the Bourgain-Brezis-Mironescu formula (I.6), we would like to get a constant such that²

$$C_{k,s} \sim \frac{1}{1-s}, \quad \text{as } s \nearrow 1.$$

In addition to this, it would be very much desirable that the dependence of $C_{k,s}$ on k is sharp, in the asymptotic regime $k \nearrow +\infty$ (exactly as in Taylor and Croke results).

Before passing to expose our results on this part, for completeness we briefly recall a couple of known estimates, very much related to the topic. The first one is the fractional version of the Faber-Krahn inequality: this reads as follows

$$\lambda_1^s(\Omega) \geq \mathcal{C}_{\text{FK},s} |\Omega|^{-\frac{2s}{N}}, \quad \text{with } \mathcal{C}_{\text{FK},s} = \lambda_1^s(B_1) |B_1|^{\frac{2s}{N}}.$$

This gives a lower bound on λ_1^s in terms of the volume of the set. Various proofs are possible, based on: classical symmetrization techniques (see for example [21, Theorem 3.5]); probabilistic techniques (see [8, Theorem 5]); symmetrization and comparison techniques for solutions of parabolic equations (see [101, Theorem 6.1]).

For the class of open *convex* subsets of \mathbb{R}^2 , R. Bañuelos, R. Latała and P. J. Méndez-Hernández in [8, Corollary 1] proved the fractional counterpart of the Hersch-Protter inequality, Namely, by means of probabilistic techniques, they obtained the following sharp lower bound

$$\lambda_1^s(\Omega) \geq \lambda_1^s(\mathcal{S}) \frac{1}{r_\Omega^{2s}}, \quad \text{where } \mathcal{S} = \mathbb{R} \times (-1, 1).$$

To the best of our knowledge, an extension of this result to higher dimensions is still not known. In the next section, we will explain how to get a lower bound of this type in every dimension, with a sub-optimal constant. This will follow as a consequence of a fractional Hardy inequality.

We refer to [51] for further spectral bounds for the fractional Laplacian.

²Throughout the thesis, the writing

$$f(s) \sim g(s), \quad \text{for } s \rightarrow s_0,$$

has to be intended in the following sense: there exists $C \in \mathbb{R} \setminus \{0\}$ such that

$$0 < \liminf_{s \rightarrow s_0} \frac{f(s)}{g(s)} \leq \limsup_{s \rightarrow s_0} \frac{f(s)}{g(s)} < +\infty.$$

I.3 Main results

The first result on this part is the following extension of the Makai-Hayman and Croke-Osserman-Taylor inequalities, to the fractional case. This is taken from [B1] (see also [B2] for the case of simply connected sets, i.e. $k = 1$).

Theorem 1 (Fractional Croke-Osserman-Taylor inequality). *Let $1/2 < s < 1$, there exists a constant $\vartheta_s > 0$ such that for every $\Omega \subseteq \mathbb{R}^2$ open, multiply connected set of order $k \in \mathbb{N} \setminus \{0\}$, we have*

$$\lambda_1^s(\Omega) \geq \frac{\vartheta_s}{k^s} \left(\frac{1}{r_\Omega} \right)^{2s}. \quad (\text{I.8})$$

Moreover, the constant ϑ_s has the following asymptotic behaviours

$$\vartheta_s \sim (2s - 1) \quad \text{for } s \searrow \frac{1}{2} \quad \text{and} \quad \vartheta_s \sim \frac{1}{1 - s} \quad \text{for } s \nearrow 1.$$

The proof will be given in Chapter 2. As explained in the previous section, we will carefully keep track of the dependence of ϑ_s on s . This will be one of the crucial points in our proof.

The previous result does not hold for $0 < s \leq 1/2$. This is optimal: in [B2], we proved that in this regime the inequality fails already for simply connected sets. Namely, we have the following

Theorem 2 (Counter-example for $0 < s \leq 1/2$). *There exists a sequence $\{\tilde{Q}_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^2$ of open bounded simply connected sets such that*

$$0 < r_{\tilde{Q}_n} \leq C, \quad \text{for every } n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow \infty} \lambda_1^s(\tilde{Q}_n) = 0, \quad \text{for every } 0 < s \leq \frac{1}{2}.$$

The proof of this fact is contained in Chapter 4.

Next, we show that the dependence of (I.8) on the parameters s and k is *optimal*, in a suitable sense. These results are achieved in [B1], and they are collected in the following statement.

Theorem 3 (Optimality). *The following facts hold:*

1. *for every $\Omega \subseteq \mathbb{R}^2$ open set, we have*

$$\limsup_{s \nearrow 1} (1 - s) \lambda_1^s(\Omega) \leq \frac{1}{2} \lambda_1(\Omega).$$

Thus, the estimate (I.8) is sharp in its dependence on $s \nearrow 1$. In particular, by taking the limit as s goes to 1 in (I.8), we get the classical Croke-Osserman-Taylor inequality, possibly with a worse constant;

2. *let $1/2 < s < 1$, there exists a sequence $\{\Omega_k\}_{k \in \mathbb{N} \setminus \{0\}} \subseteq \mathbb{R}^2$ of open sets such that Ω_k is multiply connected of order k*

$$r_{\Omega_k} \leq C \quad \text{and} \quad \limsup_{k \rightarrow \infty} k^s \lambda_1^s(\Omega_k) < +\infty.$$

Thus the estimate (I.8) is sharp in its dependence on $k \rightarrow \infty$;

3. for every $k \in \mathbb{N} \setminus \{0\}$, there exists $\Theta_k \subseteq \mathbb{R}^2$ an open multiply connected set of order k , such that

$$r_{\Theta_k} < +\infty \quad \text{and} \quad \limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2s-1} < +\infty.$$

Thus, the estimate (I.8) is sharp in its dependence on $s \searrow 1/2$.

The proofs of these results can be found in Chapter 4. Here, we just remark that the most delicate part of the last theorem is point 3. In order to prove the desired asymptotic behaviour, a very careful choice of test function has to be done, based on the *fundamental solution* for the fractional Laplacian, in dimension 1 (see [29]).

From Theorem 1, we can draw some interesting consequences, which hold for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order k and every $1/2 < s < 1$. At first, we have the following remarkable equivalence

$$\lambda_1^s(\Omega) > 0 \quad \iff \quad r_\Omega < +\infty.$$

Moreover, under these assumptions, we can prove that $\widetilde{W}_0^{s,2}(\Omega)$ coincides with the homogenous Sobolev space $\mathcal{D}_0^{s,2}(\Omega)$, given by the *completion* of $C_0^\infty(\Omega)$ with respect to the norm³

$$u \mapsto [u]_{W^{s,2}(\mathbb{R}^N)}.$$

On the contrary, for $0 < s \leq 1/2$, the sequence $\{\widetilde{Q}_n\}_n$ of Theorem 2 shows that only the implication

$$\lambda_1^s(\Omega) > 0 \quad \implies \quad r_\Omega < +\infty$$

can hold, in general. Moreover, in this range the two spaces $\widetilde{W}_0^{s,2}(\Omega)$ and $\mathcal{D}_0^{s,2}(\Omega)$ may fail to coincide.

Theorem 1 also implies a couple of *Cheeger-type inequalities*, i.e. we obtain a lower bound on $\lambda_1^s(\Omega)$ in terms of the following two *Cheeger constants*

$$h_1(\Omega) = \inf \left\{ \frac{P(E)}{|E|} : E \subseteq \Omega \text{ bounded and measurable with } |E| > 0 \right\},$$

and

$$h_s(\Omega) = \inf \left\{ \frac{P_s(E)}{|E|} : E \subseteq \Omega \text{ bounded and measurable with } |E| > 0 \right\},$$

where, for a set E , the quantities $P(E)$ and $P_s(E)$ denote its distributional perimeter and its fractional perimeter of order s , respectively:

$$\lambda_1^s(\Omega) \geq \frac{C_s}{k^s} \left(h_1(\Omega) \right)^{2s} \quad \text{and} \quad \lambda_1^s(\Omega) \geq \frac{\widetilde{C}_s}{k^s} \left(h_s(\Omega) \right)^2.$$

These results are interesting, since a general version of the fractional Cheeger inequality is still unknown. We recall that the classical version of this spectral bound is given by

$$\lambda_1(\Omega) \geq \left(\frac{h_1(\Omega)}{2} \right)^2,$$

which holds for every open set $\Omega \subseteq \mathbb{R}^N$ and every $N \geq 1$ (see for example [86, Chapter 4, Section 2]).

³Observe that on $C_0^\infty(\Omega)$ this is indeed a norm.

I.4 Outlines of the proofs

In the proof of Theorem 1 we follow the ideas introduced by Taylor in [104].

Thus, let Ω be an *open multiply connected set of order k* in \mathbb{R}^2 , we first apply a “fatness lemma” proved in Appendix B which tiles the whole space \mathbb{R}^2 with squares; hence we take an admissible function u in $\widetilde{W}_0^{s,2}(\Omega)$ and we look for a fractional Poincaré inequality for squares.

We anticipate that, the coefficient of such an inequality, depends on the s -capacity of a compact set which lies in the square. Finally we bound from below such s -capacity in terms of the length-side of the squares.

Due to the non-locality of the fractional operator, we come accross technical difficulties, and it encourages our interest in solving the problem.

Let us enter more in details in the various steps: the *fatness lemma* is a geometric result which tiles \mathbb{R}^2 with squares having fixed side length ℓ . It does not depend on the analytic problem we are studying, and thus we carefully show the proof that Taylor claimed in [104]; indeed his construction is proved for $k = 1$ only, which coincides with the class of planar simply connected sets.

Here we give a brief description of such a result: we choose ℓ to be comparable with the inradius r_Ω and with the square root \sqrt{k} ; this, combined with the topological assumption on Ω , assures that every square that intersect Ω has a “Dirichlet region” Σ , and hence we can focus our attention in one square only. Moreover, with a counting argument we prove that, in *many* squares, the projection of Σ on the same coordinate axis can be bounded from below with the length ℓ , up to a positive multiplicative constant.

As soon as we have assured the geometric result, we fix one of the “many good squares” and we observe that the admissible functions u in $\widetilde{W}_0^{s,2}(\Omega)$ do not satisfy the Dirichlet conditions on the complement of each square⁴. However, thanks to the “fatness Lemma”, a *Dirichlet region* Σ_i lies in each closed square \overline{Q}_i , and hence the s -capacity we mentioned before is $\widetilde{\text{cap}}_s(\Sigma_i; B_{2\ell}(x_i))$.

Thus we get

$$[u]_{W^{s,2}(\mathbb{R}^2)}^2 = \sum_{i \in \mathbb{Z}} \iint_{\mathbb{R}^2 \times Q_\ell(x_i)} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy \geq \sum_{i \in \mathbb{Z}} [u]_{W^{s,2}(Q_\ell(x_i))}^2 \geq \frac{1}{\ell^2} \sum_{i \in \mathbb{Z}} C_i \|u\|_{L^2(Q_\ell(x_i))}^2,$$

where $C_i = \widetilde{\text{cap}}_s(\Sigma_i; B_{2\ell}(x_i))$, up to a multiplicative positive constant.

The third (and last) step of the proof is devoted to bound from below that s -capacity in terms of ℓ without any dependence on i . In particular, with Proposition 2.1.4, we underline the connection between the analytic problem related to the s -capacity and a geometric one. That link is crucial because, with it, we can move to the geometry of the problem on the square, cleaned of the boundary conditions, that is easier to handle. Here, thanks to the construction of the “enlarging box” $B_{2\ell}(x_i)$, we show that

$$\widetilde{\text{cap}}_s(\Sigma_i, B_{2\ell}(x_i)) \geq \ell^{1-2s} \max \{ \mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma_i)), \mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma_i)) \}, \quad \text{for every } i \in \mathbb{Z},$$

where $\Pi_{\mathbf{e}_j}(\Sigma_i)$ denotes the projection of Σ_i on the coordinate axis \mathbf{e}_j . We can use the “enlarging box” thanks to a suitable extension operator that will be detailed later. Now we do not dwell in its description: we only highlight that we take care on the constant which appears. We conclude the proof by recalling the last part of the statement of the “fatness lemma”.

The connection between the two problems having different natures is obtained by taking a competitor v for the s -capacity and by decomposing its seminorm. In fact, by applying a result

⁴Due to the non-locality of the seminorm, the Dirichlet boundary condition has to be intended as *identically vanishes in the complement of the square*.

concerning the *directional fractional derivatives* which is described below, we give a lower bound of the seminorm in terms of

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{|v(x) - v(x + \rho \mathbf{e}_j)|^2}{|\rho|^{1+2s}} d\rho \right) dx.$$

Thus we decompose the variable $x = (x_1 \mathbf{e}_1, x_2 \mathbf{e}_2)$, and hence we can see the latter integral as the L^2 norm of the seminorm of v , only in the direction \mathbf{e}_j . By construction, that seminorm is a competitor for the s -capacity of a point, which is far enough from the boundaries of the interval.

Thus we have reduced the problem to dimension one. Finally we prove a *fractional Morrey-Sobolev inequality* for the line, which bounds the s -capacity of the point with a negative power of the length of the interval. This fact is valid *only* when s is greater than the threshold $1/2$. To show that Morrey-Sobolev inequality we take advantage of the linearity of our operators: indeed we exploit the *Fourier Transform* and the *Plancherel identity*.

Finally, we notice that the asymptotic behaviour of ϑ_s comes from the estimate of the s -capacity of a point in the interval: in particular the dependence

$$\vartheta_s \sim \frac{1}{1-s}, \quad \text{as } s \nearrow 1,$$

is a consequence of the Plancherel identity, while the fact that

$$\vartheta_s \sim (2s-1), \quad \text{as } s \searrow 1/2,$$

comes out when we manipulate the Fourier coefficients of the seminorm.

It remains to give some details on the *directional fractional derivatives* we mentioned before. It is a technical result that is similar to a fractional Poincaré inequality. Here the L^2 norm of the function is written as the *norm of the integral of a fractional difference quotient* for a fixed direction ω . In order to show it, we get the inspiration from [18] and [25].

For the class of planar simply connected sets, i.e. when $k = 1$, we establish such a theorem also by pursuing the idea of Hayman, which is more elementary than Taylor's one.

Thus, let us go back to the fractional Hayman proof: it can be found in the paper [B2], but here we give a brief idea of it.

The proof that Hayman did in [63] is based on two main steps: the first is a geometric result, while the second is an analytic one. We refer to the former with the wording *covering lemma with boundary disks* because it shows that every simply connected set of \mathbb{R}^2 having finite inradius can be covered with disks centered on $\partial\Omega$ whose radius is comparable with r_Ω . Moreover this family of (possibly infinity) disks can be partitioned in 36 subfamilies \mathfrak{B}_j , $j = 1, \dots, 36$, such that, taken a couple of disks in the same subfamily, their intersection is empty. The original proof of this result can be found in [63] (we report its statement in Appendix B).

Thanks to the geometric lemma, for every admissible function u in $\widetilde{W}_0^{s,2}(\Omega)$, we can fix a subfamily \mathfrak{B}_j and consider an element on this only; indeed it is straightforward that

$$[u]_{W^{s,2}(\mathbb{R}^2)}^2 \geq \frac{1}{36} \sum_{j=1}^{36} \sum_{B_r(x_i) \in \mathfrak{B}_j} \iint_{B_r(x_i) \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy.$$

Thus, the main point is to show a *Poincaré inequality for boundary disks*. We underline that the Poincaré inequality we prove has a mixed seminorm: the two integrals are valued on $B_r(x_i) \times \mathbb{R}^2$.

Here the topological assumption is crucial: since we center a disk of radius r in a point x_i of $\partial\Omega$, then the loop $\partial B_r(x_i)$ intersects the complement of Ω at least in a point. Thus we can

move to polar coordinates and, for every positive radius, we reduce the problem to dimension one. Such idea was employed in [63], where Hayman uses the crucial fact that

$$|\nabla u|^2 \geq \frac{1}{\varrho^2} |\partial_\theta u|^2, \quad (\text{I.9})$$

where (ϱ, θ) denote the usual polar coordinates. Thus he applied a one-dimensional Poincaré inequality for periodic functions vanishing in a point to the function $\theta \mapsto u(\varrho, \theta)$. He concluded by integrating over the radius ϱ .

Conversely, due to the nonlocality of the Gagliardo-Slobodeckii seminorm, the trivial inequality (I.9) does not have a fractional counterpart. We overcome such difficulty by handling the integrals valued on $B_r(x_j) \times \mathbb{R}^2$. Thus we reduce the problem to dimension 1, that is the fractional counterpart of what Hayman obtained. Finally we show a fractional Poincaré inequality for 2π -periodic functions which vanishes at least in a point.

In order to show the Poincaré inequality for $\theta \mapsto u(\varrho, \theta)$, since the function is periodic, we expand it in Fourier series. We then obtain the result, by appealing to the classical *Plancherel identity*. Finally, we point out the asymptotic behaviour as $s \searrow 1/2$ and $s \nearrow 1$ of the constant: it is obtained by manipulating the Fourier coefficients, and hence we bring it into the fractional Makai-Hayman inequality as well.

II Fractional Hardy inequality

II.1 Background

The second part of this thesis deals with the determination of the sharp constant in the fractional Hardy inequality for the class of convex sets. Here as well, let us start at first by recalling some basic facts for the classical case, in order to have a better understanding of our scopes. For an open set $\Omega \subseteq \mathbb{R}^N$, the *Hardy inequality* reads as follows

$$\int_{\Omega} |\nabla u|^2 dx \geq \lambda \int_{\Omega} \frac{|u|^2}{d_{\Omega}^2} dx, \quad \text{for every } u \in C_0^\infty(\Omega). \quad (\text{II.1})$$

From now on, $d_{\Omega}(x)$ denotes the *distance function* from the boundary of the set. In other words, this is the function defined by

$$d_{\Omega}(x) = \min_{y \in \partial\Omega} |x - y|, \quad \text{for } x \in \Omega.$$

We consider it to be extended by zero outside Ω . Hardy's inequality is probably one of the most studied functional inequality, since its first introduction in [64]. We refer the reader to [36, 91] for a comprehensive introduction to the subject. Without any attempt of completeness, we also mention the papers [35, 44, 62, 76, 78, 90, 106] for some studies on the Hardy inequality.

Observe that the Hardy inequality is nothing but a weighted version of Poincaré's inequality. More precisely, the right-hand side of (II.1) contains the L^2 norm of the function, weighted by d_{Ω}^{-2} . In particular, the two norms appearing in (II.1) now enjoy the same scaling. Accordingly, if we define the *sharp Hardy constant* by

$$\mathfrak{h}(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 dx : \int_{\Omega} \frac{|u|^2}{d_{\Omega}^2} dx = 1 \right\},$$

this is a scale invariant quantity. As usual, such an infimum is unchanged if we enlarge the class of admissible functions to the whole $W_0^{1,2}(\Omega)$. This scale invariance is responsible for some challenging behaviour of minimizing sequences for the previous variational problem: accordingly,

in general it is quite hard to know whether the previous infimum is attained or not in $W_0^{1,2}(\Omega)$. We will not study this problem here, but we refer the interested reader to [83, 84] for some results in this direction.

In order to create a connection with the first part of the thesis and motivate our interest towards this problem, we recall that for a general open set $\Omega \subseteq \mathbb{R}^N$ the two informations

$$\lambda_1(\Omega) > 0 \quad \text{and} \quad \mathfrak{h}(\Omega) > 0,$$

are not related. In other words, it may happen that for an open set Poincaré's inequality holds and Hardy's inequality does not or viceversa. However, if Ω has finite inradius and $\mathfrak{h}(\Omega) > 0$, we can automatically get a lower bound on $\lambda_1(\Omega)$, in terms of the inradius. Indeed, by observing that $d_\Omega \leq r_\Omega$, we get

$$\int_\Omega |\nabla u|^2 dx \geq \mathfrak{h}(\Omega) \int_\Omega \frac{|u|^2}{d_\Omega^2} dx \geq \frac{\mathfrak{h}(\Omega)}{r_\Omega} \int_\Omega |u|^2 dx, \quad \text{for every } u \in C_0^\infty(\Omega).$$

This in turn implies the following lower bound

$$\lambda_1(\Omega) \geq \frac{\mathfrak{h}(\Omega)}{r_\Omega^2}. \quad (\text{II.2})$$

This explains the interest in Spectral Theory for proving Hardy's inequality and for giving good lower bounds on $\mathfrak{h}(\Omega)$ (see for example [36]). On this point, we can not avoid mentioning that yet another proof of the Makai-Hayman inequality is possible, exactly by using this idea: this is due to A. Ancona, see [5, Proposition 5]. Indeed, by using conformal maps, the aforementioned *Koebe's one quarter Theorem* and the explicit knowledge of \mathfrak{h} for a half-plane (see below), he was able to prove that

$$\mathfrak{h}(\Omega) \geq \frac{1}{16},$$

for every $\Omega \subseteq \mathbb{R}^2$ open simply connected set. By spending this information into (II.2), one can get back the Makai-Hayman inequality (I.3), possibly with a worse constant.

A particular class of open sets for which \mathfrak{h} can be determined is that of open *convex* subsets of \mathbb{R}^N . In this case, if we set

$$\mathbb{H}_+^1 = (0, +\infty) \quad \text{and} \quad \mathbb{H}_+^N = \mathbb{R}^{N-1} \times (0, +\infty), \quad \text{for } N \geq 2,$$

we have the following remarkable result

$$\mathfrak{h}(\Omega) = \mathfrak{h}(\mathbb{H}_+^N) = \mathfrak{h}(\mathbb{H}_+^1) = \frac{1}{4},$$

for every open convex set $\Omega \subseteq \mathbb{R}^N$. See for example [83, Theorem 11] for this result.

For our scopes, it is useful to recall the proof of the previous result. Various proofs are possible, let us recall here the one which will be the most interesting for us: this is based on exploiting *local weak supersolutions* of the following equation

$$-\Delta u = \lambda \frac{u}{d_\Omega^2}, \quad \text{in } \Omega. \quad (\text{II.3})$$

By definition, a local weak supersolution is a function $u \in W_{\text{loc}}^{1,2}(\Omega)$, such that

$$\int_\Omega \langle \nabla u, \nabla \varphi \rangle dx \geq \lambda \int_\Omega \frac{u \varphi}{d_\Omega^2} dx, \quad \text{for every } \varphi \in C_0^\infty(\Omega), \varphi \geq 0.$$

Then, by [5, Appendix], we have the following “dual” formulation for the sharp Hardy constant

$$\mathfrak{h}(\Omega) = \sup \left\{ \lambda \geq 0 : \text{equation (II.3) admits a positive local weak supersolution} \right\},$$

which is valid for every open set, not necessarily convex. In a convex set, this result becomes extremely useful, since we have a standard way to construct the required supersolutions. Indeed, since on a convex set the distance function d_Ω is concave and verifies $|\nabla d_\Omega| = 1$ almost everywhere, one can easily see that for $0 < \alpha < 1$ the function $U_\alpha = d_\Omega^\alpha$ is a positive local weak supersolution of (II.3), with

$$\lambda = \alpha(1 - \alpha).$$

In light of the dual characterization above, we immediately get

$$\mathfrak{h}(\Omega) \geq \sup_{0 < \alpha < 1} \alpha(1 - \alpha) = \frac{1}{4},$$

for every open convex set $\Omega \subsetneq \mathbb{R}^N$ and every dimension $N \geq 1$. Observe that the optimal choice of α is given by $\alpha = 1/2$, thus $U_{1/2} = \sqrt{d_\Omega}$ is the “optimal” supersolutions in this computation. Nevertheless, such a function can not be an extremal for $\mathfrak{h}(\Omega)$, since it does not belong to $W^{1,2}(\Omega)$.

In order to complete the proof and show that the sharp constant is actually given by $1/4$, one proves at first that

$$\mathfrak{h}(\mathbb{H}_+^1) \leq \frac{1}{4}.$$

This can be done by simply using a test function of the form $x^\alpha \eta$, with $\eta \in C_0^\infty((-\infty, 1))$ and $\alpha > 1/2$, then letting α go to $1/2$. In turn, by using test functions with separate variables, one can easily prove that

$$\mathfrak{h}(\mathbb{H}_+^N) \leq \mathfrak{h}(\mathbb{H}_+^1) = \frac{1}{4}, \quad \text{for every } N \geq 2.$$

Finally, if $\Omega \subsetneq \mathbb{R}^N$ is an open convex set not coinciding with a half-space, by using a blow-up argument at a regular point of the boundary, we can get

$$\mathfrak{h}(\Omega) \leq \mathfrak{h}(\mathbb{H}_+^N) = \frac{1}{4},$$

see [83, Theorem 5]. All these steps permit to finally close the circle.

II.2 Main goal

Likewise to what we did in the first part of the introduction, we want to consider a fractional counterpart of $\mathfrak{h}(\Omega)$, for the class of convex sets. To do this, for every $0 < s < 1$, we define

$$\mathfrak{h}_s(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ [u]_{W^{s,2}(\mathbb{R}^N)}^2 : \int_\Omega \frac{|u|^2}{d_\Omega^{2s}} dx = 1 \right\}, \quad (\text{II.4})$$

and we observe that this it is still a scale invariant quantity. By definition, this is the sharp constant in the following fractional Hardy inequality

$$[u]_{W^{s,2}(\mathbb{R}^N)}^2 \geq C \int_\Omega \frac{|u|^2}{d_\Omega^{2s}} dx, \quad \text{for every } u \in C_0^\infty(\Omega). \quad (\text{II.5})$$

Observe that, by reproducing varbatim the argument that gives (II.2), we can obtain the following lower bound

$$\lambda_1^s(\Omega) \geq \frac{\mathfrak{h}_s(\Omega)}{r_\Omega^{2s}}, \quad (\text{II.6})$$

whenever Ω is such that $\mathfrak{h}_s(\Omega) > 0$ and $r_\Omega < +\infty$.

In the case of convex sets, the validity of the fractional Hardy inequality (II.5) was proved by L. Brasco and E. Cinti in [19]. Here, the authors provided the following lower bound

$$\frac{\mathcal{C}}{s(1-s)} \leq \mathfrak{h}_s(\Omega),$$

where \mathcal{C} is a positive constant, depending only on N . Their proof was quite close to the one based on supersolutions exposed above: indeed, it exploited the fact that d_Ω^s is *locally weakly superharmonic* in fractional sense, that is

$$(-\Delta)^s d_\Omega^s \geq 0, \quad \text{in } \Omega.$$

We will refine this argument below, by exploiting the local weak supersolutions of the equation

$$(-\Delta)^s u = \lambda \frac{u}{d_\Omega^{2s}}, \quad \text{in } \Omega, \quad (\text{II.7})$$

as in the classical case.

Let us now try to explain the method of supersolutions more in details: we say that u is a *positive local weak supersolution* of (II.7) if it solves

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \geq \lambda \int_\Omega \frac{u(x)}{d_\Omega(x)^{2s}} \varphi(x) dx,$$

for every non-negative $\varphi \in W^{s,2}(\mathbb{R}^N)$ with compact support in Ω .

At first we remark the paper [38] of B. Dyda, where the author computed the one-dimensional constant $\mathfrak{h}_s((-1, 1))$. In particular, he reached his aim by showing that

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{|w(x) - w(y)|^2}{|x - y|^{1+2s}} dx dy \geq \mathfrak{h}_s((-1, 1)) \int_{-1}^1 |w|^2 \left(\frac{1}{1-x} + \frac{1}{x+1} \right)^{2s} dx,$$

where

$$w(x) = (1 - x^2)^{\frac{2s-1}{2}}.$$

On the other hand, in Lemma 6.1.5, we prove that for every $-1 < \beta < 2s$, the function

$$f_\beta(x) := x^{2s-1-\beta} (1-x)^\beta, \quad \text{for } x \in (0, 1),$$

is a positive local weak supersolution of (II.7) with a coefficient $\lambda(\beta)$ depending on the parameter. Since $\beta \mapsto \lambda(\beta)$ attains its maximum in $(2s-1)/2$, we compute such value which coincides with Dyda's coefficient.

A problem similar to $\mathfrak{h}_s(\Omega)$ was studied by K. Bogdan and B. Dyda in [13, Theorem 1.1], and by M. Loss and C. Sloane in [80, Theorem 1.2]. Both the papers deal with the ‘‘regional seminorm’’, where the integrals are evaluated on $\Omega \times \Omega$ only, i.e. they are concerned with the following sharp constant

$$\mathfrak{h}_s^{\text{reg}}(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy : \int_\Omega \frac{|u|^2}{d_\Omega^{2s}} dx = 1 \right\}.$$

The paper [13] computed the sharp constant of the half-space \mathbb{H}_+^N for every $0 < s < 1$, while [80] compute the sharp constant for every open bounded convex set, under the restriction $s > 1/2$. In the ‘‘regional’’ case, we recall that such a restriction is unavoidable for bounded sets, as shown in [38, Theorem 1.1] (see also [39, Section 2]). We underline some ingredients of the proofs of both [13] and [80], because we will use them later. Bogdan and Dyda in [13] used the *method of*

supersolutions exposed in the previous subsection, suitably adapted to the fractional case. That is, they observed that for $\Omega = \mathbb{H}_+^N$ the functions

$$U_\alpha(x) = (d_{\mathbb{H}_+^N}(x))^\alpha = x_N^\alpha,$$

are positive local weak supersolutions (actually, by homogeneity, solutions) of

$$(-\Delta)^{s,\text{reg}}u = \lambda(\alpha) \frac{u}{d_{\mathbb{H}_+^N}^{2s}}, \quad \text{in } \mathbb{H}_+^N,$$

with a $\lambda(\alpha)$ which is maximal for $\alpha = (2s - 1)/2$. Observe that this is consistent with the local case $s = 1$: in this case, the maximal power becomes $\alpha = 1/2$, as seen above.

Loss and Sloane in [80] solved first the one-dimensional problem, namely they computed $\mathfrak{h}_s^{\text{reg}}$ for an interval on the real line. Then they got the higher dimensional result by exploiting a *directional decomposition formula*, reminiscent of a similar alternative proof for the classical case (see for example [36, Theorem 1.5.3])

One could argue that, by means of the simple inequality

$$[u]_{W^{s,2}(\mathbb{R}^N)}^2 \geq \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy,$$

from the results proved in [13, 80] one could trivially prove a fractional Hardy inequality for the case we are interested in. Left apart for the unnatural restriction $s > 1/2$ introduced, which should not be needed in our case, the sharpness of the estimate is lost in this process, of course.

The last reference we point out is the paper [48] due to S. Filippas, L. Moschini, A. Tertikas. They computed \mathfrak{h}_s for the class of convex sets, but under the restriction that $s \geq 1/2$. This restriction is now due to their method of proof, which in addition works for the Hilbertian case only: indeed, they exploited the so-called *Caffarelli-Silvestre extension* (see [31]), which permits to turn the problem into a local one, at the price of increasing the dimension of the ambient space. The case $0 < s < 1/2$ was left open in [48]: in this thesis, we have been able to fill this gap and compute \mathfrak{h}_s for every open convex set and the full range $0 < s < 1$.

II.3 Main results

We now describe our main contributions on this topic. The first one, taken from [B4], is a “dual” characterization for \mathfrak{h}_s which is valid for every open set $\Omega \subsetneq \mathbb{R}^N$. This generalizes to the fractional case the analogous result presented in the previous section. This characterization is in terms of *positive local weak supersolutions* of (II.7) and it reads as follows:

Theorem 4 (A characterization for \mathfrak{h}_s). *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then we have*

$$\mathfrak{h}_s(\Omega) = \sup \left\{ \lambda \geq 0 : \text{equation (II.7) admits a positive local weak supersolution} \right\}.$$

The proof of this result will be a direct consequence of the following equivalence

$$\mathfrak{h}_s(\Omega) > 0 \quad \iff \quad \begin{array}{l} \text{the equation (II.7) admits a positive} \\ \text{local weak supersolution for some } \lambda > 0. \end{array} \quad (\text{II.8})$$

We should notice that the result above can also be obtained from [50, Theorem 1.9] by P. J. Fitzsimmons, which is concerned with the more general framework of Dirichlet forms associated

to symmetric Markov processes. However, as we will explain below, our proof is different and more elementary. Moreover, the same proof applies to the more general case of

$$\mathfrak{h}_{s,p}(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy : \int_{\Omega} \frac{|u|^p}{d_{\Omega}^{sp}} dx = 1 \right\}$$

for every $0 < s < 1$ and $1 < p < +\infty$, which is not the case for [50, Theorem 1.9]. We refer to [B4] for the full statement: in this thesis, we will only focus on the Hilbertian case $p = 2$, for consistency with the first part of the work.

As a second major contribution, we compute the sharp constant \mathfrak{h}_s for *convex* sets, for every $0 < s < 1$ and every dimension. This is contained in [B5]. In order to state the result, we need some notation. Let us introduce the following constants

$$\Lambda_s := 2 \int_0^1 \frac{\left|1 - t^{\frac{2s-1}{2}}\right|^2}{(1-t)^{1+2s}} dt + \frac{1}{s}. \quad (\text{II.9})$$

and

$$C_{N,2s} := \begin{cases} (N-1) \omega_{N-1} \mathcal{I}(N-2; 2s), & \text{for } N \geq 2, \\ 1, & \text{for } N = 1, \end{cases} \quad (\text{II.10})$$

where

$$\mathcal{I}(k; \alpha) = \int_0^{+\infty} t^k (1+t^2)^{-\frac{k+2+\alpha}{2}} dt.$$

Our main result is the following

Theorem 5 (Sharp fractional Hardy inequality for convex sets). *Let $N \geq 1$, then we have:*

1. for every $0 < s < 1$

$$\mathfrak{h}_s(\mathbb{H}_+^N) = C_{N,2s} \Lambda_s;$$

2. for every $0 < s < 1$ and every $\Omega \subsetneq \mathbb{R}^N$ open convex set

$$\mathfrak{h}_s(\Omega) = \mathfrak{h}_s(\mathbb{H}_+^N).$$

In each case, the constant \mathfrak{h}_s is not attained.

We remark that, from the sharp fractional Hardy inequality for convex sets, we get a fractional Hersch-Protter-type inequality in every dimension (with a constant that might not be sharp). We can simply appeal to (II.6), which gives

$$\lambda_1^s(\Omega) \geq \frac{\mathfrak{h}_s(\Omega)}{r_{\Omega}^{2s}} = \frac{C_{N,2s} \Lambda_s}{r_{\Omega}^{2s}}.$$

Finally, we wish to mention that in the original paper [B5], we considered the more general question of determining the sharp constant

$$\mathfrak{h}_{s,p}(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy : \int_{\Omega} \frac{|u|^p}{d_{\Omega}^{sp}} dx = 1 \right\}$$

still for convex sets. For $p \neq 2$, we were able to get the analogue of item 2 of Theorem 5 only under the further restriction $sp \geq 1$ (with a proof different from those of both [48] and [80]). We remark that, on the contrary, for $p = 2$ this is not needed in our result.

II.4 Outlines of the proofs

As we said, the proof of Theorem 4 is based on the equivalence (II.8). In most of the classical references, the proof of the implication “ \implies ” is based on the Lax-Milgram Theorem. This is the case of [5] and [50], for example. Our proof of this fact is slightly different: more precisely, we use a purely variational approach in order to show existence of a supersolution. In particular, by using the *Direct Method*, we show the existence of a minimizer for the functional

$$\varphi \mapsto \frac{1}{2} [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \int_{\Omega} \frac{|\varphi|^2}{d_{\Omega}^{2s}} dx - \int_B \varphi dx,$$

and thus, by optimality, of a positive local weak solution of

$$(-\Delta)^s u = \lambda \frac{u}{d_{\Omega}^{2s}} + 1_B, \quad \text{in } \Omega,$$

for every $\lambda < \mathfrak{h}_s(\Omega)$. Here 1_B denotes the characteristic function of a ball $B \subseteq \Omega$. This existence result is quite delicate, since the only assumption on Ω is that $\mathfrak{h}_s(\Omega) > 0$. Thus, obtaining the compactness of minimizing sequences is the hard part of the business. The “subcriticality” assumption $\lambda < \mathfrak{h}_s(\Omega)$ plays a key role. On the other hand, as for the lower semicontinuity of the functional, we can circumvent the lack of convexity of the term containing the weighted L^2 norm, by using the classical Brezis-Lieb lemma in a suitable way. The choice of λ is important, here as well.

Now, let us give a glimpse of the proof of Theorem 5, in which we strongly use the method just introduced: first, we construct a family of local weak supersolutions for convex sets given by the admissible powers of the distance function in dimension 1. We observe that, when we restrict the analysis to the line, there exists only two type of convex sets: the half line and the segment. Even if the latter seems to be easier than the former due to its boundness, the seek of local weak supersolutions of the segment is harder. Indeed, when the domain is bounded, we are able to show that $(d_{(a,b)})^{\beta}$ is a local weak supersolution only if $0 \leq \beta < 2s$, where the constant $\lambda = \lambda(\beta)$ depends on the power. On the contrary, when we remove the boundness assumption on the set, we enlarge the interval of admissible β , that is $-1 < \beta < 2s$.

Afterwards, we compute the trivial study of the function $\beta \mapsto \lambda(\beta)$ and we figure out that it attains its maximum in $(2s - 1)/2$.

The last two statements combined with the scaling invariance of \mathfrak{h}_s , encourage to claim that the *largest* λ of the fractional Hardy inequality (II.5) is the same for every interval (possibly unbounded), that is

$$\lambda = \lambda \left(\frac{2s - 1}{2} \right).$$

Thus we have to pay attention when $I = (a, b)$ is bounded (without loss of generality we consider $I = (0, 1)$) and $s < 1/2$: under such assumptions we cannot show that $(d_I)^{\beta}$ is a local weak supersolution. Instead, in Lemma 6.4.1, we show a counter-example: if $s = 1/2$, for points x which are close to $1/2$ it holds

$$(-\Delta)^{\frac{1}{2}} d_I^{\beta}(x) < 0.$$

This implies that, when the convex set is the interval I and $s < 1/2$, one has to investigate deeply in the search for local weak supersolutions. To this aim we introduce the family of functions

$$f_{\beta}(x) = x^{2s-1-\beta} (1-x)^{\beta}, \quad \text{for every } x \in I,$$

and we prove that they are local weak supersolutions of the revised equation

$$(-\Delta)^s u = \lambda \frac{u}{(x(1-x))^{2s}}, \quad \text{since } -1 < \beta < 2s,$$

where $\lambda = \lambda(\beta)$ is the same as before. Thanks to the fact that $x(1-x) \leq \min\{x, 1-x\}$ in I , we reach our aim.

Moreover, whenever Ω is an interval or an half-line, we show that both d_Ω^β and f_β are not attained in $\widetilde{W}_0^{s,2}(\Omega)$, if $\beta = (2s-1)/2$. Here we exploit the property we anticipated before: d_Ω^β does not lie in $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$, if β is small enough (see Section 5.1 for the definition of such space).

Then we move to convex sets in higher dimensions by exploiting the results of the half-line and the interval. To do this we consider the β -power of the distance function $(d_\Omega)^\beta$: we first observe that, thanks to the convexity assumption on the set, for every point of $\partial\Omega$ there exists a supporting hyperplane. Thus, up to rigid motions, we can assume that $d_\Omega(x) = x_N$, for a fixed point x in Ω . Moreover, since $\beta \geq 0$ and Ω is convex,

$$(d_\Omega(y))^\beta \leq (y_N)^\beta, \quad \text{for every } y \in \Omega,$$

where we have the equality if and only if Ω is an half-space. Thus, in such case, we can consider also the negative powers β . Finally, thanks to the *reduction Formula*

$$\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x_N - y_N|^2 + |x' - y'|^2)^{\frac{N+2s}{2}}} = \frac{C_{N,2s}}{|x_N - y_N|^{1+2s}},$$

we decrease the dimension of the problem. The fact that such constant is not attained, follows from the statement in dimension 1.

Finally, we inquire into the *sharpness* of our inequality. At first, we show that the largest \mathfrak{h}_s is attained by the half-space \mathbb{H}_+^N among the class of convex sets. Here we pursue the geometric idea of [83] we mentioned before.

Then we compute the sharp constant of the half-space. To do this, we get the inspiration by the paper of R. Frank and R. Seiringer [53], where the authors proved the sharp fractional Hardy inequality $\mathbb{R}^N \setminus \{0\}$. We consider the dimension 1 and we compare the sharp constants of the interval and the half-line respectively. Finally, with a *reduction formula* similar to the one above, we generalize such a comparison between the half-space and the convex set to any dimensions.

Finally, by comparing $\mathfrak{h}_s(\Omega)$ with the explicit value of $\mathfrak{h}_s(\mathbb{H}_+^N)$, we fill the gap of the open problem with Ω different than an half-space and $s < 1/2$. Indeed, in this specific case, we are not able to show the Theorem 5 by constructing a positive local weak supersolution.

At last, we give a brief of what the reader will find in each chapter of the thesis.

III Plan of the thesis

Chapter 1 is devoted to introduce the fractional Sobolev space $\widetilde{W}_0^{s,2}(\Omega)$ and prove some of its properties needed throughout the whole manuscript. In the first section, we also collected some inequalities for functions in $\widetilde{W}_0^{s,2}(\Omega)$, for example an interpolation inequality (see Lemma 1.1.3), some Leibniz-type rules (see Lemma 1.1.7) and a fractional version of *Benguria’s hidden convexity principle* (see Lemma 1.1.8). At last, we point out the study of a special cut-off function, done in Lemma 1.1.9. The second section of this chapter is devoted to show the existence of a continuous extension operator for functions in $W^{s,2}(K)$, where K is a convex set in \mathbb{R}^N . The main novelty, with respect to statements already existing, is that we want to keep track of the dependency of the extension constant on s . Also, we need to know that the $W^{s,2}$ seminorm of the extension can be controlled in terms of the seminorm only. In the third section we show the *fractional Poincaré-Wirtinger inequality* that will be useful below. In the last section of this chapter we introduce the notion of *capacity*, and its fractional counterpart. The section ends with a Poincaré inequality for closed squares Q containing a “Dirichlet region” Σ .

Chapter 2 contains the proof of Theorem 1. In the last section of the chapter, we also collect some consequences of the main result. In particular, we introduce the *Cheeger constant* and the *fractional Cheeger constant* and we compare them with λ_1^s . Moreover, we show a comparison between λ_1^s and λ_1 raised to the power s .

In **Chapter 3** we focus on the case of planar *simply connected sets*. We essentially offer an alternative proof of Theorem 1 in the case $k = 1$, by using this time Hayman’s approach, which is more elementary than Taylor’s one. This result crucially uses a one-dimensional Poincaré inequality for periodic functions vanishing at a point, which is the first result of the same section.

In **Chapter 4** we show that our results are optimal, in a suitable sense. More precisely, we prove that our fractional version of the Croke-Osserman-Taylor inequality exhibits the correct asymptotic behaviour in the regime $s \nearrow 1$, $s \searrow 1/2$ and $k \nearrow \infty$. In particular, we can retrieve the classical Croke-Osserman-Taylor in the limit as s goes to 1, possibly with a worse constant. We also construct an open simply connected set in the plane, the *infinite complement comb*, having finite inradius and $\lambda_1^s = 0$, for every $0 < s \leq 1/2$.

Chapter 5 contains the proof of Theorem 4. To reach our aim, we need at first to introduce a further fractional Sobolev space, defined in terms of a weight function depending on the distance. We then study some of its properties, in particular we need to prove that it compactly embeds in $L^2(\Omega)$, under some minimal assumptions on Ω . In the second section of this chapter, Lemma 5.2.1 investigates the uniqueness of the minimizers for $\mathfrak{h}_s(\Omega)$, whenever they exist. Moreover, we prove in Proposition 5.2.4 that $\mathfrak{h}(\Omega)$ is certainly not attained, under some conditions on Ω . The last section is devoted to explain the *supersolution method*, and to show the proof of the main result of this chapter, which is based in two preliminary lemmas.

Chapter 6 is devoted to prove Theorem 5. We will need to distinguish the case of a half-space and that of a general open convex set. In this second case, a different proof would be needed, in order to cover the whole range $0 < s < 1$. The chapter ends with the construction of a counter-example which shows that negative powers of the distance are not supersolutions at the threshold $s = 1/2$: this suggests that the method of constructing “optimal” supersolution of the form d_Ω^β is bound to fail in the regime $2s < 1$ (where actually we need to use a different proof).

As anticipated during the introduction, this thesis contains two appendices, as well. In particular, in **Appendix A** we collect some miscellaneous stuff. Among others, we construct a

bi-Lipschitz homeomorphism from an open bounded convex set into a ball; we give an alternative expression of the constant $C_{N,2^s}$ which appears in the fractional Hardy inequality.

Appendix B contains the two geometric results needed in Chapters 2 and 3: the former is *Taylor's fatness lemma*, while the latter is *Hayman's covering lemma*.

Chapter 1

Fractional Sobolev spaces

1.1 Definitions and basic properties

As we have already mentioned in the introduction, this thesis is aimed at giving the fractional counterpart of some classical inequality. To do this, we need some definitions from the theory of fractional Sobolev spaces. We refer to [37, 43] for a brief introduction to these spaces.

Let $0 < s < 1$ and $1 < p < \infty$, for a measurable set $E \subseteq \mathbb{R}^N$ we recall the definition of the Gagliardo-Slobodeckii seminorm

$$[u]_{W^{s,p}(E)} := \left(\iint_{E \times E} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}, \quad \text{for } u \in L^1_{\text{loc}}(E).$$

Accordingly, we consider

$$W^{s,p}(E) = \left\{ u \in L^p(E) : [u]_{W^{s,p}(E)} < +\infty \right\},$$

endowed with the norm

$$\|u\|_{W^{s,p}(E)} = \|u\|_{L^p(E)} + [u]_{W^{s,p}(E)}, \quad \text{for every } u \in W^{s,p}(E).$$

Occasionally, we will need these definitions for $p = \infty$. For $0 < s < 1$, we set

$$W^{s,\infty}(E) = \left\{ u \in L^\infty(E) : [u]_{W^{s,\infty}(E)} < +\infty \right\},$$

where

$$[u]_{W^{s,\infty}(E)} := \sup_{x,y \in E, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^s}.$$

When $E \subseteq \mathbb{R}^N$ is an open set, we will also consider the classical Sobolev space

$$W^{1,p}(E) = \left\{ u \in L^p(E) : [u]_{W^{1,p}(E)} < +\infty \right\},$$

where we used the symbol

$$[u]_{W^{1,p}(E)} := \|\nabla u\|_{L^p(E)}, \quad \text{for every } u \in W^{1,p}(E).$$

The space $W^{1,p}(E)$ will be endowed with the norm

$$\|u\|_{W^{1,p}(E)} = \|u\|_{L^p(E)} + [u]_{W^{1,p}(E)}, \quad \text{for every } u \in W^{1,p}(E).$$

In the case $p = \infty$, the definition of this space does not need any further precision. Finally, for $0 < s \leq 1$ and $1 < p \leq \infty$, the symbol $\widetilde{W}_0^{s,p}(\Omega)$ will denote the closure of $C_0^\infty(\Omega)$ in the space $W^{s,p}(\mathbb{R}^N)$. By $W_{\text{loc}}^{s,p}(\mathbb{R}^N)$ we mean the collection of functions which are in $W^{s,p}(B_R)$, for every $R > 0$. The last notation can be generalized for any open set Ω in the following way: $W_{\text{loc}}^{s,p}(\Omega)$ is the space of functions $u \in L_{\text{loc}}^p(\Omega)$ such that $u \in W^{s,p}(\Omega')$ for every $\Omega' \Subset \Omega$.

For our purposes, we focus on the linear case, specifically with $p = 2$. Below, we will also consider the *Fourier Transform* of a function in $C_0^\infty(\Omega)$, whose properties are well explained in [70, Chapter VII, Section 1]. To clarify the link between fractional spaces and the Fourier Transform, we recall some basic notions on the latter.

For every $\varphi \in C_0^\infty(\mathbb{R}^N)$, the function $\mathcal{F}[\varphi]$ defined by

$$\mathcal{F}[\varphi](\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \varphi(x) e^{-i\langle x, \xi \rangle} dx, \quad \text{for } \xi \in \mathbb{R}^N,$$

is called *Fourier transform of φ* . From the inversion formula proved in [70, Chapter VII, Section 1], we can write

$$\varphi(x) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \mathcal{F}[\varphi](\xi) e^{i\langle x, \xi \rangle} d\xi, \quad \text{for } x \in \mathbb{R}^N. \quad (1.1.1)$$

Thus, an identity between the L^2 norms is valid. It is called the *Plancherel identity*, and it states that

$$\int_{\mathbb{R}^N} |\varphi|^2 dx = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} |\mathcal{F}[\varphi]|^2 d\xi$$

Moreover, if we consider the L^2 norm of the gradient, for each fixed $j = 1, \dots, N$ ¹ we integrate by parts and we get

$$\mathcal{F}[\nabla_j \varphi](\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \nabla_j \varphi(x) e^{-i\langle x, \xi \rangle} dx = -\frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} i \xi_j \varphi(x) e^{-i\langle x, \xi \rangle} dx, \quad (1.1.2)$$

where $\xi = (\xi_1, \dots, \xi_N)$. Hence

$$|\mathcal{F}[\nabla \varphi](\xi)|^2 = |\xi|^2 |\mathcal{F}[\varphi](\xi)|^2, \quad \text{for every } \xi \in \mathbb{R}^N,$$

and, from the Plancherel identity, it follows that

$$(2\pi)^N \int_{\mathbb{R}^N} |\nabla \varphi(x)|^2 dx = \int_{\mathbb{R}^N} |\xi|^2 |\mathcal{F}[\varphi]|^2 d\xi.$$

Moreover, from [70, Chapter VII, Section 9], for any $0 < s < 1$ we also have an identity for the Gagliardo-Slobodeckii seminorm, that is

$$(2\pi)^N A_s \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy = \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}[\varphi]|^2 d\xi.$$

The constant A_s is defined as

$$A_s = \left(\int_{\mathbb{R}} \frac{|e^{it} - 1|^2}{|t|^{1+2s}} dt \right)^{-1},$$

and it has the following asymptotic behaviour

$$A_s \sim 1 - s, \quad \text{for } s \nearrow 1 \quad \text{and} \quad A_s \sim s \quad \text{for } s \searrow 0.$$

¹Here, the writing $\nabla_j \varphi$ denotes the partial derivative $\frac{\partial \varphi}{\partial x_j}$ and $\langle x, \xi \rangle$ means the scalar product.

Moreover, we introduce a *weighted, local* space we will need in Chapter 5. For $0 < \beta < \infty$, we denote by $L_{2s}^\beta(\mathbb{R}^N)$ the following weighted Lebesgue space

$$L_{2s}^\beta(\mathbb{R}^N) = \left\{ u \in L_{\text{loc}}^\beta(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|^\beta}{(1+|x|)^{N+2s}} dx < +\infty \right\}. \quad (1.1.3)$$

We observe that this is a Banach space for $\beta \geq 1$, when endowed with the natural norm. It is not difficult to see that

$$L_{2s}^\beta(\mathbb{R}^N) \subset L_{2s}^{\beta'}(\mathbb{R}^N), \quad \text{for } 0 < \beta' < \beta < \infty. \quad (1.1.4)$$

It is sufficient to use that

$$\int_{\mathbb{R}^N} \frac{1}{(1+|x|)^{N+2s}} dx < +\infty, \quad \text{for every } N \geq 1, \text{ and } 0 < s < 1,$$

and then apply Jensen's inequality.

Finally, in Section 5.1, we will study the fractional Sobolev space $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ introduced in the same section.

Chapters 5 and 6 are devoted to explain how to prove an Hardy-type inequality by using the *method of supersolutions*. Here we introduce the definition of this notion.

Definition 1.1.1. *We say that $u \in W_{\text{loc}}^{s,2}(\Omega) \cap L_{2s}^1(\mathbb{R}^N)$ is a*

- *local weak supersolution of (II.7) if*

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \geq \lambda \int_{\Omega} \frac{u(x)}{d_\Omega(x)^{2s}} \varphi(x) dx, \quad (1.1.5)$$

for every non-negative $\varphi \in W^{s,2}(\mathbb{R}^N)$ with compact support in Ω ;

- *local weak solution of (II.7) if (1.1.5) holds as an equality, for every $\varphi \in W^{s,2}(\mathbb{R}^N)$ with compact support in Ω .*

Remark 1.1.2. *Under the assumptions taken on u and the test functions, the previous definition is well-posed, i.e.*

$$\frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N).$$

We also observe that if a local weak solution u belongs to $\widetilde{W}_0^{s,2}(\Omega)$, then by a density argument we can take $\varphi = u$ itself as a test function in the weak formulation.

The next result is a sort of interpolation-type inequality for smooth functions, proved in [B5]. It is useful in order to prove some Leibniz-type formulas in fractional Sobolev spaces.

Lemma 1.1.3. *Let $0 < s < 1$, then for every $\varphi \in C_0^1(\mathbb{R}^N)$ we have*

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dy \leq \frac{C}{s(1-s)} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^{2s} \|\varphi\|_{L^\infty(\mathbb{R}^N)}^{2(1-s)},$$

for some $C = C(N) > 0$.

Proof. We pick $\delta > 0$, then for $x \in \mathbb{R}^N$ we split the integral in two parts

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dy &= \int_{B_\delta(x)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus B_\delta(x)} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dy \\ &\leq \|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^2 \int_{B_\delta(x)} |x - y|^{2(1-s)-N} dy \\ &\quad + 4 \|\varphi\|_{L^\infty(\mathbb{R}^N)}^2 \int_{\mathbb{R}^N \setminus B_\delta(x)} |x - y|^{-N-2s} dy \\ &= \frac{N \omega_N}{2} \left(\frac{\|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^2}{1-s} \delta^{2(1-s)} + 4 \frac{\|\varphi\|_{L^\infty(\mathbb{R}^N)}^2}{s} \delta^{-2s} \right). \end{aligned}$$

By optimizing in $\delta > 0$, we get the desired result. \square

Now we prove some lemmas that will be useful below: we point out some sufficient conditions in order for a function to belong in $\widetilde{W}_0^{s,2}(\Omega)$.

The proof of the first is simply based on standard properties of convolutions. This shows in particular that in Definition 1.1.1 we can simply take $\varphi \in C_0^\infty(\Omega)$, considered to be 0 on the complement $\mathbb{R}^N \setminus \Omega$.

Lemma 1.1.4. *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set. If $\varphi \in W^{s,2}(\mathbb{R}^N)$ has compact support in Ω , then we have $\varphi \in \widetilde{W}_0^{s,2}(\Omega)$.*

Proof. We consider a nonnegative function $\rho \in C_0^\infty(\mathbb{R}^N)$ which support lies in B_1 . Without loss of generality we assume that

$$\int_{B_1} \rho dx = 1.$$

Moreover, for every $x \in \mathbb{R}^N$, we define the mollifier

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^N} \varphi\left(\frac{x}{\varepsilon}\right), \quad \text{where } \varepsilon > 0,$$

and we take the convolution product

$$\varphi_\varepsilon(x) := (\varphi * \rho_\varepsilon)(x), \quad \text{for every } x \in \mathbb{R}^N.$$

By construction φ_ε lies in $C^\infty(\mathbb{R}^N)$, but we have to check that its support is contained in Ω , and to do this it is useful to introduce the notation

$$\Omega_\delta := \left\{ x \in \Omega \text{ s.t. } \text{dist}(x, \partial\Omega) \geq \frac{\delta}{2} \right\}.$$

Due to the fact that φ is compactly supported in Ω , there exists $\delta > 0$ such that $\text{supp} \varphi \subseteq \Omega_\delta$. Thus, for every $x \notin \Omega_{\delta-2\varepsilon}$, it holds

$$\varphi_\varepsilon(x) = \int_{\mathbb{R}^N} \varphi(y) \rho_\varepsilon(x - y) dy = \int_{\Omega_\delta} \varphi(y) \rho_\varepsilon(x - y) dy = 0.$$

It is sufficient to consider $\varepsilon < \delta/2$ only.

Finally the strong convergence $\varphi_\varepsilon \rightarrow \varphi$ in $W^{s,2}(\mathbb{R}^N)$ is proved in [49, Lemma 11]. \square

Now we explicit a generalized *Leibniz-type rule*, which follows from the definition of the Gagliardo-Slobodeckii seminorm and the Minkowski's inequality.

Remark 1.1.5 (Leibniz-type rule). *For every $u, v \in W^{s,2}(E) \cap L^\infty(E)$ the following Leibniz-type rule holds*

$$[uv]_{W^{s,2}(E)} \leq [u]_{W^{s,2}(E)} \|v\|_{L^\infty(E)} + [v]_{W^{s,2}(E)} \|u\|_{L^\infty(E)}. \quad (1.1.6)$$

From now on, we indicate the volume of the k -dimensional open ball of radius 1 with ω_k . For $N \geq 2$, such a constant could be explicitly computed in terms of the Gamma function, see for example the proof of [52, Lemma 2.4]. For our purposes, this is not very important and we prefer not to appeal to such an explicit expression.

Lemma 1.1.6. *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set, then for every $\eta \in C_0^1(\Omega)$ and $u \in W_{\text{loc}}^{s,2}(\Omega)$, the function ηu is compactly supported in Ω and belongs to $W^{s,2}(\mathbb{R}^N)$. In particular, we have*

$$\eta u \in \widetilde{W}_0^{s,2}(\Omega).$$

Proof. We consider both η and u to be extended by 0 to $\mathbb{R}^N \setminus \Omega$. In light of Lemma 1.1.4, we only need to show that $\eta u \in W^{s,2}(\mathbb{R}^N)$. We take $\Omega'' \subseteq \Omega' \Subset \Omega$ such that the support of η is contained in Ω'' . Then we may write

$$\begin{aligned} [\eta u]_{W^{s,2}(\mathbb{R}^N)}^2 &= [\eta u]_{W^{s,2}(\Omega')}^2 + 2 \iint_{\Omega' \times (\mathbb{R}^N \setminus \Omega')} \frac{|\eta(x)u(x)|^2}{|x-y|^{N+2s}} dx dy \\ &= [\eta u]_{W^{s,2}(\Omega')}^2 + 2 \iint_{\Omega'' \times (\mathbb{R}^N \setminus \Omega')} \frac{|\eta(x)u(x)|^2}{|x-y|^{N+2s}} dx dy, \end{aligned}$$

where we used that η vanishes outside Ω'' . For the first term, by using Minkowski's inequality and Lemma 1.1.3, we can estimate it from above by means of the following Leibniz-type rule

$$\begin{aligned} [\eta u]_{W^{s,2}(\Omega')} &\leq \left(\int_{\Omega'} |u(x)|^2 \left(\int_{\Omega'} \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{\Omega'} |\eta(y)|^2 \left(\int_{\Omega'} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx \right) dy \right)^{\frac{1}{2}} \\ &\leq \left(\frac{C}{s(1-s)} \right)^{\frac{1}{2}} \|u\|_{L^2(\Omega')} \|\nabla \eta\|_{L^\infty(\mathbb{R}^N)}^s \|\eta\|_{L^\infty(\mathbb{R}^N)}^{1-s} + \|\eta\|_{L^\infty(\mathbb{R}^N)} [u]_{W^{s,2}(\Omega')} < +\infty. \end{aligned}$$

For the second term, we have

$$\begin{aligned} 2 \iint_{\Omega'' \times (\mathbb{R}^N \setminus \Omega')} \frac{|\eta(x)u(x)|^2}{|x-y|^{N+2s}} dx dy &\leq 2 \|\eta\|_{L^\infty(\mathbb{R}^N)}^2 \int_{\Omega''} |u(x)|^2 \left(\int_{\mathbb{R}^N \setminus \Omega'} \frac{dy}{|x-y|^{N+2s}} \right) dx \\ &\leq 2 \|\eta\|_{L^\infty(\mathbb{R}^N)}^2 \int_{\Omega''} |u(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_\mathfrak{d}(x)} \frac{dy}{|x-y|^{N+2s}} \right) dx \\ &= \frac{N \omega_N}{s} \frac{\|\eta\|_{L^\infty(\mathbb{R}^N)}^2}{\mathfrak{d}^{2s}} \int_{\Omega''} |u(x)|^2 dx < +\infty, \end{aligned}$$

where we set $\mathfrak{d} = \text{dist}(\Omega'', \partial\Omega') > 0$. □

The following technical result will be used in order to verify the sharpness of our constant in the fractional Hardy's inequality on the half-line $\mathbb{H}_+^1 = (0, +\infty)$ in Chapter 6.

Lemma 1.1.7. *Let $0 < s < 1$ and $M > 0$. We take $u \in W^{s,2}((0, M))$ and extend it by 0 outside $(0, M)$. We also suppose that there exist $C > 0$ and $\beta > (2s - 1)/2$ such that*

$$|u(x)| \leq C x^\beta, \quad \text{for a. e. } x \in (0, M).$$

Then for every $\eta \in C_0^\infty((-\infty, M))$, we have

$$u\eta \in \widetilde{W}_0^{s,2}(\mathbb{H}_+^1).$$

Moreover, the following estimates hold

$$[u\eta]_{W^{s,2}(\mathbb{R})}^2 \leq [u\eta]_{W^{s,2}((0,M))}^2 + \frac{1}{s} \int_0^M \frac{|u(x)\eta(x)|^2}{|x|^{2s}} dx + \frac{M^{2-2s}}{s} \|\eta'\|_{L^\infty(\mathbb{R})}^2 \|u\|_{L^2((0,M))}^2, \quad (1.1.7)$$

and

$$[u\eta]_{W^{s,2}((0,M))} \leq \|\eta\|_{L^\infty(\mathbb{R})} [u]_{W^{s,2}((0,M))} + \sqrt{\frac{C}{s(1-s)}} \|u\|_{L^2((0,M))} \|\eta'\|_{L^\infty(\mathbb{R})}^s \|\eta\|_{L^\infty(\mathbb{R})}^{1-s}, \quad (1.1.8)$$

for some $C > 0$.

Proof. We start by observing that $\widetilde{W}_0^{s,2}(\mathbb{H}_+^1)$ can be identified with the space of functions in $W^{s,2}(\mathbb{R})$ which vanish almost everywhere in $(-\infty, 0]$, thanks to [49, Theorem 6]. By construction, it is then sufficient to prove that $\eta u \in W^{s,2}(\mathbb{R})$.

It is easy to see that $u\eta \in L^2(\mathbb{R})$, hence let us focus on proving that $u\eta$ has a finite $W^{s,2}$ seminorm. By construction, this function vanishes almost everywhere outside $(0, M)$. We decompose the seminorm as follows

$$\begin{aligned} [u\eta]_{W^{s,2}(\mathbb{R})}^2 &= [u\eta]_{W^{s,2}((0,M))}^2 + 2 \int_0^M \int_{-\infty}^0 \frac{|u(x)\eta(x)|^2}{|x-y|^{1+2s}} dy dx + 2 \int_0^M \int_M^{+\infty} \frac{|u(x)\eta(x)|^2}{|x-y|^{1+2s}} dy dx \\ &= [u\eta]_{W^{s,2}((0,M))}^2 + \frac{1}{s} \int_0^M \frac{|u(x)\eta(x)|^2}{|x|^{2s}} dx + \frac{1}{s} \int_0^M \frac{|u(x)\eta(x)|^2}{(M-x)^{2s}} dx. \end{aligned}$$

In order to estimate the first term on the right-hand side, we proceed similarly as in the proof of Lemma 1.1.6, so to get

$$\begin{aligned} [u\eta]_{W^{s,2}((0,M))} &\leq \left[\int_0^M |u(x)|^2 \left(\int_0^M \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{1+2s}} dy \right) dx \right]^{\frac{1}{2}} \\ &\quad + \left[\int_0^M |\eta(y)|^2 \left(\int_0^M \frac{|u(x) - u(y)|^2}{|x-y|^{1+2s}} dx \right) dy \right]^{\frac{1}{2}} \\ &\leq \left(\frac{C}{s(1-s)} \right)^{\frac{1}{2}} \|u\|_{L^2((0,M))} \|\eta'\|_{L^\infty(\mathbb{R})}^s \|\eta\|_{L^\infty(\mathbb{R})}^{1-s} + \|\eta\|_{L^\infty((0,M))} [u]_{W^{s,2}((0,M))}. \end{aligned}$$

In the last inequality, we applied again Lemma 1.1.3. As for the other terms, we observe that

$$\int_0^M \frac{|u(x)\eta(x)|^2}{|x|^{2s}} dx,$$

is finite, thanks to the growth assumption on u . Finally, by using that $\eta \in C_0^\infty((-\infty, M))$, we have

$$|u(x)\eta(x)|^2 = |u(x)|^2 |\eta(x) - \eta(M)|^2 \leq \|\eta'\|_{L^\infty(\mathbb{R})}^2 |u(x)|^2 (M-x)^2, \quad \text{for a. e. } x \in (0, M),$$

so that we can infer

$$\int_0^M \frac{|u(x)\eta(x)|^2}{(M-x)^{2s}} dx \leq M^{2-2s} \|\eta'\|_{L^\infty(\mathbb{R})}^2 \int_0^M |u(x)|^2 dx < +\infty.$$

This completes the proof. \square

Due to the non-locality of the s -Laplace operator, it is not trivial to show the convexity property of a function. In the following result we show an hidden convexity for fractional Sobolev spaces which will be useful in Chapter 5.

Lemma 1.1.8 (Fractional hidden convexity). *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set, for every two non-negative functions $u, v \in \widetilde{W}_0^{s,2}(\Omega)$, we set*

$$\sigma = \left(\frac{1}{2} u^2 + \frac{1}{2} v^2 \right)^{\frac{1}{2}}.$$

Then $\sigma \in \widetilde{W}_0^{s,2}(\Omega)$ and there holds

$$[\sigma]_{W^{s,2}(\mathbb{R}^N)}^2 \leq \frac{1}{2} [u]_{W^{s,2}(\mathbb{R}^N)}^2 + \frac{1}{2} [v]_{W^{s,2}(\mathbb{R}^N)}^2. \quad (1.1.9)$$

Moreover, if equality holds in (1.1.9) and u, v are both positive almost everywhere in Ω , then there exists a constant C such that

$$u = C v, \quad \text{a. e. in } \Omega.$$

Proof. The proof of (1.1.9) and the identification of equality cases are contained in [54, Lemma 4.1 & Theorem 4.2]. We just show here that σ belongs to the relevant fractional Sobolev space, a fact that seems to have been overlooked in the literature. We first notice that

$$\int_{\Omega} |\sigma|^2 dx = \frac{1}{2} \int_{\Omega} u^2 dx + \frac{1}{2} \int_{\Omega} v^2 dx < +\infty,$$

and by (1.1.9) we have in particular

$$[\sigma]_{W^{s,2}(\mathbb{R}^N)} < +\infty.$$

This shows that $\sigma \in W^{s,2}(\mathbb{R}^N)$. We now consider $\{u_n\}_{n \in \mathbb{N}}, \{v_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\Omega)$ two sequences which converge respectively to u and v in $W^{s,2}(\mathbb{R}^N)$. Since u and v are positive, without loss of generality we can take u_n and v_n to be non-negative. Moreover, up to pass to a subsequence, we can suppose to have almost everywhere convergence.

We set

$$\sigma_n = \left(\frac{1}{2} \left(u_n + \frac{1}{n} \right)^2 + \frac{1}{2} \left(v_n + \frac{1}{n} \right)^2 \right)^{\frac{1}{2}} - \frac{1}{n}, \quad \text{for every } n \in \mathbb{N} \setminus \{0\},$$

and observe that $\{\sigma_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\Omega) \subseteq \widetilde{W}_0^{s,2}(\Omega)$. Moreover, σ_n converges to σ almost everywhere, as n goes to ∞ . We claim that

$$\lim_{n \rightarrow \infty} \|\sigma_n - \sigma\|_{L^2(\Omega)} = 0. \quad (1.1.10)$$

Indeed, thanks to Fatou's Lemma, it holds that

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |\sigma_n|^2 dx \geq \int_{\Omega} |\sigma|^2 dx.$$

Conversely, we observe that²

$$\sigma_n \leq \left(\frac{1}{2} u_n^2 + \frac{1}{2} v_n^2 \right)^{\frac{1}{2}}, \quad \text{for every } n \in \mathbb{N} \setminus \{0\}.$$

²This follows by noticing that the function

$$h(\varepsilon) = \left(\frac{1}{2} (a + \varepsilon)^2 + \frac{1}{2} (b + \varepsilon)^2 \right)^{\frac{1}{2}} - \varepsilon, \quad \text{for every } a, b \geq 0,$$

is monotone decreasing with respect to $\varepsilon \geq 0$, thus $h(\varepsilon) \leq h(0)$.

By raising to the square and taking the limit, we get

$$\limsup_{n \rightarrow \infty} \int_{\Omega} |\sigma_n|^2 dx \leq \limsup_{n \rightarrow \infty} \left[\frac{1}{2} \int_{\Omega} u_n^2 dx + \frac{1}{2} \int_{\Omega} v_n^2 dx \right] = \int_{\Omega} |\sigma|^2 dx.$$

These facts entail that we have convergence of the L^2 norms. By joining this with the almost everywhere convergence, we get (1.1.10) from the so-called *Brézis-Lieb Lemma* (see [27, Theorem 1]).

We also observe that $[\sigma_n]_{W^{s,2}(\mathbb{R}^N)}$ is bounded. Indeed, we can apply the convexity inequality (1.1.9) as follows

$$\begin{aligned} [\sigma_n]_{W^{s,2}(\mathbb{R}^N)}^2 &= \left[\sigma_n + \frac{1}{n} \right]_{W^{s,2}(\mathbb{R}^N)}^2 \leq \frac{1}{2} \left[u_n + \frac{1}{n} \right]_{W^{s,2}(\mathbb{R}^N)}^2 + \frac{1}{2} \left[v_n + \frac{1}{n} \right]_{W^{s,2}(\mathbb{R}^N)}^2 \\ &= \frac{1}{2} [u_n]_{W^{s,2}(\mathbb{R}^N)}^2 + \frac{1}{2} [v_n]_{W^{s,2}(\mathbb{R}^N)}^2, \end{aligned}$$

and observe that the last terms are uniformly bounded, by construction. The uniform bound on $\|\sigma_n\|_{W^{s,2}(\mathbb{R}^N)}$ and the reflexivity of the space $W^{s,2}(\mathbb{R}^N)$ entail that σ_n weakly converges, up to subsequences, to a function in $\widetilde{W}_0^{s,2}(\Omega)$, the latter being a weakly closed subspace of $W^{s,2}(\mathbb{R}^N)$. By the uniqueness of the limit, such a function must coincide with σ , which then belongs to $\widetilde{W}_0^{s,2}(\Omega)$. \square

The following lemma will be useful to construct a sequence of properly test functions in the proof of point 3 of Theorem 3. From now on, with $(\cdot)_+$ we mean the *positive part*, that is

$$(\cdot)_+ : \mathbb{R} \rightarrow [0, +\infty), \quad (t)_+ = \begin{cases} t, & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases} \quad (1.1.11)$$

Lemma 1.1.9 (A special cut-off function). *Let $1/2 < s < 1$ and let*

$$\zeta_s(x) = \left(1 - |x|^{2s-1} \right)_+, \quad \text{for } x \in \mathbb{R}.$$

Then we have

$$[\zeta_s]_{W^{s,2}(\mathbb{R})} \leq C \frac{\sqrt{2s-1}}{\sqrt{1-s}}, \quad (1.1.12)$$

with a constant $C > 0$ independent of $s \in (1/2, 1)$.

Proof. We decompose the seminorm as follows

$$\begin{aligned} [\zeta_s]_{W^{s,2}(\mathbb{R})}^2 &= \iint_{(-1,1) \times (-1,1)} \frac{\left| |x|^{2s-1} - |y|^{2s-1} \right|^2}{|x-y|^{1+2s}} dx dy \\ &\quad + \frac{1}{s} \int_{-1}^1 \frac{\left| 1 - |x|^{2s-1} \right|^2}{(1-x)^{2s}} dx + \frac{1}{s} \int_{-1}^1 \frac{\left| 1 - |x|^{2s-1} \right|^2}{(1+x)^{2s}} dx = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

In order to prove (1.1.12), we will prove that

$$\mathcal{I}_i \leq C \frac{2s-1}{1-s}, \quad \text{for } i = 1, 2, 3. \quad (1.1.13)$$

For the first term \mathcal{I}_1 , we observe that by using the symmetry of the set and of the integrand, we have

$$\mathcal{I}_1 \leq 4 \iint_{(0,1) \times (0,1)} \frac{\left| |x|^{2s-1} - |y|^{2s-1} \right|^2}{|x-y|^{1+2s}} dx dy.$$

By using (6.1.3) we will prove below with the choice $\beta = 2s - 1$ there, we can estimate the last double integral as follows

$$\mathcal{I}_1 \leq 4 \iint_{(0,1) \times (0,1)} \frac{\left| |x|^{2s-1} - |y|^{2s-1} \right|^2}{|x-y|^{1+2s}} dx dy \leq \left(\int_0^1 \frac{|1 - \tau^{2s-1}|^2}{|1 - \tau|^{1+2s}} (1 + \tau^{1-2s}) d\tau \right) \frac{4}{2s-1}.$$

We then write

$$\begin{aligned} \int_0^1 \frac{|1 - \tau^{2s-1}|^2}{|1 - \tau|^{1+2s}} (1 + \tau^{1-2s}) d\tau &= \int_0^{\frac{1}{2}} \frac{|1 - \tau^{2s-1}|^2}{|1 - \tau|^{1+2s}} (1 + \tau^{1-2s}) d\tau \\ &\quad + \int_{\frac{1}{2}}^1 \frac{|1 - \tau^{2s-1}|^2}{|1 - \tau|^{1+2s}} (1 + \tau^{1-2s}) d\tau \\ &\leq C \int_0^{\frac{1}{2}} |1 - \tau^{2s-1}|^2 (1 + \tau^{1-2s}) d\tau \\ &\quad + C \int_{\frac{1}{2}}^1 \frac{|1 - \tau^{2s-1}|^2}{|1 - \tau|^{1+2s}} d\tau =: \mathcal{I}_{1,1} + \mathcal{I}_{1,2}. \end{aligned}$$

The constant $C > 0$ can be taken independent of s . We start by estimating $\mathcal{I}_{1,2}$, which is simpler: we use the following pointwise inequality

$$a^\alpha - b^\alpha \leq \alpha b^{\alpha-1} (a - b), \quad \text{for } 0 < b \leq a, 0 < \alpha < 1,$$

which just follows from concavity of the map $\tau \mapsto \tau^\alpha$. This gives

$$\mathcal{I}_{1,2} \leq C 16^{1-s} (2s-1)^2 \int_{\frac{1}{2}}^1 (1 - \tau)^{1-2s} d\tau = C \frac{4^{1-s}}{2(1-s)} (2s-1)^2,$$

as desired. We now come to $\mathcal{I}_{1,1}$, which is the most subtle. We have to distinguish two cases: $1/2 < s < 3/4$ and $3/4 \leq s < 1$. In the first case, we set for simplicity

$$f_\tau(s) = \tau^{2s-1}, \quad \text{for } \tau > 0, s > \frac{1}{2}.$$

Then we have

$$\left| f_\tau(s) - f_\tau\left(\frac{1}{2}\right) \right| = \left| \int_{\frac{1}{2}}^s f'_\tau(t) dt \right|,$$

that is for $0 < \tau \leq 1/2$

$$|1 - \tau^{2s-1}| = 2 |\log \tau| \left| \int_{\frac{1}{2}}^s \tau^{2t-1} dt \right| \leq 2 (-\log \tau) \left(s - \frac{1}{2} \right) = (-\log \tau) (2s-1).$$

Thus we get for $1/2 < s < 3/4$

$$\mathcal{I}_{1,1} \leq C (2s-1)^2 \int_0^{\frac{1}{2}} (-\log \tau)^2 (1 + \tau^{1-2s}) d\tau \leq 2C (2s-1)^2 \int_0^{\frac{1}{2}} (-\log \tau)^2 \frac{d\tau}{\sqrt{\tau}}. \quad (1.1.14)$$

This gives the desired estimate for $1/2 < s < 3/4$, since the last integral is finite and independent of s . On the other hand, for $3/4 \leq s < 1$, we can simply estimate

$$\mathcal{I}_{1,1} \leq C \int_0^{\frac{1}{2}} (1 + \tau^{1-2s}) d\tau \leq 2C \int_0^{\frac{1}{2}} \tau^{1-2s} d\tau = \frac{C}{1-s} \left(\frac{1}{2}\right)^{2-2s} \leq \frac{1}{1-s}. \quad (1.1.15)$$

In particular, we get from (1.1.14) and (1.1.15)

$$\mathcal{I}_{1,1} \leq C \frac{(2s-1)^2}{1-s}, \quad \text{for } \frac{1}{2} < s < 1,$$

possibly for a different $C > 0$, still independent of s . By collecting the estimates for $\mathcal{I}_{1,1}$ and $\mathcal{I}_{1,2}$, we thus get (1.1.12) for \mathcal{I}_1 .

We now consider \mathcal{I}_2 and \mathcal{I}_3 . We only estimate the first one, since the estimate for the second one is similar. For $s > 1/2$, we have

$$\begin{aligned} \frac{1}{s} \int_{-1}^1 \frac{|1 - |x|^{2s-1}|^2}{(1-x)^{2s}} dx &\leq 2 \int_0^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + 2 \int_{-1}^0 \frac{(1-|x|^{2s-1})^2}{(1-x)^{2s}} dx \\ &\leq 2 \int_0^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + 2 \int_{-1}^0 (1-|x|^{2s-1})^2 dx \\ &= 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + 2 \int_0^{\frac{1}{2}} \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx \\ &\quad + 2 \int_0^1 (1-x^{2s-1})^2 dx \\ &\leq 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + 2 \cdot 4^s \int_0^{\frac{1}{2}} (1-x^{2s-1})^2 dx \\ &\quad + 2 \int_0^1 (1-x^{2s-1})^2 dx \\ &\leq 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + 2(4^s + 1) \int_0^1 (1-x^{2s-1}) dx. \end{aligned}$$

By computing the last integral, this gives in particular

$$\frac{1}{s} \int_{-1}^1 \frac{|1 - |x|^{2s-1}|^2}{(1-x)^{2s}} dx \leq 2 \int_{\frac{1}{2}}^1 \frac{(1-x^{2s-1})^2}{(1-x)^{2s}} dx + (4^s + 1) \frac{2s-1}{s}.$$

At this point, the integral in the right-hand side can be estimated as we did for $\mathcal{I}_{1,2}$ above. By proceeding as before, we get (1.1.13) for \mathcal{I}_2 (and thus for \mathcal{I}_3), as well. \square

1.2 An extension operator

In the following technical result, we explicitly construct a continuous extension operator for fractional Sobolev spaces defined on a ball. The result is certainly well-known (see for example [37, Theorem 5.4]), but here we pay particular attention to the constant appearing in the continuity estimates (1.2.1) and (1.2.2) below: indeed, these can be taken to be independent of the differentiability index s .

Lemma 1.2.1. *Let $r > 0$ and $x_0 \in \mathbb{R}^N$, there exists a linear extension operator*

$$\mathcal{E}_r : L^1(B_r(x_0)) \rightarrow L^1_{\text{loc}}(\mathbb{R}^N),$$

with the following property: for $0 < s \leq 1$ and $1 < p \leq \infty$ it maps $W^{s,p}(B_r(x_0))$ to $W^{s,p}_{\text{loc}}(\mathbb{R}^N)$. Moreover, for every $u \in W^{s,p}(B_r(x_0))$ and every $R > r$ we have³

$$\left[\mathcal{E}_r[u] \right]_{W^{s,p}(B_R(x_0))} \leq 4^{\frac{1}{p}} \left(\frac{R}{r} \right)^{\frac{4N}{p}} [u]_{W^{s,p}(B_r(x_0))}, \quad (1.2.1)$$

and

$$\left\| \mathcal{E}_r[u] \right\|_{L^p(B_R(x_0))} \leq 2^{\frac{1}{p}} \left(\frac{R}{r} \right)^{\frac{2N}{p}} \|u\|_{L^p(B_r(x_0))}. \quad (1.2.2)$$

Proof. We first prove the result at scale 1, i.e. when $r = 1$. Then we will show how to get the general result, by an easy scaling argument.

Case $r = 1$. We first consider $0 < s < 1$ and $1 < p < \infty$. The limiting cases $s = 1$ and $p = \infty$ are proved by taking the limits.

Without loss of generality, we can suppose that x_0 coincides with the origin. Then, let us recall the definition of *inversion with respect to \mathbb{S}^{N-1}* : this is the bijection $\mathcal{K} : \mathbb{R}^N \setminus \{0\} \rightarrow \mathbb{R}^N \setminus \{0\}$, given by

$$\mathcal{K}(x) = \frac{x}{|x|^2}, \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}.$$

It is easily seen that if $x \in B_R \setminus B_1$, then $\mathcal{K}(x) \in B_1 \setminus B_{1/R}$. Moreover, we have

$$\mathcal{K}^{-1}(x) = \mathcal{K}(x) \quad \text{and} \quad |\det(D\mathcal{K}(x))| = \frac{1}{|x|^{2N}}, \quad \text{for every } x \in \mathbb{R}^N \setminus \{0\}.$$

For every $u \in W^{s,2}(B_1)$, we define the extended function $\mathcal{E}_1[u]$ given by

$$\mathcal{E}_1[u](x) = \begin{cases} u(x), & \text{if } x \in B_1, \\ u(\mathcal{K}(x)) & \text{if } x \in \mathbb{R}^N \setminus B_1. \end{cases} \quad (1.2.3)$$

It is easily seen that the operator $u \mapsto \mathcal{E}_1[u]$ is linear. In order to prove that $\mathcal{E}_1[u] \in W^{s,p}_{\text{loc}}(\mathbb{R}^N)$, together with the claimed estimate (1.2.1), we take $R > 1$ and we split the seminorm of $\mathcal{E}_1[u]$ as follows

$$\begin{aligned} \left[\mathcal{E}_1[u] \right]_{W^{s,p}(B_R)}^p &= [u]_{W^{s,p}(B_1)}^p \\ &+ \iint_{(B_R \setminus B_1) \times (B_R \setminus B_1)} \frac{|u(\mathcal{K}(x)) - u(\mathcal{K}(y))|^p}{|x - y|^{N+sp}} dx dy \\ &+ 2 \iint_{B_1 \times (B_R \setminus B_1)} \frac{|u(x) - u(\mathcal{K}(y))|^p}{|x - y|^{N+sp}} dx dy. \end{aligned}$$

By performing the change of variable $z = \mathcal{K}(x)$ in the second term on the right-hand side and the change of variable $w = \mathcal{K}(y)$ in the second and third terms, we get

$$\begin{aligned} \left[\mathcal{E}_1[u] \right]_{W^{s,p}(B_R)}^p &= [u]_{W^{s,p}(B_1)}^p \\ &+ \iint_{(B_1 \setminus B_{\frac{1}{R}}) \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(z) - u(w)|^p}{|\mathcal{K}^{-1}(z) - \mathcal{K}^{-1}(w)|^{N+sp}} |\det D\mathcal{K}^{-1}(z)| |\det D\mathcal{K}^{-1}(w)| dz dw \\ &+ 2 \int_{B_1 \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(x) - u(w)|^p}{|x - \mathcal{K}^{-1}(w)|^{N+sp}} |\det D\mathcal{K}^{-1}(w)| dx dw. \end{aligned}$$

³In the case $p = \infty$, we use the convention $1/\infty = 0$.

By using the expression for the Jacobian determinant, we then obtain

$$\begin{aligned} \left[\mathcal{E}_1[u] \right]_{W^{s,p}(B_R)}^p &\leq [u]_{W^{s,p}(B_1)}^p \\ &\quad + R^{4N} \iint_{(B_1 \setminus B_{\frac{1}{R}}) \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(z) - u(w)|^p}{|\mathcal{K}^{-1}(z) - \mathcal{K}^{-1}(w)|^{N+sp}} dz dw \\ &\quad + 2R^{2N} \iint_{B_1 \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(x) - u(w)|^p}{|x - \mathcal{K}^{-1}(w)|^{N+sp}} dx dw. \end{aligned} \quad (1.2.4)$$

In order to estimate the last two integrals, it is sufficient to use that

$$|\mathcal{K}^{-1}(z) - \mathcal{K}^{-1}(w)| = \left| \frac{1}{|z|^2} z - \frac{1}{|w|^2} w \right| \geq |z - w|, \quad \text{for every } z, w \in B_1 \setminus \{0\}, \quad (1.2.5)$$

and

$$|x - \mathcal{K}^{-1}(w)| = \left| x - \frac{1}{|w|^2} w \right| \geq |x - w|, \quad \text{for every } x, w \in B_1 \setminus \{0\}. \quad (1.2.6)$$

Indeed, by taking the square, we see that (1.2.5) is equivalent to

$$\left(\frac{1}{|z|^2} - |z|^2 \right) + \left(\frac{1}{|w|^2} - |w|^2 \right) \geq 2 \left(\frac{1}{|z|^2 |w|^2} - 1 \right) \langle z, w \rangle.$$

This in turn follows from Young's inequality

$$2 \langle z, w \rangle \leq |z|^2 + |w|^2,$$

once we multiply both sides by the positive quantity

$$\left(\frac{1}{|z|^2 |w|^2} - 1 \right).$$

As for inequality (1.2.6), by taking again the square we see that the latter is equivalent to

$$\frac{1}{|w|^2} - |w|^2 \geq 2 \left(\frac{1}{|w|^2} - 1 \right) \langle x, w \rangle. \quad (1.2.7)$$

This in turn follows again from Young's inequality: more precisely, by using that $|x| < 1$, we have

$$2 \langle x, w \rangle \leq |x|^2 + |w|^2 \leq 1 + |w|^2,$$

and if we now multiply both sides by the positive quantity (here we use that $|w| < 1$)

$$\left(\frac{1}{|w|^2} - 1 \right),$$

we get (1.2.7), with some simple algebraic manipulations.

By applying the estimates (1.2.5) and (1.2.6) in (1.2.4), we finally get

$$\begin{aligned} \left[\mathcal{E}_1[u] \right]_{W^{s,p}(B_R)}^p &\leq [u]_{W^{s,p}(B_1)}^p \\ &\quad + R^{4N} \iint_{(B_1 \setminus B_{\frac{1}{R}}) \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(z) - u(w)|^p}{|z - w|^{N+sp}} dz dw \\ &\quad + 2R^{2N} \iint_{B_1 \times (B_1 \setminus B_{\frac{1}{R}})} \frac{|u(x) - u(w)|^p}{|x - w|^{N+sp}} dx dw \\ &\leq (1 + R^{4N} + 2R^{2N}) [u]_{W^{s,p}(B_1)}^p, \end{aligned}$$

which proves the estimate (1.2.1) for $0 < s < 1$.

The estimate on the L^p norm of $\mathcal{E}_1[u]$ is simpler and can be done as follows

$$\begin{aligned} \int_{B_R} |\mathcal{E}_1[u](x)|^p dx &= \int_{B_1} |u(x)|^p dx + \int_{B_R \setminus B_1} |u(\mathcal{K}(x))|^p dx \\ &\leq \int_{B_1} |u(x)|^p dx + R^{2N} \int_{B_1 \setminus B_{\frac{1}{R}}} |u(z)|^p dz \leq (1 + R^{2N}) \int_{B_1} |u(x)|^p dx. \end{aligned}$$

Hence we get the assertion (1.2.2) for $0 < s < 1$.

We now come to the case $s = 1$ and $1 < p < \infty$. We take $u \in W^{1,p}(B_1(x_0))$, thus by [43, Proposition 3.1] we have $u \in W^{s,p}(B_1(x_0))$ for every $0 < s < 1$, as well. From the previous step, we know that

$$(1-s)^{\frac{1}{p}} \left[\mathcal{E}_1[u] \right]_{W^{s,p}(B_R(x_0))} \leq 4^{\frac{1}{p}} R^{\frac{4N}{p}} (1-s)^{\frac{1}{p}} [u]_{W^{s,p}(B_1(x_0))}.$$

By using [15, Theorem 2], we get the desired result by taking the limit as s goes to 1, that is

$$\left[\mathcal{E}_1[u] \right]_{W^{1,p}(B_R(x_0))} \leq 4^{\frac{1}{p}} R^{\frac{4N}{p}} [u]_{W^{1,p}(B_1(x_0))}.$$

Finally, the case $p = \infty$ can be obtained from the last formula in display, by taking the limit as p goes to ∞ .

Case $r \neq 1$. At first, we need a notation. For every $\tau > 0$, we indicate by

$$\mathcal{T}_\tau(x) = \tau(x - x_0) + x_0, \quad \text{for every } x \in \mathbb{R}^N.$$

Then the operator \mathcal{E}_r can be simply defined as

$$\mathcal{E}_r[u] := (\mathcal{E}_1[u \circ \mathcal{T}_r]) \circ \mathcal{T}_{\frac{1}{r}}.$$

In other words, given a function $u \in L^1(B_r(x_0))$, we first scale it to a function defined on $B_1(x_0)$, then extend it with \mathcal{E}_1 and finally scale back this extension. Observe that for $x \in B_r(x_0)$, we have

$$\mathcal{E}_r[u](x) = \mathcal{E}_1[u \circ \mathcal{T}_r] \left(\frac{x - x_0}{r} + x_0 \right) = u \left(\mathcal{T}_r \left(\frac{x - x_0}{r} + x_0 \right) \right) = u(x).$$

By using the scaling properties of the norms involved, it is easy to see that this operator has the desired properties. \square

Remark 1.2.2. *Another important feature of the previous result is that, rather than the usual continuity estimate*

$$\| \mathcal{E}_r[u] \|_{W^{s,2}(B_R)} \leq C \| u \|_{W^{s,2}(B_1)},$$

for the extension operator, we obtained the more precise estimates (1.2.1) and (1.2.2).

By combining Lemma 1.2.1 with the Lemma A.2.1 in the appendix, we can get a universal linear extension operator for any $K \subseteq \mathbb{R}^N$ open bounded convex set. The control on the relevant constants is quite precise and useful for our scopes. In what follows, for every $x_0 \in K$, we introduce the following geometric quantities

$$d_K(x_0) = \min_{x \in \partial K} |x - x_0|, \quad D_K(x_0) = \max_{x \in \partial K} |x - x_0|.$$

Corollary 1.2.3. *Let $K \subseteq \mathbb{R}^N$ be an open bounded convex set and $x_0 \in K$, there exists a linear extension operator*

$$\mathcal{E}_K : L^1(K) \rightarrow L^1_{\text{loc}}(\mathbb{R}^N),$$

with the following property:

for $0 < s \leq 1$ and $1 < p \leq \infty$ it maps $W^{s,p}(K)$ to $W^{s,p}_{\text{loc}}(\mathbb{R}^N)$. Moreover, for every $u \in W^{s,p}(K)$ and every $R > 1$ we have

$$\left[\mathcal{E}_K[u] \right]_{W^{s,p}(K_R(x_0))} \leq (4 \cdot 6^{3N+sp})^{\frac{1}{p}} R^{\frac{4N}{p}} \left(\frac{D_K(x_0)}{d_K(x_0)} \right)^{\frac{6N}{p}+2s} [u]_{W^{s,p}(K)}, \quad (1.2.8)$$

and

$$\| \mathcal{E}_K[u] \|_{L^p(K_R(x_0))} \leq (2 \cdot 6^N)^{\frac{1}{p}} R^{\frac{2N}{p}} \left(\frac{D_K(x_0)}{d_K(x_0)} \right)^{\frac{2N}{p}} \|u\|_{L^p(K)}, \quad (1.2.9)$$

where

$$K_R(x_0) := R(K - x_0) + x_0 = \left\{ R(x - x_0) + x_0 : x \in K \right\}.$$

Proof. The operator \mathcal{E}_K is constructed as follows: by indicating with $\Phi_{K,x_0} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ the bi-Lipschitz homeomorphism of Lemma A.2.1, for every $u \in L^1_{\text{loc}}(K)$, we define

$$\mathcal{E}_K[u] := \left(\mathcal{E}_1[u \circ \Phi_{K,x_0}^{-1}] \right) \circ \Phi_{K,x_0},$$

where \mathcal{E}_1 is the operator of Lemma 1.2.1 with $r = 1$. In other words, we transplant u to the unit ball centered at x_0 , then we extend this function to the whole \mathbb{R}^N by means of \mathcal{E}_1 and finally compose the resulting function with Φ_{K,x_0} .

By construction, it is clear that \mathcal{E}_K is linear and such that

$$\mathcal{E}_K[u](x) = u(x), \quad \text{for } x \in K.$$

The continuity estimates (1.2.8) and (1.2.9) can now be proved from the corresponding estimates for \mathcal{E}_1 , by using the properties of Φ_{K,x_0} and Φ_{K,x_0}^{-1} encoded by Lemma A.2.1. Indeed, by writing explicitly the operator \mathcal{E}_K we compute

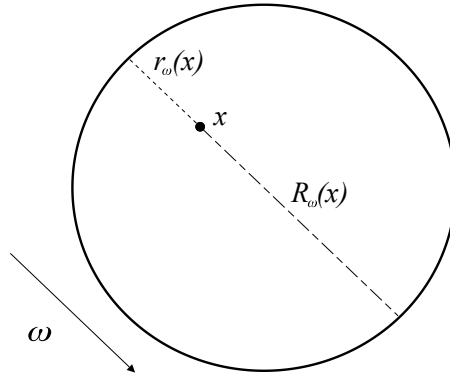
$$\begin{aligned} \left[\mathcal{E}_K[u] \right]_{W^{s,p}(K_R(x_0))} &\leq M_K L_K^{\frac{2N}{p}} \left[\mathcal{E}_1 \left[u \circ \Phi_{K,x_0}^{-1} \right] \right]_{W^{s,p}(B_R(x_0))} \\ &\leq 4^{\frac{1}{p}} M_K L_K^{\frac{N}{p}} \left(\frac{R}{r} \right)^{\frac{4N}{p}} \left[u \circ \Phi_{K,x_0}^{-1} \right]_{W^{s,p}(B_r(x_0))} \\ &\leq 4^{\frac{1}{p}} M_K^{1+\frac{2N}{p}} L_K^{\frac{2N}{p}+1} \left(\frac{R}{r} \right)^{\frac{4N}{p}} [u]_{W^{s,p}(K)}. \end{aligned}$$

The inequality (1.2.8) follows by recalling the definitions of L_K and M_K introduced in Lemma A.2.1.

Finally we prove the continuity (1.2.9) similarly to what we did for $[\mathcal{E}_K]_{W^{s,p}(K_R(x_0))}$. Indeed it holds that

$$\begin{aligned} \left\| \mathcal{E}_K[u] \right\|_{L^p(K_R(x_0))} &\leq M_K L_K^{\frac{N}{p}} \left\| \mathcal{E}_1 \left[u \circ \Phi_{K,x_0}^{-1} \right] \right\|_{L^p(B_R(x_0))} \\ &\leq 2^{\frac{1}{p}} M_K L_K^{\frac{N}{p}} \left(\frac{R}{r} \right)^{\frac{2N}{p}} \left\| u \circ \Phi_{K,x_0}^{-1} \right\|_{L^p(B_r(x_0))} \\ &\leq 2^{\frac{1}{p}} M_K^{1+\frac{N}{p}} L_K^{\frac{N}{p}+1} \left(\frac{R}{r} \right)^{\frac{2N}{p}} \|u\|_{L^p(K)}, \end{aligned}$$

and we get the claimed (1.2.9), thanks to the definitions of M_K and L_K . \square

Figure 1.1: The two quantities $R_\omega(x)$ and $r_\omega(x)$.

1.3 A Poincaré-Wirtinger inequality

In what follows, given a ball $B_r(x_0) \subseteq \mathbb{R}^N$, a point $x \in B_r(x_0)$ and a direction $\omega \in \mathbb{S}^{N-1}$, we set

$$R_\omega(x) = \sup \left\{ \varrho \in \mathbb{R} : x + \varrho\omega \in B_r(x_0) \right\},$$

and

$$r_\omega(x) = \inf \left\{ \varrho \in \mathbb{R} : x + \varrho\omega \in B_r(x_0) \right\},$$

see Figure 1.1.

The notations above make the next proposition more readable: in such result we fix a direction $\omega \in \mathbb{S}^{N-1}$ and we bound the “regional seminorm” valued on a ball $B_r(x_0) \subseteq \mathbb{R}^N$ in terms of the “regional seminorm” valued on an interval of \mathbb{R} . This result is useful in particular when we reduce the dimension of the ambient space. The idea of its proof is to fix a direction and to bound from below the seminorm on the ball by considering the points in that direction only. To do this we use an interpolation technique: namely we consider the K -functional studied by J. Bergh and J. Löfström in [9, Section 3.1], which allows us to compare the fractional “regional seminorms” above with the classical $W^{1,2}$ -norm. Hence we can use some trivial local inequality, such as $|\partial_\omega \tilde{v}| \leq |\nabla \tilde{v}|$.

Proposition 1.3.1 (Directional fractional derivatives). *Let $0 < s < 1$ and $r > 0$, for every $u \in C^1(\overline{B_r(x_0)})$ and every $\omega \in \mathbb{S}^{N-1}$, we have*

$$\int_{B_r(x_0)} \left(\int_{r_\omega(x)}^{R_\omega(x)} \frac{|u(x) - u(x + \varrho\omega)|^2}{|\varrho|^{1+2s}} d\varrho \right) dx \leq \mathcal{A} [u]_{W^{s,2}(B_r(x_0))}^2, \quad (1.3.1)$$

for some $\mathcal{A} = \mathcal{A}(N) > 0$.

Proof. Without loss of generality, we can assume that x_0 coincides with the origin. We use Lemma 1.2.1 and in particular the estimate (1.2.1) with $R = 4r$, so to get

$$\begin{aligned} \iint_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy &\geq \frac{1}{C} \iint_{B_{4r} \times B_{4r}} \frac{|\mathcal{E}_r[u](x) - \mathcal{E}_r[u](y)|^2}{|x - y|^{N+2s}} dx dy \\ &\geq \frac{1}{C} \int_{B_r} \left(\int_{B_{2r}(x)} \frac{|\mathcal{E}_r[u](x) - \mathcal{E}_r[u](y)|^2}{|x - y|^{N+2s}} dy \right) dx \\ &= \frac{1}{C} \iint_{B_r \times B_{2r}} \frac{|\mathcal{E}_r[u](x) - \mathcal{E}_r[u](x+h)|^2}{|h|^{N+2s}} dx dh, \end{aligned} \quad (1.3.2)$$

where C only depends on the dimension N . In the last identity, we used the change of variable $y = x + h$.

From now on, we will write \tilde{u} in place of $\mathcal{E}_r[u]$, for notational simplicity. We then introduce the following K -functional

$$K(t, u) = \min_{v \in W^{1,2}(B_r)} \left[\|u - v\|_{L^2(B_r)} + t [v]_{W^{1,2}(B_r)} \right], \quad \text{for } t \in [0, 2r]. \quad (1.3.3)$$

We claim that the following two estimates hold: there exist two constants $A_1, A_2 > 0$ depending on the dimension N only, such that

$$\int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t} \leq A_1 \iint_{B_r \times B_{2r}} \frac{|\tilde{u}(x) - \tilde{u}(x+h)|^2}{|h|^{N+2s}} dx dh, \quad (1.3.4)$$

and

$$\int_{B_r} \left(\int_{-2r}^{2r} \frac{|\tilde{u}(x) - \tilde{u}(x+\varrho\omega)|^2}{|\varrho|^{1+2s}} d\varrho \right) dx \leq A_2 \int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t}, \quad \text{for every } \omega \in \mathbb{S}^{N-1}. \quad (1.3.5)$$

Observe that by joining (1.3.2), (1.3.4) and (1.3.5), we would get

$$\iint_{B_r \times B_r} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \geq \frac{1}{C \cdot A_1 \cdot A_2} \int_{B_r} \left(\int_{-2r}^{2r} \frac{|\tilde{u}(x) - \tilde{u}(x+\varrho\omega)|^2}{|\varrho|^{1+2s}} d\varrho \right) dx,$$

and thus the desired conclusion (1.3.1) would follow, once observed that $R_\omega(x) \leq 2r$ and $r_\omega(x) \geq -2r$, together with the fact that $\tilde{u} = u$ on B_r . Thus we are left with establishing the validity of both (1.3.4) and (1.3.5).

In order to prove (1.3.4), we proceed exactly as in the proof of [25, Proposition 4.5], up to some necessary modifications. At first, it is useful to define

$$U(h) = \left(\int_{B_r} |\tilde{u}(x+h) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{2}}, \quad h \in B_{2r}.$$

Thus, by definition, the right-hand side of (1.3.4) can be rewritten as

$$\iint_{B_r \times B_{2r}} \frac{|\tilde{u}(x) - \tilde{u}(x+h)|^2}{|h|^{N+2s}} dx dh = \int_{B_{2r}} \frac{U(h)^2}{|h|^{N+2s}} dh.$$

We also define

$$\bar{U}(\varrho) = \int_{\partial B_\varrho} U d\mathcal{H}^{N-1}, \quad \text{for } 0 < \varrho \leq 2r.$$

By Jensen's inequality we obtain

$$\begin{aligned} \int_0^{2r} \bar{U}^2 \frac{d\varrho}{\varrho^{1+2s}} &\leq \frac{1}{N \omega_N} \int_0^{2r} \left(\int_{\partial B_\varrho} U^2 d\mathcal{H}^{N-1} \right) \frac{d\varrho}{\varrho^{N+2s}} \\ &= \frac{1}{N \omega_N} \int_{B_{2r}} \frac{U(h)^2}{|h|^{N+2s}} dh = \frac{1}{N \omega_N} \iint_{B_r \times B_{2r}} \frac{|\tilde{u}(x) - \tilde{u}(x+h)|^2}{|h|^{N+2s}} dx dh. \end{aligned} \quad (1.3.6)$$

We now take the compactly supported Lipschitz function

$$\psi(x) = \frac{N+1}{\omega_N} (1 - |x|)_+,$$

where $(\cdot)_+$ was introduced in (1.1.11). Observe that ψ has unit L^1 norm, by construction. We then define the rescaled function

$$\psi_t(x) = \frac{1}{t^N} \psi\left(\frac{x}{t}\right), \quad \text{for } 0 < t \leq 2r,$$

which is supported on $\overline{B_t}$. By observing that $\psi_t * \tilde{u} \in W^{1,2}(B_r)$, from the definition of $K(t, u)$ we have

$$K(t, u) \leq \|u - \psi_t * \tilde{u}\|_{L^2(B_r)} + t [\psi_t * \tilde{u}]_{W^{1,2}(B_r)}.$$

We estimate the two norms in the right-hand side separately: for the first one, by Minkowski's inequality and Fubini's Theorem we obtain

$$\begin{aligned} \|u - \psi_t * \tilde{u}\|_{L^2(B_r)} &= \left\| \int_{B_t} [\tilde{u}(\cdot) - \tilde{u}(\cdot - y)] \psi_t(y) dy \right\|_{L^2(B_r)} \\ &\leq \int_{B_t} \left(\int_{B_r} |\tilde{u}(x) - \tilde{u}(x - y)|^2 dx \right)^{\frac{1}{2}} \psi_t(y) dy \\ &= \int_{B_t} U(-y) \psi_t(y) dy \leq \frac{N+1}{\omega_N t^N} \int_{B_t} U(-y) dy \\ &= \frac{N(N+1)}{t^N} \int_0^t \bar{U} \varrho^{N-1} d\varrho \leq \frac{N(N+1)}{t} \int_0^t \bar{U} d\varrho. \end{aligned}$$

In the first identity we used that $\tilde{u} = u$ in B_r , in the last inequality we used that $\varrho^{N-1} \leq t^{N-1}$. For the second norm, we first observe that the Divergence Theorem gives

$$\int_{B_t} \nabla \psi_t(y) dy = 0,$$

thus we can write

$$\nabla \psi_t * \tilde{u} = (\nabla \psi_t) * \tilde{u} = \int_{B_t} \nabla \psi_t(y) [\tilde{u}(x - y) - \tilde{u}(x)] dy.$$

Thus, again Minkowski's inequality yields

$$\begin{aligned} [\psi_t * \tilde{u}]_{W^{1,2}(B_r)} &= \left\| \int_{B_t} \nabla \psi_t(y) [\tilde{u}(\cdot - y) - \tilde{u}(\cdot)] dy \right\|_{L^2(B_r)} \\ &\leq \int_{B_t} \left(\int_{B_r} |\tilde{u}(x - y) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{2}} |\nabla \psi_t(y)| dy \\ &\leq \frac{N+1}{\omega_N t^{N+1}} \int_{B_t} U(-y) dy \leq \frac{N(N+1)}{t^2} \int_0^t \bar{U} d\varrho. \end{aligned}$$

In conclusion, we have obtained

$$K(t, u) \leq \frac{2N(N+1)}{t} \int_0^t \bar{U} d\varrho, \quad \text{for every } 0 < t \leq 2r.$$

By raising to the power 2, dividing by t^{2s+1} and integrating, the previous estimate yields

$$\int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t} \leq (2N(N+1))^2 \int_0^{2r} \left(\frac{1}{t} \int_0^t \bar{U} d\varrho \right)^2 \frac{dt}{t^{1+2s}}.$$

If we now use the one-dimensional Hardy inequality (see [103, Teorema 1]) for the function $t \mapsto \int_0^t \bar{U} d\rho$, we get

$$\begin{aligned} \int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t} &\leq \left(\frac{2N(N+1)}{s+1} \right)^2 \int_0^{2r} \bar{U}^2 \frac{dt}{t^{1+2s}} \\ &\leq \frac{4N(N+1)^2}{\omega_N} \iint_{B_r \times B_{2r}} \frac{|\tilde{u}(x) - \tilde{u}(x+h)|^2}{|h|^{N+2s}} dx dh, \end{aligned}$$

where we used (1.3.6) in the second inequality. This proves (1.3.4), as desired.

The proof of (1.3.5) is similar to that of [18, Proposition B.1], but some technical modifications are needed, here as well. We take $0 < |\rho| \leq 2r$, by definition of the K -functional there exists $v_\rho \in W^{1,2}(B_r)$ such that

$$\|u - v_\rho\|_{L^2(B_r)} + |\rho| \|\nabla v_\rho\|_{L^2(B_r)} = K(|\rho|, u). \quad (1.3.7)$$

For notational simplicity, we simply write v in place of v_ρ . We also denote by \tilde{v} the extension of v given by $\mathcal{E}_r[v]$. For $\omega \in \mathbb{S}^{N-1}$ and $|\rho| \leq 2r$, we get⁴

$$\begin{aligned} \left(\int_{B_r} |\tilde{u}(x + \rho\omega) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{2}} &\leq \left(\int_{B_r} |\tilde{u}(x + \rho\omega) - \tilde{v}(x + \rho\omega) - \tilde{u}(x) + \tilde{v}(x)|^2 dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{B_r} |\tilde{v}(x + \rho\omega) - \tilde{v}(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq 2 \|\tilde{u} - \tilde{v}\|_{L^2(B_{3r})} + |\rho| \|\partial_\omega \tilde{v}\|_{L^2(B_{3r})} \\ &\leq 2 (\|\tilde{u} - \tilde{v}\|_{L^2(B_{3r})} + |\rho| \|\nabla \tilde{v}\|_{L^2(B_{3r})}). \end{aligned}$$

In the last estimate, we used the pointwise inequality $|\partial_\omega \tilde{v}| \leq |\nabla \tilde{v}|$ and the trivial estimate $|\rho| \leq 2|\rho|$. We can now use the properties of our extension operator \mathcal{E}_r , in order to replace the norms over B_{3r} with those on B_r . By Lemma 1.2.1, we have

$$\|\tilde{u} - \tilde{v}\|_{L^2(B_{3r})} = \|\mathcal{E}_r[u] - \mathcal{E}_r[v]\|_{L^2(B_{3r})} = \|\mathcal{E}_r[u - v]\|_{L^2(B_{3r})} \leq \sqrt{2} \cdot 3^N \|u - v\|_{L^2(B_r)},$$

and also

$$\|\nabla \tilde{v}\|_{L^2(B_{3r})} = \left[\mathcal{E}_r[v] \right]_{W^{1,2}(B_{3r})} \leq 2 \cdot 9^N [v]_{W^{1,2}(B_r)}.$$

This leads to

$$\left(\int_{B_r} |\tilde{u}(x + \rho\omega) - \tilde{u}(x)|^2 dx \right)^{\frac{1}{2}} \leq C (\|u - v\|_{L^2(B_r)} + |\rho| [v]_{W^{1,2}(B_r)}).$$

By combining this estimate with (1.3.7), we then obtain for $0 < |\rho| \leq 2r$

$$\int_{B_r} \frac{|\tilde{u}(x + \rho\omega) - \tilde{u}(x)|^2}{|\rho|^{1+2s}} dx \leq 4C^2 |\rho|^{-1-2s} K(|\rho|, u)^2.$$

If we now integrate with respect to ρ we get (1.3.5), as desired. The proof is now over. \square

⁴In the second inequality, we use that for every $\omega \in \mathbb{S}^{N-1}$ and every $|\rho| \leq 2r$, we have

$$\left(\int_{B_r} |\tilde{v}(x + \rho\omega) - \tilde{v}(x)|^2 dx \right)^{\frac{1}{2}} \leq |\rho| \left(\int_{B_{3r}} |\partial_\omega \tilde{v}|^2 dx \right)^{\frac{1}{2}}.$$

As a straightforward consequence of Proposition 1.3.1, we also get the following result (see also [10, Lemma A.4]).

Corollary 1.3.2. *Let $0 < s < 1$, for every $u \in C_0^\infty(\mathbb{R}^N)$ and every $\omega \in \mathbb{S}^{N-1}$, we have*

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}} \frac{|u(x) - u(x + \varrho\omega)|^2}{|\varrho|^{1+2s}} d\varrho \right) dx \leq \mathcal{A} [u]_{W^{s,2}(\mathbb{R}^N)}^2,$$

for the same constant $\mathcal{A} = \mathcal{A}(N) > 0$ appearing in (1.3.1).

The next result can be found in [86] and [94, Corollary 1]. In the latter, the estimate is slightly worse in its dependence on s , while in the former the result is not explicitly stated, but it must be extrapolated from the proof of [86, Corollary 1, page 524]. For these reasons, we prefer to provide a full proof, which in any case is different from those of the aforementioned references.

In order to show it, let us introduce the following notation. For $u \in L_{\text{loc}}^1(\mathbb{R}^N)$ and a bounded measurable set $E \subseteq \mathbb{R}^N$ with positive measure, we set

$$\text{av}(u; E) := \int_E u dx = \frac{1}{|E|} \int_E u dx,$$

the integral average of u over E .

Lemma 1.3.3 (Fractional Poincaré-Wirtinger inequality). *Let $0 < s < 1$, for every $u \in C^1(\overline{B_r(x_0)})$ we have*

$$\left\| u - \text{av}(u; B_r(x_0)) \right\|_{L^2(B_r(x_0))}^2 \leq \mathcal{M} (1-s) r^{2s} [u]_{W^{s,2}(B_r(x_0))}^2,$$

for some $\mathcal{M} = \mathcal{M}(N) > 0$.

Proof. We can suppose that $x_0 = 0$, without loss of generality. We use real interpolation techniques, as in the previous result. By combining (1.3.2) and (1.3.4), we have

$$[u]_{W^{s,2}(B_r)}^2 \geq \frac{1}{C} \int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t}, \quad (1.3.8)$$

where C depends on the dimension N only and $K(t, u)$ is still defined by (1.3.3). We now take $0 < t \leq 2r$ and $v \in W^{1,2}(B_r)$, by the triangle inequality we get

$$\begin{aligned} t \|u - \text{av}(u; B_r)\|_{L^2(B_r)} &\leq t \|u - v\|_{L^2(B_r)} + t \|v - \text{av}(v; B_r)\|_{L^2(B_r)} \\ &\quad + t \|\text{av}(v; B_r) - \text{av}(u; B_r)\|_{L^2(B_r)} \\ &\leq 2r (\|u - v\|_{L^2(B_r)} + \|\text{av}(v; B_r) - \text{av}(u; B_r)\|_{L^2(B_r)}) \\ &\quad + t \|v - \text{av}(v; B_r)\|_{L^2(B_r)}. \end{aligned}$$

By using Jensen's inequality we have

$$\|\text{av}(v; B_r) - \text{av}(u; B_r)\|_{L^2(B_r)} \leq \|u - v\|_{L^2(B_r)},$$

while by using the classical Poincaré-Wirtinger inequality we have

$$\|v - \text{av}(v; B_r)\|_{L^2(B_r)} \leq \frac{r}{\mu} [v]_{W^{1,2}(B_r)},$$

for some $\mu = \mu(N) > 0$. By keeping all these estimates together, we obtain

$$\begin{aligned} t \|u - \text{av}(u; B_r)\|_{L^2(B_r)}^2 &\leq 4r \|u - v\|_{L^2(B_r)} + \frac{tr}{\mu(B_1)} [v]_{W^{1,2}(B_r)} \\ &\leq Cr (\|u - v\|_{L^2(B_r)} + t [v]_{W^{1,2}(B_r)}), \end{aligned}$$

where $C = \max\{4, 1/\mu\}$ depends on N only. If we now take the infimum over $v \in W^{1,2}(B_r)$, we get

$$t \|u - \text{av}(u; B_r)\|_{L^2(B_r)} \leq Cr K(t, u), \quad \text{for } 0 < t \leq 2r.$$

By raising to the power 2, dividing by t^{2s+1} and integrating over $(0, 2r)$, this yields⁵

$$\|u - \text{av}(u; B_r)\|_{L^2(B_r)}^2 \frac{(2r)^{2-2s}}{2(1-s)} \leq C^2 r^2 \int_0^{2r} \left(\frac{K(t, u)}{t^s} \right)^2 \frac{dt}{t}.$$

By using this estimate in (1.3.8), we finally get the desired conclusion. \square

The following result is more general than the previous one, and it can be obtained from the latter as a corollary.

Corollary 1.3.4. *Let $0 < s < 1$ and let $E \subseteq B_R(x_0) \subseteq \mathbb{R}^N$ be a measurable set, with positive measure. There exists a constant $\mathcal{M} = \mathcal{M}(N) > 0$ such that for every $u \in W^{s,2}(B_R(x_0))$ we have*

$$\|u - \text{av}(u; E)\|_{L^2(B_R(x_0))}^2 \leq \mathcal{M}(1-s) \frac{R^N}{|E|} R^{2s} [u]_{W^{s,2}(B_R(x_0))}^2.$$

Proof. By a standard scaling argument, it is sufficient to prove the result for $R = 1$ and $x_0 = 0$.

We consider the extension $\mathcal{E}[u]$ of u to the whole \mathbb{R}^N , as in (1.2.3). By using the Minkowski's inequality and the fact that $B_1 \subsetneq B_2$, we have

$$\begin{aligned} \|u - \text{av}(u; E)\|_{L^2(B_1)}^2 &\leq \|\mathcal{E}[u] - \text{av}(u; E)\|_{L^2(B_2)}^2 \\ &\leq 2 \|\mathcal{E}[u] - \text{av}(\mathcal{E}[u]; B_2)\|_{L^2(B_2)}^2 + 2 \|\text{av}(\mathcal{E}[u]; B_2) - \text{av}(u; E)\|_{L^2(B_2)}^2. \end{aligned} \tag{1.3.9}$$

By using Jensen's inequality and the fact that $|B_2| = 2^N \omega_N$, we can estimate the second term as follows

$$\begin{aligned} \|\text{av}(\mathcal{E}[u]; B_2) - \text{av}(u; E)\|_{L^2(B_2)}^2 &= 2^N \omega_N \left| \text{av}(\mathcal{E}[u]; B_2) - \text{av}(u; E) \right|^2 \\ &= 2^N \omega_N \left| \int_E (\mathcal{E}[u](x) - \text{av}(\mathcal{E}[u]; B_2)) dx \right|^2 \\ &\leq 2^N \omega_N \int_E |\mathcal{E}[u](x) - \text{av}(\mathcal{E}[u]; B_2)|^2 dx \\ &\leq \frac{2^N \omega_N}{|E|} \|\mathcal{E}[u] - \text{av}(\mathcal{E}[u]; B_2)\|_{L^2(B_2)}^2. \end{aligned}$$

Thus from (1.3.9) we get

$$\|u - \text{av}(u; E)\|_{L^2(B_1)}^2 \leq 2 \left(1 + \frac{2^N \omega_N}{|E|} \right) \|\mathcal{E}[u] - \text{av}(\mathcal{E}[u]; B_2)\|_{L^2(B_2)}^2.$$

⁵We remark that the presence of the factor $(1-s)$ is important for our scopes. If one is not interested in keeping track of this factor, actually the proof would be much simpler, see for example [88, page 297].

We can now apply the following fractional Poincaré inequality proved in Lemma 1.3.3, and this yields

$$\|u - \text{av}(u; E)\|_{L^2(B_1)}^2 \leq 2^{1+2s} \left(1 + \frac{2^N \omega_N}{|E|}\right) \mathcal{M}(1-s) [\mathcal{E}[u]]_{W^{s,2}(B_2)}^2.$$

It is now sufficient to apply Lemma 1.1.4 with $R = 2$, to get the claimed conclusion. \square

1.4 Fractional capacity

We start with the definition of fractional capacity. Our definition is tailored on our needs, we recall that other variants of this notion are possible, see for example in [4, 97, 99, 107]. To this aim, we introduce the following notation: for every set $E \subseteq \mathbb{R}^N$, we denote the *characteristic function of E* with the symbol 1_E .

Definition 1.4.1. *Let $\Sigma \subseteq \mathbb{R}^N$ be a compact set and let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\Sigma \Subset \Omega$. For $0 < s < 1$, we define the fractional capacity of Σ of order s relative to Ω as the quantity*

$$\widetilde{\text{cap}}_s(\Sigma; \Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ [u]_{W^{s,2}(\mathbb{R}^N)}^2 : u \geq 1_\Sigma \right\}.$$

Remark 1.4.2. *By standard approximation arguments based on convolutions, in the definition of $\widetilde{\text{cap}}_s(\Sigma; \Omega)$ we can replace $C_0^\infty(\Omega)$ with Lipschitz functions having compact support in Ω . Indeed let us denote*

$$\mathbf{c}_s(\Sigma; \Omega) := \inf \left\{ [u]_{W^{s,2}(\mathbb{R}^N)}^2 : u \in \text{Lip}(\Omega), \text{supp}(u) \Subset \Omega \text{ and } u \geq 1_\Sigma \right\}.$$

Thus the inequality

$$\widetilde{\text{cap}}_s(\Sigma; \Omega) \geq \mathbf{c}_s(\Sigma; \Omega),$$

trivially follows from the inclusion of $C_0^\infty(\Omega)$ in the set of Lipschitz functions having compact support in Ω . In order to show the reverse inequality, we consider the convolution

$$u_\varepsilon(x) := (u * \rho_\varepsilon)(x), \quad \text{with} \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^N,$$

where $\rho \in C_0^\infty(\mathbb{R}^N)$ is the same mollifier we used in the proof of Lemma 1.1.4. We recall some of its properties

$$\rho \geq 0 \text{ in } \mathbb{R}^N, \quad \text{supp}(\rho) \subseteq B_1, \quad \int_{B_1} \rho(x) dx = 1.$$

Finally we define

$$v_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}, \quad v_\varepsilon(x) := \frac{u_\varepsilon(x)}{\min_{y \in \Sigma} u_\varepsilon(y)}.$$

By construction $v_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ and $v_\varepsilon \geq 1_\Sigma$. Moreover, by proceeding as we did in the proof of Lemma 1.1.4, we get that v_ε has compact support in Ω , if ε is small enough. Thus it represents a competitor for $\widetilde{\text{cap}}_s(\Sigma; \Omega)$.

Thanks to the properties of convolutions, we get that u_ε uniformly converges to u , as ε goes to 0 and thus

$$\lim_{\varepsilon \rightarrow 0} \min_{y \in \Sigma} u_\varepsilon(y) = \min_{y \in \Sigma} u(y).$$

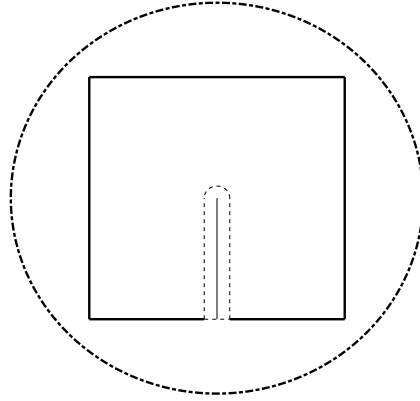


Figure 1.2: The geometric configuration of Lemma 1.4.3: we have a smooth function defined on the square, which vanishes on the dashed neighborhood of the vertical line (i.e. the set Σ). The relative fractional capacity of Σ is computed with respect to the surrounding disk.

We observe that the left-hand side is positive, thanks to the uniform convergence, while the right-hand is larger than or equal to 1. Moreover, from [49, Lemma 11] we get the convergence

$$\lim_{\varepsilon \rightarrow 0} [u_\varepsilon]_{W^{s,2}(\mathbb{R}^N)} = [u]_{W^{s,2}(\mathbb{R}^N)}.$$

This implies that

$$\widetilde{\text{cap}}_s(\Sigma; \Omega) \leq \lim_{\varepsilon \rightarrow 0} [v_\varepsilon]_{W^{s,2}(\mathbb{R}^N)} = \frac{[u]_{W^{s,2}(\mathbb{R}^N)}}{\min_{y \in \Sigma} u(y)} \leq [u]_{W^{s,2}(\mathbb{R}^N)}.$$

The inequality $\widetilde{\text{cap}}_s(\Sigma; \Omega) \leq \mathfrak{c}_s(\Sigma; \Omega)$ follows by taking the infimum over u .

In the proof of Theorem 1, we will need the following fractional Poincaré inequality for functions on a cube, which vanish in a compact subset of its closure, having positive fractional capacity with respect to the cube. This is analogous to the result of [104, Theorem A], but we will follow the approach of [86, Chapter 14], which is more suitable for our framework. In particular, we will not explicitly relate this result to eigenvalues with mixed boundary conditions, differently from [104].

Proposition 1.4.3. *Let $0 < s < 1$ and let $\Sigma \subseteq \overline{Q_r(x_0)} \subseteq \mathbb{R}^N$ be a compact set. For every $R > \sqrt{N}r$, there exists a constant $\phi(N, R/r) > 0$ such that the following Poincaré inequality holds*

$$[u]_{W^{s,2}(Q_r(x_0))}^2 \geq \left[\frac{s}{r^N} \phi \left(N, \frac{R}{r} \right) \right] \widetilde{\text{cap}}_s(\Sigma; B_R(x_0)) \|u\|_{L^2(Q_r(x_0))}^2, \quad (1.4.1)$$

for every $u \in C^\infty(\overline{Q_r(x_0)})$ with $\text{dist}(\text{supp}(u), \Sigma) > 0$. Moreover, we have

$$\lim_{t \rightarrow +\infty} \phi(N, t) = \lim_{t \searrow \sqrt{N}} \phi(N, t) = 0.$$

Proof. The proof is lengthy, though elementary. Without loss of generality we can assume $x_0 = 0$. Let $u \in C^\infty(\overline{Q_r})$ be as in the statement, we can additionally assume that

$$\int_{Q_r} |u|^2 dx = 1, \quad (1.4.2)$$

still without loss of generality. We now use the extension operator \mathcal{E}_K of Corollary 1.2.3, with the choices

$$K = Q_r \quad \text{and} \quad x_0 = 0, \quad \text{so that} \quad \frac{D_K(x_0)}{d_K(x_0)} = \sqrt{N}.$$

In order not to overburden the presentation, we will use the symbol \tilde{u} in place of $\mathcal{E}_K[u]$. By the properties of our extension operator, we get in particular that \tilde{u} is locally Lipschitz continuous and from (1.2.8) with $p = 2$ we also have

$$[\tilde{u}]_{W^{s,2}(B_R)} \leq [\tilde{u}]_{W^{s,2}(Q_R)} \leq C_N \left(\frac{R}{r}\right)^{2N} [u]_{W^{s,2}(Q_r)}. \quad (1.4.3)$$

Without loss of generality, we can further suppose that

$$\text{av}(\tilde{u}; B_R) \geq 0. \quad (1.4.4)$$

We take a Lipschitz cut-off function η such that

$$0 \leq \eta \leq 1, \quad \eta \equiv 1 \text{ in } \overline{B_{\sqrt{N}r}}, \quad \eta \equiv 0 \text{ in } \mathbb{R}^N \setminus B_{\frac{R+\sqrt{N}r}{2}}, \quad |\nabla \eta| \leq \frac{2}{R - \sqrt{N}r}.$$

and we define $\psi = (1 - \tilde{u})\eta$. By recalling Remark 1.4.2, we have that ψ is an admissible trial function for the variational problem defining $\widehat{\text{cap}}_s(\Sigma; B_R)$. By using this fact and some algebraic manipulations, we get

$$\begin{aligned} \sqrt{\widehat{\text{cap}}_s(\Sigma; B_R)} &\leq [\psi]_{W^{s,2}(\mathbb{R}^N)} \\ &= \left([\psi]_{W^{s,2}(B_R)}^2 + 2 \int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) dx \right)^{\frac{1}{2}} \\ &\leq [\psi]_{W^{s,2}(B_R)} + \sqrt{2} \left(\int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (1.4.5)$$

In turn, by using the definition of ψ and Minkowki's inequality, we have

$$\begin{aligned} [\psi]_{W^{s,2}(B_R)} &\leq \left(\int_{B_R} |\eta(x)|^2 \left(\int_{B_R} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{B_R} |1 - \tilde{u}(y)|^2 \left(\int_{B_R} \frac{|\eta(x) - \eta(y)|^2}{|x-y|^{N+2s}} dx \right) dy \right)^{\frac{1}{2}} \\ &\leq [\tilde{u}]_{W^{s,2}(B_R)} + \|1 - \tilde{u}\|_{L^2(B_R)} \sqrt{\frac{C}{s(1-s)}} \|\nabla \eta\|_{L^\infty(B_R)}^s \|\eta\|_{L^\infty(B_R)}^{1-s}, \end{aligned}$$

for some $C = C(N) > 0$. In the last inequality, we used that for every Lipschitz function φ with compact support, we have

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{N+2s}} dy \leq \frac{C}{s(1-s)} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^N)}^{2s} \|\varphi\|_{L^\infty(\mathbb{R}^N)}^{2(1-s)},$$

see Lemma 1.1.3. If we now use (1.4.3) to bound the seminorm of \tilde{u} and the properties of η , from (1.4.5) we get

$$\begin{aligned} \sqrt{\widehat{\text{cap}}_s(\Sigma; B_R)} &\leq C_N \left(\frac{R}{r}\right)^{2N} [u]_{W^{s,2}(Q_r)} + \frac{2}{(R - \sqrt{N}r)^s} \sqrt{\frac{C}{s(1-s)}} \|1 - \tilde{u}\|_{L^2(B_R)} \\ &\quad + \sqrt{2} \left(\int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) dx \right)^{\frac{1}{2}}. \end{aligned} \quad (1.4.6)$$

In order to handle the last term, we recall that ψ identically vanishes outside $B_{(R+\sqrt{N}r)/2}$. Thus, we actually have

$$\begin{aligned} \int_{B_R} |\psi(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) &= \int_{B_{\frac{R+\sqrt{N}r}{2}}} |\eta(x)|^2 |1 - \tilde{u}(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) dx \\ &\leq \int_{B_{\frac{R+\sqrt{N}r}{2}}} |1 - \tilde{u}(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \right) dx. \end{aligned}$$

We now observe that, for every $x \in B_{(R+\sqrt{N}r)/2}$ and $y \notin B_R$ we have

$$|x-y| \geq |y| - |x| \geq |y| - \frac{R + \sqrt{N}r}{2} \geq |y| - \frac{R + \sqrt{N}r}{2R} |y| = \left(\frac{R - \sqrt{N}r}{2R} \right) |y|.$$

Thus, for every $x \in B_{(R+\sqrt{N}r)/2}$, we get

$$\int_{\mathbb{R}^N \setminus B_R} \frac{dy}{|x-y|^{N+2s}} \leq \frac{N \omega_N}{2s} \left(\frac{2R}{R - \sqrt{N}r} \right)^{N+2s} \frac{1}{R^{2s}}.$$

By collecting the previous estimates, we obtain from (1.4.6)

$$\begin{aligned} \sqrt{\widehat{\text{cap}}_s(\Sigma; B_R)} &\leq C_N \left(\frac{R}{r} \right)^{2N} [u]_{W^{s,2}(Q_r)} + \frac{2}{(R - \sqrt{N}r)^s} \sqrt{\frac{C}{s(1-s)}} \|1 - \tilde{u}\|_{L^2(B_R)} \\ &\quad + \sqrt{\frac{N \omega_N}{s}} \left(\frac{2R}{R - \sqrt{N}r} \right)^{\frac{N}{2}+s} \frac{1}{R^s} \|1 - \tilde{u}\|_{L^2(B_{\frac{R+\sqrt{N}r}{2}})}. \end{aligned}$$

We need to estimate the L^2 norm of $1 - \tilde{u}$. For this, we use the triangle inequality

$$\|1 - \tilde{u}\|_{L^2(B_{\frac{R+\sqrt{N}r}{2}})} \leq \|1 - \tilde{u}\|_{L^2(B_R)} \leq \|1 - \text{av}(\tilde{u}; B_R)\|_{L^2(B_R)} + \|\text{av}(\tilde{u}; B_R) - \tilde{u}\|_{L^2(B_R)} := \mathcal{I}_1 + \mathcal{I}_2,$$

so that

$$\begin{aligned} \sqrt{\widehat{\text{cap}}_s(\Sigma; B_R)} &\leq C_N \left(\frac{R}{r} \right)^{2N} [u]_{W^{s,2}(Q_r)} + \frac{2}{(R - \sqrt{N}r)^s} \sqrt{\frac{2C}{s(1-s)}} (\mathcal{I}_1 + \mathcal{I}_2) \\ &\quad + \sqrt{\frac{N \omega_N}{s}} \left(\frac{2R}{R - \sqrt{N}r} \right)^{\frac{N}{2}+s} \frac{1}{R^s} (\mathcal{I}_1 + \mathcal{I}_2). \end{aligned} \tag{1.4.7}$$

In turn, the term \mathcal{I}_1 can be bounded by \mathcal{I}_2 . Indeed, by observing that the integrand of \mathcal{I}_1 is constant and using the normalization (1.4.2), we get

$$\begin{aligned} \mathcal{I}_1 &= \sqrt{|B_R|} |1 - \text{av}(\tilde{u}; B_R)| = \sqrt{\frac{|B_R|}{|Q_r|}} \left| \|u\|_{L^2(Q_r)} - \|\text{av}(\tilde{u}; B_R)\|_{L^2(Q_r)} \right| \\ &\leq \sqrt{\frac{|B_R|}{|Q_r|}} \|u - \text{av}(\tilde{u}; B_R)\|_{L^2(Q_r)} \leq \sqrt{\frac{|B_R|}{|Q_r|}} \mathcal{I}_2. \end{aligned}$$

Observe that we also used the condition (1.4.4) in the second identity. As for the integral \mathcal{I}_2 , by Lemma 1.3.3 we directly get

$$\mathcal{I}_2 \leq \sqrt{\mathcal{M}(1-s)} R^s [\tilde{u}]_{W^{s,2}(B_R)}.$$

Then the last term can be estimated by (1.4.3), again. By inserting these estimates in (1.4.7) we eventually conclude the proof. \square

Chapter 2

The fractional Croke-Osserman-Taylor inequality

This chapter contains the main result proved in [B1]. We will show that it is possible to extend the *Croke-Osserman-Taylor inequality* to the setting of fractional Sobolev spaces. Let us recall the following notation

$$\lambda_1^s(\Omega) = \inf_{u \in \widetilde{W}_0^{s,2}(\Omega) \setminus \{0\}} \frac{[u]_{W^{s,2}(\mathbb{R}^N)}^2}{\|u\|_{L^2(\Omega)}^2}.$$

We want to provide a lower bound on λ_1^s , for the following class of sets.

Definition. Let us indicate by $(\mathbb{R}^2)^*$ the one-point compactification of \mathbb{R}^2 , i.e. the compact space obtained by adding to \mathbb{R}^2 the point at infinity. We say that an open connected set $\Omega \subseteq \mathbb{R}^2$ is multiply connected of order k if its complement in $(\mathbb{R}^2)^*$ has k connected components. When $k = 1$, we will simply say that Ω is simply connected.

The main result of this chapter then reads as follows.

Theorem 1 (Fractional Croke-Osserman-Taylor inequality). *Let $1/2 < s < 1$, there exists a constant $\vartheta_s > 0$ such that for every $\Omega \subseteq \mathbb{R}^2$ open multiply connected set of order $k \in \mathbb{N} \setminus \{0\}$, having finite inradius, we have*

$$\lambda_1^s(\Omega) \geq \frac{\vartheta_s}{k^s} \left(\frac{1}{r_\Omega} \right)^{2s}.$$

Moreover, the constant ϑ_s has the following asymptotic behaviours

$$\vartheta_s \sim (2s - 1) \quad \text{for } s \searrow \frac{1}{2} \quad \text{and} \quad \vartheta_s \sim \frac{1}{1 - s} \quad \text{for } s \nearrow 1.$$

2.1 An estimate for the s -capacity

The following result is a particular case of the well-known fractional Morrey-type embedding in the space of continuous functions (see for example [70, Corollary 7.9.4]). For our scopes, we need a precise “quantitative” behaviour of the relevant constant, as s goes to 1 or $1/2$. Here we consider $N = 1$.

Theorem 2.1.1 (Fractional Morrey-Sobolev inequality). *For every $1/2 < s < 1$ there exists a constant $\mathfrak{m}_s > 0$ depending on s only, such that*

$$\mathfrak{m}_s [u]_{W^{s-\frac{1}{2},\infty}(\mathbb{R})}^2 \leq [u]_{W^{s,2}(\mathbb{R})}^2, \quad \text{for every } u \in C_0^\infty(\mathbb{R}). \quad (2.1.1)$$

In particular, if $a < b$ we have

$$\mathbf{m}_s \|u\|_{L^\infty((a,b))}^2 \leq (b-a)^{2s-1} [u]_{W^{s,2}(\mathbb{R})}^2, \quad \text{for every } u \in C_0^\infty((a,b)). \quad (2.1.2)$$

Moreover, the constant \mathbf{m}_s has the following asymptotic behaviour

$$\mathbf{m}_s \sim 2s - 1, \quad \text{as } s \searrow 1/2, \quad \text{and} \quad \mathbf{m}_s \sim \frac{1}{1-s}, \quad \text{as } s \nearrow 1.$$

Proof. We first observe that (2.1.2) is an easy consequence of (2.1.1). Indeed, for every $u \in C_0^\infty((a,b))$ and every $x \in (a,b)$, by (2.1.1) we would get

$$|u(x)|^2 = |u(x) - u(a)|^2 \leq \frac{1}{\mathbf{m}_s} (x-a)^{2s-1} [u]_{W^{s,2}(\mathbb{R})}^2 \leq \frac{1}{\mathbf{m}_s} (b-a)^{2s-1} [u]_{W^{s,2}(\mathbb{R})}^2,$$

as desired.

In order to establish (2.1.1), let us take $\varphi \in C_0^\infty(\mathbb{R})$. We consider its Fourier transform $\mathcal{F}[\varphi]$ and, from the inversion formula (1.1.1), for every $t, \tau \in \mathbb{R}$ we get

$$\begin{aligned} |\varphi(t) - \varphi(\tau)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\mathcal{F}[\varphi](\xi)| |e^{it\xi} - e^{i\tau\xi}| d\xi \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}[\varphi](\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi \right)^{\frac{1}{2}}. \end{aligned} \quad (2.1.3)$$

We now recall that by (1.1.2) we have

$$\int_{\mathbb{R}} |\xi|^{2s} |\mathcal{F}[\varphi](\xi)|^2 d\xi = 2\pi A_s [\varphi]_{W^{s,2}(\mathbb{R})}^2,$$

where A_s satisfies

$$A_s \sim 1-s, \quad \text{for } s \nearrow 1 \quad \text{and} \quad A_s \sim s \quad \text{for } s \searrow 0.$$

From (2.1.3), we obtain

$$|\varphi(t) - \varphi(\tau)| \leq \sqrt{A_s} \left(\int_{\mathbb{R}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi \right)^{\frac{1}{2}} [\varphi]_{W^{s,2}(\mathbb{R})}. \quad (2.1.4)$$

In order to conclude, we are only left with handling the integral on the right-hand side. For every $\alpha > 0$, we split this integral as follows

$$\int_{\mathbb{R}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi = \int_{\{|\xi| \leq \alpha\}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi + \int_{\{|\xi| > \alpha\}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi.$$

In order to estimate the low frequencies, we use the 1-Lipschitz character of $\vartheta \mapsto e^{i\vartheta}$ to infer that

$$|e^{it\xi} - e^{i\tau\xi}| \leq |t - \tau| |\xi|.$$

The high frequencies are dealt with by using that

$$|e^{it\xi} - e^{i\tau\xi}| \leq |e^{it\xi}| + |e^{i\tau\xi}| = 2.$$

These lead to

$$\begin{aligned} \int_{\mathbb{R}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi &\leq 2|t - \tau|^2 \int_0^\alpha \xi^{2-2s} d\xi + 8 \int_\alpha^{+\infty} \xi^{-2s} d\xi \\ &= \frac{2}{3-2s} |t - \tau|^2 \alpha^{3-2s} + \frac{8}{2s-1} \frac{1}{\alpha^{2s-1}}, \end{aligned}$$

which is valid for every $\alpha > 0$. We can now optimize this estimate with respect to α : indeed, the quantity on the right-hand side is minimal for¹ $\alpha = \alpha_0 = 2/|t - \tau|$. With such a choice, we get

$$\int_{\mathbb{R}} \frac{|e^{it\xi} - e^{i\tau\xi}|^2}{|\xi|^{2s}} d\xi \leq 4^{2-s} \frac{2}{(3-2s)(2s-1)} |t - \tau|^{2s-1}.$$

By inserting this estimate in (2.1.4), we finally get (2.1.1) with

$$\mathbf{m}_s = \frac{(3-2s)(2s-1)}{2 \cdot 4^{2-s} A_s},$$

which has the claimed asymptotic behaviour. \square

Remark 2.1.2. We point out the reference [100], which keeps track of the dependence on s in the one-dimensional fractional Morrey estimate, as this parameter goes to the borderline situation $s = 1/2$ (see [100, Corollary 26]). However, the asymptotic behaviour detected in this reference is sub-optimal. Moreover, the asymptotic behaviour as s goes to 1 is not taken into account. For these reasons, the estimates of [100] are not suitable for our needs.

As a straightforward consequence of the definition of capacity 1.4.1 (see also Remark 1.4.2) and the Morrey-type inequality, we have an explicit lower bound for the fractional capacity of a point. As simple as it is, this will play a crucial role in our main result.

Lemma 2.1.3 (One-dimensional capacity of a point). *Let $1/2 < s < 1$ and $x_0 \in (a, b)$. Then*

$$\widetilde{\text{cap}}_s(\{x_0\}; (a, b)) \geq (b - a)^{1-2s} \mathbf{m}_s,$$

where \mathbf{m}_s is the same constant as in Theorem 2.1.1.

Proof. Let us take $u \in C_0^\infty((a, b))$ such that $u(x_0) \geq 1$. Hence, from (2.1.2), we get

$$1 \leq |u(x_0)|^2 \leq \frac{(b - a)^{2s-1}}{\mathbf{m}_s} [u]_{W^{s,2}(\mathbb{R})}^2.$$

The thesis follows by taking the infimum over the admissible functions u . \square

In dimension $N = 2$ and for $s > 1/2$, by exploiting the fact that points on the line have positive relative fractional capacity, it is possible to give a geometric lower bound for the term

$$\widetilde{\text{cap}}_s(\Sigma; B_R(x_0)),$$

appearing in (1.4.1). We will follow the idea of [86, Chapter 3, Section 1.2, Proposition 1], which is quite close to that used by Taylor, even if the latter worked with a different notion of *capacity* coming from Potential Theory. The proof will also crucially exploits the result on “directional” fractional derivatives (Proposition 1.3.1 and Corollary 1.3.2). We still use the symbol Π_ω to denote the projection

$$\Pi_\omega(x) = x - \langle x, \omega \rangle \omega,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product.

Proposition 2.1.4. *Let $N = 2$, $1/2 < s < 1$ and let $\Sigma \Subset B_r(x_0)$ be a compact set. For every direction $\omega \in \mathbb{S}^1$, it holds that*

$$\widetilde{\text{cap}}_s(\Sigma; B_r(x_0)) \geq \frac{\mathbf{m}_s}{\mathcal{A}} \left(r \text{dist}(\Sigma, \partial B_r(x_0)) \right)^{\frac{1-2s}{2}} \mathcal{H}^1(\Pi_\omega(\Sigma)).$$

Here \mathcal{A} is the same constant as in Proposition 1.3.1 and \mathbf{m}_s is the same constant as in Theorem 2.1.1.

¹We can obviously suppose that $t \neq \tau$, otherwise there is nothing to prove.

Proof. We observe that we can assume $\mathcal{H}^1(\Pi_\omega(\Sigma)) > 0$, otherwise there is nothing to prove. We may suppose as always that $x_0 = 0$, without loss of generality.

We start by noticing that every $x \in \mathbb{R}^2$ can be written as

$$x = x' + t\omega, \quad \text{with } x' \in \Pi_\omega(\mathbb{R}^2) \text{ and } t \in \mathbb{R}.$$

We also set

$$R_\omega(x') = \sup \left\{ \varrho \in \mathbb{R} : x' + \varrho\omega \in B_r \right\} \quad \text{and} \quad r_\omega(x') = \inf \left\{ \varrho \in \mathbb{R} : x' + \varrho\omega \in B_r \right\}.$$

We take $u \in C_0^\infty(B_r)$ such that $u \geq 1_\Sigma$. By using Corollary 1.3.2 and Fubini's Theorem, we can infer

$$\begin{aligned} [u]_{W^{s,2}(\mathbb{R}^2)}^2 &\geq \frac{1}{\mathcal{A}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{|u(x) - u(x + \varrho\omega)|^2}{|\varrho|^{1+2s}} d\varrho \right) dx \\ &= \frac{1}{\mathcal{A}} \int_{\Pi_\omega(\mathbb{R}^2)} \left(\iint_{\mathbb{R} \times \mathbb{R}} \frac{|u(x' + t\omega) - u(x' + t\omega + \varrho\omega)|^2}{|\varrho|^{1+2s}} dt d\varrho \right) dx' \\ &\geq \frac{1}{\mathcal{A}} \int_{\Pi_\omega(\Sigma)} [u(x' + \cdot\omega)]_{W^{s,2}(\mathbb{R})}^2 dx'. \end{aligned} \quad (2.1.5)$$

Recalling that $u \geq 1$ on Σ , it follows that for every $x' \in \Pi_\omega(\Sigma)$ there exists t_0 such that $u(x' + t_0\omega) \geq 1$. Hence, by using the trial function

$$\psi_{x'} = u(x' + \cdot\omega) \in C_0^\infty((r_\omega(x'), R_\omega(x'))),$$

we have

$$[u(x' + \cdot\omega)]_{W^{s,2}(\mathbb{R})}^2 = [\psi_{x'}]_{W^{s,2}(\mathbb{R})}^2 \geq \widehat{\text{cap}}_s(\{t_0\}; (r_\omega(x'), R_\omega(x'))), \quad \text{for } x' \in \Pi_\omega(\Sigma),$$

by the very definition of capacity. In turn, by applying Lemma 2.1.3 in the right-hand side above, we get

$$[u(x' + \cdot\omega)]_{W^{s,2}(\mathbb{R})}^2 \geq \mathbf{m}_s \left(R_\omega(x') - r_\omega(x') \right)^{1-2s}.$$

In order to get a lower bound for the last term, we set $\ell = \text{dist}(\Sigma, \partial B_r) > 0$. Then in particular we have

$$R_\omega(x') - r_\omega(x') \geq \sqrt{r^2 - (r - \ell)^2} \geq \sqrt{r} \ell, \quad \text{for every } x' \in \Pi_\omega(\Sigma).$$

This entails that

$$[u(x' + \cdot\omega)]_{W^{s,2}(\mathbb{R})}^2 \geq \mathbf{m}_s (r \ell)^{\frac{1-2s}{2}}, \quad \text{for every } x' \in \Pi_\omega(\Sigma).$$

By spending this information in (2.1.5), we can obtain

$$[u]_{W^{s,2}(B_r)}^2 \geq \frac{\mathbf{m}_s}{\mathcal{A}} (r \ell)^{\frac{1-2s}{2}} \mathcal{H}^1(\Pi_\omega(\Sigma)).$$

The thesis follows by taking the infimum over the admissible trial functions u . \square

2.2 Proof of Theorem 1

The proof of Theorem 1 is based on three steps: the first one is to tile the plane \mathbb{R}^2 with squares of well chosen length side. This allows us to focus our attention on one square only and to consider the ‘‘regional’’ seminorm. Hence, in the second step, we apply the results proved in Section 1.4 and we obtain a fractional Poincaré inequality where the coefficient depends on the s -capacity of a specific compact. Finally, in the last step we combine the Taylor's fatness Lemma B.1.1 with the Proposition 2.1.4. Thus we bound from below the s -capacity above in terms of three elements: the inradius of the open set, the topological parameter k and the order of derivative s . The dependence on k comes from Lemma B.1.1.

Proof of Theorem 1. Without loss of generality, we can assume $r_\Omega = 1$. As in the proof of Lemma B.1.1, we denote the *integer part of \sqrt{k}* with

$$\sqrt{k} - 1 \leq \lfloor \sqrt{k} \rfloor \leq \sqrt{k},$$

and we consider the natural number $\delta = \lfloor \sqrt{k} \rfloor + 1$. Thus we take the family of squares $\{\mathcal{Q}_{i,j}\}_{(i,j) \in \mathbb{Z}^2} \subseteq \mathbb{R}^2$ given by

$$\mathcal{Q}_{i,j} := Q_{5\delta}(10\delta i, 10\delta j), \quad \text{for } (i, j) \in \mathbb{Z}^2,$$

where the notation $Q_{5\delta}(10\delta, 10\delta)$ corresponds to the one used in Lemma B.1.1. We observe that they form a tiling of the whole plane, more precisely they are pairwise disjoint and the union of their closures covers the whole \mathbb{R}^2 . We also introduce the set of indexes

$$\mathbb{Z}_\Omega^2 = \left\{ (i, j) \in \mathbb{Z}^2 : \mathcal{Q}_{i,j} \cap \Omega \neq \emptyset \right\},$$

and for every $(i, j) \in \mathbb{Z}_\Omega^2$, we indicate by $\Sigma_{i,j} \subseteq \overline{\mathcal{Q}_{i,j}} \setminus \Omega$ the compact set provided by Lemma B.1.1. By using the tiling properties of these squares, for a function $u \in C_0^\infty(\Omega)$ we have

$$\begin{aligned} [u]_{W^{s,2}(\mathbb{R}^2)}^2 &= \sum_{(i,j) \in \mathbb{Z}^2} \iint_{\mathcal{Q}_{i,j} \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy \\ &\geq \sum_{(i,j) \in \mathbb{Z}^2} \iint_{\mathcal{Q}_{i,j} \times \mathcal{Q}_{i,j}} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy = \sum_{(i,j) \in \mathbb{Z}_\Omega^2} [u]_{W^{s,2}(\mathcal{Q}_{i,j})}^2. \end{aligned}$$

For every $(i, j) \in \mathbb{Z}_\Omega^2$, we can use the fractional Poincaré inequality of Proposition 1.4.3, with the choices

$$r = 5\delta \quad \text{and} \quad R = 2r = 10\delta.$$

By setting for brevity $\mathcal{B}_{i,j} := B_{10\delta}(10\delta i, 10\delta j)$, this leads to

$$[u]_{W^{s,2}(\mathcal{Q}_{i,j})}^2 \geq \left[\frac{1}{50\delta^2} \phi(2, 2) \right] \widetilde{\text{cap}}_s(\Sigma_{i,j}; \mathcal{B}_{i,j}) \|u\|_{L^2(\mathcal{Q}_{i,j})}^2, \quad \text{for every } (i, j) \in \mathbb{Z}_\Omega^2,$$

where we also used that $s > 1/2$. We now have to estimate from below the relative fractional capacity of each compact set $\Sigma_{i,j}$. By combining Lemma B.1.1 and Proposition 2.1.4, we have

$$\begin{aligned} \widetilde{\text{cap}}_s(\Sigma_{i,j}; \mathcal{B}_{i,j}) &\geq \left(50(2 - \sqrt{2}) \right)^{\frac{1-2s}{2}} \frac{\mathbf{m}_s}{\mathcal{A}} \delta^{1-2s} \max \left\{ \mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma_{i,j})), \mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma_{i,j})) \right\} \\ &\geq \left(50(2 - \sqrt{2}) \right)^{\frac{1-2s}{2}} \frac{\mathbf{m}_s}{4\mathcal{A}} \delta^{1-2s} \sqrt{k}. \end{aligned}$$

By collecting the estimates above, we obtain

$$\begin{aligned} [u]_{W^{s,2}(\mathbb{R}^2)}^2 &\geq \left(50(2 - \sqrt{2}) \right)^{\frac{1-2s}{2}} \frac{\mathbf{m}_s \phi(2, 2)}{200\mathcal{A}} \sqrt{k} \delta^{-1-2s} \sum_{(i,j) \in \mathbb{Z}_\Omega^2} \|u\|_{L^2(\mathcal{Q}_{i,j})}^2 \\ &= \left(50(2 - \sqrt{2}) \right)^{\frac{1-2s}{2}} \frac{\mathbf{m}_s \phi(2, 2)}{200\mathcal{A}} \sqrt{k} \delta^{-1-2s} \|u\|_{L^2(\Omega)}^2, \end{aligned} \tag{2.2.1}$$

where the last identity follows by the tiling property of the family $\{\mathcal{Q}_{i,j}\}_{i,j}$. By recalling the definition of δ and using (B.1.1), we get

$$\sqrt{k} \delta^{-1-2s} \geq \sqrt{k} \left(\sqrt{k} + 1 \right)^{-1-2s} \geq \frac{1}{2^{1+2s}} \frac{1}{k^s}.$$

By the arbitrariness of $u \in C_0^\infty(\Omega)$, from (2.2.1) we get the claimed lower bound on $\lambda_1^s(\Omega)$, with

$$\vartheta_s = \frac{(50(2 - \sqrt{2}))^{\frac{1-2s}{2}} \mathbf{m}_s \phi(2, 2)}{2^{1+2s} 200 \mathcal{A}}.$$

Finally, the claimed asymptotic behaviour of ϑ_s simply follows from its definition and the properties of \mathbf{m}_s , encoded in Theorem 2.1.1. \square

2.3 Some consequences

In this section we highlight some consequences of the Theorem 1. In particular we prove that the positivity of λ_1^s and the boundedness of the inradius are two equivalent conditions, if s is “large enough”. This allows us to introduce a new fractional space $\mathcal{D}_0^{s,2}(\Omega)$, and to identify it with $\widetilde{W}_0^{s,2}(\Omega)$.

To get the section self-contained, let us introduce the set $\Theta \subsetneq \mathbb{R}^2$ we will consider in Section 4.1. Let

$$\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma^{(i)}, \quad \text{where } \Sigma^{(i)} := \{(x_1, i) \in \mathbb{R}^2 : |x_1| \geq 1\},$$

then we define $\Theta := \mathbb{R}^2 \setminus \Sigma$.

Corollary 2.3.1. *Let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k for some $k \in \mathbb{N} \setminus \{0\}$. Then we have:*

- for $1/2 < s < 1$

$$\lambda_1^s(\Omega) > 0 \quad \iff \quad r_\Omega < +\infty;$$

- for $0 < s \leq 1/2$

$$\lambda_1^s(\Omega) > 0 \quad \implies \quad r_\Omega < +\infty,$$

but

$$r_\Omega < +\infty \quad \not\implies \quad \lambda_1^s(\Omega) > 0.$$

Proof. Let $0 < s < 1$ and assume that $\lambda_1^s(\Omega) > 0$. Let $r > 0$ be such that there exists $x_0 \in \Omega$ with $B_r(x_0) \subseteq \Omega$. By using the monotonicity of λ_1^s with respect to set inclusion, we get

$$\lambda_1^s(\Omega) \leq \lambda_1^s(B_r(x_0)) = \frac{\lambda_1^s(B_1)}{r^{2s}}.$$

The previous estimate gives

$$r < \left(\frac{\lambda_1^s(B_1)}{\lambda_1^s(\Omega)} \right)^{\frac{1}{2s}}.$$

By taking the supremum over admissible r , we get $r_\Omega < +\infty$ by definition of inradius.

For the converse implication in the case $s > 1/2$, it is sufficient to apply Theorem 1. Finally, by taking Θ introduced above, we get an open set with finite inradius, but vanishing λ_1^s for $0 < s \leq 1/2$. \square

Our main results permit to compare two different Sobolev spaces, built up of functions “vanishing at the boundary”. More precisely, let us denote by $\mathcal{D}_0^{s,2}(\Omega)$ the *completion* of $C_0^\infty(\Omega)$ with respect to the norm

$$u \mapsto [u]_{W^{s,2}(\mathbb{R}^N)}, \quad \text{for every } u \in C_0^\infty(\Omega).$$

Observe that this is indeed a norm on $C_0^\infty(\Omega)$. We refer to [24] for more details on this space. We also recall that by $\widetilde{W}_0^{s,2}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{s,2}(\mathbb{R}^N)$.

We have the following

Corollary 2.3.2. *Let $1/2 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k for some $k \in \mathbb{N} \setminus \{0\}$, with finite inradius. Then*

$$\mathcal{D}_0^{s,2}(\Omega) = \widetilde{W}_0^{s,2}(\Omega).$$

On the contrary, for $0 < s \leq 1/2$ and Θ the infinite complement comb introduced above, the two spaces

$$\mathcal{D}_0^{s,2}(\Theta) \quad \text{and} \quad \widetilde{W}_0^{s,2}(\Theta),$$

can not be identified with each other.

Proof. For $1/2 < s < 1$ and an open simply connected set $\Omega \subseteq \mathbb{R}^2$, by Theorem 1 the two norms

$$[u]_{W^{s,2}(\mathbb{R}^2)} \quad \text{and} \quad \|u\|_{W^{s,2}(\mathbb{R}^2)},$$

are equivalent on $C_0^\infty(\Omega)$. This proves the first point.

As for the second statement, it is sufficient to observe that $\widetilde{W}_0^{s,2}(\Theta)$ is always continuously embedded in $L^2(\Theta)$, by its very definition. On the other hand, for $0 < s \leq 1/2$ such an embedding does not hold for $\mathcal{D}_0^{s,2}(\Theta)$, since $\lambda_1^s(\Theta) = 0$ (see Remark 4.1.2 below). \square

Remark 2.3.3. *If $k = +\infty$ the previous corollary is false even when $1/2 < s < 1$. A counterexample is $\Omega = \mathbb{R}^2 \setminus \mathbb{Z}^2$.*

We now show how Theorem 1 implies some fractional versions of the classical *Cheeger's inequality*, a fundamental result in Spectral Geometry. At this aim, for an open set $\Omega \subseteq \mathbb{R}^N$ we recall the definition of *Cheeger constant*

$$h_1(\Omega) = \inf \left\{ \frac{P(E)}{|E|} : E \subseteq \Omega \text{ bounded and measurable with } |E| > 0 \right\},$$

and *s-Cheeger constant* (for $0 < s < 1$)

$$h_s(\Omega) = \inf \left\{ \frac{P_s(E)}{|E|} : E \subseteq \Omega \text{ bounded and measurable with } |E| > 0 \right\},$$

see [21] for some properties of this constant. Here P stands for the *perimeter* of a set in the sense of De Giorgi, while P_s is the *s-perimeter* of a set, defined by

$$P_s(E) = [1_E]_{W^{s,1}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|1_E(x) - 1_E(y)|}{|x - y|^{N+s}} dx dy,$$

for any measurable set $E \subseteq \mathbb{R}^N$. Then we have the following

Corollary 2.3.4 (Fractional Cheeger inequality). *Let $1/2 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k for some $k \in \mathbb{N} \setminus \{0\}$, with finite inradius. Then we have*

$$\lambda_1^s(\Omega) \geq \frac{\vartheta_s}{k^s} \left(\frac{h_1(\Omega)}{2} \right)^{2s},$$

and

$$\lambda_1^s(\Omega) \geq \frac{\vartheta_s}{k^s} \left(\frac{\pi}{P_s(B_1)} h_s(\Omega) \right)^2.$$

where ϑ_s is the same constant as in Theorem 1.

Proof. Let $r < r_\Omega$, by definition of inradius there exists a disk $B_r(x_0) \subseteq \Omega$. By using this disk as a competitor for the minimization problem defining $h_1(\Omega)$, we get

$$h_1(\Omega) \leq \frac{2\pi r}{\pi r^2} = \frac{2}{r}.$$

By taking the supremum over admissible r , we get

$$h_1(\Omega) \leq \frac{2}{r_\Omega}.$$

By raising to the power $2s$ and using Theorem 1, we get the first inequality. The second one can be obtained in exactly the same way. \square

Finally, we have the following result, which permits to compare $\lambda_1^s(\Omega)$ and $\lambda_1(\Omega)$, for simply connected sets in the plane. We refer to [25, Theorem 6.1] and [32, Theorem 4.5] for a similar result in general dimension $N \geq 2$, under stronger regularity assumptions on the sets.

Corollary 2.3.5 (Comparison of eigenvalues). *Let $1/2 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k for some $k \in \mathbb{N} \setminus \{0\}$, with finite inradius. Then we have*

$$\frac{\alpha_s}{k^s} \left(\lambda_1(\Omega) \right)^s \leq \lambda_1^s(\Omega) \leq \beta_s \left(\lambda_1(\Omega) \right)^s,$$

where α_s, β_s are two positive constants depending on s only, such that

$$\alpha_s \sim \left(s - \frac{1}{2} \right), \quad \text{for } s \searrow \frac{1}{2}, \quad \text{and} \quad \alpha_s \sim \frac{1}{1-s}, \quad \text{for } s \nearrow 1,$$

$$\beta_s \sim \frac{1}{1-s}, \quad \text{for } s \nearrow 1.$$

Proof. The upper bound follows directly from the general result of [25, Theorem 6.1], see equation (6.1) there. From this reference, we can also extract a value for the constant β_s , which is given by

$$\beta_s = \frac{4^{1-s}}{s(1-s)} \pi.$$

For the lower bound, the proof is similar to that of Corollary 2.3.4, it is sufficient to join the estimate

$$\lambda_1(\Omega) \leq \frac{\lambda_1(B_1)}{r_\Omega^2},$$

with Theorem 1. This gives the claimed estimate, with constant

$$\alpha_s = \frac{\vartheta_s}{(\lambda_1(B_1))^s},$$

and ϑ_s is the same as in (I.8). \square

Chapter 3

The fractional Makai-Hayman inequality

In this chapter, which is based on [B2], we will give another proof of the main result of Chapter 2, in the case of *simply connected set*. Indeed, we will show that in this case one could adapt to the fractional setting the simple argument by Hayman (see [63]). Apart from being simpler than the proof based on Taylor's ideas, the adaptation to the fractional case needs a couple of technical results, which are interesting in themselves. We will thus show the following

Theorem (Fractional Makai-Hayman inequality). *Let $1/2 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open simply connected set, with finite inradius r_Ω . There exists an explicit universal constant $C_s > 0$ such that*

$$\lambda_1^s(\Omega) \geq \frac{C_s}{r_\Omega^{2s}}.$$

Moreover, C_s has the following asymptotic behaviours

$$C_s \sim \left(s - \frac{1}{2}\right), \quad \text{for } s \searrow \frac{1}{2} \quad \text{and} \quad C_s \sim \frac{1}{1-s}, \quad \text{for } s \nearrow 1.$$

3.1 An expedient Poincaré inequality

The main result of this section is a nonlocal counterpart of [63, Lemma 1] of Hayman's paper. In the proof we pay due attention to the dependence of the constant on the fractional parameter s , as always.

For a fixed $T > 0$, we recall the definition of the one-dimensional torus $\mathbb{S}_T^1 = \mathbb{R}/(T\mathbb{Z})$, endowed with the norm

$$|\theta - \varphi|_{\mathbb{S}_T^1} = \min_{k \in \mathbb{Z}} |\theta - \varphi + kT|, \quad \text{for } \theta, \varphi \in \mathbb{R},$$

that will introduce in Section A.3.

Proposition 3.1.1. *Let $1/2 < s < 1$ and $T > 0$. Let $\theta_0 \in [0, T]$, there exists a constant $\mu_s > 0$ depending on s only such that for every Lipschitz function $w : \mathbb{R} \rightarrow \mathbb{R}$ which is T -periodic and vanishing at θ_0 , we have*

$$\mu_s \left(\frac{2\pi}{T}\right)^{2s} \int_0^T |w(\theta)|^2 d\theta \leq \iint_{[0,T] \times [0,T]} \frac{|w(\theta) - w(\varphi)|^2}{|\theta - \varphi|_{\mathbb{S}_T^1}^{1+2s}} d\theta d\varphi, \quad (3.1.1)$$

Moreover, the constant μ_s has the following asymptotic behaviours

$$\mu_s \sim \left(s - \frac{1}{2}\right), \quad \text{for } s \searrow \frac{1}{2} \quad \text{and} \quad \mu_s \sim \frac{1}{1-s}, \quad \text{for } s \nearrow 1.$$

Proof. Without loss of generality, we can assume that $\theta_0 = 0$ and $T = 2\pi$. Thus, in this case we have $|\cdot|_{\mathbb{S}_{2\pi}^1} = |\cdot|_{\mathbb{S}^1}$, with the notation of Lemma A.3.1.

Thanks to the periodicity of w , we can expand it in Fourier series, i.e. we can write

$$w(\theta) = \sum_{n \in \mathbb{Z}} \widehat{w}(n) e^{in\theta}, \quad \text{where} \quad \widehat{w}(n) = \frac{1}{2\pi} \int_0^{2\pi} w(\theta) e^{-in\theta} d\theta.$$

The series is uniformly converging, thanks to the assumption on w . We will achieve the claimed result by joining the following two estimates

$$\iint_{[0, 2\pi] \times [0, 2\pi]} \frac{|w(\theta) - w(\varphi)|^2}{|\theta - \varphi|_{\mathbb{S}^1}^{1+2s}} d\theta d\varphi \geq C_{1,s} \sum_{n \in \mathbb{Z}} |n|^{2s} |\widehat{w}(n)|^2, \quad (3.1.2)$$

and

$$\int_0^{2\pi} |w(\theta)|^2 d\theta \leq C_{2,s} \sum_{n \in \mathbb{Z}} |n|^{2s} |\widehat{w}(n)|^2, \quad (3.1.3)$$

that we prove separately. This would give (3.1.1), with constant $\mu_s = C_{1,s}/C_{2,s}$. In the last part of the proof, we will then prove that such a constant has the claimed asymptotics.

Proof of (3.1.2). We proceed similarly as in the proof of [37, Proposition 3.4], with suitable adaptations. The latter deals with $W^{s,2}$ functions on \mathbb{R} and their Fourier transform.

First of all, we rewrite the Gagliardo-Slobodeckii seminorm as follows: let us apply the change of variable $h = \varphi - \theta$, so to get

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \frac{|w(\varphi) - w(\theta)|^2}{|\varphi - \theta|_{\mathbb{S}^1}^{1+2s}} d\theta d\varphi &= \int_0^{2\pi} \int_{-\theta}^{2\pi-\theta} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh \\ &= \int_0^{2\pi} \int_{-\theta}^0 \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh \\ &\quad + \int_0^{2\pi} \int_0^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh \\ &\quad - \int_0^{2\pi} \int_{2\pi-\theta}^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh. \end{aligned}$$

On the third integral, we can use that the integrand is 2π -periodic, thus we get

$$\begin{aligned} \int_0^{2\pi} \int_{2\pi-\theta}^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh &= \int_0^{2\pi} \int_{2\pi-\theta}^{2\pi} \frac{|w(\theta+h-2\pi) - w(\theta)|^2}{|h-2\pi|_{\mathbb{S}^1}^{1+2s}} d\theta dh \\ &= \int_0^{2\pi} \int_{-\theta}^0 \frac{|w(\theta+\eta) - w(\theta)|^2}{|\eta|_{\mathbb{S}^1}^{1+2s}} d\theta d\eta. \end{aligned}$$

This finally permits to infer that

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|w(\varphi) - w(\theta)|^2}{|\varphi - \theta|_{\mathbb{S}^1}^{1+2s}} d\theta d\varphi = \int_0^{2\pi} \int_0^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh.$$

By using the definition of $|\cdot|_{\mathbb{S}^1}$ explicited in (A.3.1), we can conclude that

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh &= \int_0^\pi \frac{1}{h^{1+2s}} \left(\int_0^{2\pi} |w(\theta+h) - w(\theta)|^2 d\theta \right) dh \\ &+ \int_\pi^{2\pi} \frac{1}{(2\pi-h)^{1+2s}} \left(\int_0^{2\pi} |w(\theta+h) - w(\theta)|^2 d\theta \right) dh. \end{aligned} \quad (3.1.4)$$

Now, for every h we denote by $w_h(\theta)$ the translation $w_h(\theta) = w(\theta+h)$. Thanks to the well-known properties of the Fourier coefficients, we have

$$\widehat{w}_h(n) = e^{ihn} \widehat{w}(n), \quad \text{for every } n \in \mathbb{Z}. \quad (3.1.5)$$

By using Plancherel's identity in (3.1.4) and then applying (3.1.5), we finally obtain

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh \\ = 2\pi \sum_{n \in \mathbb{Z} \setminus \{0\}} \left(\int_0^\pi \frac{|e^{ihn} - 1|^2}{h^{1+2s}} dh + \int_\pi^{2\pi} \frac{|e^{ihn} - 1|^2}{(2\pi-h)^{1+2s}} dh \right) |\widehat{w}(n)|^2. \end{aligned} \quad (3.1.6)$$

By recalling the identities (A.3.2), we have

$$|e^{ihn} - 1|^2 = 2(1 - \cos(hn)),$$

and applying the change of variable $\tau = hn$ with $n \in \mathbb{Z} \setminus \{0\}$, we can rewrite the first integral on the right-hand side of (3.1.6) as

$$\begin{aligned} \int_0^\pi \frac{|e^{ihn} - 1|^2}{h^{1+2s}} dh &= 2 \int_0^\pi \frac{1 - \cos(hn)}{h^{1+2s}} dh \\ &= 2 \int_0^{\pi n} \frac{1 - \cos \tau}{\left(\frac{\tau}{n}\right)^{1+2s}} \frac{d\tau}{n} \geq 2|n|^{2s} \int_0^\pi \frac{1 - \cos \tau}{\tau^{2s}} \frac{d\tau}{\tau}. \end{aligned}$$

For the second integral, it is sufficient to observe that by periodicity

$$\begin{aligned} \int_\pi^{2\pi} \frac{|e^{ihn} - 1|^2}{(2\pi-h)^{1+2s}} dh &= 2 \int_\pi^{2\pi} \frac{1 - \cos(hn)}{(2\pi-h)^{1+2s}} dh \\ &= 2 \int_\pi^{2\pi} \frac{1 - \cos(2\pi n - hn)}{(2\pi-h)^{1+2s}} dh \\ &= 2 \int_0^\pi \frac{1 - \cos(hn)}{h^{1+2s}} dh \geq 2|n|^{2s} \int_0^\pi \frac{1 - \cos \tau}{\tau^{2s}} \frac{d\tau}{\tau}. \end{aligned}$$

Thus, from (3.1.6) we get in particular

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|w(\theta+h) - w(\theta)|^2}{|h|_{\mathbb{S}^1}^{1+2s}} d\theta dh \geq 8\pi \left(\int_0^\pi \frac{1 - \cos \tau}{\tau^{2s}} \frac{d\tau}{\tau} \right) \sum_{n \in \mathbb{Z}} |n|^{2s} |\widehat{w}(n)|^2.$$

This finally proves (3.1.2), with constant

$$C_{1,s} = 8\pi \int_0^\pi \frac{1 - \cos \tau}{\tau^{2s}} \frac{d\tau}{\tau}.$$

Proof of (3.1.3): from Plancherel's identity, we know that

$$\frac{1}{2\pi} \int_0^{2\pi} |w(\theta)|^2 d\theta = \sum_{n \in \mathbb{Z}} |\widehat{w}(n)|^2. \quad (3.1.7)$$

By using the Fourier expansion for w and the assumption $w(0) = w(2\pi) = 0$, we can infer that

$$0 = w(0) = \sum_{n \in \mathbb{Z}} \widehat{w}(n).$$

This in turn implies that

$$|\widehat{w}(0)| = \left| \sum_{n \in \mathbb{Z} \setminus \{0\}} \widehat{w}(n) \right| \leq \sum_{n \in \mathbb{Z} \setminus \{0\}} |\widehat{w}(n)|,$$

and so we can obtain

$$\sum_{n \in \mathbb{Z}} |\widehat{w}(n)|^2 = \sum_{n \in \mathbb{Z} \setminus \{0\}} |\widehat{w}(n)|^2 + |\widehat{w}(0)|^2 \leq \sum_{n \in \mathbb{Z} \setminus \{0\}} |\widehat{w}(n)|^2 + \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} |\widehat{w}(n)| \right)^2. \quad (3.1.8)$$

We now estimate the last term in (3.1.8) by using Hölder's inequality

$$\sum_{n \in \mathbb{Z}} |a_n| |b_n| \leq \left(\sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} |b_n|^2 \right)^{\frac{1}{2}},$$

with the choices $|a_n| = 1/|n|^s$ and $|b_n| = |\widehat{w}(n)| |n|^s$. This yields

$$\begin{aligned} \sum_{n \in \mathbb{Z}} |\widehat{w}(n)|^2 &\leq \sum_{n \in \mathbb{Z} \setminus \{0\}} |\widehat{w}(n)|^2 + \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{|n|^{2s}} \right) \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{2s} |\widehat{w}(n)|^2 \right) \\ &\leq C \left(\sum_{n \in \mathbb{Z} \setminus \{0\}} |n|^{2s} |\widehat{w}(n)|^2 \right), \end{aligned}$$

where we set

$$C = 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}}.$$

Observe that this is a finite quantity, thanks to the crucial assumption $s > 1/2$. By using this estimate in (3.1.7), we then obtain the claimed inequality (3.1.3), with constant

$$C_{2,s} = 2\pi \left(1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right).$$

Asymptotic behaviour of the constant. As we said, from the above discussion we get the inequality (3.1.1), with $\mu_s = C_{1,s}/C_{2,s}$. It is easily seen that

$$\lim_{s \rightarrow (\frac{1}{2})^+} C_{1,s} = 8\pi \int_0^\pi \frac{1 - \cos \tau}{\tau} \frac{d\tau}{\tau} < +\infty,$$

while

$$\lim_{s \rightarrow (\frac{1}{2})^+} (2s - 1) C_{2,s} = 2\pi \lim_{s \rightarrow (\frac{1}{2})^+} (2s - 1) \left(1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2 s} \right) = 4\pi,$$

by using the fact that the Riemann zeta function has a simple pole with residue 1 at $z = 1$ (see [74, Section 13.2.6]). This proves that μ_s has the claimed asymptotic behaviour, as s goes to $1/2$.

As for the behaviour at $s \nearrow 1$, we observe that

$$\lim_{s \rightarrow 1^-} C_{2,s} = 2\pi \left(1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \right) = 2\pi \left(1 + \frac{\pi^2}{3} \right),$$

while

$$\begin{aligned} \lim_{s \rightarrow 1^-} (1-s) C_{1,s} &= 8\pi \lim_{s \rightarrow 1^-} (1-s) \int_0^\pi \frac{1 - \cos \tau}{\tau^{2s}} \frac{d\tau}{\tau} \\ &= 4\pi \lim_{s \rightarrow 1^-} (1-s) \int_0^\pi \tau^{2-2s} \frac{d\tau}{\tau} \\ &\quad - 4\pi \lim_{s \rightarrow 1^-} (1-s) \int_0^\pi \frac{\int_0^\tau (\tau - \ell)^2 \sin \ell \, d\ell}{\tau^{2s}} \frac{d\tau}{\tau} = 2\pi, \end{aligned}$$

where we used the third order Taylor expansion

$$f(\tau) = f(0) + f'(0)\tau + \frac{1}{2}f''(0)\tau^2 + \frac{1}{2} \int_0^\tau f'''(\ell) (\tau - \ell)^2 \, d\ell,$$

for the cosine function. This eventually leads to the conclusion of the proof. \square

Now we are ready to prove the following

Proposition 3.1.2 (Poincaré inequality for boundary disks). *Let $1/2 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open simply connected set, with $\partial\Omega \neq \emptyset$. There exists a constant $\mathcal{T}_s > 0$ depending on s only, such that for every $r > 0$ and every $x_0 \in \partial\Omega$, we have*

$$\frac{\mathcal{T}_s}{r^{2s}} \int_{B_r(x_0)} |u(x)|^2 \, dx \leq \iint_{B_r(x_0) \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} \, dx \, dy, \quad \text{for every } u \in C_0^\infty(\Omega).$$

Moreover, \mathcal{T}_s has the following asymptotic behaviours

$$\mathcal{T}_s \sim \left(s - \frac{1}{2} \right), \quad \text{for } s \searrow \frac{1}{2} \quad \text{and} \quad \mathcal{T}_s \sim \frac{1}{1-s}, \quad \text{for } s \nearrow 1.$$

Proof. Up to scaling and translating, we can assume without loss of generality that $r = 1$ and that x_0 coincides with the origin.

We split the proof in three main steps: we first show that it is sufficient to prove the claimed estimate for the boundary ring $B_1 \setminus B_{1/2}$. Then we prove such an estimate and at last we discuss the asymptotic behaviour of the constant obtained.

Step 1: reduction to a ring. Let $u \in C_0^\infty(\Omega)$, we then estimate the L^2 norm on B_1 as follows

$$\begin{aligned} \int_{B_1} |u(x)|^2 \, dx &= \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx + \int_{B_{1/2}} |u(x)|^2 \, dx \\ &\leq \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx + 2 \int_{B_{1/2}} |u(x) - \bar{u}_{B_1 \setminus B_{1/2}}|^2 \, dx + 2 \int_{B_{1/2}} |\bar{u}_{B_1 \setminus B_{1/2}}|^2 \, dx \\ &\leq \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx + 2 \int_{B_1} |u(x) - \bar{u}_{B_1 \setminus B_{1/2}}|^2 \, dx + 2 \int_{B_{1/2}} |\bar{u}_{B_1 \setminus B_{1/2}}|^2 \, dx \\ &\leq \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx + 2 \int_{B_1} |u(x) - \bar{u}_{B_1 \setminus B_{1/2}}|^2 \, dx + \frac{2}{3} \int_{B_1 \setminus B_{1/2}} |u(x)|^2 \, dx, \end{aligned}$$

where we used the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$ and Jensen's inequality. If we now apply Corollary 1.3.4 with $R = 1$ and $E = B_1 \setminus B_{1/2}$, we get

$$\int_{B_1} |u(x)|^2 dx \leq \frac{5}{3} \int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx + \frac{8}{3\pi} \mathcal{M}(1-s) [u]_{W^{s,2}(B_1)}^2.$$

Thus, in order to conclude, it is sufficient to prove that there exists a constant $C = C(s) > 0$ such that

$$\int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \leq C \iint_{B_1 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy, \quad \text{for every } u \in C_0^\infty(\Omega). \quad (3.1.9)$$

Step 2: estimate on the ring. We start with a topological observation. Since we are assuming that $0 \in \partial\Omega$ and that Ω is simply connected, we have the following crucial property

$$\partial B_\varrho \cap (\mathbb{R}^2 \setminus \Omega) \neq \emptyset, \quad \text{for every } \varrho > 0. \quad (3.1.10)$$

Indeed, if this were not true, we would have existence of a circle entirely contained in Ω and centered on the boundary of $\partial\Omega$. Such a circle could not be null-homotopic in Ω , thus contradicting our topological assumption.

In the rest of the proof, we will use polar coordinates (ϱ, θ) and we will make the slight abuse of notation of writing $u(\varrho, \theta)$. Then, in light of the property (3.1.10), for each $\varrho \in (1/2, 1)$ there exists $\theta_\varrho \in [0, 2\pi)$ such that $\theta \mapsto u(\varrho, \theta)$ must vanish at θ_ϱ . Hence, for every $1/2 < \varrho < 1$ we can apply Proposition 3.1.1 to the function $\theta \mapsto u(\varrho, \theta)$ and get

$$\int_0^{2\pi} |u(\varrho, \theta)|^2 d\theta \leq \frac{1}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \frac{|u(\varrho, \theta) - u(\varrho, \varphi)|^2}{|\theta - \varphi|_{\mathbb{S}^1}^{1+2s}} d\theta d\varphi.$$

The constant μ_s is the same as in Proposition 3.1.1 and

$$|\theta - \varphi|_{\mathbb{S}^1} := \min_{k \in \mathbb{Z}} |\theta - \varphi + 2k\pi|, \quad \text{for every } \theta, \varphi \in \mathbb{R}.$$

If we now multiply both sides by ϱ , integrate over the interval $(1/2, 1)$ and write the L^2 norm in polar coordinates, we get

$$\int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \leq \frac{1}{\mu_s} \int_{\frac{1}{2}}^1 \int_0^{2\pi} \int_0^{2\pi} \frac{|u(\varrho, \theta) - u(\varrho, \varphi)|^2}{\varrho^{1+2s} |\theta - \varphi|_{\mathbb{S}^1}^{1+2s}} \varrho d\varrho d\theta d\varphi. \quad (3.1.11)$$

Observe that we further used the fact that $\varrho \leq 1$, to let the term ϱ^{-1-2s} appear.

In order to achieve (3.1.9), we need to show that the term on the right-hand side can be estimated by a two-dimensional Gagliardo-Slobodeckii seminorm. To this aim, we follow an argument similar to that of [16, Lemma B.2]. At first, it is easily seen that

$$(\varrho |\theta - \varphi|_{\mathbb{S}^1})^{-1-2s} = (1 + 2s) \int_0^{+\infty} (t + \varrho |\theta - \varphi|_{\mathbb{S}^1})^{-2-2s} dt.$$

By inserting this in (3.1.11), we end up with

$$\begin{aligned} & \int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \\ & \leq \frac{1 + 2s}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\varrho, \varphi)|^2}{(t + \varrho |\theta - \varphi|_{\mathbb{S}^1})^{2+2s}} \varrho d\theta d\varphi d\varrho dt \\ & \leq \frac{2(1 + 2s)}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\varrho, \varphi)|^2}{(t + \varrho |\theta - \varphi|_{\mathbb{S}^1})^{2+2s}} \varrho (\varrho + t) d\theta d\varphi d\varrho dt, \end{aligned} \quad (3.1.12)$$

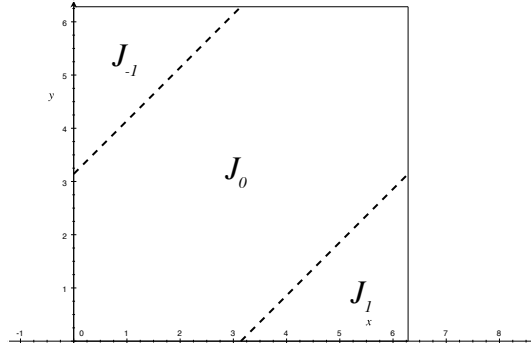


Figure 3.1: The partition of $[0, 2\pi] \times [0, 2\pi]$ needed to define the midpoint function.

where we have used that $1/2 \leq \varrho + t$. We now split the set $[0, 2\pi] \times [0, 2\pi] = J_{-1} \cup J_0 \cup J_1$, where

$$J_{-1} = \left\{ (\theta, \varphi) : \theta \in [0, \pi], \theta + \pi < \varphi \leq 2\pi \right\}, \quad J_1 = \left\{ (\theta, \varphi) : \theta \in [\pi, 2\pi], 0 \leq \varphi < \theta - \pi \right\},$$

and

$$J_0 = \left\{ (\theta, \varphi) : \theta \in [0, 2\pi], \max\{0, \theta - \pi\} \leq \varphi \leq \min\{2\pi, \theta + \pi\} \right\},$$

see Figure 3.1. Then, we define the *midpoint function* by

$$\overline{\theta\varphi} = \frac{\theta + \varphi}{2} + \ell\pi, \quad \text{if } (\theta, \varphi) \in J_\ell, \text{ with } \ell = -1, 0, 1. \quad (3.1.13)$$

Thanks to the Minkowski's inequality, we estimate the numerator in the right-hand side of (3.1.12) as follows

$$|u(\varrho, \theta) - u(\varrho, \varphi)|^2 \leq 2 |u(\varrho, \theta) - u(\varrho + t, \overline{\theta\varphi})|^2 + 2 |u(\varrho, \varphi) - u(\varrho + t, \overline{\theta\varphi})|^2.$$

As for the denominator, we observe that $|\theta - \overline{\theta\varphi}|_{\mathbb{S}^1} = |\varphi - \overline{\theta\varphi}|_{\mathbb{S}^1}$, thus we get

$$|\theta - \varphi|_{\mathbb{S}^1} = 2|\theta - \overline{\theta\varphi}|_{\mathbb{S}^1} \geq 2|e^{i\theta} - e^{i\overline{\theta\varphi}}| \quad \text{and} \quad |\theta - \varphi|_{\mathbb{S}^1} = 2|\varphi - \overline{\theta\varphi}|_{\mathbb{S}^1} \geq 2|e^{i\varphi} - e^{i\overline{\theta\varphi}}|,$$

where the inequalities come from Lemma A.3.1 in the appendix. By using this fact, the identity $|e^{i\overline{\theta\varphi}}| = 1$ and the triangle inequality again, we can estimate the denominator as

$$\begin{aligned} t + \varrho |\theta - \varphi|_{\mathbb{S}^1} &\geq t + \varrho |e^{i\theta} - e^{i\overline{\theta\varphi}}| \\ &\geq \left| \varrho (e^{i\theta} - e^{i\overline{\theta\varphi}}) - t e^{i\overline{\theta\varphi}} \right| = \left| \varrho e^{i\theta} - (\varrho + t) e^{i\overline{\theta\varphi}} \right|, \end{aligned}$$

and similarly

$$t + \varrho |\theta - \varphi|_{\mathbb{S}^1} \geq \left| \varrho e^{i\varphi} - (\varrho + t) e^{i\overline{\theta\varphi}} \right|.$$

These allow us to estimate the right-hand side in (3.1.12) in the following way:

$$\begin{aligned}
& \int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \\
& \leq \frac{4(1+2s)}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\varrho+t, \overline{\theta\varphi})|^2}{|\varrho e^{i\theta} - (\varrho+t) e^{i\overline{\theta\varphi}}|^{2+2s}} \varrho(\varrho+t) d\theta d\varphi d\varrho dt \\
& + \frac{4(1+2s)}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \varphi) - u(\varrho+t, \overline{\theta\varphi})|^2}{|\varrho e^{i\varphi} - (\varrho+t) e^{i\overline{\theta\varphi}}|^{2+2s}} \varrho(\varrho+t) d\theta d\varphi d\varrho dt \\
& = \frac{8(1+2s)}{\mu_s} \int_0^{2\pi} \int_0^{2\pi} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\varrho+t, \overline{\theta\varphi})|^2}{|\varrho e^{i\theta} - (\varrho+t) e^{i\overline{\theta\varphi}}|^{2+2s}} \varrho(\varrho+t) d\theta d\varphi d\varrho dt.
\end{aligned}$$

In the last identity we used that both multiple integrals coincide, by symmetry of the integrands. If we now make the change of variable $\tau = \varrho + t$ and use the decomposition $[0, 2\pi] \times [0, 2\pi] = J_{-1} \cup J_0 \cup J_1$, we obtain

$$\begin{aligned}
& \int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \\
& \leq \frac{8(1+2s)}{\mu_s} \sum_{\ell=-1}^1 \iint_{J_\ell} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \overline{\theta\varphi})|^2}{|\varrho e^{i\theta} - \tau e^{i\overline{\theta\varphi}}|^{2+2s}} \varrho \tau d\theta d\varphi d\varrho d\tau. \tag{3.1.14}
\end{aligned}$$

If we now denote

$$\tilde{J}_{-1} = \left\{ (\theta, \varphi) : \theta \in [0, \pi], \theta - \pi < \varphi \leq 0 \right\} \quad \text{and} \quad \tilde{J}_1 = \left\{ (\theta, \varphi) : \theta \in [\pi, 2\pi], 2\pi \leq \varphi < \theta + \pi \right\},$$

use the definition of midpoint function (3.1.13) and make the change of variables

$$\begin{aligned}
J_{-1} & \rightarrow \tilde{J}_{-1} & J_1 & \rightarrow \tilde{J}_1 \\
& \text{and} \\
(\theta, \varphi) & \mapsto (\theta, \varphi - 2\pi) & (\theta, \varphi) & \mapsto (\theta, \varphi + 2\pi),
\end{aligned}$$

we obtain from (3.1.14)

$$\int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \leq \frac{8(1+2s)}{\mu_s} \iint_{\tilde{J}_{-1} \cup J_0 \cup \tilde{J}_1} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u\left(\tau, \frac{\theta+\varphi}{2}\right)|^2}{|\varrho e^{i\theta} - \tau e^{i\frac{\theta+\varphi}{2}}|^{2+2s}} \varrho \tau d\theta d\varphi d\varrho d\tau.$$

For every $\theta \in [0, 2\pi]$, we now make the change of variable $\gamma = (\theta + \varphi)/2$, thus the above estimate becomes

$$\int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx \leq \frac{16(1+2s)}{\mu_s} \sum_{\ell=-1}^1 \iint_{\hat{J}_\ell} \int_{\frac{1}{2}}^1 \int_0^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \gamma)|^2}{|\varrho e^{i\theta} - \tau e^{i\gamma}|^{2+2s}} \varrho \tau d\theta d\gamma d\varrho d\tau, \tag{3.1.15}$$

where

$$\hat{J}_{-1} = \left\{ (\theta, \gamma) : \theta \in \left[0, \frac{\pi}{2}\right], \theta - \frac{\pi}{2} < \gamma \leq 0 \right\}, \quad \hat{J}_1 = \left\{ (\theta, \gamma) : \theta \in \left[\frac{3}{2}\pi, 2\pi\right], 2\pi \leq \gamma < \theta + \frac{\pi}{2} \right\},$$

and

$$\widehat{J}_0 = \left\{ (\theta, \gamma) : \theta \in [0, 2\pi], \max \left\{ 0, \theta - \frac{\pi}{2} \right\} \leq \gamma \leq \min \left\{ 2\pi, \theta + \frac{\pi}{2} \right\} \right\}.$$

If we now exploit the 2π -periodicity of the integrand, we have

$$\begin{aligned} & \iint_{\widehat{J}_{-1}} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \gamma)|^2}{|\varrho e^{i\theta} - \tau e^{i\gamma}|^{2+2s}} \varrho \tau d\theta d\gamma d\varrho d\tau \\ &= \iint_{\widehat{J}_{-1}} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \gamma + 2\pi)|^2}{|\varrho e^{i\theta} - \tau e^{i(\gamma+2\pi)}|^{2+2s}} \varrho \tau d\theta d\gamma d\varrho d\tau \\ &= \iint_{I_{-1}} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \varphi)|^2}{|\varrho e^{i\theta} - \tau e^{i\varphi}|^{2+2s}} \varrho \tau d\theta d\varphi d\varrho d\tau, \end{aligned} \quad (3.1.16)$$

where we set $\varphi = \gamma + 2\pi$ and

$$I_{-1} = \left\{ (\theta, \varphi) : \theta \in \left[0, \frac{\pi}{2} \right], \theta + \frac{3}{2}\pi < \varphi \leq 2\pi \right\}.$$

Similarly, we can obtain

$$\begin{aligned} & \iint_{\widehat{J}_1} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \gamma)|^2}{|\varrho e^{i\theta} - \tau e^{i\gamma}|^{2+2s}} \varrho \tau d\theta d\gamma d\varrho d\tau \\ &= \iint_{I_1} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \varphi)|^2}{|\varrho e^{i\theta} - \tau e^{i\varphi}|^{2+2s}} \varrho \tau d\theta d\varphi d\varrho d\tau, \end{aligned} \quad (3.1.17)$$

with the change of variable $\varphi = \gamma - 2\pi$ and

$$I_1 = \left\{ (\theta, \varphi) : \theta \in \left[\frac{3}{2}\pi, 2\pi \right], 0 \leq \varphi < \theta - \frac{3}{2}\pi \right\}.$$

By observing that $I_{-1} \cup \widehat{J}_0 \cup I_1 \subseteq [0, 2\pi] \times [0, 2\pi]$ and that the three sets I_{-1} , \widehat{J}_0 and I_1 are pairwise disjoint, from (3.1.15), (3.1.16) and (3.1.17) we finally obtain

$$\begin{aligned} \int_{B_1 \setminus B_{1/2}} |u(x)|^2 dx &\leq \frac{16(1+2s)}{\mu_s} \iint_{[0, 2\pi] \times [0, 2\pi]} \int_{\frac{1}{2}}^1 \int_{\varrho}^{+\infty} \frac{|u(\varrho, \theta) - u(\tau, \varphi)|^2}{|\varrho e^{i\theta} - \tau e^{i\varphi}|^{2+2s}} \varrho \tau d\theta d\varphi d\varrho d\tau \\ &\leq \frac{16(1+2s)}{\mu_s} \iint_{B_1 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy. \end{aligned}$$

This concludes the proof of (3.1.9).

Step 3: asymptotics for the constant. From **Step 1** and **Step 2**, we obtained the Poincaré inequality claimed in the statement, with constant

$$\mathcal{T}_s = \left(\frac{80(1+2s)}{3\mu_s} + \frac{8}{3\pi} \mathcal{M}(1-s) \right)^{-1}.$$

By using the asymptotics for the constant μ_s explicitied in Proposition 3.1.1, we get the desired conclusion. \square

Remark 3.1.3. *The previous result can not hold for $0 < s \leq 1/2$. Indeed, if the result were true for $0 < s \leq 1/2$, this would permit to extend the fractional Makai-Hayman inequality to this range, as well. However, this would contradict Theorem 2 below.*

3.2 Proof of the fractional Makai-Hayman inequality

Without loss of generality, we can consider $r_\Omega = 1$. We take \mathfrak{B} and $\mathfrak{B}_1, \dots, \mathfrak{B}_{36}$ to be respectively the covering of Ω and the subclasses given by Lemma B.2.1, made of balls with radius $r = 1 + \sqrt{2}$.

We take an index $i \in \{1, \dots, 36\}$, then we know that \mathfrak{B}_i is composed of (possibly) countably many disjoint balls with radius r , centered on $\partial\Omega$. We indicate by $B^{j,i}$ each of these balls.

Then, for every $u \in C_0^\infty(\Omega) \setminus \{0\}$ we have

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy \geq \sum_{B^{j,i} \in \mathfrak{B}_i} \iint_{B^{j,i} \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy. \quad (3.2.1)$$

For each ball $B^{j,i}$, we can apply Proposition 3.1.2 so to obtain that

$$\sum_{B^{j,i} \in \mathfrak{B}_i} \iint_{B^{j,i} \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy \geq \frac{\mathcal{T}_s}{(1 + \sqrt{2})^{2s}} \sum_{B^{j,i} \in \mathfrak{B}_i} \int_{B^{j,i}} |u(x)|^2 dx.$$

We insert this estimate in (3.2.1) and then sum over $i = 1, \dots, 36$. We get

$$\begin{aligned} 36 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy &\geq \sum_{i=1}^{36} \sum_{B^{j,i} \in \mathfrak{B}_i} \iint_{B^{j,i} \times \mathbb{R}^2} \frac{|u(x) - u(y)|^2}{|x - y|^{2+2s}} dx dy \\ &\geq \frac{\mathcal{T}_s}{(1 + \sqrt{2})^{2s}} \sum_{i=1}^{36} \sum_{B^{j,i} \in \mathfrak{B}_i} \int_{B^{j,i}} |u(x)|^2 dx \\ &\geq \frac{\mathcal{T}_s}{(1 + \sqrt{2})^{2s}} \int_{\Omega} |u(x)|^2 dx. \end{aligned}$$

In the last inequality we used that \mathfrak{B} is a covering of Ω . By recalling the definition of $\lambda_1^s(\Omega)$, from the previous chain of inequalities we thus get the claimed estimate, with constant

$$\mathcal{C}_s := \frac{\mathcal{T}_s}{36(1 + \sqrt{2})^{2s}}.$$

The asymptotic behaviour of \mathcal{C}_s can now be inferred from that of \mathcal{T}_s , which in turn is contained in Proposition 3.1.2.

Remark 3.2.1. For suitable classes of open sets in \mathbb{R}^N and every $0 < s < 1$, it is possible to give a Makai-Hayman-type lower bound on λ_1^s , by taking advantage of the nonlocality of the Gagliardo-Slobodeckii seminorm. More precisely, this is possible provided Ω satisfies the following mild regularity assumption: there exist¹ $\sigma > 1$ and $\alpha > 0$ such that

$$\frac{|B_{\sigma r_\Omega}(x) \setminus \Omega|}{|B_{\sigma r_\Omega}(x)|} \geq \alpha, \quad \text{for every } x \in \Omega. \quad (3.2.2)$$

Indeed, in this case for every $u \in C_0^\infty(\Omega)$ we can simply estimate

$$\begin{aligned} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy &\geq \int_{\Omega} \left(\int_{B_{\sigma r_\Omega}(x) \setminus \Omega} \frac{|u(x)|^2}{|x - y|^{N+2s}} dy \right) dx \\ &\geq \frac{1}{(\sigma r_\Omega)^{N+2s}} \int_{\Omega} |B_{\sigma r_\Omega}(x) \setminus \Omega| |u(x)|^2 dx \\ &\geq \frac{\alpha \omega_N}{(\sigma r_\Omega)^{2s}} \int_{\Omega} |u(x)|^2 dx, \end{aligned}$$

¹It is not difficult to see that this property never holds for $\sigma = 1$.

where in the last inequality we used the additional condition (3.2.2). By arbitrariness of u , we get

$$\lambda_1^s(\Omega) \geq \frac{\alpha \omega_N}{\sigma^{2s}} \frac{1}{r_\Omega^{2s}}.$$

One could observe that the additional condition (3.2.2) does not always hold for a simply connected set in the plane. Moreover, the constant obtained in this way is quite poor: first of all, it is not universal. It depends on the parameters α and σ and it deteriorates as $\sigma \searrow 1$, since in this case we must have $\alpha \searrow 0$. Secondly, it does not exhibit the correct asymptotic behaviour as s goes to 1.

Chapter 4

Optimality of the estimates

In this chapter, still based on [B1] and [B2], we will investigate the optimality of lower bound on λ_1^s , in a suitable sense. In particular, we will first show that a fractional Makai-Hayman inequality (and hence a fractional Osserman-Taylor-Croke inequality) cannot hold for $0 < s \leq 1/2$.

Moreover, we will also prove that the asymptotic behaviour of the constant ϑ_s appearing in Theorem 1 is sharp, in the two asymptotic regimes $s \searrow 1/2$ and $s \nearrow 1$.

Last, but not least, we will show that the coefficient k^{-s} in the estimate of Theorem 1 is optimal, as k goes to ∞ .

4.1 The assumption $s > 1/2$ can not be dropped

The following theorem underlines the necessity on the assumption $s > 1/2$.

Theorem 2 (Counter-example for $0 < s \leq 1/2$). *There exists a sequence $\{Q_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}^2$ of open bounded simply connected sets such that*

$$0 < r_{Q_n} \leq C, \quad \text{for every } n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow \infty} \lambda_1^s(Q_n) = 0, \quad \text{for every } 0 < s \leq \frac{1}{2}.$$

Proof. Let $0 < s \leq 1/2$ and $\{Q_\ell\}_{\ell \in \mathbb{N}} \subseteq \mathbb{R}^2$ be the sequence of open squares $Q_\ell = (-\ell, \ell)^2$, with $\ell \in \mathbb{N} \setminus \{0, 1\}$. We introduce the one-dimensional set

$$\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma^{(i)}, \quad \text{where } \Sigma^{(i)} := \{(x_1, i) \in \mathbb{R}^2 : |x_1| \geq 1\},$$

and then define, for every fixed $\ell \in \mathbb{N} \setminus \{0, 1\}$, the “cracked” square $\tilde{Q}_\ell = Q_\ell \setminus \Sigma$ (see Figure 4.1).

First of all, we observe that

$$r_{\tilde{Q}_\ell} = \frac{\sqrt{5}}{2}, \quad \text{for every } \ell \geq 2.$$

Thus, if we can show that

$$\lim_{\ell \rightarrow \infty} \lambda_1^s(\tilde{Q}_\ell) = 0, \tag{4.1.1}$$

we would automatically get the desired counter-example. We will obtain (4.1.1) by proving that

$$\lambda_1^s(\tilde{Q}_\ell) = \lambda_1^s(Q_\ell), \quad \text{for every } \ell \geq 2. \tag{4.1.2}$$

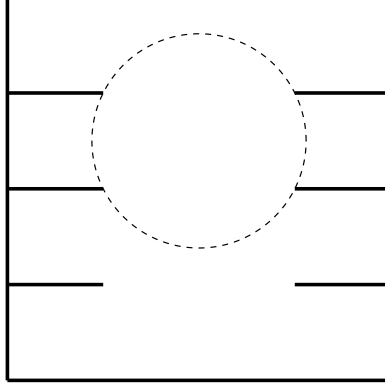


Figure 4.1: The set \tilde{Q}_ℓ with $\ell = 2$. In dashed line, a disk of maximal radius.

Indeed, if this were true, we would have

$$\lim_{\ell \rightarrow \infty} \lambda_1^s(\tilde{Q}_\ell) = \lim_{\ell \rightarrow \infty} \lambda_1^s(Q_\ell) = \lim_{\ell \rightarrow \infty} \ell^{-2s} \lambda_1^s(Q_1) = 0,$$

by the scale properties of λ_s^1 . This would prove (4.1.1), as claimed.

We are thus left with proving (4.1.2). We already know that

$$\lambda_1^s(\tilde{Q}_\ell) \geq \lambda_1^s(Q_\ell),$$

thanks to the fact that λ_1^s is monotone with respect to set inclusion. In the remaining part of the proof, we focus our attention in proving the opposite inequality.

At this aim, for every $n \in \mathbb{N} \setminus \{0\}$ we introduce the neighborhoods

$$\Sigma_{\ell,n}^{(i)} = \left\{ x \in \mathbb{R}^2 : \text{dist}(x, \Sigma^{(i)} \cap Q_\ell) \leq \frac{1}{n+1} \right\}, \quad \text{for } i \in \{-(\ell-1), \dots, \ell-1\},$$

and consider a sequence of cut-off functions $\{\varphi_n^{(i)}\}_{n \in \mathbb{N} \setminus \{0\}} \subset C_0^\infty(\Sigma_{\ell,2n}^{(i)})$ such that

$$0 \leq \varphi_n^{(i)} \leq 1, \quad \varphi_n^{(i)} \equiv 1 \text{ on } \Sigma_{\ell,4n}^{(i)}, \quad |\nabla \varphi_n^{(i)}(x)| \leq Cn,$$

for some constant $C > 0$, independent of n . Observe that by construction we have

$$\text{spt}(\varphi_n^{(i)}) \cap \text{spt}(\varphi_n^{(j)}) = \emptyset, \quad \text{for } i \neq j,$$

By using an interpolation inequality (see [22, Corollary 2.2]) and the properties of the cut-off functions, we can estimate the energy of each $\varphi_n^{(i)}$ as follows

$$\begin{aligned} [\varphi_n^{(i)}]_{W^{s,2}(\mathbb{R}^2)}^2 &\leq C \left(\int_{\Sigma_{\ell,2n}^{(i)}} |\varphi_n^{(i)}|^2 dx \right)^{1-s} \left(\int_{\Sigma_{\ell,2n}^{(i)}} |\nabla \varphi_n^{(i)}|^2 dx \right)^s \\ &\leq C |\Sigma_{\ell,2n}^{(i)}|^{1-s} |\Sigma_{\ell,2n}^{(i)}|^s n^{2s} \leq C n^{2s-1}, \end{aligned}$$

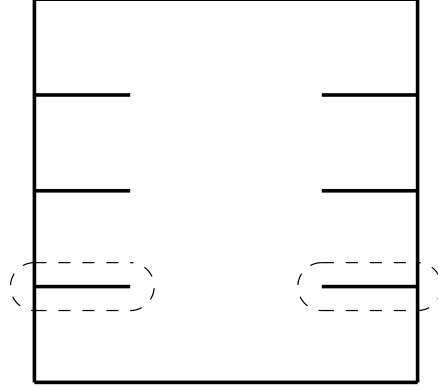


Figure 4.2: The dashed line encloses one of the set $\Sigma_{\ell,n}^{(i)}$.

for a constant $C > 0$ independent¹ of n . In particular, for every $i \in \{-(\ell-1), \dots, \ell-1\}$ we have

$$\lim_{n \rightarrow +\infty} [\varphi_n^{(i)}]_{W^{s,2}(\mathbb{R}^2)}^2 = 0, \quad \text{if } 0 < s < \frac{1}{2}, \quad (4.1.3)$$

while

$$\sup_{n \geq 1} [\varphi_n^{(i)}]_{W^{s,2}(\mathbb{R}^2)}^2 \leq C, \quad \text{if } s = \frac{1}{2}. \quad (4.1.4)$$

From now on, for ease of notation, we denote

$$\Phi_{\ell,n} = \sum_{i=-(\ell-1)}^{\ell-1} \varphi_n^{(i)} \in C_0^\infty(Q_{2\ell}).$$

Due to the different behaviours (4.1.3) and (4.1.4), we need to consider the cases $0 < s < 1/2$ and $s = 1/2$ separately.

Case $0 < s < 1/2$. For every $u \in C_0^\infty(Q_\ell) \setminus \{0\}$, we simply take

$$u_n = (1 - \Phi_{\ell,n}) u,$$

and observe that $u_n \in C_0^\infty(\tilde{Q}_\ell)$ for every $n \in \mathbb{N} \setminus \{0\}$. Since each u_n is admissible for the problem (I.5), we get

$$\sqrt{\lambda_1^s(\tilde{Q}_\ell)} \leq \frac{[u_n]_{W^{s,2}(\mathbb{R}^2)}}{\|u_n\|_{L^2(\tilde{Q}_\ell)}} \leq \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} [1 - \Phi_{\ell,n}]_{W^{s,2}(\mathbb{R}^2)}}{\|u(1 - \Phi_{\ell,n})\|_{L^2(\tilde{Q}_\ell)}}, \quad (4.1.5)$$

where in the last inequality we have used the Leibniz-type rule (1.1.6) and the fact $|1 - \Phi_{\ell,n}| \leq 1$. We now observe that

$$\lim_{n \rightarrow \infty} \|u(1 - \Phi_{\ell,n})\|_{L^2(\tilde{Q}_\ell)} = \|u\|_{L^2(Q_\ell)},$$

¹Observe that such a constant depends on ℓ , through the length of the set $\Sigma^{(i)} \cap Q_\ell$. However this is not a problem, since in this part ℓ is now fixed.

which follows from a standard application of the Lebesgue Dominated Convergence Theorem, together with the properties of $\Phi_{\ell,n}$. Moreover, it holds

$$\lim_{n \rightarrow \infty} [1 - \Phi_{\ell,n}]_{W^{s,2}(\mathbb{R}^2)} = 0.$$

This simply follows by using the definition of $\Phi_{\ell,n}$, the triangle inequality and (4.1.3). By using these two limits in (4.1.5), we get

$$\sqrt{\lambda_1^s(\tilde{Q}_\ell)} \leq \lim_{n \rightarrow \infty} \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} [1 - \Phi_{\ell,n}]_{W^{s,2}(\mathbb{R}^2)}}{\|u(1 - \Phi_{\ell,n})\|_{L^2(\tilde{Q}_\ell)}} = \frac{[u]_{W^{s,2}(\mathbb{R}^2)}}{\|u\|_{L^2(Q_\ell)}}.$$

By arbitrariness of $u \in C_0^\infty(Q_\ell) \setminus \{0\}$, we get

$$\lambda_1^s(\tilde{Q}_\ell) \leq \lambda_1^s(Q_\ell).$$

and thus the desired conclusion (4.1.2).

Borderline case $s = 1/2$. This is more delicate, we can not use directly the sequence $\{\Phi_{\ell,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ to construct an approximation of $u \in C_0^\infty(Q_\ell)$. Indeed, by owing to (4.1.4), we can now guarantee that $\{\Phi_{\ell,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ only converges weakly to 0 in $W^{1/2,2}(\mathbb{R}^2)$ as n goes to ∞ , up to a subsequence.

In order to “boost” such a sequence, we make a suitable application of *Mazur’s Lemma* (see for example [79, Theorem 2.13]). More precisely, we define the sequence $\{D^s \Phi_{\ell,n}\}_{n \in \mathbb{N} \setminus \{0\}} \subset L^2(\mathbb{R}^2 \times \mathbb{R}^2)$, given by

$$D^s \Phi_{\ell,n}(x, y) = \frac{\Phi_{\ell,n}(x) - \Phi_{\ell,n}(y)}{|x - y|^{1+\frac{1}{2}}}.$$

By construction, we have that

$$\|D^s \Phi_{\ell,n}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)} = [\Phi_{\ell,n}]_{W^{\frac{1}{2},2}(\mathbb{R}^2)} \leq C,$$

and $\{D^s \Phi_{\ell,n}\}_{n \in \mathbb{N} \setminus \{0\}}$ converges weakly to 0 in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$, up to a subsequence. Thanks to Mazur’s Lemma, we can enforce this weak convergence to the strong one, by passing to a sequence of convex combinations. More precisely, we know that for every $n \in \mathbb{N} \setminus \{0\}$ there exists

$$\{t_m(n)\}_{m=1}^n \subset [0, 1], \quad \text{such that} \quad \sum_{m=1}^n t_m(n) = 1,$$

and such that the new sequence made of convex combinations

$$\widetilde{D^s \Phi_{\ell,n}}(x, y) = \sum_{m=1}^n t_m(n) D^s \Phi_{\ell,m}(x, y),$$

strongly converges in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ to 0, as n goes to ∞ . Observe that by construction we have

$$\begin{aligned} \|\widetilde{D^s \Phi_{\ell,n}}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 &= \left\| \sum_{m=1}^n t_m(n) D^s \Phi_{\ell,m} \right\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left| \sum_{m=1}^n t_m(n) \frac{\Phi_{\ell,m}(x) - \Phi_{\ell,m}(y)}{|x - y|^{1+\frac{1}{2}}} \right|^2 dx dy \\ &= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{|\sum_{m=1}^n t_m(n) \Phi_{\ell,m}(x) - \sum_{m=1}^n t_m(n) \Phi_{\ell,m}(y)|^2}{|x - y|^3} dx dy. \end{aligned}$$

Thus, if we set

$$\tilde{\Phi}_{\ell,n} = \sum_{m=1}^n t_m(n) \Phi_{\ell,m} \in C_0^\infty(Q_{2\ell}),$$

the previous observations give that

$$\lim_{n \rightarrow \infty} [\tilde{\Phi}_{\ell,n}]_{W^{\frac{1}{2},2}(\mathbb{R}^2)}^2 = \lim_{n \rightarrow \infty} \|D^s \tilde{\Phi}_{\ell,n}\|_{L^2(\mathbb{R}^2 \times \mathbb{R}^2)}^2 = 0. \quad (4.1.6)$$

Moreover, by using the fractional Poincaré inequality with $s = 1/2$ for the open bounded set $Q_{2\ell}$, we also have

$$\lim_{n \rightarrow \infty} \|\tilde{\Phi}_{\ell,n}\|_{L^2(Q_{2\ell})}^2 \leq \frac{1}{\lambda_1^{\frac{1}{2}}(Q_{2\ell})} \lim_{n \rightarrow \infty} [\tilde{\Phi}_{\ell,n}]_{W^{\frac{1}{2},2}(\mathbb{R}^2)}^2 = 0. \quad (4.1.7)$$

We take as in the previous case $u \in C_0^\infty(Q_\ell) \setminus \{0\}$. In order to approximate u with functions compactly supported in \tilde{Q}_ℓ , we now define

$$\tilde{u}_n = (1 - \tilde{\Phi}_{\ell,n}) u.$$

We observe that this function belongs to $C_0^\infty(\tilde{Q}_\ell)$. Indeed, observe that

$$\Phi_{\ell,m}(x) = 1, \quad \text{for every } x \in \Sigma_{\ell,4m}^{(i)}, \quad i \in \{-(\ell-1), \dots, \ell-1\} \text{ and } m \in \{1, \dots, n\},$$

thus in particular

$$\tilde{\Phi}_{\ell,n}(x) = \sum_{m=1}^n t_m(n) \Phi_{\ell,m}(x) = \sum_{m=1}^n t_m(n) = 1, \quad \text{for every } x \in \Sigma_{\ell,4n}^{(i)}, \quad i \in \{-(\ell-1), \dots, \ell-1\},$$

thanks to the fact that

$$\Sigma_{\ell,4n}^{(i)} \subseteq \Sigma_{\ell,4m}^{(i)}, \quad \text{for } m \in \{1, \dots, n\}.$$

Clearly, we still have

$$|1 - \tilde{\Phi}_{\ell,n}| \leq 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|\tilde{u}_n\|_{L^2(\tilde{Q}_\ell)} = \|u\|_{L^2(Q_\ell)}. \quad (4.1.8)$$

The second fact in (4.1.8) can be proved by observing that

$$\begin{aligned} \left| \int_{\tilde{Q}_\ell} |\tilde{u}_n|^2 dx - \int_{Q_\ell} |u|^2 dx \right| &= \left| \int_{\tilde{Q}_\ell} |u|^2 [1 - \tilde{\Phi}_{\ell,n}]^2 dx \right| \\ &\leq \int_{\tilde{Q}_\ell} |u|^2 [1 - |1 - \tilde{\Phi}_{\ell,n}|^2] dx \\ &\leq 2 \int_{\tilde{Q}_\ell} |u|^2 [1 - |1 - \tilde{\Phi}_{\ell,n}|] dx \leq 2 \|u\|_{L^\infty(Q_\ell)}^2 \int_{\tilde{Q}_\ell} |\tilde{\Phi}_{\ell,n}| dx, \end{aligned}$$

and then using (4.1.7).

We can now use \tilde{u}_n as a competitor for the variational problem defining $\lambda_1^s(\tilde{Q}_\ell)$ and proceed exactly as in the case $0 < s < 1/2$, by using (4.1.6) and (4.1.8). This finally concludes the proof. \square

Remark 4.1.1. *The previous example displays the pivotal role of fractional s -capacity, in the failure of the Makai-Hayman inequality for $0 < s \leq 1/2$. Indeed, the range $0 < s \leq 1/2$ is precisely the one for which lines have zero fractional s -capacity. This implies that, by removing a finite number of segments from an open set, the first eigenvalue λ_1^s remains unchanged, while*

this operation heavily affects the inradius. However, even if this is the ultimate reason for such a failure, in the proof above we preferred to give an elementary construction, without explicitly appealing to the properties of capacities.

We point out that, for practical reasons, our sequence $\{Q_n\}_{n \in \mathbb{N}}$ is given by a square with side length $2n$, from which a periodical array of segments is removed. If we scale this sequence by a factor $1/n$, we could produce another sequence contradicting the fractional Makai-Hayman, with the additional property of being equi-bounded.

Remark 4.1.2. With the notation above, we obtain in particular that the infinite complement comb $\Theta := \mathbb{R}^2 \setminus \Sigma$ is an open simply connected set such that

$$r_\Theta = \frac{\sqrt{5}}{2} \quad \text{and} \quad \lambda_1^s(\Theta) = 0, \quad \text{for } 0 < s \leq \frac{1}{2}.$$

Indeed, by domain monotonicity and (4.1.1), we have

$$0 \leq \lambda_1^s(\Theta) \leq \lim_{\ell \rightarrow \infty} \lambda_1^s(\tilde{Q}_\ell) = 0.$$

4.2 Sharp behaviour of the constant

The next result shows that the estimate (I.8) is sharp, apart from the evaluation of the absolute constant².

Theorem 3 (Optimality). *The following facts hold:*

1. for every $\Omega \subseteq \mathbb{R}^2$ open set, we have

$$\limsup_{s \nearrow 1} (1-s) \lambda_1^s(\Omega) \leq \frac{1}{2} \lambda_1(\Omega).$$

Thus, the estimate (I.8) is sharp in its dependence on $s \nearrow 1$. In particular, by taking the limit as s goes to 1 in (I.8), we get the classical Croke-Osserman-Taylor inequality, possibly with a worse constant;

2. let $1/2 < s < 1$, there exists a sequence $\{\Omega_k\}_{k \in \mathbb{N} \setminus \{0\}} \subseteq \mathbb{R}^2$ of open sets such that Ω_k is multiply connected of order k

$$r_{\Omega_k} \leq C \quad \text{and} \quad \limsup_{k \rightarrow \infty} k^s \lambda_1^s(\Omega_k) < +\infty.$$

Thus the estimate (I.8) is sharp in its dependence on $k \rightarrow \infty$;

3. for every $k \in \mathbb{N} \setminus \{0\}$, there exists $\Theta_k \subseteq \mathbb{R}^2$ an open multiply connected set of order k , such that

$$r_{\Theta_k} < +\infty \quad \text{and} \quad \limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2s-1} < +\infty.$$

Thus, the estimate (I.8) is sharp in its dependence on $s \searrow 1/2$.

²This is a quotation from Taylor's paper, see [104, page 452].

4.2.1 Proof of point 1. of Theorem 3

For every $\Omega \subseteq \mathbb{R}^2$ open set, we have

$$\limsup_{s \nearrow 1} (1-s) \lambda_1^s(\Omega) \leq \frac{1}{2} \lambda_1(\Omega).$$

Thus, the estimate (I.8) is sharp in its dependence on $s \nearrow 1$. In particular, by taking the limit as s goes to 1 in (I.8), we get the classical Croke-Osserman-Taylor inequality, possibly with a worse constant;

Proof. This is a straightforward consequence of the *Bourgain-Brezis-Mironescu formula*. Indeed, for every $\Omega \subseteq \mathbb{R}^2$ open set, let $u \in C_0^\infty(\Omega) \setminus \{0\}$. Then by [43, Corollary 3.20] we have

$$\lim_{s \nearrow 1} (1-s) [u]_{W^{s,2}(\mathbb{R}^2)}^2 = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx.$$

This implies that

$$\limsup_{s \nearrow 1} (1-s) \lambda_1^s(\Omega) \leq \lim_{s \nearrow 1} \frac{(1-s) [u]_{W^{s,2}(\mathbb{R}^2)}^2}{\|u\|_{L^2(\Omega)}^2} = \frac{1}{2} \frac{\int_{\Omega} |\nabla u|^2 dx}{\|u\|_{L^2(\Omega)}^2}.$$

By taking the infimum over $C_0^\infty(\Omega) \setminus \{0\}$, we get

$$\limsup_{s \nearrow 1} (1-s) \lambda_1^s(\Omega) \leq \frac{1}{2} \lambda_1(\Omega),$$

as claimed. Thus, by multiplying both sides of (I.8) by the factor $(1-s)$, using the previous property and the asymptotic behaviour of ϑ_s , we get back the classical Croke-Osserman-Taylor estimate, in the limit as s goes to 1. \square

4.2.2 Proof of point 2. of Theorem 3

We need at first the following

Lemma 4.2.1. *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^2$ be an open set. Then for every $\{x_0, \dots, x_m\} \subset \Omega$, we have*

$$\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\}) = \lambda_1^s(\Omega).$$

Proof. We may suppose that the points $\{x_0, \dots, x_m\}$ are distinct. We first observe that

$$\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\}) \geq \lambda_1^s(\Omega),$$

since $\Omega \setminus \{x_0, \dots, x_m\} \subset \Omega$. In order to prove the converse implication, we set

$$\varepsilon_0 = \frac{1}{2} \min_{i,j \in \{0,m\}} \{|x_i - x_j| : i \neq j\}.$$

Then we take a cut-off function $\eta \in C_0^\infty(B_1)$ such that

$$\eta \equiv 1 \text{ in } B_{\frac{1}{2}}, \quad 0 \leq \eta \leq 1, \quad |\nabla \eta| \leq C,$$

and define for every $0 < \varepsilon < \varepsilon_0$

$$\Psi_\varepsilon(x) = \sum_{i=0}^m \eta\left(\frac{x - x_i}{\varepsilon}\right).$$

We now take $u \in C_0^\infty(\Omega) \setminus \{0\}$ and observe that $u(1 - \Psi_\varepsilon)$ is a feasible trial function for the variational problem which defines $\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})$. Thus, by using Minkowski's inequality, we get for every $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} \sqrt{\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})} &\leq \frac{[u(1 - \Psi_\varepsilon)]_{W^{s,2}(\mathbb{R}^2)}}{\|u(1 - \Psi_\varepsilon)\|_{L^2(\Omega)}} \\ &\leq \frac{[u]_{W^{s,2}(\mathbb{R}^2)} \|1 - \Psi_\varepsilon\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} [\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)}}{\|u(1 - \Psi_\varepsilon)\|_{L^2(\Omega)}} \quad (4.2.1) \\ &= \frac{[u]_{W^{s,2}(\mathbb{R}^2)} + \|u\|_{L^\infty(\mathbb{R}^2)} [\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)}}{\|u(1 - \Psi_\varepsilon)\|_{L^2(\Omega)}}. \end{aligned}$$

By applying the Dominated Convergence Theorem, we easily get that

$$\lim_{\varepsilon \rightarrow 0} \|u(1 - \Psi_\varepsilon)\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)}.$$

As for the second term in the numerator, we observe that by Minkowski's inequality again, we have

$$[\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)} = \left[\sum_{i=0}^m \eta \left(\frac{\cdot - x_i}{\varepsilon} \right) \right]_{W^{s,2}(\mathbb{R}^2)} \leq (m+1) \varepsilon^{1-s} [\eta]_{W^{s,2}(\mathbb{R}^2)}.$$

We also used the scaling properties of the fractional seminorm. This in turn implies that

$$\lim_{\varepsilon \rightarrow 0} [\Psi_\varepsilon]_{W^{s,2}(\mathbb{R}^2)} = 0.$$

Thus, by taking the limit as ε goes to 0 in (4.2.1), we end up with

$$\sqrt{\lambda_1^s(\Omega \setminus \{x_0, \dots, x_m\})} \leq \frac{[u]_{W^{s,2}(\mathbb{R}^2)}}{\|u\|_{L^2(\Omega)}}.$$

By arbitrariness of u , we get the desired conclusion. \square

Remark 4.2.2. *The previous result is a particular case of the following general fact: removing sets with zero fractional capacity does not alter the relevant fractional Sobolev space. Consequently, fractional Poincaré constants are insensitive to removal of these sets. We refer for example to [4, Proposition 2.6 & Corollary 2.7] for this general result.*

Proof of point 2. of Theorem 3. The sequence $\{\Omega_k\}_{k \in \mathbb{N} \setminus \{0,1\}}$ is then constructed as follows: for every $k \in \mathbb{N} \setminus \{0,1\}$, we set

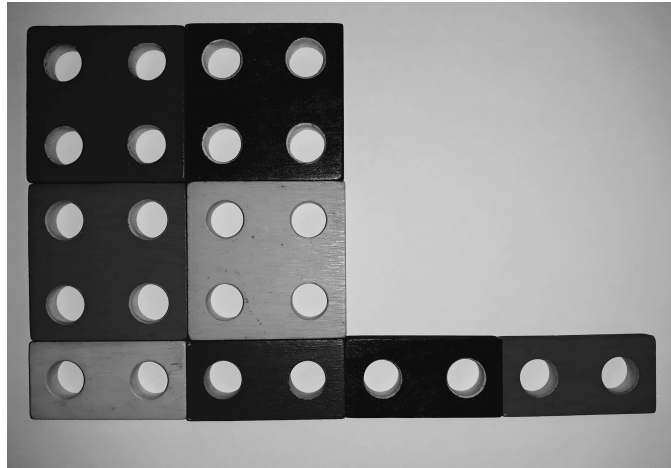
$$n_k = \lfloor \sqrt{k-1} \rfloor \quad \text{and} \quad m_k = (k-1) - n_k^2.$$

Then, we take the set

$$\text{Shell}_k = \left([0, n_k] \times [0, n_k] \right) \setminus \bigcup_{i,j=0}^{n_k-1} \left\{ \left(i + \frac{1}{2}, j + \frac{1}{2} \right) \right\}, \quad \text{for } k \geq 2,$$

which consists of a square with n_k^2 equally spaced points removed. More precisely, we remove the centers of the squares

$$[i, i+1] \times [j, j+1], \quad \text{for } i, j = 0, \dots, n_k - 1.$$

Figure 4.3: The set Ω_k of Theorem 3 point 2, for $k = 25$.

We also introduce the set

$$\text{Slug}_k = \left([0, m_k] \times [-1, 0] \right) \setminus \bigcup_{i=0}^{m_k-1} \left\{ \left(i + \frac{1}{2}, -\frac{1}{2} \right) \right\},$$

which consists of an horizontal strip of width 1 and length m_k , from which we removed the centers of the squares

$$[i, i+1] \times [-1, 0], \quad \text{for } i = 0, \dots, m_k - 1.$$

Finally, we define the open bounded set

$$\Omega_k = \text{int}(\text{Shell}_k \cup \text{Slug}_k), \quad \text{for every } k \geq 2,$$

i. e. the interior points of the union of Shell_k and Slug_k (see Figure 4.3). By construction, we have that Ω_k is multiply connected of order k . Moreover, we have

$$r_{\Omega_k} \leq \frac{\sqrt{2}}{2}, \quad \text{for every } k \geq 2,$$

and

$$\Omega_k \supseteq \text{int}(\text{Shell}_k) = \left((0, n_k) \times (0, n_k) \right) \setminus \bigcup_{i,j=0}^{n_k-1} \left\{ \left(i + \frac{1}{2}, j + \frac{1}{2} \right) \right\}.$$

By using the monotonicity of λ_1^s with respect to set inclusion and Lemma 4.2.1 for $\text{int}(\text{Shell}_k)$, we can then infer

$$\lambda_1^s(\Omega_k) \leq \lambda_1^s \left((0, n_k) \times (0, n_k) \right) = n_k^{-2s} \lambda_1^s \left((0, 1) \times (0, 1) \right).$$

By recalling the definition of n_k , this finally gives the desired result. \square

4.2.3 Proof of point 3. of Theorem 3

For every $k \in \mathbb{N} \setminus \{0\}$, there exists $\Theta_k \subseteq \mathbb{R}^2$ an open multiply connected set of order k , such that

$$r_{\Theta_k} < +\infty \quad \text{and} \quad \limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2s-1} < +\infty.$$

Thus, the estimate (I.8) is sharp in its dependence on $s \searrow 1/2$.

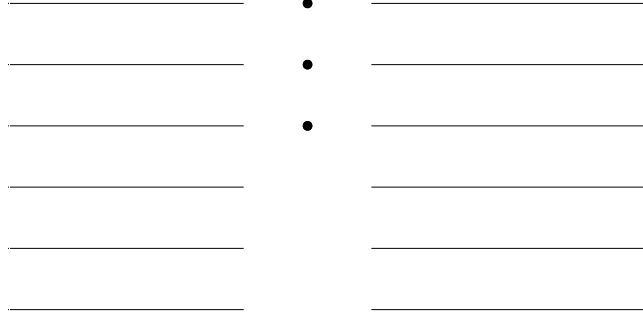


Figure 4.4: The set Θ_k for $k = 4$ of Theorem 3 point 3: it has been obtained by removing the black dots from Θ .

Proof. We divide the proof in various steps, for ease of presentation.

Step 1: construction of the set. We define

$$\Sigma = \bigcup_{i \in \mathbb{Z}} \Sigma^{(i)}, \quad \text{where } \Sigma^{(i)} := \{(x_1, i) \in \mathbb{R}^2 : |x_1| \geq 1\},$$

and then consider the *infinite complement comb* $\Theta = \mathbb{R}^2 \setminus \Sigma$ of Remark 4.1.2. The set Θ_k of the statement is then constructed by simply removing $k - 1$ distinct points from Θ , i.e. we set

$$\Theta_k = \Theta \setminus \{(0, i) : i = 1, \dots, k - 1\},$$

see Figure 4.4. By construction, we have that Θ_k is multiply connected of order k , with finite inradius. Thus, by Theorem 1 we have $\lambda_1^s(\Theta_k) > 0$, for every $s > 1/2$. We claim that

$$\limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\Theta_k)}{2s - 1} < +\infty. \quad (4.2.2)$$

Step 2: one-dimensional reduction. Here we need the following result.

Lemma 4.2.3. *Let $0 < s < 1$ and let $A \subseteq \mathbb{R}$ be an open set. Then we have*

$$\lambda_1^s(A \times \mathbb{R}) \leq \alpha_s \lambda_1^s(A), \quad \text{where } \alpha_s = \int_{\mathbb{R}} \frac{dt}{(1+t^2)^{\frac{2+2s}{2}}}. \quad (4.2.3)$$

Proof. We proceed as in the proof of [52, Lemma 2.4]. For every $x \in \mathbb{R}^2$, we will use the notation $x = (x_1, x_2)$. We take $u \in C_0^\infty(A) \setminus \{0\}$ and $\varphi \in C_0^\infty(\mathbb{R}) \setminus \{0\}$. We first observe that by Fubini's Theorem

$$\|u\varphi\|_{L^2(A \times \mathbb{R})} = \|u\|_{L^2(A)} \|\varphi\|_{L^2(\mathbb{R})}.$$

We then estimate the fractional seminorm of $u\varphi$. By Minkowski's inequality, we have

$$\begin{aligned} [u\varphi]_{W^{s,2}(\mathbb{R}^2)} &\leq \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1)|^2 \frac{|\varphi(y_1) - \varphi(y_2)|^2}{|x - y|^{2+2s}} dx dy \right)^{1/2} \\ &\quad + \left(\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\varphi(y_2)|^2 \frac{|u(x_1) - u(x_2)|^2}{|x - y|^{2+2s}} dx dy \right)^{1/2}. \end{aligned}$$

By using Fubini's Theorem, we have

$$\begin{aligned} & \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1)|^2 \frac{|\varphi(y_1) - \varphi(y_2)|^2}{|x - y|^{2+2s}} dx dy \\ &= \int_{\mathbb{R}} |u(x_1)|^2 \left(\iint_{\mathbb{R} \times \mathbb{R}} |\varphi(y_1) - \varphi(y_2)|^2 \left(\int_{\mathbb{R}} \frac{dx_2}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{\frac{2+2s}{2}}} \right) dy_1 dy_2 \right) dx_1. \end{aligned}$$

By using a changing of variable, we get

$$\int_{\mathbb{R}} \frac{dx_2}{((x_1 - x_2)^2 + (y_1 - y_2)^2)^{\frac{2+2s}{2}}} = \frac{\alpha_s}{|y_1 - y_2|^{1+2s}}.$$

Thus we obtain

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(x_1)|^2 \frac{|\varphi(y_1) - \varphi(y_2)|^2}{|x - y|^{2+2s}} dx dy = \alpha_s \|u\|_{L^2(A)}^2 [\varphi]_{W^{s,2}(\mathbb{R})}^2.$$

With a similar computation, we also get

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} |\varphi(y_2)|^2 \frac{|u(x_1) - u(x_2)|^2}{|x - y|^{2+2s}} dx dy = \alpha_s \|\varphi\|_{L^2(\mathbb{R})}^2 [u]_{W^{s,2}(\mathbb{R})}^2.$$

Thus, from the variational definition of $\lambda_1^s(A \times \mathbb{R})$, we get

$$\begin{aligned} \sqrt{\lambda_1^s(A \times \mathbb{R})} &\leq \frac{[u\varphi]_{W^{s,2}(\mathbb{R}^2)}}{\|u\varphi\|_{L^2(\omega \times \mathbb{R})}} \leq \sqrt{\alpha_s} \frac{\|u\|_{L^2(A)} [\varphi]_{W^{s,2}(\mathbb{R})} + \|\varphi\|_{L^2(\mathbb{R})} [u]_{W^{s,2}(\mathbb{R})}}{\|u\|_{L^2(A)} \|\varphi\|_{L^2(\mathbb{R})}} \\ &= \sqrt{\alpha_s} \left(\frac{[\varphi]_{W^{s,2}(\mathbb{R})}}{\|\varphi\|_{L^2(\mathbb{R})}} + \frac{[u]_{W^{s,2}(\mathbb{R})}}{\|u\|_{L^2(A)}} \right). \end{aligned}$$

By taking the infimum over u and φ , recalling that $\lambda_1^s(\mathbb{R}) = 0$, we get the desired conclusion \square

In particular, from the previous result with $A = \mathbb{R} \setminus \mathbb{Z}$, we get that

$$\lambda_1^s(\Theta_k) \leq \lambda_1^s(\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Z})) \leq \alpha_s \lambda_1^s(\mathbb{R} \setminus \mathbb{Z}).$$

In the first inequality we used that

$$\mathbb{R} \times (\mathbb{R} \setminus \mathbb{Z}) \subset \Theta_k.$$

From its definition (4.2.3), it is easy to see that α_s varies continuously with respect to $s \in [0, 1]$. Thus, in order to prove (4.2.2), it will be sufficient to establish that

$$\limsup_{s \searrow \frac{1}{2}} \frac{\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})}{2s - 1} < +\infty. \quad (4.2.4)$$

Step 3: choice of the trial functions. In order to prove (4.2.4), we will need to carefully construct a suitable family of s -depending trial functions, which provides an upper bound on $\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})$ with the correct asymptotic behaviour. For every

$$n \in \mathbb{N} \setminus \{0\}, \quad s > 1/2 \quad \text{and} \quad 0 < \varepsilon < \frac{1}{10},$$

we consider the trial function $u_n \varphi_{n,s,\varepsilon}$, where:

- $u_n \in C_0^\infty((-n, n))$ has the form

$$u_n(x) = u\left(\frac{x}{n}\right),$$

for some $u \in C_0^\infty((-1, 1))$ such that $\|u\|_{L^2((-1, 1))} = 1$;

- the *multiple funnel-type* cut-off function $\varphi_{n,s,\varepsilon} \in W_{\text{loc}}^{s,2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ has the form

$$\varphi_{n,s,\varepsilon} = 1 - \sum_{j=-n}^n \zeta_s\left(\frac{x-j}{\varepsilon}\right),$$

where ζ_s is the function given by

$$\zeta_s(x) = \left(1 - |x|^{2s-1}\right)_+.$$

Thanks to Lemma 1.1.6, we see that

$$u_n \varphi_{n,s,\varepsilon} \in \widetilde{W}_0^{s,2}((-n, n)) \subset \widetilde{W}_0^{s,2}(\mathbb{R} \setminus \mathbb{Z}).$$

Thus it is a feasible trial function. By using again Minkowski's inequality, this yields

$$\sqrt{\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})} \leq \frac{[u_n]_{W^{s,2}(\mathbb{R})} + \|u_n\|_{L^\infty((-n, n))} [\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})}}{\|u_n \varphi_{n,\varepsilon,s}\|_{L^2((-n, n))}}.$$

Step 4: estimate of the quotient. Let us start by handling the terms at the numerator. We consider at first the terms containing u_n , which are simpler. By recalling the definition of u_n , we have

$$[u_n]_{W^{s,2}(\mathbb{R})} = n^{\frac{1}{2}-s} [u]_{W^{s,2}(\mathbb{R})}.$$

The last term can be estimated by using the interpolation inequality [22, Corollary 2.2], which gives

$$[u]_{W^{s,2}(\mathbb{R})} \leq \sqrt{\frac{C}{s(1-s)}} \|u\|_{L^2((-1, 1))}^{1-s} \|u'\|_{L^2((-1, 1))}^s,$$

for some $C > 0$ independent of s . This guarantees that we have

$$[u_n]_{W^{s,2}(\mathbb{R})} \leq n^{\frac{1}{2}-s} \sqrt{\frac{C}{s(1-s)}} \|u'\|_{L^2((-1, 1))}^s. \quad (4.2.5)$$

The term with the L^∞ norm is easy to handle, since we simply have

$$\|u_n\|_{L^\infty((-n, n))} = \|u\|_{L^\infty((-1, 1))}. \quad (4.2.6)$$

The term containing the cut-off is the most delicate one. In order to estimate it, we observe that

$$\sum_{j=-n}^n \zeta_s\left(\frac{x-j}{\varepsilon}\right) = \max_{j=-n, \dots, n} \zeta_s\left(\frac{x-j}{\varepsilon}\right),$$

thanks to the fact that all the functions involved in the sum have disjoint support. We can then use the sub-modularity of the Gagliardo-Slobodeckii seminorm (see [58, Theorem 3.2 & Remark

3.3]) and obtain

$$\begin{aligned} [\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})} &= \left[\sum_{j=-n}^n \zeta_s \left(\frac{\cdot - j}{\varepsilon} \right) \right]_{W^{s,2}(\mathbb{R})} \\ &= \left[\max_{j=-n,\dots,n} \zeta_s \left(\frac{\cdot - j}{\varepsilon} \right) \right]_{W^{s,2}(\mathbb{R})} \\ &\leq \left(\sum_{j=-n}^n \left[\zeta_s \left(\frac{\cdot - j}{\varepsilon} \right) \right]_{W^{s,2}(\mathbb{R})}^2 \right)^{\frac{1}{2}} = \sqrt{2n+1} \varepsilon^{\frac{1}{2}-s} [\zeta_s]_{W^{s,2}(\mathbb{R})}. \end{aligned}$$

In order to conclude, the key fact is a very precise asymptotic estimate of the last term, as s goes to $1/2$. This is contained in Lemma 1.1.9, which permits to infer

$$[\varphi_{n,\varepsilon,s}]_{W^{s,2}(\mathbb{R})} \leq C \sqrt{2n+1} \varepsilon^{\frac{1}{2}-s} \sqrt{2s-1}, \quad \text{for } \frac{1}{2} < s < \frac{3}{4}, \quad (4.2.7)$$

with $C > 0$ not depending on s .

We now pass to examine the denominator. In this case, we have

$$\|u_n \varphi_{n,\varepsilon,s}\|_{L^2((-n,n))} = n^{\frac{1}{2}} \left(\int_{-1}^1 |u(y)|^2 \left(1 - \sum_{j=-n}^n \zeta_s \left(\frac{ny-j}{\varepsilon} \right) \right)^2 dy \right)^{\frac{1}{2}} \geq n^{\frac{1}{2}} \|u\|_{L^2(A_\varepsilon)}, \quad (4.2.8)$$

where

$$A_\varepsilon = (-1, 1) \setminus \bigcup_{j=-n}^n \left(\frac{j-\varepsilon}{n}, \frac{j+\varepsilon}{n} \right).$$

Step 5: conclusion. By collecting the estimates (4.2.5), (4.2.6), (4.2.7) and (4.2.8), we obtain

$$\begin{aligned} \sqrt{\frac{\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})}{2s-1}} &\leq \frac{n^{\frac{1}{2}-s} \sqrt{\frac{C}{s(1-s)}} \|u'\|_{L^2((-1,1))}^s + C \|u\|_{L^\infty((-1,1))} \sqrt{2n+1} \varepsilon^{\frac{1}{2}-s} \sqrt{2s-1}}{n^{\frac{1}{2}} \sqrt{2s-1} \|u\|_{L^2(A_\varepsilon)}} \\ &\leq \sqrt{\frac{C}{s(1-s)}} \frac{\|u'\|_{L^2((-1,1))}^s}{\|u\|_{L^2(A_\varepsilon)}} \frac{n^{-s}}{\sqrt{2s-1}} + C \frac{\|u\|_{L^\infty((-1,1))}}{\|u\|_{L^2(A_\varepsilon)}} \sqrt{3} \varepsilon^{\frac{1}{2}-s}. \end{aligned}$$

It is now important to make a clever choice of the parameters n and ε : we take them to be

$$\varepsilon = \left(\frac{1}{10} \right)^{\frac{1}{2s-1}} \quad \text{and} \quad n = \left(\left\lfloor \frac{1}{2s-1} \right\rfloor + 1 \right)^2.$$

Observe that with these choices we have

$$\lim_{s \searrow \frac{1}{2}} \varepsilon = 0 \quad \text{and} \quad \varepsilon^{\frac{1}{2}-s} = \sqrt{10},$$

and

$$\lim_{s \searrow \frac{1}{2}} \frac{n^{-s}}{\sqrt{2s-1}} \leq \lim_{s \searrow \frac{1}{2}} (2s-1)^{2s-\frac{1}{2}} = 0,$$

where we also used (B.1.1). Moreover, by using the Dominated Convergence Theorem, we also have

$$\lim_{s \searrow \frac{1}{2}} \|u\|_{L^2(A_\varepsilon)} = \|u\|_{L^2((-1,1))} = 1.$$

These facts finally enable us to conclude that

$$\limsup_{s \searrow \frac{1}{2}} \sqrt{\frac{\lambda_1^s(\mathbb{R} \setminus \mathbb{Z})}{2s-1}} \leq \sqrt{30} C \|u\|_{L^\infty((-1,1))} < +\infty.$$

The proof is now over. □

Chapter 5

A fractional Hardy inequality

In this chapter, taken from [B4] and [B5], we set up the ingredients to determine the sharp constant in the fractional Hardy inequality, for the class of open convex sets. The latter will be the scope of the next chapter. The results of this chapter have a general character and do not need the convexity assumption on the open sets. We also remark that, in the original papers [B4] and [B5], we studied the problem for the whole range $1 < p < +\infty$, that is we considered

$$\mathfrak{h}_{s,p}(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy : \int_{\Omega} \frac{|u|^p}{d_{\Omega}^{sp}} dx = 1 \right\}.$$

where d_{Ω} denotes the distance function defined as

$$d_{\Omega}(x) = \inf_{y \in \partial\Omega} |x - y|, \quad \text{for all } x \in \Omega,$$

extended by 0 outside Ω . Here, we will stick to the Hilbertian case $p = 2$ only.

In the case $p = 2$, we will simply write $\mathfrak{h}_s(\Omega)$, in place of $\mathfrak{h}_{s,2}(\Omega)$. We recall that, if such a value is positive, then the following fractional Hardy inequality holds

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \geq \lambda \int_{\Omega} \frac{|u|^2}{d_{\Omega}^{2s}} dx,$$

with $\lambda = \mathfrak{h}_s(\Omega)$. We will need at first to link the constant \mathfrak{h}_s to the *local weak supersolutions* of the equation

$$(-\Delta)^s u = \lambda \frac{u}{d_{\Omega}^{2s}}, \quad \text{in } \Omega,$$

where $\lambda \geq 0$ (see Definition 1.1.1). This will be the content of the *supersolution method*, contained in Section 5.3. Observe that this equation naturally appears as the Euler-Lagrange equation of the minimization problem defining $\mathfrak{h}_s(\Omega)$.

5.1 A weighted fractional Sobolev space

In the proof of Theorem 4 below, we will crucially exploit a suitable weighted fractional Sobolev space, whose definition is inspired by [5, Appendix]. In order to define such space, we recall the space $L_{2s}^2(\mathbb{R}^N)$ introduced in (1.1.3).

Definition. Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set, we define

$$\mathcal{X}^{s,2}(\Omega; d_{\Omega}) := \left\{ u \in L_{2s}^2(\mathbb{R}^N) : [u]_{W^{s,2}(\mathbb{R}^N)} < +\infty \text{ and } \frac{u}{d_{\Omega}^s} \in L^2(\Omega) \right\},$$

endowed with the norm

$$\|u\|_{\mathcal{X}^{s,2}(\Omega;d_\Omega)} := [u]_{W^{s,2}(\mathbb{R}^N)} + \left(\int_{\Omega} \frac{|u|^2}{d_\Omega^{2s}} dx \right)^{\frac{1}{2}}, \quad \text{for every } u \in \mathcal{X}^{s,2}(\Omega;d_\Omega).$$

Then we define $\mathcal{X}_0^{s,2}(\Omega;d_\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $\mathcal{X}^{s,2}(\Omega;d_\Omega)$.

Remark 5.1.1. We observe that if the open set $\Omega \subsetneq \mathbb{R}^N$ is such that $\mathfrak{h}_s(\Omega) > 0$, then by a simple density argument we can assure that Hardy's inequality holds in $\mathcal{X}_0^{s,2}(\Omega;d_\Omega)$, as well. Namely we have

$$\mathfrak{h}_s(\Omega) \int_{\Omega} \frac{|u|^2}{d_\Omega^{2s}} dx \leq [u]_{W^{s,2}(\mathbb{R}^N)}^2, \quad \text{for every } u \in \mathcal{X}_0^{s,2}(\Omega;d_\Omega).$$

Accordingly, this implies that in this case

$$u \mapsto [u]_{W^{s,2}(\mathbb{R}^N)},$$

defines an equivalent norm on $\mathcal{X}_0^{s,2}(\Omega;d_\Omega)$.

For the ease of readability, it is useful to recall the definition of the following functional space

$$L_{2s}^\beta(\mathbb{R}^N) = \left\{ u \in L_{\text{loc}}^\beta(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|u(x)|^\beta}{(1+|x|)^{N+2s}} dx < +\infty \right\}.$$

Proposition 5.1.2. Let $0 < s < 1$. Let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then

$$\mathcal{X}^{s,2}(\Omega;d_\Omega) \subset W_{\text{loc}}^{s,2}(\Omega) \cap L_{2s}^1(\mathbb{R}^N),$$

and we have the estimate

$$\left(\int_{\mathbb{R}^N} \frac{|u|^2}{(1+|x|)^{N+2s}} dx \right)^{\frac{1}{2}} \leq C_\Omega \|u\|_{\mathcal{X}^{s,2}(\Omega;d_\Omega)}, \quad \text{for every } u \in \mathcal{X}^{s,2}(\Omega;d_\Omega). \quad (5.1.1)$$

Moreover, $\mathcal{X}^{s,2}(\Omega;d_\Omega)$ and $\mathcal{X}_0^{s,2}(\Omega;d_\Omega)$ are Banach spaces.

Proof. The first fact is straightforward, by also taking into account the definition of local weak supersolution introduced in (1.1.4).

We prove the estimate (5.1.1). We take a ball $B_R(x_0) \Subset \Omega$ such that $B_{2R}(x_0) \Subset \Omega$, as well. We then write

$$\begin{aligned} [u]_{W^{s,2}(\mathbb{R}^N)} &= \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\geq \left(\iint_{B_R(x_0) \times (\mathbb{R}^N \setminus B_{2R}(x_0))} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\geq \left(\int_{B_R(x_0)} \left(\int_{\mathbb{R}^N \setminus B_{2R}(x_0)} \frac{|u(y)|^2}{|x - y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \\ &\quad - \left(\int_{B_R(x_0)} |u(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_{2R}(x_0)} \frac{1}{|x - y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}}, \end{aligned}$$

thanks to Minkowski's inequality. By observing that

$$|x - y| \geq \frac{1}{2} |y - x_0|, \quad \text{for every } x \in B_R(x_0), y \notin B_{2R}(x_0),$$

we have

$$\begin{aligned} \int_{B_R(x_0)} |u(x)|^2 \left(\int_{\mathbb{R}^N \setminus B_{2R}(x_0)} \frac{1}{|x - y|^{N+2s}} dy \right) dx &\leq \frac{N \omega_N 2^N}{2s} R^{-2s} \int_{B_R(x_0)} |u|^2 dx \\ &\leq \frac{N \omega_N 2^{N-1}}{s} R^{-2s} \int_{\Omega} \frac{|u|^2}{d_{\Omega}^{2s}} dx \|d_{\Omega}\|_{L^{\infty}(B_R(x_0))}^{2s}. \end{aligned}$$

This implies that we have

$$\left(\int_{B_R(x_0)} \left(\int_{\mathbb{R}^N \setminus B_{2R}(x_0)} \frac{|u(y)|^2}{|x - y|^{N+2s}} dy \right) dx \right)^{\frac{1}{2}} \leq C \|u\|_{\mathcal{X}^{s,2}(\Omega; d_{\Omega})}, \quad (5.1.2)$$

for a constant $C = C(N, s, \Omega, B_R(x_0)) > 0$. We now use that

$$|x - y| \leq 2|x_0 - y|, \quad \text{for every } x \in B_R(x_0), y \notin B_{2R}(x_0),$$

together with the fact that

$$|x_0 - y| \leq |x_0| + |y| \leq (1 + |x_0|)(1 + |y|).$$

By using these in (5.1.2), we get

$$\left(\int_{\mathbb{R}^N \setminus B_{2R}(x_0)} \frac{|u(y)|^2}{(1 + |y|)^{N+2s}} dy \right)^{\frac{1}{2}} \leq C \|u\|_{\mathcal{X}^{s,2}(\Omega; d_{\Omega})},$$

possibly with a different constant $C = C(N, s, \Omega, B_R(x_0)) > 0$. The proof of estimate (5.1.1) is almost over: it is now sufficient to add on both sides of the previous estimate the term

$$\left(\int_{B_{2R}(x_0)} \frac{|u(y)|^2}{(1 + |y|)^{N+2s}} dy \right)^{\frac{1}{2}}.$$

Then by using that

$$\begin{aligned} \int_{B_{2R}(x_0)} \frac{|u(y)|^2}{(1 + |y|)^{N+2s}} dy &\leq \int_{B_{2R}(x_0)} |u(y)|^2 dy \leq \int_{\Omega} \frac{|u|^2}{d_{\Omega}^{2s}} dy \|d_{\Omega}\|_{L^{\infty}(B_{2R}(x_0))}^{2s} \\ &\leq C \|u\|_{\mathcal{X}^{s,2}(\Omega; d_{\Omega})}^2, \end{aligned}$$

for some $C = C(s, \Omega, B_{2R}(x_0)) > 0$, we eventually get the desired conclusion.

We prove the second part of the statement. We first observe that it is sufficient to prove that $\mathcal{X}^{s,2}(\Omega; d_{\Omega})$ is a Banach space. We take $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{X}^{s,2}(\Omega; d_{\Omega})$ to be a Cauchy sequence. Then we get that this is a Cauchy sequence in the Banach space $L^2(\Omega; d_{\Omega}^{-2s})$ and that

$$\{D^s u_n\}_{n \in \mathbb{N}} \quad \text{where } D^s \varphi(x, y) := \frac{\varphi(x) - \varphi(y)}{|x - y|^{\frac{N}{2} + s}},$$

is a Cauchy sequence in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. This follows from the fact that¹

$$[u_n]_{W^{s,2}(\mathbb{R}^N)} = \|D^s u_n\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)}.$$

Moreover, according to (5.1.1), the sequence $\{u_n\}_{n \in \mathbb{N}}$ is also a Cauchy sequence in the Banach space $L^2_{2s}(\mathbb{R}^N)$. The last fact implies that there exists $u \in L^2_{2s}(\mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{|u_n - u|^2}{(1 + |x|)^{N+2s}} dx = 0.$$

In particular, up to a subsequence, we can suppose that u_n converges to u almost everywhere in \mathbb{R}^N . By using the completeness of $L^2(\Omega; d_\Omega^{-2s})$, we get similarly the existence of $\tilde{u} \in L^2(\Omega; d_\Omega^{-2s})$ such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{|u_n - \tilde{u}|^2}{d_\Omega^{2s}} dx = 0.$$

By uniqueness of the limit, we must have $u = \tilde{u}$ almost everywhere in Ω . Finally, by using that $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ is a Banach space, we get that there exists $\phi \in L^2(\mathbb{R}^N \times \mathbb{R}^N)$ such that

$$\lim_{n \rightarrow \infty} \|D^s u_n - \phi\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)} = 0.$$

This in particular would imply that

$$\lim_{n \rightarrow \infty} D^s u_n(x, y) = \phi(x, y), \quad \text{for a. e. } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

up to a subsequence. On the other hand, by using the almost everywhere convergence of u_n previously inferred, we also obtain that

$$\lim_{n \rightarrow \infty} D^s u_n(x, y) = D^s u(x, y), \quad \text{for a. e. } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

By using the uniqueness of the limit, we get at the same time that

$$[u]_{W^{s,2}(\mathbb{R}^N)} < +\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \|D^s u_n - D^s u\|_{L^2(\mathbb{R}^N \times \mathbb{R}^N)} = \lim_{n \rightarrow \infty} [u_n - u]_{W^{s,2}(\mathbb{R}^N)} = 0.$$

This concludes the proof. \square

In the next technical lemma, we show that the summability of a *negative* power of the distance implies certain geometric properties of the open set.

Lemma 5.1.3. *Let $N \geq 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that*

$$\int_{\Omega} \frac{1}{d_\Omega^\alpha} dx < +\infty,$$

for some $\alpha > 0$. Then we must have $\alpha < N$. Moreover, we have the estimates

$$r_\Omega \leq \left(\frac{2^\alpha}{\omega_N} \int_{\Omega} \frac{1}{d_\Omega^\alpha} dx \right)^{\frac{1}{N-\alpha}} \quad \text{and} \quad |\Omega| \leq \left(\frac{2^\alpha}{\omega_N} \right)^{\frac{\alpha}{N-\alpha}} \left(\int_{\Omega} \frac{1}{d_\Omega^\alpha} dx \right)^{\frac{N}{N-\alpha}}, \quad (5.1.3)$$

where r_Ω is defined in (I.2).

¹We used $D^s \varphi$ to see the Gagliardo-Slobodeckii seminorm as a L^2 norm also in the proof of Theorem 2

Proof. We take $x_0 \in \Omega$ and consider the open ball $B_r(x_0)$ with radius $r = d_\Omega(x_0)$. This implies that

$$B_r(x_0) \subseteq \Omega \quad \text{and} \quad \partial B_r(x_0) \cap \partial\Omega \neq \emptyset.$$

Let us call \tilde{x}_0 a point contained in this intersection. By observing that

$$d_\Omega(x) \leq |x - \tilde{x}_0|, \quad \text{for every } x \in B_r(x_0),$$

we get

$$+\infty > \int_\Omega \frac{1}{d_\Omega^\alpha} dx \geq \int_{B_r(x_0)} \frac{1}{|x - \tilde{x}_0|^\alpha} dx.$$

By using spherical coordinates, we see that the last integral diverges for $\alpha \geq N$. Thus we get the first statement.

In order to get the claimed estimates, we go on by estimating from below the last integral as follows

$$+\infty > \int_\Omega \frac{1}{d_\Omega^\alpha} dx \geq \frac{1}{2^\alpha r^\alpha} \int_{B_r(x_0)} dx = \frac{\omega_N}{2^\alpha} r^{N-\alpha} = \frac{\omega_N}{2^\alpha} d_\Omega(x_0)^{N-\alpha}.$$

Since $\alpha < N$ from the first part of the proof, we can take the supremum on $x_0 \in \Omega$ and get that the distance function is actually bounded. Moreover, we obtain the first estimate in (5.1.3), thus in particular the inradius is finite. In turn, by using this fact we get

$$\int_\Omega \frac{1}{d_\Omega^\alpha} dx \geq \frac{1}{r_\Omega^\alpha} \int_\Omega dx = \frac{|\Omega|}{r_\Omega^\alpha},$$

which shows that the volume is finite, as well, together with the second estimate in (5.1.3). This concludes the proof. \square

As a straightforward consequence of Lemma 5.1.3, we get the following

Lemma 5.1.4. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then for $2s \geq N$ the unique constant function contained in $\mathcal{X}^{s,2}(\Omega; d_\Omega)$ is the null one.*

The same conclusion holds also for $2s < N$, if we additionally suppose that $|\Omega| = +\infty$.

In the next result we compare the two spaces $\widetilde{W}_0^{s,2}(\Omega)$ and $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$, under the assumption that the Hardy inequality is valid.

Proposition 5.1.5. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that $\mathfrak{h}_s(\Omega) > 0$. Then we have*

$$\widetilde{W}_0^{s,2}(\Omega) \subseteq \mathcal{X}_0^{s,2}(\Omega; d_\Omega), \quad (5.1.4)$$

and the inclusion is continuous. Moreover, if we assume that $r_\Omega < +\infty$, then

$$\widetilde{W}_0^{s,2}(\Omega) = \mathcal{X}_0^{s,2}(\Omega; d_\Omega),$$

and

$$\varphi \mapsto [\varphi]_{W^{s,2}(\mathbb{R}^N)}, \quad (5.1.5)$$

is an equivalent norm on this space. Finally, if we further require that $|\Omega| < +\infty$, then we have the continuous embedding

$$\mathcal{X}_0^{s,2}(\Omega; d_\Omega) \hookrightarrow L^2(\Omega),$$

and this is compact, as well.

Proof. By recalling Remark 5.1.1, we know that (5.1.5) is an equivalent norm on $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$. Since we trivially have

$$[\varphi]_{W^{s,2}(\mathbb{R}^N)} \leq \|\varphi\|_{W^{s,2}(\mathbb{R}^N)}, \quad \text{for every } \varphi \in C_0^\infty(\Omega),$$

the continuous inclusion (5.1.4) easily follows.

We now assume that $r_\Omega < +\infty$. In conjunction with Hardy's inequality and recalling the definition of inradius (I.2), this yields

$$\int_\Omega |\varphi|^2 dx \leq r_\Omega^{2s} \int_\Omega \frac{|\varphi|^2}{d_\Omega^{2s}} dx \leq \frac{r_\Omega^{2s}}{\mathfrak{h}_s(\Omega)} [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2, \quad \text{for every } \varphi \in C_0^\infty(\Omega).$$

Thus we get that

$$\varphi \mapsto \|\varphi\|_{\mathcal{X}^{s,2}(\mathbb{R}^N)} \quad \text{and} \quad \varphi \mapsto \|\varphi\|_{W^{s,2}(\mathbb{R}^N)},$$

are equivalent norms on $C_0^\infty(\Omega)$, again thanks to Remark 5.1.1. Then the claimed identity of the two closures immediately follows. The last statement is now an easy consequence of the same property for the space $\widetilde{W}_0^{s,2}(\Omega)$, which is well-known. \square

Remark 5.1.6. *Under the sole assumption that $\mathfrak{h}_s(\Omega) > 0$, in general we have*

$$\widetilde{W}_0^{s,2}(\Omega) \subset \mathcal{X}_0^{s,2}(\Omega; d_\Omega) \quad \text{and} \quad \widetilde{W}_0^{s,2}(\Omega) \neq \mathcal{X}_0^{s,2}(\Omega; d_\Omega),$$

contrary to what was incorrectly stated in [73, Theorem 5.1] for the local case $s = 1$. As an example of the fact that the two spaces do not coincide, it is sufficient to take any open set $\Omega \subsetneq \mathbb{R}^N$ such that

$$\mathfrak{h}_s(\Omega) > 0 \quad \text{and} \quad \inf_{\varphi \in C_0^\infty(\Omega)} \left\{ [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 : \int_\Omega |\varphi|^2 dx = 1 \right\} = 0.$$

For example, we can take Ω to be a half-space. In such a case, we have by construction

$$\widetilde{W}_0^{s,2}(\Omega) \hookrightarrow L^2(\Omega),$$

while

$$\mathcal{X}_0^{s,2}(\Omega; d_\Omega) \not\hookrightarrow L^2(\Omega).$$

A consequence of Proposition 5.1.5 is the following compactness result for the space $\mathcal{X}_0^{s,2}(\Omega)$, under minimal assumptions on the open set Ω . Its proof can be easily generalized to the case $1 < p < +\infty$, see [B4, Theorem 3.8].

Theorem 5.1.7. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that $\mathfrak{h}_s(\Omega) > 0$. Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ be such that*

$$[u_n]_{W^{s,2}(\mathbb{R}^N)}^2 \leq M, \quad \text{for every } n \in \mathbb{N},$$

for some $M > 0$. Then there exist a function $u \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ and subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} u_{n_k}(x) = u(x), \quad \text{for a. e. } x \in \Omega.$$

Moreover, for every $\Omega' \Subset \Omega$, we also have

$$\lim_{k \rightarrow \infty} \|u_{n_k} - u\|_{L^2(\Omega')} = 0,$$

up to a possible further subsequence.

Proof. We need to distinguish two cases: either $|\Omega| < +\infty$ or $|\Omega| = +\infty$.

Case 1: Ω has finite volume. This is the easiest case: here the result plainly follows from Proposition 5.1.5. We also observe that the last statement actually holds in a stronger form, since we can infer convergence in $L^2(\Omega)$.

Case 2: Ω has infinite volume. We still use the notation $D^s\varphi$ for a measurable function, as in Proposition 5.1.2. Thus, by assumption, we get that $\{D^s u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. This entails that, up to a subsequence, it is weakly converging in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$. For simplicity, we do not relabel the subsequence. Let us call ϕ such a limit. We may apply Mazur's Lemma (see [79, Theorem 2.13]) and get that for every $n \in \mathbb{N}$ there exists

$$\{t_\ell(n)\}_{\ell=0}^n \subset [0, 1], \quad \text{such that} \quad \sum_{\ell=0}^n t_\ell(n) = 1,$$

and such that the new sequence made of convex combinations

$$\tilde{\phi}_n(x, y) = \sum_{\ell=0}^n t_\ell(n) D^s u_\ell(x, y),$$

strongly converges in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ to ϕ , as n goes to ∞ . Observe that by construction we have

$$\sum_{\ell=0}^n t_\ell(n) D^s u_\ell = D^s \left(\sum_{\ell=0}^n t_\ell(n) u_\ell \right),$$

and

$$\tilde{u}_n := \sum_{\ell=0}^n t_\ell(n) u_\ell \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega),$$

since the latter is a vector space. This proves that $\{D^s \tilde{u}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ and this, in turn, implies that $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$, thanks to Remark 5.1.1. By using that $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ is a Banach space, we get that $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ converges in this space to a limit function $u \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$. In particular, we must have

$$D^s u = \phi.$$

We now want to prove that $\{u_n\}_{n \in \mathbb{N}}$ converges almost everywhere on \mathbb{R}^N to the function u , up to a subsequence. We first observe that all the elements of $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ vanish almost everywhere in $\mathbb{R}^N \setminus \Omega$, by construction. Thus we only need to prove convergence almost everywhere in Ω .

We denote by $\{\Omega_k\}_{k \in \mathbb{N}}$ an exhausting sequence for Ω , made of bounded open subsets with smooth boundary: in other words

$$\Omega_k \Subset \Omega, \quad \Omega_k \Subset \Omega_{k+1} \text{ for every } k \in \mathbb{N} \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} \Omega_k = \Omega,$$

see [34, Proposition 8.2.1]. We preliminarily observe that, thanks to the assumption $\mathfrak{h}_s(\Omega) > 0$, we have for every $k, n \in \mathbb{N}$

$$\int_{\Omega_k} |u_n|^2 dx \leq \|d_\Omega\|_{L^\infty(\Omega_k)}^{2s} \int_{\Omega_k} \frac{|u_n|^2}{d_\Omega^{2s}} dx \leq \frac{1}{\mathfrak{h}_s(\Omega)} \|d_\Omega\|_{L^\infty(\Omega_k)}^{2s} [u_n]_{W^{s,2}(\mathbb{R}^N)}^2 \leq C_k M,$$

which entails that $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in each $W^{s,2}(\Omega_k)$. By using the compactness of the embedding $W^{s,2}(\Omega_k) \hookrightarrow L^2(\Omega_k)$ (see for example [37, Theorem 7.1]) and a diagonal

argument, we can obtain existence of a function $U \in W_{\text{loc}}^{s,2}(\Omega)$ and of a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\lim_{k \rightarrow \infty} u_{n_k}(x) = U(x), \quad \text{for a. e. } x \in \Omega.$$

We then extend U to be 0 outside Ω : by using Fatou's Lemma and the almost everywhere convergence, we get

$$[U]_{W^{s,2}(\mathbb{R}^N)}^2 \leq \liminf_{k \rightarrow \infty} [u_{n_k}]_{W^{s,2}(\mathbb{R}^N)}^2 \leq M.$$

By further using Hardy's inequality and (5.1.1), we also get

$$\int_{\Omega} \frac{|U|^2}{d_{\Omega}^{2s}} dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} \frac{|u_{n_k}|^2}{d_{\Omega}^{2s}} dx \leq \frac{M}{\mathfrak{h}_s(\Omega)},$$

and

$$\begin{aligned} \left(\int_{\mathbb{R}^N} \frac{|U|^2}{(1+|x|)^{N+2s}} dx \right)^{\frac{1}{2}} &\leq \liminf_{k \rightarrow \infty} \left(\int_{\mathbb{R}^N} \frac{|u_{n_k}|^2}{(1+|x|)^{N+2s}} dx \right)^{\frac{1}{2}} \\ &\leq C_{\Omega} \liminf_{k \rightarrow \infty} \left[[u_{n_k}]_{W^{s,2}(\mathbb{R}^N)} + \left(\int_{\Omega} \frac{|u_{n_k}|^2}{d_{\Omega}^{2s}} dx \right)^{\frac{1}{2}} \right] \leq \tilde{C}. \end{aligned}$$

This shows that

$$U \in \mathcal{X}^{s,2}(\Omega; d_{\Omega}).$$

We now observe that from the first part of the proof, by uniqueness of the weak limit we must have

$$D^s u = D^s U, \quad \text{a. e. in } \mathbb{R}^N \times \mathbb{R}^N.$$

By recalling the definition of D^s , this in turn implies that there exists a constant $c \in \mathbb{R}$ such that

$$u = U + c, \quad \text{a. e. in } \mathbb{R}^N.$$

By using that $\mathcal{X}^{s,2}(\Omega; d_{\Omega})$ is a vector space, the function constantly equal to c must belong to $\mathcal{X}^{s,2}(\Omega; d_{\Omega})$. In light of Lemma 5.1.4, we get that $c = 0$ and thus the desired conclusion holds. \square

5.2 Some preliminary results

This section is devoted to inquire the existence (or the non-existence) of minimizers for $\mathfrak{h}_s(\Omega)$.

Lemma 5.2.1. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. If $\mathfrak{h}_s(\Omega)$ admits a non-trivial minimizer $u \in \widetilde{W}_0^{s,2}(\Omega)$, then this has constant sign in Ω and $u \neq 0$ almost everywhere in Ω . Moreover, the minimizer is unique, up to the choice of the sign and it is a weak solution of (II.7), with $\lambda = \mathfrak{h}_s(\Omega)$.*

Proof. Let us suppose that (II.4) admits a minimizer $u \in \widetilde{W}_0^{s,2}(\Omega)$, in particular this implies that $\mathfrak{h}_s(\Omega) > 0$. We observe that

$$\left| |a| - |b| \right| \leq |a - b|, \quad \text{for every } a, b \in \mathbb{R},$$

and the inequality is strict, whenever $a b < 0$. This yields

$$\mathfrak{h}_s(\Omega) \leq [|u|]_{W^{s,2}(\mathbb{R})}^2 \leq [u]_{W^{s,2}(\mathbb{R})}^2 = \mathfrak{h}_s(\Omega),$$

and thus it must result

$$u(x)u(y) \geq 0, \quad \text{for a. e. } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

This shows that u has constant sign almost everywhere in Ω . Without loss of generality, we can suppose that u is non-negative.

We then observe that u must be a minimizer of the following functional

$$\mathcal{G}(\varphi) := \frac{1}{2} [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\mathfrak{h}_s(\Omega)}{2} \int_{\Omega} \frac{|\varphi|^2}{d_{\Omega}^{2s}} dx, \quad \text{for every } \varphi \in \widetilde{W}_0^{s,2}(\Omega),$$

as well. Indeed, by definition of $\mathfrak{h}_s(\Omega)$, we have $\mathcal{G}(\varphi) \geq 0$ for every admissible function and $\mathcal{G}(u) = 0$. Moreover, u is non-trivial, due to the normalization on the weighted L^2 norm.

By minimality, we get that u must be a non-trivial non-negative weak solution of the Euler-Lagrange equation, which is given by (II.7) with $\lambda = \mathfrak{h}_s(\Omega)$. By the minimum principle (see [20, Theorem A.1]), we directly obtain that $u > 0$ almost everywhere in Ω , if the latter is connected. If Ω has more than one connected component, the same conclusion can be drawn by proceeding as in [23, Proposition 2.6], thanks to the nonlocality of the operator.

We now show the uniqueness for the positive minimizer of $\mathfrak{h}_s(\Omega)$. For this, it is sufficient to exploit Lemma 1.1.8. Let us take $u, v \in \widetilde{W}_0^{s,2}(\Omega)$ two positive minimizers of $\mathfrak{h}_s(\Omega)$ and set

$$\sigma = \left(\frac{1}{2} u^2 + \frac{1}{2} v^2 \right)^{\frac{1}{2}},$$

Thanks to the fractional hidden convexity (1.1.9), we get that $\sigma \in \widetilde{W}_0^{s,2}(\Omega)$ is still a minimizer for $\mathfrak{h}_s(\Omega)$. Thus (1.1.9) holds as an identity. By the last statement of Lemma 1.1.8, this means that there exists a constant C such that

$$u = C v, \quad \text{a. e. in } \Omega.$$

Finally, the normalization on the weighted norm implies that $C = 1$ and this concludes the proof. \square

Remark 5.2.2. *In the local case, the uniqueness of an extremal for \mathfrak{h}_s (provided it exists) can be found for example in [84, Proposition 3.2]. Differently from [84], here we found useful to rely on a hidden convexity argument, rather than on Picone's inequality.*

For the second result, we first need the following

Definition 5.2.3. *Let $\Omega \subsetneq \mathbb{R}^N$ be an open set. We say that $\partial\Omega$ is locally continuous at $x_0 \in \partial\Omega$ if there exist:*

- an open N -dimensional hyper-rectangle Q_{δ_0, δ_1} centered at the origin, defined by

$$Q_{\delta_0, \delta_1} = (-\delta_0, \delta_0)^{N-1} \times (-\delta_1, \delta_1), \quad \text{with } \delta_0, \delta_1 > 0;$$

- a linear isometry $\mathcal{O} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\mathcal{O}(x_0) = 0$;
- a continuous function $\Psi : (-\delta_0, \delta_0)^{N-1} \rightarrow (-\delta_1, \delta_1)$;

such that

$$Q_{\delta_0, \delta_1} \cap \mathcal{O}(\Omega) = \left\{ x = (x', x_N) \in Q_{\delta_0, \delta_1} : \Psi(x') < x_N < \delta_1 \right\},$$

and

$$Q_{\delta_0, \delta_1} \cap \mathcal{O}(\partial\Omega) = \left\{ x = (x', x_N) \in Q_{\delta_0, \delta_1} : x_N = \Psi(x') \right\}.$$

Roughly speaking, this means that $\partial\Omega$ coincides with the graph of a continuous function, in a small rectangular neighborhood of x_0 .

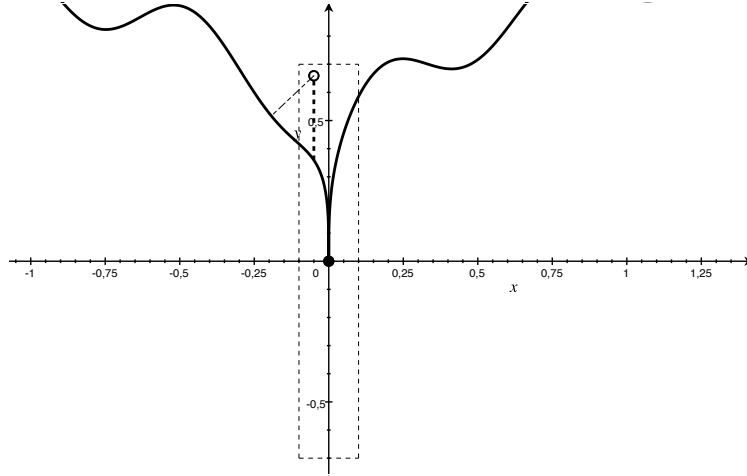


Figure 5.1: For (x', x_N) around a continuity point for the boundary, the “vertical” distance $x_N - \Psi(x')$ (in bold dashed line) is always larger than its distance from the boundary.

The following result will be important in the case of convex sets, where the assumption on $\partial\Omega$ comes for free.

Proposition 5.2.4. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set, which is locally continuous at a point $x_0 \in \partial\Omega$. Let us suppose that there exists a positive local weak supersolution u of (II.7) with $\lambda = \mathfrak{h}_s(\Omega)$, such that*

$$u \geq \frac{1}{C} d_\Omega^{\frac{2s-1}{2}}, \quad \text{in } \Omega. \quad (5.2.1)$$

Then the infimum $\mathfrak{h}_s(\Omega)$ is not attained.

Proof. We first show that for such a set, we have

$$1/d_\Omega \notin L^1(\Omega). \quad (5.2.2)$$

At this aim, we can assume without loss of generality that

$$x_0 = (0, \dots, 0) \quad \text{and} \quad \mathcal{O} = \text{Id},$$

so that

$$Q_{\delta_0, \delta_1}(x_0) \cap \Omega = \left\{ x = (x', x_N) \in Q_{\delta_0, \delta_1}(x_0) : \Psi(x') < x_N < \delta_1 \right\}.$$

We then observe that (see Figure 5.1)

$$d_\Omega(x) \leq |x_N - \Psi(x')| = (x_N - \Psi(x')), \quad \text{for every } x = (x', x_N) \in Q_{\delta_0, \delta_1}(x_0) \cap \Omega.$$

This implies that

$$\int_\Omega \frac{1}{d_\Omega} dx \geq \int_{Q_{\delta_0, \delta_1}(x_0) \cap \Omega} \frac{1}{d_\Omega} dx \geq \int_{(-\delta_0, \delta_0)^{N-1}} \left(\int_{\Psi(x')}^{\delta_1} \frac{1}{x_N - \Psi(x')} dx_N \right) dx'.$$

By observing that the last integral is diverging, we get (5.2.2).

We now argue by contradiction and suppose that $v \in \widetilde{W}_0^{s,2}(\Omega)$ is a minimizer for $\mathfrak{h}_s(\Omega)$. This in particular implies that $\mathfrak{h}_s(\Omega) > 0$. By Lemma 5.2.1, we can suppose that v is positive. We then take a sequence $\{v_n\}_{n \in \mathbb{N}} \subseteq C_0^\infty(\Omega)$ approximating v in $W^{s,2}(\mathbb{R}^N)$. Without loss of

generality, we can take each v_n to be non-negative and suppose that they converge to v almost everywhere, as well. We then insert in the weak formulation of the equation for u the test function

$$\varphi = \frac{v_n^2}{u},$$

which is admissible thanks to Lemma 1.1.6 and (5.2.1). This leads to

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))}{|x - y|^{N+sp}} \left(\frac{v_n(x)^2}{u(x)} - \frac{v_n(y)^2}{u(y)} \right) dx dy \geq \mathfrak{h}_s(\Omega) \int_{\Omega} \frac{v_n^2}{d_{\Omega}^{2s}} dx. \quad (5.2.3)$$

We now set

$$\mathcal{R}(v_n, u) := |v_n(x) - v_n(y)|^2 - (u(x) - u(y)) \left(\frac{v_n(x)^2}{u(x)} - \frac{v_n(y)^2}{u(y)} \right),$$

and observe that, thanks to the discrete Picone's inequality shown in the appendix (see Lemma A.1.1), this is always a non-negative quantity. With the previous notation, from equation (5.2.3) we get

$$\mathfrak{h}_s(\Omega) \int_{\Omega} \frac{v_n^2}{d_{\Omega}^{2s}} dx + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\mathcal{R}(v_n, u)}{|x - y|^{N+2s}} dx dy \leq [v_n]_{W^{s,2}(\mathbb{R}^N)}^2.$$

We now pass to the limit in the previous estimate and use Fatou's Lemma on the second term on the left-hand side: this yields

$$\mathfrak{h}_s(\Omega) \int_{\Omega} \frac{v^2}{d_{\Omega}^{2s}} dx + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\mathcal{R}(v, u)}{|x - y|^{N+2s}} dx dy \leq [v]_{W^{s,2}(\mathbb{R}^N)}^2.$$

By recalling that v is a solution for the variational problem (II.4) related to the Hardy inequality, the previous estimate gives

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\mathcal{R}(v, u)}{|x - y|^{N+2s}} dx dy = 0.$$

Since by Lemma A.1.1 we have $\mathcal{R}(v, u) \geq 0$ almost everywhere, this in turn implies that

$$0 = \mathcal{R}(v, u) = |v(x) - v(y)|^2 - (u(x) - u(y)) \left(\frac{v(x)^2}{u(x)} - \frac{v(y)^2}{u(y)} \right), \quad \text{for a. e. } (x, y) \in \Omega \times \Omega.$$

By using the equality cases in the discrete Picone inequality, it follows that there exists a constant $C > 0$ such that

$$u = C v, \quad \text{a. e. in } \Omega.$$

This fact and the assumption (5.2.1) imply in particular that

$$v \geq \frac{1}{C} d_{\Omega}^{\frac{2s-1}{2}}, \quad \text{in } \Omega,$$

possibly for a different constant $C > 0$. By minimality of v , it follows

$$+\infty > [v]_{W^{s,2}(\mathbb{R}^N)}^2 = \mathfrak{h}_s(\Omega) \int_{\Omega} \frac{|v|^2}{d_{\Omega}^{2s}} dx \geq \frac{\mathfrak{h}_s(\Omega)}{C^2} \int_{\Omega} \frac{1}{d_{\Omega}} dx.$$

This finally gives a contradiction with (5.2.2). \square

5.3 The method of supersolutions

The main result of this chapter is the following theorem, taken from [B4]. It provides a dual formulation of $\mathfrak{h}_s(\Omega)$, in terms of *positive local weak supersolutions* of (II.7).

Theorem 4. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then we have*

$$\mathfrak{h}_s(\Omega) = \sup \left\{ \lambda \geq 0 : \text{equation (II.7) admits a positive local weak supersolution} \right\}.$$

In order to show it, we need the following preliminary results.

Lemma 5.3.1. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set. Then:*

- (i) *if there exists $\lambda \geq 0$ such that the equation (II.7) admits a positive local weak supersolution u , then $\lambda \leq \mathfrak{h}_s(\Omega)$;*
- (ii) *in particular, if u is a positive weak solution in $\widetilde{W}_0^{s,2}(\Omega)$, then $\lambda = \mathfrak{h}_s(\Omega)$ and u is a minimizer for $\mathfrak{h}_s(\Omega)$.*

Proof. In order to prove (i), for every $\eta \in C_0^\infty(\Omega)$, we test the weak formulation with

$$\varphi = \frac{|\eta|^2}{\varepsilon + u},$$

where $\varepsilon > 0$. We observe that this is a feasible test function, thanks to Lemma 1.1.6. By using the *discrete Picone inequality* (see [53, Lemma 2.6] or [20, Proposition 4.2]), we obtain

$$\begin{aligned} \lambda \int_{\Omega} \frac{u}{d_{\Omega}^{2s}} \frac{|\eta|^2}{\varepsilon + u} dx &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y)) \left(\frac{|\eta|^2}{\varepsilon + u}(x) - \frac{|\eta|^2}{\varepsilon + u}(y) \right)}{|x - y|^{N+2s}} dx dy \\ &\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^2}{|x - y|^{N+2s}} dx dy \leq [\eta]_{W^{s,2}(\mathbb{R}^N)}^2. \end{aligned}$$

In the last inequality we used that

$$[|\eta|]_{W^{s,2}(\mathbb{R}^N)}^2 \leq [\eta]_{W^{s,2}(\mathbb{R}^N)}^2, \quad (5.3.1)$$

and the inequality is strict, unless η has constant sign almost everywhere (see the proof of Lemma 5.2.1). By taking the limit as ε goes to 0 on the left-hand side, using that u is positive on Ω and the arbitrariness of $\eta \in C_0^\infty(\Omega)$, this finally gives that $\lambda \leq \mathfrak{h}_s(\Omega)$, as desired.

In order to prove point (ii), we observe that if $u \in \widetilde{W}_0^{s,2}(\Omega)$, we can test the weak formulation of the equation with the solution itself. This yields

$$[u]_{W^{s,2}(\mathbb{R}^N)}^2 = \lambda \int_{\Omega} \frac{u^2}{d_{\Omega}^{2s}} dx.$$

On the other hand, by definition of $\mathfrak{h}_s(\Omega)$, we know that

$$\mathfrak{h}_s(\Omega) \int_{\Omega} \frac{u^2}{d_{\Omega}^{2s}} dx \leq [u]_{W^{s,2}(\mathbb{R}^N)}^2.$$

This shows that $\mathfrak{h}_s(\Omega) \leq \lambda$. Since the reverse inequality holds from (i), we conclude that it must result $\lambda = \mathfrak{h}_s(\Omega)$. \square

In the next lemma, we will use the weighted space $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ studied in Section 5.1.

Lemma 5.3.2. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open set such that*

$$\mathfrak{h}_s(\Omega) > 0.$$

Then for every $0 \leq \lambda < \mathfrak{h}_s(\Omega)$ there exists a positive local weak supersolution $u_\lambda \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ of the equation (II.7). More precisely, the function u_λ is a weak solution of the equation

$$(-\Delta)^s u = \lambda \frac{u}{d_\Omega^{2s}} + 1_B, \quad \text{in } \Omega, \quad (5.3.2)$$

where $B \Subset \Omega$ is a fixed ball.

Proof. We first observe that, for every $\varphi \in \mathcal{X}_0^{s,2}(\Omega, d_\Omega)$, we have

$$\int_B |\varphi| dx \leq |B|^{\frac{1}{2}} \|d_\Omega\|_{L^\infty(B)}^s \left(\int_\Omega \frac{|\varphi|^2}{d_\Omega^{2s}} dx \right)^{\frac{1}{2}} \leq |B|^{\frac{1}{2}} \|d_\Omega\|_{L^\infty(B)}^s \left(\frac{1}{\mathfrak{h}_s(\Omega)} \right)^{\frac{1}{2}} [\varphi]_{W^{s,2}(\mathbb{R}^N)},$$

thanks to Hölder's inequality, the definition of $\mathfrak{h}_{s,2}(\Omega)$ and the fact that Hardy's inequality holds in $\mathcal{X}_0^{s,2}(\Omega; d_\Omega)$, as well (see Remark 5.1.1). This shows that we have the continuous embedding $\mathcal{X}_0^{s,2}(\Omega; d_\Omega) \hookrightarrow L^1(B)$, for every $B \Subset \Omega$ as in the statement.

Let $0 \leq \lambda < \mathfrak{h}_{s,2}(\Omega)$, we consider the functional

$$\mathfrak{F}_\lambda(\varphi) = \frac{1}{2} [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \int_\Omega \frac{|\varphi|^2}{d_\Omega^{2s}} dx - \int_B \varphi dx, \quad \text{for every } \varphi \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega).$$

We will construct the desired supersolution as a minimizer of the following problem

$$m(\lambda) := \inf_{\varphi \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)} \mathfrak{F}_\lambda(\varphi).$$

We first notice that by Hardy's inequality we have, for every $\varphi \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$

$$\begin{aligned} \mathfrak{F}_\lambda(\varphi) &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\mathfrak{h}_s(\Omega)} \right) [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \int_B \varphi dx \\ &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\mathfrak{h}_s(\Omega)} \right) [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{1}{2\varepsilon} \int_B d_\Omega^{2s} dx - \frac{\varepsilon}{2} \int_B \frac{|\varphi|^2}{d_\Omega^{2s}} dx \\ &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\mathfrak{h}_s(\Omega)} \right) [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{1}{2\varepsilon} \int_B d_\Omega^{2s} dx - \frac{\varepsilon}{2} \frac{1}{\mathfrak{h}_s(\Omega)} [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2, \end{aligned}$$

with $\varepsilon > 0$, where we also used Young's inequality. In particular, by choosing

$$\varepsilon = \frac{\mathfrak{h}_s(\Omega) - \lambda}{2},$$

we can infer that

$$\mathfrak{F}_\lambda(\varphi) \geq c_1 [\varphi]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{1}{C_1}, \quad \text{for every } \varphi \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega), \quad (5.3.3)$$

where $c_1 > 0$ and $C_1 > 0$ do not depend on φ . This in particular shows that $m(\lambda) > -\infty$.

Let us now take a minimizing sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ such that

$$\mathfrak{F}_\lambda(u_n) \leq m(\lambda) + \frac{1}{n+1}, \quad \text{for every } n \in \mathbb{N}.$$

By appealing to (5.3.3), we get in particular that there exists a constant $M > 0$ such that

$$[u_n]_{W^{s,2}(\mathbb{R}^N)}^2 \leq M, \quad \text{for every } n \in \mathbb{N}.$$

By applying Theorem 5.1.7, we can infer existence of $u \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$ such that the sequence converges almost everywhere in \mathbb{R}^N and such that

$$\int_B u_n dx = \int_B u dx + o(1), \quad \text{as } n \rightarrow \infty,$$

up to a subsequence. Observe that by construction we have

$$m(\lambda) + \frac{1}{n+1} \geq \frac{1}{2} [u_n]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \int_\Omega \frac{|u_n|^2}{d_\Omega^{2s}} dx - \int_B u_n dx \geq m(\lambda),$$

which in particular implies that

$$\frac{1}{2} [u_n]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \int_\Omega \frac{|u_n|^2}{d_\Omega^{2s}} dx - \int_B u_n dx = m(\lambda) + o(1), \quad \text{as } n \rightarrow \infty. \quad (5.3.4)$$

By applying the Brézis-Lieb Lemma (see [27, Theorem 1] and also [26, Lemma 2.2]), we get

$$\frac{\lambda}{2} \int_\Omega \frac{|u_n|^2}{d_\Omega^{2s}} dx = \frac{\lambda}{2} \int_\Omega \frac{|u|^2}{d_\Omega^{2s}} dx + \frac{\lambda}{2} \int_\Omega \frac{|u_n - u|^2}{d_\Omega^{2s}} dx + o(1), \quad \text{as } n \rightarrow \infty,$$

and

$$[u_n]_{W^{s,2}(\mathbb{R}^N)}^2 = [u]_{W^{s,2}(\mathbb{R}^N)}^2 + [u_n - u]_{W^{s,2}(\mathbb{R}^N)}^2 + o(1), \quad \text{as } n \rightarrow \infty.$$

By inserting these informations in (5.3.4), we obtain

$$\mathfrak{F}_\lambda(u) + \frac{1}{2} [u_n - u]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{\lambda}{2} \int_\Omega \frac{|u_n - u|^2}{d_\Omega^{2s}} dx = m(\lambda) + o(1), \quad \text{as } n \rightarrow \infty.$$

We can now use Hardy's inequality for the function $u_n - u \in \mathcal{X}_0^{s,2}(\Omega; d_\Omega)$. Thanks to the choice of λ , it holds that

$$\mathfrak{F}_\lambda(u) \leq m(\lambda) + o(1), \quad \text{as } n \rightarrow \infty,$$

and by taking the limit as n goes to ∞ , we finally get that u is a minimizer.

By minimality, we get that u must be non-negative. Indeed, by using (5.3.1) and observing that

$$- \int_B u dx \geq - \int_B |u| dx,$$

we have

$$\mathfrak{F}_\lambda(u) \geq \mathfrak{F}_\lambda(|u|).$$

Moreover, the inequality sign in the latter is strict, unless u has constant sign almost everywhere. By virtue of the inequality for the integral on B , we get that such a sign must be non-negative, i.e. we must have $u \geq 0$ almost everywhere in Ω , as claimed.

Additionally, by minimality u is a weak solution of the Euler-Lagrange equation (5.3.2). This in particular proves that $u \not\equiv 0$, thanks to the presence of the term 1_B . Observe that (see Proposition 5.1.2)

$$\mathcal{X}_0^{s,2}(\Omega) \subset W_{\text{loc}}^{s,2}(\Omega) \cap L_{2s}^1(\mathbb{R}^N),$$

thus u is a local weak supersolution. Finally, by using the minimum principle, we get that u is positive on Ω (we can proceed as in the proof of Lemma 5.2.1, for example). \square

By joining the previous two technical results, we finally get the characterization of the sharp fractional s -Hardy constant stated in Theorem 4. Thus, we are now ready for the

Proof of Theorem 4

We first observe that the set of admissible λ is non-empty: indeed, it always contains $\lambda = 0$. To see this, it is sufficient to observe that any positive constant function is a local weak solution of

$$(-\Delta)^s u = 0, \quad \text{in } \Omega,$$

which is (II.7) for $\lambda = 0$.

In order to prove the claimed identity, we first consider the case $\mathfrak{h}_s(\Omega) = 0$. Then, the previous discussion and Lemma 5.3.1 imply that the set of admissible λ is actually given by the singleton $\{0\}$. Thus the conclusion holds.

In the case $\mathfrak{h}_s(\Omega) > 0$, again by Lemma 5.3.1, we have that $\mathfrak{h}_s(\Omega) \geq \lambda$ for every λ such that (II.7) admits a positive local weak supersolution. On the other hand, from Lemma 5.3.2 we have that for every $\varepsilon > 0$ if we take

$$\mathfrak{h}_s(\Omega) - \varepsilon < \lambda < \mathfrak{h}_s(\Omega),$$

then (II.7) admits a positive local weak supersolution. This concludes the proof. \square

Chapter 6

The sharp fractional Hardy inequality for convex sets

In this chapter, still based on [B5], we will determine the sharp constant for the fractional Hardy inequality

$$\mathfrak{h}_s(\Omega) = \inf_{u \in C_0^\infty(\Omega)} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy : \int_{\Omega} \frac{|u|^2}{d_{\Omega}^{2s}} dx = 1 \right\},$$

whenever $\Omega \subsetneq \mathbb{R}^N$ is an open convex set. We have already mentioned that in [B5] we tackled the more general case of $\mathfrak{h}_{s,p}(\Omega)$, for $1 < p < \infty$ and $s \in (0, 1)$. More precisely, we determined the sharp constant $\mathfrak{h}_{s,p}$ for:

- general convex sets, under the restriction $sp \geq 1$;
- half-spaces, without any additional restriction on p and s .

Here, as in the previous chapter, we will focus on the case $p = 2$ and show that in this case, we can completely solve the problem, without the additional restriction $2s \geq 1$. Let us first introduce some notation: for every $0 < s < 1$, we denote

$$\Lambda_s := 2 \int_0^1 \frac{|1 - t^{\frac{2s-1}{2}}|^2}{(1-t)^{1+2s}} dt + \frac{1}{s}.$$

As we will see below, this will coincide with $\mathfrak{h}_s(\mathbb{H}_+^1)$, where \mathbb{H}_+^1 denotes the half-line $(0, +\infty)$. For every $k \in \mathbb{N}$ and $\alpha \geq 0$, we set

$$\mathcal{I}(k; \alpha) = \int_0^{+\infty} t^k (1+t^2)^{-\frac{k+2+\alpha}{2}} dt.$$

and we recall the following dimensional constant, defined in the introduction

$$C_{N,2s} := \begin{cases} (N-1)\omega_{N-1}\mathcal{I}(N-2; 2s), & \text{for } N \geq 2, \\ 1, & \text{for } N = 1. \end{cases}$$

An alternative expression for this constant is given in Lemma A.4.1 below. Finally, we recall the standard notation for the half-space

$$\mathbb{H}_+^1 = (0, +\infty), \quad \mathbb{H}_+^N = \mathbb{R}^{N-1} \times (0, +\infty), \text{ for } N \geq 2.$$

We can now state the main result of this chapter.

Theorem 5 (Sharp fractional Hardy inequality for convex sets). *Let $N \geq 1$, then we have:*

1. for every $0 < s < 1$

$$\mathfrak{h}_s(\mathbb{H}_+^N) = C_{N,2s} \Lambda_s;$$

2. for every $0 < s < 1$ and every $\Omega \subsetneq \mathbb{R}^N$ open convex set

$$\mathfrak{h}_s(\Omega) = \mathfrak{h}_s(\mathbb{H}_+^N).$$

In each case, the constant \mathfrak{h}_s is not attained.

The previous result is obtained by combining Theorems 6.3.2 and 6.3.4 below. We first notice that point 1 could be also obtained by using [52, Theorem 1.1] for the “regional” case and then observing that

$$[u]_{W^{s,2}(\mathbb{H}_+^N)}^2 = [u]_{W^{s,2}(\mathbb{R}^N)}^2 - \frac{C_{N,2s}}{s} \int_{\mathbb{H}_+^N} \frac{|u|^2}{d_{\mathbb{H}_+^N}^{2s}} dx, \quad \text{for every } u \in C_0^\infty(\mathbb{H}_+^N),$$

as in [13]. Here, we first construct a suitable local weak supersolution of

$$(-\Delta)^s u = \lambda \frac{u}{d_\Omega^{2s}}, \quad \text{in } \Omega,$$

in dimension 1, for a half-line and an interval. Afterwards, we show how to reduce the higher-dimensional case to dimension one. Then, in Section 6.3, we will prove that the constant obtained by this method is the sharp one.

6.1 Construction of supersolutions in dimension 1

The aim of this section is the construction of a local weak supersolution for the half-line.

In what follows, for $t > 0$ we use the notation

$$I_\varepsilon(t) := \left(\frac{t}{1+\varepsilon}, (1+\varepsilon)t \right), \quad \text{for } 0 < \varepsilon \ll 1. \quad (6.1.1)$$

Moreover, for a fixed $\beta \in \mathbb{R}$, we set

$$U_\beta(t) := t^\beta, \quad \text{for } t \in \mathbb{H}_+^1,$$

and we extend it by 0 to the complement of \mathbb{H}_+^1 . In particular, in the borderline case $\beta = 0$, this has to be intended as the characteristic function of \mathbb{H}_+^1 .

The next result collects some properties of U_β which will be useful in the sequel.

Lemma 6.1.1. *Let $0 < s < 1$. For every $\beta \in \mathbb{R}$ we have $U_\beta \in W_{\text{loc}}^{s,p}(\mathbb{H}_+^1)$. Moreover, U_β has the following further properties:*

- for $(2s-1)/2 < \beta$, we have $U_\beta \in W^{s,2}((0, M))$, for every $M > 0$;
- for $-1 < \beta < 2s$, we have $U_\beta \in L_{2s}^1(\mathbb{R})$.

Proof. We observe that U_β is locally Lipschitz on \mathbb{H}_+^1 , for every $\beta \in \mathbb{R}$. This easily implies that $U_\beta \in W_{\text{loc}}^{s,2}(\mathbb{H}_+^1)$.

Let us now suppose that $\beta > (2s - 1)/2$. From the fact that $U_\beta \in W_{\text{loc}}^{s,2}(\mathbb{H}_+^1)$, we get that for every $0 < \varepsilon < M$ we have

$$\int_\varepsilon^M \int_\varepsilon^M \frac{|U_\beta(t) - U_\beta(y)|^2}{|t - y|^{1+2s}} dt dy < +\infty.$$

We show that this is uniformly bounded with respect to ε . For $\beta > s$ this is straightforward, it is sufficient to use that U_β is either β -Hölder continuous (for $s < \beta < 1$) or even Lipschitz continuous (for $\beta \geq 1$) on $[0, M]$.

We thus assume $(2s - 1)/2 < \beta \leq s$. By using the definition of U_β , Fubini's Theorem and the change of variable $y = \tau t$, we get

$$\begin{aligned} \int_\varepsilon^M \int_\varepsilon^M \frac{|U_\beta(t) - U_\beta(y)|^2}{|t - y|^{1+2s}} dt dy &= \int_\varepsilon^M \left(\int_{\frac{\varepsilon}{t}}^{\frac{M}{t}} \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau \right) t^{2\beta-2s} dt \\ &= \int_\varepsilon^M \left(\int_{\frac{\varepsilon}{t}}^1 \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau \right) t^{2\beta-2s} dt \\ &\quad + \int_\varepsilon^M \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau \right) t^{2\beta-2s} dt. \end{aligned} \tag{6.1.2}$$

We now observe that

$$\int_{\frac{\varepsilon}{t}}^1 \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau \leq \int_0^1 \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau < +\infty.$$

For second integral, we observe that

$$\frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} \sim \frac{1}{\tau^{1+2s-2\beta}}, \quad \text{for } \tau \rightarrow +\infty,$$

and the last function is integrable on $[1, +\infty)$, for $\beta < s$. Thus we get

$$\int_1^{\frac{M}{t}} \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau \leq \int_1^{+\infty} \frac{|1 - \tau^\beta|^2}{|1 - \tau|^{1+2s}} d\tau < +\infty.$$

This discussion entails that

$$\int_\varepsilon^M \int_\varepsilon^M \frac{|U_\beta(t) - U_\beta(y)|^2}{|t - y|^{1+2s}} dt dy \leq C \int_\varepsilon^M t^{2\beta-2s} dt = C \frac{M^{2\beta-2s+1} - \varepsilon^{2\beta-2s+1}}{2\beta - 2s + 1},$$

and the last quantity is bounded as ε goes to 0, thanks to the fact that $\beta > (2s - 1)/2$. We thus proved the claimed property of U_β , for $(2s - 1)/2 < \beta < s$.

We still miss the borderline case $\beta = s$. From (6.1.2), we can infer

$$\int_\varepsilon^M \int_\varepsilon^M \frac{|U_s(t) - U_s(y)|^2}{|t - y|^{1+2s}} dt dy \leq \int_\varepsilon^M \left(\int_0^1 \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt + \int_\varepsilon^M \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt.$$

The first integral on the right-hand side is uniformly bounded in ε , but now we have to pay attention to the fact that

$$\lim_{t \rightarrow 0^+} \int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau = +\infty.$$

We can proceed as follows: we write

$$\int_{\varepsilon}^M \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt = \int_{\varepsilon}^{\frac{M}{2}} \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt + \int_{\frac{M}{2}}^M \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt$$

and observe that for $0 < t < M/2$, we have

$$\int_{\varepsilon}^{\frac{M}{2}} \left(\int_1^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt \leq \frac{M}{2} \int_1^2 \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau + \int_{\varepsilon}^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt$$

and, at last

$$\int_{\varepsilon}^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \frac{|1 - \tau^s|^2}{|1 - \tau|^{1+2s}} d\tau \right) dt \leq 2^{1+2s} \int_{\varepsilon}^{\frac{M}{2}} \left(\int_2^{\frac{M}{t}} \tau^{2s-1-2s} d\tau \right) dt = 2^{1+2s} \int_{\varepsilon}^M \log \left(\frac{M}{2t} \right) dt.$$

The last integral is uniformly bounded, as ε goes to 0. This finally proves that $U_s \in W^{s,2}((0, M))$.

Finally, we observe that

$$U_{\beta} \in L_{\text{loc}}^1(\mathbb{R}) \iff \beta > -1,$$

and

$$\int_{\mathbb{R}} \frac{U_{\beta}}{(1 + |t|)^{1+2s}} dt = \int_0^{+\infty} \frac{t^{\beta}}{(1 + t)^{1+2s}} dt < +\infty \iff -1 < \beta(p - 1) < sp.$$

This concludes the proof. \square

Remark 6.1.2. For later reference, we observe that in the previous proof for

$$\frac{2s - 1}{2} < \beta < s,$$

we proved the following upper bound

$$[U_{\beta}]_{W^{s,2}((0,M))}^2 \leq \left(\int_0^1 \frac{|1 - \tau^{\beta}|^2}{|1 - \tau|^{1+2s}} d\tau + \int_1^{+\infty} \frac{|1 - \tau^{\beta}|^2}{|1 - \tau|^{1+2s}} d\tau \right) \frac{M^{2\beta-2s+1}}{2\beta - 2s + 1}.$$

By making the change of variable $\tau = 1/\xi$ in the second integral, this can also be rewritten as

$$[U_{\beta}]_{W^{s,2}((0,M))}^2 \leq \left(\int_0^1 \frac{|1 - \tau^{\beta}|^2}{|1 - \tau|^{1+2s}} \left(1 + \tau^{2\beta-2s-1} \right) d\tau \right) \frac{M^{2\beta-2s+1}}{2\beta - 2s + 1}. \quad (6.1.3)$$

We have used the last estimate also in the proof of Lemma 1.1.9.

In the next result, we compute the fractional Laplacian of order s for U_{β} . This generalizes [71, Lemma 3.1] to the case $\beta \neq s$.

Proposition 6.1.3. Let $0 < s < 1$. For every $-1 < \beta < 2s$ the function U_{β} is a local weak solution of (II.7) in \mathbb{H}_+^1 , with

$$\lambda = \lambda(\beta) = 2 \int_0^1 \frac{1 - t^{\beta}}{(1 - t)^{1+2s}} \left(1 - t^{2s-1-\beta} \right) dt + \frac{1}{s}. \quad (6.1.4)$$

Moreover, if we define the family of functions on \mathbb{H}_+^1 by

$$F_{\varepsilon}(t) = 2 \int_{\mathbb{R} \setminus I_{\varepsilon}(t)} \frac{U_{\beta}(t) - U_{\beta}(y)}{|t - y|^{1+2s}} dy, \quad \text{for } 0 < \varepsilon < 1, \quad (6.1.5)$$

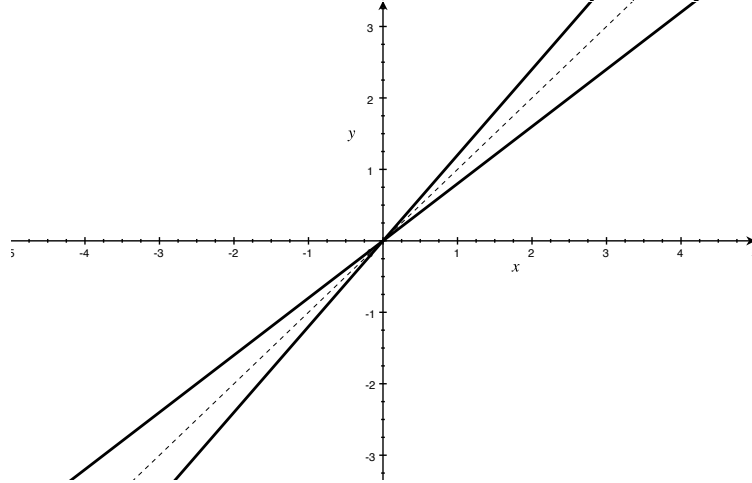


Figure 6.1: The set \mathcal{O}_ε is the conical region “centered” around the line $y = t$.

where $I_\varepsilon(t)$ is defined by (6.1.1), we get that this converges to

$$F_0(t) = \lambda(\beta) \frac{U_\beta(t)}{t^{2s}},$$

uniformly on compact subsets of \mathbb{H}_+^1 , as ε goes to 0.

Proof. Let us take $\varphi \in C_0^\infty(\mathbb{H}_+^1)$, we observe that by the Dominated Convergence Theorem we have

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^{1+2s}} dt dy = \lim_{\varepsilon \rightarrow 0^+} \iint_{(\mathbb{R} \times \mathbb{R}) \setminus \mathcal{O}_\varepsilon} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^{1+2s}} dt dy,$$

where

$$\mathcal{O}_\varepsilon = \left\{ (t, y) \in \mathbb{R} \times \mathbb{R} : \min \left\{ \frac{t}{1+\varepsilon}, (1+\varepsilon)t \right\} \leq y \leq \max \left\{ \frac{t}{1+\varepsilon}, (1+\varepsilon)t \right\} \right\},$$

see Figure 6.1. For every $0 < \varepsilon < 1$, by proceeding as in [19, Lemma 2.3], we have

$$\frac{(U_\beta(t) - U_\beta(y))}{|t - y|^{1+2s}} \varphi(t) \in L^1((\mathbb{R} \times \mathbb{R}) \setminus \mathcal{O}_\varepsilon).$$

Thus we can use Fubini’s Theorem and a change of variable, to write

$$\iint_{(\mathbb{R} \times \mathbb{R}) \setminus \mathcal{O}_\varepsilon} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^{1+2s}} dt dy = 2 \int_0^{+\infty} \left(\int_{\mathbb{R} \setminus I_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^{1+2s}} dy \right) \varphi(t) dt.$$

Observe that we used that φ is compactly supported on \mathbb{H}_+^1 . By recalling the definition (6.1.5), up to now we have obtained

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^{1+2s}} dt dy = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} F_\varepsilon(t) \varphi(t) dt, \quad (6.1.6)$$

for every $\varphi \in C_0^\infty(\mathbb{H}_+^1)$. We now manipulate this quantity, for a fixed $0 < \varepsilon < 1$: by recalling that U_β identically vanishes in $(-\infty, 0]$, for $t > 0$ we have

$$\begin{aligned} F_\varepsilon(t) &= 2 \int_{\mathbb{R} \setminus I_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^{1+2s}} dy \\ &= 2 \int_{\mathbb{H}_+^1 \setminus I_\varepsilon(t)} \frac{t^\beta - y^\beta}{|t - y|^{1+2s}} dy + 2 \int_{-\infty}^0 \frac{t^\beta}{|t - y|^{1+2s}} dy. \end{aligned}$$

The second integral can be directly computed: this gives

$$\int_{-\infty}^0 \frac{t^\beta}{|t-y|^{1+2s}} dy = \frac{1}{2s} \frac{U_\beta(t)}{t^{2s}},$$

where we used the definition of $U_\beta(t)$. For the first integral in the definition of F_ε , by performing the change of variable $y = \tau t$, we obtain

$$\begin{aligned} \int_{\mathbb{H}_+^1 \setminus I_\varepsilon(t)} \frac{t^\beta - y^\beta}{|t-y|^{1+2s}} dy &= \frac{t^\beta}{t^{2s}} \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + \frac{t^\beta}{t^{2s}} \int_{1+\varepsilon}^{+\infty} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau \\ &= \frac{U_\beta(t)}{t^{2s}} \left(\int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + \int_{1+\varepsilon}^{+\infty} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau \right), \end{aligned}$$

again thanks to the definition of U_β . Thus we have obtained

$$F_\varepsilon(t) = \lambda_\varepsilon(\beta) \frac{U_\beta(t)}{t^{2s}}, \quad \text{for every } t \in \mathbb{H}_+^1, 0 < \varepsilon < 1, \quad (6.1.7)$$

where

$$\lambda_\varepsilon(\beta) = 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + 2 \int_{1+\varepsilon}^{+\infty} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + \frac{1}{s}.$$

By inserting this in (6.1.6), we have

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y))(\varphi(t) - \varphi(y))}{|t-y|^{1+2s}} dt dy = \left(\lim_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(\beta) \right) \int_{\mathbb{R}} \frac{U_\beta(t)}{t^{2s}} \varphi(t) dt. \quad (6.1.8)$$

To conclude the proof, we only need to show that for $\lambda(\beta)$ defined by (6.1.4), we have

$$\lambda(\beta) = \lim_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(\beta), \quad \text{for every } -1 < \beta < s.$$

We first observe that the case $\beta = 0$ is simple: in this case we have

$$t^\beta - 1 = 0, \quad \text{for } t \in (0, 1),$$

and thus we directly get

$$\lambda(0) = \lambda_\varepsilon(0) = \frac{1}{s}.$$

We can thus suppose that $\beta \neq 0$. By recalling the definition of $\lambda_\varepsilon(\beta)$ above and performing the change of variable $\tau = 1/\zeta$ in the second integral, we get for $0 < \varepsilon < 1$

$$\begin{aligned} \lambda_\varepsilon(\beta) &= 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\zeta^{-\beta}}{|1-\zeta^{-1}|^{1+2s}} \frac{d\zeta}{\zeta^2} + \frac{1}{s} \\ &= 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} d\tau + 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{\zeta^\beta - 1}{|\zeta - 1|^{1+2s}} \zeta^{2s-1-\beta} d\zeta + \frac{1}{s} \\ &= 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1-\tau^\beta}{|1-\tau|^{1+2s}} \left(1 - \tau^{2s-1-\beta} \right) d\tau + \frac{1}{s}. \end{aligned}$$

On the other hand, by a Taylor expansion, we have

$$\frac{1-\tau^\beta}{|1-\tau|^{1+2s}} \left(1 - \tau^{2s-1-\beta} \right) \sim \beta (2s-1-\beta) (1-\tau)^{1-2s}, \quad \text{for } \tau \nearrow 1,$$

which shows that

$$\frac{1 - \tau^\beta}{|1 - \tau|^{1+2s}} \left(1 - \tau^{2s-1-\beta}\right) \in L^1((0, 1)).$$

These facts permit to establish that

$$\lambda(\beta) = \lim_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(\beta), \quad \text{for every } -1 < \beta < 2s,$$

thus from (6.1.8) we get that U_β is a local weak solution of the claimed equation.

The last statement about the convergence of F_ε is an easy consequence of formula (6.1.7). \square

The next result investigates some properties of the function $\lambda(\beta)$ defined in (6.1.4). This in particular permits to single out a special solution, among all the functions U_β : this corresponds to the choice

$$\beta = \frac{2s - 1}{2}.$$

Indeed, for this function, the constant λ is *the largest possible*. This fact is comparable to what Bogdan and Dyda did in [13, Theorem 1] for the “regional seminorm”. In their proof, they did a similar discussion in order to find the *largest* constant.

Proposition 6.1.4. *For every $0 < s < 1$, let us consider the function*

$$\beta \mapsto \lambda(\beta), \quad \text{defined by (6.1.4) on the interval } (-1, 2s).$$

Then this has the following properties:

1. *it is symmetric with respect to the point $(2s - 1)/2$. In particular, we have*

$$\lambda(s - 1) = \lambda(s) = 0;$$

2. *it is monotonically decreasing for $\beta > (2s - 1)/2$ and monotonically increasing for $\beta < (2s - 1)/2$. In particular, we have*

$$\lambda(\beta) \leq \lambda\left(\frac{2s - 1}{2}\right) = 2 \int_0^1 \frac{\left|1 - t^{\frac{2s-1}{2}}\right|^2}{(1-t)^{1+2s}} dt + \frac{1}{s}.$$

Proof. Is easy to check that

$$\lambda(2s - 1 - \beta) = \lambda(\beta), \quad \text{for every } -1 < \beta < 2s. \quad (6.1.9)$$

Moreover, the fact that $\lambda(s) = 0$ has been proved in [71, Lemma 3.1]. Accordingly, thanks to (6.1.9), $\lambda(s - 1) = 0$ too, and it concludes the proof of 1.

Now we show the statement 2. We proceed similarly as in [20, Lemma B.1], but making a more complete study. Thanks to the symmetry shown in the statement 1, we consider $\beta \geq (2s - 1)/2$ only. For every $0 < t < 1$, we define the function

$$g(\beta) := (1 - t^\beta) \left(1 - t^{2s-1-\beta}\right), \quad \text{for } -1 < \beta < 2s.$$

We discuss the monotonicity of such a function. We first observe that $g(0) = 0$. Let us take $\beta \neq 0$ and differentiate g , we have

$$g'(\beta) = t^{2s-1-\beta} \log t (1 - t^\beta) - t^\beta \log t \left(1 - t^{2s-1-\beta}\right),$$

We now discuss separately the cases $\beta < 0$ and $\beta > 0$. We start with the latter. By observing that $\log t < 0$, we get

$$\begin{aligned} g'(\beta) > 0 &\iff t^{2s-1-\beta}(1-t^\beta) - t^\beta(1-t^{2s-1-\beta}) < 0 \\ &\iff t^{2s-1-\beta} < t^\beta \\ &\iff t^{2s-1-2\beta} < 1. \end{aligned}$$

Since $0 < t < 1$ and $\beta \geq (2s-1)/2$, the last requirement is never attained. This implies that:

- if $2s < 1$, then g is monotone decreasing on the whole interval $(-1, 2s)$ and thus

$$g(\beta) \leq g(0) = 0, \quad \text{for every } -1 < \beta < 2s;$$

- if $2s \geq 1$, then g is monotone decreasing on $((2s-1)/2, 2s)$. In particular, it is maximal at $\beta = (2s-1)/2$ and thus

$$g(\beta) \leq g\left(\frac{2s-1}{2}\right) = \left(1 - t^{\frac{2s-1}{2}}\right)^2, \quad \text{for every } -1 < \beta < 2s,$$

that is nonnegative.

This permits to infer that

$$\max_{-1 < \beta < 2s} g(\beta) \leq \left(1 - t^{\frac{2s-1}{2}}\right)^2.$$

and such a maximal value is uniquely attained at $\beta = (2s-1)/2$. By recalling that by definition

$$\lambda(\beta) = 2 \int_0^1 \frac{g(\beta)}{(1-t)^{1+2s}} dt + \frac{1}{s},$$

the properties of λ claimed in \mathcal{Q} follow from the above detailed discussion on g . \square

We now consider the interval I and we construct a local weak supersolution of (II.7). In what follows, we use the notation

$$I = (0, 1) \quad \text{and} \quad d_I(t) = \min\{t, 1-t\}, \quad \text{for } t \in I.$$

We first observe that the function $U_\beta = (d_I)^\beta$ may fail be a local weak supersolution of the equation (II.7) if $\beta < 0$: see Lemma 6.4.1 below. This implies that we need to seek a different family of supersolutions.

We reach our goal by using the previous results and the properties of the *fractional Kelvin transform*. Indeed, we can “transplant” supersolutions on \mathbb{H}_+^1 to construct suitable supersolutions in a bounded interval. We refer for example to [14] and [98, Appendix A] for the definition and properties of the fractional Kelvin transform, in connection with the fractional Laplacian.

Lemma 6.1.5. *Let $0 < s < 1$ and let $-1 < \beta < 2s$. We consider the function defined by*

$$f_\beta(t) := t^{2s-1-\beta}(1-t)^\beta, \quad \text{for } t \in I,$$

extended by 0 to the complement of I . Then this is a positive local weak solution of the equation

$$(-\Delta)^s u = \lambda(\beta) \frac{u}{(t(1-t))^{2s}}, \quad \text{in } I.$$

In particular, it is a positive local weak supersolution of the equation (II.7), with $\lambda = \lambda(\beta)$.

Proof. We first notice that the last part of the statement easily follows from the fact that

$$t(1-t) \leq \min\{t, 1-t\}, \quad \text{for } t \in I.$$

Let us focus on proving that f_β is a solution of the claimed equation. It is easily seen that $f_\beta \in W_{\text{loc}}^{s,2}(I) \cap L_{2s}^1(\mathbb{R})$, under the assumption $-1 < \beta < 2s$. By still using the notation of the previous section, we see that

$$f_\beta(t) = t^{2s-1} U_\beta \left(\frac{1}{t} - 1 \right), \quad \text{for } t \in I.$$

Thus, f_β coincides with the *fractional Kelvin transform* of the “shifted” function

$$x \mapsto U_\beta(x-1),$$

defined on the half-line $(1, +\infty)$ and extended by 0 to its complement. Then the proof of the statement above consists in computing the fractional Laplacian of such a Kelvin transform. For every $\varphi \in C_0^\infty((0, 1))$, we write

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(f_\beta(t) - f_\beta(\tau)) (\varphi(t) - \varphi(\tau))}{|t - \tau|^{1+2s}} dt d\tau \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{\left(|t|^{2s-1} U_\beta \left(\frac{1}{t} - 1 \right) - |\tau|^{2s-1} U_\beta \left(\frac{1}{\tau} - 1 \right) \right) (\varphi(t) - \varphi(\tau))}{|t - \tau|^{1+2s}} dt d\tau. \end{aligned}$$

We then make the change of variable $t = 1/x$ and $\tau = 1/y$, so to get

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(f_\beta(t) - f_\beta(\tau)) (\varphi(t) - \varphi(\tau))}{|t - \tau|^{1+2s}} dt d\tau \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{(|x|^{1-2s} U_\beta(x-1) - |y|^{1-2s} U_\beta(y-1)) \left(\varphi\left(\frac{1}{x}\right) - \varphi\left(\frac{1}{y}\right) \right)}{|x-y|^{1+2s}} |x|^{2s-1} |y|^{2s-1} dx dy \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{(|y|^{2s-1} U_\beta(x-1) - |x|^{2s-1} U_\beta(y-1)) \left(\varphi\left(\frac{1}{x}\right) - \varphi\left(\frac{1}{y}\right) \right)}{|x-y|^{1+2s}} dx dy. \end{aligned}$$

We now observe that we have the following pointwise identity

$$\begin{aligned} & (|y|^{2s-1} U_\beta(x-1) - |x|^{2s-1} U_\beta(y-1)) \left(\varphi\left(\frac{1}{x}\right) - \varphi\left(\frac{1}{y}\right) \right) \\ &= (U_\beta(x-1) - U_\beta(y-1)) \left(|x|^{2s-1} \varphi\left(\frac{1}{x}\right) - |y|^{2s-1} \varphi\left(\frac{1}{y}\right) \right) \\ &\quad - \left(U_\beta(x-1) \varphi\left(\frac{1}{x}\right) - U_\beta(y-1) \varphi\left(\frac{1}{y}\right) \right) (|x|^{2s-1} - |y|^{2s-1}). \end{aligned}$$

This implies that

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(f_\beta(t) - f_\beta(\tau)) (\varphi(t) - \varphi(\tau))}{|t - \tau|^{1+2s}} dt d\tau \\ &= \iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(x-1) - U_\beta(y-1)) \left(|x|^{2s-1} \varphi\left(\frac{1}{x}\right) - |y|^{2s-1} \varphi\left(\frac{1}{y}\right) \right)}{|x-y|^{1+2s}} dx dy \\ &\quad - \iint_{\mathbb{R} \times \mathbb{R}} \frac{\left(U_\beta(x-1) \varphi\left(\frac{1}{x}\right) - U_\beta(y-1) \varphi\left(\frac{1}{y}\right) \right) (|x|^{2s-1} - |y|^{2s-1})}{|x-y|^{1+2s}} dx dy. \end{aligned}$$

The second integral on the right-hand side vanishes, thanks to the fact that the function $x \mapsto |x|^{2s-1}$ is a local weak solution of

$$(-\Delta)^s u = 0, \quad \text{in } \mathbb{R} \setminus \{0\},$$

(see [29, Theorem 2.3]), once we observe that

$$x \mapsto U_\beta(x-1) \varphi\left(\frac{1}{x}\right),$$

is an element of $C_0^\infty((1, +\infty))$.

We can now use the equation solved by $x \mapsto U_\beta(x-1)$: indeed, by Proposition 6.1.3 for every $\psi \in C_0^\infty((1, +\infty))$ we have

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(x-1) - U_\beta(y-1)) (\psi(x) - \psi(y))}{|x-y|^{1+2s}} dx dy = \lambda(\beta) \int_1^{+\infty} \frac{U_\beta(x-1)}{(x-1)^{2s}} \psi(x) dx.$$

In particular, by choosing

$$\psi(x) = |x|^{2s-1} \varphi\left(\frac{1}{x}\right),$$

and observing that this belongs to $C_0^\infty((1, +\infty))$ if $\varphi \in C_0^\infty((0, 1))$, we get

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(x-1) - U_\beta(y-1)) \left(|x|^{2s-1} \varphi\left(\frac{1}{x}\right) - |y|^{2s-1} \varphi\left(\frac{1}{y}\right) \right)}{|x-y|^{1+2s}} dx dy \\ &= \lambda(\beta) \int_1^{+\infty} \frac{U_\beta(x-1)}{(x-1)^{2s}} |x|^{2s-1} \varphi\left(\frac{1}{x}\right) dx. \end{aligned}$$

Finally, by changing back variable $x = 1/t$ in the last integral, we get with simple manipulations

$$\lambda(\beta) \int_1^{+\infty} \frac{U_\beta(x-1)}{(x-1)^{2s}} |x|^{2s-1} \varphi\left(\frac{1}{x}\right) dx = \lambda(\beta) \int_0^1 \frac{f_\beta(t)}{t^{2s} (1-t)^{2s}} \varphi(t) dt.$$

This concludes the proof. \square

Remark 6.1.6. *The previous result has been greatly inspired to us by the reading of [38]. More precisely, in [38, Lemma 2.1] it is computed the fractional Laplacian of order s of the function*

$$w(x) = (1-x^2)^\beta, \quad \text{for } x \in (-1, 1). \quad (6.1.10)$$

In [38] the equation obtained is similar, though a bit different: the computation uses the Kelvin transformation, as well, even if in a slightly implicit fashion. In other words, in [38] the function w is not displayed as the conformal transplantation of a solution on the half-line: in this respect, we believe that our proof above has its own interest.

In order to compare our function f_β with Dyda's one (6.1.10), we observe that by making the change of variable $x = 2t - 1$, we get

$$W(t) = w(2t-1) = (1-(2t-1)^2)^\beta = 4^\beta t^\beta (1-t)^\beta, \quad \text{for } t \in I.$$

Up to the unessential multiplicative factor 4^β , we see that Dyda's function coincides with ours if and only if

$$2s-1-\beta = \beta \quad \text{i. e.} \quad \beta = \frac{2s-1}{2}.$$

Incidentally, we notice that this is the value of β which makes $\lambda(\beta)$ the largest possible, by Proposition 6.1.4.

Remark 6.1.7. By recalling Proposition 6.1.4 point 1, from Lemma 6.1.5 we get in particular that, with the choices $\beta = s$ and $\beta = s - 1$, the two functions

$$t \mapsto t^{s-1} (1-t)^s \quad \text{and} \quad t \mapsto t^s (1-t)^{s-1},$$

are locally weakly s -harmonic on I . This is a particular case of a result contained in [69], see also [40, case (b), page 428]. We owe this remark to the kind courtesy of Bartłomiej Dyda.

6.2 Construction of supersolutions in dimension $N \geq 2$

In what follows, for an open set $\Omega \subsetneq \mathbb{R}^N$ we will use the shortcut notations

$$U_\beta := d_\Omega^\beta,$$

and we extend it by 0 to the complement $\mathbb{R}^N \setminus \Omega$. In particular, in the borderline case $\beta = 0$, U_β has to be intended as the characteristic function of Ω .

Lemma 6.2.1. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open convex set. For every $0 \leq \beta < 2s$ we have*

$$U_\beta \in W_{\text{loc}}^{s,2}(\Omega) \cap L_{2s}^1(\mathbb{R}^N).$$

If Ω is a half-space, then this property is still true for $-1 < \beta < 2s$.

Proof. The fact that $U_\beta \in W_{\text{loc}}^{s,2}(\Omega)$ easily follows from its local Lipschitz character. In order to conclude, we need to prove that

$$\int_{\mathbb{R}^N} \frac{U_\beta}{(1+|x|)^{N+2s}} dx < +\infty, \quad \text{if } \beta < 2s.$$

For $\beta \geq 0$, it is sufficient to fix $x_0 \in \partial\Omega$ and observe that (recall that U_β vanishes outside Ω)

$$U_\beta(x) \leq |x - x_0|^\beta, \quad \text{for every } x \in \mathbb{R}^N.$$

We then obtain

$$\int_{\mathbb{R}^N} \frac{U_\beta}{(1+|x|)^{N+2s}} dx \leq \int_{\mathbb{R}^N} \frac{|x - x_0|^\beta}{(1+|x|)^{N+2s}} dx.$$

It is easily seen that the last integral converges if $\beta < 2s$.

Finally, if Ω is a half-space, we can suppose without loss of generality that

$$\Omega = \mathbb{H}_N^+ = \mathbb{R}^{N-1} \times (0, +\infty).$$

The case $N = 1$ is already contained in Lemma 6.1.1, thus we consider $N \geq 2$. We take $-1 < \beta < 0$ and we decompose

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{U_\beta}{(1+|x|)^{N+2s}} dx &= \int_{\mathbb{H}_+^N} \frac{x_N^\beta}{(1+|x|)^{N+2s}} dx \\ &= \int_{x \in \mathbb{H}_+^N : x_N \geq 1} \frac{x_N^\beta}{(1+|x|)^{N+2s}} dx \\ &\quad + \int_{\mathbb{R}^{N-1}} \left(\int_0^1 \frac{x_N^\beta}{\left(1 + \sqrt{x_N^2 + |x'|^2}\right)^{N+2s}} dx_N \right) dx' \\ &\leq \int_{x \in \mathbb{H}_+^N : x_N \geq 1} \frac{1}{(1+|x|)^{N+2s}} dx \\ &\quad + \int_{\mathbb{R}^{N-1}} \frac{dx'}{(1+|x'|)^{N+2s}} \left(\int_0^1 x_N^\beta dx_N \right). \end{aligned}$$

Thanks to the choice of β , the last integral is finite. \square

For every $k \in \mathbb{N}$ and $\alpha > 0$, we recall that we set

$$\mathcal{I}(k; \alpha) = \int_0^{+\infty} t^k (1+t^2)^{-\frac{k+2+\alpha}{2}} dt.$$

Then we observe that for $N \geq 2$ and every $m > 0$, by using the $(N-1)$ -dimensional spherical coordinates and a change of variable, we have

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \frac{dy'}{(m^2 + |x' - y'|^2)^{\frac{N+2s}{2}}} &= \frac{1}{m^{N+2s}} \int_{\mathbb{R}^{N-1}} \frac{dy'}{\left(1 + \frac{|x' - y'|^2}{m^2}\right)^{\frac{N+2s}{2}}} \\ &= \frac{m^{N-1}}{m^{N+2s}} \int_{\mathbb{S}^{N-1}} \int_0^{+\infty} \frac{t^{N-2}}{(1+t^2)^{\frac{N+2s}{2}}} dt d\mathcal{H}^{N-1}(\omega) \\ &= \frac{(N-1)\omega_{N-1}}{m^{1+2s}} \mathcal{I}(N-2; 2s). \end{aligned} \quad (6.2.1)$$

From now on, we still denote by $\lambda(\beta)$ the constant given by (6.1.4), while $C_{N,2s}$ is defined in (II.10). We refer to Remark 6.2.3 below, for a comment about the sharpness of the restriction $\beta \geq 0$.

Theorem 6.2.2. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open convex set. Then:*

1. *if $0 \leq \beta < 2s$, the function U_β is a local weak supersolution of (II.7), with $\lambda = C_{N,2s} \lambda(\beta)$;*
2. *if Ω is a half-space and $-1 < \beta < 2s$, the function U_β is a local weak solution of (II.7), still with $\lambda = C_{N,2s} \lambda(\beta)$.*

Proof. We will use a simple geometric construction, already exploited in the proof of [19, Proposition 3.2], in conjunction with the formula (6.2.1). We take $x \in \Omega$ and let $\bar{x} \in \partial\Omega$ be a point such that

$$d_\Omega(x) = |x - \bar{x}|.$$

Since Ω is convex, there exists a supporting hyperplane for it at the point \bar{x} . Without loss of generality, we can suppose that such a supporting hyperplane coincides with

$$\partial\mathbb{H}_+^N = \mathbb{R}^{N-1} \times \{0\},$$

and thus

$$x = (x', x_N) \text{ with } x_N > 0, \quad \bar{x} = (x', 0) \quad \text{and} \quad d_\Omega(x) = x_N.$$

Moreover, we suppose that $\Omega \subseteq \mathbb{H}_+^N$. We now observe that for every other $y = (y', y_N) \in \Omega$, by convexity it results

$$d_\Omega(y) \leq y_N,$$

see Figure 6.2. By using this fact and recalling that U_β vanishes in the complement of Ω , we actually have for $\beta \geq 0$

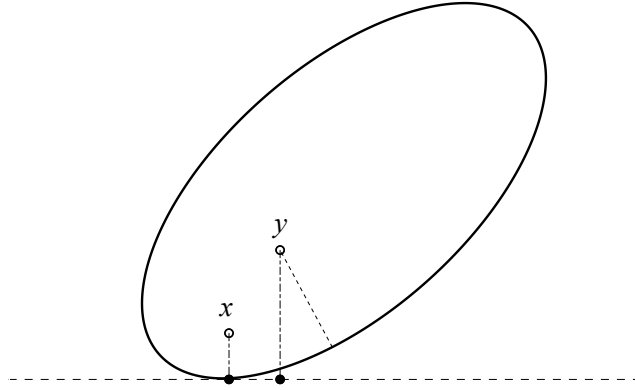
$$U_\beta(y) \leq (y_N)_+^\beta, \quad \text{for almost every } y = (y', y_N) \in \mathbb{R}^N.$$

In particular, we thus get that for almost every $y \in \mathbb{R}^N$ and $\beta \geq 0$

$$U_\beta(x) - U_\beta(y) \geq (x_N)_+^\beta - (y_N)_+^\beta. \quad (6.2.2)$$

For every $0 < \varepsilon \ll 1$ and for every $x \in \mathbb{R}^N$, we introduce the following slab

$$\mathcal{K}_\varepsilon(x) = \left\{ y \in \mathbb{R}^N : \min \left\{ \frac{x_N}{1+\varepsilon}, (1+\varepsilon)x_N \right\} \leq y_N \leq \max \left\{ \frac{x_N}{1+\varepsilon}, (1+\varepsilon)x_N \right\} \right\}.$$

Figure 6.2: The supporting hyperplane for Ω at \bar{x} .

Recalling that $x_N > 0$, we now use (6.2.2): we obtain for $\beta \geq 0$ and $0 < \varepsilon \ll 1$

$$\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy \geq \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{(x_N)_+^\beta - (y_N)_+^\beta}{|x - y|^{N+2s}} dy. \quad (6.2.3)$$

If $N \geq 2$, the last integral can be written as

$$\begin{aligned} \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy &\geq \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{(x_N)_+^\beta - (y_N)_+^\beta}{|x - y|^{N+2s}} dy \\ &= \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \left((x_N)_+^\beta - (y_N)_+^\beta \right) \\ &\quad \times \left(\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x_N - y_N|^2 + |x' - y'|^2)^{\frac{N+2s}{2}}} \right) dy_N, \end{aligned}$$

where $I_\varepsilon(x_N)$ is the same interval as in (6.1.1). If we now use (6.2.1) with $m = |x_N - y_N|$, we get

$$\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|x_N - y_N|^2 + |x' - y'|^2)^{\frac{N+2s}{2}}} = \frac{C_{N,2s}}{|x_N - y_N|^{1+2s}}.$$

Thus, we obtain from (6.2.3)

$$\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy \geq C_{N,2s} \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \frac{(x_N)_+^\beta - (y_N)_+^\beta}{|x_N - y_N|^{1+2s}} dy_N.$$

By recalling that we set $C_{1,2s} = 1$, the above formula obviously holds for $N = 1$, as well: actually, it coincides with (6.2.3).

By the definition (6.1.5) and the identity (6.1.7), we have for $x_N > 0$

$$2 \int_{\mathbb{R} \setminus I_\varepsilon(x_N)} \frac{(x_N)_+^\beta - (y_N)_+^\beta}{|x_N - y_N|^{1+2s}} dy_N = F_\varepsilon(x_N) = \lambda_\varepsilon(\beta) \frac{x_N^\beta}{x_N^{2s}} = \lambda_\varepsilon(\beta) \frac{U_\beta(x)}{d_\Omega(x)^{2s}}.$$

Moreover, we recall that (see the proof of Proposition 6.1.3)

$$\lambda_\varepsilon(\beta) = 2 \int_0^{\frac{1}{1+\varepsilon}} \frac{1 - \tau^\beta}{|1 - \tau|^{1+2s}} (1 - \tau^{2s-\beta}) d\tau + \frac{1}{s},$$

and

$$\lim_{\varepsilon \rightarrow 0^+} \lambda_\varepsilon(\beta) = \lambda(\beta). \quad (6.2.4)$$

Thus we have

$$2C_{N,2s} \int_{\mathbb{R}^N \setminus I_\varepsilon(x_N)} \frac{(x_N)_+^\beta - (y_N)_+^\beta}{|x_N - y_N|^{1+2s}} dy_N = C_{N,2s} \lambda_\varepsilon(\beta) \frac{U_\beta(x)}{d_\Omega(x)^{2s}}.$$

This in turn leads to

$$2 \int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy \geq C_{N,2s} \lambda_\varepsilon(\beta) \frac{U_\beta(x)}{d_\Omega(x)^{2s}}.$$

We take $\varphi \in C_0^\infty(\Omega)$ non-negative, multiply the previous inequality by $\varphi(x)$ and integrate over Ω . We get

$$2 \int_\Omega \left(\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy \right) \varphi(x) dx \geq C_{N,2s} \lambda_\varepsilon(\beta) \int_\Omega \frac{U_\beta(x)}{d_\Omega(x)^{2s}} \varphi(x) dx. \quad (6.2.5)$$

On the other hand, we have

$$\frac{U_\beta(x) - U_\beta(y) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \in L^1(\mathbb{R}^N \times \mathbb{R}^N),$$

thanks to Lemma 6.2.1. Thus by the Dominated Convergence Theorem, we get

$$\begin{aligned} & \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(U_\beta(x) - U_\beta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\ &= \lim_{\varepsilon \rightarrow 0^+} \iint_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathcal{C}_\varepsilon} \frac{(U_\beta(x) - U_\beta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy, \end{aligned}$$

where

$$\mathcal{C}_\varepsilon = \left\{ (x, y) \in \mathbb{R}^N \times \mathbb{R}^N : y \in \mathcal{K}_\varepsilon(x) \right\}.$$

Moreover, for every $0 < \varepsilon \ll 1$ we have

$$\frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} \varphi(x) \in L^1((\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathcal{C}_\varepsilon).$$

Thus with a simple change of variables, we get

$$\begin{aligned} & \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(U_\beta(x) - U_\beta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \\ &= \lim_{\varepsilon \rightarrow 0^+} \iint_{(\mathbb{R}^N \times \mathbb{R}^N) \setminus \mathcal{C}_\varepsilon} \frac{(U_\beta(x) - U_\beta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy. \quad (6.2.6) \\ &= 2 \lim_{\varepsilon \rightarrow 0^+} \int_\Omega \left(\int_{\mathbb{R}^N \setminus \mathcal{K}_\varepsilon(x)} \frac{U_\beta(x) - U_\beta(y)}{|x - y|^{N+2s}} dy \right) \varphi(x) dx. \end{aligned}$$

Observe that we also used Fubini's Theorem for every fixed $0 < \varepsilon \ll 1$, in order to arrive at the last integral. By joining (6.2.5), (6.2.6) and (6.2.4), we finally get

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(U_\beta(x) - U_\beta(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} dx dy \geq C_{N,2s} \lambda(\beta) \int_\Omega \frac{U_\beta(x)}{d_\Omega(x)^{2s}} \varphi(x) dx,$$

which is the desired conclusion for $\beta \geq 0$.

In order to prove the second statement, we first observe that if Ω is a half-space, we can assume for simplicity that

$$\Omega = \mathbb{H}_+^N.$$

Then, in the case $N = 1$, the statement has been proved in Proposition 6.1.3. For $N \geq 2$, it suffices to observe that we have equalities everywhere in the previous argument, even for $\beta < 0$, provided it is an admissible exponent. \square

Remark 6.2.3 (Optimality of Theorem 6.2.2). *As a consequence of Proposition 6.1.4, we have that*

$$C_{N,2s} \lambda(\beta) \leq C_{N,2s} \lambda\left(\frac{2s-1}{2}\right), \quad \text{for every } -1 < \beta < 2s.$$

Thus, even in the more general case of a convex subset Ω , the choice

$$\beta = \frac{2s-1}{2},$$

still produces a supersolution of (II.7), which has the largest possible λ , among supersolutions of this type. However, it should be noticed that, in light of Theorem 6.2.2, such a choice is now feasible only for

$$\frac{2s-1}{2} \geq 0 \quad \text{i. e.} \quad 2s \geq 1,$$

unless Ω is a half-space. Moreover, if the convex set Ω is not a half-space, such a result is optimal in the following sense: already in the borderline case $2s = 1$, the function U_β with $\beta < 0$ is not a supersolution of (II.7). See Lemma 6.4.1 below for a simple counter-example.

6.3 Sharpness

The aim of this section is to show the sharpness of the constant we computed before. Here we still use the notation

$$\mathbb{H}_+^1 = (0, +\infty) \quad \text{and} \quad \mathbb{H}_+^N = \mathbb{R}^{N-1} \times (0, +\infty), \quad \text{for } N \geq 2.$$

We start with the following general fact.

Proposition 6.3.1. *Let $0 < s < 1$ and let $\Omega \subsetneq \mathbb{R}^N$ be an open convex set. Then we have*

$$\mathfrak{h}_s(\Omega) \leq \mathfrak{h}_s(\mathbb{H}_+^N).$$

Proof. In dimension $N = 1$, we suppose $\Omega \subsetneq \mathbb{R}$ to be a bounded interval. Thanks to the scaling invariant property of \mathfrak{h}_s , we can assume that $\Omega = I = (0, 1)$. We take $\psi \in C_0^\infty(\mathbb{H}_+^1)$ and define the rescaled function

$$\psi_\varepsilon(t) = \psi\left(\frac{t}{\varepsilon}\right).$$

We observe that for $\varepsilon > 0$ sufficiently small, we have $\psi_\varepsilon \in C_0^\infty(I)$. We compute

$$[\psi_\varepsilon]_{W^{s,2}(\mathbb{R})}^2 = \varepsilon^{1-2s} [\psi]_{W^{s,2}(\mathbb{R})}^2,$$

and

$$\int_I \frac{|\psi_\varepsilon|^2}{d_I^{2s}} dt = \int_0^1 \frac{\left|\psi\left(\frac{t}{\varepsilon}\right)\right|^2}{(\min\{t, 1-t\})^{2s}} dt = \varepsilon^{1-2s} \int_0^{\frac{1}{\varepsilon}} \frac{|\psi(\tau)|^2}{(\min\{\tau, \varepsilon^{-1}-\tau\})^{2s}} d\tau.$$

By recalling that ψ is compactly supported, for $0 < \varepsilon \ll 1$ we have

$$\min \{ \tau, \varepsilon^{-1} - \tau \} = \tau, \quad \text{for } \tau \text{ in the support of } \psi.$$

In conclusion, we get for every $\psi \in C_0^\infty(\mathbb{H}_+^1)$

$$\mathfrak{h}_s(I) \leq \lim_{\varepsilon \rightarrow 0^+} \frac{[\psi_\varepsilon]_{W^{s,2}(\mathbb{R})}^2}{\int_I \frac{|\psi_\varepsilon|^2}{d_I^{2s}} dt} = \lim_{\varepsilon \rightarrow 0^+} \frac{[\psi]_{W^{s,2}(\mathbb{R})}^2}{\int_0^{\frac{1}{\varepsilon}} \frac{|\psi(\tau)|^2}{\tau^{2s}} d\tau} = \frac{[\psi]_{W^{s,2}(\mathbb{R})}^2}{\int_0^{+\infty} \frac{|\psi(\tau)|^2}{\tau^{2s}} d\tau}.$$

By arbitrariness of ψ , this gives the claimed inequality.

For the case $N \geq 2$, we can repeat the same proof of [83, Theorem 5], which deals with the local case, up to some very minor modifications. We just recall that the proof in [83] is based on a scaling argument as in the one-dimensional case exposed above, together with the fact that a convex set admits a tangent hyperplane at almost every boundary point. \square

6.3.1 The case of the half-space

For an half-space, we can determine the sharp Hardy constant.

Theorem 6.3.2. *Let $0 < s < 1$ and $N \geq 1$. Then we have*

$$\mathfrak{h}_s(\mathbb{H}_+^N) = C_{N,2s} \Lambda_s,$$

where Λ_s and $C_{N,2s}$ are defined by (II.9) and (II.10), respectively. Moreover, such a constant is not attained.

Proof. By combining Theorem 4 and Theorem 6.2.2 for $\Omega = \mathbb{H}_+^N$, we immediately obtain

$$\mathfrak{h}_s(\mathbb{H}_+^N) \geq C_{N,2s} \lambda(\beta), \quad \text{for every } -1 < \beta < 2s.$$

Moreover, by Proposition 6.1.4, we know that the right-hand side is maximal for $\beta = (2s - 1)/2$ and thus

$$\mathfrak{h}_s(\mathbb{H}_+^N) \geq C_{N,2s} \lambda\left(\frac{2s - 1}{2}\right) = C_{N,2s} \Lambda_s.$$

In order to prove that the right-hand side actually gives the sharp constant, we distinguish two cases: $N = 1$ and $N \geq 2$. We will show that the latter reduces to the former: this is quite a standard fact for the Hardy inequality, but we prefer to give the details, since some non-trivial computations are needed. For the case $N = 1$, we will use a slightly different family of trial functions with respect to [52, 53]: this permits to treat the cases $s < 1/2$ and $s \geq 1/2$ at the same time.

Sharpness: case $N = 1$. We need to prove that

$$\mathfrak{h}_s(\mathbb{H}_+^1) \leq \Lambda_s.$$

We take a cut-off function $\psi \in C_0^\infty((-\infty, 2))$ such that

$$0 \leq \psi \leq 1, \quad \psi \equiv 1, \text{ on } \in [0, 1], \quad |\psi'| \leq \tilde{C},$$

and we use the trial function

$$\phi_\beta = U_\beta \psi, \quad \text{with } \frac{2s - 1}{2} < \beta < s.$$

According to Lemma 6.1.1 and Lemma 1.1.7, this function belongs to $\widetilde{W}_0^{s,2}(\mathbb{H}_+^1)$. In light of the estimate (1.1.7) shown in the first chapter and the properties of the cut-off, we get

$$\mathfrak{h}_s(\mathbb{H}_+^1) \leq \frac{[\phi_\beta]_{W^{s,2}(\mathbb{R})}^2}{\int_0^{+\infty} \frac{|\phi_\beta(x)|^2}{x^{2s}} dx} \leq \frac{[U_\beta \psi]_{W^{s,2}((0,2))}^2}{\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx} + \frac{1}{s} + \frac{2^{3-2s}}{2s} \widetilde{C}^2 \frac{\|U_\beta\|_{L^2((0,2))}^2}{\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx}. \quad (6.3.1)$$

We evaluate separately the two quotients on the right-hand side. For the first one, by using the estimate (1.1.8) proved at the beginning of the thesis, we have

$$\frac{[U_\beta \psi]_{W^{s,2}((0,2))}}{\left(\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx\right)^{\frac{1}{2}}} \leq \frac{[U_\beta]_{W^{s,2}((0,2))}}{\left(\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx\right)^{\frac{1}{2}}} + \left(\frac{C}{s(1-s)}\right)^{\frac{1}{2}} \widetilde{C}^s \frac{\|U_\beta\|_{L^2((0,2))}}{\left(\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx\right)^{\frac{1}{2}}}.$$

Moreover, by recalling the definition of U_β , we note that

$$\lim_{\beta \rightarrow (\frac{2s-1}{2})^+} \int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx = \lim_{\beta \rightarrow (\frac{2s-1}{2})^+} \int_0^2 \frac{\psi^2}{x^{2s-2\beta}} dx = +\infty,$$

thus the denominator diverges as β goes to $(2s-1)/2$. Coming back to (6.3.1), this entails that

$$\begin{aligned} \mathfrak{h}_s(\mathbb{H}_+^1) &\leq \limsup_{\beta \rightarrow (\frac{2s-1}{2})^+} \left[\frac{[U_\beta \psi]_{W^{s,2}((0,2))}^2}{\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx} + \frac{1}{s} + \frac{2^{2-2s}}{2s} \widetilde{C}^2 \frac{\|U_\beta\|_{L^2((0,2))}^2}{\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx} \right] \\ &\leq \limsup_{\beta \rightarrow (\frac{2s-1}{2})^+} \frac{[U_\beta]_{W^{s,2}((0,2))}^2}{\int_0^2 \frac{(U_\beta \psi)^2}{x^{2s}} dx} + \frac{1}{s} \leq \limsup_{\beta \rightarrow (\frac{2s-1}{2})^+} \frac{[U_\beta]_{W^{s,2}((0,2))}^2}{\int_0^1 \frac{(U_\beta)^2}{x^{2s}} dx} + \frac{1}{s}. \end{aligned}$$

We claim that

$$\limsup_{\beta \rightarrow (\frac{2s-1}{2})^+} \frac{[U_\beta]_{W^{s,2}((0,2))}^2}{\int_0^1 \frac{(U_\beta)^2}{x^{2s}} dx} \leq 2 \int_0^1 \frac{|1 - t^{\frac{2s-1}{2}}|^2}{(1-t)^{1+2s}} dt, \quad (6.3.2)$$

this would conclude the proof, by recalling the definition (II.9) of Λ_s . By using the form of U_β we have

$$\int_0^1 \frac{(U_\beta)^2}{x^{2s}} dx = \int_0^1 x^{2\beta-2s} dt = \frac{1}{2\beta - 2s + 1}.$$

Thus in order to prove (6.3.2), we just need to show that

$$\limsup_{\beta \rightarrow (\frac{2s-1}{2})^+} (2\beta - 2s + 1) [U_\beta]_{W^{s,2}((0,2))}^2 \leq 2 \int_0^1 \frac{|1 - t^{\frac{2s-1}{2}}|^2}{(1-t)^{1+2s}} dt.$$

By recalling the estimate (6.1.3) from Remark 6.1.2, we have

$$[U_\beta]_{W^{s,2}((0,2))}^2 \leq \left(\int_0^1 \frac{|1 - t^\beta|^2}{|1 - t|^{1+2s}} \left(1 + t^{2s-2\beta-1}\right) dt \right) \frac{2^{2\beta-2s+1}}{2\beta - 2s + 1}.$$

Hence, by taking the limit as β goes to $(2s-1)/2$ and using the Dominated Convergence Theorem, we get (6.3.2), as desired. This proves the sharpness for $N = 1$.

Sharpness: case $N \geq 2$. We will show that this can be reduced to the previous case, by proceeding as in [52, Theorem 1.1] and [89, Proposition 3.2]. Let $\eta \in C_0^\infty((0, +\infty))$ and $\chi \in C_0^\infty((-1, 1)^{N-1})$, we use the test function

$$\varphi = \chi_M(x') \eta(x_N), \quad \text{where } \chi_M(x') := \frac{M^{\frac{1-N}{2}}}{\|\chi\|_{L^2(\mathbb{R}^{N-1})}} \chi\left(\frac{x'}{M}\right),$$

for some $M > 0$. Observe that by construction the function χ_M has compact support on $(-M, M)^{N-1}$ and unit L^2 norm. We thus obtain

$$\mathfrak{h}_s(\mathbb{H}_+^N) \leq \frac{[\eta \chi_M]_{W^{s,2}(\mathbb{R}^N)}^2}{\int_{\mathbb{H}_+^N} \frac{(\eta \chi_M)^2}{x_N^{2s}} dx} = \frac{[\eta \chi_M]_{W^{s,2}(\mathbb{R}^N)}^2}{\int_0^{+\infty} \frac{\eta^2}{x_N^{2s}} dx_N},$$

where in the last identity we used Fubini's Theorem and the properties of χ_M . In order to estimate the seminorm, we first use Minkowski's inequality

$$\begin{aligned} [\eta \chi_M]_{W^{s,2}(\mathbb{R}^N)} &\leq \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_M(x')|^2 |\eta(x_N) - \eta(y_N)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\quad + \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(y_N)|^2 |\chi_M(x') - \chi_M(y')|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}, \end{aligned}$$

and we focus separately on the two integrals on the right-hand side. For the first integral, we use Fubini's Theorem and the identity (6.2.1), so to get

$$\begin{aligned} &\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\chi_M(x')|^2 |\eta(x_N) - \eta(y_N)|^2}{|x - y|^{N+2s}} dx dy \\ &= C_{N,2s} \left(\int_{\mathbb{R}^{N-1}} |\chi_M(x')|^2 dx' \right) \left(\iint_{\mathbb{R} \times \mathbb{R}} \frac{|\eta(x_N) - \eta(y_N)|^2}{|x_N - y_N|^{1+2s}} dx_N dy_N \right) = C_{N,2s} [\eta]_{W^{s,2}(\mathbb{R})}^2. \end{aligned}$$

On the other hand, by using a computation similar to (6.2.1), we have

$$\int_{\mathbb{R}} \frac{1}{(|x_N - y_N|^2 + |x' - y'|^2)^{\frac{N+2s}{2}}} dx_N = \frac{2 \int_0^{+\infty} (1+t^2)^{-\frac{N+2s}{2}} dt}{|x' - y'|^{N-1+2s}} = \frac{C}{|x' - y'|^{N-1+2s}}.$$

Thus, it holds

$$\begin{aligned} &\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(y_N)|^2 |\chi_M(x') - \chi_M(y')|^2}{|x - y|^{N+2s}} dx dy \\ &= C \int_{\mathbb{R}} |\eta(y_N)|^2 dy_N \left(\iint_{\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}} \frac{|\chi_M(x') - \chi_M(y')|^2}{|x' - y'|^{N-1+2s}} dx' dy' \right) \\ &= C \frac{\|\eta\|_{L^2(\mathbb{R})}^2}{\|\chi\|_{L^2(\mathbb{R}^{N-1})}^2} \frac{[\chi]_{W^{s,2}(\mathbb{R}^{N-1})}^2}{M^{2s}}. \end{aligned}$$

In the last identity we used the definition of χ_M and a change of variable. Then, it follows that

$$\left(\mathfrak{h}_s(\mathbb{H}_+^N) \right)^{\frac{1}{2}} \leq (C_{N,2s})^{\frac{1}{2}} \frac{[\eta]_{W^{s,2}(\mathbb{R})}}{\left(\int_0^{+\infty} \frac{\eta^2}{x_N^{2s}} dx \right)^{\frac{1}{2}}} + C^{\frac{1}{2}} \frac{\|\eta\|_{L^2(\mathbb{R})}}{M^s \|\chi\|_{L^2(\mathbb{R}^{N-1})}} \frac{[\chi]_{W^{s,2}(\mathbb{R}^{N-1})}}{\left(\int_0^{+\infty} \frac{\eta^2}{x_N^{2s}} dx \right)^{\frac{1}{2}}}$$

By letting M go to $+\infty$ and thanks to the arbitrariness of $\eta \in C_0^\infty((0, +\infty))$, we obtain

$$\mathfrak{h}_s(\mathbb{H}_+^N) \leq C_{N,2s} \mathfrak{h}_s(\mathbb{H}_+^1) = C_{N,2s} \Lambda_s,$$

as desired. The last identity follows from the sharpness for $N = 1$.

The fact that $\mathfrak{h}_s(\mathbb{H}_+^N)$ is not attained follows directly from Proposition 5.2.4, since by Theorem 6.2.2 we found a local weak solution of (II.7) with $\lambda = \mathfrak{h}_s(\mathbb{H}_+^N)$, of the form

$$u = d_{\mathbb{H}_+^N}^{\frac{2s-1}{2}}.$$

The proof is over. \square

6.3.2 The case of convex sets

We first highlight the following consequence of Lemma 6.1.5. The resulting inequality is the same as [38, Corollary 1.3] by Bartłomiej Dyda.

Proposition 6.3.3. *Let $0 < s < 1$ and let $a < b$ be two real numbers. We have the following one-dimensional Hardy-type inequality*

$$\Lambda_s \int_a^b |u|^2 \left[\frac{1}{t-a} + \frac{1}{b-t} \right]^{2s} dt \leq [u]_{W^{s,2}(\mathbb{R})}^2, \quad (6.3.3)$$

for every $u \in \widetilde{W}_0^{s,2}((a,b))$. In particular, we have

$$\mathfrak{h}_s((a,b)) = \Lambda_s.$$

and such a constant is not attained.

Proof. Firstly, we observe that \mathfrak{h}_s is scaling invariant, so that

$$\mathfrak{h}_s(\Omega) = \mathfrak{h}_s(t\Omega + x_0), \quad \text{for every } t > 0, x_0 \in \Omega.$$

Hence we can assume $(a,b) = (0,1) = I$.

Then the proof of the inequality (6.3.3) is the same as in [38] and it follows the same lines as that of Lemma 5.3.1: it is sufficient to take

$$f_\beta(t) := t^{2s-1-\beta} (1-t)^\beta, \quad \text{with } \beta = \frac{2s-1}{2},$$

observe that by Lemma 6.1.5 this weakly solves

$$(-\Delta)^s u = \Lambda_s \frac{u}{(t(1-t))^{2s}}, \quad \text{in } I,$$

and then make a suitable application of the discrete Picone inequality.

As for the sharp fractional Hardy constant, by observing that

$$\frac{1}{t} + \frac{1}{1-t} \geq \frac{1}{\min\{t, 1-t\}} = \frac{1}{d_I(t)}, \quad \text{for } t \in I,$$

we easily get from (6.3.3) that

$$\mathfrak{h}_s((0,1)) \geq \Lambda_s.$$

The reverse inequality can be obtained from Proposition 6.3.1 and Theorem 6.3.2. Finally, the fact that the sharp constant is not attained can be inferred again from Proposition 5.2.4. \square

By combining the previous one-dimensional result with a decomposition of the Gagliardo-Slobodeckii seminorm taken from [80] (see also [36, Chapter 1, Section 5]), we can finally compute the sharp fractional Hardy constant of a convex set.

This complements [48, Theorem 5, points (i) & (ii)], where the case $0 < s < 1/2$ was left open.

Theorem 6.3.4. *Let $0 < s < 1$, then for every $\Omega \subsetneq \mathbb{R}^N$ open convex set, we have*

$$\mathfrak{h}_s(\Omega) = C_{N,2s} \Lambda_s,$$

where Λ_s and $C_{N,2s}$ are defined by (II.9) and (II.10), respectively. Moreover, such a constant is not attained.

Proof. By joining Proposition 6.3.1 and Theorem 6.3.2, we already know that

$$\mathfrak{h}_s(\Omega) \leq \mathfrak{h}_s(\mathbb{H}_+^N) = C_{N,2s} \Lambda_s. \quad (6.3.4)$$

In order to prove the reverse estimate, it is sufficient to reproduce verbatim the proof of [80, Theorem 1.1] for convex sets, by replacing the $W^{s,2}$ seminorm on Ω there with that on \mathbb{R}^N . In particular, we state the following *reduction formula*

$$\begin{aligned} & \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ &= \frac{1}{2} \int_{\mathbb{S}^{N-1}} \int_{\{h \in \mathbb{R}^N : \langle h, \sigma \rangle = 0\}} \left(\iint_{\mathbb{R} \times \mathbb{R}} \frac{|u(h + t\sigma) - u(h + \tau\sigma)|^2}{|t - \tau|^{1+2s}} dt d\tau \right) dh d\mathcal{H}^{N-1}(\sigma), \end{aligned} \quad (6.3.5)$$

where dh denotes the $(N - 1)$ -dimensional Lebesgue measure on the hyperplane $\{h \in \mathbb{R}^N : \langle h, \sigma \rangle = 0\}$. To show it we apply the translation $z = y - x$ and we decompose z by using polar coordinates as follows

$$\begin{aligned} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(x + z)|^2}{|z|^{N+2s}} dh dx \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_0^{+\infty} \frac{|u(x) - u(x + \rho\sigma)|^2}{\rho^{N+2s}} d\rho d\mathcal{H}^{N-1}(\sigma) dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}} \frac{|u(x) - u(x + r\sigma)|^2}{|r|^{N+2s}} dr d\mathcal{H}^{N-1}(\sigma) dx. \end{aligned}$$

Hence we split $x = t\sigma + h$, where h is a vector orthogonal to $\sigma \in \mathbb{S}^{N-1}$, and we take $\tau = t + r$. Thanks to the Fubini's Theorem, we get the claimed (6.3.5).

By starting from this, it is sufficient to use the one-dimensional Hardy inequality of Proposition 6.3.3 for the integral on $\mathbb{R} \times \mathbb{R}$, in place of the inequality of [80, Theorem 2.1].

This would lead to

$$[u]_{W^{s,2}(\mathbb{R}^N)}^2 \geq \Lambda_s \left(\frac{1}{2} \int_{\mathbb{S}^{N-1}} |\sigma_N|^{2s} d\mathcal{H}^{N-1}(\sigma) \right) \int_{\Omega} \frac{|u|^2}{d_{\Omega}^{2s}} dx.$$

By using Lemma A.4.1 in the appendix, we conclude the proof. \square

Remark 6.3.5. *For an open convex set Ω , under the assumption $1/2 \leq s < 1$ only, we can obtain the identity*

$$\mathfrak{h}_s(\Omega) = C_{N,2s} \Lambda_s,$$

also by using the supersolutions d_Ω^β . Indeed, the inequality (6.3.4) still works, while the reverse inequality can be computed similarly to what we did in the proof of Theorem 6.3.2. In particular, thanks to the additional assumption on s , we combine Theorem 4 and Theorem 6.2.2 and we obtain

$$\mathfrak{h}_s(\mathbb{H}_+^N) \geq C_{N,2s} \lambda(\beta), \quad \text{for every } -1 < \beta < 2s.$$

The claimed inequality follows from the optimization on β proved in Proposition 6.1.4.

Finally, the fact that $\mathfrak{h}_s(\Omega)$ is not attained follows directly from Proposition 5.2.4, since by Theorem 6.2.2 we found a local weak supersolution of (II.7) with $\lambda = \mathfrak{h}_s(\Omega)$, having the form

$$u = d_\Omega^{\frac{2s-1}{2}}.$$

Finally, as a direct consequence of Theorem 6.3.4, we highlight the following spectral lower bound for convex sets, which is very much in the spirit of the first part of the thesis. This can be seen as a fractional counterpart of the Hersch-Protter inequality. The constant in this estimate should not be expected to be sharp.

Corollary 6.3.6. *Let $0 < s < 1$ and let $\Omega \subseteq \mathbb{R}^N$ be an open convex set with finite inradius r_Ω . Then*

$$\lambda_1^s(\Omega) \geq C_{N,2s} \Lambda_s \frac{1}{r_\Omega^{2s}}.$$

6.4 Negative powers of the distance in the borderline case $s = 1/2$

Let us take $-1 < \beta < 0$ and let I be the interval $(0, 1)$. Thus we consider the function

$$U_\beta(t) = d_I(t)^\beta = (\min\{t, 1-t\})^\beta, \quad \text{for } t \in I = (0, 1),$$

and we extend it by 0 outside I . In the following lemma we want to estimate its fractional Laplacian of order $1/2$.

Lemma 6.4.1. *Under the assumptions above, we have*

$$(-\Delta)^{\frac{1}{2}} U_\beta \leq \beta H(t) U_\beta(t) + \frac{2}{t(1-t)} U_\beta(t), \quad \text{in } I, \quad (6.4.1)$$

in weak sense, where H is the continuous function on $I \setminus \{1/2\}$ defined by

$$H(t) = -\frac{2}{t(1-t)} + \frac{2}{d_I(t)} \log \left(\frac{4t(1-t)}{(1-2t)^2} \right).$$

This has the following properties:

- it is symmetric with respect to $1/2$, i.e.

$$H(t) = H(1-t), \quad \text{for } t \in I \setminus \left\{ \frac{1}{2} \right\};$$

- it belongs to $L_{\text{loc}}^q(I)$ for every $1 \leq q < \infty$;
- it satisfies

$$H(t) \sim -4 \log \left(\frac{1}{2} - t \right)^2, \quad \text{for } t \rightarrow \frac{1}{2},$$

thus the right-hand side of (6.4.1) diverges to $-\infty$ as t approaches $1/2$.

In particular, in this case the function U_β is not even locally weakly $(1/2)$ -superharmonic on I .

Proof. Observe that $U_\beta \in W_{\text{loc}}^{s,2}(I) \cap L_1^1(\mathbb{R})$, under the assumption $-1 < \beta < 0$. We first show that U_β satisfies (6.4.1) in $I \setminus \{1/2\}$. Let us take $\varphi \in C_0^\infty(I \setminus \{1/2\})$ non-negative, then there exists $0 < \delta_0 < 1/4$ such that its support is contained in the set

$$\mathcal{I}_{\delta_0} = \left[\delta_0, \frac{1}{2} - \delta_0 \right] \cup \left[\frac{1}{2} + \delta_0, 1 - \delta_0 \right].$$

For every $\varepsilon > 0$ and $t \in I$, we set

$$\mathcal{J}_\varepsilon(t) = (t - \varepsilon, t + \varepsilon) \quad \text{and} \quad \mathcal{D}_\varepsilon = \left\{ (t, y) \in \mathbb{R} \times \mathbb{R} : t - \varepsilon \leq y \leq t + \varepsilon \right\}.$$

By using Fubini's Theorem and a change of variable, we can write as usual

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} dt dy \\ &= \lim_{\varepsilon \rightarrow 0} \iint_{(\mathbb{R} \times \mathbb{R}) \setminus \mathcal{D}_\varepsilon} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} dt dy \\ &= 2 \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \left(\int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^2} dy \right) \varphi(t) dt \\ &= 2 \lim_{\varepsilon \rightarrow 0} \int_{\mathcal{I}_{\delta_0}} \left(\int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^2} dy \right) \varphi(t) dt. \end{aligned} \tag{6.4.2}$$

We first observe that, by using that $U_\beta(t) = U_\beta(1 - t)$, we have for $t \in \mathcal{I}_{\delta_0}$

$$\begin{aligned} \int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^2} dy &= \int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(1 - t) - U_\beta(1 - y)}{|t - y|^2} dy \\ &= \int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(1 - t) - U_\beta(1 - y)}{|(1 - t) - (1 - y)|^2} dy = \int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(1-t)} \frac{U_\beta(1 - t) - U_\beta(\tau)}{|(1 - t) - \tau|^2} d\tau. \end{aligned}$$

This shows that it is sufficient to consider $t \in [\delta_0, 1/2 - \delta_0]$. For almost every $t \in [\delta_0, 1/2 - \delta_0]$ and every $0 < \varepsilon < \delta_0$, we have¹

$$\begin{aligned} \int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t - y|^2} dy &= \int_0^{t-\varepsilon} \frac{t^\beta - y^\beta}{|t - y|^2} dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta - y^\beta}{|t - y|^2} dy + \int_{\frac{1}{2}}^1 \frac{t^\beta - (1 - y)^\beta}{(y - t)^2} dy \\ &\quad + \int_1^{+\infty} \frac{t^\beta}{(y - t)^2} dy + \int_{-\infty}^0 \frac{t^\beta}{(t - y)^2} dy \\ &= \int_0^{t-\varepsilon} \frac{t^\beta - y^\beta}{|t - y|^2} dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta - y^\beta}{|t - y|^2} dy + \int_{\frac{1}{2}}^1 \frac{t^\beta - (1 - y)^\beta}{(y - t)^2} dy \\ &\quad + \frac{U_\beta(t)}{(1 - t)} + \frac{U_\beta(t)}{t}. \end{aligned}$$

To estimate the remaining integrals, we use the ‘‘above tangent’’ property for the convex function $\tau \mapsto \tau^\beta$, to infer that

$$t^\beta - y^\beta \leq \beta t^{\beta-1} (t - y), \quad \text{for } y \in (0, t - \varepsilon) \cup \left(t + \varepsilon, \frac{1}{2} \right),$$

¹Observe that by construction $t - \varepsilon > 0$ and $t + \varepsilon < 1/2$.

and

$$t^\beta - (1-y)^\beta \leq \beta t^{\beta-1} (t+y-1), \quad \text{for } y \in \left(\frac{1}{2}, 1\right).$$

These yield

$$\begin{aligned} & \int_0^{t-\varepsilon} \frac{t^\beta - y^\beta}{|t-y|^2} dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t^\beta - y^\beta}{|t-y|^2} dy + \int_{\frac{1}{2}}^1 \frac{t^\beta - (1-y)^\beta}{(y-t)^2} dy \\ & \leq \beta t^{\beta-1} \int_0^{t-\varepsilon} \frac{t-y}{|t-y|^2} dy + \beta t^{\beta-1} \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t-y}{|t-y|^2} dy + \beta t^{\beta-1} \int_{\frac{1}{2}}^1 \frac{(t+y-1)}{(y-t)^2} dy. \end{aligned}$$

The last integrals can be explicitly computed. We have

$$\int_0^{t-\varepsilon} \frac{t-y}{|t-y|^2} dy + \int_{t+\varepsilon}^{\frac{1}{2}} \frac{t-y}{|t-y|^2} dy = \log t - \log\left(\frac{1}{2} - t\right),$$

and

$$\begin{aligned} \int_{\frac{1}{2}}^1 \frac{(t+y-1)}{(y-t)^2} dy &= t \int_{\frac{1}{2}}^1 \frac{1}{(y-t)^2} dy + \int_{\frac{1}{2}}^1 \frac{y-1}{(y-t)^2} dy \\ &= t \left[\left(\frac{1}{2} - t\right)^{-1} - (1-t)^{-1} \right] \\ &\quad - \frac{1}{2} \left(\frac{1}{2} - t\right)^{-1} + \left[\log(1-t) - \log\left(\frac{1}{2} - t\right) \right]. \end{aligned}$$

This finally gives that for $t \in [\delta_0, 1/2 - \delta_0]$, we have

$$\int_{\mathbb{R} \setminus \mathcal{J}_\varepsilon(t)} \frac{U_\beta(t) - U_\beta(y)}{|t-y|^{1+2s}} dy \leq \beta t^{\beta-1} G(t) + \frac{U_\beta(t)}{t(1-t)}, \quad (6.4.3)$$

where we set for simplicity

$$\begin{aligned} G(t) &= \left[\log t - \log\left(\frac{1}{2} - t\right) \right] + t \left[\left(\frac{1}{2} - t\right)^{-1} - (1-t)^{-1} \right] \\ &\quad - \frac{1}{2} \left(\frac{1}{2} - t\right)^{-1} + \left[\log(1-t) - \log\left(\frac{1}{2} - t\right) \right]. \end{aligned}$$

With simple manipulations, we see that this can be also written as

$$G(t) = -\frac{1}{1-t} + \log\left(\frac{4t(1-t)}{(1-2t)^2}\right),$$

and thus this is a continuous function on $(0, 1/2)$ such that

$$G \in L_{\text{loc}}^q((0, 1/2]), \quad \text{for every } 1 \leq q < \infty \quad \text{and} \quad \lim_{t \rightarrow (\frac{1}{2})^-} G(t) = +\infty,$$

because of the logarithmic term. If we now define

$$H(t) := 2 \frac{G(t)}{t}, \quad \text{for } t \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad H(t) := H(1-t), \quad \text{for } t \in \left(\frac{1}{2}, 1\right),$$

from (6.4.2) and (6.4.3), by recalling that φ is non-negative we finally get

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} dt dy \leq \int_I \left[\beta H(t) + \frac{2}{t(1-t)} \right] U_\beta(t) \varphi(t) dt.$$

This shows that U_β is a local weak subsolution of the the claimed equation (6.4.1), at least in the open set $I \setminus \{1/2\}$.

In order to show that U_β is a local weak subsolution on the whole interval I , it is sufficient to use a standard trick to² “fill the hole”. We take $\varphi \in C_0^\infty(I)$ non-negative and for every natural number $n \geq 5$ we take $\psi_n \in C^\infty(\bar{I})$ such that

$$\psi_n \equiv 1 \text{ on } \bar{I} \setminus \left[\frac{1}{2} - \frac{2}{n}, \frac{1}{2} + \frac{2}{n} \right], \quad \psi_n \equiv 0 \text{ on } \left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n} \right],$$

and

$$0 \leq \psi_n \leq 1, \quad |\psi_n'| \leq Cn.$$

The seminorm of ψ_n can be estimated by using its properties and an interpolation inequality (see [22, Corollary 2.2]), i. e.

$$\begin{aligned} [\psi_n]_{W^{\frac{1}{2},2}(I)}^2 &= [1 - \psi_n]_{W^{\frac{1}{2},2}(I)}^2 \leq C \left(\int_I |\psi_n'|^2 dt \right)^{\frac{1}{2}} \left(\int_I |1 - \psi_n|^2 dt \right)^{\frac{1}{2}} \\ &= C \left(\int_{\frac{1}{2} - \frac{2}{n}}^{\frac{1}{2} + \frac{2}{n}} |\psi_n'|^2 dt \right)^{\frac{1}{2}} \left(\int_{\frac{1}{2} - \frac{2}{n}}^{\frac{1}{2} + \frac{2}{n}} |1 - \psi_n|^2 dt \right)^{\frac{1}{2}} \leq \tilde{C}, \end{aligned}$$

where \tilde{C} does not depend on n . This in particular implies that the sequence $\{\Psi_n\}_{n \geq 5} \subseteq L^2(I \times I)$ defined by

$$\Psi_n(t, y) := \frac{\psi_n(t) - \psi_n(y)}{|t - y|}, \quad \text{for a. e. } (t, y) \in I \times I, \quad (6.4.4)$$

is bounded in $L^2(I \times I)$, since by construction

$$\|\Psi_n\|_{L^2(I \times I)} = [\psi_n]_{W^{\frac{1}{2},2}(I)}.$$

Thus, up to a subsequence, it converges weakly in $L^2(I \times I)$. Thanks to the properties of ψ_n , such a limit function must coincide with the null one.

The test function $\varphi \psi_n$ belongs to $C_0^\infty(I \setminus \{1/2\})$ and is non-negative. From the first part we get

$$\begin{aligned} &\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) \psi_n(t) - \varphi(y) \psi_n(y))}{|t - y|^2} dt dy \\ &\leq \int_I \left[\beta H(t) + \frac{2}{t(1-t)} \right] U_\beta(t) \varphi(t) \psi_n(t) dt. \end{aligned} \quad (6.4.5)$$

We wish to pass to the limit in (6.4.5), as n goes to ∞ : for the right-hand side, it is easily seen that

$$\lim_{n \rightarrow \infty} \int_I \left[\beta H(t) + \frac{2}{t(1-t)} \right] U_\beta(t) \varphi(t) \psi_n(t) dt = \int_I \left[\beta H(t) + \frac{2}{t(1-t)} \right] U_\beta(t) \varphi(t) dt,$$

²Of course, this will be possible because in \mathbb{R} points have zero fractional capacity of order $s \leq 1/2$, i.e. they are removable sets.

by the Dominated Convergence Theorem. As for the left-hand side, we split the integral as follows:

$$\begin{aligned} & \iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) \psi_n(t) - \varphi(y) \psi_n(y))}{|t - y|^2} dt dy \\ &= \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) \psi_n(t) - \varphi(y) \psi_n(y))}{|t - y|^2} dt dy \\ &+ 2 \iint_{I' \times (\mathbb{R} \setminus I'')} \frac{(U_\beta(t) - U_\beta(y)) \varphi(t) \psi_n(t)}{|t - y|^2} dt dy, \end{aligned}$$

where $I' \subseteq I'' \subseteq I$ and I' contains the support of φ . For the last integral we can easily pass to the limit as n goes to ∞ , for the first one we proceed as follows

$$\begin{aligned} & \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) \psi_n(t) - \varphi(y) \psi_n(y))}{|t - y|^2} dt dy \\ &= \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} \frac{\psi_n(t) + \psi_n(y)}{2} dt dy \\ &+ \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\psi_n(t) - \psi_n(y))}{|t - y|^2} \frac{\varphi(t) + \varphi(y)}{2} dt dy. \end{aligned}$$

By using that

$$\frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} \in L^1(I'' \times I''),$$

and the properties of ψ_n , we get that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} \frac{\psi_n(t) + \psi_n(y)}{2} dt dy \\ &= \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} dt dy, \end{aligned}$$

again thanks to the Dominated Convergence Theorem. Finally, the last integral is the most delicate one: with the notation (6.4.4), we can write

$$\frac{(U_\beta(t) - U_\beta(y)) (\psi_n(t) - \psi_n(y))}{|t - y|^2} \frac{\varphi(t) + \varphi(y)}{2} = \Phi(t, y) \Psi_n(t, y),$$

where

$$\Phi(t, y) = \frac{(U_\beta(t) - U_\beta(y))}{|t - y|} \frac{\varphi(t) + \varphi(y)}{2} \in L^2(I'' \times I'').$$

The last property follows from the fact that U_β is locally Lipschitz on I . Thus, by using the weak convergence of $\{\Psi_n\}_{n \geq 5}$ previously inferred, we get

$$\lim_{n \rightarrow \infty} \iint_{I'' \times I''} \frac{(U_\beta(t) - U_\beta(y)) (\psi_n(t) - \psi_n(y))}{|t - y|^2} \frac{\varphi(t) + \varphi(y)}{2} dt dy = 0.$$

Finally, we obtain that we can pass to the limit in (6.4.5) as n goes to ∞ and obtain

$$\iint_{\mathbb{R} \times \mathbb{R}} \frac{(U_\beta(t) - U_\beta(y)) (\varphi(t) - \varphi(y))}{|t - y|^2} dt dy \leq \int_I \left[\beta H(t) + \frac{2}{t(1-t)} \right] U_\beta(t) \varphi(t) dt,$$

for every $\varphi \in C_0^\infty(I)$ non-negative, as desired. \square

Appendix A

Miscellaneous stuff

This appendix is devoted to show some technical stuff that have been useful in some points of the thesis.

A.1 A trial inequality

We recall the following discrete version of Picone's inequality, taken from [20, Proposition 4.2] (see also [53, Lemma 2.6]). We explicitly state the equality cases.

Lemma A.1.1 (Discrete Picone's inequality). *For every $a, b > 0$ and $c, d \geq 0$ we have*

$$(a - b) \left(\frac{c^2}{a} - \frac{d^2}{b} \right) \leq |c - d|^2.$$

Moreover, equality holds if and only if

$$\frac{c}{a} = \frac{d}{b}.$$

Proof. We first observe that if $c = 0$, the inequality is equivalent to

$$(b - a) \frac{d^2}{b} \leq d^2,$$

that is trivially true, actually with the strict inequality sign.

We then suppose $c \neq 0$: we first observe that the left-hand side can be rewritten as

$$(a - b) \left(\frac{c^2}{a} - \frac{d^2}{b} \right) = c^2 \left(1 - \frac{b}{a} \right) \left(1 - \left(\frac{d}{c} \right)^2 \left(\frac{a}{b} \right) \right).$$

If we introduce the shortcut notation

$$t = \frac{b}{a} \quad \text{and} \quad A = \frac{d}{c},$$

we then get that the claimed inequality is equivalent to

$$(1 - t) \left(1 - \frac{A^2}{t} \right) \leq |1 - A|^2, \quad \text{for every } t > 0, A \geq 0.$$

It is not difficult to see that the function

$$\Phi(t) = (1 - t) \left(1 - \frac{A^2}{t} \right),$$

is monotone increasing for $t < A$ and monotone decreasing for $t > A$. The choice $t = A$ thus corresponds to the *unique* maximum point, for which we have

$$\Phi(t) \leq \Phi(A) = |1 - A|^2.$$

This concludes the proof. \square

A.2 A bi-Lipschitz homeomorphism

In this section, we show that every open bounded convex set is bi-Lipschitz homeomorphic to a ball. This is probably a well-known result, but for our purposes we needed a precise description of the bi-Lipschitz constants. The result is taken from [B1]. In what follows, for every open bounded convex set $K \subseteq \mathbb{R}^N$ and every $x_0 \in K$, we denote

$$d_K(x_0) = \min_{x \in \partial K} |x - x_0|, \quad D_K(x_0) = \max_{x \in \partial K} |x - x_0|.$$

Lemma A.2.1. *Let $K \subseteq \mathbb{R}^N$ be an open bounded convex set and $x_0 \in K$. There exists a bi-Lipschitz homeomorphism $\Phi_{K,x_0} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ with the following properties:*

- $\Phi_{K,x_0}(x_0) = x_0$ and $\Phi_{K,x_0}(r(K - x_0) + x_0) = B_r(x_0)$, for every $r > 0$;
- Φ_{K,x_0} is L_K -Lipschitz with

$$L_K = \frac{2}{d_K(x_0)};$$

- Φ_{K,x_0}^{-1} is M_K -Lipschitz with

$$M_K = D_K(x_0) \left(2 + \frac{D_K(x_0)}{d_K(x_0)} \right).$$

Moreover, we have

$$\left(\frac{1}{M_K} \right)^N \leq |\det \nabla \Phi_{K,x_0}(x)| \leq (L_K)^N, \quad \text{for a. e. } x \in \mathbb{R}^N. \quad (\text{A.2.1})$$

and

$$\left(\frac{1}{L_K} \right)^N \leq |\det \nabla \Phi_{K,x_0}^{-1}(x)| \leq (M_K)^N, \quad \text{for a. e. } x \in \mathbb{R}^N. \quad (\text{A.2.2})$$

Proof. For notational simplicity, we omit to indicate the subscript x_0 everywhere. We define at first the *Minkowski functional of K centered at x_0* , i.e.

$$j_K(x) = \inf \left\{ \lambda > 0 : x \in \lambda(K - x_0) + x_0 \right\}.$$

We recall that this is a Lipschitz function, which verifies the following homogeneity property

$$j_K(t(x - x_0) + x_0) = t j_K(x), \quad \text{for every } x \in \mathbb{R}^N, t > 0. \quad (\text{A.2.3})$$

We also observe that by construction it holds

$$j_K(x) < r \quad \text{if and only if} \quad x \in r(K - x_0) + x_0,$$

and that

$$j_K(x) = r \quad \text{if and only if} \quad x \in r(\partial K - x_0) + x_0.$$

Moreover, j_K is Lipschitz and it holds

$$|j_K(x) - j_K(y)| \leq \frac{1}{d_K(x_0)} |x - y|, \quad \text{for every } x, y \in \mathbb{R}^N. \quad (\text{A.2.4})$$

Last, but not least, we have the following lower bound

$$j_K(x) = |x - x_0| j_K \left(\frac{x - x_0}{|x - x_0|} + x_0 \right) \geq \frac{|x - x_0|}{D_K(x_0)}. \quad (\text{A.2.5})$$

Then we define Φ_K as follows

$$\Phi_K(x_0) = x_0, \quad \Phi_K(x) = \frac{x - x_0}{|x - x_0|} j_K(x) + x_0, \quad \text{if } x \in \mathbb{R}^N \setminus \{x_0\}.$$

Thanks to the properties of the Minkowski functional, we have that Φ_K is injective. In order to verify that Φ_K is bijective, let us take $y \in \mathbb{R}^N \setminus \{x_0\}$. We then define

$$\bar{x} = \frac{|y - x_0|}{j_K(y)} (y - x_0) + x_0, \quad (\text{A.2.6})$$

we claim that $\Phi_K(\bar{x}) = y$. Indeed, by construction we have

$$\Phi_K(\bar{x}) = \frac{\bar{x} - x_0}{|\bar{x} - x_0|} j_K(\bar{x}) + x_0 = \frac{y - x_0}{|y - x_0|} j_K \left(\frac{|y - x_0|}{j_K(y)} (y - x_0) + x_0 \right) + x_0.$$

From property (A.2.3) we get

$$\Phi_K(\bar{x}) = \frac{y - x_0}{|y - x_0|} j_K \left(\frac{|y - x_0|}{j_K(y)} (y - x_0) + x_0 \right) + x_0 = \frac{y - x_0}{|y - x_0|} j_K(y) \frac{|y - x_0|}{j_K(y)} + x_0 = y,$$

as desired. This shows that Φ_K is bijective and from (A.2.6) we get

$$\Phi_K^{-1}(y) = \frac{|y - x_0|}{j_K(y)} (y - x_0) + x_0, \quad \text{for } y \in \mathbb{R}^N \setminus \{x_0\}.$$

Thanks to the properties of the Minkowski functional, it is easily seen that

$$\Phi_K(r(K - x_0) + x_0) = B_r(x_0), \quad \text{for every } r > 0.$$

We now claim that both Φ_K and its inverse are Lipschitz continuous. We start with Φ_K : we take $x, y \in \mathbb{R}^N \setminus \{x_0\}$ and, without loss of generality, we can suppose that $|y - x_0| \leq |x - x_0|$. By the triangle inequality we get

$$\begin{aligned} |\Phi_K(x) - \Phi_K(y)| &\leq j_K(y) \left| \frac{x - x_0}{|x - x_0|} - \frac{y - x_0}{|y - x_0|} \right| + |j_K(x) - j_K(y)| \\ &\leq j_K(y) \frac{|x - y|}{\sqrt{|x - x_0| |y - x_0|}} + |j_K(x) - j_K(y)|, \end{aligned} \quad (\text{A.2.7})$$

where we used that

$$\begin{aligned} \left| \frac{x - x_0}{|x - x_0|} - \frac{y - x_0}{|y - x_0|} \right|^2 &= 2 - 2 \frac{\langle x - x_0, y - x_0 \rangle}{|x - x_0| |y - x_0|} \\ &\leq \frac{|x - x_0|^2 + |y - x_0|^2}{|x - x_0| |y - x_0|} - 2 \frac{\langle x - x_0, y - x_0 \rangle}{|x - x_0| |y - x_0|} = \frac{|x - y|^2}{|x - x_0| |y - x_0|}, \end{aligned}$$

thanks to Young's inequality. By using (A.2.4) and the fact that $|y - x_0| \leq |x - x_0|$, we get from (A.2.7)

$$|\Phi_K(x) - \Phi_K(y)| \leq \frac{1}{d_K(x_0)} \left[|y - x_0| \frac{|x - y|}{\sqrt{|x - x_0| |y - x_0|}} + |x - y| \right] \leq \frac{2}{d_K(x_0)} |x - y|.$$

This proves the claimed Lipschitz regularity of Φ_K .

We now turn our attention to the inverse function Φ_K^{-1} . We proceed in a similar way: we take $x, y \in \mathbb{R}^N \setminus \{x_0\}$ and we can suppose that $|y - x_0| \leq |x - x_0|$. Then by the triangle inequality

$$\begin{aligned} |\Phi_K^{-1}(x) - \Phi_K^{-1}(y)| &\leq \frac{1}{j_K(x)} \left| |x - x_0| (x - x_0) - |y - x_0| (y - x_0) \right| \\ &\quad + |y - x_0|^2 \left| \frac{1}{j_K(x)} - \frac{1}{j_K(y)} \right|. \end{aligned}$$

By using (A.2.5) and observing that

$$\left| |x - x_0| (x - x_0) - |y - x_0| (y - x_0) \right| \leq (|x - x_0| + |y - x_0|) |x - y| = 2 |x - x_0| |x - y|,$$

we get that

$$\begin{aligned} |\Phi_K^{-1}(x) - \Phi_K^{-1}(y)| &\leq \frac{2 |x - x_0|}{j_K(x)} |x - y| + \frac{|y - x_0|^2}{j_K(x) j_K(y)} |j_K(x) - j_K(y)| \\ &\leq 2 D_K(x_0) |x - y| + D_K(x_0)^2 \frac{|y - x_0|^2}{|x - x_0| |y - x_0|} \frac{|x - y|}{d_K(x_0)} \\ &\leq 2 D_K(x_0) |x - y| + D_K(x_0)^2 \frac{|x - y|}{d_K(x_0)} \\ &= D_K(x_0) \left(2 + \frac{D_K(x_0)}{d_K(x_0)} \right) |x - y|. \end{aligned}$$

This gives the desired Lipschitz estimate for Φ_K^{-1} , as well.

Finally, the two-sided estimates (A.2.1) and (A.2.2) are a standard consequence of the Lipschitz estimates on Φ_K and Φ_K^{-1} , in conjunction with the Area Formula for Lipschitz functions and Rademacher's Theorem. \square

A.3 Two equivalent distances on \mathbb{S}^1

In what follows, we introduce two equivalent distances on the one-dimensional torus $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$. The first one is the following

$$|\alpha|_{\mathbb{S}^1} := \min_{k \in \mathbb{Z}} |\alpha + 2k\pi|, \quad \text{for every } \alpha \in \mathbb{R}.$$

We observe that in particular for $\alpha \in [0, 2\pi]$ this quantity is given by

$$|\alpha|_{\mathbb{S}^1} = \begin{cases} \alpha, & \text{if } 0 \leq \alpha \leq \pi, \\ 2\pi - \alpha, & \text{if } \pi < \alpha \leq 2\pi. \end{cases} \quad (\text{A.3.1})$$

Lemma A.3.1. *We have*

$$\frac{2}{\pi} |\theta - \varphi|_{\mathbb{S}^1} \leq |e^{i\theta} - e^{i\varphi}| \leq |\theta - \varphi|_{\mathbb{S}^1}, \quad \text{for every } \theta, \varphi \in \mathbb{R}.$$

Moreover, both inequalities are sharp.

Proof. We first observe that we can write

$$\begin{aligned} |e^{i\theta} - e^{i\varphi}| &= |e^{i\varphi}| |e^{i(\theta-\varphi)} - 1| \\ &= |e^{i(\theta-\varphi)} - 1| \\ &= \sqrt{(1 - \cos(\theta - \varphi))^2 + \sin^2(\theta - \varphi)} = 2 \left| \sin \left(\frac{\theta - \varphi}{2} \right) \right|, \end{aligned} \tag{A.3.2}$$

thanks to standard trigonometric formulas. In order to conclude the proof, it is sufficient to prove that

$$\frac{2}{\pi} |\alpha|_{\mathbb{S}^1} \leq 2 \left| \sin \left(\frac{\alpha}{2} \right) \right| \leq |\alpha|_{\mathbb{S}^1}, \quad \text{for every } \alpha \in \mathbb{R}. \tag{A.3.3}$$

It is easily seen that both functions

$$\alpha \mapsto |\alpha|_{\mathbb{S}^1} \quad \text{and} \quad \alpha \mapsto \left| \sin \left(\frac{\alpha}{2} \right) \right|,$$

are 2π -periodic, thus is it sufficient to prove (A.3.3) for $\alpha \in [0, 2\pi]$. We thus seek for the maximum and the minimum on $[0, 2\pi]$ of the function

$$\alpha \mapsto 2 \frac{|\sin(\alpha/2)|}{|\alpha|_{\mathbb{S}^1}},$$

extended by continuity to the whole interval. By keeping in mind (A.3.1), on $[0, 2\pi]$ this function can be rewritten as

$$\alpha \mapsto \begin{cases} 2 \frac{\sin(\alpha/2)}{\alpha}, & \text{if } 0 \leq \alpha \leq \pi, \\ 2 \frac{\sin(\alpha/2)}{2\pi - \alpha}, & \text{if } \pi \leq \alpha \leq 2\pi, \end{cases} = \begin{cases} \frac{\sin(\alpha/2)}{\alpha/2}, & \text{if } 0 \leq \alpha \leq \pi, \\ \frac{\sin(\pi - \alpha/2)}{\pi - \alpha/2}, & \text{if } \pi \leq \alpha \leq 2\pi. \end{cases}$$

By recalling that the *sinc function* $t \mapsto (\sin t)/t$ is monotone decreasing on the interval $[0, \pi/2]$, in light of the above discussion we now easily obtain

$$\frac{2}{\pi} \leq 2 \frac{|\sin(\alpha/2)|}{|\alpha|_{\mathbb{S}^1}} \leq 1.$$

This gives (A.3.3), thus concluding the proof. \square

A.4 The constant $C_{N,2s}$ in the fractional Hardy inequality

In the next simple result, we write $C_{N,2s}$ is an alternative way. This permits to compare the constant we obtained in the proof of Theorem 6.3.4, with that of [80].

Lemma A.4.1. *Let $N \geq 2$. Then we have*

$$C_{N,2s} = \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\sigma_N|^{2s} d\mathcal{H}^{N-1}(\sigma).$$

Proof. We first observe that by using spherical coordinates, we have

$$\int_{B_1(0)} |x_N|^{2s} dx = \int_{\mathbb{S}^{N-1}} |\sigma_N|^{2s} \left(\int_0^1 \varrho^{N-1+2s} d\varrho \right) d\mathcal{H}^{N-1}(\sigma) = \frac{1}{N+2s} \int_{\mathbb{S}^{N-1}} |\sigma_N|^{2s} d\mathcal{H}^{N-1}(\sigma).$$

Thus, in order to conclude, it is sufficient to show that

$$\frac{N+2s}{2} \int_{B_1(0)} |x_N|^{2s} dx = (N-1) \omega_{N-1} \int_0^{+\infty} t^{N-2} (1+t^2)^{-\frac{N+2s}{2}} dt.$$

We compute again the integral on the left-hand side, this time by using cylindrical coordinates, i.e. we write

$$\begin{aligned} \int_{B_1(0)} |x_N|^{2s} dx &= \int_{-1}^1 \left(\int_{\{x \in B_1(0) : x_N = \tau\}} d\mathcal{H}^{N-1} \right) |\tau|^{2s} d\tau = \omega_{N-1} \int_{-1}^1 (1-\tau^2)^{\frac{N-1}{2}} |\tau|^{2s} d\tau \\ &= 2\omega_{N-1} \int_0^1 (1-\tau^2)^{\frac{N-1}{2}} \tau^{2s} d\tau. \end{aligned}$$

We now use the change of variable $\tau = (1+t^2)^{-1/2}$. This yields

$$\begin{aligned} \int_{B_1(0)} |x_N|^{2s} dx &= 2\omega_{N-1} \int_0^{+\infty} (1+t^2)^{-s} \frac{t^{N-1}}{(1+t^2)^{\frac{N-1}{2}}} \frac{t}{(1+t^2)^{\frac{3}{2}}} dt \\ &= 2\omega_{N-1} \int_0^{+\infty} t^N (1+t^2)^{-\frac{N+2s}{2}} dt. \end{aligned}$$

In turn, by using an integration by parts, we get

$$\begin{aligned} \int_0^{+\infty} t^N (1+t^2)^{-\frac{N+2s}{2}} dt &= \left[-\frac{1}{N+2s} t^{N-1} (1+t^2)^{-\frac{N+2s}{2}} \right]_0^{+\infty} \\ &\quad + \frac{1}{N+2s} (N-1) \int_0^{+\infty} t^{N-2} (1+t^2)^{-\frac{N+2s}{2}} dt. \end{aligned}$$

This concludes the proof. □

Appendix B

Geometric Lemmas

B.1 Taylor's fatness lemma

In the proof of the fractional Croke-Osserman-Taylor inequality, we crucially exploited a “fatness” property of the complement of planar open sets with finite inradius and given topology. In particular the optimal dependence on the parameter k , as k goes to $+\infty$, comes from this estimate. This result is contained in the proof of [104, Theorem 2]. Since in [104] the full proof appears only for the case $k = 1$, in [B1, Lemma 2.1] we provided all the details for the general case $k \geq 1$.

We recall that $\Pi_{\mathbf{e}_i}$ stands for the projection on the coordinate axis orthogonal to \mathbf{e}_i . We also recall that, with the writing

$$\alpha - 1 \leq \lfloor \alpha \rfloor \leq \alpha, \quad \text{for every } \alpha \in \mathbb{R}, \quad (\text{B.1.1})$$

we mean the *integer part* of a real number α .

Lemma B.1.1 (Taylor's fatness lemma). *Let $k \in \mathbb{N} \setminus \{0\}$ and let $\Omega \subseteq \mathbb{R}^2$ be an open multiply connected set of order k , with finite inradius. Let \mathcal{Q} be an open square with side length $10(\lfloor \sqrt{k} \rfloor + 1)r_\Omega$, whose sides are parallel to the coordinate axes. Then there exists a compact set $\Sigma \subseteq \mathcal{Q} \setminus \Omega$ such that*

$$\max \left\{ \mathcal{H}^1(\Pi_{\mathbf{e}_1}(\Sigma)), \mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma)) \right\} \geq \frac{\sqrt{k}}{4} r_\Omega. \quad (\text{B.1.2})$$

Proof. Let us set $\delta = \lfloor \sqrt{k} \rfloor + 1$, for notational simplicity. By dilating and translating, there is no loss of generality in assuming $r_\Omega = 1$ and

$$\mathcal{Q} = Q_{5\delta}(0) = (-5\delta, 5\delta) \times (-5\delta, 5\delta).$$

We can suppose that $\mathcal{Q} \cap \Omega \neq \emptyset$, otherwise the proof is trivial: it would be sufficient to take $\Sigma = \overline{\mathcal{Q}}$ to get the desired conclusion.

We then fix the following set of $4\delta^2$ centers

$$P_{j,m} = \left(-5\delta + \frac{5}{2} + 5j, 5\delta - \frac{5}{2} - 5m \right), \quad \text{for } j, m = 0, \dots, 2\delta - 1,$$

and take accordingly the two family of squares and disks, given by

$$B_{\frac{3}{2}}(P_{j,m}) \subseteq Q_{\frac{5}{2}}(P_{j,m}), \quad \text{for } j, m = 0, \dots, 2\delta - 1.$$

We observe that by construction we have

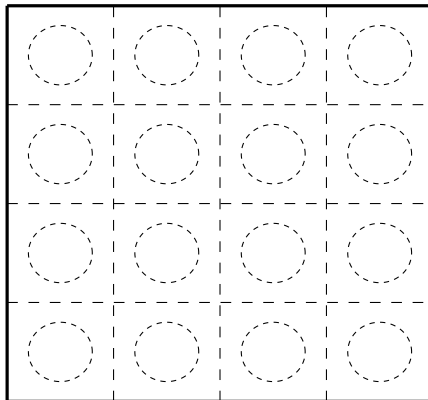


Figure B.1: The construction of disks and squares in the proof of Lemma B.1.1, for the cases $k = 1$, $k = 2$ or $k = 3$ (i.e. $\delta = 2$). Each disk contains at least a point belonging to $\mathbb{R}^2 \setminus \Omega$. The *reliable* squares are those for which such a point can be “connected” to the boundary of the “cell” containing it, with a continuum lying outside of Ω .

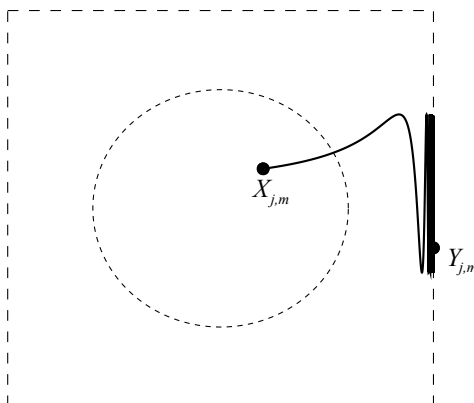


Figure B.2: A zoom on a reliable square $Q_{5/2}(P_{j,m})$. The bold line corresponds to a continuum which connects the point $X_{j,m}$ to the boundary of the “cell”, lying outside of Ω .

$$\text{dist}\left(B_{\frac{3}{2}}(P_{j,m}), \partial Q_{\frac{5}{2}}(P_{j,m})\right) = 1, \quad \text{for every } j, m = 0, \dots, 2\delta - 1. \quad (\text{B.1.3})$$

Since $r_\Omega = 1$, our open set Ω can not entirely contain an open disk with radius larger than 1. Thus, we have that each disk $B_{3/2}(P_{j,m})$ must intersect the complement $\mathbb{R}^2 \setminus \Omega$. Let us select a point $X_{j,m} \in B_{3/2}(P_{j,m}) \setminus \Omega$. We will say that a square $Q_{5/2}(P_{j,m})$ is:

- *unreliable* if for every continuum $K \subseteq \overline{Q_{\frac{5}{2}}(P_{j,m})} \setminus \Omega$ such that $X_{j,m} \in K$, we have

$$K \cap \partial Q_{\frac{5}{2}}(P_{j,m}) = \emptyset;$$

- *reliable* if there exists a continuum $K_{j,m} \subseteq \overline{Q_{\frac{5}{2}}(P_{j,m})} \setminus \Omega$ such that $X_{j,m} \in K_{j,m}$ and

$$K_{j,m} \cap \partial Q_{\frac{5}{2}}(P_{j,m}) \neq \emptyset.$$

We observe that every unreliable square must contain at least a connected component of $(\mathbb{R}^2)^* \setminus \Omega$. Thus, by definition of multiply connected set of order k , the unreliable squares can be at most k . Thus, if we set

$$\mathcal{N} = \left\{ (j, m) : Q_{\frac{5}{2}}(P_{j,m}) \text{ is reliable} \right\},$$

we get¹

$$\#\mathcal{N} \geq 4\delta^2 - k = 4\left(\lfloor \sqrt{k} \rfloor + 1\right)^2 - k \geq 3\left(\lfloor \sqrt{k} \rfloor + 1\right)^2 = 3\delta^2.$$

That is, our square \mathcal{Q} contains at least $3\delta^2$ reliable squares. We want to work with these squares and their continua $K_{j,m}$ defined above. By construction, we have

$$K_{j,m} \subseteq \overline{\mathcal{Q}} \setminus \Omega.$$

We are ready to construct the compact set Σ of the statement: this is given by²

$$\Sigma = \bigcup_{(j,m) \in \mathcal{N}} K_{j,m}.$$

We need to show that its projections along the coordinate axes satisfy (B.1.2). At this aim, we first observe that $K_{j,m}$ is a connected set, containing both the point $X_{j,m} \in B_{3/2}(P_{j,m})$ and a point $Y_{j,m} \in \partial Q_{5/2}(P_{j,m})$. By recalling (B.1.3), we have that

$$|X_{j,m} - Y_{j,m}| \geq 1.$$

Moreover, we have that at least one of the two quantities

$$|\Pi_{\mathbf{e}_1}(X_{j,m}) - \Pi_{\mathbf{e}_1}(Y_{j,m})| \quad \text{or} \quad |\Pi_{\mathbf{e}_2}(X_{j,m}) - \Pi_{\mathbf{e}_2}(Y_{j,m})|,$$

is larger than or equal to 1 (recall that all the squares involved have sides parallel to the coordinate axes). By using this fact, together with the fact that both projections

$$\Pi_{\mathbf{e}_i}(K_{j,m}), \quad \text{for } i = 1, 2,$$

coincide with a segment containing both $\Pi_{\mathbf{e}_i}(X_{j,m})$ and $\Pi_{\mathbf{e}_i}(Y_{j,m})$, we can finally assure that at least one of the two projections of $K_{j,m}$ has a length larger than or equal to 1. In order to

¹We denote by $\#$ the cardinality of a discrete set.

²We notice that this union is not necessarily a disjoint one.

conclude, we need to take care of the possible overlaps in these projections. Let us denote by $J_1, J_2 \in \mathbb{N}$ the following numbers

$$J_i = \#\{K_{j,m} : \mathcal{H}^1(\Pi_{\mathbf{e}_i}(K_{j,m})) \geq 1\}, \quad \text{for } i = 1, 2.$$

According to the previous discussion, we have

$$J_1 + J_2 \geq 3\delta^2 \quad \text{and thus in particular} \quad \max\{J_1, J_2\} \geq \delta^2.$$

Without loss of generality, we can suppose that $J_2 \geq J_1$. This implies that there are at least δ^2 “good” projections, i.e. projections with length at least 1, on the first coordinate axis. We need to estimate the number of such projections, modulo possible overlaps: observe that for every fixed $m \in \{0, \dots, 2\delta - 1\}$, the array of squares

$$\overline{Q_{0,m}(P_{0,m})}, \dots, \overline{Q_{2\delta-1,m}(P_{2\delta-1,m})},$$

all have the same projection. Thus the number of distinct projections is at least

$$\frac{\delta^2}{2\delta} = \frac{\delta}{2}.$$

As a technical and annoying fact, we record that this could fail to be a natural number. However, if we set

$$\Lambda_k = \begin{cases} 1, & \text{for } k \in \{1, 2, 3\}, \\ \frac{\lfloor \sqrt{k} \rfloor}{2}, & \text{for } k \geq 4 \text{ such that } \lfloor \sqrt{k} \rfloor \text{ is even,} \\ \frac{\lfloor \sqrt{k} \rfloor - 1}{2}, & \text{for } k \geq 4 \text{ such that } \lfloor \sqrt{k} \rfloor \text{ is odd,} \end{cases}$$

we have

$$\frac{\delta}{2} \geq \Lambda_k.$$

Thus we have at least Λ_k projections on the first coordinate axis, each having length at least 1. This in turn yields

$$\mathcal{H}^1(\Pi_{\mathbf{e}_2}(\Sigma)) \geq \Lambda_k.$$

Finally, by observing that $\Lambda_k \geq \sqrt{k}/4$, we get the claimed conclusion. \square

B.2 Hayman’s covering lemma

In adapting Hayman’s proof to the fractional case, we needed the following covering Lemma, whose proof can be found in [63, Lemma 2]. The result in [63] is stated for bounded sets: accordingly, the relevant covering is made of a *finite* number of balls. However, a closer inspection of the proof in [63] easily shows that the same result still holds by removing the boundedness assumption. In this case, the covering could be made of countably infinitely many balls: this is still enough for our purposes. We omit the proof, since it is exactly the same as in [63].

Lemma B.2.1 (Hayman’s covering lemma). *Let $\Omega \subseteq \mathbb{R}^2$ be an open set, with finite inradius r_Ω . Then there exist at most countably many distinct points $\{z_n\}_{n \in \mathbb{N}} \subseteq \partial\Omega$ such that the family of disks*

$$\mathfrak{B} = \{B_r(z_n)\}_{n \in \mathbb{N}}, \quad \text{with } r = r_\Omega (1 + \sqrt{2}),$$

is a covering of Ω . Moreover, \mathfrak{B} can be split in at most 36 subfamilies $\mathfrak{B}_1, \dots, \mathfrak{B}_{36}$ such that

$$B_r(z_n) \cap B_r(z_m) = \emptyset \quad \text{if } B_r(z_n), B_r(z_m) \in \mathfrak{B}_k, \text{ with } m \neq n,$$

for every $k = 1, \dots, 36$.

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