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Gradient Maps Associated with Actions of Real Reductive Groups

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Abstract

This thesis is concerned with the study of the action of a real reductive group G on a real submanifold X of a Kähler manifold (Z, ω) which is the restriction of a holomorphic action of a complex reductive Lie group $U^{\mathbb{C}}$. Let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$ acting on Z and preserving ω. We assume that the U-action extends holomorphically to an action of the complexified group $U^{\mathbb{C}}$ and the U-action on Z is Hamiltonian. Then there exists a U-equivariant momentum map $\mu : Z \to \mathfrak{u}$. If $G \subset U^{\mathbb{C}}$ is a closed subgroup such that the Cartan decomposition $U^{\mathbb{C}} = U\exp(i\mathfrak{u})$ induces a Cartan decomposition $G = K\exp(\mathfrak{p})$, where $K = U \cap G$, $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Lie algebra of G, there is a corresponding gradient map $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$. Given an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$, we obtain a Morse like function $f = \frac{1}{2} \parallel \mu_{\mathfrak{p}} \parallel^2$ on X. We study some properties of the gradient map $\mu_{\mathfrak{p}}$ and its norm square function f and analyse the G-action on X using the gradient map.

Introduction

This thesis is concerned with the study of Hamiltonian action of a complex reductive group on a Kähler manifold. The momentum map helps to analyze the geometric properties of this action [5, 36, 55, 30]. The aim of the present work is to investigate a class of actions of real reductive Lie groups on real submanifolds of a Kähler manifold using gradient map techniques. Some of the original results have already been published; some other results have been collected in a preprint and submitted for publication, and others have not yet been submitted for publication.

We consider a Kähler manifold (Z, ω) with an holomorphic action of a complex reductive group $U^{\mathbb{C}}$, where $U^{\mathbb{C}}$ is the complexification of a compact Lie group U with Lie algebra $\mathfrak{u}$. We also assume ω is U-invariant and that there is a U-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$. By definition, for any $\xi \in \mathfrak{u}$ and $z \in Z$, $d\mu^\xi = i_{\xi_Z}\omega$, where $\mu^\xi(z) := \mu(z)(\xi)$ and ξ_Z denotes the fundamental vector field induced on Z by the action of U, i.e.,

$$\xi_Z(z) := \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)z$$

(Section 1.3.) Recently, the momentum map has been generalized to the following settings [44, 45, 46]. We say that a subgroup G of $U^{\mathbb{C}}$ is compatible if G is closed and the map $K \times \mathfrak{p} \to G$, $(k, \beta) \mapsto k \exp(\beta)$ is a diffeomorpism where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. The Lie algebra $\mathfrak{u}^{\mathbb{C}}$ of $U^{\mathbb{C}}$ is the direct sum $\mathfrak{u} \oplus i\mathfrak{u}$. It follows that G is compatible with the Cartan decomposition of $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$, K is a maximal compact subgroup of G with Lie algebra $\mathfrak{k}$ and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (see Section 1.4).

The inclusion $i\mathfrak{p} \hookrightarrow \mathfrak{u}$ induces by restriction, a K-equivariant map $\mu_{i\mathfrak{p}} : Z \to (i\mathfrak{p})^*$. Choose and fix an $\mathrm{Ad}(U^{\mathbb{C}})$-invariant inner product of the Euclidean type B on the Lie algebra $\mathfrak{u}^{\mathbb{C}}$, see [22, Section 3.2], [60, Definition 3.2.4] and also [50, Section 2.1] for the analogue in the algebraic GIT. Such an inner product will automatically induce a well-defined inner product on any maximal compact subgroup U' of $U^{\mathbb{C}}$.

Let $\langle\cdot,\cdot\rangle$ denote the real part of B. Then $\langle\cdot,\cdot\rangle$ is positive definite on $i\mathfrak{u}$, negative definite on $\mathfrak{u}$, $\langle\mathfrak{u}, i\mathfrak{u}\rangle = 0$ and finally the multiplication by i satisfies $\langle i\cdot, i\cdot\rangle = -\langle\cdot,\cdot\rangle$. In order to simplify the notation we replace consideration of $\mu_{i\mathfrak{p}}$ by that of $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$, where

$$\mu_{\mathfrak{p}}^{\beta}(x) := \langle\mu_{\mathfrak{p}}(x), \beta\rangle := \langle i\mu(x), \beta\rangle = -\langle\mu(x), -i\beta\rangle = \mu^{-i\beta}(x).$$

The map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ is K-equivariant and $\operatorname{grad}\mu_{\mathfrak{p}}^{\beta} = \beta_Z$ for any $\beta \in \mathfrak{p}$. Here the grad is computed with respect to the Riemannian metric induced by the Kähler structure. The map $\mu_{\mathfrak{p}}$ is called the G-gradient map associated with μ (Section 1.6).

For a G-stable locally closed real submanifold X of Z, we consider $\mu_{\mathfrak{p}}$ as a mapping $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$. The norm square of $\mu_{\mathfrak{p}}$ is defined as $f(x) := \frac{1}{2} \parallel \mu_{\mathfrak{p}}(x) \parallel^2; \quad x \in X$.

Assume that X is connected and compact. In [45] the authors proved the existence of a smooth G-invariant stratification of X using the norm square of the gradient map and they studied its properties. A strategy to analyze the G-action on X is to view f as a generalized Morse function. The norm square of $\mu_{\mathfrak{p}}$ is in general far from being Morse-Bott and its critical set may be very complicated. But can we have a particular case when the norm square will be Morse-Bott? We find this question to be true for two orbits variety. If the action of G on X has two orbits, then X is called a two orbit variety. Akhiezer [2], Brion [21], Feldmüller [29] among others have studied two orbit complex varieties extensively. S. Cupit-Foutou [26] obtained the classification of a complex algebraic variety on which a reductive complex algebraic group acts with two orbits. Applying standard Morse-theoretic results in [54] and [45], we prove that the norm square is Morse-Bott and obtain information on the cohomology and K-equivariant cohomology of a two orbit variety (Theorem 4.4.1), generalizing [32].

A central ingredient to prove this result is the Ness Uniqueness Theorem which asserts that any two critical points of f in the same G-orbit in fact belong to the same K-orbit (Theorem 2.2.8). Moreover, although we do not assume that X is real analytic manifold, we point out that for any G-invariant compact and connected submanifold of Z, the Lojasiewicz gradient inequality holds for the norm square. Therefore, the limit of the negative gradient flow exists and it is unique and any G-orbit collapses to a single K-

orbit (Theorem 2.2.11). We use the original ideas from [30] in a different context. By the stratification theorem, we have

$$\{p \in X : \overline{G \cdot p} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\} = \{p \in X : \lim_{t \to +\infty} \varphi_t(p) \in \mu_{\mathfrak{p}}^{-1}(0)\} = S_G(\mu_{\mathfrak{p}}^{-1}(0)),$$

where $\varphi_t(p)$ is the flow of the vector field -grad$f$. Then, there exist a K-equivariant strong deformation of $S_G(\mu_{\mathfrak{p}}^{-1}(0))$ onto the set $\mu_{\mathfrak{p}}^{-1}(0)$ (Theorem 2.1.6). Hence no analyticity assumption is necessary in the statement of the retraction Theorem answering Question 1 in [47, p.219].

Biliotti and Ghigi [9] proved a convexity theorem along orbits in a very general setting using only so-called Kempf-Ness function (see Section 2.2 for details on Kempf-Ness funtion). The behaviour of the corresponding gradient map is encoded in the Kempf-Ness function. Recently, Biliotti [7] gave a new proof of the Hilbert-Mumford criterion for real reductive Lie groups stressing the properties of the Kempf-Ness functions. He shows that the Kempf-Ness function is Morse-Bott and it is convex along geodesics for the action of a linear group on $\mathbb{P}(V)$ where V is a finite dimensional vector space. We prove this result in a general setting (Theorem 2.2.7).

We also investigate closed orbits of the parabolic subgroups of G if G is connected. If $\beta \in \mathfrak{p}$ set

$$\begin{aligned} G^{\beta+} &:= \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists}\} \\ R^{\beta+} &:= \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) = e\} \qquad \mathfrak{r}^{\beta+} := \bigoplus_{\lambda > 0} V_\lambda(\operatorname{ad}\beta). \\ G^{\beta} &= \{g \in G : \operatorname{Ad}(g)(\beta) = \beta\} \end{aligned}$$

Note that $\mathfrak{g}^{\beta+} = \mathfrak{g}^{\beta} \oplus \mathfrak{r}^{\beta+}$. It is well-known that if G is connected, $G^{\beta+}$ is a parabolic subgroup of G with Lie algebra $\mathfrak{g}^{\beta+}$ and every parabolic subgroup of G arises as $G^{\beta+}$ for some $\beta \in \mathfrak{p}$. $R^{\beta+}$ is connected and it is the unipotent radical of $G^{\beta+}$. G^{β} is a Levi factor of $G^{\beta+}$ (see section 1.5 for more details). We prove the following result under the assumption that G is connected.

Theorem. *(see Theorem 3.0.4). Let $\beta \in \mathfrak{p}$. Then:*

- *if $G^{\beta+}\cdot x$ is compact, then $\mathcal{O}=G\cdot x$ is compact and $G^{\beta+}\cdot x$ is a finite union of connected components of $\max_{\mathcal{O}}(\beta)=\left\{p\in\mathcal{O}:\ \max_{z\in\mathcal{O}}\mu_{\mathfrak{p}}^{\beta}=\mu_{\mathfrak{p}}^{\beta}(p)\right\}$;*
- *if $\mathcal{O}$ is a compact G-orbit, then $\max_{\mathcal{O}}(\beta)$ is a finite union of compact $G^{\beta+}$-orbits.*

In particular, the number of compact $G^{\beta+}$-orbits is equal or bigger than the number of compact G-orbits.

Observe that $\mu_{\mathfrak{p}}(\mathcal{O})$ is a K-orbit but it is not true in general that $\mu_{\mathfrak{p}}$ defines a diffeomporhism between $\mathcal{O}$ and $\mu_{\mathfrak{p}}(\mathcal{O})$, without the assumption that G is a complex reductive group. Therefore, Theorem 1.2 in [15, pag. 582] does not apply in our context.

Suppose U is a compact connected Lie group. Let $\xi\in\mathfrak{u}$. The standard notation for parabolic subgroups of $U^{\mathbb{C}}$, see for instance [54], is given by

$$U^{\mathbb{C}}(\xi)=\{g\in U^{\mathbb{C}}:\ \lim_{t\to-\infty}\exp(it\xi)g\exp(-it\xi)\,\text{exists}\,\}.$$

It is well-known that $U^{\mathbb{C}}(\xi)$ is connected and it contains a Borel subgroup, that is, a maximal solvable subgroup of $U^{\mathbb{C}}$ [1]. Hence, if $\beta\in i\mathfrak{u}$, then $U^{\mathbb{C}}(-i\beta)$ corresponds to $(U^{\mathbb{C}})^{\beta+}$ in our notation. If $\tilde{\mathcal{O}}$ is a compact orbit of $U^{\mathbb{C}}$ then it is a complex U-orbit [45] and so a flag manifold [35]. Since

$$\max_{\tilde{\mathcal{O}}}(\beta)=\left\{p\in\tilde{\mathcal{O}}:\ \max_{z\in\tilde{\mathcal{O}}}\mu^{-i\beta}=\mu^{-i\beta}(p)\right\},$$

it follows that $\max_{\tilde{\mathcal{O}}}(\beta)$ is connected [5, 36]. Hence the following result, see also [13], holds.

Corollary. *(See Corollary 3.0.4.1). The number of compact $(U^{\mathbb{C}})^{\beta+}$-orbits is equal to the number of compact $U^{\mathbb{C}}$-orbits. Moreover, any closed $(U^{\mathbb{C}})^{\beta+}$-orbit arises as $\max_{\tilde{\mathcal{O}}}(\beta)$, where $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit.*

Assume that G is a real form of $U^{\mathbb{C}}$. Assume there exists $p\in Z$ such that $X=U^{\mathbb{C}}\cdot p$ is compact. If Z is compact, then the $U^{\mathbb{C}}$-orbit throughout the maximum of the norm square function $\parallel\mu\parallel^2$ is a compact orbit and so it is a flag manifold [45]. It is well-known that G has a unique closed orbit $\mathcal{O}$ in X. This is an old result of Wolf [70], see also [45]. In this setting, we prove the following result.

Theorem. *(See Theorem 3.0.8). The set* $\max_{\mathcal{O}}(\beta)$ *is the unique compact orbit of* $G^{\beta+}$ *acting on* X*. This orbit is connected and it is a* $(K^\beta)^o$*-orbit.*

As a consequence of the proof we obtain the following result. Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. If $\pi_{\mathfrak{a}} : \mathfrak{p} \to \mathfrak{a}$ is the orthogonal projection, then $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ is the corresponding $A = \exp(\mathfrak{a})$ gradient map.

Proposition. *(See Proposition 3.0.7). Let* X *be a compact orbit of* $U^{\mathbb{C}}$ *and* $\mathfrak{a} \subset \mathfrak{p}$ *be an Abelian subalgebra. Then* $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$*.*

It is well-known that both $\mu_{\mathfrak{a}}(X)$ and $\mu_{\mathfrak{a}}(\mathcal{O})$ are polytope [45]. The above result tells us that $\mathcal{O}$ captures much of the informations of $\mu_{\mathfrak{a}}$. Note that if $\mathfrak{a} \subset \mathfrak{p}$ is a maximal Abelian subalgebra then by a beautiful Theorem of Kostant [58], keeping in mind that $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}(\mathcal{O})$ is a K-orbit in $\mathfrak{p}$, the set $\mu_{\mathfrak{a}}(X)$ is the convex hull of an orbit of the Weyl group $W(\mathfrak{g}, \mathfrak{a}) = \{\mathrm{Ad}(k) : k \in K, \mathrm{Ad}(k)(\mathfrak{a}) = \mathfrak{a}\}$ (see [56] for more details on Weyl group).

Results on convexity theorems are obtained. Although there is no counterexample, we do not know if the Abelian Convexity Theorem holds for any G-invariant connected submanifold X (see for instance [14, 9, 46] for more details on the subject). If G-action on X has a unique closed orbit $\mathcal{O}$, then we prove that $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$ and so a polytope (Theorem 4.1.2). This result is new also if $G = U^{\mathbb{C}}$ and $X = Z$. This means that $\mathcal{O}$ captures all the information of the A-gradient map. As an application, we prove that the Abelian convexity Theorem holds for a two orbits variety (Theorem 4.4.2).

If Z is connected and compact, and $X \subset Z$ is a A-stable closed coisotropic submanifold, then we prove $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(Z)$ (Theorem 4.2.4) and so it is a polytope as well. More precisely, there exists an open and dense subset W of X such that for any $p \in W$, we have $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\overline{A \cdot p})$.

The norm square f defines a stratification on X (Section 2.1). Non-Abelian covexity result essentially use the stratification theorem [46, 54]. In a situation where the convexity result holds, the minimal stratum, the stratum where the normed square attains its minimum is unique. The uniqueness of the minimal stratum holds if $X = Z$ and $G = U^{\mathbb{C}}$. In this situation, a general convexity results are available [36, 54].

In a general setting, there is still a minimal stratum and this stratum is always open. However, there could be more than one minimal strata and other open strata that are not minimal. Heinzner P and Schützdeller P proved the convexity result in a projective setup [46]. We find a special case where the minimal stratum is the unique open stratum. More precisely, we assume $G \subset U^{\mathbb{C}}$ is a real form. This means that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}}$. Then we prove the following result.

Theorem. *(See Theorem 4.3.2). Suppose X is a G-invariant compact connected Lagrangian submanifold of Z such that $X \subset \mu_{\mathfrak{k}}^{-1}(0)$. Then there exists a unique open stratum in X. Moreover, this stratum is dense in X.*

Since X is Lagrangian, $\mu_{\mathfrak{k}}$ restricted to X is constant and the image lies in the center of the Lie algebra of K. Hence, if G has a finite fundamental group, then K has a finite fundamental group and so K is semisimple and the center of K is finite. Summing up, the condition $X \subset \mu_{\mathfrak{k}}^{-1}(0)$ is always satisfied if G has finite fundamental group.

Let $\mathfrak{a}$ be a maximal Abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{a}_+$ a positive Weyl chamber of $\mathfrak{a}$. Using Theorem 4.3.2, we prove the following non-Abelian convexity theorem

Theorem. *(See Theorem 4.3.3). Suppose X is a G-invariant compact connected Lagrangian submanifold of Z such that $X \subset \mu_{\mathfrak{k}}^{-1}(0)$. The set $\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+$ is a convex polytope.*

In the last chapter, we study the Hilbert Mumford criterion for semistability and polystability points associated with the actions of real reductive groups on real submanifolds of Kähler manifolds. In the classical case of a group action on a Kähler manifold these characterization are given by Mundet i Riera [65, 64], Teleman [4], Kapovich, Leeb and Millson [53] and probably many others. Indeed any of these ideas go back as far as Mumford [66, §2.2], where systematic presentation of notions of stability theory in the non-algebraic Kählerian geometry of complex reductive Lie groups and the relationships between these notions are given.

Semistable (polystable) point in X can be identified by the position of the G-orbit with respect to $\mu_{\mathfrak{p}}^{-1}(0)$. A point $x \in X$ is polystable if its G-orbits intersect the level set $\mu_{\mathfrak{p}}^{-1}(0)$ ($G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$) and semistable if the closure of its G-orbits intersects $\mu_{\mathfrak{p}}^{-1}(0)$ ($\overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$). The set of semistable points associated with the critical points of f is studied in great details in [45, 47]. Our aim is to develop a Geometrical Invariant Theory for actions of real Lie groups on real submanifolds of a Kähler manifold, generalizing results of Heinzner-Huckleberry-Loose [39, 40, 41, 42] and Sjamaar [68].

On the other hand, the semistability (polystability) condition can be checked using a numerical criterion in term of a function called maximal weight, which can be regarded as a Kählerian version of the Hilbert Criterion in Geometric Invariant Theorem. We refer to this numerical condition as analytic stability condition. Biliotti and Zedda [16], see also [10], gave a systematic treatment of the stability theory for an action of a real non-compact reductive Lie group G on a compact connected real manifold X using the maximal weight function and apply this settings to the action of G on measures of X. We introduce a large class of actions of G on X following Teleman [4], called *energy complete action*. As Teleman pointed out, the energy completeness condition gives the natural framework for the stability theory in non-algebraic complex geometry. For such actions, we prove that the semistability condition, respectively polystability condition, is equivalent to analytic semistability condition (Theorem 5.3.4), respectively analytic polystability condition (Theorem 5.3.6), extending the results due to Teleman [4] and Biliotti and Zedda [16], respectively [30, Theorem 7.4]. We also characterize the semistable and polystable points in terms of one-parameter subgroups (Corollary 5.3.4.1 and Corollary 5.3.6.1).

One main ingredients in the proofs of these results is the moment weight inequality (Theorem 5.2.7). We give a proof of this inequality applying the idea in [30] which was due to Xiuxiong Chen [23] to our case.

The thesis is organised as follow. Except for the first chapter, each chapter represents original results. There is a brief introduction at the beginning of each chapter and this introduction includes the organization of each Chapter.

In Chapter 1, basic notions of Convex geometry, Morse theory, Hamiltonian action on Kähler Manifold, compatible subgroup, parabolic subgroup and some properties of Gradient map are collected.

In Chapter 2, the norm square of the gradient map and the Kempf-Ness function associated with the gradient map are studied. The results of this chapter are obtained in a joint work with Biliotti [17].

In Chapter 3, the closed orbit of parabolic subgroup are studied. The result of this chapter is obtained in a joint work with Biliotti [18] which has already been published in Nagoya Mathematical Journal.

Chapter 4, concerns the convexity properties of the gradient map applying most of the results obtained in chapter 2. Both the Abelian and non-Abelian cases are studied. The results of this chapter is a work by the author [69] and a joint work with Biliotti [17].

Chapter 5, concerns the Hilbert Mumford criterion for semistability and polystability points associated with the actions of real reductive groups on real submanifolds of Kähler manifolds. Most of the results of this chapter are obtained in a joint work with Biliotti [19]. The paper has been accepted for publication in The Journal of Geometric Analysis.

Table of Contents

Chapter 1

Preliminaries

1.1 Convex geometry

Let V be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$. The set $H_\alpha(v) = \{u \in V : \langle u, v \rangle = \alpha\}$ with $v \neq 0$ and $\alpha \in \mathbb{R}$ is a hyperplane of V. $H_\alpha^-(v) = \{u \in V : \langle u, v \rangle \leq \alpha\}$ and $H_\alpha^+(v) = \{u \in V : \langle u, v \rangle \geq \alpha\}$ are two closed halfspaces bounded by $H_\alpha(v)$. For $x, y \in V$, let $[x, y] := \{(1-\theta)x + \theta y : 0 \leq \theta \leq 1\}$ denote the closed segment joining x and y.

A vector $v \in V$ is an affine combination of points $v_1, \cdots, v_k \in V$ if there exists $\theta_1, \cdots, \theta_k \in \mathbb{R}$ such that $v = \sum_{i=1}^k \theta_i v_i$ and $\sum_{i=1}^k \theta_i = 1$. For $A \subset V$, affine hull of A is the set of all affine combinations of elements of A and it is the smallest affine subspace of V containing A. We recall some basic definitions.

Definition 1.1.1. *Let $A \subset V$ be any subset of V.*

- *A is convex if for every $x, y \in A$, $[x, y] \in A$.*
- *A is a convex cone if A is convex, non empty and closed under multiplication by non negative real numbers.*

Definition 1.1.2. *The dimension of a nonempty convex subset $A \subset V$ denoted by* $\dim A$ *is the dimension of the smallest affine subset of V containing A.*

A point $v \in V$ is a convex combination of the points $v_1, \cdots, v_k \in V$ if there exists $\theta_1, \cdots, \theta_k \in \mathbb{R}$ with $\theta_i \geq 0$ for all $i = 1, \cdots, k$ such that $v = \sum_{i=1}^k \theta_i v_i$ and $\sum_{i=1}^k \theta_i = 1$. For $A \subset V$, the set of all convex combinations of any finitely many elements of A is called the *convex hull* of A denoted by $\operatorname{conv}(A)$. If A is convex, then $\operatorname{conv}(A) = A$. For any set $A \subset V$, $\operatorname{conv}(A)$ is the intersection of all convex subsets of V containing A.

Let $E \subset V$ be a compact convex subset.

Definition 1.1.3. *Let $x, y \in E$.*

- *The relative interior of E, denoted by* $\operatorname{relint} E$, *is the interior of E in its affine hull.*
- *A face of E is a convex subset F of E such that if* $\operatorname{relint}[x, y] \cap F \neq \emptyset$, *then $[x, y] \subset F$.*
- *The extreme points of E denoted by* $\operatorname{ext} E$ *are the points $x \in E$ such that $\{x\}$ is a face.*

By a theorem of Minkowski [67, Corollary 1.4.5], E is the convex hull of its extremal points. The faces of E are closed [67, p. 62]. The empty set and E itself are faces of E. The other faces are called *proper*.

Definition 1.1.4. *The support function of E is defined by the function $h_E : V \to \mathbb{R}$, $h_E(u) = \max\{\langle x, u \rangle : x \in E\}$.*

If $u \neq 0$, the hyperplane $H_E(u) := \{x \in E : \langle x, u \rangle = h_E(u)\}$ is called the *supporting hyperplane* of E for u.

The set

$$F_u(E) := E \cap H_E(u) \tag{1.1}$$

is a face and it is called the *exposed face* of E defined by u.

Intuitively, the meaning of the support function is simple. For instance, consider a nonempty closed convex set $E \subset \mathbb{R}^n$. Then for a unit vector $u \in S^{n-1} \cap \operatorname{dom} h_E$, the supporting function $h_E(u)$ is the signed distance of the support plane to E with exterior normal vector u from the origin; the distance is negative if and only if u points into the open half-space containing the origin.

The following properties follows from the definition of the support function;

- $h_E = h_x$ if and only if $E = \{x\}$, $h_{E+r}(u) = h_E(u) + \langle r, u\rangle$ for all $r \in V$, $h_E(\lambda u) = \lambda h_E(u)$ for all λ non negative and $h_E(u+v) \leq h_E(u) + h_E(v)$. These implies that h_E is a convex function if $E \neq V$.
- $h_E \leq h_G$ if and only if $E \subset G$.

In general not all faces of a convex subset are exposed. For instance, consider the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces.

If F is a face of a convex set E, then $\operatorname{ext} F = F \cap \operatorname{ext} E$. Indeed, it is immediate that $F \cap \operatorname{ext} E \subset \operatorname{ext} F$ and the converse follows from the definition of a face.

Lemma 1.1.1. *[15, Lemma 2] If G is a compact group and V is a representation space of G define*

$$\rho : V \to V^G \qquad \rho(v) := \int_G gx\, dg$$

where dg denotes the Haar measure on G. Then $V = V^G \oplus \ker \rho$. If $x \in V$ and $x = x_0 + x_1$ in this decomposition, then

a) $G \cdot x = x_0 + G \cdot x_1$;

b) $\operatorname{conv}(G \cdot x) = x_0 + \operatorname{conv}(G \cdot x_1)$;

c) x_0 *is the unique fixed point of* G *contained in* $\operatorname{conv}(G \cdot x)$;

d) $x_0 \in \operatorname{relint} \operatorname{conv}(G \cdot x)$.

Proof. That $V = V^G \oplus \ker \rho$ follows from the fact that $\operatorname{Im} \rho = V^G$ and $\rho^2 = \rho$. (a) and (b) are immediate. Since $x_0 = \rho(x)$, it follows from the definition of ρ that $x_0 \in \operatorname{conv}(G \cdot x)$.

If $y \in \operatorname{conv}(G \cdot x)$ is another fixed point, then $y_0 = x_0$ and $y_1 \in \ker \rho \cap V^G$. Hence $y_1 = 0$ and $y = x_0$. This proves (c). By Theorem 1.1.6 there is a unique face $F \subset \operatorname{conv}(G \cdot x)$ such that $x_0 \in \operatorname{relint} F$. Since $\operatorname{conv}(G \cdot x)$ is G-invariant and x_0 is fixed by G, also F is G-invariant, and hence also $\operatorname{ext} F$. Since $\operatorname{ext} F \subset \operatorname{ext}(\operatorname{conv}(G \cdot x)) = G \cdot x$, it follows that $\operatorname{ext} F = G \cdot x$ and hence that $F = \operatorname{conv}(G \cdot x)$. ☐

The following results about a compact convex set E and its faces are recalled from [13]. Their proofs are included for completeness.

Lemma 1.1.2. *If $F \subset E$ is an exposed face, the set $\mathcal{F}_F := \{u \in V : F = F_u(E)\}$ is a convex cone. If G is a compact subgroup of $O(V)$ that preserves both E and F, then $\mathcal{F}_F$ contains a fixed point of G.*

Proof. Let $u_1, u_2 \in \mathcal{F}_F$ and $\theta_1, \theta_2 \geq 0$. Set $u = \theta_1 u_1 + \theta_2 u_2$. We need to prove that if at least one of θ_1, θ_2 is strictly positive, then $F = F_u(E)$. Suppose $\theta_1 > 0$. Then $h_E(u) \leq \theta_1 h_E(u_1) + \theta_2 h_E(u_2)$. If $f \in F$, then

$$\begin{aligned} \langle f, u \rangle &= \theta_1 \langle f, u_1 \rangle + \theta_2 \langle f, u_2 \rangle \\ &= \theta_1 h_E(u_1) + \theta_2 h_E(u_2). \end{aligned}$$

This imlies that $h_E(u) = \theta_1 h_E(u_1) + \theta_2 h_E(u_2)$ and $F \subset F_u(E)$. On the other hand, if $f \in F_u(E)$, then

$$0 = h_E(u) - \langle f, u \rangle = \theta_1 (h_E(u_1) - \langle f, u_1 \rangle) + \theta_2 (h_E(u_2) - \langle f, u_2 \rangle).$$

Since $\theta_1 > 0$, $h_E(u_1) - \langle f, u_1 \rangle = 0$ and so $f \in F_{u_1}(E) = F$. Thus $F = F_u(E)$. This proves the first fact. To prove the second, pick any vector $u \in \mathcal{F}_F$ and apply Lemma 1.1.1 to the orbit $G \cdot u \subset \mathcal{F}_F$: this gives a G-invariant $\bar{u} \in \mathcal{F}_F$. ☐

Lemma 1.1.3. *If E is a compact convex set and $F \subsetneq E$ is a face, then* $\dim F < \dim E$.

Proof. If $\dim F = \dim E$, then $\operatorname{relint} F$ is open in the affine span of E, so $\operatorname{relint} F \subset \operatorname{relint} E$. By the previous theorem this implies that $F = E$. ☐

Lemma 1.1.4. *If E is a compact convex set and $F \subset E$ is a face, then there is a chain of faces $F_0 = F \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = E$ which is maximal, in the sense that for any i there is no face of E strictly contained between F_{i-1} and F_i.*

Proof. The result is trivial for $F = E$. Suppose $F \subsetneq E$. Set $F_0 := F$. If there is no face strictly contained between F_0 and E, set $F_1 = E$. Otherwise we have a chain $F_0 \subsetneq F_1 \subsetneq F_2 = E$. If this is not maximal, we can further refine it. Repeating this step we get a chain with $k+1$ elements. Since $\dim F_{i-1} < \dim F_i$, $k \leq n$. Hence the chain after at most n steps is maximal. If $F = E$, the result is trivial. $\square$

Lemma 1.1.5. *If E is a convex subset of $\mathbb{R}^n$, $M \subset \mathbb{R}^n$ is an affine subspace and $F \subset E$ is a face, then $F \cap M$ is a face of $E \cap M$.*

Proof. If $x, y \in E \cap M$ and $\operatorname{relint}[x, y] \cap F \cap M \neq \emptyset$ then $[x, y] \subset F$ since F is a face, but $[x, y]$ is also contained in M since M is affine. So $[x, y] \subset F \cap M$ as desired. $\square$

Theorem 1.1.6 ([67, p. 62])**.** *If E is a compact convex set and F_1, F_2 are distinct faces of E, then $\operatorname{relint} F_1 \cap \operatorname{relint} F_2 = \emptyset$. If G is a nonempty convex subset of E which is open in its affine hull, then $G \subset \operatorname{relint} F$ for some face F of E. Therefore E is the disjoint union of the relative interiors of its faces.*

Definition 1.1.5. *A convex polytope in V is the convex hull of a finite (possibly empty) subset of V.*

A polytope is the intersection of finitely many closed halfspaces [67, p. 96]. If $A = \operatorname{conv}\{v_i \cdots, v_k\}$, then the extreme points of A are among the points $v_i \cdots, v_k$. This implies that polytopes are precisely the convex bodies with finitely many extreme points. If $H_A(\cdot)$ is a support plane of A, then $H_A(\cdot) \cap A = \operatorname{conv}(H_A(\cdot) \cap \{v_i \cdots, v_k\})$. This implies that each support set of a polytope is itself a polytope. Each proper face of a polytope is a support set [67, p. 95].

Proposition 1.1.7. *Let $C_1 \subset C_2$ be two compact convex set of V. Assume that for any $\beta \in V$ we have*

$$\max_{y \in C_1} \langle y, \beta \rangle = \max_{y \in C_2} \langle y, \beta \rangle.$$

Then $C_1 = C_2$.

Proof. We may assume without loss of generality that the affine hull of C_2 is V. Assume by contradiction that $C_1 \subsetneq C_2$. Since C_1 and C_2 are both compact, it follows that there exists $p \in \partial C_1$ such that $p \in \overset{o}{C_2}$. Since every face of a compact convex set is contained in an exposed face [67], there exists $\beta \in V$ such that

$$\max_{y \in C_1} \langle y, \beta \rangle = \langle p, \beta \rangle.$$

This means the linear function $x \mapsto \langle x, \beta \rangle$ restricted on C_2 achieves its maximum at an interior point which is a contradiction. □

1.2 Morse Theory

In this section, we briefly review the concept of Morse theory. The convexity results discussed in this work strongly rely on Morse Theory. We only recall important facts from [37] and [54] needed to understand the results discussed in this work. The reader can take a look at those materials for more details of some facts that are taken for granted in this section.

Throughout this section, M will denote a compact connected smooth manifold of dimension n. Let $f \in C^\infty(M)$ be a real smooth function on M. A point $p \in M$ is a critical point of f if $df_p = 0$ in T_pM.

Let C denotes the set of critical points of f and $H_p(f) : T_pM \times T_pM \to \mathbb{R}$ denotes the Hessian of f at p which is a symmetric bilinear form defined by $H_p(f)(X_p, Y_p) := X(Y(f))(p)$, for vector fields X, Y on M. At critical points, the Hessian is well-defined and symmetric. In fact, $X(Y(f))(p) - Y(X(f))(p) = d_pf([X,Y]) = 0$, since $df_p = 0$. This implies that $X(Y(f))(p) = Y(X(f))(p)$. Moreover, if $X_p = 0$, then by definition of vector field, $X(Y)(f))(p) = 0$. If $Y_p = 0$, then $X(Y(f))(p) = 0$ since $X(Y(f))(p) = Y(X(f))(p)$.

A point $p \in C$ is said to be nondegenerate if $H_p(f)$ is a nondegenerate symmetric bilinear form.

Definition 1.2.1. *A function $f \in C^\infty(M)$ is called Morse function if all critical points of f are nondegenerate.*

The Morse index of a critical point p of f is the dimension of the negative space of $H_p(f)$. Let m and k be the number of positive and negative eigenvalues of $H_p(f)$ counted by multiplicity respectively. If f is a Morse function, then by nondegeneracy, $\dim M = n = m + k$.

Lemma 1.2.1. *(Morse Lemma [62, Lemma 2.2]). Let $f \in C^\infty(M)$ be a Morse function. If $p \in M$ is a critical point of f with Morse index k, then there exist a neighbourhood U of p and a chart $\varphi : U \to \mathbb{R}^n$ satisfying $\varphi(p) = 0$ such that*

$$(f \circ \varphi^{-1})(x_1, \cdots x_n) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

in a neighbourhood of $0 \in \mathbb{R}^n$.

It follows from the Morse lemma that if $f \in C^\infty(M)$ is a Morse function, the critical set of f is a closed discrete subset of M, hence f has only finitely many critical points.

Fix a Riemannian metric g on M and define the gradient vector field ∇f of a function $f \in C^\infty(M)$ by $g(\nabla f, X) = df(X)$ for all vector field X on M. Let x_t be the negative gradient flow generated by $-\nabla f$.

Definition 1.2.2. *For $p \in M$, the stable manifold S_p of $-\nabla f$ at p is defined as*

$$S_p := \{q \in M; \lim_{t\to\infty} x_t(q) = p\}.$$

Lemma 1.2.2. *[37, Lemma 2.3] The dimension of S_p is m.*

Theorem 1.2.3. *[62] Let f be a Morse function on M. Then the manifold decomposes as*

$$M = \bigsqcup_{p\in C} S_p. \tag{1.2}$$

The decomposition 1.2 is called the Morse decomposition of M.

Example 1.2.1. *Let* $M = S^2$, *for* $v \neq 0$, $f(x) = \langle x, v\rangle$ *is a smooth function on* M. $\{\frac{v}{\|v\|}, \frac{-v}{\|v\|}\}$ *is the critical set and the stratification is given as*

$$S_{min} = S^2 \setminus \left\{ \frac{v}{\| v \|} \right\}, \quad S_{max} = \left\{ \frac{v}{\| v \|} \right\}.$$

Remark 1.2.4. *Suppose we are in a situation where* m *is even for all* $p \in C$, *then* f *has a unique local maximum and a unique local minimum.* [5, 37]

Definition 1.2.3. *Let* M *be a compact manifold, and let* $f \in C^\infty(M)$. *Then* f *is a Morse-Bott function if the critical set* C *of* f *is a smooth submanifold of* M *and for each* $p \in C$, $H_p(f)$ *induces a nondegenerate bilinear form on the quotient space* T_pM/T_pC. *i.e., on each* $p \in C$, $H_p(f)$ *is nondegenerate in normal directions.*

For any $q \in M$, let

$$B_\epsilon(q) := \{m \in M : d(q, m) \leq \epsilon, \epsilon > 0\}; \ d \text{ denotes the geodesic distance.}$$

We define a subset $\omega(p)$ of M as

$$\omega(p) := \{q \in M : B_\epsilon(q) \cap x_t \neq \emptyset \ \forall \epsilon > 0, \forall t \in [T, \infty), T \in \mathbb{R}\}.$$

Let $f \in C^\infty(M)$ be a Morse-Bott function and C' a connected component of C. The stable manifold of $-\nabla f$ at C' is defined as

$$S_{C'} := \{p \in M : \omega(p) \subset C'\}.$$

Theorem 1.2.5. *(Morse-Bott). Let* $f \in C^\infty$ *be a Morse-Bott function, and let* $\mathcal{C}$ *be the set of connected components of* C. *Then*

$$M = \bigsqcup_{C' \in \mathcal{C}} S_{C'}. \tag{1.3}$$

The decomposition in 1.3 above is referred to as Morse Stratification of M.

Example 1.2.2. *Consider* $M = S^2$ *and define a function* $f(x, y, z) = \frac{1}{2}z^2$ *on* M. *The critical set of* f *is given as* $\{(x, y, z) \in M; x^2 + y^2 = 1, z = 0\} \cup \{e_3, -e_3\}$ *which is the equator of the sphere union north pole and south pole. One can check that* f *is a Morse-Bott function, and* $S_{max} = \{N, S\}$ *and* $S_{min} = S^2 \setminus \{N, S\}$.

Remark 1.2.6. *In Morse-Bott theory, if the* $\dim S_{C'}$ *is even for all* $C' \in \mathcal{C}$ *and* M *is connected, then* f *attains a local minimum (maximum) at a unique component* C' *[37], [5].*

Kirwan shows in [54, Appendix 10] that the Morse theory can be extended to a larger class of functions.

Definition 1.2.4. *A function* $f \in C^\infty(M)$ *is called a minimally degenerate function if the following holds.*

- *The critical set of* f *on* M *is a finite union of disjoint closed subsets* $\boldsymbol{C} \subset C$ *on which* f *takes a constant value. The subsets* $\boldsymbol{C}$ *are called critical subsets of* f*. If the critical set of* f *is well-behaved we can take* $\boldsymbol{C}$ *to be its connected components.*

- *For every such* $\boldsymbol{C}$*, there is a submanifold* $M_{\boldsymbol{C}}$ *of* M *containing* $\boldsymbol{C}$ *with orientable normal bundle in* M*, such that*

    a) the restriction of f *to* $M_{\boldsymbol{C}}$ *takes its minimal value on* $\boldsymbol{C}$*,*

    b) for all $p \in M_{\boldsymbol{C}}$*, the Hessian* $H_p(f)$ *is positive semidefinite on* $T_pM_{\boldsymbol{C}} \subset T_pM$*, and there is no subspace of* T_pM *strictly containing* $T_pM_{\boldsymbol{C}}$*, on which* $H_p(f)$ *is positive semidefinite.*

A submanifold $M_{\mathbf{C}}$ above is called a minimising submanifold for f along $\mathbf{C}$. Note that if f is Morse-Bott, then it is a minimally degenerate function with $M_{\mathbf{C}} = S_{C'}$.

If f is minimally degenerated, f defines a smooth stratification on M as follows.

Theorem 1.2.7. *[54, Theorem 10.4]. Let* $f \in C^\infty(M)$ *be a minimally degenerate function on* M*. Suppose that the gradient of* f *is tangent to the minimising manifold* $M_{\boldsymbol{C}}$*. Then*

$$M = \bigsqcup_{\boldsymbol{C}} S_{\mathbf{C}}, \tag{1.4}$$

where $S_{\mathbf{C}} := \{p \in M : \omega(p) \subset \mathbf{C}\}$*. The strata* $S_{\boldsymbol{C}}$ *are smooth, and the minimising manifolds* $M_{\boldsymbol{C}}$ *are open neighbourhoods of* $\boldsymbol{C}$ *in* $S_{\boldsymbol{C}}$*.*

If the $\dim S_{\mathbf{C}}$ is even for all $\mathbf{C}$, then f has a local minimum along unique set $\mathbf{C}$.

1.3 Hamiltonian Action on Kähler Manifold

Let U be a compact Lie group with Lie algebra $\mathfrak{u}$. The commutator ideal of $\mathfrak{u}$ is

$$[\mathfrak{u}, \mathfrak{u}] = \{\text{linear combinations of } [\xi_1, \xi_2] \ \forall\, \xi_1, \xi_2 \in \mathfrak{u}\}.$$

Definition 1.3.1.

- *A compact Lie group U is semisimple if $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}]$.*
- *A Lie group T is a torus if it is isomorphic to the product $S^1 \times \cdots \times S^1 = (S^1)^n$. These are the compact connected abelian Lie groups [6, Page 85]. $T \subset U$ is called maximal torus if $T = T'$ for any other torus $T' \subset U$ such that $T \subset T'$.*

Theorem 1.3.1. *(Maximal Torus Theorem)[6, Theorem 4.1] Let U be a connected, compact Lie group.*

i There exists a maximal torus T in U.

ii Any two maximal tori in U are conjugate.

iii Every element of U is contained in a maximal torus.

Let $U^{\mathbb{C}}$ be the complexification of a compact Lie group U in the sense of [51]. The group $U^{\mathbb{C}}$ is a complex reductive group with Lie algebra $\mathfrak{u}^{\mathbb{C}}$ and contains U as a maximal compact subgroup [24].

Let (Z, ω) be a compact connected Kähler manifold, where ω is a symplectic form, J is an integrable complex structure such that $\omega(\cdot, J\cdot)$ is a Riemannian metric. We assume that $U^{\mathbb{C}}$ acts holomorphically on Z and ω is U-invariant. For any $\xi \in \mathfrak{u}^{\mathbb{C}}$ and $z \in Z$, ξ_Z denotes the fundamental vector field induced on Z by the action of $U^{\mathbb{C}}$ given as

$$\xi_Z(z) := \frac{d}{dt}\Big|_{t=0} \exp(t\xi) \cdot z.$$

Definition 1.3.2. *The U-action on (Z, ω) is Hamiltonian if there exists a map $\mu : Z \to \mathfrak{u}^*$, where $\mathfrak{u}^*$ is the dual of $\mathfrak{u}$, satisfying the following conditions;*

- *For any $\xi \in \mathfrak{u}$ and $z \in Z$, the function $\mu^\xi : Z \to \mathbb{R}$ defined as $\mu^\xi(z) := \mu(z)(\xi)$ is such that*
$$d\mu^\xi = i_{\xi_Z}\omega.$$
i.e, μ^ξ is a Hamiltonian function for ξ_Z.

- *μ is equivariant with respect to the U-action on Z and the coadjoint action of U on $\mathfrak{u}^*$. i.e., $\mu(gz) = Ad^*(g)(\mu(z))$; $g \in U, z \in Z$, $(Ad^*(g) := (Ad(g^{-1}))^*)$.*

The map μ satisfying the above conditions is called a momentum map.

We now discuss the existence and uniqueness of the momentum map. Let $C^k := \Lambda^k\mathfrak{u}^*$ be an alternating k-linear maps. $C^1 = \mathfrak{u}^*$. Let $\delta_k : C^{k-1} \to C^k$ be a linear operator defined by
$$\delta c(\xi_0, \cdots, \xi_k) = \sum_{i<j}(-1)^{i+j}c([\xi_i, \xi_j], \xi_0, \cdots, \bar{\xi_i}, \cdots, \bar{\xi_j}, \cdots, \xi_k).$$
This linear operator satisfies $\delta^2 = 0$. The Lie algebra cohomology groups of $\mathfrak{u}$ are the cohomology groups of the complex $0 \to^{\delta_0} C^0 \to^{\delta_1} C^1 \to^{\delta_2} \cdots$ such that
$$H^k(\mathfrak{u}; \mathbb{R}) := \frac{\ker \delta_k}{\operatorname{im} \delta_{k-1}}.$$
If $c \in \mathfrak{u}^*$, then $\delta_0 c(\xi_0, \xi_1) = -c([\xi_0, \xi_1])$. This implies that $\delta_0 c = 0$ if and only if c vanishes on $[\mathfrak{u}, \mathfrak{u}]$, hence,
$$H^1(\mathfrak{u}, \mathbb{R}) = [\mathfrak{u}, \mathfrak{u}]^0$$
where $[\mathfrak{u}, \mathfrak{u}]^0 \subset \mathfrak{u}^*$ is the annihilator of $[\mathfrak{u}, \mathfrak{u}]$.

Theorem 1.3.2. *(Whitehead Lemmas[3]). U is semisimple if and only if $H^1(\mathfrak{u}, \mathbb{R}) = 0$.*

Z can be given a symplectic structure by choosing the ω, the Kähler form on Z and consider Z as a symplectic manifold.

Theorem 1.3.3. *[3]. Let U be a compact connected Lie group. If $H^1(\mathfrak{u}; \mathbb{R}) = 0$, then momentum maps for Hamiltonian U-actions on Z are unique.*

Corollary 1.3.3.1. *In general, momentum maps are unique up to a constant $c \in H^1(\mathfrak{u}; \mathbb{R})$. i.e., if $\mu : Z \to \mathfrak{u}^*$ is a momentum map, then given any $c \in H^1(\mathfrak{u}; \mathbb{R})$, $\mu + c$ is another momentum map.*

We have the following special cases:

a) If U is semisimple, then the U-action on Z is Hamiltonian and the momentum map is unique.

b) If U is Abelian, then the momentum map, if it exists, is unique up to constants $c \in \mathfrak{u}^*$.

We choose and fix an $\mathrm{Ad}(U^{\mathbb{C}})$ inner product of Euclidean type on the Lie algebra $\mathfrak{u}^{\mathbb{C}}$ [22, Section 3.2] and [61, Definition 3.2.4]. One can use for instance an embedding of U in $\mathrm{U}(n)$ and its extension $U^{\mathbb{C}} \longrightarrow \mathrm{GL}(n, \mathbb{C})$. Then

$$B : \mathfrak{u}^{\mathbb{C}} \times \mathfrak{u}^{\mathbb{C}} \longrightarrow \mathbb{C}, \qquad (X, Y) \mapsto \mathrm{Tr}(XY)$$

is an $\mathrm{Ad}(U^{\mathbb{C}})$ inner product of Euclidean type on the Lie algebra $\mathfrak{u}^{\mathbb{C}}$.

Let $\langle \cdot, \cdot \rangle$ denote the real part of B. Then, keeping in mind that $\mathfrak{u} \subset \mathrm{u}(n)$, $\langle \cdot, \cdot \rangle$ is positive-definite on $i\mathfrak{u}$, negative-define on $\mathfrak{u}$, $\langle \mathfrak{u}, i\mathfrak{u} \rangle = 0$ and $\langle i\cdot, i\cdot \rangle = -\langle \cdot, \cdot \rangle$. In particular we may identify $\mathfrak{u}$ and $\mathfrak{u}^*$ by means of $-\langle \cdot, \cdot \rangle$ and so one can think of the momentum map as $\mathfrak{u}$-valued map. With respect to $-\langle \cdot, \cdot \rangle$, the momentum map $\mu : Z \to \mathfrak{u}$ interchanges the U-action on Z with the adjoint action of U on $\mathfrak{u}$. i.e., the second condition becomes $\mu(gz) = \mathrm{Ad}(g)(\mu(z))$; $g \in U, z \in Z$. If $\xi \in \mathfrak{u}$, then $\mu^{\xi}(z) := \langle \mu(z), \xi \rangle$ is smooth and its gradient is given by $J\xi_Z$.

Example 1.3.1. *Consider the standard $U(n)$-action on $(\mathbb{C}^n, -\frac{i}{2} \sum dz_j \wedge d\bar{z}_j)$ given by normal matrix-vector multiplication. The Lie algebra $\mathfrak{u}(n)$ is the set of skew-hermitian matrices $\mathcal{X} = V + iW$. Identify $\mathfrak{u}(n)$ with its dual $\mathfrak{u}(n)^*$ via $\langle A, B \rangle = Tr(AB)$. This action is Hamiltonian and the momentum map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ is given as*

$$\mu(z) = \frac{i}{2} z z^*.$$

Indeed, Let $\mathcal{X} \in \mathfrak{u}(n)$. Then, $\mathcal{X}^* = \overline{\mathcal{X}}^T = -\mathcal{X}$, and

$$-V - iW = -\mathcal{X} = \overline{\mathcal{X}}^T = V^T - iW^T.$$

This implies that $V = -V^T$ *and* $W = W^T$*. The fundermental vector field induced on* $\mathbb{C}^n$ *by* $\mathcal{X}$ *is given as*

$$\begin{aligned}
\mathcal{X}_{\mathbb{C}^n} &= \frac{d}{dt}|_{t=0} \exp(t\mathcal{X})z \\
&= \mathcal{X} \exp(t\mathcal{X})z|_{t=0} \\
&= (V + iW) \exp(0(V + iW))z \quad \textit{identifying } \mathbb{C}^n \textit{ with } \mathbb{R}^{2n} \\
&= \begin{pmatrix} V & -W \\ W & V \end{pmatrix} z.
\end{aligned}$$

Let $x + iy = z \in \mathbb{C}^n$*. The component of the momentum map along* $\mathcal{X}$ *is given as*

$$\mu^{\mathcal{X}}(z) = \langle \mu(z), \mathcal{X} \rangle = \frac{i}{2} \operatorname{tr}(zz^* \mathcal{X}),$$

where tr *is the trace of a matrix. Since the trace is invariant under the cyclic permutations and the trace of a complex number is the number itself,*

$$\mu^{\mathcal{X}}(z) = \frac{i}{2} \operatorname{tr}(zz^* \mathcal{X}) = \frac{i}{2} \operatorname{tr}(z^* \mathcal{X} z) = \frac{i}{2}(z^* \mathcal{X} z).$$

It is an easy exercise to see that

$$\mu^{\mathcal{X}}(z) = \frac{1}{2} z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} z.$$

We need to show that $d\mu^{\mathcal{X}} = i_{\mathcal{X}_{\mathbb{C}^n}} \omega$ *and* $\mu(gz) = Ad(g)(\mu(z))$; $g \in U(n), z \in \mathbb{C}^n$*. Let* Y *be a vector field on* $\mathbb{C}^n$ *and set* $Y_z = \tilde{z} \in \mathbb{R}^{2n}$*, we have*

$$\begin{aligned}
i_{\mathcal{X}_{\mathbb{C}^n}} \omega(Y)(z) &= \omega(\mathcal{X}_{\mathbb{C}^n}, Y)(z) \\
&= -\frac{i}{2} \sum dz_j \wedge d\bar{z}_j (\mathcal{X}_{\mathbb{C}^n}, Y)(z).
\end{aligned}$$

Identify $\mathbb{C}^n$ *with* $\mathbb{R}^{2n}$ *so that a symplectic form* $-\frac{i}{2} \sum dz_j \wedge d\bar{z}_j$ *becomes* $\sum_{j=i}^n dx_j \wedge dy_j$*. By calculation, we have*

$$i_{\mathcal{X}_{\mathbb{C}^n}} \omega(Y)(z) = z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z}.$$

On the other hand, by calculation and nothing that Y_z *can be written as* $\frac{d}{dt}|_{t=0}z + t\tilde{z}$*, one can show that*

$$\begin{aligned} d\mu^{\mathcal{X}}(Y)(z) &= Y_z(\mu^{\mathcal{X}}) \\ &= z^T \begin{pmatrix} -W & -V \\ V & -W \end{pmatrix} \tilde{z} = i_{\mathcal{X}_{\mathbb{C}^n}}\omega(Y)(z). \end{aligned}$$

Let $g \in U(n)$*.* $g^* = g^{-1}$ *and so,*

$$\begin{aligned} \mu(gz) &= \frac{i}{2}gz(gz)^* \\ &= \frac{i}{2}gzz^*g^* \\ &= \frac{i}{2}gzz^*g^{-1} \\ &= g(\frac{i}{2}zz^*)g^{-1} \\ &= \mathrm{Ad}_g(\mu(z)). \end{aligned}$$

Proposition 1.3.4. *Let* $H \subset U$ *be a Lie subgroup with Lie algebra* $\mathfrak{h}$*. Let* $\pi : \mathfrak{u} \to \mathfrak{h}$ *be orthogonal projection onto* $\mathfrak{h}$*. If the* U*-action on* Z *is Hamiltonian, then, the restricted action of* H *on* Z *is also Hamiltonian and the momentum map is given as* $\mu_{\mathfrak{h}} = \pi \circ \mu$*.*

Proposition 1.3.5. *Let* $H \subset U(n)$ *be a Lie subgroup and* $\mu_{\mathfrak{h}} : \mathbb{C}^n \to \mathfrak{h}$ *be the momentum map as given in the above Proposition. Then*

$$\mu_{\mathfrak{h}}([z]) := \frac{\mu(z)}{\parallel z \parallel^2}$$

is a momentum map for the H *action on* $\mathbb{C}P^n$*.*

From now on, we assume that U is connected and denote the fixed point set of U by $Z^U := \{z \in Z : U \cdot z = z\}$. Let T be a maximal torus of U with Lie algebra $\mathfrak{t}$. The momentum map associated to torus actions on Z have been studied by Atiyah [5] and Guillemin and Sternberg [36]. It is well known that every connected component of the fixed point set of a torus action on Z is a complex submanifold of Z and the induced action of the torus on its normal bundle in Z has no nonzero fixed vectors. Using this fact Atiyah proved the following result.

Lemma 1.3.6. *For any $\xi \in \mathfrak{u}$, μ^{ξ} is Morse-Bott and has only critical manifolds of even index.*

The set $Z^{\xi} := \{z \in Z : \xi_Z(z) = 0\}$ is a submanifold of Z (possibly disconnected) fixed by the one-parameter subgroup generated by ξ and invariant under the T-action.

Atiyah [5] and Guillemin and Sternberg [36] showed that the image of the fixed point set of T on Z under the momentum map associated with the T-action is a finite set of points in $\mathfrak{t}^*$.

Theorem 1.3.7. *(Abelian Convexity theorem [5], [36].) The set $\mu(Z^T)$ is finite and $\mu(Z)$ is the convex hull of $\mu(Z^T)$ and so it is a convex polytope.*

Atiyah [5] refined Theorem 1.3.7 by describing the behaviour of the momentum map on the orbits of $T^{\mathbb{C}}$.

Theorem 1.3.8. *[5] If $z \in Z$, then $\mu(\overline{T^{\mathbb{C}} \cdot z}) = conv(\overline{T^{\mathbb{C}} \cdot z} \cap Z^T)$.*

Atiyah [5] suggested that the convexity of $\mu(T^{\mathbb{C}} \cdot x)$ for $x \in Z$ could be used to give an alternative proof of Theorem 1.3.7 by showing that there always exists an orbit $T^{\mathbb{C}} \cdot z$ such that $\mu(\overline{T^{\mathbb{C}} \cdot z}) = \mu(Z)$. Duistermaat [28] proved that the set of points $z \in Z$ for which $\mu(\overline{T^{\mathbb{C}} \cdot z}) = \mu(Z)$ is non-empty and dense. Biliotti and Ghigi [9] gave an alternative proof of this result and also show that the set is open. This new approach adds to the understanding of some basic results in the subject.

Let $\| \cdot \|$ be the norm induced by the $\mathrm{Ad}(U)$-invariant scalar product fixed on $\mathfrak{u}$.

Definition 1.3.3. *The function $F : Z \to \mathbb{R}$ given by*

$$F(z) := \frac{1}{2} \parallel \mu(z) \parallel^2$$

for $z \in Z$, is called the norm square of the momentum map.

The critical point of F is given below.

Lemma 1.3.9. *[30] The gradient of F is given by*

$$\nabla F = J\mu(z)_Z$$

for $z \in Z$. Hence, $z \in Z$ is a critical point of F if and only if $\mu(z)_Z = 0$.

The norm square F of the momentum map is in general not Morse-Bott. In fact, the critical set C of F has singularities in general so that F cannot be a Morse-Bott function. Nevertheless, Kirwan defines a smooth stratification of X by showing that F is a minimally degenerate function on Z [54].

Let x_t denote the flow of $-\nabla F$. Let S_C be a stable manifold associated with the component C of the critical set of F where F takes constant value (see Section 1.2). Duistermaat has shown that the limit of the negative gradient flow of F through a point $z \in Z$ is a single point.

Theorem 1.3.10. *[59, 1.1]. The following holds under the above assumption.*

- *for each $z \in Z$, the limit set of the trajectory x_t is a single point.*
- *for each connected component C of critical points of F, the map*

  $$\varphi : [0, \infty] \times S_C \to C, \quad (t, z) \mapsto x_t(z)$$

  is a deformation retraction.

A Weyl chamber decomposition of M in $\mathfrak{t}$ is a fundamental domain for the coadjoint action of the compact Lie group on $\mathfrak{t}^*$. So, the intersection of the image of a momentum map with a Weyl chamber determines this image completely, since the image of a momentum map is invariant under the coadjoint action.

Guillemin and Sternberg conjectured in [36] that the intersection of image of the momentum map associated with the actions of compact Lie group on compact symplectic manifolds with any Weyl chamber in the dual of the Lie algebra of a maximal torus is a convex polytope. They prove this result in the case when the group is a torus. Kirwan [55] gave a prove of this conjecture in a more general settings. More precisely, Kirwan proved the following.

Theorem 1.3.11. *(Non-Abelian Convexity). Let $\mu : Z \to \mathfrak{u}$ be the momentum map for the U-action on Z and suppose $T \subset U$ is a maximal torus with Lie algebra $\mathfrak{t}$ and $\mathfrak{t}_+$ is a positive Weyl chamber. Then $\mu(Z) \cap \mathfrak{t}_+$ is a polytope.*

The proof of this theorem relies on the stratification of Z defined by the norm square. The fundamental fact, in this case, is that the stratum corresponding to the minimum is open and it is the unique open stratum. This is the main point to prove the non-Abelian convexity result.

Heizner P. and Huckleberry A. [43] proved the convexity properties of the momentum map in a non-compact Kählerian settings. Let $Z_{\max} := \{z \in Z : \dim(U \cdot x) \text{ is maximal}\}$.

Theorem 1.3.12. *[43]. For a G-irreducible complex G-space Z, the set $\mu(Z_{max}) \cap \mathfrak{t}_+$ is convex.*

Another important application of momentum map is the Kähler reduction.

Theorem 1.3.13. *(Kähler Reduction): Suppose U acts freely on $\mu^{-1}(0)$. Then, the orbit space $\mu^{-1}(0)/U$ is a manifold and there is a Kähler form ω_0 on $\mu^{-1}(0)/U$. $(\mu^{-1}(0)/U, \omega_0)$ is called Kähler reduction.*

Factorization problem for group actions in both algebraic geometry and complex geometry is an important subject. Defining the notion of good quotient for the $U^{\mathbb{C}}$-action on Z has been greatly studied in the literature. A naive example will be the orbit space $Z/U^{\mathbb{C}}$ but the orbit space is not always Hausdorff. For instance, consider $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acting on $\mathbb{C}^n$; $\lambda \mapsto$ multiplication by λ. The orbit space $\mathbb{C}^n/\mathbb{C}^* = \mathbb{C}P^{n-1} \sqcup \{0\}$ is not Hausdorff. Indeed, the only open set containing the point $\{0\}$ is the whole space. However, $\mathbb{C}^n \setminus \{0\}/\mathbb{C}^* = \mathbb{C}P^{n-1}$. In this case the point 0 was removed which can be considered as the 'bad orbit'. We can therefore restrict the action on a "big" $U^{\mathbb{C}}$-invariant subset $M \subset Z$ obtained by removing some "bad" $U^{\mathbb{C}}$-orbits to obtain a "good quotient" $Z//U^{\mathbb{C}}$. Another application of momentum map is to define the $U^{\mathbb{C}}$-invariant subset M.

Definition 1.3.4. *A point $z \in Z$ is called*

- *μ-polystable if $U^{\mathbb{C}} \cdot z \cap \mu^{-1}(0) \neq \emptyset$.*
- *μ-semistable if $\overline{U^{\mathbb{C}} \cdot z} \cap \mu^{-1}(0) \neq \emptyset$.*
- *μ-stable if $U^{\mathbb{C}} \cdot z \cap \mu^{-1}(0) \neq \emptyset$ and $U^{\mathbb{C}}_z := \{g \in U^{\mathbb{C}} | gz = z\}$ is discrete.*

Let Z^{ss}, Z^{ps} denotes sets of μ-semistable and μ-polystable respectively. Z^{ss} is open in Z. The closure in Z^{ss} of every μ-semistable $U^{\mathbb{C}}$-orbit contains a unique μ-polystable orbit.

Theorem 1.3.14. *(Heinzner-Huckleberry-Loose). There is a good quotient $Z^{ss} \to Q$ such that*

- *The induced morphism $\mu^{-1}(0)/U \to Q$ is a homeomorphism.*
- *$z_1, z_2 \in Z^{ss}$ are equivalent if and only if $\mu^{-1}(0) \cap \overline{U^{\mathbb{C}} \cdot z_1} \cap \overline{U^{\mathbb{C}} \cdot z_2} \neq \emptyset$.*

The focus of this work is to prove most of the results presented in this section in a more general setting and introduce new results in this setting. Instead of considering the action of $U^{\mathbb{C}}$ on Z, we will consider the action of a closed subgroup satisfying some compatibility condition. This notion is discussed in section 1.4.

1.4 Compatible subgroups

Let U be a compact connected Lie group with Lie algebra $\mathfrak{u}$. Let $U^{\mathbb{C}}$ be the complexification of U with Lie algebra $\mathfrak{u}^{\mathbb{C}}$. There is a Cartan decomposition

$$\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} + i\mathfrak{u}$$

with a conjugation map $\theta : \mathfrak{u}^{\mathbb{C}} \to \mathfrak{u}^{\mathbb{C}}$, $\beta + i\xi \mapsto \beta - i\xi$, $\beta, \xi \in \mathfrak{u}$. We also use θ to denote the corresponding antiholomorphic involution of $U^{\mathbb{C}}$. The map $f : U \times i\mathfrak{u} \to U^{\mathbb{C}}$; $f(u, \beta) = u\exp(\beta)$ is a diffeomorphism. The decomposition $U^{\mathbb{C}} = U \exp(i\mathfrak{u})$ is referred to as the Cartan decomposition of $U^{\mathbb{C}}$ [56].

Definition 1.4.1. *A closed real subgroup G of $U^{\mathbb{C}}$ is compatible with the Cartan decomposition $U^{\mathbb{C}} = U \exp i\mathfrak{u}$ if the map $K \times \mathfrak{p} \to G$, $(k, \beta) \mapsto k \exp \beta$ is a diffeomorpism, where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G.*

The restriction of f to $K \times \mathfrak{p}$ is then a diffeomorphism onto G. It follows that K is a maximal compact subgroup of G and that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of K. The following Lemma asserts that G has finitely many connected component.

Lemma 1.4.1. *[25]. If $G \subset U^{\mathbb{C}}$ is a closed θ-invariant subgroup, then G is compatible if and only if it has only finitely many connected components.*

Proof. Suppose G is compatible. Then it is θ-invariant and retracts onto K, a maximal compact subgroup of G. Since a compact Lie group has a finitely many connected components, it follows that G has finitely many connected components. Conversely suppose G has finitely many connected component. Let G^0 be a connected component of G and G/G^0 is finite. Since G is closed, $f(K \times \mathfrak{p})$ is a closed subset of G. Since G is θ-invariant, $f(K \times \mathfrak{p})$ has the same dimension as G and is therefore also open. Therefore it contains G^0 and is a union of connected components of G. Given $g \in G$, let $g = u \exp \beta$ with $u \in U$ and $\beta \in i\mathfrak{u}$. Then $g\theta(g^{-1}) = u \exp(\beta)\theta(\exp(-\beta))u^{-1}) = \exp(2 \operatorname{Ad}(u)(\beta))$. But G/G^0 is finite, this means that there is a natural number $N > 0$ such that $\left(g\theta(g^{-1})\right)^N = \exp(2N \operatorname{Ad}(u)(\beta)) \in G^0 \subset f(K \times \mathfrak{p})$. Hence $\operatorname{Ad}(u)(\beta) \in \mathfrak{p}$, $u = u \exp(-\beta)u^{-1}u \exp(\beta) = \exp(- \operatorname{Ad}(u)\beta)g \in G \cap U = K$ and $\beta \in \mathfrak{p}$.

□

Corollary 1.4.1.1. *[15, Lemma 7a]. If $G \subset U^{\mathbb{C}}$ is a compatible subgroup, and $H \subset G$ is closed and θ-invariant, then H is compatible if and only if H has only finitely many connected components.*

Example 1.4.1. *$SL(n, \mathbb{R}) \subset SL(n, \mathbb{C}) = U(n) \exp(i\mathfrak{su}(n))$ is a compatible subgroup. Indeed, the Cartan decomposition of $SL(n, \mathbb{C})$ is given by the polar decomposition of complex matrices. Let $A \in \mathrm{SL}(n, \mathbb{R})$. Then $P = \sqrt{AA^*} = \sqrt{AA^t}$ is a positive define symmetric matrice and $AP^{-1} \in \mathrm{SO}(n)$. Therefore $\mathrm{SO}(n)$ is a maximal compact subgroup of $\mathrm{SL}(n, \mathbb{R})$ and the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ decomposed as $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{sym}_0(n)$, where $\mathfrak{sym}_0(n)$ denotes the set of symmetric matrices of trace zero.*

Any compact Lie group U can be embedded in the general linear group $\mathrm{Gl}(n, \mathbb{C})$ for some n which induces a closed embedding of $U^{\mathbb{C}}$. Then, any compatible subgroup is a closed linear group. Moreover $\mathfrak{g}$ is a real reductive Lie algebra and so, $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Let G_{ss} denote the analytic subgroup whose Lie algebra is $[\mathfrak{g}, \mathfrak{g}]$. Then G_{ss} is closed and

G^0, the connected component of the identity of G, is given by $G^0 = Z(G)^0 \cdot G_{ss}$ [56, p. 442]. Since $[\mathfrak{g}, \mathfrak{g}]$ is θ-invariant and G_{ss} is connected, then G_{ss} is also θ-invariant. Since it is also closed, by Lemma 1.4.1.1 G_{ss} is compatible.

Let H be a Lie group with Lie algebra $\mathfrak{h}$ and $\beta \in \mathfrak{h}$. Set

$$\mathfrak{h}^\beta := \{\eta \in \mathfrak{h} : [\eta, \beta] = 0\} \quad \text{and} \quad H^\beta = \{g \in H : \operatorname{Ad} g(\beta) = \beta\}$$

Let $A \subset \mathfrak{p}$ be any subset of $\mathfrak{p}$.

Lemma 1.4.2. *[56, Proposition 7.25]. If $G \subset U^{\mathbb{C}}$ is a compatible subgroup. Then G^A is compatible. Indeed, $G^A = K^A \exp(\mathfrak{p}^A)$, where $K^A = K \cap G^A$ and $\mathfrak{p}^A = \{x \in \mathfrak{p} : [x, A] = 0\}$.*

1.5 Parabolic Subgroup

Given $\beta \in \mathfrak{p}$, we define

$$G^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) \text{ exists}\}$$

$$R^{\beta+} := \{g \in G : \lim_{t \to -\infty} \exp(t\beta) g \exp(-t\beta) = e\}$$

$$\mathfrak{r}^{\beta+} := \bigoplus_{\lambda > 0} V_\lambda(\mathrm{ad}\beta)$$

$$G^{\beta-} : \{g \in G : \lim_{t \to +\infty} \exp(t\beta) g \exp(-t\beta) = e \text{ exists}\}$$

$$R^{\beta-} := \{g \in G : \lim_{t \to +\infty} \exp(t\beta)$$

$$\mathfrak{r}^{\beta-} := \bigoplus_{\lambda < 0} V_\lambda(\mathrm{ad}\beta).$$

Note that $G^{\beta-} = G^{-\beta+} = \theta(G^{\beta+})$, $\theta(R^{\beta+}) = R^{\beta-}$ and $G^\beta = G^{\beta+} \cap G^{\beta-}$. If G is connected, the group $G^{\beta+}$, respectively $G^{\beta-}$, is a parabolic subgroup of G with unipotent radical $R^{\beta+}$, respectively $R^{\beta-}$ and Levi factor G^β and any parabolic subgroup is given by $G^{\beta+}$ for some $\beta \in \mathfrak{p}$ [15]. $R^{\beta+}$ is connected with Lie algebra $\mathfrak{r}^{\beta+}$. Hence $R^{\beta-}$ is connected with Lie algebra $\mathfrak{r}^{\beta-}$. The parabolic subgroup $G^{\beta+}$ is the semidirect product of G^β with $R^{\beta+}$ and we have the projection $\pi^{\beta+} : G^{\beta+} \longrightarrow G^\beta$, $\pi^{\beta+}(g) = \lim_{t \mapsto -\infty} exp(t\beta) g \exp(-t\beta)$.

Analogously, $G^{\beta-}$ is the semidirect product of G^{β} with $R^{\beta-}$ and we have the projection $\pi^{\beta-} : G^{\beta-} \longrightarrow G^{\beta}$, $\pi^{\beta-}(g) = \lim_{t\mapsto+\infty} exp(t\beta)g\exp(-t\beta)$. In particular, $\mathfrak{g}^{\beta+} = \mathfrak{g}^{\beta} \oplus \mathfrak{r}^{\beta+}$, respectively $\mathfrak{g}^{\beta-} = \mathfrak{g}^{\beta} \oplus \mathfrak{r}^{\beta-}$. The following result is well-known.

Proposition 1.5.1. *For any $\beta \in \mathfrak{p}$, we have $G = KG^{\beta+}$.*

Proof. If G is connected, the result is well-known, see for instance [15, Lemma 9] and [45, Lemma 4.1]. Since $G = KG^{o}$, it follows that $G = KG^{o} = K(G^{o})^{\beta+} = KG^{\beta+}$, concluding the proof. ☐

Let $\xi \in \mathfrak{u}$. The standard notation for parabolic subgroups of complex reductive groups, see for instance [54], is given by

$$U^{\mathbb{C}}(\xi) = \{g \in U^{\mathbb{C}} : \lim_{t\mapsto-\infty} \exp(i\xi)g\exp(-i\xi)\,\text{exists}\,\}.$$

If U is connected, it is well-known that $U^{\mathbb{C}}(\xi)$ is connected and it contains a Borel subgroup, that it a maximal solvable subgroup of $U^{\mathbb{C}}$ [1]. Note that if $\beta \in i\mathfrak{u}$, then $U^{\mathbb{C}}(-i\beta)$ corresponds to $(U^{\mathbb{C}})^{\beta+}$ in our notation.

Remark 1.5.2. *One major difference between the complex and real case we are considering in this project is that the parabolic subgroup of $U^{\mathbb{C}}$ is connected while it is not always so for a parabolic subgroup of G. For instance,*

$$\left\{ \begin{pmatrix} z & \omega \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^{*}, \omega \in \mathbb{C} \right\}$$

is parabolic in $SL(2,\mathbb{C})$ and it is connected. The Levi factor is given as

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^{*}, \right\}$$

and the Unipotent radical is given as

$$\left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} : \omega \in \mathbb{C} \right\}.$$

While

$$\left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x \in \mathbb{R}^*, y \in \mathbb{R} \right\}$$

is parabolic in $SL(2,\mathbb{R})$ *and it is not connected. The Levi factor*

$$\left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in \mathbb{R}^*, \right\}$$

has two connected components and the Unipotent radical is given as

$$\left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} : y \in \mathbb{R} \right\}.$$

1.6 Gradient map

Let U be a compact Lie group with Lie algebra $\mathfrak{u}$ and $U^{\mathbb{C}}$ its complexification. Let (Z,ω) be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that U preserves ω and that there is a U-equivariant momentum map $\mu : Z \to \mathfrak{u}^*$.

Let $\langle \cdot, \cdot \rangle$ denote the real part of B as defined in Section 1.3. We can identify $\mathfrak{u}$ and $\mathfrak{u}^*$ by means of $-\langle \cdot, \cdot \rangle$ and consider the momentum map as $\mathfrak{u}$-valued map.

Let $G \subset U^{\mathbb{C}}$ be a compatible subgroup of $U^{\mathbb{C}}$. Then we have $\mathfrak{p} \subset i\mathfrak{u}$. If $z \in Z$, then the orthogonal projection of $i\mu(z)$ onto $\mathfrak{p}$ with respect to $\langle \cdot, \cdot \rangle$ defines a K-equivariant map $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$. [44, 45, 46]. In other words, we define $\mu_{\mathfrak{p}}$ requiring that for any $\beta \in \mathfrak{p}$, we have

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}(z), \beta \rangle = \langle i\mu(z), \beta \rangle = -\langle \mu(z), i\beta \rangle = \mu^{-i\beta}. \tag{1.5}$$

Let $(\, , \,)$ be the Kähler metric associated with ω, i.e. $(v,w) = \omega(v, Jw)$ for all $z \in Z$ and $v, w \in T_z Z$, where J denotes the complex structure on TZ. Then β_Z is the gradient of $\mu_{\mathfrak{p}}^{\beta}$. Indeed,

$$\mathrm{grad}\mu_{\mathfrak{p}}^{\beta} = \mathrm{grad}\mu^{-i\beta} = J(-i\beta_Z) = \beta_Z.$$

Definition 1.6.1. *The map* $\mu_{\mathfrak{p}} : Z \longrightarrow \mathfrak{p}$ *is called a gradient map and it is such that*

- $\mu_{\mathfrak{p}}(kz) = \mathrm{Ad}(k)(\mu_{\mathfrak{p}}(z)), \forall z \in Z;$

- *for any $\beta \in \mathfrak{p}$, let $\mu_{\mathfrak{p}}^{\beta} \in C^{\infty}(Z)$, Then $\mu_{\mathfrak{p}}^{\beta}(z) = \mu^{-i\beta}(z)$ and grad$\mu_{\mathfrak{p}}^{\beta} = \beta_Z$, where the gradient is computed with respect to the Riemannian metric induced by ω.*

Let X be a G-invariant locally closed real submanifold of Z and denote the restriction of $\mu_{\mathfrak{p}}$ to X by $\mu_{\mathfrak{p}}$. Then $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ is a K-equivariant map such that grad$\mu_{\mathfrak{p}}^{\beta} = \beta_X$, where the gradient is computed with respect to the induced Riemannian metric on X that is also denote by $(\cdot, \cdot)$. Similarly, $\perp$ denotes perpendicularity relative to the Riemannian metric on X.

Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. Then $A = \exp(\mathfrak{a})$ is a compatible Lie group. The A-gradient map on X associated to μ is given by $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$, where $\pi_{\mathfrak{a}} : \mathfrak{p} \longrightarrow \mathfrak{a}$ denotes the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$.

Example 1.6.1. *Consder the action of a special linear group $SL(n, \mathbb{C})$ on a complex projective space $\mathbb{C}P^n$. $SL(n, \mathbb{R}) \subset SL(n, \mathbb{C})$ is compatible (Example1.4.1) and $\mathbb{R}P^n \subset \mathbb{C}P^n$ is $SL(n, \mathbb{R})$-invariant. One may check that the associated gradient map is given as*

$$\mu_{\mathfrak{sym}_0(n)}([x]) = \frac{1}{2}\left(\frac{xx^t}{\parallel x \parallel^2} - \frac{1}{n} Id_n\right)$$

Remark 1.6.1. *If $\mathfrak{g} \subset \mathfrak{u}^{\mathbb{C}}$ is compatible with respect to the Cartan decomposition $\mathfrak{u}^{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$, then $\langle \cdot, \cdot \rangle_{\mathfrak{g}\times\mathfrak{g}}$ is non-degenerate. Hence $\mathfrak{u}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$, where $\mathfrak{g}^{\perp} = \{v \in \mathfrak{u}^{\mathbb{C}} : \langle v, \mathfrak{g} \rangle = 0\}$.*

Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. Maximal abelian subalgebra of $\mathfrak{p}$ is conjugate via $\mathrm{Ad}(K)$ in the sense that if $\mathfrak{a}'$ is another maximal abelian subalgebra of $\mathfrak{p}$, then there exists $k \in K$ such that $\mathrm{Ad}(k)(\mathfrak{a}') = \mathfrak{a}$. k can be chosen in $K_{ss} = K \cap G_{ss}$. Therefore, $\mathfrak{p} = \bigcup_{k \in K_{ss}} \mathrm{Ad}(k)\mathfrak{a}$. Let $A = \exp(\mathfrak{a})$ be a subgroup of G with Lie algebra $\mathfrak{a}$.

Theorem 1.6.2. *[56, Theorem 7.39] G can be decomposed as $G = KAK$.*

Proof. If G is connected, then the result is well-known [56]. Since $G = KG^o$, it follows that $G = K(K^oAK^o) = KAK$. □

For any subspace $\mathfrak{m}$ of $\mathfrak{g}$ and $x \in X$, let

$$\mathfrak{m} \cdot x := \{\xi_X(x) : \xi \in \mathfrak{m}\} \quad \text{and} \quad \mathfrak{p}_x := \{\xi \in \mathfrak{p} : \xi_X(x) = 0\}.$$

We will now recall some of the properties of the gradient map.

Lemma 1.6.3. *Let $x \in X$ and let $\beta \in \mathfrak{p}$. Then either $\beta_X(x) = 0$ or the function $t \mapsto \mu_{\mathfrak{p}}^{\beta}(exp(t\beta) \cdot x)$ is strictly increasing.*

Proof. Let $f(t) = \mu_{\mathfrak{p}}^{\beta}(\exp(t\beta) \cdot x) = \langle \mu_{\mathfrak{p}}(\exp(t\beta) \cdot x), \beta \rangle$. Then

$$f'(t) = (\beta_X(\exp(t\beta) \cdot x), \beta_X(\exp(t\beta) \cdot x) \geq 0.$$

But for $f(0) = f(1)$, we must have $f'(0) = 0$ which implies that $\beta \in \mathfrak{p}_x$. Otherwise, $f(t)$ is strictly increasing. $\square$

Lemma 1.6.4. *Let $x \in X$. Then*

$$ker\ d\mu_{\mathfrak{p}}(x) = (\mathfrak{p} \cdot x)^{\perp}$$

Proof. From (1.5), $v \in \ker\, d\mu_{\mathfrak{p}}(x)$ if and only if for all $\beta \in \mathfrak{p}$

$$\begin{aligned}
&\langle d\mu_{\mathfrak{p}}(x)(v), \beta \rangle = 0 \\
&\iff d\mu_{\mathfrak{p}}^{\beta}(v) = 0 \\
&\iff \langle \beta_X(x), v \rangle = 0.
\end{aligned}$$

$\square$

Lemma 1.6.5. *Let $x \in X$. The following are equivalent:*

a) $d\mu_{\mathfrak{p}} : T_x X \to \mathfrak{p}$ *is onto*

b) $d\mu_{\mathfrak{p}} : \mathfrak{p} \cdot x \to \mathfrak{p}$ *is onto*

c) $\dim \mathfrak{p} \cdot x = \dim \mathfrak{p}$

Proof.

$$\ker\, d\mu_{\mathfrak{p}}(x) = (\mathfrak{p} \cdot x)^{\perp},$$

it follows that $d\mu_{\mathfrak{p}}(x)$ is surjective if and only if $d\mu_{\mathfrak{p}} : \mathfrak{p} \cdot x \to \mathfrak{p}$ is surjective and so by dimensional reason if and only if $\dim \mathfrak{p} \cdot x = \dim \mathfrak{p}$ concluding the proof.

$\square$

Lemma 1.6.6. *If $x \in X$ and $\beta \in \mathfrak{p}$, then $\mu_{\mathfrak{p}}(exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(x)$ if and only if $\beta_X(x) = 0$.*

Proof. Suppose $\mu_{\mathfrak{p}}(\exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(x)$. Define the curve $x : [0, 1] \to Z$ by $x(t) = \exp(t\beta) \cdot x$ and set $y(t) = \langle \mu_{\mathfrak{p}}(x(t)), \beta\rangle$. Then, $\dot{x} = \beta_X(x)$, $y(0) = y(1)$ and

$$\begin{aligned} y'(t) &= \frac{d}{dt}\langle \mu(x(t)), -i\beta\rangle \\ &= \langle \beta_X(x), \beta_X(x)\rangle = (\beta_X(x(t)), \beta_X(x(t))) \\ &= \| \beta_X(x(t)) \|^2 \geq 0. \end{aligned}$$

We must have $y'(t) = 0$ for all $t \in [0, 1]$ and this happens if and only if $\beta_X(x(t)) = 0$. This implies that $\exp(\beta) \cdot x = x$. ☐

Corollary 1.6.6.1. *If $G = exp(\mathfrak{a})$, $\mathfrak{a} \subset \mathfrak{p}$ an Abelian subalgebra of $\mathfrak{p}$. Then*

$$G \cdot x \cap \mu_{\mathfrak{a}}^{-1}(0) = \{x\}$$

The next property asserts that two points in the zero set of the gradient map are equivalent under G if and only if they are equivalent under K.

Lemma 1.6.7. *Let $x, y \in \mu_{\mathfrak{p}}^{-1}(0)$. If $y \in G \cdot x$, then $y \in K \cdot x$, i. e., $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) = K \cdot x$*

Proof. Let $g = k\exp(\beta) \in G$, where $k \in K$ and $\beta \in \mathfrak{p}$ be such that $y = gx$. Then $\mu_{\mathfrak{p}}(k\exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(\exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(x) = 0$. By lemma (1.6.6), we have $y = k(\exp(\beta)) \cdot x = k \cdot x$. ☐

Using lemma (1.6.6), we can also show that the stabilizer G_x of $x \in \mu_{\mathfrak{p}}^{-1}(0)$ is compatible with respect to the Cartan decomposition of $U^{\mathbb{C}}$. This is stated and proved below.

Lemma 1.6.8. *Given $x \in X$. If $\mu_{\mathfrak{p}}(x) = 0$, then*

$$G_x = K_x \ exp \ \mathfrak{p}_x \backsimeq K_x \times \mathfrak{p}_x$$

Proof. Pick $g \in G_x$. Then $g = k\exp(\beta)$ and $k\exp(\beta) \cdot x = x$. Hence $0 = \mu_{\mathfrak{p}}(k\exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(\exp(\beta) \cdot x) = \mu_{\mathfrak{p}}(x)$. By lemma (1.6.6), $\beta \in \mathfrak{p}_x$. This implies that $k \in K_x$ and so $G_x = K_x \exp \mathfrak{p}_x$. ☐

We recall the Slice Theorem, see [45]. For any Lie group G, a closed subgroup H and any set S with an H-action, the G-bundle over G/H associated with the H-principal bundle $G \to G/H$ is denoted by $G \times^H S$. This is the orbit space of the H-action on $G \times S$ given by $h \cdot (g, s) = (gh^{-1}, h \cdot s)$ where $g \in G$, $s \in S$ and $h \in H$. The H-orbit of (g, s), considered as a point in $G \times^H S$, is denoted by $[g, s]$.

Theorem 1.6.9. *[Slice Theorem [45, Thm. 3.1]] If $x \in X$ and $\mu_{\mathfrak{p}}(x) = 0$, there are a G_x-invariant decomposition $T_xX = \mathfrak{g} \cdot x \oplus W$, open G_x-invariant subsets $S \subset W$, $\Omega \subset X$ and a G-equivariant diffeomorphism $\Psi : G \times^{G_x} S \to \Omega$, such that $0 \in S, x \in \Omega$ and $\Psi([e, 0]) = x$.*

We can relax the condition that $\mu_{\mathfrak{p}}(x) = 0$ and show that there is a slice for all $x \in X$ as follow. For any $x \in X$, let $\mu_{\mathfrak{p}}(x) := \beta \in \mathfrak{p}$. By Lemma 1.4.2 G^β is compatible and

$$G^\beta = K^\beta \exp(\mathfrak{p}^\beta),$$

where $K^\beta = K \cap G^\beta = \{g \in K : \mathrm{Ad}(g)(\beta) = \beta\}$ and $\mathfrak{p}^\beta = \{v \in \mathfrak{p} : [v, \beta] = 0\}$. It is well known that $U^{i\beta}$ is a compact Lie group [49] and its complexification is given by $(U^{\mathbb{C}})^{i\beta} = (U^{i\beta})^{\mathbb{C}}$. The $U^{i\beta}$-action on Z is Hamiltonian with momentum map $\mu_{\mathfrak{u}^{i\beta}} : Z \longrightarrow \mathfrak{u}^{i\beta}$ given by $\pi_{\mathfrak{u}^{i\beta}} \circ \mu$, where $\pi_{\mathfrak{u}^{i\beta}} : \mathfrak{u} \longrightarrow \mathfrak{u}^{i\beta}$ is the orthogonal projection of $\mathfrak{u}$ onto $\mathfrak{u}^{i\beta}$. Now, $G^\beta \subset (U^{\mathbb{C}})^{i\beta}$ is compatible and $\pi_{\mathfrak{p}^\beta} \circ \mu_{\mathfrak{p}}$ is the G^β-gradient map, where $\pi_{\mathfrak{p}^\beta} : \mathfrak{p} \longrightarrow \mathfrak{p}^\beta$ is the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{p}^\beta$.

Corollary 1.6.9.1. *If $x \in X$ and $\mu_{\mathfrak{p}}(x) = \beta$, there are a G^β-invariant decomposition $T_xX = \mathfrak{g}^\beta \cdot x \oplus W$, open G^β-invariant subsets $S \subset W$, $\Omega \subset X$ and a G^β-equivariant diffeomorphism $\Psi : G^\beta \times^{G_x} S \to \Omega$, such that $0 \in S, x \in \Omega$ and $\Psi([e, 0]) = x$.*

Proof. $U^{i\beta}$-action on $\mathfrak{u}^{i\beta}$ fixes $i\beta$. Hence, $\mu_{\mathfrak{u}^{i\beta}}$ defines a $U^{i\beta}$-equivariant shifted momentum map $\widehat{\mu_{\mathfrak{u}^{i\beta}}} = Z \to \mu_{\mathfrak{u}^{i\beta}}$; $\widehat{\mu_{\mathfrak{u}^{i\beta}}} := \mu_{\mathfrak{u}^{i\beta}} + i\beta$. The corollary then follows by applying Theorem 1.6.9 to the action of G^β with the associated gradient map $\widehat{\mu_{\mathfrak{p}^\beta}} : X \to \mathfrak{p}^\beta$; $\widehat{\mu_{\mathfrak{p}^\beta}} = \mu_{\mathfrak{p}^\beta} - \beta$, where $\mu_{\mathfrak{p}^\beta}$ denotes the projection of $\mu_{\mathfrak{p}}$ onto $\mathfrak{p}^\beta$. □

If $\beta \in \mathfrak{p}$, then β_X is a vector field on X, i.e. a section of TX. For $x \in X$, the differential is a map $T_xX \to T_{\beta_X(x)}(TX)$. If $\beta_X(x) = 0$, there is a canonical splitting

$T_{\beta_X(x)}(TX) = T_xX \oplus T_xX$. Accordingly $d\beta_X(x)$ splits into a horizontal and a vertical part. The horizontal part is the identity map. We denote the vertical part by $d\beta_X(x)$. It belongs to $\operatorname{End}(T_xX)$. Let $\{\varphi_t = \exp(t\beta)\}$ be the flow of β_X. There is a corresponding flow on TX. Since $\varphi_t(x) = x$, the flow on TX preserves T_xX and it is given by $d\varphi_t(x) \in \mathrm{Gl}(T_xX)$. Thus we get a linear $\mathbb{R}$-action on T_xX with infinitesimal generator $d\beta_X(x)$.

Corollary 1.6.9.2. *If $\beta \in \mathfrak{p}$ and $x \in X$ is a critical point of $\mu_{\mathfrak{p}}^{\beta}$, then there are open invariant neighbourhoods $S \subset T_xX$ and $\Omega \subset X$ and an $\mathbb{R}$-equivariant diffeomorphism $\Psi : S \to \Omega$, such that $0 \in S, x \in \Omega$, $\Psi(0) = x$. (Here $t \in \mathbb{R}$ acts as $d\varphi_t(x)$ on S and as φ_t on Ω.)*

Proof. Since $\exp : \mathfrak{p} \longrightarrow G$ is a diffeomorphism onto the image, the subgroup $H := \exp(\mathbb{R}\beta)$ is closed and so it is compatible. Hence, it is enough to apply Corollary 1.6.9.1 to the H-action on X and the value at x of the corresponding gradient map. ☐

Assume now that $\beta \in \mathfrak{p}$ and that $x \in \operatorname{Crit}(\mu_{\mathfrak{p}}^{\beta})$. Let $D^2\mu_{\mathfrak{p}}^{\beta}(x)$ denote the Hessian, which is a symmetric operator on T_xX such that

$$(D^2\mu_{\mathfrak{p}}^{\beta}(x)v, v) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0}(\mu_{\mathfrak{p}}^{\beta} \circ \gamma)$$

where γ is a smooth curve, $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. Denote by V_- (respectively V_+) the sum of the eigenspaces of the Hessian of $\mu_{\mathfrak{p}}^{\beta}$ corresponding to negative (resp. positive) eigenvalues. Denote by V_0 the kernel. Since the Hessian is symmetric we get an orthogonal decomposition

$$T_xX = V_- \oplus V_0 \oplus V_+. \tag{1.6}$$

Let $\alpha : G \to X$ be the orbit map: $\alpha(g) := gx$. The differential $d\alpha_e$ is the map $\xi \mapsto \xi_X(x)$.

Proposition 1.6.10. *If $\beta \in \mathfrak{p}$ and $x \in \operatorname{Crit}(\mu_{\mathfrak{p}}^{\beta})$ then*

$$D^2\mu_{\mathfrak{p}}^{\beta}(x) = d\beta_X(x).$$

Moreover $d\alpha_e(\mathfrak{r}^{\beta\pm}) \subset V_{\pm}$ and $d\alpha_e(\mathfrak{g}^{\beta}) \subset V_0$. If X is G-homogeneous these are equalities.

Proof. The first statement is proved in [45, Prop. 2.5]. Denote by $\rho : G_x \to T_x X$ the isotropy representation: $\rho(g) = dg_x$. Observe that α is G_x-equivariant where G_x acts on G by conjugation, hence $d\alpha_e$ is G_x-equivariant, where G_x acts on $\mathfrak{g}$ by the adjoint representation and on $T_x X$ by the isotropy representation. Since $\beta_X(x) = 0$, $\exp(t\beta) \in G_x$ for any t and $d\alpha_e$ is $\mathbb{R}$-equivariant. Therefore it interchanges the infinitesimal generators of the $\mathbb{R}$-actions, i.e. $d\alpha_e \circ \mathrm{ad}\beta = d\beta_X = D^2\mu_{\mathfrak{p}}^{\beta}(x)$. The required inclusions follow. If G acts transitively on X we must have $T_x X = d\alpha_e(\mathfrak{g})$. Hence the three inclusions must be equalities. □

Corollary 1.6.10.1. *For every $\beta \in \mathfrak{p}$, $\mu_{\mathfrak{p}}^{\beta}$ is a Morse-Bott function.*

Proof. Let $X^\beta := \{x \in X : \beta_X(x) = 0\}$. Corollary 1.6.9.2 implies that X^β is a smooth submanifold. Since $T_x X^\beta = V_0$ for $x \in X^\beta$, the first statement of Proposition 1.6.10 shows that the Hessian is nondegenerate in the normal directions. □

Corollary 1.6.10.2. *If X is G-homogenous then G^β-orbits are open and closed in* $\mathrm{Crit}\, \mu_{\mathfrak{p}}^{\beta}$.

Proof. Since $T_x \mathrm{Crit}\, \mu_{\mathfrak{p}}^{\beta} = T_x G^\beta \cdot x$ for $x \in \mathrm{Crit}\, \mu_{\mathfrak{p}}^{\beta}$, the result follows. □

Proposition 1.6.11. *X^β is G^β-invariant. Moreover, if $x \in X^\beta$ and $g \in G^\beta$, then $\mu_{\mathfrak{p}}^{\beta}(x) = \mu_{\mathfrak{p}}^{\beta}(gx)$.*

Proof. Let $x \in X^\beta$ and let $g \in G^\beta$. Since $\beta_X(gx) = (\mathrm{d}g)_x(\beta_X(x))$, it follows $gx \in X^\beta$. By Lemma 1.4.2, $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$. Hence $g = k \exp(\xi)$, where $k \in K^\beta$ and $\xi \in \mathfrak{p}^\beta$. By the K-equivariance of the G-gradient map we have $\mu_{\mathfrak{p}}^{\beta}(gx) = \mu_{\mathfrak{p}}^{\beta}(\exp(\xi)x)$. Let $y(t) = \mu_{\mathfrak{p}}^{\beta}(\exp(t\xi)x)$. Then

$$\dot{y}(t) = (\beta_X(\exp(t\xi)x), \beta_X(\exp(t\xi)x)).$$

By the first part of the proof, $\beta_X(\exp(\xi)x) = 0$, and so $\dot{y}(t) = 0$. This implies $\mu_{\mathfrak{p}}^{\beta}(gx) = \mu_{\mathfrak{p}}^{\beta}(\exp(\xi)x) = \mu_{\mathfrak{p}}^{\beta}(x)$, and so the result follows. □

Proposition 1.6.12. *The restriction $(\mu_{\mathfrak{p}})_{|_{X^\beta}}$ takes value on $\mathfrak{p}^\beta$ and so it coincides with= $(\mu_{\mathfrak{p}^\beta})_{|_{X^\beta}}$.*

Proof. If $G = U^{\mathbb{C}}$ then the proof is easy to check. Indeed, let $x \in Z^{\beta} = Z^{i\beta}$. Since $\exp(ti\beta)x = x$, by the U-equivariance of the momentum map, we have

$$\mu(x) = \mu(\exp(ti\beta)x) = \mathrm{Ad}(\exp(ti\beta))(\mu(x)).$$

Taking the limit we get $[\mu(x), i\beta] = 0$. In the general case it suffices to prove that $i\mu(x) \in i\mathfrak{u}^{\beta}$ implies that its orthogonal projection onto $\mathfrak{p}$ belongs to $\mathfrak{p}^{\beta}$.

Since $\beta \in \mathfrak{g}$, it follows that $\mathrm{ad}(\beta)$ leaves $\mathfrak{g}$ invariant. By remark 1.6.1, it follows that $\mathfrak{u}^{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}^{\perp}$ with respect to $\langle \cdot, \cdot \rangle$ and so $\mathrm{ad}(\beta)$ leaves also invariant $\mathfrak{g}^{\perp}$. Moreover, keeping in mind that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal decomposition with respect to $\langle \cdot, \cdot \rangle$, it follows that

$$\mathfrak{g}^{\perp} = \mathfrak{k}^{\perp} \cap \mathfrak{u} \oplus \mathfrak{p}^{\perp} \cap i\mathfrak{u}.$$

Suppose $i\mu(x) = a + b \in i\mathfrak{u}^{\beta}$, where $a \in \mathfrak{p}$ and $b \in \mathfrak{p}^{\perp} \cap i\mathfrak{u}$. Then $b \in \mathfrak{g}^{\perp}$ and

$$0 = [\beta, i\mu(x)] = [\beta, a] + [\beta, b],$$

and so $[\beta, a] \in \mathfrak{g}$ and $[\beta, b] \in \mathfrak{g}^{\perp}$. This implies $[\beta, b] = 0$ concluding the proof. $\square$

Suppose X is compact. Let $c_1 > \cdots > c_r$ be the critical values of $\mu_{\mathfrak{p}}^{\beta}$. The corresponding level sets of $\mu_{\mathfrak{p}}^{\beta}$, $C_i := (\mu_{\mathfrak{p}}^{\beta})^{-1}(c_i)$ are submanifolds which are union of components of $\mathrm{Crit}(\mu_{\mathfrak{p}}^{\beta})$. The function $\mu_{\mathfrak{p}}^{\beta}$ defines a gradient flow generated by its gradient which is given by β_X. By Theorem 1.6.9, it follows that for any $x \in X$ the limit:

$$x_{\infty}(x) := \lim_{t \to +\infty} \exp(t\beta)x,$$

exists. Let us denote by S_i^{β} the *unstable manifold* of the critical component C_i for the gradient flow of $\mu_{\mathfrak{p}}^{\beta}$:

$$S_i^{\beta} := \{x \in X : x_{\infty}(x) \in C_i\}. \tag{1.7}$$

Applying Theorem 1.6.9, we have the following well-known decomposition of X into unstable manifolds with respect to $\mu_{\mathfrak{p}}^{\beta}$.

Theorem 1.6.13. *In the above assumption, we have*

$$X = \bigsqcup_{i=1}^{r} S_i^\beta, \tag{1.8}$$

and for any i, the map:

$$(x_\infty)|_{S_i} : S_i^\beta \to C_i,$$

is a smooth fibration with fibres diffeomorphic to $\mathbb{R}^{l_i}$ where l_i is the index (of negativity) of the critical submanifold C_i

Let $\beta \in \mathfrak{p}$ and let

$$X^\beta = \{z \in X : \beta_X(z) = 0\}.$$

By Corollary 1.6.9.2, X^β is a smooth, possibly disconnected, submanifold of X.

Chapter 2

Norm square and Kempf-Ness Function

In this chapter, we study the norm square of the gradient map and the Kempf-Ness function associated with the gradient map. The results of this chapter are obtained in a joint work with Biliotti [17].

In section 2.1, we study the properties of the norm square. Without the assumption that X is real analytic manifold, we prove that for any G-invariant compact and connected submanifold of Z, the Lojasiewicz gradient inequality holds for the norm square (Theorem 2.1.3). Stratifications of the norm square is recalled in section 2.1.1. By the stratification theorem, we have

$$\{p \in X : \overline{G \cdot p} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\} = \{p \in X : \lim_{t \to +\infty} \varphi_t(p) \in \mu_{\mathfrak{p}}^{-1}(0)\} = S_G(\mu_{\mathfrak{p}}^{-1}(0)),$$

where $\varphi_t(p)$ is the flow of the vector field -grad$f$. Then, there exist a K-equivariant strong deformation of $S_G(\mu_{\mathfrak{p}}^{-1}(0))$ onto the set $\mu_{\mathfrak{p}}^{-1}(0)$ (Theorem 2.1.6). Hence no analyticity assumption is necessary in the statement of the retraction Theorem answering Question 1 in [47, *p*.219]. We prove a well-known result that the stratum corresponding to the minimum of the norm square is open (Theorem 2.1.5).

In section 2.2, we study Kempf-Ness function. The behaviour of gradient map is encoded in the Kempf-Ness function associated with it. Recently, Biliotti [7] shows that the Kempf-Ness function is Morse-Bott and it is convex along geodesics for the action of a linear group on $\mathbb{P}(V)$ where V is a finite dimensional dimensional vector space. We prove this result in a general setting (Theorem 2.2.7). Theorem 2.2.7 generalizes the one proved in [30].

We apply Theorem 2.1.3 to prove the Ness Uniqueness Theorem which asserts that any two critical points of f in the same G-orbit in fact belong to the same K-orbit (Theorem 2.2.8). Therefore, the limit of the negative gradient flow exists and it is unique and any G-orbit collapses to a single K-orbit (Theorem 2.2.11).

2.1 The Norm Square of the Gradient Map

Let (Z,ω) be a Kähler manifold, U a compact and connected Lie group with Lie algebra $\mathfrak{u}$ and $U^{\mathbb{C}}$ the complexification of U. Suppose $U^{\mathbb{C}}$ acts holomorphically on Z with a momentum map $\mu : Z \to \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a connected compatible subgroup. Hence $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of G and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. Let X be a G-stable locally closed real submanifold of Z and let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ be the corresponding G-gradient map. We assume throughout this section that X is compact and connected. Let $\parallel \cdot \parallel$ denote the norm function associated to the fixed $\mathrm{Ad}(K)$-invariant scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$. Define the function $f : X \to \mathbb{R}$ by

$$f(x) := \frac{1}{2} \parallel \mu_{\mathfrak{p}}(x) \parallel^2, \qquad \text{for} \quad x \in X. \tag{2.1}$$

In this section, the critical points of this function will be of central importance.

Lemma 2.1.1. *The gradient of f is given by*

$$\nabla f(x) = \beta_X(x), \quad \beta := \mu_{\mathfrak{p}}(x) \in \mathfrak{p} \quad \textit{and} \quad x \in X. \tag{2.2}$$

Hence, $x \in X$ is a critical point of f if and only if $\beta_X(x) = 0$.

Proof. Define a curve $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma'(0) = v \in T_xX$.

$$\begin{aligned} df(x)v &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \\ &= \frac{1}{2}\left.\frac{d}{dt}\right|_{t=0} \langle \mu_{\mathfrak{p}}(\gamma(t)), \mu_{\mathfrak{p}}(\gamma(t))\rangle \\ &= \langle d\mu_{\mathfrak{p}}(\gamma(t))\gamma'(t), \mu_{\mathfrak{p}}(\gamma(t))\rangle|_{t=0} \\ &= \langle d\mu_{\mathfrak{p}}(x)v, \mu_{\mathfrak{p}}(x)\rangle = \langle \beta_X(x), v\rangle, \quad \beta = \mu_{\mathfrak{p}}(x). \end{aligned}$$

Hence, $\nabla f(x) = \beta_X(x)$. □

Corollary 2.1.1.1. *Let $x \in X$ and set $\beta := \mu_{\mathfrak{p}}(x)$. The following are equivalent.*

a) $\beta_X(x) = 0$,

b) $d\mu_{\mathfrak{p}}^{\xi}(x) = 0$, $\xi \in \mathfrak{p}$,

c) $df(x) = 0$.

For the remaining part of this work, we fix $\beta = \mu_{\mathfrak{p}}(x)$. The negative gradient flow line of f through $x \in X$ is the solution of the differential equation

$$\begin{cases} \dot{x}(t) = -\beta_X(x(t)), \quad t \in \mathbb{R} \\ x(0) = x. \end{cases}$$

The G-orbits are invariant under the gradient flow.

Lemma 2.1.2. *Let $g : \mathbb{R} \to G$ be the unique solution of the differential equation*

$$\begin{cases} g^{-1}\dot{g}(t) = \beta_X(x(t)) \\ g(0) = e, \quad \text{where } e \text{ is the identity of } G. \end{cases}$$

Then,

$$x(t) = g^{-1}(t)x$$

for all $t \in \mathbb{R}$.

Proof. Define $y : \mathbb{R} \to X$ by

$$y(t) = g^{-1}(t)x.$$

Since $\dot{\overline{g^{-1}}} = -g^{-1}\dot{g}g^{-1}$ and $g^{-1}\dot{g} = \beta_X(x)$, it follows that

$$\dot{y} = -g^{-1}\dot{g}g^{-1}x = -\beta_X(g^{-1}x) = -\beta_X(y(t))$$

and

$$y(0) = (g(0))^{-1}x = e^{-1}x = x.$$

Hence $x(t) = y(t) = g^{-1}(t)x$ for all $t \in \mathbb{R}$.

$\square$

The proof of the following Theorem is based on the Lojasiewicz gradient inequality, which holds in general for analytic gradient flows. A proof for the case of an action of a complex reductive group is given in [30].

Theorem 2.1.3. *Let $x_0 \in X$ and $x : \mathbb{R} \to X$ the negative gradient flow line of f through x_0. There exist positive constants α, C, ψ, and $\frac{1}{2} < \gamma < 1$ such that*

$$x_\infty := \lim_{t\to\infty} x(t)$$

exists. Moreover, there exist a constant $T > 0$ such that for any $t > T$,

$$\begin{aligned} d(x(t), x_\infty) &\leq \int_t^\infty |\dot{x}(s)|ds \\ &\leq \frac{\alpha}{1-\gamma}(f(x(t)) - f(x_\infty))^{1-\gamma} \\ &\leq \frac{C}{(t-T)^\psi}. \end{aligned}$$

Proof. Let $X = Z$. Using the Marle-Guiliemin-Sternberg local normal form, the moment map is locally real analytic. Since $\mu_\mathfrak{p} = \pi_\mathfrak{p} \circ i\mu$ where $\pi_\mathfrak{p} : i\mathfrak{u} \to \mathfrak{p}$ is the orthogonal projection. Then $\mu_\mathfrak{p}$ is locally real analytic. This implies that $f = \frac{1}{2} \parallel \mu_\mathfrak{p} \parallel^2 : Z \to \mathbb{R}$ satisfies the Lojasiewicz gradient inequality. By Lemma 2.1.1, the gradient of $f : Z \to \mathbb{R}$ coincide with the gradient of $f : X \to \mathbb{R}$. Hence $f|_X$ also satisfies Lojasiewicz gradient inequality: there exists constants $\delta > 0$, $\alpha > 0$, and $\frac{1}{2} < \gamma < 1$ such that, for every critical value a of f and every $x \in X$,

$$|f(x) - a| < \delta \qquad \Longrightarrow |f(x) - a|^{\gamma} \leq \alpha |\nabla f(x)|. \tag{2.3}$$

Let $x : \mathbb{R} \to X$ be a nonconstant negative gradient flow line of f.

$$a = \lim_{t\to\infty} f(x(t))$$

is a critical value of f. Choose a constant $T > 0$ such that $a < f(x(t)) < a + \delta$ for $t \geq T$. Then, for $t \geq T$,

$$\frac{d}{dt}(f(x) - a)^{1-\gamma} = (1-\gamma)(f(x) - a)^{-\gamma}|\nabla f(x)| \geq \frac{1-\gamma}{\alpha}|\dot{x}|.$$

Integrating the inequality over the interval $[t, \infty)$ gives

$$\int_t^{\infty} |\dot{x}(s)| ds \leq \frac{\alpha}{1-\gamma}(f(x(t)) - a)^{1-\gamma} \qquad \text{for } \ t \geq T. \tag{2.4}$$

This shows that

$$x_{\infty} := \lim_{t\to\infty} x(t)$$

exists and it is a critical point of f and hence satisfies $\mu_{\mathfrak{p}}(x_{\infty})_X = 0$.

Set $\xi(t) = (f(x(t)) - a)^{1-2\gamma}$.

$$\dot{\xi}(t) = (2\gamma - 1)(f(x(t)) - a)^{-2\gamma}|\nabla f(x(t))|^2 \geq \frac{2\gamma - 1}{\alpha^2} \qquad \text{for } \ t \geq T.$$

Which implies that

$$\xi(t) \geq \frac{2\gamma - 1}{\alpha^2}(t - T) \qquad \text{for } \ t \geq T.$$

Hence

$$(f(x(t)) - a)^{1-\gamma} = \xi(t)^{-\frac{1-\gamma}{2\gamma-1}} \leq \left(\frac{2\gamma - 1}{\alpha^2}(t - T)\right)^{-\frac{1-\gamma}{2\gamma-1}} \qquad \text{for } \ t \geq T.$$

Thus

$$\frac{\alpha}{1-\gamma}(f(x(t)) - a)^{1-\gamma} \leq \frac{c}{(t-T)^{\psi}}, \quad \psi := \frac{1-\gamma}{2\gamma - 1}, \quad c := \frac{\alpha}{1-\gamma}\left(\frac{\alpha^2}{2\gamma - 1}\right)^{\psi}$$

and by (2.4) the result follows. □

2.1.1 Stratifications of the Norm Square of the Gradient map.

We recall the stratification theorem for actions of reductive group. For details see [45].

Given a maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$, we pick $\mathfrak{a}_+ \subset \mathfrak{a}$ a positive Weyl-chamber. Let $f : X \to \mathbb{R}$ be the norm square of the gradient map $\mu_\mathfrak{p}$. i.e.,

$$f(x) := \frac{1}{2} \parallel \mu_\mathfrak{p}(x) \parallel^2,$$

where $\parallel \cdot \parallel$ denotes the norm function. Let C denote the critical set of f, $\mathfrak{B} := \mu_\mathfrak{p}(C)$ and $\mathfrak{B}_+ := \mathfrak{B} \cap \mathfrak{a}_+$.

Let $X^{ss} := \{x \in X : \overline{G \cdot x} \cap \mu_\mathfrak{p}^{-1}(0) \neq \emptyset\}$, where X is a compact G-invariant subset of Z. For $\beta \in \mathfrak{B}_+$, following the notation introduced in [45], set

$$X|_{\|\beta\|^2} := \{x \in X : \overline{\exp(\mathbb{R}\beta) \cdot x} \cap (\mu_\mathfrak{p}^\beta)^{-1}(\parallel \beta \parallel^2) \neq \emptyset\}$$

$$X^\beta := \{x \in X : \beta_X(x) = 0\}$$

$$X^\beta|_{\|\beta\|^2} := X^\beta \cap X|_{\|\beta\|^2}$$

$$X^{\beta+}|_{\|\beta\|^2} := \{x \in X|_{\|\beta\|^2} : \lim_{t \to -\infty} \exp(t\beta) \cdot x \text{ exists and it lies in } X^\beta|_{\|\beta\|^2}\}$$

The set $X^{\beta+}|_{\|\beta\|^2}$ is $G^{\beta+}$-invariant. Let $\mu_{\mathfrak{p}^\beta}$ be a gradient map of the G^β-action on $X^{\beta+}|_{\|\beta\|^2}$. Since β is in the center of $\mathfrak{g}^\beta$, we have a shifted gradient map

$$\widehat{\mu_{\mathfrak{p}^\beta}} := \mu_{\mathfrak{p}^\beta} - \beta.$$

Let

$$S^{\beta+} := \{x \in X^{\beta+}|_{\|\beta\|^2} : \overline{G^\beta \cdot x} \cap \mu_{\mathfrak{p}^\beta}^{-1}(\beta) \neq \emptyset\}.$$

$S^{\beta+}$ coincides with the set of semistable points of G^β in $X^{\beta+}|_{\|\beta\|^2}$ after shifting.

Definition 2.1.1. *The β-stratum of X is given by $S_\beta := G \cdot S^{\beta+}$.*

Theorem 2.1.4. *(Stratification Theorem)[45, 7.3]. Suppose X is a compact G-invariant submanifold of Z. Then $\mathfrak{B}_+$ is finite and*

$$X = \bigsqcup_{\beta \in \mathfrak{B}_+} S_\beta.$$

Moreover

$$\overline{S_\beta} \subset S_\beta \cup \bigcup_{|\gamma|>|\beta|} S_\gamma.$$

From the proof of Theorem 2.1.3, it was observed that f satisfies the Lojasiewicz's gradient inequality. As an application, we prove a well-known result that the stratum corresponding to the minimum of f is open.

Theorem 2.1.5. *If $\beta \in \mathfrak{B}$ is such that $\frac{1}{2} \parallel \beta \parallel^2$ is a minimum value of f. Then the corresponding stratum S_β is open in X.*

Proof. From the proof of Theorem 2.1.3, there exist constants $\delta > 0$, $\alpha > 0$, and $\frac{1}{2} < \gamma < 1$ such that, for every critical value a of f and every $x \in X$,

$$|f(x) - a| < \delta \qquad \Longrightarrow |f(x) - a|^\gamma \leq \alpha |\triangledown f(x)|. \tag{2.5}$$

In particular, if x is critical point, then $f(x) = a$. Since X is compact, by Theorem 2.1.4,

$$X = \bigsqcup_{i=1}^{k} S_{\beta_i},$$

where $\beta_i \in \mathfrak{B}_+$. By the stratification theorem, we may assume that $\beta = \beta_1$ and so, $\parallel \beta_j \parallel > \parallel \beta \parallel$ for any $j = 2, \cdots, k$. Let

$$0 < \delta' < \min\left(\delta, \frac{\parallel \beta_2 \parallel^2 - \parallel \beta \parallel^2}{2}, \cdots, \frac{\parallel \beta_k \parallel^2 - \parallel \beta \parallel^2}{2}\right).$$

Let

$$U = \{x \in X : |f(x) - \parallel \beta \parallel^2 | < \delta'\}.$$

Let $x_0 \in U$ and let $x(t)$ the gradient flow of $-\nabla f$ through x_0. Therefore,

$$\begin{aligned} f(x(t)) \leq f(x_0) \leq |f(x_0) - \parallel \beta \parallel^2 | + \parallel \beta \parallel^2 & \\ &< \delta' + \parallel \beta \parallel^2 \\ &< \frac{\parallel \beta_j \parallel^2 - \parallel \beta \parallel^2}{2} + \parallel \beta \parallel^2 \quad j = 2, \cdots, k \\ &< \frac{\parallel \beta_j \parallel^2 + \parallel \beta \parallel^2}{2} \leq \parallel \beta_j \parallel^2, \quad j = 2, \cdots, k. \end{aligned}$$

Therefore, $f(x_\infty) = \parallel \beta \parallel^2$. This implies that $U \subset S_\beta$ and so, using standard arguments of the gradient flow, S_β is open. $\square$

As in [59], we have the following deformation retraction.

Theorem 2.1.6 (Retraction Theorem)**.** *Let $\beta \in \mathfrak{a}_+$ be a critical value of f. Let S_β be the stratum associated to β. Let $x_t(x)$ denote the gradient flow of $-\nabla f$. Then there exists a K-equivariant strong deformation retraction of S_β onto $S_\beta \cap \mu_{\mathfrak{p}}^{-1}(K \cdot \beta)$ given by*

$$[0, \infty] \times S_\beta \to S_\beta \cap \mu_{\mathfrak{p}}^{-1}(K \cdot \beta), \quad (t, p) \mapsto x_t(p),$$

and

$$x_\infty(p) = \lim_{t \to +\infty} x_t(p).$$

2.1.2 Hessian of Norm Square

If $v \in T_pX$, then $|v| = \sqrt{(v, v)}$, where $(\cdot, \cdot)$ is the scalar product induced by the Kähler form ω. The following proposition give the Hessian of f.

Proposition 2.1.7. *[45, Prop. 2.5, 2]. Let $v \in T_xX$ be an eigenvector of $\beta \in \mathfrak{p}_x$ with eigenvalue $\lambda(\beta)$. Let $\gamma(t)$ be a smooth curve in X with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Then if x is a critical point of f,*

$$\frac{d^2}{dt^2}(f \circ \gamma)(0) = \lambda(\beta)|v|^2 + \parallel d\mu_{\mathfrak{p}}(x)v \parallel^2 \tag{2.6}$$

The Hessian of f at critical points satisfies the following:

Proposition 2.1.8. *[45, Prop. 6.6] Let $x \in C$ be a critical point of $f : X \to \mathbb{R}$ and let S_β be the associated stratum. Let $H_x(f)$ denote the Hessian of f at x. Then*

a) $H_x(f) = 0$ *on* $T_x(K \cdot x)$,

b) $H_x(f) > 0$ *on* $\mathfrak{p}^\beta \cdot x + \mathfrak{r}^{\beta+} \cdot x$, *where* $\mathfrak{r}^{\beta+}$ *is the Lie algebra of* $R^{\beta+}$,

c) $H_x(f) \geq 0$ *on* $T_x(S_\beta) = \mathfrak{g} \cdot x + T_x(S^{\beta+}) = \mathfrak{k} \cdot x + T_x(S^{\beta+})$ *and*

d) $H_x(f) < 0$ *on* $T_x(S_\beta)^\perp = (\mathfrak{g} \cdot x)^\perp \cap (T_x(S^{\beta+}))^\perp = (\mathfrak{k} \cdot x)^\perp \cap (T_x(S^{\beta+}))^\perp$.

Remark 2.1.9. *[45, 6.7] The tangent space $T_x(G \cdot x)$ decompose to $T_x(G \cdot x) = T_x(K \cdot x) \oplus \mathfrak{p}^{\beta} \cdot x \oplus \mathfrak{r}^{\beta+} \cdot x$. This follows from the decomposition $G = KG^{\beta+}$, $G^{\beta+} = G^{\beta} R^{\beta+}$, the identity $K \cap G^{\beta+} = K^{\beta}$ and the fact that G^{β} acts on X^{β} whereas $R^{\beta+}$ acts on the fibers of $\mathfrak{p}^{\beta+}$. Thus the behaviour of $H_x(f)$ on $T_x(G \cdot x)$ is precisely described by Proposition 2.1.8.*

The following result is proved in [45] (see Corollary 6.12 and Corollary 6.13 p. 21).

Proposition 2.1.10. *Let $z \in X$.*

- *if $f(z) = max_{x \in G \cdot z} f(x)$, Then $G \cdot z = K \cdot z$ and so it is closed orbit.*
- *if $G \cdot z$ is compact in X, then $G \cdot z = K \cdot z$*

2.2 Kempf-Ness Function

Given G a real reductive group which acts smoothly on Z; $G = K\exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Let X be a G-invariant locally closed submanifold of Z. As Mundet pointed out in [63], there exists a function $\Phi : X \times G \to \mathbb{R}$, such that

$$\langle \mu_{\mathfrak{p}}(x), \xi \rangle = \frac{d}{dt}\bigg|_{t=0} \Phi(x, \exp(t\xi)), \qquad \xi \in \mathfrak{p},$$

and satisfying the following conditions:

a) For any $x \in X$, the function $\Phi(x, .)$ is smooth on G.

b) The function $\Phi(x, .)$ is left-invariant with respect to K, i.e., $\Phi(x, kg) = \Phi(x, g)$.

c) For any $x \in X$, $v \in \mathfrak{p}$ and $t \in \mathbb{R}$;

$$\frac{d^2}{dt^2} \Phi(x, \exp(tv)) \geq 0.$$

Moreover:

$$\frac{d^2}{dt^2} \Phi(x, \exp(tv)) = 0$$

if and only if $\exp(\mathbb{R}v) \subset G_x$.

d) For any $x \in X$, and any $g, h \in G$;

$$\Phi(x, hg) = \Phi(x, g) + \Phi(gx, h).$$

This equation is called the cocycle condition. It follows that $\Phi(x, e) = 0$, where e is the identity element of G. The proof is given in [16].

The function $\Phi : X \times G \to \mathbb{R}$ is called the Kempf-Ness function for (X, G, K)

Let $M = G/K$. Equip G with the unique left invariant Riemannian metric which agree with the scalar product $\langle \xi_1 + \beta_1, \xi_2 + \beta_2 \rangle := -\langle \xi_1, \xi_2 \rangle + \langle \beta_1, \beta_2 \rangle$ where $\xi_1, \xi_2 \in \mathfrak{k}$ and $\beta_1, \beta_2 \in \mathfrak{p}$. We recall that $\langle \cdot, \cdot \rangle$ is negative define on $\mathfrak{k}$. This metric is $\mathrm{Ad}(K)$-invariant, and so, it induces a G-invariant Riemannian metric of nonpositive curvature on M. M is a symmetric space of non-compact type [20]. Let $\pi : G \to M$ be the projection onto the right cosets of G and ∇ denote the Levi-Civita connection on M. Let $x \in X$. By (b), $\Phi(x, kg) = \Phi(x, g)$. Hence, the function $\Phi_x : G \to \mathbb{R}$ given by $\Phi_x(g) := \Phi(x, g^{-1})$ descends to a function $\Phi_x : M \to \mathbb{R}$ denoted by the same symbol and called the Kempf-Ness function at x. First, we compute the differential of Φ at any point in a given direction.

Lemma 2.2.1. *For $x \in X$, the differential of $\Phi_x : M \to \mathbb{R}$ is given as*

$$d(\Phi_x)_{\pi(g)}(v_x) = -\langle \mu_{\mathfrak{p}}(g^{-1}x), \xi \rangle,$$

where, $v_x(g) = (\mathrm{d}\pi)_g \left((\mathrm{d}L_g)_e(\xi) \right)$ and $\xi \in \mathfrak{p}$. Therefore, $\nabla \Phi_x(\pi(g)) = -(\mathrm{d}\pi)_g \left(dL_g(\mu_{\mathfrak{p}}(g^{-1}x)) \right)$.

Proof. Let $\pi(g) \in M$, $\xi \in \mathfrak{p}$ and $v_x \in T_{\pi(g)}G/K$. There exist $\xi \in \mathfrak{p}$ such that

$$v_x = \left.\frac{d}{dt}\right|_{t=0} g\exp(t\xi)K.$$

Take

$$\gamma(t) = \pi(g\exp(t\xi)), \qquad t \in [a, b],\ \xi \in \mathfrak{p}.$$

Then $v_x = (d\pi)_g((dL_g)(\xi))$,

$$\begin{aligned}
d(\Phi_x)_{\pi(g)}(v_x) &= \frac{d}{dt}\bigg|_{t=0} \Phi(x, \gamma(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi(x, \pi(g\exp(t\xi)) \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi(x, \exp(-t\xi)g^{-1}) \qquad \text{(by the deifintion of } \Phi) \\
&= \frac{d}{dt}\bigg|_{t=0} [\Phi(x, g^{-1}) + \Phi(g^{-1}x, \exp(-t\xi))] \qquad \text{(by condition (d))} \\
&= \frac{d}{dt}\bigg|_{t=0} \Phi(g^{-1}x, \exp(-t\xi)) \\
&= -\langle \mu_{\mathfrak{p}}(g^{-1}x), \xi \rangle.
\end{aligned}$$

This implies that

$$d(\Phi_x)_{\pi(g)}(v_x) = -\langle \mu_{\mathfrak{p}}(g^{-1}x), \xi \rangle.$$

□

Let $\Phi_x : G \to \mathbb{R}$, be the Kempf-Ness function at x. Define $\varphi_x : G \to G \cdot x$ as follows

$$\varphi_x(g) = g^{-1}x.$$

Lemma 2.2.2. *The map* φ_x *intertwines the gradient of* $\Phi_x : G \to \mathbb{R}$ *and the gradient of* f. *i.e.,* $\forall g \in G$

$$d(\varphi_x)_g \nabla \Phi_x = \nabla f((\varphi_x(g)).$$

Proof. Let $\beta \in \mathfrak{p}$. Since

$$\varphi_x(g\exp(t\beta)) = \exp(-t\beta)g^{-1}x,$$

we have

$$(d\Phi_x)_g(dL_g(\beta)) = -\beta_X(g^{-1}x).$$

The result follow from taking $\beta = \mu_{\mathfrak{p}}(x)$. □

Remark 2.2.3. *A smooth curve* $\gamma(t) := \pi \circ g : \mathbb{R} \to M$ *is a negative gradient flow line of* $\Phi_x : M \to \mathbb{R}$ *if and only if the smooth curve* $g : \mathbb{R} \to M$ *satisfies* $g^{-1}\dot{g} = \mu_{\mathfrak{p}}(g^{-1}x)$.

Indeed, from Lemma 2.2.1,

$$\nabla \Phi_x(\pi(g)) = -d_{\pi(g)}(dL_g(\mu_{\mathfrak{p}}(g^{-1}x))$$

and by Lemma 2.2.2, $\varphi_x : G \to \mathbb{R}$ *interwines the gradient of* Φ_x *with* ∇f. *Therefore, the gradient flow of* Φ_x *is such that* $g^{-1}\dot{g} = \mu_{\mathfrak{p}}(g^{-1}x)$. *On the other hand if* $g : \mathbb{R} \to G$ *satisfies* $g^{-1}\dot{g} = \mu_{\mathfrak{p}}(g^{-1}x)$, *the curve* $\gamma = \pi \circ g(t)$ *is a geodesic and*

$$\frac{d}{dt}(\Phi_x \circ \gamma) = -\langle \mu_{|}(g^{-1}x), g^{-1}\dot{g} \rangle.$$

We recall some result from Riemannian geometry. We refer the reader to Appendix A in [30] for further details. Suppose M is an Hadamard manifold, i.e., connected, complete, simply-connected with non-positive curvature. Let ∇ denote the Levi-Civita connection on M and $d : M \times M \to [0, \infty)$ the distance function associated to the Riemannian metric. Let $\gamma_1, \gamma_2 : [a, b] \to M$ be smooth curves and for each $t \in [a, b]$, $\gamma(s, t) : [0, 1] \to M$ be the unique geodesic such that

$$\gamma(0, t) = \gamma_1(t), \qquad \gamma(1, t) = \gamma_2(t).$$

Define the function $\rho : [a, b] \to [0, \infty)$ by

$$\rho(t) := d(\gamma_1(t), \gamma_2(t)).$$

We recall the following results. The reader can see for instance, Appendix in [30] for their proofs.

Lemma 2.2.4. *[30, Lemma A.2] Suppose* $f : M \to \mathbb{R}$ *is a smooth function that is convex along geodesics. Let* $\gamma_1, \gamma_2 : \mathbb{R} \to M$ *be the negative gradient flow lines of* f, *and let* γ *and* ρ *be as defined above. Then* ρ *is nonincreasing and, if* $\rho(t) \neq 0$, *then*

$$\dot{\rho}(t) = -\frac{1}{\rho(t)} \int_0^1 \frac{\partial^2}{\partial s^2}(f \circ \gamma)(s, t) ds. \tag{2.7}$$

Lemma 2.2.5. *[30, Lemma A.3] Let* $\gamma_1, \gamma_2 : I \subset \mathbb{R} \to M$ *be smooth curves such that* γ_0 *is a geodesic. If* $\gamma_0(t) \neq \gamma_1(t)$, *then*

$$\dot{\rho}(t) \geq -|\nabla \dot{\gamma}_1(t)|.$$

If $\gamma_0(t) = \gamma_1(t)$, *then*

$$\frac{d\rho(t)}{dt^{\pm}} = \pm|\dot{\gamma}_0(t) - \dot{\gamma}_1(t)|.$$

Lemma 2.2.6. *[30, Lemma A.4] For all* $p \in M$, *all* $u, v \in T_pM$, *and for all* $t \geq 1$,

$$\frac{d(\exp_p(tu), \exp_p(tv))}{t} \geq d(\exp_p(u), \exp_p(v)) \geq |u - v|.$$

We now show that the Kempf-Ness function is a Morse-Bott function and it is convex along geodesics.

Theorem 2.2.7. *Let* $\Phi_x : M \to \mathbb{R}$. *Then*

a) Φ_x *is a Morse-Bott function and it is convex along geodesics.*

b) If $\gamma : \mathbb{R} \to M$ *is a negative gradient flow of* Φ_x, *then,*

$$\lim_{t\to\infty} \Phi_x(\gamma(t)) = inf_{x\in M}\Phi_x.$$

Proof. (a): By lemma 2.2.1, $g \in G$ is a critical point of Φ_x if and only if

$$\mu_{\mathfrak{p}}(g^{-1}x) = 0.$$

$$\mathrm{Crit}(\Phi_x) = \{\pi(g) \in M : \mu_{\mathfrak{p}}(g^{-1}x) = 0\}. \tag{2.8}$$

The next is to show that the $\mathrm{Crit}(\Phi_x)$ is a submanifold of M. To do this, the Hessian of the function is computed along geodesics. The geodesic on M passing through $\pi(g)$ in the direction $v = d\pi_g g\xi$ has the form $\pi(g\exp(t\xi))$ [16]. Hence, M is complete and by the Hadamard theorem,

$$\mathfrak{p} \to M, \qquad \xi \mapsto \pi(g\exp(\xi))$$

is a diffeomorphism. This implies that $\pi(g\exp(\xi)) \in \mathrm{Crit}(\Phi_x)$ if and only if $\mu_{\mathfrak{p}}(g^{-1}\exp(-\xi)x) = 0$.

$$\mathrm{Hess}(\Phi_x) = d^2(\Phi_x)_{\pi(g)}(v), \qquad \pi(g) \in \mathrm{Crit}(\Phi_x)$$

$$\begin{aligned} d^2(\Phi_x)_{\pi(g)}(v) &= \frac{d^2}{dt^2}\bigg|_{t=0} \Phi(x,\gamma(t)) \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} \Phi(x,\pi(g\exp(-t\xi)) \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} \Phi(x,g\exp(-t\xi)K) \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} \Phi(x,\exp(t\xi)g^{-1}) \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} [\Phi(x,g^{-1})+\Phi(g^{-1}x,\exp(t\xi))] \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} \Phi(g^{-1}x,\exp(t\xi) \geq 0 \qquad \text{(by condition (c))} \end{aligned}$$

Moreover,

$$\frac{d^2}{dt^2}\bigg|_{t=0} \Phi(g^{-1}x,\exp(t\xi)) = 0$$

if and only if $\exp(t\xi) \subset G_{g^{-1}x}$. Hence,

$$\mathrm{Crit}(\Phi_x) = \{\pi(g\exp t\xi) \in M : \exp t\xi \subset G_{g^{-1}x}\},$$

which is a submanifold and the kernel of the Hessian. Therefore, Φ is a Morse-Bott function and since the Hessian is non-negative along geodesics, it is convex along geodesics.

(b). Let γ_1, γ_2 be negative gradient flow of Φ_x. There exists $g_1, g_2 : \mathbb{R} \to G$ such that $\gamma_1 = \pi(g_1(t))$ and $\gamma_2 = \pi(g_2(t))$. Let $\beta : \mathbb{R} \to \mathfrak{p}$ and $k : \mathbb{R} \to K$ be such that $g_2(t) = g_1(t)\exp(\beta(t))k(t)$. Define $H : \mathbb{R} \times \mathbb{R} \to M$ by

$$H(t,s) = \pi(g_1(t)\exp(s\beta(t))).$$

The curve $s \mapsto H(t,s)$ is the unique geodesic joining γ_1 and γ_2. By Lemma 2.2.4 the function $\rho(t) = d_M(\gamma_1(t), \gamma_2(t)) = \parallel \beta(t) \parallel$ is nonincreasing.

Assume that $\mathrm{Crit}(\Phi_x)$ is not empty. Hence we may suppose that $\mu_{\mathfrak{p}}(g_1(0)^{-1}x) = 0$. This implies that the curve γ_1 is constant. Since ρ is nonincreasing, the image of γ_2 is contained in a compact subset of M. Since Φ_x is Morse-Bott, then γ_2 converges to a critical point of Φ_x. This implies that if the critical manifold of Φ_x is nonempty, then Φ_x has a global minimal and every negative flow line of Φ_x converges to a critical point. Now suppose $\mathrm{Crit}(\Phi_x)$ is empty. Assume by contradiction that

$$a := \lim_{t\to\infty} \Phi_x(\gamma_1(t)) \geq \inf_M \Phi_x.$$

Then, $\Phi_x(\gamma_1(t))$ is bounded from below. We can choose γ_2 such that $\Phi(\gamma_2(0)) < a$. Since the function $\rho = \parallel \beta(t) \parallel$ is nonincreasing, then there exists a constant $C > 0$ such that $\rho(t) = \parallel \beta(t) \parallel \leq C$. Hence,

$$\begin{aligned} \left.\frac{d}{ds}\right|_{s=0} \Phi(H(t,s)) &= (d\Phi_x)_{\gamma_1(t)}(\dot{H}(t,0)) \\ &= -\langle \mu_{\mathfrak{p}}(g_1(t)^{-1}x), \beta(t)\rangle \\ &\geq - \parallel \mu_{\mathfrak{p}}(g_1(t)^{-1}x) \parallel \parallel \beta(t) \parallel \\ &\geq -C \parallel \mu_{\mathfrak{p}}(g_1(t)^{-1}x) \parallel . \end{aligned}$$

Since for a fixed t, the function $s \to \Phi_x(H(t,s))$ is convex, this implies that the derivative $\frac{d}{ds}\Phi(H(t,s))$ increases. It follows that

$$\begin{aligned} \Phi_x(\gamma_2(t)) &= \Phi_x(H(t,1)) \\ &= \Phi_x(H(t,0)) + \int_0^1 \frac{d}{ds}\Phi(H(t,s))\mathrm{d}s \\ &\geq \Phi_x(\gamma_1(t)) - C \parallel \mu_{\mathfrak{p}}(g_1(t)^{-1}x) \parallel . \end{aligned}$$

Since the function $\Phi_x(\gamma_1(t))$ is bounded below and $\frac{d}{dt}\Phi_x(\gamma_1(t)) = - \parallel \mu_{\mathfrak{p}}(g_1(t)^{-1}x) \parallel^2$, there exists a sequence $t_i \to \infty$ such that $\lim_{i\to\infty} \parallel \mu_{\mathfrak{p}}(g_1(t_i)^{-1}x) \parallel^2 = 0$. This implies that

$$\lim_{i\to\infty} \Phi_x(\gamma_2(t_i)) \geq \lim_{i\to\infty} \Phi_x(\gamma_1(t_i)) = a.$$

This is a contradiction since by assumption $\Phi_x(\gamma_2(t)) < a$ and so $\lim_{i\to\infty} \Phi_x(\gamma_1(t)) < a$.

□

The following result asserts that any critical points of f in the same G-orbit in fact belong to the same K-orbit. We use original ideas from [30] in a different context.

Theorem 2.2.8. *Let $x_0, x_1 \in X$ be critical points of the norm square f. Then*

$$x_1 \in G \cdot x_0 \implies x_1 \in K \cdot x_0.$$

Proof. Since $x_0, x_1 \in X$ are critical points of f, then by Lemma 2.1.1

$$\mu_{\mathfrak{p}}(x_0)_X = 0, \qquad \mu_{\mathfrak{p}}(x_1)_X = 0 \tag{2.9}$$

Suppose $x_1 \in G \cdot x_0$. Let $g_0 \in G$ such that

$$x_1 = g_0^{-1} x_0$$

and $g, h : \mathbb{R} \to G$ be defined by

$$g(t) := \exp(t\mu_{\mathfrak{p}}(x_0)), \qquad h(t) := g_0 \exp(t\mu_{\mathfrak{p}}(x_1)).$$

Since $\mu_{\mathfrak{p}}(x_0)_X = 0$, $g(t)^{-1}x_0 = \exp(-t\mu_{\mathfrak{p}}(x_0))x_0 = x_0$. Similarly $h(t)^{-1}x_0 = \exp(-t\mu_{\mathfrak{p}}(x_1))g_0^{-1}x_0 = g_0^{-1}x_0 = x_1$ for all t. Thus $g(t)$ and $h(t)$ satisfy the differential equation $g^{-1}\dot{g} = \mu_{\mathfrak{p}}(g^{-1}x_0)$. These implies that the curves $\gamma_1 := \pi \circ g$ and $\gamma_2 := \pi \circ h$ are geodesics and are negative gradient flow lines of the Kempf-Ness function. Define $\xi(t) \in \mathfrak{p}$ and $k(t) \in K$ so that

$$h(t) := g(t)\exp(\xi(t))k(t).$$

$$x_1 = h(t)^{-1}g(t)x_0 = k(t)^{-1}\exp(-\xi(t))x_0.$$

Let

$$\rho(t) := d_M(\gamma_1(t), \gamma_2(t)) = \| \xi(t) \|, \quad \text{for all t.}$$

If $\rho \equiv 0$, then $\xi(t) = 0$ for all t and

$$x_1 = k(t)^{-1}x_0$$

and this means that $x_1 \in K \cdot x_0$. Otherwise, for each t, let $\gamma(s,t) : [0,1] \to M$ defined by

$$\gamma(s,t) := \pi(g(t)\exp(s\xi(t)))$$

be the unique geodesic. Note that $\gamma(0,t) = \gamma_1(t)$ and $\gamma(1,t) = \gamma_1(t)$.

By equation 2.7, we have

$$\dot\rho(t) = -\frac{1}{\rho(t)}\int_0^1 \frac{\partial^2}{\partial s^2}\Phi_{x_0}(g(t)\mathrm{exp}(s\xi(t))\mathrm{d}s \tag{2.10}$$

$$= -\frac{1}{\rho(t)}\int_0^1 (\xi(t)_X(\mathrm{exp}(s\xi(t))x_0), \xi(t)_X(\mathrm{exp}(s\xi(t))x_0)\mathrm{d}s. \tag{2.11}$$

Choose a sequence $t_n \to \infty$ such that the limits

$$\lim_{n\to\infty}\dot\rho(t_n) = 0, \quad \xi_\infty := \lim_{n\to\infty}\xi(t_n), \quad k_\infty := \lim_{n\to\infty}k(t_n)$$

exist. By (2.11) and since $\lim_{n\to\infty}\dot\rho(t_n) = 0$, $\xi_{\infty X}(x_0) = 0$. Then,

$$x_1 = \lim_{n\to\infty}k(t_n)^{-1}\mathrm{exp}(-\xi(t_n))x_0 = k_\infty^{-1}\mathrm{exp}(-\xi_\infty)x_0 = k_\infty^{-1}x_0.$$

Hence $x_1 \in K\cdot x_0$ □

The following theorems are well-known if $X = Z$ and $G = U^{\mathbb{C}}$ [30, 54].

Theorem 2.2.9. *Let $x_0 \in X$ and $x : \mathbb{R} \to X$ the negative gradient flow line of f through x_0. Define $x_\infty := \lim_{t\to\infty} x(t)$. Then*

$$\parallel \mu_{\mathfrak{p}}(x_\infty) \parallel = inf_{g\in G} \parallel \mu_{\mathfrak{p}}(gx_0) \parallel .$$

Moreover, the K-orbit of x_∞ depends only on the G-orbit of x_0.

Proof. The limit x_∞ exists by Theorem 2.1.3. The solution of the negative gradient flow line of f through x_0 by Lemma 2.1.2 is given by

$$x(t) = g(t)^{-1}x_0,$$

where $g : \mathbb{R} \to G$ is the solution of

$$\begin{cases} g^{-1}\dot g(t) = \beta_X(x(t)) \\ g(0) = e, \quad \text{where } e \text{ is the identity of } G. \end{cases}$$

Fix an element $g_0 \in G$ and let $y : \mathbb{R} \to X$ and $h : \mathbb{R} \to G$ be the solutions of the differential equations

$$\dot y = -\beta_X(y(t)), \qquad y(0) = g_0^{-1}x_0,$$

and

$$h^{-1}\dot{h} = \beta_X(y(t)), \qquad h(0) = g_0.$$

Define $\xi(t) \in \mathfrak{p}$ and $k(t) \in k$ by

$$h(t) =: g(t)\exp(\xi(t))k(t).$$

By Lemma 2.1.2,

$$y(t) = h(t)^{-1}x_0 = k(t)^{-1}\exp(-\xi(t))x(t), \quad \forall t \in \mathbb{R}.$$

Let $d_M : M \times M \to [0, \infty)$ be the distance function of the Riemannian metric on M. $\gamma_1 := \pi \circ g$ and $\gamma_2 := \pi \circ h$ are geodesics and are negative gradient flow lines of the so called Kempf-Ness function. Since M is simply connected with nonpositive sectional curvature. Then

$$d_M(\gamma_1(t), \gamma_2(t)) = \| \xi(t) \|$$

is nonincreasing. Hence there exist a sequence $t_n \to \infty$ such that the limits

$$\xi_\infty := \lim_{n\to\infty} \xi(t_n), \quad k_\infty := \lim_{n\to\infty} k(t_n)$$

exist. Hence

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} k(t)^{-1}\exp(-\xi(t))x(t) = k_\infty^{-1}\exp(-\xi_\infty)x_\infty.$$

Which implies that y_∞ and x_∞ are critical points of the normed sqaure of the gradient map belonging to the same G-orbit. Hence they belong to the same K-orbit by Theorem 2.2.8 and therefore,

$$\| \mu_{\mathfrak{p}}(x_\infty) \| = \| \mu_{\mathfrak{p}}(y_\infty) \| \leq \| \mu_{\mathfrak{p}}(g_0^{-1}x_0) \| .$$

□

Theorem 2.2.10. . *Let $x_0 \in X$ and*

$$m := inf_{g\in G} \| \mu_{\mathfrak{p}}(gx_0) \| .$$

Then

$$x, y \in \overline{G \cdot x_0}, \quad \| \mu_{\mathfrak{p}}(x) \| = \| \mu_{\mathfrak{p}}(y) \| = m \qquad \Longrightarrow\ y \in K \cdot x.$$

Proof. The solution of the negative gradient flow line of f through x_0 is given by

$$x(t) = g(t)^{-1}x_0,$$

Fix $g_0 \in G$ and we know that the limit x_∞ of $x(t)$ exists. Then by Theorem 2.2.9,

$$x_\infty \in \overline{G \cdot x}, \quad \| \mu_{\mathfrak{p}}(x_\infty) \| = m.$$

Let $x \in \overline{G \cdot x}$ such that $\| \mu_{\mathfrak{p}}(x) \| = m$.

Choose a sequence $g_n \in G$ such that

$$x = \lim_{n\to\infty} g_n^{-1}x_0$$

and define $y_n : \mathbb{R} \to X$ and $x_i \in X$ by

$$\dot{y_n} = -\beta_X(y_n), \quad y_n(0) = g_n^{-1}x_0, \quad x_n := \lim_{t\to\infty} y_n(t).$$

Then from the estimate of Theorem 2.1.3, there exists a constant $c > 0$ such that, for n sufficiently large,

$$d(x_n, g_n^{-1}x_0) \leq \int_0^\infty |\dot{y_n}(t)| dt \leq c(\| \mu_{\mathfrak{p}}(g_n^{-1}x_0) \|^2 - m^2)^{1-\alpha}.$$

Since

$$m = \| \mu_{\mathfrak{p}}(x) \| = \lim_{n\to\infty} \| \mu_{\mathfrak{p}}(g_n^{-1}x_0) \|,$$

which implies that $x = \lim_{n\to\infty} x_n$ and $x_n \in K \cdot x_\infty$ for all n by Theorem 2.2.9. Therefore, $x \in K \cdot x_\infty$ because the group orbit $K \cdot x_\infty$ is compact. □

Let $x \in X$ be a critical point of f and $y : \mathbb{R} \to M$ be the unique solution of the equation

$$\dot{y} = -\beta_X(y(t)), \quad y(0) = y_0 \in X; \quad \beta = \mu_{\mathfrak{p}}(y).$$

We define the stable manifold of the critical set $K \cdot x$ by

$$S(K \cdot x) := \{y_0 \in X : \lim_{t\to\infty} y(t) = kx \quad \text{for some} \quad k \in K\}. \tag{2.12}$$

By Theorem 2.1.3, X is the union of these stable manifolds and each stable manifold is a union of G-orbits by Theorems 2.2.9 and 2.2.10. The stable manifolds of the gradient flow have a structure close to a stratification by stable manifolds corresponding to a Morse-Bott function.

By Theorems 2.1.3, 2.2.9 and 2.2.10, we have the following result. This result generalises Theorem 5.2 proved by Jablonski for the G-gradient map of a projective representation [52].

Theorem 2.2.11. *Let $x \in X$ be a critical point of f and $S(K \cdot x) \subset X$ be as defined above. The following holds:*

a) $X = \bigcup_{x \in Crit(f)} S(K \cdot x)$.

b) Let $y_o \in X$. Then $y_0 \in S(K \cdot x)$ if and only if

$$x \in \overline{G \cdot y_0}, \quad \parallel \mu_{\mathfrak{p}}(x) \parallel = inf_{g \in G} \parallel \mu_{\mathfrak{p}}(gy_0) \parallel \tag{2.13}$$

c) $S(K \cdot x)$ is a union of G-orbits.

Proof. (a) follows by Theorem 2.1.3.

To proof (b); let $y : \mathbb{R} \to M$ be the unique solution of the equation

$$\dot{y} = -\beta_X(y(t)), \quad y(0) = y_0 \in X; \quad \beta = \mu_{\mathfrak{p}}(y)$$

and $y_\infty := \lim_{t \to \infty} y(t)$. Then, by Lemma 2.1.2 and Theorem 2.2.9, we have

$$y_\infty \in \overline{G \cdot y_0}, \quad \parallel \mu_{\mathfrak{p}}(y_\infty) \parallel = \inf_{g \in G} \parallel \mu_{\mathfrak{p}}(gy_0) \parallel . \tag{2.14}$$

From (2.12), $y_0 \in S(K \cdot x)$ if and only if $y_\infty \in K \cdot x$. Thus $y_0 \in S(K \cdot x)$ implies (2.13). Conversely, if y_0 satisfies (2.13), then it follows from (2.14) and Theorem 2.2.10 that $y_\infty \in K \cdot x$ and hence, $y_0 \in S(K \cdot x)$.

To proof (c); From (ii) and the uniqueness in Theorem 2.2.9 that $S(K \cdot x)$ is a union of G-orbits. □

We conclude this chapter with a result (Proposition 2.2.15) that will be needed in the last section of this work (Section 5.7). The result will not be needed anywhere else in this work. The proof follows the idea in [30]. Recall that $M = G/K$ is a symmetric space of non-compact type. Let $\pi : G \to M$ be the canonical projection.

Let $x : \mathbb{R} \to X$ be the solution of the negative gradient flow of the norm square of the gradient map through x_0, $g : \mathbb{R} \to G$ be such that $x(t) = g(t)^{-1}x_0$ and $\gamma(t) = \pi \circ g(t)$. The limit $x_\infty = \lim_{t\to\infty} x(t)$ exists by Theorem 2.1.3. Define $\beta : \mathbb{R} \to \mathfrak{p}; t \mapsto \beta(t)$ and $k : \mathbb{R} \to \mathfrak{p}; t \mapsto k(t)$ by

$$g(t) =: \exp(-\beta(t))k(t).$$

We want to show that the limit

$$\beta_\infty(t) := \lim_{t\to\infty} \frac{\beta(t)}{t}$$

exists in $\mathfrak{p}$..

Let $\beta = \mu_{\mathfrak{p}}(x(t))$. For every $t > 0$ by Lemma 2.1.2 $g(t)^{-1}\dot{g}(t) = \beta_X(x(t))$.

Lemma 2.2.12. *Under the above assumption,*

$$|\nabla\dot{\gamma}| = |d\mu_{\mathfrak{p}}(x)\beta_X(x(t))|. \tag{2.15}$$

Moreover, there exist positive constants c and α such that for all $t > 0$,

$$\int_t^\infty |d\mu_{\mathfrak{p}}(x(r))\beta_X(x(r))|dr \leq \frac{c}{t^\alpha}. \tag{2.16}$$

Proof. Since $\beta = \mu_{\mathfrak{p}}(x(t)) \in \mathfrak{p}$, then $\dot{\beta} = d\mu_{\mathfrak{p}}(x(t))\dot{x}$ and we know from Section 2.1 that $\dot{x} = -\beta_X(x(t))$. So, by [30, Theorem C.1i], bearing in mind that $\mathrm{Re}(g^{-1}\dot{g})$ corresponds to the component of $g^{-1}\dot{g}$ along $\mathfrak{k}$ in our case, $\nabla\dot{\gamma} = d\pi_g(dL_g(d\mu_{\mathfrak{p}}(x)\dot{x})) = -d\pi_g(dL_g(d\mu_{\mathfrak{p}}(x)\beta_X(x(t))))$. Hence, (2.15) follow. (2.16) follow from Theorem 2.1.3. ☐

For $0 \leq s < t$, let

$$H_{s,t} : [s,t] \to M$$

be given by

$$H_{s,t}(r) := \pi\left(g(s)\exp\left(-\frac{r-s}{t-s}\beta(s,t)\right)\right) \quad \forall s \leq r \leq t, \tag{2.17}$$

where $\beta(s,t) \in \mathfrak{p}$ and $k(s,t) \in K$ are chosen such that $g(t) = g(s)\exp(-\beta(s,t))k(s,t)$. $H_{s,t}$ is a geodesic connecting the points $\gamma(t)$ and $\gamma(s)$. i.e

$$H_{s,t}(s) = \gamma(s), \quad H_{s,t}(t) = \gamma(t).$$

For $0 \leq s < t$, define the function $\rho_{s,t} : [s,t] \to \mathbb{R}_{\geq 0}$ by

$$\rho_{s,t}(r) := d_M(H_{s,t}(r), \gamma(r)) \quad \forall s \leq r \leq t.$$

Lemma 2.2.13. *Let c and α be positive constants such that (2.16) holds and fix three real numbers r_0, s, t. Assume $0 \leq s < r_0 < t$ and $\rho_{s,t}(r_0) \neq 0$. Then $\dot{\rho}_{s,t}(r_0) \leq \frac{c}{r_0^\alpha}$.*

Proof. Let r_1 be the smallest real number bigger than r_0 such that $\rho_{s,t}(r_1) = 0$. Then

$$\rho_{s,t}(r_1) = 0; \quad \forall r_1 \in (r_0, t] \quad \text{and } \rho_{s,t}(r) \neq 0; \quad \forall r \in [r_0, r_1).$$

By Lemma 2.2.5 and 2.2.12,

$$-\ddot{\rho}_{s,t}(r) \leq |\nabla \dot{\gamma}(r)| = |d\mu_{\mathfrak{p}}(x(r))\xi_X(x(r)|, \quad \forall r \in [r_0, r_1).$$

Integrating the inequality over the interval $r_0 \leq r < r_1$ gives

$$\begin{aligned}
\dot{\rho}_{s,t}(r_0) &= \frac{d\rho_{s,t}}{dr^-}(r_1) - \int_{r_0}^{r_1} \ddot{\rho}_{s,t}(r)dr \\
&= -|\dot{\gamma}(r_1) - \dot{H}_{s,t}(r_1)| - \int_{r_0}^{r_1} \ddot{\rho}_{s,t}(r)dr \\
&\leq \int_{r_0}^{r_1} |d\mu_{\mathfrak{p}}(x(r))\xi_X(x(r)|dr \\
&\leq \frac{c}{r_0^\alpha}.
\end{aligned}$$

□

Lemma 2.2.14. *Let c and α be positive constants such that (2.16) holds. There exist a positive constant C such that*

$$\rho_{s,t}(r) \leq C(r^{1-\alpha} - s^{1-\alpha}), \tag{2.18}$$

for all real numbers r, s, t such that $0 \leq s \leq r \leq t$.

Proof. Suppose $\alpha > 0$ is as in above and fix $r_1 \in (s,t)$. Suppose without loss of generality that $\rho_{s,t}(r_1) \neq 0$ and choose $r_0 \in [s, r_1)$ with $\rho_{s,t}(r_0) = 0$, and $\rho_{s,t}(r) \neq 0$ for $r \in (r_0, r_1]$. Integrate $\dot{\rho}_{s,t}(r) \leq \frac{c}{r^\alpha}$ over the interval $(r_0, r_1]$ to obtain

$$\rho_{s,t}(r_1) = \int_{r_0}^{r_1} \dot{\rho}_{s,t}(r)dr \leq \int_{r_0}^{s} \frac{c}{r^\alpha}dr = \frac{c}{1-\alpha}(r_1^{1-\alpha} - s^{1-\alpha}).$$

2.18 follows by taking $C = \frac{c}{1-\alpha}$. □

Proposition 2.2.15. *Let* $s \geq 0$. *Then*

$$\left| \frac{\beta(s,t')}{t'-s} - \frac{\beta(s,t)}{t-s} \right| \leq \frac{C}{t^\alpha} \quad \forall t' \geq t \geq s+1.$$

This implies that the limit

$$\beta_\infty(s) := \lim_{t\to\infty} \frac{\beta(s,t)}{t-s}$$

exists in $\mathfrak{p}$.

Proof. the geodesics $H_{s,t}$ and $H_{s,t'}$ intersect at $\gamma(s) = H_{s,t}(s) = H_{s,t'}(s)$. Hence it follows from (2.17), (2.18) and Lemma 2.2.6 that for $t' \geq t \geq s+1$,

$$\begin{aligned}
\left| \frac{\beta(s,t')}{t'-s} - \frac{\beta(s,t)}{t-s} \right| &= |\dot{H}_{s,t}(s) - \dot{H}_{s,t'}(s)| \\
&\leq \frac{d_M(H_{s,t}(t), H_{s,t'}(t))}{t-s} \\
&= \frac{\rho_{s,t'}(t)}{t-s} \\
&\leq C\frac{t^{1-\alpha} - s^{1-\alpha}}{t-s} \\
&\leq \frac{C}{t^\alpha}.
\end{aligned}$$

□

Chapter 3

Closed Orbit of Parabolic Subgroups

In this chapter, we study closed orbits of parabolic subgroups. The result of this chapter is obtained in a joint work with Biliotti [18] which has already been accepted in Nagoya Mathematical Journal.

Let (Z, ω) be a Kähler manifold, U a compact and connected Lie group with Lie algebra $\mathfrak{u}$ and $U^{\mathbb{C}}$ the complexification of U. Suppose $U^{\mathbb{C}}$ acts holomorphically on Z with a momentum map $\mu : Z \to \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a closed connected compatible subgroup. Hence $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of G and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. Let X be a connected G-stable submanifold of Z and let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ the corresponding G-gradient map.

Remarks: Let $Q \subset G$ be a parabolic subgroup. The following facts are easy to check:

a) If $Q \cdot p$ is compact, then $G \cdot p$ is closed since $G = KQ$;

b) let $\mathcal{O}$ be a compact G-orbit. By Proposition 2.1.10, it follows that $\mathcal{O} = G \cdot p = K \cdot p$. Since $\mu_{\mathfrak{p}}$ is K-equivariant, the restricted gradient map $\mu_{\mathfrak{p}} : K \cdot p \to K \cdot \mu_{\mathfrak{p}}(x)$ is a smooth K-equivariant submersion.

Let $\beta \in \mathfrak{p}$ and let

$$Y = \{z \in X : \max_{x \in X} \mu_{\mathfrak{p}}^{\beta}(x) = \mu_{\mathfrak{p}}^{\beta}(z)\}.$$

Assume that Y is not empty. By Corollary 1.6.9.2, Y is a smooth, possibly disconnected, submanifold of X.

Lemma 3.0.1. *Y is $G^{\beta+}$ invariant.*

Proof. Let $g \in G$ and let $\xi \in \mathfrak{p}$. It is easy to check that

$$(\mathrm{d}g)_p(\xi_X) = (\mathrm{Ad}(g)(\xi))_X(gp),$$

and so G^β preserves X^β. We claim that Y is G^β-stable. In fact, $G^\beta = K^\beta \exp(\mathfrak{p}^\beta)$ and Y is K^β-invariant by K-invariant property of the gradient map. For each $y \in Y$, let $\xi \in \mathfrak{p}^\beta$ and let $\gamma(t) = \exp(t\xi) \cdot y$. Since $\beta_X(\gamma(t)) = 0$ it follows that $\mu_{\mathfrak{p}}^\beta(\gamma(t))$ is constant and so $\exp(t\xi) \cdot y \in Y$. Now, $G^{\beta+} = G^\beta R^{\beta+}$ where $R^{\beta+}$ is connected and the unipotent radical of $G^{\beta+}$. By Proposition 1.6.10, $\mathfrak{r}^{\beta+} \subset V_+$ and so $\mathfrak{r}^{\beta+} \cdot z \subset G_z$ for all $z \in Y$. Since $R^{\beta+}$ is connected, this implies $R^{\beta+}$ does not act on Y and the result follows. $\square$

Lemma 3.0.2. *Let $\mathcal{O}$ be a compact G-orbit which is a K-orbit. Let $\beta \in \mathfrak{p}$. If $x \in \mathcal{O}$ is a local maximum of $\mu_{\mathfrak{p}}^\beta : \mathcal{O} \to \mathbb{R}$, then x is a global maximum of $\mu_{\mathfrak{p}}^\beta : \mathcal{O} \to \mathbb{R}$.*

Proof. If $x \in \mathcal{O}$ is a local maximum of $\mu_{\mathfrak{p}}^\beta$, then $\mu_{\mathfrak{p}}(x)$ is a local maximum of the height function

$$K \cdot \mu_{\mathfrak{p}}(x) \to \mathbb{R}, \quad z \mapsto \langle z, \beta \rangle.$$

But it was noted in the proof of Proposition 3.9 in [15] that $\langle \cdot, \beta \rangle$ has only global maximum when restricted to $K \cdot \mu_{\mathfrak{p}}(x)$. Then a local maximum is a global maximum and this implies that $\mu_{\mathfrak{p}}(x)$ is a global maximum of the height function $\langle \cdot, \beta \rangle$. Since

$$\max_{\mathcal{O}}(\beta) = \max\{\langle z, \beta \rangle,\ z \in K \cdot \mu_{\mathfrak{p}}(x)\},$$

x is a global maximum of $\mu_{\mathfrak{p}}^\beta$. $\square$

Proposition 3.0.3. *Let $p \in \mathcal{O}$ be such that $G^{\beta+} \cdot p$ is closed. Then $G^{\beta+} \cdot p$ is a finite union of connected components of* $\max_{\mathcal{O}}(\beta)$.

Proof. Since $G^{\beta+} \cdot p$ is compact, $\mu_{\mathfrak{p}}^\beta|_{G^{\beta+} \cdot p}$ has a maximum. Let $q \in G^{\beta+} \cdot p$ denote a maximum of $\mu_{\mathfrak{p}}^\beta|_{G^{\beta+} \cdot p}$. By Proposition 1.6.10, q is a $R^{\beta+}$ fixed point. Applying again, Proposition 1.6.10, keeping in mind that $\mathcal{O}$ is G homogeneous, q is a local maximum of

$\mu_{\mathfrak{p}}^{\beta} : \mathcal{O} \to \mathbb{R},$. By Lemma 3.0.2, q is a global maximum of $\mu_{\mathfrak{p}}^{\beta}$. By Lemma 3.0.1, the unipotent group $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$ and $G^{\beta+} \cdot p \subset \max_{\mathcal{O}}(\beta)$.

Let $x \in \max_{\mathcal{O}}(\beta)$, By Proposition 1.6.10 and Corollary 1.6.10.1, keeping in mind that $\mathcal{O}$ is G homogeneous and $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$, it follows that $T_x \max_{\mathcal{O}}(\beta) = T_x G^{\beta} \cdot x$. By Lemma 3.0.1 $(G^{\beta})^o$ preserves any connected component of $\max_{\mathcal{O}}(\beta)$. Moreover, the restriction of $\mu_{\mathfrak{p}}$ to any connected component of $\max_{\mathcal{O}}(\beta)$ defines the gradient map of $(G^{\beta})^o$, see [45]. By Proposition 2.1.10 it follows that $(G^{\beta})^o$ has a closed orbit on any connected component of $\max_{\mathcal{O}}(\beta)$. Since any $(G^{\beta})^o$ orbit is open in $\max_{\mathcal{O}}(\beta)$, it follows that the connected component of $\max_{\mathcal{O}}(\beta)$ containing x is $(G^{\beta})^o$ homogeneous. The connected components of G^{β} are finite and intersect the connected components of K^{β}. Therefore, keeping in mind that $R^{\beta+}$ acts trivially on $\max_{\mathcal{O}}(\beta)$, $G^{\beta+} \cdot x$ is a finite union of connected components of $\max_{\mathcal{O}}(\beta)$. The same result holds for $G^{\beta+} \cdot p$, concluding the proof. $\square$

Corollary 3.0.3.1. *Let* $x \in \max_{\mathcal{O}}(\beta)$. *Then* $G^{\beta+} \cdot x$ *is closed and a finite union of connected components of* $\max_{\mathcal{O}}(\beta)$.

Summing up, we have proved our first main result.

Theorem 3.0.4. *Let* $\beta \in \mathfrak{p}$. *Then:*

- *if* $G^{\beta+} \cdot x$ *is compact, then* $\mathcal{O} = G \cdot x$ *is compact and* $G^{\beta+} \cdot x$ *is a finite union of connected components of* $\max_{\mathcal{O}}(\beta)$:
- *if* $\mathcal{O}$ *is a compact* G*-orbit, then* $\max_{\mathcal{O}}(\beta)$ *is a finite union of compact* $G^{\beta+}$*-orbits.*

In particular, the number of compact $G^{\beta+}$*-orbits is equal or bigger than the number of compact* G*-orbits.*

Let $Q \subset U^{\mathbb{C}}$ be a parabolic subgroup. There exists $\beta \in i\mathfrak{u}$ such that $Q = (U^{\mathbb{C}})^{\beta+}$. If $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit, then it is a complex U-orbit and so a flag manifold [35]. By definition of the gradient map,

$$\max_{\tilde{\mathcal{O}}}(\beta) = \max\{p \in \tilde{\mathcal{O}} : \langle \mu(p), -i\beta \rangle = \max_{p \in \tilde{\mathcal{O}}} \mu^{-i\beta}\}$$

and so it is connected [5, 36]. This means that $(U^{\mathbb{C}})^{\beta+}$ has a unique closed orbit in $\tilde{\mathcal{O}}$.

Corollary 3.0.4.1. *The number of compact $(U^{\mathbb{C}})^{\beta+}$-orbits is equal to the number of compact $U^{\mathbb{C}}$-orbits. Any closed $(U^{\mathbb{C}})^{\beta+}$-orbit arises as $\max_{\tilde{\mathcal{O}}}(-i\beta)$, where $\tilde{\mathcal{O}}$ is a compact $U^{\mathbb{C}}$-orbit.*

Assume that G is a real form of $U^{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and so $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^{\mathbb{C}}$ (see [47, 56]). Assume that there exists $x \in Z$ such that $U^{\mathbb{C}} \cdot x$ is compact. Then $U^{\mathbb{C}} \cdot x = U \cdot x$ and so a flag manifold. The following result is essentially an old Theorem of Wolf [70], see also [45].

Theorem 3.0.5 (Wolf)**.** *There exists a unique compact G-orbit in $U^{\mathbb{C}} \cdot x$.*

Proof. Let G_{ss} denote the connected subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Then G_{ss} is closed, compatible and $G = Z(G)^o \cdot G_{ss}$, where $Z(G)^o$ is the connected component of the center [56, p.442]. By Proposition 2.1.10, G has a closed orbit in $U^{\mathbb{C}} \cdot x$. The center of U does not act on $U \cdot x$ and G_{ss} is a real from of $(U^{\mathbb{C}})_{ss} = (U_{ss})^{\mathbb{C}}$. By a Theorem of Wolf [70], G_{ss} has a unique closed orbit in $U^{\mathbb{C}} \cdot x$. On the other hand, by Proposition 2.1.10 it follows that G_{ss} has a closed orbit on any closed orbit of G. Therefore, G has a unique closed orbit in $U^{\mathbb{C}} \cdot x$. □

Let $\mathcal{O}$ denote the unique compact G-orbit in $U^{\mathbb{C}} \cdot x$. Let $\beta \in \mathfrak{p}$. We denote by

$$\max_{U^{\mathbb{C}} \cdot x}(\beta) = \left\{p \in U^{\mathbb{C}} \cdot x : \, \mu_{\mathfrak{p}}^{\beta}(p) = \max_{p \in U^{\mathbb{C}} \cdot x} \mu_{\mathfrak{p}}^{\beta}\right\}$$

Lemma 3.0.6. *For any $\beta \in \mathfrak{p}$, $\max_{U^{\mathbb{C}} \cdot x}(\beta) \cap \mathcal{O} \neq \emptyset$. Hence*

$$\max_{U^{\mathbb{C}} \cdot x}(\beta) \cap \mathcal{O} = \max_{\mathcal{O}}(\beta).$$

Proof. Since

$$\max_{U^{\mathbb{C}} \cdot x}(\beta) = \max\left\{p \in U^{\mathbb{C}} \cdot x : \, \langle \mu(p), -i\beta \rangle = \max_{p \in U^{\mathbb{C}} \cdot x} \mu^{-i\beta}\right\},$$

by Corollary 3.0.4.1 it follows that $\max_{U^{\mathbb{C}} \cdot x}(\beta)$ is the unique closed orbit of $(U^{\mathbb{C}})^{\beta+}$. Since

$$G^{\beta+} = G \cap (U^{\mathbb{C}})^{\beta+}$$

we get $R^{\beta+} \subset R((U^{\mathbb{C}})^{\beta+})$ and so $R^{\beta+}$ acts trivially on $\max_{U^{\mathbb{C}}\cdot x}\mu_{\mathfrak{p}}^{\beta}$. Applying the same arguments of Proposition 3.0.3, it follows that $G^{\beta+}$ has a closed orbit in $\max_{U^{\mathbb{C}}\cdot x}\mu_{\mathfrak{p}}^{\beta}$. By Theorem 3.0.5 we get $\max_{U^{\mathbb{C}}\cdot x}(\beta) \cap \mathcal{O} \neq \emptyset$ and the result follows. □

As a consequence, we obtain the following result.

Proposition 3.0.7. *Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra. Then*

$$\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O}).$$

Proof. It is well-known that both $\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x)$ and $\mu_{\mathfrak{a}}(\mathcal{O})$ are polytope [45, 58]. Since $\mu_{\mathfrak{a}}(\mathcal{O}) \subset \mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x)$, applying the above Lemma and Proposition 1.1.7, we get $\mu_{\mathfrak{a}}(U^{\mathbb{C}} \cdot x) = \mu_{\mathfrak{a}}(\mathcal{O})$. □

Now we are ready to prove the next result.

Theorem 3.0.8. *The set $\max_{\mathcal{O}}(\beta)$ is the unique compact orbit of $G^{\beta+}$ in $\mathcal{O}$. Moreover, it is connected and a $(K^{\beta})^{o}$-orbit.*

Proof. Let $(G^{\beta})^{o}_{ss}$ denote the connected subgroup whose Lie algebra is $[\mathfrak{g}^{\beta}, \mathfrak{g}^{\beta}]$. It is closed, semisimple and compatible [56]. By Lemma 3.0.1 it preserves any connected components of $\max_{\mathcal{O}}(\beta)$. By Proposition 2.1.10, $(G^{\beta})^{o}_{ss}$ has a closed orbit on any connected component $\max_{\mathcal{O}}(\beta)$. On the other hand, $\max_{U^{\mathbb{C}}\cdot x}(\beta)$ is connected and, by Proposition 1.6.10, is a closed orbit of $(U^{\mathbb{C}})^{\beta}$. Note that $(G^{\beta})^{o}_{ss}$ is a real form of $(U^{\beta})_{ss}$ and $\max_{U^{\mathbb{C}}\cdot x}(\beta)$, keeping in mind that it is a flag manifold and the center of $(U^{\mathbb{C}})^{\beta}$ does not act on it, is a compact $(U^{\beta})_{ss}$ orbit. Applying a Theorem of Wolf [70] it follows that $(G^{\beta})^{o}_{ss}$ has a unique closed orbit in $\max_{U^{\mathbb{C}}\cdot x}(\beta)$. Since both $R^{\beta+}$ and the center of $(G^{\beta})^{o}$ act trivially on $\max_{U^{\mathbb{C}}\cdot x}(\beta)$, the unique closed orbit of $(G^{\beta})^{o}_{ss}$ is contained in a closed orbit of $G^{\beta+}$ and so it is contained in $\mathcal{O}$. By Theorem 3.0.4, this orbit is contained in $\max_{\mathcal{O}}(\beta)$. Since $(G^{\beta})^{o}_{ss}$ preserves $\max_{\mathcal{O}}(\beta)$ and it has a closed orbit on any connected component of $\max_{\mathcal{O}}(\beta)$, it follows that $\max_{\mathcal{O}}(\beta)$ is connected. This means $\max_{\mathcal{O}}(\beta)$ is the unique closed orbit of $G^{\beta+}$. In particular, keeping in mind $\max_{\mathcal{O}}(\beta)$ is $(G^{\beta})^{o}$ homogeneous, applying Proposition 2.1.10 we get $\max_{\mathcal{O}}(\beta)$ is a $(K^{\beta})^{o}$-orbit concluding the proof. □

Chapter 4

Convexity Properties of Gradient map

In this chapter, we study the convexity properties of the gradient map applying most of the results obtained in chapter 2. Both the Abelian and non-Abelian cases are studied. The results of this chapter is a joint work with Biliotti [17] and by Windare [69]. We assume U is connected, G is connected and X is compact and connected throughout this chapter.

In section 4.1, we prove the Abelian convexity theorem. If the G-action on X has a unique closed orbit $\mathcal{O}$, then we prove that $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$ and so a polytope (Theorem 4.1.2). This result is new also if $G = U^{\mathbb{C}}$ and $X = Z$. This means that $\mathcal{O}$ captures all the information of the A-gradient map. In section 4.1.1, we obtain the Abelian convexity theorem from the non-Abelian convexity result.

In section 4.2, we continue to investigate convexity properties of gradient map. If Z is connected and compact, and $X \subset Z$ is a A-stable compact coisotropic submanifold, then we prove $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(Z)$ (Theorem 4.2.4) and so it is a polytope as well. More precisely, there exists an open and dense subset W of X such that for any $p \in W$, we have $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\overline{A \cdot p})$.

In section 4.3, we study the convexity property for G-action on a Lagrange submanifold. Suppose $G \subset U^{\mathbb{C}}$ is a real form, we show that there exists a unique open stratum in X if X is a G-invariant compact Lagrangian submanifold of Z such that $X \subset \mu_{\mathfrak{k}}^{-1}(0)$ (Theorem 4.3.2.) Therefore, the non-Abelian convexity result holds in this case (Theorem

4.3.3).

We conclude the chapter with the study of two orbits variety in section 4.4. Applying standard Morse-theoretic results in [54] and [45], we prove that the norm square is Morse-Bott for two orbits variety and obtain information on the cohomology and K-equivariant cohomology of X (Theorem 4.4.1), generalizing [32]. As an application of Theorem 4.1.2, we prove that the Abelian convexity Theorem holds for a two orbits variety (Theorem 4.4.2).

4.1 The Abelian Convexity Theorem

Let (Z, ω) be a Kähler manifold, U a compact and connected Lie group with Lie algebra $\mathfrak{u}$ and $U^{\mathbb{C}}$ the complexification of U. Suppose $U^{\mathbb{C}}$ acts holomorphically on Z with a momentum map $\mu : Z \to \mathfrak{u}$. Let $G \subset U^{\mathbb{C}}$ be a connected compatible subgroup. Hence $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of G and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G. Let X be a G-stable locally closed compact and connected real submanifold of Z and let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ be the corresponding G-gradient map. Let $\beta \in \mathfrak{p}$ and let

$$Y = \{z \in X : \max_{x \in X} \mu_{\mathfrak{p}}^{\beta} = \mu_{\mathfrak{p}}^{\beta}(z)\}.$$

Proposition 4.1.1. *Y contains a closed orbit of $G^{\beta+}$ which coincides with a K^{β} orbit.*

Proof. By Lemma 3.0.1, $(G^{\beta})^o$ preserves any connected component of Y and the restriction of $\mu_{\mathfrak{p}}$ on any connected component defines a gradient map with respect to $(G^{\beta})^o$ [45]. By Corollary 6.11 in [45] pag. 21, $(G^{\beta+})^o$ has closed orbit which coincides with a $(K^{\beta})^o$ orbit. Since G^{β} has a finite number of connected components and any connected component of G^{β} intersects K^{β}, it follows that G^{β} has a closed orbit which coincides with a K^{β} orbit. This is also a closed orbit of $G^{\beta+}$ since $R^{\beta+}$ acts freely on Y by the Linearization theorem proved in [44], see also Proposition 1.6.12, concluding the proof. $\square$

Let $\mathfrak{a} \subset \mathfrak{p}$ be an Abelian subalgebra of $\mathfrak{p}$ and let $\pi_{\mathfrak{a}} : \mathfrak{p} \to \mathfrak{a}$ be the orthogonal projection onto $\mathfrak{a}$. Then $\mu_{\mathfrak{a}} := \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ is the gradient map associated to $A = \exp(\mathfrak{a})$.

Denote $P = \mathrm{Conv}(\mu_{\mathfrak{a}}(X))$. It is well-known, see for instance [46, Prop. 3] and [12, Prop. 3.1], that P is is the convex hull of $\mu_{\mathfrak{a}}(X^A)$, where $X^A = \{p \in X : A \cdot p = p\}$. Since $\mu(X^A)$ is finite, then the convex hull of $\mu_{\mathfrak{a}}(X)$ is a polytope.

Suppose that the G action on X has a unique closed orbit, which is a K orbit [45] denoted by $\mathcal{O}$. Let $\mathfrak{a}'$ be a maximal Abelian subalgebra containing $\mathfrak{a}$. Since $\mu_{\mathfrak{a}}(\mathcal{O}) = \pi_{\mathfrak{a}}(\mu_{\mathfrak{a}'}(\mathcal{O}))$, by a Theorem of Kostant [58] it follows that $\mu_{\mathfrak{a}}(\mathcal{O})$ is a polytope.

Theorem 4.1.2. *Suppose the G-action on X has a unique closed orbit $\mathcal{O}$. Then $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$ and so it is a convex polytope.*

Proof. Let σ be a face of P. Since P is a polytope, any face is exposed [67]. Therefore, there exists $\xi \in \mathfrak{a}$ such that

$$\sigma = \{\alpha \in P : \max_{\gamma \in P} \langle \gamma, \xi \rangle = \langle \alpha, \xi \rangle\}.$$

Then

$$Y = \mu_{\mathfrak{a}}^{-1}(\sigma) = \{z \in X : \max_{x \in X} \mu_{\mathfrak{p}}^{\xi}(x) = \mu_{\mathfrak{p}}^{\xi}(z)\}.$$

In fact, it is easy to see that $Y \subset \mu_{\mathfrak{a}}^{-1}(\sigma)$. Suppose $z \in \mu_{\mathfrak{a}}^{-1}(\sigma)$, then $\mu_{\mathfrak{a}}(z) \in \sigma$ and $\max_{\gamma \in P} \langle \gamma, \xi \rangle = \langle \mu_{\mathfrak{a}}(z), \xi \rangle = \mu_{\mathfrak{p}}^{\xi}(z)$. Hence, $z \in Y$. By Lemma 3.0.1, Y is $G^{\xi+}$-invariant. By Proposition 4.1.1, let $z \in Y$ be such that $G^{\xi+} \cdot z$ is closed. But $G = KG^{\xi+}$, then $G \cdot z$ is closed. Since the action of G on X has a unique closed orbit, $G \cdot z = \mathcal{O}$. Therefore,

$$\max_{x \in P} \langle x, \xi \rangle = \max_{x \in \mu_{\mathfrak{a}}(\mathcal{O})} \langle x, \xi \rangle.$$

By Proposition 1.1.7, $\mu_{\mathfrak{a}}(\mathcal{O}) = P$. Hence, $\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(\mathcal{O})$ is a polytope. □

Remark 4.1.3. *In the above assumption, applying the main Theorem in [12], the convex hull of $\mu_{\mathfrak{p}}(X)$ coincides with the convex hull of $\mu_{\mathfrak{p}}(\mathcal{O})$. Hence the convex hull of $\mu_{\mathfrak{p}}(X)$ is the convex hull of a K-orbit in $\mathfrak{p}$ and so a polar orbitope [15].*

4.1.1 Abelian convexity from Non-Abelian convexity

The non-Abelian convexity theorem implies the Abelian convexity theorem. This is the purpose of this section.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian, $\mathfrak{a}_+$ positive Weyl Chamber. If $\lambda \in \mathfrak{a}_+$, we denote by

$$\Delta_\lambda = \mathrm{Conv}\{w\lambda : w \in W\},$$

where $W = \frac{N_k(\mathfrak{a})}{C_k(\mathfrak{a})}$ is the Weyl-Group. If $G = U^{\mathbb{C}}$, then the following result is proved in [34]. The authors applied Kirwan's Theorem [55] for the action $U \times \mathbb{T}$ on the cotangent bundle T^*U, where $\mathbb{T}$ is a maximal torus of U. Our proof uses a result of Gichev [31].

Theorem 4.1.4. *If $S \subset \mathfrak{a}_+$ is a convex subset, then,*

$$S^\# = \bigcup \{\Delta_\lambda : \lambda \in S\} \tag{4.1}$$

is convex subset of $\mathfrak{a}$.

Proof. Let

$$\Delta_0 = \{(\lambda, \mu) \in \mathfrak{a}_+ \times \mathfrak{a} : \mu \in \Delta_\lambda\}.$$

We claim that Δ_0 is convex. Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Delta_0$

$$t(\lambda_1, \mu_1) + (1-t)(\lambda_2, \mu_2) = (t\lambda_1 + (1-t)\lambda_2, t\mu_1 + (1-t)\mu_2)$$

Now, from [31] we have

$$\Delta_{(t\lambda_1+(1-t)\lambda_2)} = \Delta_{t\lambda_1} + \Delta_{(1-t)\lambda_2} = t\Delta_{\lambda_1} + (1-t)\Delta_{\lambda_2}$$

and so

$$t\mu_1 + (1-t)\mu_2 \in \Delta_{(t\lambda_1+(1-t)\lambda_2)}.$$

This shows that Δ_0 is convex. Let

$$\pi_1 : \mathfrak{a}_+ \times \mathfrak{a} \to \mathfrak{a}_+$$

and

$$\pi_2 : \mathfrak{a}_+ \times \mathfrak{a} \to \mathfrak{a}.$$

Then,

$$S^\# = \pi_2(\pi_1^{-1}(S) \cap \Delta_0)$$

and so it is convex. □

Theorem 4.1.5. *Let $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$ be the gradient map. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra and let $\mu_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ be the corresponding gradient map. If $\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+$ is convex, then*

$$\mu_{\mathfrak{a}}(X) = (\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+)^{\#}, \tag{4.2}$$

and so convex.

Proof. The action of K on $\mathfrak{p}$ is polar and $\mathfrak{a}$ is a section [27]. Moreover, if $x \in \mathfrak{p}$, then

$$K \cdot x \cap \mathfrak{a}_+ = \{\lambda\}.$$

By a beautiful Theorem of Kostant [58], $\pi_{\mathfrak{a}}(K \cdot x) = \Delta_\lambda$ and so a polytope. Therefore

$$\mu_{\mathfrak{a}}(X) = \{\mu \in \mathfrak{a} : \mu \in \Delta_\lambda, \quad \text{where} \quad \lambda \in \mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+\} = (\mu_{\mathfrak{p}}(X) \cap \mathfrak{a}_+)^{\#}.$$

By the above Theorem, $\mu_{\mathfrak{a}}(X)$ is convex. □

4.2 Convexity results of Gradient map on coistropic submanifold

In this section, we continue to investigate Abelian convexity properties of the gradient map. We give a new proof of the convexity property of the gradient map for $X = Z$, avoiding the Linearization Theorem.

Theorem 4.2.1. *Suppose (Z, ω) is a connected and compact Kähler manifold. Then*

$$\mu_{\mathfrak{a}} : Z \to \mathfrak{a}$$

is a convex polytope.

Proof. Let $\xi \in \mathfrak{a}$ and $\{t_n\}_{n \in \mathbb{N}}$ be a sequence such that $t_n \to \infty$. Denote by $g_n = \exp(t_n \xi)$. By Theorem 2 in [8], up to passing to a subsequence, there exist a proper analytic subset U of Z such that

$$g_n : Z \setminus U \to Z, \quad g_n \to \varphi_\infty,$$

where φ_∞ is non-dominant meromorphic map. Note that $Z \setminus U$ is connected and Zariski open. Since $g_n = \exp(t_n \xi)$, it follows $\varphi_\infty(Z \setminus U) \subset Z^\xi$ where

$$Z^\xi = \{x \in Z : \xi_Z(x) = 0\}.$$

The vector $J\xi$ is a Killing vector field and so Z^ξ is smooth, possibly disconnected, submanifold of Z [57]. Since $Z \setminus U$ s connected, $\varphi_\infty(Z \setminus U)$ is contained in a connected components of Z^ξ which we denoted by Z_0.

Set $\mu_{\mathfrak{a}}^\xi := \langle \mu_{\mathfrak{a}}, \xi \rangle$. $\mu_{\mathfrak{a}}^\xi(Z_0)$ is constant. We claim that

$$\mu_{\mathfrak{a}}^\xi(Z_0) = \max_{z \in Z} \mu_{\mathfrak{a}}^\xi.$$

Indeed, let $x_0 \in Z$:

$$\mu_{\mathfrak{a}}^\xi(x_0) = c = \max_{z \in Z} \mu_{\mathfrak{a}}^\xi.$$

Let $\epsilon > 0$. There exist neighbourhood N of x_0 such that

$$\mu_{\mathfrak{a}}^\xi(N) \subset (c - \epsilon, c].$$

$(Z \setminus U) \cap N \neq \emptyset$. Pick $p \in (Z \setminus U) \cap N$. Then

$$c - \epsilon \leq \mu_{\mathfrak{a}}^\xi(p) \leq \mu_{\mathfrak{a}}^\xi(\varphi_\infty(p)).$$

Therefore

$$\mu_{\mathfrak{a}}^\xi(Z_0) = c = \max_{z \in Z} \mu_{\mathfrak{a}}^\xi.$$

Let $P = \mathrm{Conv}(\mu_{\mathfrak{a}}(Z))$. By a Theorem of Atiyah [5], see also [9, 47], for any $p \in Z$ we have

$$\mu_{\mathfrak{a}}(\overline{A \cdot p}) = \mathrm{Conv}(\mu_{\mathfrak{a}}(Z^A \cap \overline{A \cdot p})) \subset \mu_{\mathfrak{a}}(p) + \mathfrak{a}_p, \tag{4.3}$$

where $Z^A = \{z \in Z : A \cdot z = z\}$ and $\mathfrak{a}_p$ is the Lie algebra of A_p. Since Z is compact, the set Z^A has finitely many connected components. By (4.3), it follows that $\mu_{\mathfrak{a}}(Z^A)$ is finite and $P = \mathrm{Conv}(\mu_{\mathfrak{a}}(Z^A))$. This implies that P is a polytope.

Let $x_0, x_1, \cdots, x_n \in P$ be verticies. Since P is a polytope any face is exposed. Then there exist $\xi_0, \xi_1, \cdots, \xi_n \in \mathfrak{a}$ such that

$$x_i = \{\theta \in P : \langle \theta, \xi_i \rangle = \max_{y \in P} \langle y, \xi_i \rangle, i = 0, 1, \cdots, n\}.$$

Denote $c_i = \langle x_i, \xi_i \rangle$. There exists $U_0, \cdots, U_n$ proper analytic subset and and a $t_N \to +\infty$ such that $\lim_{N\to\infty} \exp(t_N \xi_i)$ exists in $Z \setminus U_i$. Moreover, if $z_i \in Z \setminus U_i$, then

$$\lim_{N\to\infty} \exp(t_N \xi_i) \cdot z_i \in (\mu_{\mathfrak{a}}^{\xi_i})^{-1}(c_i) = \mu_{\mathfrak{a}}^{-1}(X_i).$$

Now,

$$(Z \setminus U_0) \cap (Z \setminus U_1) \cap \cdots \cap (Z \setminus U_n) = Z \setminus (U_0 \cup U_1 \cup \cdots \cup U_n) \neq \emptyset,$$

then

$$\overline{A \cdot p} \cap \mu_{\mathfrak{a}}^{-1}(x_i) \neq \emptyset,$$

whenever $p \in Z \setminus (U_0 \cup U_1 \cup \cdots \cup U_n)$ for any $i = 0, \cdots, n$. By Atiyah-Theorem, see [9, 45],

$$\mu_{\mathfrak{a}}(\overline{A \cdot p}) = \mathrm{Conv}(\mu_{\mathfrak{a}}(Z^A) \cap \overline{A \cdot p}) = P.$$

Therefore

$$\mu_{\mathfrak{a}}(Z) = P.$$

$\square$

Corollary 4.2.1.1. *In the above setting, the following hold true:*

a) $\{p \in Z : \mu_{\mathfrak{a}}(\overline{A \cdot p}) = \mu_{\mathfrak{a}}(Z)\}$ *contains an open and dense subset of* Z.

b) Any local maximum of $\mu_{\mathfrak{a}}^{\xi}$ *is a global maximum. Indeed, we have proved that the unstable manifold of the critical component* c_0 *corresponding to the maximum is Zariski open.*

We now prove the convexity property of the gradient map when X is a connected, compact coisotropic submanifold of (Z, ω).

Definition 4.2.1. *A submanifold* $X \subset (Z, \omega)$ *is coisotropic if for any* $p \in X$, *we have*

$$(T_pX)^{\perp_\omega} \subset T_pX.$$

Since (Z, ω) is Kähler ,

$$(T_pX)^{\perp_\omega} = J((T_pX)^\perp).$$

Lemma 4.2.2. *If X is coisotropic, then for any $p \in X$, we have*

$$T_pX + J(T_pX) = T_pZ.$$

Proof.

$$J((T_pX)^\perp) \subset T_pX.$$

Applying J we have

$$(T_pX)^\perp \subset J(T_pX).$$

And so

$$T_pX + J(T_pX) = T_pZ.$$

$\square$

Lemma 4.2.3. *Let X be a A-invariant compact connected coistropic submanifold of (Z, ω). Let $\xi \in \mathfrak{a}$. Then*

$$max_{p\in X}\mu_{\mathfrak{a}}^\xi = max_{z\in Z}\mu_{\mathfrak{a}}^\xi.$$

Moreover, the unstable manifold associated to the maximum of $\mu_{\mathfrak{a}}^\xi$ is open and dense.

Proof. Let W_0^ξ be the unstable manifold of the critical component C_0 satisfying $\mu_{\mathfrak{a}}^\xi(C_0) = c_0$. Assume that C_0 corresponds to a Local maximum. Since

$$\nabla\mu_{\mathfrak{a}}^\xi|_X = \nabla\mu_{\mathfrak{a}}^\xi,$$

it follows that

$$W_0^\xi = \bar{W_0^\xi} \cap X,$$

where $\bar{W_0^\xi}$ is the unstable manifold in Z of the critical components $\bar{C}_0$ such that

$$\mu_{\mathfrak{a}}^\xi(\bar{C}_0) = \mu_{\mathfrak{a}}^\xi(C_0) = c_0.$$

By a Linearization theorem in [44], $\bar{W}_0^\xi$ is a complex manifold and W_0^ξ is open in X. Let $p \in W_0^\xi$. Since

$$T_p W_0^\xi = T_p X \subset T_p \bar{W}_0^\xi,$$

it follows that

$$T_p X + J(T_p X) \subset T_p \bar{W}_0^\xi.$$

By Lemma 4.2.2, $\bar{W}_0^\xi$ is open. Since $\mu_{\mathfrak{a}}^\xi : Z \longrightarrow \mathbb{R}$ is Morse-Bott of even index, it follows that $\mu_{\mathfrak{a}}^\xi : Z \longrightarrow \mathbb{R}$ has a unique local maximum and so $\bar{W}_0^\xi$ is open and dense. Therefore $\mu_{\mathfrak{a}} : X \longrightarrow \mathbb{R}$ has also a unique local maximum. This proves

$$\max_{p\in X} \mu_{\mathfrak{a}}^\xi = \max_{z\in Z} \mu_{\mathfrak{a}}^\xi.$$

Since $\mu_{\mathfrak{a}}^\xi : X \longrightarrow \mathbb{R}$ is Morse-Bott, applying Theorem 1.6.13 we get that, the unstable manifolds different from W_0^ξ have codimension at least one. This implies that W_0^ξ is also open and dense in X.

□

Theorem 4.2.4. *If X is a A-invariant compact connected coistropic submanifold of (Z, ω). Then*

$$\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(Z),$$

and so a polytope. Moreover, there exists an open and dense subset W of X such that for any $p \in W$, we have

$$\mu_{\mathfrak{a}}(X) = \overline{\mu_{\mathfrak{a}}(A \cdot p)}.$$

Proof. Let $\xi \in \mathfrak{a}$, by Lemma 4.2.3,

$$\max_{p\in X} \mu_{\mathfrak{a}}^\xi = \max_{z\in Z} \mu_{\mathfrak{a}}^\xi,$$

and the unstable manifold associated to the maximum of $\mu_{\mathfrak{a}}^\xi$ is open and dense. By Proposition 1.1.7,

$$\mu_{\mathfrak{a}}(X) = \mu_{\mathfrak{a}}(Z).$$

By Proposition 3.1 in [9], the set

$$\{p \in X : \mu_{\mathfrak{a}}(\overline{A \cdot p}) = \mu_{\mathfrak{a}}(X)\}$$

is open and dense, concluding the proof. $\square$

4.3 Lagrange submanifold

The result of this section is obtained by Windare in [69]. The norm square f define a stratification on X (section 2.1). Non-Abelian covexity result essentially use the stratification theorem [46, 54]. In a situation where the convexity result holds, the minimal stratum, the stratum where the normed square attains its minimum is unique. The uniqueness of the minimal stratum holds if $X = Z$ and $G = U^{\mathbb{C}}$. In this situation, a general convexity results are available [36, 54].

In a general setting, there is still a minimal stratum and this stratum is always open. However, there could be more than one minimal strata and other open strata that are not minimal. Heinzner P and Schützdeller P proved the convexity result in a projective setup [46].

In this section, we find a special case where the minimal stratum is the unique open stratum in X.

Suppose as above that the action $U^{\mathbb{C}} \times Z \to Z$ is holomorphic, U preserves ω and that there is a U-equivariant momentum map $\mu : Z \to \mathfrak{u}$. Let the function $F : Z \to \mathbb{R}$ defined by

$$F(z) := \frac{1}{2} \parallel \mu(z) \parallel^2, \qquad \text{for} \quad z \in Z \tag{4.4}$$

denote the norm square of the momentum map.

Suppose $G \subset U^{\mathbb{C}}$ is a real form. This means that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ implies $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ and the momentum map μ decompose to $\mu = \mu_{\mathfrak{k}} \oplus -i\mu_{\mathfrak{p}}$. We fix a G-invariant compact Lagrangian submanifold X of Z such that $X \subset \mu_{\mathfrak{k}}^{-1}(0)$ and let $f : X \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{2} \parallel \mu_{\mathfrak{p}}(x) \parallel^2, \qquad \text{for} \quad x \in X \tag{4.5}$$

denote the norm square of the gradient map for the G-action on X.

Note that the condition $X \subset \mu_{\mathfrak{k}}^{-1}(0)$ is always satisfied if G has finite fundamental group. Indeed, since X is Lagrangian, $\mu_{\mathfrak{k}}$ restricted to X is constant and the image lies in the center of the Lie algebra of K. Hence, if G has a finite fundamental group, then K has a finite fundamental group and so K is semisimple and the center of K is finite.

Lemma 4.3.1. *$x \in X$ is a critical point of f if and only if x is a critical point of F.*

Proof. Since μ decompose to $\mu = \mu_{\mathfrak{k}} \oplus \mu_{\mathfrak{p}}$ and $X \subset \mu_{\mathfrak{k}}^{-1}(0)$ then, $\mu|_X = \mu_{\mathfrak{p}}|_X$. This implies that for $x \in X$, the negative gradient flow of f through x coincides with the gradient flow of F through x and so, the result follows. $\square$

Lemma 4.3.1 asserts that for $\beta \in \mathfrak{p}$, $K \cdot \beta$ is a critical orbit of $f_{\mathfrak{p}}$ if and only if $U \cdot i\beta$ is a critical orbit of f.

Theorem 4.3.2. *Suppose X is a G-invariant compact connected Lagrangian submanifold of Z such that $X \subset \mu_{\mathfrak{k}}^{-1}(0)$. Then there exists a unique open stratum in X. Moreover, this stratum is dense in X.*

Proof. It is well known that the strata associated with F are locally closed submanifolds of Z of even dimension [54]. Since it is impossible to disconnect a manifold by removing submanifolds of codimension at least two, there must be a unique open stratum. Let $\overline{S_{-i\beta}}$ be the stratum associated with the minimum of F for the $U^{\mathbb{C}}$-action on Z. $\overline{S_{-i\beta}}$ is the unique open stratum of F. We set $S_\beta = \overline{S_{-i\beta}} \cap X$ which is a stratum of f by Lemma 4.3.1. We claim that S_β is the minimal stratum of f. Indeed, For $p \in S_\beta$, $T_pS_\beta = T_pX \subset T_p\overline{S_{-i\beta}}$ and $J(T_pX) \subset T_p\overline{S_{-i\beta}}$. This means that $T_pS_\beta \oplus J(T_pS_\beta) = T_pZ$. This implies that S_β is open. Suppose it is not the unique open stratum. Then for each open stratum $S_{\widetilde{\beta}}$, there exists $\overline{S_{-i\widetilde{\beta}}}$ such that $S_{\widetilde{\beta}} = \overline{S_{-i\widetilde{\beta}}} \cap X$. Then $\overline{S_{-i\widetilde{\beta}}}$ is open. But $\overline{S_{-i\beta}}$ is unique. It follows that S_β is the unique open strata associated with f.

Since X is compact, there are finitely many strata [45, 8.5]. But X has exactly one open stratum and the other strata are codimension at least one. This implies that the union of these other strata has empty interior. Thus, the open stratum S_β is dense. $\square$

Let $\beta \in \mathfrak{p}$. The orbit $U \cdot \beta$ can be identified with the coadjoint orbit $U \cdot (-i\beta) \subset \mathfrak{u}$ which is a complex flag manifold. There is an induced $U^{\mathbb{C}}$-action on $U \cdot \beta$ and the map $\mu_\beta : Z \times U \cdot \beta \to \mathfrak{u}$; $(x, \xi) \mapsto \mu(z) - Ad_g\xi$ with $g \in U$ is the moment map for the U-action on $Z \times U \cdot \beta$. There is a corresponding Gradient map called the shifting of $\mu_\mathfrak{p}$ with respect to β given as $\mu_{\mathfrak{p},\beta} : X \times K \cdot \beta \to \mathfrak{p}, (x, \xi) \mapsto \mu_\mathfrak{p} - Ad_k\beta$.

By a result in [33], $K \cdot \beta$ is a Lagrangian submanifold of $U \cdot \beta$. Hence, $X \times K \cdot \beta$ is a Lagrangian submanifold of $Z \times U \cdot \beta$ and $X \times K \cdot \beta \subset \mu_{\mathfrak{k},\beta}^{-1}(0)$.

Corollary 4.3.2.1. *In the above assumption, the norm square of the gradient map $\mu_{\mathfrak{p},\beta}$ has a unique open stratum which is dense.*

Let $\mathfrak{a}$ be the maximal Abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{a}_+$ a positive Weyl chamber of $\mathfrak{a}$. Using Theorem 4.3.2, the following non-Abelian convexity theorem holds.

Theorem 4.3.3. *Suppose X is a G-invariant compact connected Lagrangian submanifold of Z such that $X \subset \mu_\mathfrak{k}^{-1}(0)$. The set $\mu_\mathfrak{p}(X) \cap \mathfrak{a}_+$ is a convex polytope.*

Proof. The proof follows the geometric idea used in Kirwan's proof of convexity result in [55]. The proof used the fact that if a subset D of a Euclidean vector space is a finite union of convex polytopes and is not convex, then for any sufficiently small $r > 0$ there exists a point $\beta \in \mathfrak{a}$ such that the closed ball centered at β with radius r meets D in precisely two points.

$\mu_\mathfrak{p}(X) \cap \mathfrak{a}_+$ is a finite union of convex polytopes [46, 8.5]. Suppose $\mu_\mathfrak{p}(X) \cap \mathfrak{a}_+$ is not convex. There is a point $\beta \in \mathfrak{a}_+$ and $r > 0$, such that the boundary of the closed ball $B_r(\beta)$ meets $\mu_\mathfrak{p}(X) \cap \mathfrak{a}_+$ in at least two points. This implies that the function $\parallel \mu_{\mathfrak{p},\beta} \parallel^2$ attains its minimum value in two or more points. Suppose ξ_1 and ξ_2 are two of the minimum points of $\parallel \mu_{\mathfrak{p},\beta} \parallel^2$, then since $\xi_1 \neq \xi_2$ and keeping in mind that $K \cdot \xi_1 \cap \mathfrak{a}_+ = \{\xi_1\}$, by the stratification theorem [45], see also Theorem 2.1.4, we have two disjoint open strata S_{ξ_1} and S_{ξ_2} and this is a contradiction of the above lemma. $\square$

4.4 Two orbits variety

In this section we investigate two orbits variety.

Definition 4.4.1. *Let X be a compact and connected G-stable submanifold of (Z, ω). We say that X is a two orbit variety if G-action on X has two orbits.*

Example 4.4.1. *$SL(n, \mathbb{R})$ acting diagonally on $\mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R})$ is a two orbit variety. The two orbits are $\{(p, q) \in \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R}) : p = q\}$ and $\{(p, q) \in \mathbb{P}^n(\mathbb{R}) \times \mathbb{P}^n(\mathbb{R}) : p \neq q\}$.*

The norm square f has a maximum and a minimum. By the stratification theorem, Theorem 2.1.4, keeping in mind that the strata are G-invariant, X is the union of a closed G-orbit $S_{\beta_{max}}$, where the norm square achieves the maximum, and an open G-orbit $S_{\beta_{min}}$, the stratum relative to the minimum of the norm square. We then show that f is a Morse-Bott function.

Theorem 4.4.1. *If G acts on X with two orbits, then*

a) the function $f : X \to \mathbb{R}$ given by

$$f(x) := \frac{1}{2} \parallel \mu_{\mathfrak{p}}(x) \parallel^2 \qquad \text{for} \quad x \in X.$$

is Morse-Bott; it has only two connected critical submanifolds given by the closed G-orbit $S_{\beta_{max}}$, the stratum associated with the maximum of f and by a K-orbit $S_{\beta_{min}}$, the stratum associated with the minimum of f.

b) The Poincaré polynomial $P_X(t)$ of X satisfies

$$P_X(t) = t^k \cdot P_{S_{\beta_{max}}}(t) + P_{S_{\beta_{min}}}(t) - (1 + t)R(t),$$

where k is the real codimension of $S_{\beta_{max}}$ in X and $R(t)$ is a polynomial with positive integer coefficients. In particular $\chi(X) = \chi(S_{\beta_{max}}) + \chi(S_{\beta_{min}})$;

c) The K-equivariant Poincaré series of X is given by

$$P^K_X(t) = t^k \cdot P^K_{S_{\beta_{max}}}(t) + P^K_{S_{\beta_{min}}}(t).$$

Proof. We first proof (a). Consider the function f and its critical set C. f is non constant on X; in fact if f is constant, then every point of X is a maximum point and in view of proposition 2.1.10, all G-orbit would be closed.

By Theorem 2.2.8, we have that $S_{\beta_{min}}$ consist of a single K-orbit, and so it is connected.

Since f realizes its maximum value at any critical point x belonging to $S_{\beta_{max}}$, then by Proposition 2.1.8d

$$H_x(f) < 0 \quad \text{on} \quad T_x(S_{\beta_{max}})^{\perp}.$$

Now we show that the Hessian of f at a critical point x belonging to $S_{\beta_{min}}$ is non degenerate in the normal direction. Set $\mu_{\mathfrak{p}}(x) = \beta_{min} = \beta$.

Suppose $\beta \neq 0$. By Remark 2.1.9,

$$T_x S_{\beta_{min}} = T_x(G \cdot x) = T_x(K \cdot x) \oplus \mathfrak{p}^{\beta} \cdot x \oplus \mathfrak{r}^{\beta+} \cdot x,$$

where $\mathfrak{r}^{\beta+}$ is the Lie algebra of $R^{\beta+}$. By Proposition 2.1.8b,

$$H_x(f) > 0 \quad \text{on} \quad \mathfrak{p}^{\beta} \cdot x \oplus \mathfrak{r}^{\beta+} \cdot x.$$

Since $H_x(f) \geq 0$, it follows that

$$T_x(G \cdot x) = T_x(K \cdot x) \oplus^{\perp} (\mathfrak{p}^{\beta} \cdot x \oplus \mathfrak{r}^{\beta+} \cdot x)$$

Suppose $\beta = 0$. Let $x = x_{min}$. By Theorem 2.2.8, $\mu_{\mathfrak{p}}^{-1}(0) = K \cdot x$.

$$\ker \ d\mu_{\mathfrak{p}}(x) = (\mathfrak{p} \cdot x)^{\perp}$$

By Proposition 2.1.7,

$$H_x(f)|_{(\mathfrak{p} \cdot x)} > 0.$$

$$T_x X = T_x(G \cdot x).$$

Since $K \cdot x_{min} \subset \ker \ d\mu_{\mathfrak{p}}(x_{min})$, it follows that

$$T_x(G \cdot x) = K \cdot x + \mathfrak{p} \cdot x = T_x X$$

$$H_x(f)|_{(K\cdot x)} = 0.$$

$$T_xX = K \cdot x \oplus^{\perp} \mathfrak{p} \cdot x.$$

By dimensional reason, $H_x(f)$ is non degenerate.

These show that $H_x(f)$ is non degenerate. Hence f is Morse-Bott. The statements in (b) and (c) follow from the general theory in [54]. $\square$

Now, we investigate the convexity property of gradient map of a two orbit variety. Since X has a unique closed orbit, we derive the following result.

Theorem 4.4.2. *If X is a two orbits variety, then $\mu_{\mathfrak{a}}(X)$ is a polytope.*

Proof. A two orbits variety have one closed orbit while the other is open. Hence, it has a unique closed orbit and the result follows from Theorem 4.1.2. $\square$

Chapter 5

Stability, analytic stability for real reductive Lie groups

Let (Z, ω) be a connected Kähler manifold, U a compact (we do not assume that U is connected) Lie group with Lie algebra $\mathfrak{u}$ and $U^{\mathbb{C}}$ the complexification of U. Suppose $U^{\mathbb{C}}$ acts holomorphically on Z with a momentum map $\mu : Z \to \mathfrak{u}$.

In this chapter, we study the Hilbert Mumford criterion for semistability and polystability points associated with the actions of real reductive groups on real submanifolds of Kähler manifolds. Most of the results of this chapter are obtained in a joint work with Biliotti [19] and has been accepted for publication on The Journal of Geometric Analysis.

In section 5.1, we define the stability conditions and the maximal weight associated with the gradient map and some of its properties are given.

In section 5.2 we introduce and discuss the class of energy complete actions and we prove the moment-weight inequality (Theorem 5.2.7).

In Section 5.3 we introduce the notions of analytic semistability and polystability. we prove that the semistability condition, respectively polystability condition, is equivalent to analytic semistability condition (Theorem 5.3.4), respectively analytic polystability condition (Theorem 5.3.6). We also characterize the semistable and polystable points in terms of one-parameter subgroups (Corollary 5.3.4.1 and Corollary 5.3.6.1).

In Section 5.5 we compare the set of semistable points of different compatible subgroups. Roughly speaking, we prove that if $H \subset G$ is compatible then the set of the semistable points with respect to G is contained in the set of semistable points of H. Given a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, we have $X_{\mu_{\mathfrak{p}}}^{ss} = \bigcap_{k\in K} kX_{\mu_{\mathfrak{a}}}^{ss}$. Finally, if $Z_{\mu}^{ss} \neq \emptyset$, then $Z_{\mu_{\mathfrak{p}}}^{ss}$ is open, connected and dense.

In Section 5.6, following Teleman [4], we introduce the notion of symplectization of the $U^{\mathbb{C}}$ action with respect to G.

In the last section 5.7, we characterize the stability conditions in term of the Kempf-Ness function under the assumption that X is compact (Theorems 5.7.2, 5.7.3 and 5.7.4). We conclude the section with the Hilbert criterion for semistability condition with the compactness of X (Theorem 5.7.5).

5.1 Stability and Maximal Weight Function

Let $G \subset U^{\mathbb{C}}$ be a closed compatible subgroup. $G = K\exp(\mathfrak{p})$, where $K := G \cap U$ is a maximal compact subgroup of G and $\mathfrak{p} := \mathfrak{g} \cap i\mathfrak{u}$; $\mathfrak{g}$ is the Lie algebra of G.

Suppose $X \subset Z$ is a G-stable locally closed connected real submanifold of Z with the gradient map $\mu_{\mathfrak{p}} : X \to \mathfrak{p}$. We recall that by G_x and K_x, we denote the stabilizer subgroup of $x \in X$ with respect to the G-action and the K-action respectively and by $\mathfrak{g}_x$ and $\mathfrak{k}_x$ their respective Lie algebras.

Definition 5.1.1. *Let $x \in X$. Then:*

a) x is stable if $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ and $\mathfrak{g}_x$ is conjugate to a Lie subalgebra of $\mathfrak{k}$.

b) x is polystable if $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.

c) x is semistable if $\overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.

We denote by $X_{\mu_{\mathfrak{p}}}^{s}$, $X_{\mu_{\mathfrak{p}}}^{ss}$, $X_{\mu_{\mathfrak{p}}}^{ps}$ the set of stable, respectively semistable, polystable, points. It follows directly from the definitions above that the conditions are G-invariant in the sense that if a point satisfies one of the conditions, then every point in its orbit satisfies the same condition, and for stability, recall that $\mathfrak{g}_{gx} = \mathrm{Ad}(g)(\mathfrak{g}_x)$.

One may define a relation $\sim$ on $X^{ss}_{\mu_\mathfrak{p}}$ where $x \sim y$ if $\mu_\mathfrak{p}^{-1}(0) \cap \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$. This relation is indeed an equivalence relation [44] and we denote the corresponding quotient by $X^{ss}_{\mu_\mathfrak{p}}//G$ and call it the topological Hilbert quotient of $X^{ss}_{\mu_\mathfrak{p}}$ by the action of G. Let $\pi : X^{ss}_{\mu_\mathfrak{p}} \longrightarrow X^{ss}_{\mu_\mathfrak{p}}//G$ denote the quotient map. The results of Heinzner-Schwarz-Stötzel [45, Quotient Theorem p.164] and [44, 47], show the following theorem.

Theorem 5.1.1. *The subsets $X^s_{\mu_\mathfrak{p}}$, $X^{ss}_{\mu_\mathfrak{p}}$ are open subset in X. Moreover, the topological Hilbert quotient $X^{ss}_{\mu_\mathfrak{p}}//G$ has the following properties.*

a) Every fiber contains a unique closed G-orbit. Any other orbit in the fiber has strictly larger dimension.

b) The closure of every G-orbit in a fiber of π contains the closed G-orbit.

c) every fiber of π intersects $\mu_\mathfrak{p}^{-1}(0)$ in a unique K-orbit which lies in the unique closed G-orbit.

d) the inclusion $\mu_\mathfrak{p}^{-1}(0) \hookrightarrow X$ induces a homeomorphism $\mu_\mathfrak{p}^{-1}(0)/K \cong X^{ss}_{\mu_\mathfrak{p}}//G$.

Therefore $X^{ss}_{\mu_\mathfrak{p}}//G$ can be identified with the space of polystable orbits. On the other hand the set of polystable points is in general neither open nor closed. The following result establishes a relation between the Kempf-Ness function and the polystability condition.

Proposition 5.1.2. *Let $x \in X$ and let $g \in G$. The following conditions are equivalent:*

a) $\mu_\mathfrak{p}(gx) = 0$.

b) g is a critical point of $\Phi(x, \cdot)$.

c) $g^{-1}K$ is a critical point of Φ_x.

Proof. Lemma 2.2.1 proves (a) is equivalent to (c). Since $\Phi(x, \cdot)$ is K-invariant it follows that (b) is equivalent to (c), concluding the proof. □

Proposition 5.1.3. *Let $x \in X$.*

- *If x is polystable, then G_x is reductive.*
- *If x is stable, then G_x is compact.*

Proof. Assume $\mu_{\mathfrak{p}}(x) = 0$. By Lemma 1.6.8, G_x is compatible. Therefore if x is stable, respectively polystable, then G_x is compact, respectively reductive. Since $G_{gx} = gG_xg^{-1}$, the result follows. ☐

5.1.1 Maximal Weight Function

In this section, we introduce the numerical invariants $\lambda(x, \beta)$ associated to an element $x \in X$ and $\beta \in \mathfrak{p}$.

For any $t \in \mathbb{R}$, define $\lambda(x, \beta, t) = \langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta \rangle$.

$$\lambda(x, \beta, t) = \langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta \rangle = \frac{d}{dt}\Phi(x, \exp(t\beta)),$$

where $\Phi : X \times G \to \mathbb{R}$ is the Kempf-Ness function. By the properties of the Kempf-Ness function,

$$\frac{d}{dt}\lambda(x, \beta, t) = \frac{d^2}{dt^2}\Phi(x, \exp(t\beta)) \geq 0.$$

This means that $\lambda(x, \beta, t)$ is a non decreasing function as a function of t.

Definition 5.1.2. *The maximal weight of $x \in X$ in the direction of $\beta \in \mathfrak{p}$ is the numerical value*

$$\lambda(x, \beta) = \lim_{t \to \infty} \lambda(x, \beta, t) \in \mathbb{R} \cup \{\infty\}.$$

From the proof of Lemma 1.6.3 we have

$$\frac{d}{dt}\lambda(x, \beta, t) = \parallel \beta_X(\exp(t\beta)x) \parallel^2,$$

and so,

$$\lambda(x, \beta, t) = \langle \mu_{\mathfrak{p}}(x), \beta \rangle + \int_0^t \parallel \beta_X(\exp(s\beta)x) \parallel^2 \,\mathrm{ds}.$$

For any $x \in X$ and $\beta \in \mathfrak{p}$, we consider the curve $c_x^\beta : [0, +\infty) \longrightarrow X$ defined by $c_x^\beta(t) = \exp(t\beta)x$. The energy functional of the curve c_x^β is given by

$$E(c_x^\beta) = \int_0^{+\infty} \parallel \beta_X(\exp(t\beta)x) \parallel^2 \mathrm{dt}.$$

Thus,

$$\lambda(x, \beta) = \lambda(x, \beta, 0) + E(c_x^\beta), \tag{5.1}$$

Lemma 5.1.4. *Let $x \in X$ and let $\beta \in \mathfrak{p}$. Then the function $(0, \infty) \to \mathbb{R} : t \mapsto t^{-1}\Phi(x, \exp(t\beta))$ is nondecreasing and*

$$\lambda(x, \beta) = \lim_{t \to \infty} \frac{\Phi(x, \exp(t\beta))}{t}.$$

Proof. Since

$$\frac{d}{dt}\Phi(x, \exp(t\beta)) = \lambda(x, \beta, t),$$

we have

$$\Phi(x, \exp(t\beta)) = \int_0^t \lambda(x, \beta, s) ds.$$

Since $\lambda(x, \beta, s)$ is nondecreasing and by definition of maximal weight, we have

$$\lambda(x, \beta) = \lim_{t \to \infty} \frac{1}{t} \int_0^t \lambda(x, \beta, s) ds = \lim_{t \to \infty} \frac{\Phi(x, \exp(t\beta))}{t}.$$

That the function $t \mapsto t^{-1}\Phi(x, \exp(t\beta))$ is nondecreasing follows from the fact that the function $t \mapsto \Phi(x, \exp(t\beta))$ is convex. $\square$

The following Lemma will be needed [4]. We include the proof for completeness.

Lemma 5.1.5. *Let V be a subspace of $\mathfrak{p}$. The following are equivalent for a point $x \in X$:*

a) The map $\Phi(x, \cdot)$ is linearly proper on V, i.e. there exist positive constants C_1 and C_2 such that:

$$\parallel v \parallel \leq C_1 \Phi(x, \exp(v)) + C_2, \quad \forall v \in V.$$

b) $\lambda(x, \beta) > 0 \ \forall \beta \in V \backslash \{0\}$.

Proof. (a) implies (b): By the inequality in (a), for any $\beta \in V$, $t \in \mathbb{R}$, $t \parallel \beta \parallel \leq C_1\Phi(x, \exp(t\beta)) + C_2$. This shows that

$$\frac{d}{dt}\Phi(x, \exp(t\beta)) = \lambda(x, \beta, t) > 0,$$

for $\beta \neq 0$ and sufficiently large t. Since the function $s \mapsto \lambda(x, \beta, s)$ is nondecreasing, keeping in mind the definition of $\lambda(x, \beta)$, (b) holds.

Suppose $\lambda(x, \beta) > 0, \forall \beta \in V\backslash\{0\}$. Suppose by contradiction that there are no positive constants (C_1, C_2) for which the inequality in (a) holds. Then that will implies that there is a sequence $(\beta_n)_n$ in V such that

$$n\Phi(x, \exp(\beta_n)) + n^2 < \parallel \beta_n \parallel . \tag{5.2}$$

Observe that $\lim_{n\to\infty} \parallel \beta_n \parallel = \infty$, otherwise, $(\beta_n)_n$ would have a bounded subsequence $(\beta_{n_m})_m$. It will then follow that $(\Phi(x, \exp(\beta_{n_m}))_m$ would also be bounded which will contradict (5.2).

From (5.2), we have

$$\frac{\Phi(x, \exp(\beta_n))}{\parallel \beta_n \parallel} + \frac{n^2}{\parallel \beta_n \parallel} < \frac{1}{n}.$$

In particular,

$$\frac{\Phi(x, \exp(\beta_n))}{\parallel \beta_n \parallel} < \frac{1}{n}.$$

Set $a_n = \parallel \beta_n \parallel$, $b_n = \frac{\beta_n}{\|\beta_n\|}$, and choose $t_0 \in \mathbb{R}$. By the convexity property of Φ,

$$\begin{aligned}
&\Phi(x, \exp(a_n b_n)) \geq \Phi(x, \exp(t_0 b_n)) + (a_n - t_0)\frac{d}{dt}\bigg|_{t=0} \Phi(x, \exp(t b_n)) \\
\Longrightarrow\ & \Phi(x, \exp(\beta_n)) \geq \Phi(x, \exp(t_0 b_n)) + (a_n - t_0)\lambda(x, b_n, t_0) \quad \forall a_n \geq t_0 \\
\Longrightarrow\ & \frac{\Phi(x, \exp(t_0 b_n)) + (a_n - t_0)\lambda(x, b_n, t_0)}{\parallel \beta_n \parallel} \leq \frac{\Phi(x, \exp(\beta_n))}{\parallel \beta_n \parallel} < \frac{1}{n}.
\end{aligned}$$

We get,

$$\frac{\Phi(x, \exp(t_0 b_n))}{\parallel \beta_n \parallel} + \left(1 - \frac{t_0}{\parallel \beta_n \parallel}\right)\lambda(x, b_n, t_0) < \frac{1}{n} \tag{5.3}$$

The sequence $(b_n)_n$ has a subsequence which converges to, say $b_0 \in V$ with $\parallel b_0 \parallel = 1$. Taking the limit of (5.3), we have $\lambda(x, b_0, t_0) \leq 0$. But this implies that $\lambda(x, b_0) \leq 0$, which contradicts the assumption. □

We conclude this section proving a numerical criterium for the stability condition. We start with the following Lemma.

Lemma 5.1.6. *Let $x \in X$ be such that $\mu_{\mathfrak{p}}(x) = 0$. Then $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$ and $\lambda(x, \beta) = 0$ if and only if $\beta_X(x) = 0$. In particular, x is stable if and only if $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$.*

Proof. Let $\beta \in \mathfrak{p}$ and let

$$\rho(t) : \mathbb{R} \longrightarrow \mathbb{R}, \qquad t \mapsto \Phi(x, \exp(t\beta)).$$

ρ is a convex function and

$$\lambda(x, \beta) \geq \rho'(t) = \langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta \rangle.$$

Since $\rho'(0) = \langle \mu_{\mathfrak{p}}(x), \beta \rangle = 0$, keeping in mind that $\rho'(t)$ is nondecreasing, it follows that $\lambda(x, \beta) \geq 0$ and $\lambda(x, \beta) = 0$ if and only if $\rho'(t) = 0$ and hence if and only if $\beta_X(x) = 0$. By Proposition 1.6.8, G_x is compatible. Hence x is stable if and only if $G_x \subset K$ and so if and only if $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$. □

Theorem 5.1.7. *Let $x \in X$. Then x is stable if and only if $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$.*

Proof. Assume $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$. By Lemma 5.1.5, keeping in mind that $\exp : \mathfrak{p} \longrightarrow G/K$ is a diffeomorphism, it follows that Φ_x is an exhaustion. Hence Φ_x has a minimum and so a critical point. By Proposition 5.1.2, x is polystable. Hence there exists $g \in G$ such that $\mu_{\mathfrak{p}}(gx) = 0$. We claim that $\Phi(gx, \cdot)$ is linearly properly on $\mathfrak{p}$. Indeed, by cocycle condition, we get

$$\Phi(gx, \exp(\xi)) = \Phi(x, \exp(\xi)g) - \Phi(x, g).$$

Write $\exp(\xi)g = k(\xi)\exp(\theta(\xi))$. Then $\Phi(gx, \exp(\xi)) = \Phi(x, \exp(\theta(\xi)) - \Phi(x, g)$. In [65], the author proves that there exist A_1 and A_2 such that $\parallel \xi \parallel^2 \leq A_1 \parallel \theta(\xi) \parallel^2 + A_2$. Since $\Phi(x, \cdot)$ is linearly proper on $\mathfrak{p}$, it follows that $\Phi(gx, \cdot)$ is linearly proper on $\mathfrak{p}$ as well. By Lemma 5.1.5 and 5.1.6, it follows that gx is stable and so x is stable as well.

Assume that x is stable. Then there exists $g \in G$ such that $\mu_{\mathfrak{p}}(gx) = 0$. By Lemma 5.1.6, $\lambda(gx, \beta) > 0$ for any $\beta \in \mathfrak{p}\backslash\{0\}$. By Lemma 5.1.5, $\Phi(gx, \cdot)$ is linearly proper on $\mathfrak{p}$. As in the previous part of the proof, one has $\Phi(x, \cdot)$ is linearly proper on $\mathfrak{p}$ and so $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p}\backslash\{0\}$. ☐

5.2 Energy Complete

Let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ denote the G-gradient map associated with the momentum map $\mu : Z \longrightarrow \mathfrak{u}$.

Definition 5.2.1. *The G-action on X is called energy complete if for any $x \in X$ and for any $\beta \in \mathfrak{p}$, If $E(c_x^\beta) < \infty$ then $\lim_{t \to \infty} \exp(t\beta)x$ exists.*

By Corollary 1.6.9.2, if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\mapsto+\infty} t_n = \infty$ and $\lim_{n\mapsto\infty} \exp(t_n\beta)x$ converges then the curve $c_x(t) = \exp(t\beta)x$ has a limit as $t \mapsto \infty$. The proof of the following result is similar to [4, Proposition 3.9].

Proposition 5.2.1.

a) If X is compact, then the G-action is energy complete;

b) if G acts on a complex vector space (V, h), where h is a K-invariant Hermitian scalar product, then the G-action is energy complete;

c) if $G \subset \mathrm{SL}(n, \mathbb{R})$ is a closed and compatible, then the G-action on $\mathbb{R}^n$ is energy complete.

Proposition 5.2.2. *Let $x \in X$ and let $\beta \in \mathfrak{p}$. Then $\lambda(x, \beta) < +\infty$, if and only if $E(c_x^\beta) < \infty$. Moreover, if $E(c_x^\beta) < \infty$, denoting by $y = \lim_{t\mapsto+\infty} \exp(t\beta)x$, then $y \in X^\beta$ and $\lambda(x, \beta) = \langle \mu_{\mathfrak{p}}(y), \beta\rangle$.*

Proof. Since

$$\lambda(x, \beta) = \langle \mu_{\mathfrak{p}}(z), \beta\rangle + E(c_x^\beta),$$

it follows that $\lambda(x,\beta) < +\infty$ if and only if $E(c_x^\beta) < +\infty$. Assume that $E(c_x^\beta) < +\infty$. Let $y = \lim_{t\mapsto+\infty} \exp(t\beta)x$. Since β_X is the gradient of $\mu_{\mathfrak{p}}^\beta$, it follows that $y \in X^\beta$ and

$$\lambda(x,\beta) = \lim_{t\mapsto+\infty} \langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta\rangle = \langle \mu_{\mathfrak{p}}(y), \beta\rangle.$$

□

We claim that the maximal weight satisfies a G-equivariant property. We start with the following Lemma.

Lemma 5.2.3. *Let c_x^β and let $g \in G^{\beta-}$. If $E(c_x^\beta) < +\infty$, then $E(c_{gx}^\beta) < +\infty$.*

Proof. Since $E(c_x^\beta) < +\infty$, it follows that $\lim_{t\mapsto+\infty} \exp(t\beta)x$ exists. The element $g \in G^{\beta-}$ and so the $\lim_{t\mapsto+\infty} \exp(t\beta)\, g \exp(-t\beta)$ exists. This implies that

$$\lim_{t\mapsto+\infty} \exp(t\beta)gx = \lim_{t\mapsto+\infty} \big(\exp(t\beta)\, g \exp(-t\beta)\big) \exp(t\beta)x \text{ exists.}$$

By the definition of the maximal weight, it follows that $\lambda(gx,\beta) < +\infty$. By Proposition 5.2.2, we get $E(c_{gx}^\beta) < +\infty$, concluding the proof. □

Lemma 5.2.4. *Let $g \in G^{\beta-}$. Then $\lambda(gx,\beta) = \lambda(x,\beta)$.*

Proof. Assume $E(c_x^\beta) < +\infty$. Then $\lim_{t\mapsto+\infty} \exp(t\beta)x = y$ and $y \in X^\beta$. Since $g \in G^{-\beta}$, it follows that $\lim_{t\mapsto+\infty} \exp(t\beta)\, g \exp(-t\beta)$ exists and it belongs to G^β. Hence

$$\lim_{t\mapsto+\infty} \exp(t\beta)gx = \lim_{t\mapsto+\infty} \big(\exp(t\beta)\, g \exp(-t\beta)\big) \exp(t\beta)x = g_o y,$$

for some $g_o \in G^\beta$. By Proposition 1.6.11, it follows that

$$\lambda(x,\beta) = \langle \mu_{\mathfrak{p}}(y), \beta\rangle = \langle \mu_{\mathfrak{p}}(g_o y), \beta\rangle = \lambda(gx,\beta).$$

If $E(c_x^\beta) = \infty$, then $\lambda(x,\beta) = +\infty$. By Lemma 5.2.3, $E(c_{gx}^\beta) = +\infty$ as well and so $\lambda(gx,\beta) = +\infty$. □

Let $g \in G$. By Proposition 1.5.1, $g = kh$, where $h \in G^{\beta-}$ and $k \in K$. The following two propositions are similar to the results stated in [4, Propostion 2.11].

Proposition 5.2.5. *In the above assumption, we have*

$$\lambda(gx, \beta) = \lambda(x, \mathrm{Ad}(k^{-1})(\beta)).$$

Proof. By the above Lemma, it is enough to prove that $\lambda(kx, \beta) = \lambda(x, \mathrm{Ad}(k^{-1})(\beta))$. Since $\mu_{\mathfrak{p}}(\exp(t\beta)kx) = k\mu_{\mathfrak{p}}(\exp(t\mathrm{Ad}(k^{-1})(\beta)))$, it follows

$$\lambda(kx, \beta) = \lim_{t \mapsto +\infty} \langle \mu_{\mathfrak{p}}(\exp(t\mathrm{Ad}(k^{-1})(\beta)x), \mathrm{Ad}(k^{-1})(\beta)\rangle = \lambda(x, \mathrm{Ad}(k^{-1})(\beta)).$$

□

Proposition 5.2.6. *If $(x_n, \beta_n)_n$ converges to (x, β) then $\lambda(x, \beta) \leq \liminf_{n \to \infty} \lambda(x_n, \beta_n)$.*

Proof. We prove by contradiction. If the statement was false there would exist $\epsilon > 0$ and a subsequence $(x_{n_m}, \beta_{n_m})_m$ of $(x_n, \beta_n)_n$ such that the limit $\lim_{m \to \infty} \lambda(x_{n_m}, \beta_{n_m})$ exists, finite and $\lambda(x, \beta) \geq \lim_{m \to +\infty} \lambda(x_{n_m}, \beta_{n_m}) + \epsilon$. We can choose sufficiently large t such that $\lambda(x, \beta, t) \geq \lim_{m \to +\infty} \lambda(x_{n_m}, \beta_{n_m}) + \frac{\epsilon}{2}$. However, since $\lambda(x_{n_m}, \beta_{n_m}) \geq \lambda(x_{n_m}, \beta_{n_m}, t)$, (because $\lambda(x, \beta, t)$ is an increasing function) and $(x, \beta) \mapsto \lambda(x, \beta, t)$ is continuous on $X \times \mathfrak{p}$, we have

$$\lim_{m \to +\infty} \lambda(x_{n_m}, \beta_{n_m}) \geq \lim_{m \to +\infty} \lambda(x_{n_m}, \beta_{n_m}, t) = \lambda(x, \beta, t) \geq \lim_{m \to +\infty} \lambda(x_{n_m}, \beta_{n_m}) + \frac{\epsilon}{2}.$$

which is not possible. □

The following result is the moment weight inequality which is an important ingredient to prove our results. We give a proof of this inequality applying the idea in [30, 23] which was due to Xiuxiong Chen to our case.

Theorem 5.2.7. *(Moment-Weight Inequality). For every $x \in X$, $\beta \in \mathfrak{p} \backslash \{0\}$ and $g \in G$,*

$$\frac{-\lambda(x, \beta)}{\parallel \beta \parallel} \leq \parallel \mu_{\mathfrak{p}}(gx) \parallel .$$

Proof. Let $x \in X$ and $\beta \in \mathfrak{p} \backslash \{0\}$, and $g \in G$. If $\lambda(x, \beta) = +\infty$, then the result follows. Assume that $\lambda(x, \beta) < +\infty$. By Proposition 5.2.2, $\lim_{t \mapsto +\infty} \exp(t\beta)x$ exists.

Define $\alpha : [0, \infty) \to \mathfrak{p}$ and $k : [0, \infty) \to K$ such that

$$\exp(\alpha(t))k(t) = \exp(t\beta)g^{-1}. \tag{5.4}$$

We claim that

$$\lim_{t\mapsto+\infty}\frac{\alpha(t)}{\parallel\alpha(t)\parallel}=\frac{\beta}{\parallel\beta\parallel}.$$

To prove this, observe that $\exp(-\alpha(t))\exp(t\beta)g^{-1}\in K$. Then from Lemma C.2 in [30], there is a positive constant c such that $\parallel t\beta-\alpha(t)\parallel\leq c$ for all $t\in[0,\infty)$. Therefore, for any $t\in(0,+\infty)$, we have

$$\begin{aligned}\left\|\frac{\beta}{\parallel\beta\parallel}-\frac{\alpha(t)}{\parallel\alpha(t)\parallel}\right\| &= \left\|\frac{t\beta}{t\parallel\beta\parallel}-\frac{\alpha(t)}{t\parallel\beta\parallel}+\frac{\alpha(t)}{t\parallel\beta\parallel}-\frac{\alpha(t)}{\parallel\alpha(t)\parallel}\right\| \\ &\leq\frac{\parallel t\beta-\alpha(t)\parallel}{t\parallel\beta\parallel}+\parallel\alpha(t)\parallel\left|\frac{1}{t\parallel\beta\parallel}-\frac{1}{\parallel\alpha(t)\parallel}\right| \\ &=\frac{\parallel t\beta-\alpha(t)\parallel}{t\parallel\beta\parallel}+\frac{|\parallel t\beta\parallel-\parallel\alpha(t)\parallel|}{t\parallel\beta\parallel} \\ &\leq\frac{2c}{t\parallel\beta\parallel}.\end{aligned}$$

Therefore

$$\lim_{t\to+\infty}\frac{\alpha(t)}{\parallel\alpha(t)\parallel}=\frac{\beta}{\parallel\beta\parallel}.$$

For any $t\in(0,+\infty)$, applying Lemma 1.6.3, the function

$$s\mapsto g(s)=\langle\mu_{\mathfrak{p}}(\exp\left(sk^{-1}(t)\alpha(t)k(t)\right)gx),k^{-1}(t)\alpha(t)k(t)\rangle$$

is nondecreasing. In particular

$$g(0)=\langle\mu_{\mathfrak{p}}(gx),k^{-1}(t)\alpha(t)k(t)\rangle\leq g(1)=\langle\mu_{\mathfrak{p}}\left(\exp(k^{-1}(t)\alpha(t)k(t))gx\right),k^{-1}(t)\alpha(t)k(t)\rangle.$$

Therefore, keeping in mind that $\exp(k^{-1}(t)\alpha(t)k(t))=k^{-1}(t)\exp(t\beta)$ and $\mu_{\mathfrak{p}}$ is K-equivariant, we have

$$\begin{aligned}-\parallel\mu_{\mathfrak{p}}(gx)\parallel &\leq\parallel\alpha(t)\parallel^{-1}\langle\mu_{\mathfrak{p}}(gx),k^{-1}(t)\alpha(t)k(t)\rangle \\ &\leq\parallel\alpha(t)\parallel^{-1}\langle\mu_{\mathfrak{p}}(\exp(k^{-1}(t)\alpha(t)k(t))gx),k^{-1}(t)\alpha(t)k(t)\rangle \\ &=\parallel\alpha(t)\parallel^{-1}\langle\mu_{\mathfrak{p}}(k^{-1}(t)\exp(t\beta)x),k^{-1}(t)\alpha(t)k(t)\rangle \\ &=\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\frac{\alpha(t)}{\parallel\alpha(t)\parallel}\rangle \\ &=\left\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\frac{\alpha(t)}{\parallel\alpha(t)\parallel}-\frac{\beta}{\parallel\beta\parallel}+\frac{\beta}{\parallel\beta\parallel}\right\rangle \\ &=\parallel\beta\parallel^{-1}\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\beta\rangle+\left\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\frac{\alpha(t)}{\parallel\alpha(t)\parallel}-\frac{\beta}{\parallel\beta\parallel}\right\rangle.\end{aligned}$$

Since $\lim_{t \mapsto +\infty} \exp(t\beta)x$ exists, taking the limit $t \to +\infty$, we have

$$- \parallel \mu_{\mathfrak{p}}(gx) \parallel \leq \lim_{t \to +\infty} \frac{\langle \mu_{\mathfrak{p}}(\exp(t\beta)x), \beta \rangle}{\parallel \beta \parallel} = \frac{\lambda(x, \beta)}{\parallel \beta \parallel}.$$

Hence,

$$\frac{-\lambda(x, \beta)}{\parallel \beta \parallel} \leq \parallel \mu_{\mathfrak{p}}(gx) \parallel$$

□

As an application of the general moment-weight inequality, we have the following:

Theorem 5.2.8. *If $x \in X$ is semistable, then $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$*

Proof. If $x \in X$ is semistable, then $\inf_G \parallel \mu_{\mathfrak{p}}(gx) \parallel = 0$. Suppose by contradiction that there exists a $\beta \in \mathfrak{p} \backslash \{0\}$ such that $\lambda(x, \beta) < 0$. Then by Theorem 5.2.7, for any $g \in G$, we have

$$0 < \frac{-\lambda(x, \beta)}{\parallel \beta \parallel} \leq \parallel \mu_{\mathfrak{p}}(gx) \parallel$$

and so $\inf_G \parallel \mu_{\mathfrak{p}}(gx) \parallel > 0$. which is a contradiction. Therefore $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$. □

5.3 Analytic semistability and Polystability

Definition 5.3.1. *A point $x \in X$ is called:*

a) analytically stable if $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$;

b) analytically semi-stable if $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$;

c) analytically polystable if $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$ and the condition $\lambda(x, \beta) = 0$ holds if and only if $\lim_{t \mapsto +\infty} \exp(t\beta) \in G \cdot x$.

Theorem 5.1.7 proves that x is stable if and only if x is analytically stable. From now on, we assume that the G-action on X is energy complete. What follows are the Hilbert numerical criteria for polystability and semistability under the assumption that X is energy complete. We begin with the following Lemma.

Lemma 5.3.1. *Let* $x \in X$ *and* $\beta, \alpha \in \mathfrak{p}$ *be such that* $[\beta, \alpha] = 0$. *Suppose that limits* $y := \lim_{t \to +\infty} \exp(t\beta)x$, $z := \lim_{t \to +\infty} \exp(t\alpha)y$ *exists. By the energy completeness property the lemma will be needed only when these limits exist. Then there exists* $\delta > 0$ *such that for* $0 < \epsilon < \delta$, *we have*

$$\lim_{t \to +\infty} \exp(t(\beta + \epsilon\alpha))x = z.$$

Proof. Fix $x \in X$. Then $y \in X^\beta$ and $z \in X^\alpha \cap X^\beta$. Let $\mathfrak{a} = \operatorname{span}(\alpha, \beta)$ and $A = \exp(\mathfrak{a})$. Since the exponential map is a diffeomorphism restricted on $\mathfrak{p}$, it follows that A is a closed and compatible subgroup of G. Then z is fixed by A and by the linearization theorem, Corollary 1.6.9.2, there exists A-invariant open subsets $\Omega \subset X$ and $S \subset T_zX$ and a A-equivariant diffeomorphism $\varphi : S \to \Omega$ such that $0 \in S$, $z \in \Omega$, $\varphi(0) = z$, $d\varphi_0 = id_{T_zX}$. Since $z = \lim_{t \to +\infty} \exp(t\alpha)y$, there is t_0 such that $\exp(t_0\alpha)y \in \Omega$. Since Ω is A-invariant, we get $y \in \Omega$ and also, $x \in \Omega$. Thus we can study all the limits in the linearization S. Hence, keeping in mind Proposition 1.6.10, we may assume that $\Omega = \mathbb{R}^n$, α, β are symmetric matrices of order n satisfying $[\alpha, \beta] = 0$. From now on, the proof is similar to one given in [22, pag. 1036]. For sake of completeness we give the proof.

The matrices α and β are simultaneously diagonalizable. Decompose $V = \bigoplus_{\lambda \in \operatorname{Spec}(\beta)} V_\lambda$. This means $\beta_{|V_\lambda} = \lambda_k \operatorname{Id}_{V_{\lambda_k}}$. Since $\lim_{t \mapsto +\infty} \exp(t\beta)x = y$, it follows that $x = v_0 + v_1$, where $v_0 \in V_0$ and v_1 is the sum of some eigenvetors corresponding to negative eigenvalues. Therefore $\lim_{t \mapsto +\infty} \exp(t\beta)x = v_0 + \lim_{t \mapsto +\infty} \exp(t\beta)v_1$ and $\lim_{t \mapsto +\infty} \exp(t\beta)v_1 = 0$. This implies $v_0 = y$.

Let

$$\delta = \min\left\{ \frac{-\lambda}{2|\mu|} : \lambda \in \operatorname{Spec}(\beta) \cap (0, -\infty),\ \mu \in \operatorname{Spec}(\alpha)\backslash\{0\} \right\}.$$

If $\lambda \in \operatorname{Spec}(\beta) \cap (0, -\infty)$, then

$$V_\lambda = W_0 \cap V_\lambda \bigoplus_{\mu \in \operatorname{Spec}(\alpha)\backslash\{0\}} (V_\lambda \cap W_\mu),$$

where $W_0 = \operatorname{Ker} \alpha$ and $\alpha_{|W_\mu} = \mu \operatorname{Id}_{W_\mu}$. Let $\epsilon < \delta$ and let $v \in V_\lambda$. Then $v = w_0 + \sum_{\mu \in \operatorname{Spec}(\alpha)\backslash\{0\}} w_\mu$ and so

$$(\alpha + \epsilon\beta)v = \lambda w_0 + \sum_{\mu \in \operatorname{Spec}(\alpha)\backslash\{0\}} (\lambda + \epsilon\mu) w_\mu$$

with $\lambda+\epsilon\mu<0$ for every $\mu\in\operatorname{Spec}(\alpha)\backslash\{0\}$. Therefore $\lim_{t\mapsto+\infty}\exp(t(\beta+\epsilon\alpha))v=0$. This holds for any $\lambda\in\operatorname{Spec}(\beta)\cap(-\infty,0)$. Now, keeping in mind that $x=y+v_1$, where v_1 is the sum of eigenvectors of β associated to negative eigenvalues, we have

$$\begin{aligned}\lim_{t\mapsto+\infty}\exp(t(\beta+\epsilon\alpha))x&=\lim_{t\mapsto+\infty}\exp(t(\beta+\epsilon\alpha))(y+v_1)\\&=\lim_{t\mapsto+\infty}\exp(t\epsilon\alpha))y+\lim_{t\mapsto+\infty}\exp(t(\beta+\epsilon\alpha))v_1\\&=\lim_{t\mapsto+\infty}\exp(t\alpha))y\\&=z.\end{aligned}$$

□

Lemma 5.3.2. *Let $x\in X$ be an analytically semistable point. Let $\beta\in\mathfrak{p}$ be such that $\lambda(x,\beta)=0$. Let $y=\lim_{t\mapsto+\infty}\exp(t\beta)x$. Then $\lambda(y,\alpha)\geq 0$ for any $\alpha\in\mathfrak{p}^\beta$.*

Proof. Suppose by contradiction, there exists $\alpha\in\mathfrak{p}^\beta$ with $\lambda(y,\alpha)<0$. By Proposition 5.2.2,

$$\lim_{t\to+\infty}\exp(t\alpha)y=z$$

exists. Let $A=\exp(\operatorname{span}(\beta,\alpha))$, $[\beta,\alpha]=0$ by the choice of α. Since $y\in X^\beta$ and the flow $\exp(t\alpha$ preserves X^β, it follows that $z\in X^A$, where $X^A=\{x\in X: A\cdot x=x\}$.

By Lemma 5.3.1, for all sufficiently small $\epsilon>0$,

$$\lim_{t\to+\infty}\exp(t(\beta+\epsilon\alpha))x=z.$$

Hence,

$$\begin{aligned}\lambda(x,\beta+\epsilon\alpha)&=\lim_{t\to+\infty}\langle\mu_{\mathfrak{p}}(\exp(t(\beta+\epsilon\alpha)x)),\beta+\epsilon\alpha\rangle\\&=\langle\mu_{\mathfrak{p}}(z),\beta+\epsilon\alpha\rangle\\&=\mu_{\mathfrak{p}}^\beta(z)+\epsilon\langle\mu_{\mathfrak{p}}(z),\alpha\rangle\\&=\mu_{\mathfrak{p}}^\beta(z)+\epsilon\lim_{t\to+\infty}\langle\mu_{\mathfrak{p}}(\exp(t\alpha)y),\alpha\rangle=\mu_{\mathfrak{p}}^\beta(z)+\epsilon\lambda(y,\alpha).\end{aligned}$$

But by the choice of β,

$$0=\lambda(x,\beta)=\lim_{t\to+\infty}\langle\mu_{\mathfrak{p}}(\exp(t\beta)x),\beta\rangle=\langle\mu_{\mathfrak{p}}(y),\beta\rangle=\mu_{\mathfrak{p}}^\beta(y)$$

and $\mu_{\mathfrak{p}}^{\beta}$ is constant along the curve $\exp(t\alpha)y$. This implies $\mu_{\mathfrak{p}}^{\beta}(z) = 0$. Hence,

$$\lambda(x, \beta + \epsilon\alpha) = \epsilon\lambda(y, \alpha) < 0,$$

which contradicts the analytic semistability of x. ☐

Lemma 5.3.3. *Let $x \in X^{\beta}$. If x is G^{β}-semistable, polystable or stable then x is G-semistable.*

Proof. Let $x \in X^{\beta}$. Assume $\overline{G^{\beta} \cdot x} \cap \mu_{\mathfrak{p}^{\beta}}^{-1}(0) \neq \emptyset$. Then there exists a sequence $g_n \in G^{\beta}$ such that $\mu_{\mathfrak{p}}(g_n x) \mapsto 0$. By Proposition 1.6.12 it follows $\mu_{\mathfrak{p}}(g_n x) = \mu_{\mathfrak{p}^{\beta}}(g_n x) \mapsto 0$ and so the result follows. ☐

Theorem 5.3.4. *Let $x \in X$. The following conditions are equivalent:*

(1) x is semistable.

(2) $\mathrm{Inf}_G \parallel \mu(gx) \parallel = 0$.

(3) x is analytically semistable.

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (3) follows by Theorem 5.2.8.
(3) $\Rightarrow$ (1). If $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p} \backslash \{0\}$, then by Theorem 5.1.7, x is stable and hence semistable. Assume there exists $\beta \in \mathfrak{p} \backslash \{0\}$ such that $\lambda(x, \beta) = 0$. By Proposition 5.2.2

$$\lim_{t \to +\infty} \exp(t\beta)x = y,$$

$y \in \overline{G \cdot x}$ and $\beta_X(y) = 0$.

Let $X^{\beta} = \{z \in X : \beta_X(z) = 0\}$. X^{β} is disjoint, union of closed submanifold of X. By Proposition 1.6.11, G^{β} preserves X^{β}. By Lemma 5.3.3 if y is $(G^{\beta})^o$ stable, then y is semistable and so x is semistable as well.

Let Y be the connected component of X^{β} containing y. Now, $\mathfrak{g}^{\beta} = \mathfrak{k}^{\beta} \oplus \mathfrak{p}^{\beta}$ and K^{β} preserves $\mathfrak{p}^{\beta}$. Since $K^{\beta} \cdot \beta = \beta$, it follows that we can write $\mathfrak{p}^{\beta} = \mathrm{span}(\beta) \oplus \mathfrak{p}'$, where $\mathfrak{p}'$

is a K^β-invariant subspace of $\mathfrak{p}^\beta$. Now, keeping in mind that $(G^\beta)^o = Z((G^\beta)^o)^o (G^\beta)^o_{ss}$, $Z((G^\beta)^o)^o$ is compatible and $\exp(t\beta) \in Z((G^\beta)^o)^o$, it follows that

$$(G^\beta)^o = \exp(\mathbb{R}\beta)H,$$

where H is a closed, connected and compatible Lie group of $(G^\beta)^o$ with Lie algebra $\mathfrak{h} = \mathfrak{k}^\beta \oplus \mathfrak{p}^{'}$. In particular,

$$H = (K^\beta)^o \exp(\mathfrak{p}')$$

and $\mathfrak{g}^\beta = \mathrm{span}(\beta) \oplus \mathfrak{h}$. Consider the H-action on Y. By Lemma 5.3.2, $\lambda(y, \beta'') \geq 0$ for every $\beta'' \in \mathfrak{p}^\beta$. We separate the two cases:

a) $\lambda(y, \beta') > 0, \forall \beta' \in \mathfrak{p}' \backslash \{0\}$.

b) There exists $\beta_1 \in \mathfrak{p}'$ such that $\lambda(y, \beta_1) = 0$.

Assume (a) holds. We claim that y is stable with respect to $(G^\beta)^\circ$.

Let $\Phi_y : \mathfrak{p}^\beta \to \mathbb{R}; \xi \mapsto \Phi(y, \exp(\xi))$ be the associated Kempf-Ness function. By Lemma 5.1.5, Φ_y is linearly proper on $\mathfrak{p}^{'}$. This implies that $\Phi(y, \exp(.))$ is bounded from below on $\mathfrak{p}^{'}$. Let

$$m = \inf_{\xi \in \mathfrak{p}'} \Phi(y, \exp(\xi)).$$

We claim that

$$m = \inf_{\xi \in \mathfrak{p}^\beta} \Phi(y, \exp(\xi)). \tag{5.5}$$

Indeed, $\xi \in \mathfrak{p}^\beta$ can be written as $\xi = \xi_1 + \xi_2$; $\xi_1 \in \mathrm{span}(\beta)$, $\xi_2 \in \mathfrak{p}^{'}$, $[\xi_1, \xi_2] = 0$. By the cocycle condition of the Kempf-Ness function, keeping in mind that $\exp(\xi_1)y = y$, we have

$$\Phi(y, \exp(\xi)) = \Phi(y, \exp(\xi_2 + \xi_1)) = \Phi(y, \exp(\xi_2)\exp(\xi_1)) = \Phi(y, \exp(\xi_1)) + \Phi(y, \exp(\xi_2)).$$

We claim that $\Phi(y, \exp(\xi_1)) = 0$. Indeed, let $s(t) = \Phi(y, \exp(t\beta))$. Applying again the cocycle condition, keeping in mind that $\exp(t\beta)y = y$, one can check that $s(t)$ is a linear function. Therefore $s(t) = at$, for some $a \in \mathbb{R}$. On the other hand

$$0 = \lambda(x, \beta) = \langle \mu_{\mathfrak{p}}(y), \beta \rangle = \lambda(y, \beta) = \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Phi(y, \exp(t\beta)) = a.$$

Therefore,

$$\Phi(y, \exp(\xi)) = \Phi(y, \exp(\xi_2)).$$

This proves (5.5). This means Φ_y has a critical point. By Proposition 5.1.2, there exists $g \in (G^\beta)^\circ$ such that $\mu_{\mathfrak{p}^\beta}(gy) = \mu_{\mathfrak{p}}(gy) = 0$. Hence,

$$\lim_{t\to+\infty} \exp(t\beta)gx = g \lim_{t\to+\infty} \exp(t\beta)x = gy \in \overline{G\cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0).$$

Suppose (b) holds. Let $\beta_1 \in \mathfrak{p}'\backslash\{0\}$ be such that $\lambda(y, \beta_1) = 0$. Since $\mathfrak{p}^\beta = \mathrm{span}(\beta)\oplus\mathfrak{p}'$, it follows that $[\beta, \beta_1] = 0$ and $\mathfrak{a}_1 := \mathrm{span}(\beta, \beta_1)$ has dimension 2. By energy completeness,

$$\lim_{y\to+\infty} \exp(t\beta_1)y = y_1 \in Y$$

exists, $(\beta_1)_X(y_1) = 0$ and $y_1 \in \overline{G\cdot y}$. Let Y_1 be the connected component of Y^{β_1} containing y_1. We may split $\mathfrak{p}' = \mathrm{span}(\beta_1)\oplus\mathfrak{p}''$ as $(K^{\mathfrak{a}_1})$-modules. As before, $H_{\beta_1} = (K^{\mathfrak{a}_1})^o \exp(\mathfrak{p}'')$ is a compatible Lie subgroup of G^β with Lie algebra $\mathfrak{h}_{\beta_1} = \mathfrak{k}^{\mathfrak{a}_1}\oplus\mathfrak{p}''$. The H_{β_1}-action on X preserves Y_1. Hence, one can repeat the above procedure for the H_{β_1}-action on Y_1. On the other hand, if $\mathfrak{a}\subset\mathfrak{p}$ is a maximal Abelian subalgebra, then $\dim(\mathfrak{a})$ is an invariant of the K-action on $\mathfrak{p}$ [27, 56]. This means that the above procedure will iterate at most $\dim(\mathfrak{a})$. This shows that $\overline{G\cdot x}\cap\mu_{\mathfrak{p}}^{-1}(0)\neq\emptyset$. □

Corollary 5.3.4.1. *Let $x \in X$. Then x is semistable if and only if there exists $\xi\in\mathfrak{p}$ and $g \in (G^\xi)^o$ such that $\lim_{t\mapsto+\infty}\exp(t\xi)gx \in \mu_{\mathfrak{p}}^{-1}(0)$.*

We now consider the polystable condition.

Let $\nu\in\mathfrak{g}$. We may split $\nu = \nu_{\mathfrak{k}} + \nu_{\mathfrak{p}} \in \mathfrak{k}\oplus\mathfrak{p}$. The following Lemma is proved in [30] for $G = U^{\mathbb{C}}$.

Lemma 5.3.5. *Let $x \in X$ and let $\beta\in\mathfrak{p}$. If $\beta_X(x) = 0$, then*

$$\langle\mu_{\mathfrak{p}}(x), \beta\rangle = \langle\mu_{\mathfrak{p}}(gx), (\mathrm{Ad}(g)(\beta))_{\mathfrak{p}}\rangle,$$

for any $g\in G$.

Proof. If $g = k$, then the result follows from the K-equivariance property of the gradient map. Hence we may assume $g = \exp(\nu)$. Let $g(t) = \exp(t\nu)$ and let $x(t) = g(t)x$. Let $\langle \cdot, \cdot \rangle$ denote the real part of the fixed $\mathrm{Ad}(U^{\mathbb{C}})$-invariant inner product of Euclidean type on $\mathfrak{u}^{\mathbb{C}}$. We claim that

$$f(t) = \langle \mu_{\mathfrak{p}}(g(t)x), (\mathrm{Ad}(g(t))(\beta))_{\mathfrak{p}} \rangle,$$

is constant. Let $\xi(t) = \mathrm{Ad}(g(t))(\beta)$. Then

$$\dot{\xi}(t) = [\nu, \xi(t)],$$

and so $\dot{\xi_{\mathfrak{p}}}(t) = [\nu, \xi_{\mathfrak{k}}(t)]$. Therefore

$$\dot{f}(t) = \langle (\mathrm{d}\mu_{\mathfrak{p}})_{x(t)}(\nu_X(x(t)), \xi(t)_{\mathfrak{p}} \rangle + \langle \mu_{\mathfrak{p}}(x(t)), [\nu, \xi_{\mathfrak{k}}(t)] \rangle.$$

The first term is given by

$$\begin{aligned} \mathrm{d}\mu_{\mathfrak{p}}^{\xi(t)_{\mathfrak{p}}}(\nu_X(x(t)) &= \mathrm{d}\mu^{-i\xi(t)_{\mathfrak{p}}}(\nu_X(x(t))) \\ &= \omega\big(- J\left((\xi_{\mathfrak{p}})_X(x(t)) \right), \nu_X(x(t)) \big) \\ &= \omega\big((\xi_{\mathfrak{p}})_X(x(t))), J(\nu_X(x(t))) \big). \end{aligned}$$

The second term is given by

$$\begin{aligned} \langle \mu_{\mathfrak{p}}(x(t)), [\nu, \xi_{\mathfrak{k}}(t)] \rangle &= \langle i\mu(x(t)), [\nu, \xi_{\mathfrak{k}}(t)] \rangle \\ &= -\langle \mu(x(t)), [-i\nu, \xi_{\mathfrak{k}}(t)] \rangle. \end{aligned}$$

Now, $\xi_{\mathfrak{k}} - i\nu \in \mathfrak{k} \oplus i\mathfrak{p} \subset \mathfrak{u}$. Using the U-equivariant property of the momentum map, keeping in mind that we think the momentum map as $\mathfrak{u}$-valued map by means of the $\mathrm{Ad}(U)$-invariant scalar product $-\langle \cdot, \cdot \rangle$ on $\mathfrak{u}$, we have

$$\begin{aligned} -\langle \mu(x(t)), [-i\nu, \xi_{\mathfrak{k}}(t)] \rangle &= -\left.\frac{\mathrm{d}}{\mathrm{ds}}\right|_{s=0} \langle \mu(x(t)), \mathrm{Ad}(\exp(-si\nu))(\xi_{\mathfrak{k}}(t)) \rangle \\ &= -\left.\frac{\mathrm{d}}{\mathrm{ds}}\right|_{s=0} \langle \mu(\mathrm{Ad}(\exp(si\nu))x(t)), \xi_{\mathfrak{k}}(t)) \rangle \\ &= \omega\big((\xi_{\mathfrak{k}})_X(x(t)), J\nu_X(x(t)) \big). \end{aligned}$$

Therefore, keeping in mind that $\xi_X(x(t)) = (\mathrm{d}g(t))_x(\beta_X(x)) = 0$, we get

$$\begin{aligned}\dot{f}(t) &= \omega\big((\xi_{\mathfrak{p}})_X(x(t)), J(\nu_X(x(t)))\big) + \omega\big((\xi_{\mathfrak{k}})_X(x(t)), J(\nu_X(x(t))\big) \\ &= \omega\big(((\xi_X(x(t)))\,, J(\nu_X(x(t)))\big) \\ &= 0.\end{aligned}$$

This implies $f(1) = f(0)$ and the result follows. $\square$

Corollary 5.3.5.1. *Let $x \in X$ be polystable. Then $\lambda(x, \beta) \geq 0$ for any $\beta \in \mathfrak{p}$. Moreover, $\lambda(x, \beta) = 0$ if and only if $\lim_{t\mapsto+\infty} \exp(t\xi) \in G \cdot x$.*

Proof. Assume that $\mu_{\mathfrak{p}}(x) = 0$. By Lemma 5.1.6 $\lambda(x, \beta) \geq 0$ and $\lambda(x, \beta) = 0$ if and only if $\beta_X(x) = 0$ and so $\lim_{t\mapsto+\infty} \exp(t\beta)x = x$.

Assume $\lim_{t\mapsto+\infty} \exp(t\beta)x = gx$, for some $g \in G$. Then $\beta_X(gx) = 0$ and

$$\lambda(x, \beta) = \langle \mu_{\mathfrak{p}}(gx), \beta\rangle.$$

By Lemma 5.3.5, we have

$$\lambda(x, \beta) = \langle \mu_{\mathfrak{p}}(gx), \beta\rangle = \langle \mu_{\mathfrak{p}}(x), \mathrm{Ad}(g^{-1})(\beta)_{\mathfrak{p}}\rangle = 0.$$

Let $y = gx$. Let $\beta \in \mathfrak{p}$. Then $g = hk$, where $h \in G^{\beta-}$ and $k \in K$. By Proposition 5.2.5, we have

$$\lambda(gx, \beta) = \lambda(x, \mathrm{Ad}(k^{-1})(\beta)) \geq 0.$$

By the above step, $\lambda(gx, \beta) = 0$ if and only if $\lim_{t\mapsto+\infty} \exp(t\mathrm{Ad}(k^{-1})(\beta))x \in G \cdot x$. Since

$$\exp(t\beta)gx = \big(\exp(t\beta)\, h \exp(-t\beta)\big)k \exp(t\mathrm{Ad}(k^{-1})(\beta))x,$$

keeping in mind that $h \in G^{\beta-}$, it follows that $\lambda(gx, \beta) = 0$ if and only if $\lim_{t\mapsto+\infty} \exp(t\beta)x \in G \cdot x$. $\square$

Corollary 5.3.5.2. *If $x \in X$ is analytically polystable then for any $g \in G$, gx is analytically polystable as well.*

Proof. Let $y = gx$. Let $\beta \in \mathfrak{p}$. Then $g = hk$, where $h \in G^{\beta-}$ and $k \in K$. By Proposition 5.2.5, we have

$$\lambda(gx, \beta) = \lambda(x, \mathrm{Ad}(k^{-1})(\beta)) \geq 0.$$

Therefore $\lambda(gx, \beta) = 0$ if and only if $\lim_{t\mapsto+\infty} \exp(t\mathrm{Ad}(k^{-1}))x \in G \cdot x$. Since

$$\exp(t\beta)gx = \big(\exp(t\beta)\, h \exp(-t\beta)\big) k \exp(t\mathrm{Ad}(k^{-1})(\beta))x,$$

we get that $\lambda(gx, \beta) = 0$ if and only if $\lim_{t\mapsto+\infty} \exp(t\beta)gx \in G \cdot x$. ☐

Theorem 5.3.6. *Let $x \in X$. Then x is analytically polystable if and only if x is polystable.*

Proof. By Corollary 5.3.5.1, if $x \in X$ is polystable then x is analytically polystable.

Assume that $x \in X$ is analytically polystable. If $\lambda(x, \beta) > 0$ for any $\beta \in \mathfrak{p}\backslash\{0\}$, then x is stable and so the result is proved. Assume that $\lambda(x, \beta) = 0$ for some $\beta \in \mathfrak{p}\backslash\{0\}$. Then $\lim_{t\mapsto+\infty} \exp(t\beta)x = y \in G\cdot x$ and $\beta_X(y) = 0$. By Corollary 5.3.5.2, y is analytically polystable. By Lemma 5.3.3, if y is G^β-stable then x is G-polystable. As in the proof of Theorem 5.3.4, we may decompose $\mathfrak{p}^\beta = \mathrm{span}(\beta) \oplus \mathfrak{p}'$ as K^β-modules and we consider the compatible subgroup $H = (K^\beta)^o \exp(\mathfrak{p}')$ of G^β. H preserves the connected component Y of X^β containing y. By Lemma 5.3.2 $\lambda(y, \beta') \geq 0$ for any $\beta' \in \mathfrak{p}^\beta$. We separate the two cases:

a) $\lambda(y, \beta') > 0, \forall \beta \in \mathfrak{p}'\backslash\{0\}$

b) There exists $\beta_1 \in \mathfrak{p}'$ such that $\lambda(y, \beta_1) = 0$.

If (a) holds then as in the previous proof, y is $(G^\beta)^o$ stable and so there exists $g \in (G^\beta)^o$ such that $\mu_{\mathfrak{p}^\beta}(gx) = \mu_{\mathfrak{p}}(gx) = 0$. In particular,

$$\lim_{t\mapsto+\infty} \exp(t\beta)gx \in G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0).$$

Otherwise, there exists $\beta_1 \in \mathfrak{p}'\backslash\{0\}$ such that $\lambda(y, \beta_1) = 0$. Then

$$\lim_{y\to\infty} \exp(t\beta_1)y = y_1 \in G \cdot x,$$

$(\beta_1)_X(y_1) = 0$ and $y_1 \in G \cdot x$. By Corollary 5.3.5.2, y_1 is analytically polystable. Let Y_1 be the connected component of Y^{β_1} containing y_1. Let $\mathfrak{p}' = \mathrm{span}(\beta_1) \oplus \mathfrak{p}''$ be a splitting of $K^{\mathfrak{a}_1}$-modules. As in the proof of Theorem 5.3.4, we repeat the procedure for the action of the compatible subgroup $H_{\beta_1} = (K^{\mathfrak{a}_1})^o \exp(\mathfrak{p}'')$ on Y_1, where $\mathfrak{a}_1 = \mathrm{span}(\beta, \beta_1)$. Since the dimension of a maximal Abelian subalgebra contained in $\mathfrak{p}$ is an invariant of the K-action on $\mathfrak{p}$ [27, 56], the above procedure will iterate at most $\dim(\mathfrak{a})$. This shows that $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$. $\square$

Corollary 5.3.6.1. *Let $x \in X$. Then x is polystable if and only if there exists $\xi \in \mathfrak{p}$ and $g \in (G^\xi)^o$ such that $\lim_{t \mapsto +\infty} \exp(t\xi) g x \in G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0)$.*

5.4 Linear examples

Let $Z = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{m+n})$. We consider the natural action of $\mathrm{GL}(n, \mathbb{C})$ on Z: $(g, L) := L \circ g^{-1}$. We fix the $\mathrm{Ad}(\mathrm{GL}(n, \mathbb{C}))$-invariant inner product of Euclidean type on $\mathfrak{gl}(n, \mathbb{C})$ given by $B(X, Y) = \mathrm{Tr}(XY)$. The maximal compact subgroup $\mathrm{U}(n)$ of $\mathrm{GL}(n, \mathbb{C})$ acts in a Hamiltonian fashion on Z with momentum map

$$\mu(L) = \frac{\mathbf{i}}{2}\Big(L^* \circ L - h\mathrm{Id}_n\Big),\ h \in \mathbb{R},$$

see for instance [22, p. 1043]. Therefore

$$\mu^{-1}(0)/\mathrm{U}(n) = \begin{cases} \mathrm{Gr}_n(\mathbb{C}^{m+n}) & h > 0 \\ \{0\} & h = 0 \\ \emptyset & h < 0 \end{cases},$$

where $\mathrm{Gr}_n(\mathbb{C}^{m+n})$ denotes the Grassmannian of the n dimensional subspaces of $\mathbb{C}^{n+m}$. Assume $h > 0$. If L is injective, then L is polystable. Indeed, $L^* \circ L = P^2$, where P is a positive Hermitian endomorphism of $\mathbb{C}^n$ and $S = L \circ P^{-1}$ satisfies $S^* \circ S = \mathrm{Id}_{\mathbb{C}^n}$. Therefore $g = \sqrt{h}P \in \mathrm{GL}(n, \mathbb{C})$, $g \cdot L = \sqrt{h}S$ and so $\mu(\sqrt{h}S) = 0$. Since the stabilizer of L is trivial, it follows that L is stable. If $\mathrm{Ker}\, L \neq \{0\}$, then it is easy to check that L is not semistable. Indeed, as in [22, p. 1043], let $\beta \in \mathbf{i}\mathfrak{u}(n)$ be such that $V_s \subset \operatorname{Ker} L$, where

$$V_s = \bigoplus_{\substack{\lambda \,\in\, \mathrm{Spec}(\beta) \\ \lambda < 0}} V_\lambda,$$

and $h\mathrm{Tr}(\beta) < 0$. Then $\lambda(L, \beta) < 0$.

$\mathrm{GL}(n,\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{C})$ is compatible. Indeed, $\mathrm{GL}(n,\mathbb{R}) = \mathrm{O}(n)\exp(\mathfrak{p})$, where $\mathrm{O}(n) = \mathrm{GL}(n,\mathbb{R}) \cap \mathrm{U}(n)$ and $\mathfrak{p} = \mathfrak{g} \cap iu(n) = \mathrm{Sym}(n)$, i.e., the set of the symmetric matrices of order n. $\mathrm{GL}(n,\mathbb{R})$ leaves $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{m+n}) \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{m+n})$ invariant. The associated $\mathrm{GL}(n,\mathbb{R})$-gradient map is given by

$$\mu_{\mathfrak{p}}(L) = \frac{1}{2}\Big(-L^T \circ L + h\mathrm{Id}_n\Big),\ h \in \mathbb{R}.$$

Therefore,

$$\mu_{\mathfrak{p}}^{-1}(0)/\mathrm{O}(n) = \begin{cases} \mathrm{Gr}_n(\mathbb{R}^{m+n}) & h > 0 \\ \{0\} & h = 0 \\ \emptyset & h < 0 \end{cases},$$

where $\mathrm{Gr}_n(\mathbb{R}^{m+n})$ denotes the Grassmannian of the n dimensional subspaces of $\mathbb{R}^{m+n}$. If $h > 0$, then it is easy to check that L is stable if and only if L is injective. As in the previous example, one can check that if $\operatorname{Ker} L \neq \{0\}$, then L is not semistable.

The $\mathrm{GL}(n,\mathbb{R})$-gradient map associated to the $\mathrm{GL}(n,\mathbb{R})$ action on $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{m+n})$ is given by

$$\mu_{\mathfrak{p}}(L) = \frac{1}{2}\Big(-A + h\mathrm{Id}_n\Big),\ h \in \mathbb{R},$$

where $A = \mathrm{Re}(L^* \circ L)$. Indeed, since $L^* \circ L$ is Hermitian, we have $L^* \circ L = A + \mathbf{i}C$, where A is a symmetric matrix and C is anti-symmetric matrix. Since $\langle A, \mathbf{i}C \rangle = 0$, it follows that the orthogonal projection of $L^* \circ L$ onto $\mathfrak{p}$ is given by A. Assume that $h > 0$. Let $g \in \mathrm{GL}(n,\mathbb{R})$. Then $(L \circ g^{-1})^* \circ (L \circ g^{-1}) = (g^{-1})^T \circ (L^* \circ L) \circ g^{-1} = (g^{-1})^T A g^{-1} + \mathbf{i}\left((g^{-1})^T C g^{-1}\right)$. This implies that L is polystable if and only if $\mathrm{Re}(L^* \circ L)$ is a positive-define symmetric matrix. If $\mathrm{Re}(L^* \circ L)$ is not injective, then one can check that L is not semistable. Indeed, let $\beta \in \mathfrak{p}$ be such that $V_s \subset \operatorname{Ker} \mathrm{Re}(L^* \circ L)$, where

$$V_s = \bigoplus_{\substack{\lambda \,\in\, \mathrm{Spec}(\beta) \\ \lambda < 0}} V_\lambda,$$

and $h\mathrm{Tr}(\beta) < 0$. Then $\lambda(L, \beta) < 0$.

5.5 Semistable points

In this section, we continue to study the semistable set of the gradient map using the result in the previous section. Therefore we always assume that the G-action on X is energy complete. Let $\mu_{\mathfrak{p}} : X \longrightarrow \mathfrak{p}$ denote the G-gradient map associated with the momentum map $\mu : Z \longrightarrow \mathfrak{u}$. Let $\mathfrak{a} \subset \mathfrak{p}$ a maximal Abelian subalgebra and let $A = \exp(\mathfrak{a})$. Then $\mu_{\mathfrak{a}} : X \longrightarrow \mathfrak{a}$ given by $\pi_{\mathfrak{a}} \circ \mu_{\mathfrak{p}}$ is the A-gradient map associated to μ, where $\pi_{\mathfrak{a}} : \mathfrak{p} \longrightarrow \mathfrak{a}$ is the orthogonal projection of $\mathfrak{p}$ onto $\mathfrak{a}$. Let

$$X^{ss}_{\mu_{\mathfrak{p}}} := \{x \in X : \overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset\}$$

$$X^{ss}_{\mu_{\mathfrak{a}}} := \{x \in X : \overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(0) \neq \emptyset\}.$$

Let $\mathfrak{a}' \subset \mathfrak{p}$ be another maximal Abelian subalgebra. Since the K-action on $\mathfrak{p}$ is polar and both $\mathfrak{a}$ and $\mathfrak{a}'$ are section then there exists $k \in K$ such that $\mathrm{Ad}(k)(\mathfrak{a}) = \mathfrak{a}'$ [56]. In particular,

$$A' = \exp(\mathfrak{a}') = \exp(\mathrm{Ad}(k)(\mathfrak{a})) = k \exp(\mathfrak{a}) k^{-1} = kAk^{-1}.$$

Lemma 5.5.1. *The A'-gradient map $\mu_{\mathfrak{a}'} : X \longrightarrow \mathfrak{a}'$ associated to μ is such that*

$$\mu_{\mathfrak{a}'} = \mathrm{Ad}(k) \circ \mu_{\mathfrak{a}} \circ k^{-1}, \quad k \in K$$

Proof. Let $\xi' \in \mathfrak{a}'$. Then, there exist $k \in K$ such that $\xi' = \mathrm{Ad}(k)(\xi)$ where $\xi \in \mathfrak{a}$. So that by the K-equivariant property of the gradient map,

$$\begin{aligned} \langle \mu_{\mathfrak{p}}(x), \xi' \rangle &= \langle \mu_{\mathfrak{p}}(k^{-1}x), \xi \rangle \\ &= \langle \mathrm{Ad}(k)(\mu_{\mathfrak{p}}(k^{-1}x)), \xi' \rangle. \end{aligned}$$

The result follow.

$\square$

Lemma 5.5.2. *There exists $k \in K$ such that*

$$kX^{ss}_{\mu_{\mathfrak{a}}} = X^{ss}_{\mu_{\mathfrak{a}'}}.$$

Proof. Since $A' = kAk^{-1}$, and by Lemma 5.5.1, $\mu_{\mathfrak{a}'}(x) = 0$ if and only if $\mu_{\mathfrak{a}}(k^{-1}x) = 0$. This implies that $\mu_{\mathfrak{a}'}^{-1}(0) = k\mu_{\mathfrak{a}}^{-1}(0)$. The sequence $(\exp(\xi_n)x)_n$ converges in $\mu_{\mathfrak{a}}^{-1}(0)$ if and only if $(k\exp(\xi_n)x)_n$ converges in $\mu_{\mathfrak{a}'}^{-1}(0)$. But

$$k\exp(\xi_n)x = k\exp(\xi_n)k^{-1}kx = \exp(\mathrm{Ad}(k)(\xi_n))kx.$$

This implies that $\overline{A \cdot x} \cap \mu_{\mathfrak{a}}^{-1}(0) \neq \emptyset$ if and only if $\overline{A' \cdot (kx)} \cap \mu_{\mathfrak{a}'}^{-1}(0) \neq \emptyset$. Hence $kX^{ss}_{\mu_{\mathfrak{a}}} = X^{ss}_{\mu_{\mathfrak{a}'}}$. ☐

Proposition 5.5.3. $X^{ss}_{\mu_{\mathfrak{p}}} = \bigcap_{k\in K} kX^{ss}_{\mu_{\mathfrak{a}}}$.

Proof. By Theorem 5.3.4, $x \in X^{ss}_{\mu_{\mathfrak{p}}}$ if and only if $\lambda(x,\beta) \geq 0$. Since the K-action on $\mathfrak{p}$ is polar, by [44, 13.1]

$$\mathfrak{p} = \bigcup_{k\in K} \mathrm{Ad}(k)(\mathfrak{a}).$$

Then by Lemma 5.5.2, we have

$$\begin{aligned} x \in X^{ss}_{\mu_{\mathfrak{p}}} &\iff \lambda(x,\cdot)|_{\mathrm{Ad}(k)(\mathfrak{a})} \geq 0 \quad \forall k \in K \\ &\iff x \in X^{ss}_{\mu_{\mathfrak{a}'}}, \quad \mathfrak{a}' = \mathrm{Ad}(k)(\mathfrak{a})\ \forall k \in K \\ &\iff kx \in X^{ss}_{\mu_{\mathfrak{a}}} \quad \forall k \in K. \end{aligned}$$

Therefore, $X^{ss}_{\mu_{\mathfrak{p}}} = \bigcap_{k\in K} kX^{ss}_{\mu_{\mathfrak{a}}}$. ☐

Proposition 5.5.4. *Let $H \subset G$ be a compatible subgroup such that $H = K'\exp(\mathfrak{p}')$. Then*

$$X^{ss}_{\mu_{\mathfrak{p}}} \subset X^{ss}_{\mu_{\mathfrak{p}'}}.$$

Proof. Let $x \in X^{ss}_{\mu_{\mathfrak{p}}}$. By Theorem 5.3.4 it follows that $\lambda(x,\beta) \geq 0$ for any $\beta \in \mathfrak{p}$. Hence $\lambda(x,\beta) \geq 0$ for any $\beta \in \mathfrak{p}' \subset \mathfrak{p}$ and so, applying again Theorem 5.3.4, we have $x \in X^{ss}_{\mu_{\mathfrak{p}'}}$. ☐

Theorem 5.5.5. *Let* (Z, ω) *be a compact connected Kähler manifold and let* $G \subset U^{\mathbb{C}}$ *be a compatible subgroup. If* $Z_\mu^{ss} \neq \emptyset$*, then* $Z_{\mu_\mathfrak{p}}^{ss}$ *is an open dense connected subset of* Z*.*

Proof. By Proposition 5.5.4, $Z_\mu^{ss} \subset Z_{\mu_\mathfrak{p}}^{ss}$. Since Z_μ^{ss} is open and dense in Z, it follows that $Z_{\mu_\mathfrak{p}}^{ss}$ is open and dense. Suppose

$$Z_{\mu_\mathfrak{p}}^{ss} = \Upsilon_1 \sqcup \Upsilon_2,$$

where Υ_1 and Υ_2 are open. Then $Z_\mu^{ss} \subset \Upsilon_1$ or $Z_\mu^{ss} \subset \Upsilon_2$. If $Z_\mu^{ss} \subset \Upsilon_1$, this implies that $\Upsilon_2 \subset Z \setminus Z_\mu^{ss}$. Since Z_μ^{ss} is open and dense, then $(Z \setminus Z_\mu^{ss})^0 = \emptyset$. This implies that $\Upsilon_2 = \emptyset$. The case $Z_\mu^{ss} \subset \Upsilon_2$ is similar. Therefore, Z_μ^{ss} is connected, open and dense. □

5.6 Final Remark

Let (Z, ω) be a Kähler manifold and $U^{\mathbb{C}}$ acts holomorphically on Z with a U-equivariant momentum map $\mu : Z \to \mathfrak{u}$. The stabilities conditions depend on the choice of a maximal compact subgroup U of $U^{\mathbb{C}}$, a U-invariant Kähler metric and the momentum map μ. It is well-known that two maximal compact subgroups are conjugate. Let $G \subset U^{\mathbb{C}}$ be compatible. Let $g \in G$ and let $U' = gUg^{-1}$. Then $\omega_g = (g^{-1})^\star \omega$ is a Kähler form and U' preserves ω_g. Since B is $\mathrm{Ad}(U^{\mathbb{C}})$-invariant, B restricted to $\mathfrak{u}'$, respectively $\mathbf{i}\mathfrak{u}'$, is negative-define, respectively positive-define.

Lemma 5.6.1. *The* U'*-action on* (Z, ω_g) *is Hamiltonian with momentum map* $\mu' : Z \longrightarrow \mathfrak{u}'$*, given by* $\mu' = \mathrm{Ad}(g) \circ \mu \circ g^{-1}$*.*

Proof. We prove that μ' is U'-equivariant. Let $h \in U$. Then

$$\begin{aligned} \mu'(ghg^{-1}x) &= \mathrm{Ad}(g)(\mu(hg^{-1}x)) \\ &= \mathrm{Ad}(g)\big(\mathrm{Ad}(h)(\mu(g^{-1}x)\big) \\ &= \mathrm{Ad}(ghg^{-1})\big(\mathrm{Ad}(g)(\mu(g^{-1}x))\big) \\ &= \mathrm{Ad}(ghg^{-1})(\mu'(x))) \end{aligned}$$

Let $\xi \in \mathfrak{u}$. Then

$$\begin{aligned}(\mu')^{\mathrm{Ad}(g)(\xi)}(z) &= -\langle \mu'(z), \mathrm{Ad}(g)(\xi)\rangle \\ &= -\langle \mu(g^{-1}z), \xi\rangle,\end{aligned}$$

and so $\mathrm{d}(\mu')^{\mathrm{Ad}(g)(\xi))} = \mathrm{d}\mu^{\xi} \circ \mathrm{d}g^{-1}$. Therefore

$$\begin{aligned}\mathrm{d}(\mu')_z^{\mathrm{Ad}(g)(\xi))} &= \omega(\xi_Z(g^{-1}z), \mathrm{d}g^{-1}\cdot) \\ &= \omega(\mathrm{d}g^{-1}\big((\mathrm{Ad}(g)(\xi)_Z(z)\big), \mathrm{d}g^{-1}\cdot) \\ &= \omega_g(\mathrm{Ad}(g)(\xi)_Z(z), \cdot).\end{aligned}$$

□

Let X be a G-invariant submanifold. Let g denote the Riemannian metric induced by the Kähler form ω. For any $g \in G$, we have a triple $(U', (g^{-1})^{\star}\mathsf{g}, \mu')$, where $U' = gUg^{-1}$. Note that G is also compatible with respect to the Cartan decomposition $U^{\mathbb{C}} = U' \exp(\mathfrak{p}')$. Indeed, $G = gGg^{-1} = K' \exp(\mathfrak{p}')$, where $K' = gKg^{-1} = G \cap U'$ and $\mathfrak{p}' = \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{g} \cap \mathbf{i}\mathfrak{u}'$. The associated gradient map $\mu' : X \longrightarrow \mathfrak{p}'$ is the orthogonal projection of $i\mu'$ onto $\mathfrak{p}'$ with respect to B. One can check

$$\mu_{\mathfrak{p}'} = \mathrm{Ad}(g) \circ \mu_{\mathfrak{p}} \circ g^{-1}.$$

Following Teleman [4], we say that such triples define a *symplectization* of the $U^{\mathbb{C}}$-action on Z with respect to G. A priori the concept of energy completeness condition depends on the choice of a triple. The following result shows that the notion of energy completeness does not depend on the triple chosen.

Lemma 5.6.2. *If (U, g, μ) is energy complete then $(U', (g^{-1})^{\star}\mathsf{g}, \mu')$ is energy complete as well.*

Proof. Let $\xi \in \mathfrak{g}$ and let $x \in X$. We recall that c_x^{ξ} denotes the curve $c_x^{\xi}(t) = \exp(t\xi)x$. Let $\xi \in \mathfrak{p}$. Since

$$c_x^{\mathrm{Ad}(g)(\xi)} = g \circ c_{g^{-1}x}^{\xi},$$

it follows that the energy of $c_x^{\mathrm{Ad}(g)(\xi)}$ with respect to $(g^{-1})^\star \mathsf{g}$ coincides with the energy of $c_{g^{-1}x}^{\xi}$ with respect to g. Moreover, the limit $\lim_{t\mapsto +\infty} c_x^{\mathrm{Ad}(g)(\xi)}(t)$ exists if and only if the limit $\lim_{t\mapsto+\infty} c_{g^{-1}x}^{\xi}(t)$ does. $\square$

We claim that the stable, polystable and semistable conditions do not depend on the triple chosen. It is a consequence of the following easy Lemma.

Lemma 5.6.3. *Let $x \in X$. Then $x \in \mu_{\mathfrak{p}'}^{-1}(0)$ if and only if $g^{-1}x \in \mu_{\mathfrak{p}}^{-1}(0)$.*

Proof. $\mu_{\mathfrak{p}'}(x) = 0$ if and only if for any $\xi \in \mathfrak{p}$ we have $\langle(\mu_{\mathfrak{p}'}(x), \mathrm{Ad}(g)(\xi)\rangle = 0$ hence if and only if $\langle \mu_{\mathfrak{p}}(g^{-1}x), \xi\rangle = 0$ and so the result follows. $\square$

Corollary 5.6.3.1. *Let $x \in X$. Then*

a) $G \cdot x \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ if and only if $G \cdot x \cap (\mu_{\mathfrak{p}'})^{-1}(0) \neq \emptyset$;

b) $\overline{G \cdot x} \cap \mu_{\mathfrak{p}}^{-1}(0) \neq \emptyset$ if and only if $\overline{G \cdot x} \cap (\mu_{\mathfrak{p}'})^{-1}(0) \neq \emptyset$.

The norm square gradient map depends on the choice of a triple (U, g, μ). Indeed,

$$f(x) = \frac{1}{2}\langle \mu_{\mathfrak{p}}(x), \mu_{\mathfrak{p}}(x)\rangle.$$

Let $g \in G$ and let $(U', (g^{-1})^\star \mathsf{g}, \mu')$ another triple. We denote by f' the norm square of $\mu_{\mathfrak{p}'}$. Since $\langle\cdot,\cdot\rangle$ is $\mathrm{Ad}(G)$-invariant, it follows that $f'(x) = f(g^{-1}x)$. Moreover, x is a critical point of f' if and only if $g^{-1}x$ is a critical point of f. This implies that $K \cdot \beta$ is a critical orbit of f if and only if $K' \cdot \mathrm{Ad}(g)(\beta)$ is a critical orbit of f'. From now on, we assume that X is compact and connected. Then the negative gradient flow line of the norm square is defined in all the real line. Moreover, if $x : \mathbb{R} \longrightarrow X$ denotes the negative gradient flow line, then $\lim_{t\mapsto+\infty} x(t)$ exists [17]. It is a straightforward computation that the gradient of $f'(x)$ with respect to $(g^{-1})^\star \mathsf{g}$ is given by $(\mathrm{d}g)_{g^{-1}x}\big(\mathrm{grad}\, f(g^{-1}x)\big)$. Therefore, if $x(t)$ is the negative gradient flow line of f, then $g(x(t))$ is the negative gradient flow line of f'.

Let $x_\infty = \lim_{t\mapsto+\infty} x(t)$ and let $\beta = \mu_{\mathfrak{p}}(x_\infty)$. Then $\mu_{\mathfrak{p}'}(gx_\infty) = \mathrm{Ad}(g)(\beta)$ and $f(x_\infty) = f'(gx_\infty)$. By [17, Theorem 4.9], the stratum of associated to $K \cdot \beta$ with respect to f coincides with the stratum associated to $K' \cdot \mathrm{Ad}(g)(\beta)$ with respect to f'. Summing up we have proved the following result.

Theorem 5.6.4. *If X is connected and compact, then the decomposition of X in strata associated to critical orbits of the norm square gradient map does not depend on the triple chosen.*

5.7 Stability Conditions in Term of Kempf-Ness Function

In this section, we characterise the stability conditions in term of the Kempf-Ness function if X is compact. For simplicity, G is assumed to be connected. The section is concluded with the Hilbert criterion for semistability condition with the compactness of X. The discussion in this section is a work in progress.

The following Theorem characterises the semistability condition in term of the limit of a negative gradient flow lines of the norm square of the gradient map.

Theorem 5.7.1. *Suppose X is compact. Let $x_0 \in X$ and $x : \mathbb{R} \to X$ be the solution of the negative gradient flow lines of the norm square through x_0. Define*

$$x_\infty := \lim_{t\to\infty} x(t).$$

x_0 is semistability if and only if $\mu_\mathfrak{p}(x_\infty) = 0$.

Proof. Since X is compact, x_0 is semistable if and only if $\inf_G |\mu_\mathfrak{p}(gx_0)| = 0$. By Theorem 2.2.9, $|\mu_\mathfrak{p}(x_\infty)| = \inf_G |\mu_\mathfrak{p}(gx_0)|$. The result follows. $\square$

Fix $x_0 \in X$ and let $\Phi_{x_0} : M \to \mathbb{R}$ be the Kempf-Ness function. Let $x : \mathbb{R} \to X$ be the solution of the negative gradient flow of the norm square of the gradient map through x_0, $g : \mathbb{R} \to G$ be such that $\gamma(t) = \pi \circ g(t)$, where $\pi : G \to M$ is the projection. The limit $x_\infty = \lim_{t\to\infty} x(t)$ exists by Theorem 2.1.3 and by Theorem 2.2.9, we have that $|\mu_\mathfrak{p}(x_\infty)| = \inf_G |\mu_\mathfrak{p}(gx_0)|$. The theorem below characterizes the semistability of an element in terms of the Kempf-Ness function.

Theorem 5.7.2. *Suppose X is compact. Let $x \in X$ and $\Phi_x : M \to \mathbb{R}$ be the Kempf-Ness function of x. Then x is semistable if and only if Φ_x is bounded below.*

Proof. Suppose $x_0 \in X$ is semistable. Then by Theorem 5.7.1, $\mu_{\mathfrak{p}}(x_\infty) = 0$. By the Lojaseiwicz gradient inequality for $f = \frac{1}{2} \parallel \mu_{\mathfrak{p}} \parallel^2$ there exists constants $T, \alpha > 0$ and a constant $c \in (\frac{1}{2}, 1)$ such that

$$|f(x) - f(x_\infty)|^c \leq \alpha |\nabla f(x)|.$$

Since $\mu_{\mathfrak{p}}(x_\infty) = 0$, then $f(x_\infty) = \frac{1}{2} \parallel \mu_{\mathfrak{p}}(x_\infty) \parallel^2 = 0$ and so,

$$|\mu_{\mathfrak{p}}(x)|^2 = 2|f(x)| \leq 2|f(x)|^c \leq C|\nabla f(x)| = C|\beta_X(x)| = C|\dot{x}(t)|, \quad C = 2\alpha, \beta = \mu_{\mathfrak{p}}(x) \quad \forall t \geq T.$$

The function $t \mapsto |\dot{x}(t)|$ is integrable over the positive real axis and so is the function $t \mapsto |\mu_{\mathfrak{p}}(x(t))|^2$ but by Remark 2.2.3

$$-\frac{d}{dt}(\Phi_{x_0} \circ \gamma)(t) = |\mu_{\mathfrak{p}}(x(t))|^2 \leq C|\dot{x}(t)|.$$

This implies that the limit

$$a := \lim_{t \to \infty} \Phi_{x_0}(\gamma(t))$$

exists in $\mathbb{R}$ and by Theorem 2.2.7, $a = \inf_M \Phi_{x_0}$ and so Φ_{x_0} is bounded below.

We proof the other direction by contradiction. Suppose x_0 is unstable. Then $\mu_{\mathfrak{p}}(x_\infty) \neq 0$. By Remark 2.2.3 and Theorem 2.2.9

$$\frac{d}{dt}(\Phi_{x_0} \circ \gamma) = -\langle \mu_{\mathfrak{p}}(g^{-1}x_0), g^{-1}\dot{g} \rangle = -|\mu_{\mathfrak{p}}(x(t))|^2 \leq -|\mu_{\mathfrak{p}}(x_\infty)|^2.$$

This implies that

$$\Phi_{x_0} \circ \gamma \leq -t|\mu_{\mathfrak{p}}(x_\infty)|^2 \quad \forall t \geq 0.$$

Thus Φ_{x_0} is unbounded below. □

Theorem 5.7.3. *Suppose X is compact. Let $x \in X$ and $\Phi_x : M \to \mathbb{R}$ be the Kempf-Ness function of x. Then x is polystable if and only if Φ_x has a critical point.*

Proof. The result follows from the fact that $\pi(g)$ is a critical point of Φ_{x_0} if and only if $\mu_{\mathfrak{p}}(g^{-1}x_0) = 0$. □

Theorem 5.7.4. *Suppose X is compact. Let $x \in X$ and $\Phi_x : M \to \mathbb{R}$ be the Kempf-Ness function of x. Then x is stable if and only if Φ_x is bounded below and proper.*

Proof. Suppose Φ_{x_0} is bounded below and proper. x_0 is semistable by Theorem 5.7.2. From the definition of Kempf-Ness function, we have $\Phi_{x_0}(\gamma(0)) = 0$. Let

$$c := \inf_M \Phi_{x_0} \in (-\infty, 0].$$

$\Phi_{x_0} \circ \gamma : \mathbb{R} \to \mathbb{R}$ is nonincreasing. Since Φ_{x_0} is proper, then $\Phi_{x_0}^{-1}([c, 0])$ is compact and $\gamma(t) \in \Phi_{x_0}^{-1}([c, 0])$ for all $t \in [0, \infty)$. There exists a sequence $t_i \to \infty$ such that $\gamma(t_i)$ converges.Therefore, Φ_{x_0} has a critical point and by Theorem 5.7.3, x_o is polystable. Let $g \in G$ be such that $\mu_{\mathfrak{p}}(gx_0) = 0$. Suppose by contradiction, x_o is not stable. Then $\mathfrak{g}_{x_0}$ is not conjugate to a subspace of $\mathfrak{k}$. This means that there exists $\xi \in \mathfrak{g}_{x_0}$ such that $\xi \neq 0$, hence there exists $\alpha \in \mathfrak{g}_{gx_0}$ such that $\alpha \neq 0$. But $\mathfrak{g}_{gx_0} = \mathfrak{k}_{gx_0} + \mathfrak{p}_{gx_0}$. Let $\beta \in \mathfrak{p} \setminus \{0\}$ such that $\beta_X(gx_0) = 0$. Then $\exp(t\beta) \cdot gx_0 = gx_0$ and so $\mu_{\mathfrak{p}}(\exp(t\beta) \cdot gx_0) = \mu_{\mathfrak{p}}(gx_0) = 0$. Therefore, the curve

$$\gamma(t) := \pi(g^{-1}\exp(-t\beta))$$

consists of critical points of Φ_{x_0} and hence, by Theorem 2.2.7, $\Phi_{x_0}(\gamma(t)) = c$ for all t. This implies that $\Phi_{x_0}^{-1}(c)$ is not compact which contradict the assumption that Φ_{x_0} is proper. Hence, x_0 is stable.

Suppose x_0 is stable. By Theorem 5.7.2, Φ_{x_0} is bounded below and we only need to show that it is proper. We can assume that $\mu_{\mathfrak{p}}(x_0) = 0$. Let $q := \pi(e) \in M$ where e is the identity element of G. Let $p = \pi(g) \in M \setminus \{q\}$. Choose $\beta \in \mathfrak{p}$ and $k \in K$ such that $g = \exp(\beta)k$. Then, $\beta \neq 0$ and let $y(t) := \exp(t\beta)x_0$, and $z(t) := \Phi_{x_0}(\pi(\exp(-t\beta)))$. The function $z : \mathbb{R} \to \mathbb{R}$ is convex and

$$\dot{z} = \langle \mu_{\mathfrak{p}}(y(t)), \beta \rangle, \quad \ddot{z} = |\beta_X(y(t))|^2.$$

Since $y(0) = x_0$, we have $z(0) = 0$, $\dot{z}(0) = \langle \mu_{\mathfrak{p}}(y(0)), \beta \rangle = \langle \mu_{\mathfrak{p}}(x_0), \beta \rangle = 0$ and $\ddot{z}(0) = |\beta_X(x_0)|^2 > 0$. Hence, $z(t) > 0$ for every $t \in \mathbb{R} \setminus 0$ and in particular, $\Phi_{x_0}(p) = z(-1) > 0$. This shows that q is the unique point at which Φ_{x_0} attains its minimum $\min_M \Phi_{x_0} = 0$.

For every r positive, let $q \in B_r \subset M$ be the ball of radius r centered at q. Let $\delta := \inf_{\partial B_1} \Phi_{x_0} > 0$. Then, by convexity, we have $d(q, p) \geq 1$ implies that $\Phi_{x_0}(p) \geq \delta d_M(q, p)$

for all $p \in M$. Therefore, for every $c > 0$,

$$c \geq \delta \implies \Phi_{x_0}^{-1}([0,c]) \subset B_{\frac{c}{\delta}}.$$

This shows that, for every positive c, $\Phi_{x_0}^{-1}([0,c])$ is closed and bounded, and hence compact since M is complete. Therefore, Φ_{x_0} is proper. This conclude the proof. ☐

Theorem 5.7.5. *Suppose X is compact. Then $x \in X$ is semistable if and only if $\lambda(x,\beta) \geq 0$ for any $\beta \in \mathfrak{p} \setminus \{0\}$.*

Remark 5.7.6. *Theorem 5.7.5 also hold if we just assume that the gradient map is proper even if X is not compact. The main required assumption is the existence of the stratification of X in order for semistability condition to be equivalent to a condition on the infimum of the Kempf-ness function.*

Proof. Suppose $x \in X$ is semistable. Since X is compact, $x \in X$ is semistable if and only if $\inf_G |\mu_{\mathfrak{p}}(gx_0)| = 0$. Then by Theorem 5.2.8, $\lambda(x,\beta) \geq 0$ for any $\beta \in \mathfrak{p} \setminus \{0\}$.

Let $x_0 \in X$ be such that $\lambda(x_0,\beta) \geq 0$ for any $\beta \in \mathfrak{p} \setminus \{0\}$. Suppose by contradiction that x_0 is such that

$$m := \inf_G |\mu_{\mathfrak{p}}(gx_0)| > 0.$$

Let $x : \mathbb{R} \to X$ be the solution of the negative gradient flow of the norm square of the gradient map through x_0, $g : \mathbb{R} \to G$ be such that $x(t) = g(t)^{-1}x_0$ and $\gamma(t) = \pi \circ g(t)$, where $\pi : G \to M$ is the projection. The limit $x_\infty = \lim_{t\to\infty} x(t)$ exists by Theorem 2.1.3. Define $\beta : \mathbb{R} \to \mathfrak{p}; t \mapsto \beta(t)$ and $k : \mathbb{R} \to \mathfrak{p}; t \mapsto k(t)$ by

$$g(t) =: \exp(-\beta(t))k(t).$$

The limit

$$\beta_\infty := \lim_{t\to\infty} \frac{\beta(t)}{t}$$

exists by Proposition 2.2.15. By Theorem 2.2.9, $|\mu_{\mathfrak{p}}(x_\infty)| = m$. Fix a real number $s \geq 0$ and let $H_{s,\infty} : [s,\infty) \to M$ be a geodesic define as

$$H_{s,\infty}(r) := \pi(g(s)\exp((s-r)\beta_\infty(s))) = \lim_{t\to\infty} H_{s,t}(r) \quad \text{for} \quad r \geq s.$$

By (2.18),

$$d_M(\gamma(r), H_{s,t}(r)) = \rho_{s,t}(r) \leq C(r^{1-\alpha} - s^{1-\alpha}) \quad \text{for} \quad s \leq r \leq t.$$

As $t \to \infty$,

$$d_M(\gamma(r), H_{s,\infty}(r)) \leq C(r^{1-\alpha} - s^{1-\alpha}) \quad \text{for} \quad r \geq s \geq 0. \tag{5.6}$$

Now the Kempf-Ness function is globally Lipschitz continuous with Lipschitz constant $L := \sup_{g\in G}|\mu_{\mathfrak{p}}(gx_0)|$. Hence it follows that

$$|\Phi_{x_0}(\gamma(t)) - \Phi_{x_0}(H_{s,\infty}(t))| \leq LC(r^{1-\alpha} - s^{1-\alpha}) \quad \text{for} \ \ t \geq s \geq 0. \tag{5.7}$$

Integrate the equation

$$\frac{d}{dr}\Phi_{x_0}(\gamma(r)) = -|\mu_{\mathfrak{p}}(x(r))|^2$$

to obtain

$$\Phi_{x_0}(\gamma(t)) = \Phi_{x_0}(\gamma(s)) - \int_s^t |\mu_{\mathfrak{p}}(x(r))|^2 dr. \tag{5.8}$$

By Lemma 5.1.4, (5.7) and (5.8), then

$$\begin{aligned}
\lambda(x(s), \beta_\infty(s)) &= \lim_{t\to\infty} \frac{\Phi_{x_0}(H_{s,\infty}(s+t))}{t} \\
&= \lim_{t\to\infty} \frac{\Phi_{x_0}(H_{s,\infty}(t)) - \Phi_{x_0}(\gamma(s))}{t-s} \\
&= \lim_{t\to\infty} \frac{\Phi_{x_0}(\gamma(t)) - \Phi_{x_0}(\gamma(s))}{t-s} \\
&= -\lim_{t\to\infty} \frac{1}{t-s} \int_s^t |\mu_{\mathfrak{p}}(x(r))|^2 dr \\
&= -|\mu_{\mathfrak{p}}(x_\infty)|^2 \\
&= -m^2.
\end{aligned}$$

Let $\beta(s,t)$ be as in Proposition 2.2.15. By the choice of $\beta(s,t)$,

$$\begin{aligned}
\frac{|\beta(s,t)|}{t-s} = \frac{d_m(\gamma(s), \gamma(t))}{t-s} \\
&\leq \frac{1}{t-s} \int_s^t |\dot{\gamma}(r)| dr \\
&= \frac{1}{t-s} \int_s^t |\mu_{\mathfrak{p}}(x(r))| dr.
\end{aligned}$$

So that,

$$|\beta_\infty(s)| = \lim_{t\to\infty} \frac{|\beta(s,t)|}{t-s} \leq \lim_{t\to\infty} \frac{1}{t-s} \int_s^t |\mu_{\mathfrak{p}}(x(r))| dr = |\mu_{\mathfrak{p}}(x_\infty)| = m.$$

Also, by Theorem 5.2.7,

$$m^2 = -\lambda(x(s), \beta_\infty(s)) \leq |\beta_\infty(s)| \inf_{g\in G} |\mu_{\mathfrak{p}}(gx_0)| = m|\beta_\infty(s)| \implies m \leq \beta_\infty(s).$$

Therefore, $|\beta_\infty(s)| = m$. Hence,

$$\lambda(x_0, \beta_\infty) = -m^2 < 0, \quad |\beta_\infty| = m,$$

which contradict the assumption on $\lambda(x, \beta)$. This concludes the proof of the theorem.

□

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