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Gradient continuity for $p(x)$-Laplacian systems under minimal conditions on the exponent

Paolo Baroni

Department of Mathematical, Physical and Computer Sciences, University of Parma, I-43124 Parma, Italy

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Abstract

We consider solutions of $p(x)$-Laplacian systems with coefficients and we show that their gradient is continuous provided that the variable exponent has distributional gradient belonging to the Lorentz-Zygmund space $L^{n,1}\log L$ and that the gradient of the coefficient belongs to the Lorentz space $L^{n,1}$. The result is new since the use of the sharp Sobolev embedding in rearrangement invariant spaces does not ensure the unique (up to now) known assumption for such result, namely the log-Dini continuity of $p(\cdot)$ and the plain Dini continuity of the coefficient. Our approach relies on perturbation arguments and allows to slightly improve results in dimension two even for the case where $p(\cdot)$ is constant.

Keywords: Non-uniformly elliptic systems; $p(x)$-Laplacian; Gradient continuity

1. Introduction

We consider weak solutions to the $p(x)$-Laplacian system with coefficients

$$\operatorname{div}\Big[a(x)|Du|^{p(x)-2}Du\Big]=0, \qquad \text{in } \Omega\subset\mathbb{R}^n, n\geq 2, \tag{1.1}$$

Ω bounded domain, defined for functions in $W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$, $N\geq 1$ (see Section 2 for the necessary definitions) and we start by supposing the variable exponent $p:\Omega\to\mathbb{R}$ measurable and satisfying the basic assumptions

E-mail address: paolo.baroni@unipr.it.

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$$1 < \gamma_1 \le p(\cdot) \le \gamma_2 < \infty, \tag{1.2}$$

while we require the coefficient $a : \Omega \to \mathbb{R}$ to be measurable and bounded, above and away from zero:

$$a \in L^\infty(\Omega), \qquad 0 < \nu \le a(\cdot) \le L < \infty. \tag{1.3}$$

Problems with $p(x)$-growth are one of the main models embraced by the more general class of so-called problems with (p, q)-growth whose origin goes back to the work of Marcellini starting in the late eighties [30–33]; their peculiarity, namely the fact that the nonuniform ellipticity is mild (see the nice description via pointwise and nonlocal ellipticity ratios in [21]), has ensured then a prominent position as an active research argument almost constantly for the last twenty years. The literature on $p(x)$-problems is therefore too wide to even attempt to make a reasonable list of selected references; we only recommend [25] for a somehow already outdated survey and the seminal contributions [2,24,1,16] proving, respectively $C^{0,\alpha}$ for some $\alpha \in (0, 1)$, $C^{0,\alpha}$ for all $\alpha \in (0, 1)$ and $C^{1,\beta}$ for some $\beta \in (0, 1)$ regularity for local solutions to (1.1); all these results have as common background (1.2)-(1.3), but different assumptions need to be further considered on both the regularity of the coefficient and the variable exponent, the latter in terms of the behavior for $\rho \approx 0$ of the quantity

$$\omega_{\log}(\rho) = \omega_p(\rho) \log\left(\frac{1}{\rho}\right). \tag{1.4}$$

Here $\omega_p : [0, 1] \to [0, \gamma_2 - \gamma_1]$ is a modulus of continuity for $p(\cdot)$, that is a concave function with $\omega_p(0) = 0$, continuous in zero and such that

$$|p(x) - p(y)| \le \omega_p(|x - y|) \qquad \text{for all } x, y \in \Omega \text{ with } |x - y| \le 1. \tag{1.5}$$

More in detail, supposing (1.2)-(1.3), one has the following schematic description of the regularity of (local) solutions to (1.1) in terms of the behavior of $\omega_{\log}(\cdot)$ and of $\omega_a(\cdot)$ as $\rho \searrow 0$ (being $\omega_a : [0, 1] \to [0, L - \nu]$ a modulus of continuity for $a(\cdot)$, if any - simply adapt the definition in (1.5)):

- $\limsup_{\rho \searrow 0} \omega_{\log}(\rho) < \infty \implies u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, $|Du|^{p(\cdot)} \in L^{1+\delta_0}_{\mathrm{loc}}(\Omega)$ for some constants α, $\delta_0 \in (0, 1)$ depending on the data;
- $\limsup_{\rho \searrow 0} \big(\omega_{\log}(\rho) + \omega_a(\rho)\big) = 0 \implies u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ for every $\alpha \in (0, 1)$;
- $\omega_{\log}(\rho) + \omega_a(\rho) \le c\,\rho^\gamma$ for some $\gamma \in (0, 1)$ and $c > 0 \implies u \in C^{1,\beta}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ for some exponent $\beta \in (0, 1)$ depending on the data.

A natural but interesting borderline result, of particular interest in our context and lying in between the second and the third result above, has been recently obtained by Ok in [34] (see also [35]): if the exponent $p(\cdot)$ is log-Dini continuous and the coefficient $a(\cdot)$ is Dini continuous, then $u \in C^1_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Dini continuity is a classical and almost ubiquitous assumptions in borderline cases of the regularity theory and consists in the fact that the modulus of continuity of the function one considers is integrable in zero with respect to the measure $d\rho/\rho$; in other words, $a(\cdot)$ is Dini continuous if

$$\int_0^1 \omega_a(\rho)\,\frac{d\rho}{\rho} < \infty.$$

The variable exponent $p(\cdot)$ is said to be log-Dini continuous if $\omega_{\log}(\cdot)$ is Dini continuous, that is, if

$$\int_0^1 \omega_p(\rho)\log\left(\frac{1}{\rho}\right)\frac{d\rho}{\rho} < \infty; \tag{1.6}$$

it is evident a parallel between the regularity of the coefficient and that, corrected by a logarithmic factor, of the exponent. This fact is also described in [4]: notice that in [4] the differential operator is slightly different however a simple heuristic explanation based on Taylor expansions justifies the formal similitude of the two. This formal relation continues to hold true also in the case of the assumption of this paper, even if in a different setting: see (1.7) and (1.8)-(1.9). The Dini and log-Dini assumptions are extensively used in every other aspect related to gradient continuity, for instance gradient potential estimates, see [7,10,11]. An interesting variant, mixing the Dini condition with a modulus of continuity for the integral oscillation and strictly related to the approach developed in this paper, can be found in [22,23] related to solutions to linear equations.

We prove here gradient continuity for local solutions to (1.1) under a new integral assumption on the regularity of both the coefficient and the variable exponent. More precisely, we suppose (1.2)-(1.3) and moreover we assume that

$$a, p \in W^{1,1}(\Omega) \qquad \text{with} \qquad Da \in L^{n,1}(\Omega;\mathbb{R}^n), \quad Dp \in L^{n,1}\log L(\Omega;\mathbb{R}^n). \tag{1.7}$$

We recall that Da belongs to the Lorentz space $L^{n,1}(\Omega;\mathbb{R}^n)$ if

$$\int_0^\infty \left|\{x\in\Omega : |Da(x)|>\lambda\}\right|^{\frac{1}{n}}\,d\lambda < \infty; \tag{1.8}$$

such space, besides being fundamental as borderline rearrangement invariant space between the classic Sobolev and Morrey embeddings, has attracted lot of attention in the last years as significant, differential-operator invariant borderline space ensuring gradient continuity for solutions; see for instance [8,12,17,21,28,34,35] for details in several contexts, from non-uniformly general elliptic operators to parabolic and variational ones. In view of a characterization by O'Neil, Dp belongs to the Lorentz-Zygmund space $L^{n,1}\log L(\Omega;\mathbb{R}^n)$ if

$$\int_0^\infty \left|\{x\in\Omega : |Dp(x)|>\lambda\}\right|^{\frac{1}{n}}\log^+\lambda\,d\lambda < \infty, \tag{1.9}$$

where $\log^+\lambda = \max\{\log\lambda, 0\}$ is the positive part of $\log\lambda$; very roughly, it can be seen as a space encoding a logarithmic correction in the decay of the measure of the super-levels sets defining $L^{n,1}$, and this is needed in order to locally rebalance the non-uniform ellipticity of the operator (compare with [8,17,19,21,26,34,35]).

An important point we want to stress is that the sharp, generalized Sobolev's embedding by Cianchi & Pick (see [13,14]) ensures that functions in $W^1L^{n,1}\log L$ are continuous (see (1.10) below and notice that it therefore makes sense to mention the pointwise value of $p(\cdot)$), but it *does not* guarantee that their modulus of continuity satisfies (1.6). Even more dramatically, functions in $L^{n,1}$, while being continuous, are not uniformly equicontinuous, in the sense that it is not possible to guarantee the embedding of $W^1L^{n,1}$ into any space of uniformly continuous functions sharing the same modulus of continuity; see [14, Remark 3.6]. Therefore, since these results are known to be optimal, Theorems 1.1-1.2 do not follow from continuity properties for $a(\cdot)$, $p(\cdot)$ inferred via Sobolev's embedding and their proofs require an ad-hoc approach. For X a rearrangement invariant space, we shall sometimes denote with W^1X the space of functions whose distributional gradients (better, their components) belong to X.

Anyway, not everything is lost: the aforementioned result shows (for details we refer [3]) that if the variable exponent $p(\cdot)$ has gradient belonging to $L^{n,1}\log L \subset L^{n,1}$, then it has a modulus of continuity $\omega_p(\cdot)$ satisfying

$$\limsup_{\rho\searrow 0} \omega_{\log}(\rho) \le c(n)\|p\|_{W^1L^{n,1}\log L(\Omega)} = c(n,p(\cdot)) \tag{1.10}$$

(we are not interested here in Sobolev-Lorentz-Zygmund norms, defined in terms of rearrangements), at least if $\partial\Omega$ is Lipschitz; therefore, the sole use of (generalized) Sobolev's embedding for $p(\cdot)$ ensures that solutions to (1.1) under the assumptions (1.2)-(1.3) and (1.7) have gradient that is higher integrable, that is, there exists a small positive constant $\delta_0 > 0$, depending on $n, N, \gamma_1, \gamma_2, L/\nu$ and $\tilde{L}$, such that

$$|Du|^{p(\cdot)} \in L^{1+\delta_0}_{\mathrm{loc}}(\Omega) \tag{1.11}$$

(see Paragraph 2.5 for more details). In this paper we want to show that a more careful analysis[1] leads to a much better regularity result for the gradient:

Theorem 1.1. *Let $u \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$ be a weak solution to the system (1.1); suppose that (1.2), (1.3) and (1.7) hold. Then Du is locally bounded: there exists a radius R_0 depending on* $\texttt{data}$, *$p(\cdot), a(\cdot)$ and $\||Du|^{p(\cdot)}\|_{L^1(\Omega)}$ such that if $B_{2R}(x_0) \subset \Omega$ is a ball with $R \le R_0$, then*

$$\sup_{B_R(x_0)} |Du| \le c \fint_{B_{2R}(x_0)} \big(1+|Du|\big)\,dx, \tag{1.12}$$

for a constant c depending on the $\texttt{data}$.

(see (2.1) for the meaning of $\texttt{data}$) and

Theorem 1.2. *Let $u \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$ be as in Theorem 1.1. Then Du coincides almost everywhere in Ω with a continuous function.*

[1] The results and the techniques of this paper were first announced in the online seminar available at https://www2.karlin.mff.cuni.cz/~pick/2022-01-13-baroni.mp4.

Enlarging for a moment the perspective from which we describe our results, the study of problems with coefficients having assumption of Sobolev-Lorentz type is attracting more and more interest in the very last years, even for problems satisfying classic growth assumptions; in [21] it is shown that solutions to uniformly elliptic vectorial problems of the type

$$\operatorname{div}\Big(b(x)\frac{\varphi'(|Du|)}{|Du|}Du\Big) = f \qquad \text{in } \Omega \subset \mathbb{R}^n \tag{1.13}$$

are Lipschitz (and therefore C^1, after computation of standard flavor) regular if $f \in L^{n,1}(\Omega)$ and $|Db|$ belongs to

$$\begin{cases} L^{n,1}(\Omega) & \text{if } n \geq 3 \\ L^2(\log L)^\gamma(\Omega) \quad \gamma > 2, & \text{if } n = 2 \end{cases}.$$

Here the scalar positive function φ' has growth of Orlicz type and the coefficient b is elliptic, that is, it satisfies the assumptions in (1.3); for $p \geq 1$, $\gamma \in \mathbb{R}$, $L^p \log^\gamma L(\Omega)$ is the space of measurable functions $f : \Omega \to \mathbb{R}$ such that $|f|^p \log^\gamma(e + |f|)$ belongs to $L^1(\Omega)$. Note that $L^2(\log L)^\gamma \subsetneq L^{2,1}$ (see [36, Theorem 9.5.14]); therefore our result slightly improves [21, Theorem 1.8] in dimension 2 when $\varphi(t) = t^p$, $p > 1$; a perturbation approach similar to ours would lead to the result also for the more general growth conditions considered in [21].

We complete this introductory chapter stressing that an approach similar to that of this paper has been applied by the author to a borderline case of so-called double phase problems, see [3,6]. Double phase problems are problems of the Calculus of the Variations of the form

$$u \in W^{1,1}(\Omega) \mapsto \int_\Omega \big[F(Du) + s(x)G(Du)\big]\,dx, \qquad s(x) \geq 0,$$

with the peculiarity that G grows faster than F at infinity; they are the object of a substantial amount of research nowadays and for particular choices of F, G, they share many aspects of regularity with $p(x)$ problems, see [3,6,4,11,20,37]. Also in this setting borderline cases are often difficult and require subtle arguments, see for instance [18,19] and compare with [5,15] and [26,27]. We are convinced that the perturbative approach developed in this paper could find some applications leading to a deeper understanding of borderline cases in this research area and, as consequence, a deeper insight for the general theory; for more detail we again refer to [3,6,11,26,27,37].

Technical novelty of the paper

The approach we follow in this paper is of perturbative type: using appropriate liftings, we classically relate the regularity of solutions of (1.1) to the good, known $C^{1,\beta}$-regularity of solutions to systems with p-Laplacian growth, see (2.13)-(3.6). The novelty is twofold: the localization around the average of the variable functions $p(\cdot)$, $a(\cdot)$ and the use of the Sobolev-Poincaré inequality. The former replaces the classic freezing of the variable functions $a(\cdot)$ and $p(\cdot)$ in the center of the ball considered (see essentially all the references in the bibliography regarding $p(x)$ problems); the use of the latter is allowed by the use of Hölder inequality with conjugate exponents $(1 + 1/\delta_0, 1 + \delta_0)$, where $\delta_0 \approx 0$ is the higher integrability exponent from (1.11), and this replaces the classic use of Hölder's inequality in the form $L^\infty - L^1$. Needless to say, we need to heavily use the fact that the gradient is higher integrable: this follows from (1.10), see (1.11). These two facts allow to make use of Lemma 2.1, enconding an independently interesting discretization of Lorentz-Zygmund spaces. Our approach also requires some localization effort (averages change at different steps of usual iteration procedures along sequences of shrinking balls) and we decided to solve the problem by performing the iteration at L^1 level: the linearized estimate we are able to get (see for instance (4.6)) could be useful in future advances in the theory and we think this could justify our effort. See [3] for a different solution to this localization problem.

To the best of our knowledge, this is the first time such a perturbative argument is carried on, despite being inspired by the techniques in [28].

Extension to local solutions and more general structures

In order to avoid unessential complication we suppose that the solution u is globally integrable, that is, $|Du|^{p(\cdot)} \in L^1(\Omega)$; accordingly, we set

$$M = \int_\Omega \left(1 + |Du|^{p(x)}\right) dx. \tag{1.14}$$

Clearly all the forthcoming results are local in nature and therefore it is possible to suppose u local solution (that is, $|Du|^{p(\cdot)} \in L^1_{\mathrm{loc}}(\Omega)$) and relax in an appropriate local way the assumptions in (1.7). Easy, minor modifications of the current proof would lead to the same results in this case.

Possible extensions of our results involve more general differential, or variational, structures of $p(x)$-type. We prefer to focus here on the simple model case (1.1) and leave such extensions to future contributions.

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2. Preliminaries

2.1. Notation

In this paper we are going to denote by c a positive constant possibly varying from line to line; special occurrences will be denoted by $c_1, c_*, \bar{c}$ or the like. All such constants will always be *larger or equal than one*; moreover relevant dependencies on parameters will be emphasized in parentheses, i.e., $c_1 \equiv c_1(n, p)$ means that c_1 depends on n and p. By $\texttt{data}$ we denote the set of parameters

$$\{n, N, \gamma_1, \gamma_2, L/\nu, \tilde{L}\} \tag{2.1}$$

(with $\tilde{L}$ that is going to be defined in (2.2)) so that by writing $c(\texttt{data})$ we shall mean that the constant depends on $n, N, \gamma_1, \gamma_2, L/\nu$ and $\tilde{L}$. The dependencies of the radii on $p(\cdot)$ and $a(\cdot)$ will all derive only from the use of (1.10) and Corollary 2.2. We denote by

$$B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$$

the open ball with center x_0 and radius $R > 0$; when not important, or clear from the context, we shall omit denoting the center just denoting $B_R \equiv B_R(x_0)$. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable set with positive, finite measure $|\mathcal{B}| > 0$, and with $g\colon \mathcal{B} \to \mathbb{R}^\ell$, $\ell \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average. Moreover, oscillation of g on $\mathcal{B}$ is defined as

$$\operatorname*{osc}_{\mathcal{B}} g = \sup_{x,y\in\mathcal{B}} |g(x) - g(y)|.$$

A great importance will have the (L^1) excess: for $g, \mathcal{B}$ as above, it is defined by

$$\fint_{\mathcal{B}} \big|g - (g)_{\mathcal{B}}\big|\,dx;$$

note that by triangle's inequality we have

$$\fint_{\mathcal{B}} \big|g - (g)_{\mathcal{B}}\big|\,dx \leq 2\fint_{\mathcal{B}} |g - \xi|\,dx \qquad \text{for all } \xi \in \mathbb{R}^\ell,$$

a property we are going to use very often.

For $x \geq 0$, $\gamma \in \mathbb{R}$, we denote by $\log^\gamma(e+x)$ the quantity $[\log(e+x)]^\gamma$ and $\log^\gamma(t) = |\log(t)|^\gamma$ for $t > 0$. We use the agreement that $\mathbb{N}$ is the set $\{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; by an interval in $\mathbb{N}_0$ we mean the intersection of an interval in $\mathbb{R}$ with $\mathbb{N}_0$. We use the notation $\chi_{(-\infty,2)}(p)$ for the characteristic function of the set $(-\infty, 2)$, that is

$$\chi_{(-\infty,2)}(p) = \begin{cases} 0 & \text{if } p \geq 2 \\ 1 & \text{if } p < 2 \end{cases};$$

similarly for $\chi_{[t,+\infty)}$ with $t \in \mathbb{R}$. For $1 < p < \infty$ we denote with p' the Hölder conjugate of p; for $1 < p < n$ with p^* its Sobolev conjugate and for $p > 1$ with p_* the quantity you can see below:

$$p' = \frac{p}{p-1}, \qquad p^* = \frac{np}{n-p}, \qquad p_* = \frac{np}{n+p};$$

notice that $p^* = q$ if and only if $p = q_*$ (if $p < n$).

2.2. Function spaces, variable exponents and weak solutions

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega;\mathbb{R}^\ell)$, $\ell \geq 1$, for a variable exponent as in (1.2), is defined as the space of measurable functions $f:\Omega\to\mathbb{R}^\ell$ such that $|Du|^{p(\cdot)} \in L^1(\Omega)$; it is endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\}.$$

We define $W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$ as the space of weakly differentiable functions whose distributional derivatives belong to $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}_0(\Omega;\mathbb{R}^N)$ as the closure of $C^\infty_c(\Omega;\mathbb{R}^N)$ with respect to the norm $\|u\|_{W^{1,p(\cdot)}} = \|u\|_{L^{p(\cdot)}} + \|Du\|_{L^{p(\cdot)}}$; local variants of such spaces are defined in the usual way. Note that the extension to zero outside Ω of a function in $W^{1,p(\cdot)}_0(\Omega;\mathbb{R}^N)$ belongs to $W^{1,p(\cdot)}$ of any superset of Ω.

By a weak solution to (1.1) we mean a function $u \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$ such that

$$\int_\Omega \langle a(x)|Du|^{p(x)-2}Du, D\varphi\rangle\, dx = 0 \qquad \text{for all } \varphi \in C^\infty_c(\Omega;\mathbb{R}^{nN})$$

and, by density, for every $\varphi \in W^{1,p(\cdot)}_0(\Omega;\mathbb{R}^{nN})$. $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{nN}$ (or better the Frobenius product).

We shall denote for a ball $B_R \equiv B_R(x_0) \subset \Omega$ (its center will always be clear from the context) and for radii $r \leq R$

$$p_r^+ = \sup_{B_r(x_0)} p(\cdot), \qquad p_r^- = \inf_{B_r(x_0)} p(\cdot).$$

As a consequence of (1.10) we can assume that there exists $\tilde{L} > 0$ such that

$$\sup_{\rho\in(0,1]} \omega_p(\rho)\log\left(\frac{1}{\rho}\right) \leq \tilde{L}, \tag{2.2}$$

a fact that we are going to use often. We also remark that, using (2.2), we have that if $\rho \leq 1$ then

$$\rho^{-\omega_p(\rho)} = e^{\omega_p(\rho)\log(\frac{1}{\rho})} \leq c(\tilde{L}). \tag{2.3}$$

2.3. Discretization of Lorentz-Zygmund spaces

For a function $f : B_R(x_0) \subset \mathbb{R}^n \to \mathbb{R}^\ell$, $\ell \in \mathbb{N}$, the sum

$$S_{q,\beta}(f)(x_0,R) := \sum_{j=0}^\infty R_j \log^\beta\left(\frac{1}{R_j}\right)\Big(\fint_{B_j(x_0)} |f|^q\, dx\Big)^{\frac{1}{q}}, \quad q \in (1,n),\, \beta \in \mathbb{R}, \tag{2.4}$$

where

$$R \in (0,1), \qquad \sigma \in (0,1), \qquad R_j = \sigma^j R, \quad B_j(x_0) = B_{R_j}(x_0), \qquad j \in \mathbb{N}_0,$$

will take on great importance in view of the following lemma; see [3] and compare with [28, Lemma 1].

Lemma 2.1. *Let $f \in L^{n,1}\log^\beta L(B_R(x_0))$ for $\beta \in \{0,1\}$ being $R \le 1$ and let $q \in (1,n)$ and $\sigma \in (0,1)$; then*

$$S_{q,\beta}(f)(x_0, R) \le c(n,q,\sigma)\|f\|_{L^{n,1}\log^\beta(B_R(x_0))}.$$

As a consequence, since the Lorentz norm $L^{n,1}\log^\beta L$ is defined in terms of an integral, by absolute continuity (cf. [36, Paragraph 9.9]) it follows that we can make, taking the initial radius R sufficiently small, the sum $S_{q,\beta}(f)(x_0)$ small (locally) uniformly in x_0.

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^n$ and $f \in L^{n,1}\log^\beta L(\Omega)$ for $\beta \in \{0,1\}$; let moreover $\sigma \in (0,1)$ and $q \in (1,n)$ be fixed. For every $K \Subset \Omega$ and $\varepsilon > 0$, there exists a radius $R_\varepsilon > 0$ depending on n, q, σ, f and ε such that if $R \in (0, R_\varepsilon]$ and $R < \operatorname{dist}(\partial\Omega, K)$, then*

$$\sup_{x_0 \in K} S_{q,\beta}(f)(x_0, R) \le \varepsilon.$$

2.4. Auxiliary vector fields

We will work often with a classic nonlinear expression of the gradient encoding in a precise way the monotonicity properties of the differential operator considered. In detail, for $p \in [\gamma_1, \gamma_2]$, we set

$$V_p(\xi) = |\xi|^{\frac{p-2}{2}}\xi, \quad \xi \in \mathbb{R}^\ell.$$

For $\xi_1, \xi_2 \in \mathbb{R}^\ell$ and $p(x) > 1$ we have the estimate

$$\frac{1}{c}\left|V_{p(x)}(\xi_1) - V_{p(x)}(\xi_2)\right|^2 \le \left\langle |\xi_1|^{p(x)-2}\xi_1 - |\xi_2|^{p(x)-2}\xi_2, \xi_1 - \xi_2 \right\rangle, \tag{2.5}$$

for a constant $c \equiv c(\gamma_1, \gamma_2) \ge 1$. A basic property of the map $V_p(\cdot)$ is the following local bi-Lipschitz character: indeed, for $\xi_1, \xi_2 \in \mathbb{R}^\ell$ and $p > 1$ it holds

$$\frac{1}{c}\left(|\xi_1| + |\xi_2|\right)^{p-2}|\xi_2 - \xi_1|^2 \le \left|V_p(\xi_2) - V_p(\xi_1)\right|^2 \le c\left(|\xi_1| + |\xi_2|\right)^{p-2}|\xi_2 - \xi_1|^2. \tag{2.6}$$

The constant c here depends only on p and we stress that for $p \in [\gamma_1, \gamma_2]$ it can be replaced by one depending only on γ_1 and γ_2; in other words, in case of a function $p : \Omega \to [\gamma_1, \gamma_2]$, estimate (2.6) can be written with $p = p(x)$ and the constant c will depend only on γ_1 and γ_2. As a consequence, if $2 \le p \le \gamma_2 < \infty$ then

$$|\xi_1 - \xi_2|^p \le c(\gamma_2)|V_p(\xi_2) - V_p(\xi_1)|^2 \tag{2.7}$$

while if $1 < \gamma_1 \le p < 2$ then

$$|\xi_1-\xi_2|\le c\,|V_p(\xi_1)-V_p(\xi_2)|^{\frac{2}{p}}+c\,|\xi_1|^{\frac{2-p}{2}}|V_p(\xi_1)-V_p(\xi_2)|, \tag{2.8}$$

for a constant depending only on γ_1. For these properties, see for instance [7,28,34].

2.4.1. Logs

We have the following useful properties of the logarithm we are often going to use without explicit reference:

$$\begin{cases}\log\left(\frac{x}{\ell}\right)\le(1-\log\ell)\log x & \text{for every } x\ge e \text{ and for all } \ell\in(0,1];\\ \log(e+x^\sigma)\le 1+\max\{1,\sigma\}\log(e+x) & \text{for all } x\ge 0 \text{ and } \sigma>0;\\ \log(e+xy)\le\log(e+x)+\log(e+y) & \text{for all } x,y\ge 0\\ \log(e+Ax)\le A\log(e+x) & \text{for all } x\ge 0 \text{ and } A\ge 1.\end{cases} \tag{2.9}$$

The proofs are very simple, see for instance [6].

The following lemma is an appropriate version of a well-known estimate for the treatment of such operators; for the simple proof in this general form see [6, Lemma 2.1].

Lemma 2.3. *Let $\tilde\varsigma>1$, $\sigma,\beta,\theta\ge 0$ and let f be a positive function in $L^{\tilde\varsigma}(B_r)$ for some ball $B_r\equiv B_r(x_0)$ with radius $r\le e^{-1}$. Then there exists a constant c depending on n,β,σ,θ and $\tilde\varsigma$ such that*

$$\fint_{B_r} f\log^\beta\left(e+f^\sigma\right)dx\le c\left(1+r^\theta\,\|f\|_{L^1(B_r)}\right)^\beta\log^\beta\left(\frac{1}{r}\right)\left(\fint_{B_r} f^{\tilde\varsigma}\,dx\right)^{1/\tilde\varsigma}.$$

2.5. Reference estimates for solutions to (1.1)

We consider here solutions to (1.1) under the assumptions (1.2)-(1.3) and (2.2) that, we recall, holds by embedding as a consequence of our main assumption (1.7).

The basic assumptions deduced by Sobolev's embedding, despite not allowing to get the results stated in our main Theorems 1.1-1.2, still allow to catch some gradient regularity in terms of its higher integrability. As a reference for the following result we suggest [7,9,34,38] even if we are going to sketch a proof, since we need the local estimate (2.10) in a particular form not present in the literature.

Theorem 2.4. *Let $E\subset\mathbb{R}^n$ and $u\in W^{1,p(\cdot)}(E;\mathbb{R}^N)$ be a solution to (1.1) under the assumptions (1.2)-(1.3) and (2.2). Then there exists an exponent $\delta_0\in(0,1)$, depending on the* `data` *such that $|Du|^{p(\cdot)}\in L^{1+\delta_0}_{\mathrm{loc}}(E)$; moreover, for*

$$\mathcal{M}=\int_E\left(1+|Du|^{p(x)}\right)dx,$$

there exists a threshold $\bar R\equiv\bar R(n,p(\cdot),\mathcal{M})\le 1$ such that the local estimate

$$\left(\fint_{B_R(x_0)} |Du|^{p(x)(1+\delta_0)}\, dx \right)^{\frac{1}{1+\delta_0}} \le c \left(\fint_{B_{2R}(x_0)} \big(1+|Du|\big)\, dx \right)^{p(\bar{x})} \tag{2.10}$$

holds if $R \le \bar{R}$ and $B_{2R}(x_0) \subset E$, for a constant c depending on data *and for every $\bar{x} \in B_{2R}(x_0)$. In particular we also have*

$$\bar{R} = \frac{1}{4} \min\left\{ \frac{1}{\mathcal{M}}, R_1, 1 \right\} \qquad \textit{with} \qquad \sup_{\rho \in (0,2R_1]} \omega_p(\rho) \le \frac{1}{c(n)}. \tag{2.11}$$

Proof. The local estimate

$$\left(\fint_{B_R(x_0)} |Du|^{p(x)(1+\delta_0)}\, dx \right)^{\frac{1}{1+\delta_0}} \le c \fint_{B_{2R}(x_0)} \big(1+|Du|\big)^{p(x)}\, dx \tag{2.12}$$

and, as customary consequence of the self-improving character of the reverse Hölder inequalities,

$$\left(\fint_{B_R(x_0)} |Du|^{p(x)(1+\delta_0)}\, dx \right)^{\frac{1}{1+\delta_0}} \le c \left(\fint_{B_{2R}(x_0)} \big(1+|Du|\big)\, dx \right)^{p^+_{2R}}$$

are well-founded for every ball $B_{2R}(x_0) \subset E$ with radius $R \le \bar{R}$ ($\bar{R}, \delta_0 > 0$ and the constant c as in the statement) and their proof is quite standard, see for instance [34, Theorem 3.1] for a transparent form. To prove (2.10) we need to show that we can bound

$$\left(\fint_{B_{2R}(x_0)} \big(1+|Du|\big)\, dx \right)^{p^+_{2R} - p(\bar{x})}$$

by a constant independent of R; we have, with $c \equiv c(n, \gamma_1, \gamma_2)$ and if $p^+_{2R} - p(\bar{x}) > 0$

$$\begin{aligned} \left(\fint_{B_{2R}(x_0)} \big(1+|Du|\big)\, dx \right)^{p^+_{2R} - p(\bar{x})} &\le c \left(\frac{M}{(2R)^n} \right)^{p^+_{2R} - p(\bar{x})} \\ &\le c\, (2R)^{-(n+1)\omega_p(2R)} \le c(n, \gamma_1, \gamma_2, \tilde{L}) \end{aligned}$$

thanks to (2.11) and (2.3). □

2.6. Excess decay estimates for reference problems

We consider the classic p-Laplacian system

$$\operatorname{div}\left[|Dw|^{p-2} Dw \right] = 0 \qquad \text{in } E \subset \mathbb{R}^n, \tag{2.13}$$

for $p > 1$. It is well known that solutions to (2.13) are locally $C^{1,\alpha}$ regular: see [29, Theorem 3.1] for the following precise form of the excess decay.

Theorem 2.5. *Let $w \in W^{1,p}(E;\mathbb{R}^N)$ be a weak solution to (2.13). There exist constants $c_{ph} \geq 1, \alpha_0 \in (0,1), \sigma_0 \in (0,1]$, depending on n, N and p, such that for every ball $B_R(x_0) \subset E$ and for all $\sigma \in (0, \sigma_0]$*

$$\operatorname*{osc}_{B_{\sigma R}(x_0)} Dw \leq c_{ph}\, \sigma^{\alpha_0} \fint_{B_R(x_0)} \left| Dw - (Dw)_{B_R(x_0)} \right| dx.$$

3. Comparison results

We start this section by fixing a solution $u \in W^{1,p(\cdot)}(\Omega;\mathbb{R}^N)$ to (1.1) and a ball $B_R \equiv B_R(x_0) \subset \Omega$ with radius R smaller than a threshold R_0 whose value will be reduced several times in the course of the proof; all the balls we are going to work with in this section will have the same center x_0. We start setting $R_0 = \min\{\bar{R}, e^{-1}\}$, where $\bar{R} \equiv \bar{R}(n, p(\cdot), M)$ is the radius appearing in Theorem 2.4 for $\mathcal{M} = M$ as defined in (1.14); in particular we have $R_0 \leq 1$ so (2.2) is at our disposal. Moreover, we denote for radii $r \leq R$

$$\bar{a}_r := (a)_{B_r} = \fint_{B_r} a(x)\, dx, \qquad \bar{p}_r := (p)_{B_r} = \fint_{B_r} p(x)\, dx. \tag{3.1}$$

We also set, for $q \in (1, n)$

$$\mathfrak{A}_{r,q} = r\left(\fint_{B_r} |Da|^q\, dx\right)^{\frac{1}{q}}, \qquad \mathfrak{P}_{r,q} = r\log\left(\frac{1}{r}\right)\left(\fint_{B_r} |Dp|^q\, dx\right)^{\frac{1}{q}}. \tag{3.2}$$

We immediately stress that we are going to work with the following additional assumption (later we will show how to guarantee this by appropriately reducing the value of R_0):

$$\mathfrak{A}_{r,q} \leq 1, \quad \mathfrak{P}_{r,q} \leq 1 \qquad \text{if } r \leq R_0. \tag{3.3}$$

Next we further reduce the value of the R_0 so that

$$\omega_p(R_0) \leq \frac{\gamma_1}{2\max\{2, \gamma_1'\}} \cdot \frac{\delta_0}{2+\delta_0} \leq \min\left\{\frac{\gamma_1 \delta_0}{4}, \frac{\gamma_1}{2\max\{2, \gamma_1'\}} \cdot \frac{\delta_0}{2+\delta_0}\right\}, \tag{3.4}$$

where δ_0 is the positive constant appearing in the higher-integrability result of Theorem 2.4; this makes R_0 ultimately depend only on `data`, $p(\cdot)$ and M.

First we are going to consider the solution to the comparison problem

$$\begin{cases} \operatorname{div}\left[\bar{a}_{R/2}|Dv|^{p(x)-2}Dv\right] = 0 & \text{in } B_{R/2} \\ v = u & \text{on } \partial B_{R/2} \end{cases}; \tag{3.5}$$

later we are going to consider the solution to the problem with standard p-Laplacian growth

$$\begin{cases} \operatorname{div}\Big[|Dw|^{\bar{p}_{R/4}-2}Dw\Big]=0 & \text{in } B_{R/4} \\ w=v & \text{on } \partial B_{R/4} \end{cases}. \tag{3.6}$$

Notice that both problems are well posed and have a unique solution: the boundary datum of (3.5) is a function in $W^{1,p(\cdot)}(B_{R/2};\mathbb{R}^N)$ that is exactly the energy space of the differential operator in $(3.5)_1$, while for the second we observe that the local higher integrability result in Theorem 2.4 implies that $v \in W^{1,p(\cdot)(1+\delta_0)}(B_{R/4};\mathbb{R}^N)$ and the smallness assumption of the radius (3.4) ensures that for all $x \in B_{R/4}$

$$\bar{p}_{R/4} - \inf_{B_{R/4}} p(\cdot) \le \omega_p(R_0) \le \gamma_1\delta_0 \qquad \Longrightarrow \qquad \bar{p}_{R/4} \le p(x)(1+\delta_0)$$

and therefore

$$W^{1,p(\cdot)(1+\delta_0)}(B_{R/4};\mathbb{R}^N) \subset W^{1,\bar{p}_{R/4}}(B_{R/4};\mathbb{R}^N); \tag{3.7}$$

again monotonicity methods apply. In particular, by density, the weak formulation of the system (3.5) can be tested with the function $\varphi = u - v \in W_0^{1,p(\cdot)}(B_{R/2};\mathbb{R}^N)$ and that of (3.6) with the function $\varphi = v - w \in W_0^{1,\bar{p}_{R/4}}(B_{R/4};\mathbb{R}^N)$.

3.1. Basic results

Note the solving (3.5) is equivalent to finding the minimizer of the energy $v \mapsto \int_{B_{R/2}} |Dv|^{p(x)}/p(x)\,dx$ in the Dirichlet class $u + W_0^{1,p(\cdot)}(B_{R/2};\mathbb{R}^N)$; thus

$$\fint_{B_{R/2}} |Dv|^{p(x)}\,dx \le c(\gamma_1,\gamma_2) \fint_{B_{R/2}} |Du|^{p(x)}\,dx. \tag{3.8}$$

Moreover, since the differential operator in (3.5) satisfies (1.2), (1.3) and (2.2), the higher integrability result of Theorem 2.4 applies to Dv too with the same constant and exponent. However, the critical radius for which the local estimate (2.10) holds depends on $\mathcal{M} = \int_{B_{R/2}}(1+|Dv|)\,dx$. Anyway, due to the previous inequality (3.8) and the explicit monotonicity of $\bar{R}$ with respect to the energy (2.11), we can possibly reduce the value of $\bar{R}$ (and therefore of R_0) so that Theorem 2.4 is at our disposal for both Du and Dv with a constant depending on `data` and for radii smaller than the threshold $R_0 \equiv R_0(n, p(\cdot), M)$. In particular

$$\begin{aligned} \fint_{B_{R/4}} |Dv|^{p(x)(1+\delta_0)}\,dx &\le c\Big(\fint_{B_{R/2}} \big(1+|Dv|\big)^{p(x)}\,dx\Big)^{1+\delta_0} \\ &\le c\Big(\fint_{B_{R/2}} \big(1+|Du|\big)^{p(x)}\,dx\Big)^{1+\delta_0} \\ &\le c\Big(\fint_{B_R} \big(1+|Du|\big)\,dx\Big)^{p(\bar{x})(1+\delta_0)} \end{aligned} \tag{3.9}$$

holds true for every $\bar{x} \in B_R$; c here depends only on the $\texttt{data}$. We used (2.12) for Dv, (3.8) and (2.10) for Du.

We can finally start by deriving a first comparison estimate.

Lemma 3.1 *(Comparison I). Let $v \in u + W_0^{1,p(\cdot)}(B_{R/2};\mathbb{R}^N)$ be the solution to (3.5); there exists an exponent*

$$q = q(\texttt{data}) < n \tag{3.10}$$

such that

$$\fint_{B_{R/2}} \left|V_{p(x)}(Du) - V_{p(x)}(Dv)\right|^2 dx \le c\,\mathfrak{A}^2_{R/2,q}\Big(\fint_{B_R} (1+|Du|)\,dx\Big)^{p(\bar{x})} \tag{3.11}$$

holds true for a constant c depending only on the $\texttt{data}$ and for every $\bar{x} \in B_R$.

Proof. We subtract the weak formulations of the systems solved by u and v and we test such difference with $\varphi = u - v$. We have, after some simple computations

$$\begin{aligned} I &= \fint_{B_{R/2}} \left\langle \bar{a}_{R/2}|Du|^{p(x)-2}Du - \bar{a}_{R/2}|Dv|^{p(x)-2}Dv,\, Du - Dv\right\rangle dx \\ &= \fint_{B_{R/2}} \left\langle \bar{a}_{R/2}|Du|^{p(x)-2}Du - a(x)|Du|^{p(x)-2}Du,\, Du - Dv\right\rangle dx \\ &\le \fint_{B_{R/2}} |\bar{a}_{R/2} - a(x)||Du|^{p(x)-1}|Du - Dv|\,dx = II. \end{aligned} \tag{3.12}$$

We use the monotonicity in (2.5) and (1.3) to estimate from below

$$I \ge \frac{\bar{a}_{R/2}}{c}\fint_{B_{R/2}} \left|V_{p(x)}(Du) - V_{p(x)}(Dv)\right|^2 dx \ge \frac{1}{c}\fint_{B_{R/2}} \left|V_{p(\cdot)}(Du) - V_{p(\cdot)}(Dv)\right|^2 dx.$$

To estimate II we need to distinguish two cases:

In the **case $\boldsymbol{p(x) \ge 2}$** we can estimate pointwise, using Young's inequality for $\varepsilon \in (0,1)$ to be chosen later and (2.6)

$$\begin{aligned} ii :&= |\bar{a}_{R/2} - a(x)||Du|^{p(x)-1}|Du - Dv| \\ &= |\bar{a}_{R/2} - a(x)||Du|^{\frac{p(x)}{2}}\big[|Du| + |Dv|\big]^{\frac{p(x)-2}{2}}|Du - Dv| \\ &\le \varepsilon\left|V_{p(x)}(Du) - V_{p(x)}(Dv)\right|^2 + c(\varepsilon,\gamma_2)|a(x) - \bar{a}_{R/2}|^2|Du|^{p(x)}. \end{aligned}$$

If on the other hand **$\boldsymbol{p(x) < 2}$** using (2.8), twice Young's inequality and (1.3)

$$
\begin{aligned}
ii &\le c|\bar{a}_{R/2} - a(x)||Du|^{p(x)-1}\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|^{\frac{2}{p(x)}} \\
&\qquad\qquad + c|\bar{a}_{R/2} - a(x)||Du|^{\frac{p(x)}{2}}\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big| \\
&\le \varepsilon\big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|^2 + c_\varepsilon|a(x) - \bar{a}_{R/2}|^2|Du|^{p(x)}
\end{aligned}
$$

with c depending on γ_1, L and ε; notice that $[p(x)]' > 2$ in this case and therefore

$$
|a(x) - \bar{a}_{R/2}|^{[p(x)]'} \le L^{\frac{2-\gamma_1}{\gamma_1-1}}|a(x) - \bar{a}_{R/2}|^2.
$$

Combining the two cases we get

$$
II \le \varepsilon \fint_{B_{R/2}} \big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|^2\,dx + c_\varepsilon \fint_{B_{R/2}} |a(x) - \bar{a}_{R/2}|^2|Du|^{p(x)}\,dx.
$$

At this point we estimate the last integral: for $\delta_0 \in (0,1)$ the higher integrability exponent of Theorem 2.4,

$$
\begin{aligned}
&\fint_{B_{R/2}} |a(x) - \bar{a}_{R/2}|^2|Du|^{p(x)}\,dx \\
&\qquad \le \Big(\fint_{B_{R/2}} |a(x) - \bar{a}_{R/2}|^{2(1+\delta_0)'}\,dx\Big)^{\frac{1}{(1+\delta_0)'}}\Big(\fint_{B_{R/2}} |Du|^{p(x)(1+\delta_0)}\,dx\Big)^{\frac{1}{1+\delta_0}};
\end{aligned}
$$

we use now Sobolev-Poincaré's inequality recalling that $\bar{a}_{R/2}$ is the average of $a(\cdot)$ on $B_{R/2}$ and that, belonging $a(\cdot)$ in particular to $W^{1,n}(B_{R/2})$, it is possible to choose an arbitrarily large value for the exponent of $a(\cdot) - \bar{a}_{R/2}$. This is to say, we can select $q = q(n, \delta_0) < n$ such that

$$
2(1+\delta_0)' = q^* = \frac{nq}{n-q} \quad\Longleftrightarrow\quad q = \Big[2\frac{1+\delta_0}{\delta_0}\Big]_* = \frac{2n(1+\delta_0)}{\delta_0(n+2)+2} \tag{3.13}
$$

so that

$$
\Big(\fint_{B_{R/2}} |a(x) - \bar{a}_{R/2}|^{2(1+\delta_0)'}\,dx\Big)^{\frac{1}{(1+\delta_0)'}} \le c(n,\delta_0)\Big(R^q \fint_{B_{R/2}} |Da|^q\,dx\Big)^{\frac{2}{q}} = c\,\mathfrak{A}^2_{R/2,q}.
$$

We complete the estimate for II using the local estimate from Theorem 2.4

$$
II \le \varepsilon \fint_{B_{R/2}} \big|V_{p(x)}(Du) - V_{p(x)}(Dv)\big|^2\,dx + c_\varepsilon \mathfrak{A}^2_{R/2,q}\Big(\fint_{B_R} (1+|Du|)\,dx\Big)^{p(\bar{x})}
$$

c_ε depends on the `data` and ε. Inserting this estimate into (3.12) together with the estimate for I, choosing ε sufficiently small and reabsorbing gives (3.11). □

Lemma 3.2 *(Comparison I localized). Let $v \in u + W_0^{1,p(\cdot)}(B_{R/2};\mathbb{R}^N)$ be as in Lemma 3.1 and let $\bar{x} \in B_R$ and $\rho \le R/4$; then*

$$\fint_{B_{R/4}} \left|V_{\bar{p}_\rho}(Du) - V_{\bar{p}_\rho}(Dv)\right|^2 dx \le c\left[\mathfrak{A}^2_{R,q} + \left(\frac{R}{\rho}\right)^{\frac{n}{q}-1}\mathfrak{P}^2_{R,q}\right]\left(\fint_{B_R}\left(1+|Du|\right)dx\right)^{p(\bar{x})} \tag{3.14}$$

holds true for a constant c depending only on the data *and for (a possibly different than that in (3.10)) constant $q \in (1,n)$, still depending on the* data.

Proof. Let us denote $F = |Du(x)| + |Dv(x)|$, $\bar{p} = \bar{p}_\rho$ and $\omega = \omega_p(R)$ in short; suppose $F \ne 0$. The trivial estimate (we use triangle's inequality here)

$$F^{\bar{p}-2}|Du(x) - Dv(x)|^2 \le F^{p(x)-2}|Du(x) - Dv(x)|^2 + F^{\frac{p(x)}{2}}\left|F^{\bar{p}-p(x)} - 1\right|F^{\frac{p(x)-2}{2}}|Du(x) - Dv(x)|$$

together with (2.6) and Young's inequality implies that

$$\left|V_{\bar{p}}(Du(x)) - V_{\bar{p}}(Dv(x))\right|^2 \le c\left|V_{p(x)}(Du(x)) - V_{p(x)}(Dv(x))\right|^2 + c\,F^{p(x)}\left|F^{\bar{p}-p(x)} - 1\right|^2$$

with $c \equiv c(\gamma_1,\gamma_2)$. Now we notice that the mean value theorem yields

$$\left|F^{\bar{p}-p(x)} - 1\right| = |p(x) - \bar{p}|F^{\lambda_x(\bar{p}-p(x))}|\log F| \tag{3.15}$$

with $\lambda_x \in (0,1)$. Now we estimate the quantity $F^{p(x)+2\lambda_x(\bar{p}-p(x))}\log^2 F$ distinguishing the two cases $F = F(x) \in (0,e)$ and $F \ge e$: in the first one

$$F^{p(x)+2\lambda_x(\bar{p}-p(x))}\log^2 F \le c(\gamma_1,\gamma_2)$$

thanks to the estimate

$$\sup_{t\in(0,e)} t^\sigma|\log t| \le \frac{1}{ae} + e^b \qquad \text{for all } \sigma \in [a,b], \tag{3.16}$$

with $0 < a < b$ (simply consider the cases $t \in (0,1]$ an $t \in [1,e)$) and the fact

$$\left|\lambda_x\left(\bar{p} - p(x)\right)\right| \le \omega_p(R_0) \le \frac{\gamma_1}{4}$$

- we are using (3.4) here. If $F \ge e$, using the bound above,

$$F^{p(x)+2\lambda_x(\bar{p}-p(x))}\log F \le F^{p(x)+2\omega}\log^2\left(e + F^{p(x)+2\omega}\right).$$

Merging these two cases gives

$$F^{p(x)}\left|F^{\bar{p}-p(x)} - 1\right|^2 \le c(\gamma_1,\gamma_2)|p(x) - \bar{p}|^2\left(1 + F^{p(x)+2\omega}\right)\log^2\left(e + 1 + F^{p(x)+2\omega}\right);$$

averaging the previous inequality over $B_{R/4}$ and using Hölder's inequality with conjugate exponents $(1+\delta, 1+1/\delta)$ for $\delta \in (0,1)$ to be chosen gives

$$\fint_{B_{R4}} F^{p(x)}\left|F^{\bar{p}-p(x)}-1\right|^2 dx \le c(\gamma_1,\gamma_2)\Big(\fint_{B_{R/4}} |p(x)-\bar{p}|^{2(1+\frac{1}{\delta})}\,dx\Big)^{\frac{\delta}{1+\delta}} \times$$
$$\times\Big(\fint_{B_{R/4}} \left(1+F^{p(x)+2\omega}\right)^{1+\delta}\log^{2(1+\delta)}\left(e+1+F^{p(x)+2\omega}\right)dx\Big)^{\frac{1}{1+\delta}}. \tag{3.17}$$

Now, for δ_0 the higher integrability exponent appearing in Theorem 2.4, we notice that choosing δ, depending only on δ_0 and sufficiently small, we have (after setting $\tilde{\omega}=2\omega/\gamma_1$):

$$(1+\tilde{\omega})(1+\delta)\le(1+\tilde{\omega})(1+\delta)^2\le\Big(1+\frac{\delta_0}{2}\Big)(1+\delta)^2\le 1+\delta_0 \tag{3.18}$$

taking also into account (3.4). In order to estimate the last integral in (3.17) with the correct dependence of the exponent we estimate, we use (3.9)-(2.12), to infer

$$\begin{aligned}
\fint_{B_{R/4}} \left(1+F^{p(x)+2\omega}\right)^{1+\delta}dx &\le \fint_{B_{R/4}} \left(1+F^{p(x)(1+\tilde{\omega})}\right)^{1+\delta}dx \\
&\le (\gamma_2,\delta_0)\fint_{B_{R/4}} \left(1+F^{p(x)(1+\delta_0)}\right)dx \\
&\le c\fint_{B_{R/4}} \left(1+|Du|^{p(x)(1+\delta_0)}+|Dv|^{p(x)(1+\delta_0)}\right)dx \\
&\le c(\texttt{data})\Big(\fint_{B_{R/2}} (1+|Du|)^{p(x)}\,dx\Big)^{1+\delta_0} \\
&\le c\,R^{-n(1+\delta_0)}M^{1+\delta_0}\le c\,R^{-(n+1)(1+\delta_0)}.
\end{aligned} \tag{3.19}$$

In light of Lemma 2.3 with the choices $f=(1+F^{p(x)+\omega})^{1+\delta}$, $\beta=1+\delta$, $\sigma=1/(1+\delta)$, $\theta=(n+1)\delta_0+1$, $\tilde{\varsigma}=1+\delta$ and the previous estimate (3.19) we have now

$$\fint_{B_{R/4}} \left(1+F^{p(x)+2\omega}\right)^{1+\delta}\log^{1+\delta}\left(e+1+F^{p(x)+2\omega}\right)dx$$
$$\le c\Big(1+R^{(n+1)(1+\delta_0)}\fint_{B_{R/4}} \left(F^{p(x)+\omega}\right)^{1+\delta}dx\Big)^{1+\delta}\times$$
$$\times\log^{1+\delta}\Big(\frac{4}{R}\Big)\Big(\fint_{B_{R/4}} \left(1+F^{p(x)+2\omega}\right)^{(1+\delta)^2}dx\Big)^{\frac{1}{1+\delta}}$$

$$\leq c \log^{1+\delta}\Big(\frac{1}{R}\Big)\Big(\fint_{B_{R/4}} \big(1+F^{p(x)(1+\tilde{\omega})}\big)^{(1+\delta)^2}\,dx\Big)^{\frac{1}{1+\delta}}$$

and the constant depends on $\mathtt{data}$, δ and δ_0 - that is, on $\mathtt{data}$ only. Now we can continue to estimate the integral on the right-hand side using (3.18) similarly as what done in (3.19):

$$\begin{aligned}\fint_{B_{R/4}} \big(1+F^{p(x)(1+\tilde{\omega})}\big)^{(1+\delta)^2}\,dx &\leq c\Big(\fint_{B_{R/4}} \big(1+F^{p(x)(1+\delta_0)}\big)\,dx\Big)^{\frac{(1+\tilde{\omega})(1+\delta)^2}{1+\delta_0}} \\ &\leq c\Big(\fint_{B_{R/2}} \big(1+|Du|^{p(x)}\big)\,dx\Big)^{(1+\tilde{\omega})(1+\delta)} \\ &\leq c\Big(\fint_{B_R} \big(1+|Du|\big)\,dx\Big)^{p(\bar{x})(1+\delta)}.\end{aligned} \tag{3.20}$$

We justify now the last inequality: we have

$$\Big(\fint_{B_{R/2}} \big(1+|Du|^{p(x)}\big)\,dx\Big)^{\tilde{\omega}(1+\delta)} \leq \Big(\frac{M}{R^n}\Big)^{\omega_p(R)2\frac{1+\delta_0}{\gamma_1}} \leq c(\gamma_1,\delta_0) \tag{3.21}$$

using (2.3) as in the proof of Theorem 2.4; we also used (2.10). The constant in (3.20), finally and in view of this, depends on $\mathtt{data}$ and M. Plugging (3.20) into (3.17) gives

$$\begin{aligned}&\fint_{B_{R/4}} F^{p(x)}\big|F^{\bar{p}-p(x)}-1\big|^2\,dx \\ &\qquad\leq c\,\log^2\Big(\frac{1}{R}\Big)\Big(\fint_{B_{R/4}} |p(x)-\bar{p}|^{2(1+\frac{1}{\delta_0})}\,dx\Big)^{\frac{\delta_0}{1+\delta_0}}\Big(\fint_{B_R}\big(1+|Du|\big)\,dx\Big)^{p(\bar{x})}\end{aligned}$$

with c depending on $\mathtt{data}$ and M. Now, as in (3.13), for $q=q(n,\delta_0)<n$ such that

$$2\Big(1+\frac{1}{\delta_0}\Big)=q^* \quad\Longleftrightarrow\quad q=\Big[2\Big(1+\frac{1}{\delta_0}\Big)\Big]_* = \frac{2n(1+\delta_0)}{\delta_0(n+2)+2}$$

we have, enlarging the integral over B_ρ:

$$\begin{aligned}\Big(\fint_{B_{R/4}} |p(x)-\bar{p}|^{2(1+\frac{1}{\delta_0})}\,dx\Big)^{\frac{\delta_0}{1+\delta_0}} &= \Big(\fint_{B_{R/4}} |p(x)-(p)_{B_\rho}|^{q^*}\,dx\Big)^{\frac{2}{q^*}} \\ \leq 2\Big(\fint_{B_{R/4}} |p(x)-(p)_{B_{R/4}}|^{q^*}\,dx\Big)^{\frac{2}{q^*}} &+2\Big(\fint_{B_\rho} |p(x)-(p)_{B_{R/4}}|^{q^*}\,dx\Big)^{\frac{2}{q^*}}\end{aligned}$$

$$\leq c(n,\delta_0)\Big(1+\frac{R}{\rho}\Big)^{\frac{2n}{q^*}}\Big(R^q \fint_{B_{R/4}} |Dp|^q\,dx\Big)^{\frac{1}{q}}.$$

Therefore

$$\fint_{B_{R/4}} F^{p(x)}\big|F^{\bar p-p(x)}-1\big|^2\,dx \leq c\Big(\frac{R}{\rho}\Big)^{2(\frac{n}{q}-1)}\mathfrak{P}^2_{q,R/4}\Big(\fint_{B_R}\big(1+|Du|\big)\,dx\Big)^{p(\bar x)}$$

We conclude the proof noticing that (see (2.9))

$$\mathfrak{A}_{R/2,q} \leq c(n,q)\mathfrak{A}_{R,q}, \qquad \mathfrak{P}_{R/4,q}\leq c(n,q)\mathfrak{P}_{R,q}. \quad \square$$

Now we derive a second comparison estimate for the solution of (3.6).

Lemma 3.3 *(Comparison II). Let $w \in W^{1,\bar p_{R/4}}(B_{R/4};\mathbb{R}^N)$ be the solution to (3.6); there exists an exponent*

$$q = q(\texttt{data}) < n$$

such that

$$\fint_{B_{R/4}} \big|V_{\bar p_{R/4}}(Dv)-V_{\bar p_{R/4}}(Dw)\big|^2\,dx \leq c\,\mathfrak{P}^2_{R,q}\Big(\fint_{B_R}\big(1+|Du|\big)\,dx\Big)^{p(\bar x)} \tag{3.22}$$

holds true for a constant c depending only on the $\texttt{data}$ and for every $\bar x \in B_R(x_0)$.

Proof. We use in the proof the short notation $\bar p := \bar p_{R/4}$. Similarly as before we subtract the weak formulations of the systems solved by v and w and we use as a test function $\varphi = v - w$; this function is allowed thanks to the discussion after (3.6), see in particular (3.7).

$$\begin{aligned} I &= \fint_{B_{R/4}} \big\langle |Dv|^{\bar p-2}Dv - |Dw|^{\bar p-2}Dw, Dv-Dw\big\rangle\,dx \\ &= \fint_{B_{R/4}} \big\langle |Dv|^{\bar p-2}Dv - |Dv|^{p(x)-2}Dv, Dv-Dw\big\rangle\,dx \\ &\leq \fint_{B_{R/4}} \big||Dv|^{p(x)-\bar p}-1\big||Dv|^{\bar p-1}|Dv-Dv|\,dx = II. \end{aligned}$$

Using again (2.5) we get

$$I \geq \frac{1}{\bar c(\gamma_1,\gamma_2)}\fint_{B_{R/4}} \big|V_{\bar p}(Dv)-V_{\bar p}(Dw)\big|^2\,dx. \tag{3.23}$$

In order to get an estimate for II we estimate similarly to what done for (3.15): for almost every $x \in B_{R/4}$ with $|Dv(x)| \neq 0$ it holds

$$\big||Dv(x)|^{p(x)-\bar{p}} - 1\big| = |p(x) - \bar{p}||Dv|^{\lambda_x(p(x)-\bar{p})}\big|\log|Dv|\big|$$

for $\lambda_x \in (0, 1)$. Now we distinguish two cases: if $\bar{\boldsymbol{p}} \geq \mathbf{2}$ then

$$|Dv - Dv| \leq c(\gamma_2)|Dv|^{\frac{2-\bar{p}}{2}}\big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|$$

by (2.6) and using Young's inequality we have

$$\begin{aligned} II &\leq \fint_{B_{R/4}} |p(x) - \bar{p}||Dv|^{\frac{\bar{p}}{2}+\lambda_x(p(x)-\bar{p})}\big|\log|Dv|\big|\big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|\,dx \\ &\leq c \fint_{B_{R/4}} |p(x) - \bar{p}|^2|Dv|^{\bar{p}+2\lambda_x(p(x)-\bar{p})}\log^2|Dv|\,dx \\ &\qquad + \frac{1}{2\bar{c}} \fint_{B_{R/4}} \big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|^2\,dx. \end{aligned}$$

If $\bar{\boldsymbol{p}} < \mathbf{2}$ on the other hand we use (2.8) and Young's inequality twice so that

$$\begin{aligned} II &\leq c(\gamma_1) \fint_{B_{R/4}} |p(x) - \bar{p}||Dv|^{\bar{p}-1+\lambda_x(p(x)-\bar{p})}\big|\log|Dv|\big|\big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|^{\frac{2}{\bar{p}}}\,dx \\ &\quad + c(\gamma_1) \fint_{B_{R/4}} |p(x) - \bar{p}||Dv|^{\frac{\bar{p}}{2}+\lambda_x(p(x)-\bar{p})}\big|\log|Dv|\big|\big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|\,dx \\ &\leq c \fint_{B_{R/4}} |p(x) - \bar{p}|^{\bar{p}'}|Dv|^{\bar{p}+\lambda_x\bar{p}'(p(x)-\bar{p})}\log^{\bar{p}'}|Dv|\,dx \\ &\qquad + c \fint_{B_{R/4}} |p(x) - \bar{p}|^2|Dv|^{\bar{p}+\lambda_x 2(p(x)-\bar{p})}\log^2|Dv|\,dx \\ &\qquad + \frac{1}{2\bar{c}} \fint_{B_{R/4}} \big|V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)\big|^2\,dx = III + IV + V \end{aligned}$$

using again, twice, Young's inequality.

Now we pointwise estimate **in both cases**, distinguishing the two cases $|Dv(x)| \in (0, e)$ and $|Dv(x)| \geq e$: in the first one, for $t = 2, \bar{p}'$

$$|Dv(x)|^{\bar{p}+t\lambda_x(p(x)-\bar{p})}\log^t|Dv(x)| \leq c(\gamma_1, \gamma_2)$$

thanks to the estimate (3.16), since using (3.4) we infer

$$\left|t\lambda_x(p(x)-\bar{p})\right| \le t\,\omega_{p(\cdot)}(R/4) \le \max\{2,\gamma_1'\}\,\omega_{p(\cdot)}(R_0) \le \frac{\gamma_1}{2}.$$

In the second one, when $|Dv(x)| \ge e$, using the previous bound and denoting in short $\omega = \omega_p(R/4)$

$$|Dv(x)|^{\bar{p}+t\lambda_x(p(x)-\bar{p})}\log^t|Dv(x)| \le \big(1+|Dv(x)|\big)^{\bar{p}+t\omega}\log^t\big(e+(1+|Dv(x)|)^{\bar{p}+t\omega}\big).$$

Thus $III + IV$ is bounded by

$$c\sum_{t\in\{2,\bar{p}'\}}\chi_{[2,+\infty)}(t)\fint_{B_{R/4}}|p(x)-\bar{p}|^t\big(1+|Dv|\big)^{\bar{p}+t\omega}\log^t\big(e+(1+|Dv|)^{\bar{p}+t\omega}\big)\,dx.$$

We estimate in the same way the two averaged integrals: being δ_0 the positive constant from Theorem 2.4, we use Hölder's inequality with conjugate exponents $(1+\delta_0/4, 1+4/\delta_0)$ and then Lemma 2.3 with the choices $f=(1+|Dv|)^{(\bar{p}+t\omega)(1+\delta_0/4)}$, $\beta = t(1+\delta_0/4)$, $\sigma=(1+\delta_0/4)^{-1}$, $\theta=(n+1)\delta_0/2+1$ and $\tilde{\varsigma}=\frac{1+\delta_0/2}{1+\delta_0/4}$ so that

$$\begin{aligned}
&\fint_{B_{R/4}}|p(x)-\bar{p}|^t\big(1+|Dv|\big)^{\bar{p}+t\omega}\log^t\big(e+(1+|Dv|)^{\bar{p}+t\omega}\big)\,dx\\
&\quad\le R^t\Big(\fint_{B_{R/4}}\Big|\frac{p(x)-\bar{p}}{R/4}\Big|^{t(1+\frac{4}{\delta_0})}dx\Big)^{\frac{\delta_0}{4+\delta_0}}\times\\
&\qquad\times\Big(\fint_{B_{R/4}}\big(1+|Dv|\big)^{(\bar{p}+t\omega)(1+\frac{\delta_0}{4})}\log^{t(1+\frac{\delta_0}{4})}\big(e+(1+|Dv|)^{\bar{p}+t\omega}\big)\,dx\Big)^{\frac{4}{4+\delta_0}}\\
&\quad\le c\,R^t\log^t\Big(\frac{1}{R}\Big)\Big(\fint_{B_{R/4}}\Big|\frac{p(x)-\bar{p}}{R}\Big|^{t(1+\frac{4}{\delta_0})}dx\Big)^{\frac{\delta_0}{4+\delta_0}}\times\\
&\qquad\times\Big(1+R^{(n+1)(1+\frac{\delta_0}{2})}\fint_{B_{R/4}}\big(1+|Dv|\big)^{(\bar{p}+t\omega)(1+\frac{\delta_0}{4})}dx\Big)^t\times\\
&\qquad\times\Big(\fint_{B_{R/4}}\big(1+|Dv|\big)^{(\bar{p}+t\omega)(1+\frac{\delta_0}{2})}dx\Big)^{\frac{2}{2+\delta_0}}
\end{aligned}\tag{3.24}$$

for $t\in\{2,\bar{p}'\}$; similarly as described in the proof of Lemma 3.2, we can take the constant depending only on the $\texttt{data}$ (up to possibly further reduce the value of the radius $R_0(n,p(\cdot),M)$).

For the first integrals appearing in the addendi of the display above, we can choose $q \equiv q(\texttt{data})$ large enough so that

$$t\Big(1+\frac{4}{\delta_0}\Big)=q^* \quad\Longleftrightarrow\quad q=\Big[\frac{t(4+\delta_0)}{\delta_0}\Big]_* = \frac{nt(4+\delta_0)}{(t+n)\delta_0+4t}<n$$

(remember that t can only take the values 2 and $\bar{p}'$ and therefore $t > 1$); hence

$$\Big(\fint_{B_{R/4}} \Big| \frac{p(x)-\bar{p}}{R/4} \Big|^{\frac{t(4+\delta_0)}{\delta_0}} dx \Big)^{\frac{\delta_0}{4+\delta_0}} \le c(\mathtt{data}) \Big(\fint_{B_R} |Dp|^q \, dx \Big)^{\frac{t}{q}} \tag{3.25}$$

for the choice of $q = q(t)$ made above. Clearly we can take an exponent q valid for the two cases simply choosing the largest of the ones corresponding to the choices $t = 2$ and $t = \bar{p}'$. For the second integrals we can estimate (with the compact notation $p^- = p^-_{R/4}$)

$$\fint_{B_{R/4}} \big(1+|Dv|\big)^{(\bar{p}+t\omega)(1+\frac{\delta_0}{2})} dx \le \fint_{B_{R/4}} \big(1+|Dv|\big)^{p(x)\frac{\bar{p}+t\omega}{p^-}(1+\frac{\delta_0}{2})} dx$$

and we notice that thanks to (3.4) it holds

$$\frac{\bar{p}+t\omega}{p^-}\Big(1+\frac{\delta_0}{2}\Big) \le \Big(1+\frac{\max\{2,\gamma_1'\}}{\gamma_1}\omega_p(R/4)\Big)\Big(1+\frac{\delta_0}{2}\Big) \le \Big(1+\frac{\delta_0}{2+\delta_0}\Big)\Big(1+\frac{\delta_0}{2}\Big) \le 1+\delta_0;$$

therefore for $\bar{x} \in B_R$

$$\begin{aligned} \Big(\fint_{B_{R/4}} \big(1+|Dv|\big)^{(\bar{p}+t\omega)(1+\frac{\delta_0}{2})} dx \Big)^{\frac{2}{2+\delta_0}} &\le c \Big(\fint_{B_{R/4}} \big(1+|Dv|\big)^{p(x)(1+\delta_0)} dx \Big)^{\frac{\bar{p}+t\omega}{p^-}\frac{1}{1+\delta_0}} \\ &\le c \Big(\fint_{B_{R/2}} \big(1+|Dv|\big)^{p(x)} dx \Big)^{\frac{\bar{p}+t\omega}{p^-}} \\ &\le c \min\Big\{ \Big(\fint_{B_R} \big(1+|Du|\big) dx \Big)^{p(\bar{x})}, R^{-(n+1)} \Big\} \end{aligned} \tag{3.26}$$

for $c \equiv c(\mathtt{data})$ since we can use (3.8)-(2.10) and estimate (see the similar (3.21))

$$\Big(\fint_{B_{R/2}} (1+|Du|)^{p(x)} dx \Big)^{\frac{\bar{p}+t\omega}{p^-}-1} \le \Big(\fint_{B_{R/2}} (1+|Du|)^{p(x)} dx \Big)^{\frac{t+1}{\gamma_1}\omega} \le c.$$

Merging the estimates from (3.23) to (3.26) and recalling the notation in (3.2) we come up with

$$\begin{aligned} \frac{1}{\bar{c}} \fint_{B_{R/4}} |V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)|^2 dx \le I \le II \le \frac{1}{2\bar{c}} \fint_{B_{R/4}} |V_{\bar{p}}(Dv) - V_{\bar{p}}(Dw)|^2 dx \\ + c \Big[\mathfrak{P}^2_{R,q} + \chi_{(-\infty,2)}(\bar{p}) \mathfrak{P}^{\bar{p}'}_{R,q} \Big] \Big(\fint_{B_R} \big(1+|Du|\big) dx \Big)^{p(\bar{x})} \end{aligned}$$

and this, after reabsorbing and taking into account (3.3), concludes the proof. □

Remark 3.1. We can suppose, possibly enlarging the smaller of the two, that the two exponents defined in Lemmas 3.2 and 3.3 do coincide.

We make explicit a consequence of a simple estimate in the previous proof; notice that the assumption (3.30) here is not needed (and this is the reason why (3.28) will be used *before* (3.31), in order to ensure that (3.30) holds).

Lemma 3.4 *(Rough comparison). Let w be the solution to (3.6) as in Lemma 3.3. For every $\varepsilon_1 \in (0,1)$ there exists a constant $M_1 \equiv M_1(\mathtt{data}, \varepsilon_1) \geq 1$ such that if*

$$\fint_{B_R} |Du|\, dx \leq \lambda, \qquad \mathfrak{A}_{R,q} + \mathfrak{P}_{R,q} \leq \frac{1}{M_1} \tag{3.27}$$

for some $\lambda \geq 1$, then

$$\fint_{B_{R/4}} |Du - Dw|\, dx \leq \varepsilon_1 \lambda. \tag{3.28}$$

Proof. We denote $\bar{p} = \bar{p}_{R/4}$; merging (3.14) and (3.22) for appropriate choices of $\rho, \bar{x}$, then using (3.27) yields

$$\begin{aligned}\fint_{B_{R/4}} \left|V_{\bar{p}}(Du) - V_{\bar{p}}(Dw)\right|^2 dx &\leq c\Big[\mathfrak{A}_{R,q} + \mathfrak{P}_{R,q}\Big]^2 \Big(\fint_{B_R} \big(1 + |Du|\big)\, dx\Big)^{\bar{p}} \\ &\leq \frac{\tilde{c}(\mathtt{data})}{M_1^2} \lambda^{\bar{p}} \end{aligned}\tag{3.29}$$

and this, in view of (2.7), gives (3.28) with $M_1^2 = c(\gamma_2)\tilde{c}/\varepsilon_1^{\gamma_2}$ if $\bar{p} \geq 2$ ($c(\gamma_2)$ is the constant appearing in (2.7)). If $\bar{p} < 2$ we use (2.8), Hölder's inequality and the reverse Hölder's inequality (2.10):

$$\begin{aligned}\fint_{B_{R/4}} |Du - Dw|^{\bar{p}}\, dx &\leq c \fint_{B_{R/4}} \left|V_{\bar{p}}(Du) - V_{\bar{p}}(Dw)\right|^2 dx \\ &\quad + c\Big(\fint_{B_{R/4}} |Du|^{\bar{p}}\, dx\Big)^{\frac{2-\bar{p}}{2}} \Big(\fint_{B_{R/4}} \left|V_{\bar{p}}(Du) - V_{\bar{p}}(Dw)\right|^2 dx\Big)^{\frac{\bar{p}}{2}} \\ &\leq \frac{\tilde{\tilde{c}}(\mathtt{data})}{M_1^2} \lambda^{\bar{p}} + \frac{\tilde{\tilde{c}}(\mathtt{data})}{M_1^{\bar{p}}} \Big(\fint_{B_R} \big(1 + |Du|\big)\, dx\Big)^{\bar{p}\frac{2-\bar{p}}{2}} \lambda^{\bar{p}\frac{\bar{p}}{2}}\end{aligned}$$

again using (3.29), and this gives (3.28) for $M_1^{\gamma_1} = 2\tilde{\tilde{c}}/\varepsilon_1^{\gamma_1}$. We choose the value

$$M_1 = \max\left\{\left(\frac{c(\gamma_2)\tilde{c}}{\varepsilon_1^{\gamma_2}}\right)^{\frac{1}{2}}, \left(\frac{2\tilde{\tilde{c}}}{\varepsilon_1^{\gamma_1}}\right)^{\frac{1}{\gamma_1}}\right\}$$

to conclude the proof. □

In this paper we want to follow a similar, but different, route with respect to that taken in [3]. We have different comparison estimates involving two different V-functions and we do not want here to localize such estimates at every step of the iteration; we better linearize those estimates in order to be able to perform the iteration at L^1-level. Therefore the following refined comparison lemma will be necessary: for a constant $\sigma \in (0, 1/4)$ to be chosen, suppose that $B_{\sigma^{-1}R} \equiv B_{\sigma^{-1}R}(x_0) \subset \Omega$ and let

$$\tilde{v} \in u + W^{1,p(\cdot)}(B_{\sigma^{-1}R/2}), \qquad \tilde{w} \in v + W^{1,\bar{p}_{\sigma^{-1}R/2}}(B_{\sigma^{-1}R/4})$$

be the solutions to (3.5)-(3.6) with $B_{\sigma^{-1}R}$ replacing B_R. Suppose moreover $\sigma^{-1}R \le R_0$ for $R_0 \equiv R_0(\mathtt{data}, p(\cdot), M)$ the constant defined at the beginning of Section 3 and subsequently reduced. Lemma 3.5 can be seen as a quantitative version of Lemma 3.4.

Lemma 3.5 *(Linearized comparison). Suppose that there exist constants $A, \lambda \ge 1$ such that*

$$\frac{\lambda}{A} \le \inf_{B_{R/4}} |D\tilde{w}| \qquad \text{and} \qquad \fint_{B_{\sigma^{-1}R}} |Du|\, dx \le \lambda; \tag{3.30}$$

then

$$\fint_{B_{R/4}} |Du - Dw|\, dx \le c\Big[\mathfrak{A}_{\sigma^{-1}R,q} + \mathfrak{P}_{\sigma^{-1}R,q}\Big]\lambda \tag{3.31}$$

holds true for a constant c depending only on $\mathtt{data}$, A and σ; $q < n$ is the exponent mentioned in Remark 3.1.

Proof. We again denote $\bar{p} = \bar{p}_{R/4}$ and we start noticing that Lemma 3.2 for appropriate choices of ρ and $\bar{x}$ yields

$$\fint_{B_{R/4}} \left|V_{\bar{p}}(Du) - V_{\bar{p}}(Dv)\right|^2 dx \le c\Big[\mathfrak{A}^2_{R,q} + \mathfrak{P}^2_{R,q}\Big]\lambda^{\bar{p}} \tag{3.32}$$

and

$$\fint_{B_{\sigma^{-1}R/4}} \left|V_{\bar{p}}(Du) - V_{\bar{p}}(D\tilde{v})\right|^2 dx \le c\Big[\mathfrak{A}^2_{\sigma^{-1}R,q} + \mathfrak{P}^2_{\sigma^{-1}R,q}\Big]\lambda^{\bar{p}};$$

moreover merging the previous estimate and (3.22) we have (3.29) and

$$\fint_{B_{R/4}} \left|V_{\bar p}(Du)-V_{\bar p}(D\tilde w)\right|^2 dx \le c(n,\sigma) \fint_{B_{\sigma^{-1}R/4}} \left|V_{\bar p}(Du)-V_{\bar p}(D\tilde w)\right|^2 dx$$
$$\le c\Big[\mathfrak{A}^2_{\sigma^{-1}R,q}+\mathfrak{P}^2_{\sigma^{-1}R,q}\Big]\lambda^{\bar p}. \tag{3.33}$$

The four constants depend on $\texttt{data}$ and σ; we also used (3.30). In order to prove (3.31), we separately consider the cases where $\bar{\boldsymbol p} > \mathbf{2}$ and $\bar{\boldsymbol p} < \mathbf{2}$ (being the case $\bar{\boldsymbol p} = \mathbf{2}$ a trivial consequence of (3.29), since $V_2(\xi)=\xi$). In the first one, using our assumption (3.30)

$$\fint_{B_{R/4}} |Du-Dw|^{\bar p'}\,dx \le c(\gamma_2, A)\lambda^{(2-\bar p)\bar p'} \fint_{B_{R/4}} |D\tilde w|^{(\bar p-2)\bar p'}|Du-Dw|^{\bar p'}\,dx;$$

then we use triangle's inequality several times to estimate

$$\fint_{B_{R/4}} |D\tilde w|^{(\bar p-2)\bar p'}|Du-Dw|^{\bar p'}\,dx \le c(\gamma_2)\fint_{B_{R/4}} |Dv|^{(\bar p-2)\bar p'}|Du-Dv|^{\bar p'}\,dx$$
$$+c(\gamma_2)\fint_{B_{R/4}} |Dv|^{(\bar p-2)\bar p'}|Dv-Dw|^{\bar p'}\,dx + c(\gamma_2)\fint_{B_{R/4}} |Du-Dv|^{(\bar p-2)\bar p'}|Du-Dw|^{\bar p'}\,dx$$
$$+c(\gamma_2)\fint_{B_{R/4}} |Du-D\tilde w|^{(\bar p-2)\bar p'}|Du-Dw|^{\bar p'}\,dx = c\big[I+II+III+IV\big].$$

Since $\bar p>2$, due to (2.6), as after (3.23)

$$|Dv|^{(\bar p-2)\bar p'}|Du-Dv|^{\bar p'} \le c\left|V_{\bar p}(Du)-V_{\bar p}(Dv)\right|^{\bar p'}|Dv|^{\frac{\bar p-2}{2}\bar p'};$$

therefore, using Holder's inequality with conjugate exponents $(2(\bar p-1)/\bar p, 2(\bar p-1)/(\bar p-2))$, (3.32), (3.26) and (3.30)

$$I \le \Big(\fint_{B_{R/4}} \left|V_{\bar p}(Du)-V_{\bar p}(Dv)\right|^2 dx\Big)^{\frac{\bar p'}{2}}\Big(\fint_{B_{R/4}} |Dv|^{\bar p}\,dx\Big)^{\frac{\bar p-2}{2(\bar p-1)}}$$
$$\le c\Big[\mathfrak{A}^{\bar p'}_{R,q}+\mathfrak{P}^{\bar p'}_{R,q}\Big]\lambda^{\bar p\frac{\bar p'}{2}}\Big(\fint_{B_R}(1+|Du|)\,dx\Big)^{\bar p\frac{\bar p-2}{2(\bar p-1)}} \le c\Big[\mathfrak{A}^{\bar p'}_{R,q}+\mathfrak{P}^{\bar p'}_{R,q}\Big]\lambda^{\bar p}$$

with c depending on $\texttt{data}$ and σ, since $\lambda\ge 1$. Similarly, using this time (3.22) and again (3.26)-(3.30)

$$II \le \Big(\fint_{B_{R/4}} \left|V_{\bar p}(Dv)-V_{\bar p}(Dw)\right|^2 dx\Big)^{\frac{\bar p'}{2}}\Big(\fint_{B_{R/4}} |Dv|^{\bar p}\,dx\Big)^{\frac{\bar p-2}{2(\bar p-1)}} \le c\Big[\mathfrak{A}^{\bar p'}_{R,q}+\mathfrak{P}^{\bar p'}_{R,q}\Big]\lambda^{\bar p}.$$

Finally, we use Hölder's inequality, (2.7), (3.32) to estimate the first integral and (3.29) for the second in III:

$$III \le \Big(\fint_{B_{R/4}} |Du - Dv|^{\bar p}\,dx \Big)^{\frac{\bar p-2}{\bar p-1}} \Big(\fint_{B_{R/4}} |Du - Dw|^{\bar p}\,dx \Big)^{\frac{1}{\bar p-1}} \le c\Big[\mathfrak{A}^2_{\sigma^{-1}R,q} + \mathfrak{P}^2_{\sigma^{-1}R,q}\Big]\lambda^{\bar p}.$$

Similarly, using (3.29)-(3.33)

$$IV \le \Big(\fint_{B_{R/4}} |Du - D\tilde w|^{\bar p}\,dx \Big)^{\frac{\bar p-2}{\bar p-1}} \Big(\fint_{B_{R/4}} |Du - Dw|^{\bar p}\,dx \Big)^{\frac{1}{\bar p-1}} \le c\Big[\mathfrak{A}^2_{\sigma^{-1}R,q} + \mathfrak{P}^2_{\sigma^{-1}R,q}\Big]\lambda^{\bar p}.$$

This essentially concludes the proof in the super-quadratic case, up to some algebraic manipulations, after noticing that $\mathfrak{A}^2_{\sigma^{-1}R,q} \le \mathfrak{A}^{\bar p'}_{\sigma^{-1}R,q}$ thanks to (3.3), and similarly for $\mathfrak{P}_{\sigma^{-1}R,q}$.

If, on the other hand, $\bar{\boldsymbol{p}} < \mathbf{2}$, we use (2.8) and Hölder's inequality:

$$\begin{aligned} \fint_{B_{R/4}} |Du - Dw|^{\bar p}\,dx \le c \fint_{B_{R/4}} \big|V_{\bar p}(Du) - V_{\bar p}(Dw)\big|^2\,dx \\ + c\Big(\fint_{B_{R/4}} |Du|^{\bar p}\,dx \Big)^{\frac{2-\bar p}{2}} \Big(\fint_{B_{R/4}} \big|V_{\bar p}(Du) - V_{\bar p}(Dw)\big|^2\,dx \Big)^{\frac{\bar p}{2}}. \end{aligned}$$

Next we plug in the estimate (3.29) obtaining

$$\begin{aligned} \fint_{B_{R/4}} |Du - Dw|^{\bar p}\,dx &\le c\Big[\mathfrak{A}^2_{R,q} + \mathfrak{P}^2_{R,q}\Big]\lambda^{\bar p} + c\Big[\mathfrak{A}^{\bar p}_{R,q} + \mathfrak{P}^{\bar p}_{R,q}\Big]\lambda^{\frac{\bar p^2}{2}} \Big(\fint_{B_{R/2}} |Du|^{\bar p}\,dx \Big)^{\frac{2-\bar p}{2}} \\ &\le c\Big[\mathfrak{A}_{R,q} + \mathfrak{P}_{R,q}\Big]^{\bar p}\lambda^{\bar p}, \end{aligned}$$

were we again used the higher integrability (2.10) and (3.3) thanks to $\bar p < 2$. □

4. Excess decay and conclusion

Once having at hand the comparison estimates of the previous section, the proof follows the lines of the similar ones in [28,3]; we however re-propose it to highlight several different points.

As a first result of this section, we show how the previous comparison estimates translate into a precise decay inequality for the L^1-excess. We start from a ball $B_R(x_0) \subset \Omega$ with $R \le R_0$, with R_0 defined in the previous section as a function of data, $p(\cdot)$ and M and we immediately stress we are going to further decrease the value of R_0 in a way depending only on data, $p(\cdot), a(\cdot)$ and M.

For $\sigma \in (0, 1/4)$ (that we are going to choose explicitly soon, see (4.9)), we define

$$R_j = \sigma^j R, \qquad \theta B_j = B_{\theta R_j}(x_0), \qquad j \in \mathbb{N}_0, \quad \theta > 0; \tag{4.1}$$

accordingly we call v_j the comparison function defined for $B_{R/2} = B_j/2$ in (3.5) and w_j the solution to (3.6) over $B_j/4$. We also define, q as in Remark 3.1,

$$a_j = \left| \fint_{B_j} Du\, dx \right|, \qquad E_j = \fint_{B_j} \left| Du - (Du)_{B_j} \right| dx,$$

$$\mathfrak{A}_j = \mathfrak{A}_{R_j,q} = R_j \Big(\fint_{B_j} |Da|^q\, dx \Big)^{\frac{1}{q}}, \qquad \mathfrak{P}_j = \mathfrak{P}_{R_j,q} = R_j \log\Big(\frac{1}{R_j}\Big) \Big(\fint_{B_j} |Dp|^q\, dx \Big)^{\frac{1}{q}}. \tag{4.2}$$

Notice that with this notation, for $j \in \mathbb{N}$ we have $\sigma^{-1} R_j \le R \le R_0$; therefore the pre-requisite for Lemma 3.5 is satisfied and we can prove the following lemma (see [34, Lemma 4.2], [10, After (4.14)], [29, Lemma 8.5] for related results).

Lemma 4.1. *Let $j \in \mathbb{N}$, $\varepsilon_2, \varepsilon_3 \in (0,1)$ and $A \ge 1$.*

- *(Qualitative excess smallness) There exists a threshold $\sigma_1 \in (0, 1/4)$, depending on* $\texttt{data}$ *and ε_2, such that for any $\sigma \in (0, \sigma_1]$ there exists a large constant $M_2 \equiv M_2(\texttt{data}, \varepsilon_2, \sigma) \ge 1$ such that if*

$$\fint_{B_j} |Du|\, dx \le \lambda, \qquad \mathfrak{A}_{j-1} + \mathfrak{P}_{j-1} \le \frac{1}{M_2} \tag{4.3}$$

  for some $\lambda \ge 1$, then

$$\fint_{B_{j+1}} \left| Du - (Du)_{B_{j+1}} \right| dx \le \varepsilon_2 \lambda. \tag{4.4}$$

- *(Quantitative excess smallness) There exists a threshold $\sigma_2 \equiv \sigma_2(\texttt{data}, \varepsilon_3, A) \in (0, 1/4)$ such that for $\sigma \in (0, \sigma_2]$ there exists $M_3 \equiv M_3(\texttt{data}, \sigma, A) \ge 1$ such that if*

$$\frac{\lambda}{A} \le \fint_{B_{j+1}} |Du|\, dx, \qquad \fint_{B_{j-1}} |Du|\, dx \le \lambda, \qquad \mathfrak{A}_{j-1} + \mathfrak{P}_{j-1} \le \frac{1}{M_3} \tag{4.5}$$

  for some $\lambda \ge 1$, then

$$\fint_{B_{j+1}} \left| Du - (Du)_{B_{j+1}} \right| dx \le \varepsilon_3 \fint_{B_j} \left| Du - (Du)_{B_j} \right| dx + c_{ld} \Big[\mathfrak{A}_{j-1} + \mathfrak{P}_{j-1} \Big] \lambda \tag{4.6}$$

  with c_{ld} depending on $\texttt{data}$, *A and σ.*

Proof. Using several times a standard property of the excess and being $\sigma \le \sigma_0$ (the constant appearing in Theorem 2.5)

$$\begin{aligned}
&\fint_{B_{j+1}} \left|Du-(Du)_{B_{j+1}}\right| dx \\
&\quad \le 2 \fint_{B_{j+1}} \left|Du-(Dw_j)_{B_{j+1}}\right| dx \\
&\quad \le 2 \fint_{B_{j+1}} \left|Dw_j-(Dw_j)_{B_{j+1}}\right| dx + 2 \fint_{B_{j+1}} |Du-Dw_j|\, dx \\
&\quad \le 4c_{ph}\,(4\sigma)^{\alpha_0} \fint_{B_j/4} \left|Dw_j-(Du)_{B_j}\right| dx + 2(4\sigma)^{-n} \fint_{B_j/4} |Du-Dw_j|\, dx \\
&\quad \le 4^{n+1}c_{ph}\,(4\sigma)^{\alpha_0} \fint_{B_j} \left|Du-(Du)_{B_j}\right| dx + 2\sigma^{-n} \fint_{B_j/4} |Du-Dw_j|\, dx. \qquad (4.7)
\end{aligned}$$

To prove (4.4) we observe that if $\sigma_1 \le \sigma_0$ is such that $4^{n+2}c_{ph}\,(4\sigma_1)^{\alpha_0} \le \varepsilon_2$ then, using (4.3)

$$4^{n+1}c_{ph}\,(4\sigma)^{\alpha_0} \fint_{B_j} \left|Du-(Du)_{B_j}\right| dx \le 2\cdot 4^{n+1}c_{ph}\,(4\sigma_1)^{\alpha_0} \fint_{B_j} |Du|\, dx \le \frac{\varepsilon_2}{2}\lambda$$

for every $\sigma \le \sigma_1$; now we take M_2 as the constant given by Lemma 3.4 for the choice $\varepsilon_1 = \varepsilon_2/[4\sigma^{-n}]$ and we have (4.4).

To prove (4.6) we take

$$\sigma_2 \le \min\left\{\sigma_0, \left(\frac{1}{4^{n+6}c_{ph}A}\right)^{\frac{1}{\alpha_0}}, \frac{1}{4}\left(\frac{\varepsilon_3}{4^{n+1}c_{ph}}\right)^{\frac{1}{\alpha_0}}, \frac{1}{8}\right\}$$

(c_{ph} and α_0 are from Theorem 2.5) and for $\sigma \le \sigma_2$ let M_3 be the constant M_1 from Lemma 3.4 corresponding to the choice $\varepsilon_1 = (4\sigma^2)^n/[2A]$: using our assumption, (3.28) and triangle's inequality we have

$$\begin{aligned}
\frac{\lambda}{A} \le \fint_{B_{j+1}} |Du|\, dx &\le \frac{1}{(4\sigma^2)^n} \fint_{B_{j-1}/4} |Du-Dw_{j-1}|\, dx + \fint_{B_{j+1}} |Dw_{j-1}|\, dx \\
&\le \frac{\lambda}{2A} + \fint_{B_{j+1}} |Dw_{j-1}|\, dx
\end{aligned}$$

and therefore

$$\fint_{B_{j+1}} |Dw_{j-1}|\, dx \ge \frac{\lambda}{2A}.$$

Thus there exists a point $\bar{x} \in B_{j+1}$ such that (remember that $|Dw_{j-1}|$ is continuous) $|Dw_{j-1}(\bar{x})| \ge \lambda/[3A]$ and

$$\fint_{B_{j-1}/4} |Dw_{j-1}|\,dx \le 4^n \fint_{B_{j-1}} |Du|\,dx + \fint_{B_{j-1}/4} |Du - Dw_{j-1}|\,dx \le 4^{n+1}\lambda.$$

Collecting this information and using Theorem 2.5 yields

$$\begin{aligned}\inf_{B_j/4} |Dw_{j-1}| &\ge |Dw_{j-1}(\bar{x})| - \operatorname*{osc}_{B_j/4} Dw_{j-1}\\ &\ge \frac{\lambda}{3A} - 2c_{ph}\,\sigma^{\alpha_0} \fint_{B_{j-1}/4} |Dw_{j-1}|\,dx \ge \frac{\lambda}{3A} - 4^{n+2}c_{ph}\,\sigma_2^{\alpha_0}\lambda \ge \frac{\lambda}{4A}\end{aligned}$$

and we can use Lemma 3.5 (with $B_R = B_j$ and $4A$ replacing A) to bound the second term appearing in the right-hand side of (4.7); for the first term is simply note that the coefficient of the excess is smaller than ε_3 due to our choice of σ_2. □

We also recall the notation introduced in (2.4): in this setting we have, for q the constant of Remak 3.1

$$S_{q,0}(Da, R)(x_0) := \sum_{j=0}^{\infty} \mathfrak{A}_j, \qquad S_{q,1}(Dp, R)(x_0) := \sum_{j=0}^{\infty} \mathfrak{P}_j.$$

4.1. Gradient boundedness

We here suppose that $x_0 \in \Omega$ is such that the limit

$$\lim_{\rho \searrow 0} \fint_{B_\rho(x_0)} Du\,dx$$

exists; notice that this holds for a.e. $x_0 \in \Omega$ by Lebesgue's differentiation theorem. We want to prove that

$$\lim_{j\to+\infty} \left| \fint_{B_j} Du\,dx \right| \le \lambda := \bar{c} \fint_{B_R} \big(1 + |Du|\big)\,dx \tag{4.8}$$

with $\bar{c} \ge 1$ depending on $\mathtt{data}$ to be chosen (see (4.9)); this would ensure the local boundedness of the gradient thanks to Lebesgue's differentiation theorem and the local estimate (1.12) would follow by a standard covering argument.

In order to prove (4.8) we choose $\varepsilon_3 = 1/4$, $A = 80$ in (4.5), take the corresponding constant σ_2 depending on $\mathtt{data}$ and choose

$$\sigma^{\alpha_0} = \min\left\{ \frac{1}{240 \cdot 4^{n+2} c_{ph}}, \sigma_2^{\alpha_0} \right\}, \qquad \bar{c} = 200\sigma^{-3n} \tag{4.9}$$

with c_{ph}, α_0 the constants appearing in Theorem 2.5; note that also $\bar{c}, \sigma$ both depend only on $\mathtt{data}$. Next we fix the value of M_3 in (4.5) corresponding to this choice of σ; then we take

$\varepsilon_1 = (4\sigma)^2/160$ and the corresponding constant M_1 from Lemma 3.4. We also take M_2 corresponding to $\varepsilon_2 = \sigma^{2n}/80$ in (4.4). Thanks to Lemma 2.1 and the absolute continuity of the Lorentz-Zygmund norm, we then reduce the radius R_0, depending on $\texttt{data}$, $p(\cdot)$ and $a(\cdot)$ so that

$$S_{q,0}(Da)(x_0,R)+S_{q,1}(Dp)(x_0,R)=\sum_{j=0}^{\infty}\mathfrak{A}_j+\sum_{j=0}^{\infty}\mathfrak{P}_j\le\min\left\{\frac{1}{M_1},\frac{1}{M_2},\frac{1}{M_3},\frac{\sigma^n}{16c_{ld}}\right\}$$

$$\Longrightarrow\qquad \mathfrak{A}_j+\mathfrak{P}_j\le\min\left\{\frac{1}{M_1},\frac{1}{M_2},\frac{1}{M_3},\frac{1}{4c_{ld}}\right\}\quad\text{for all } j\in\mathbb{N}_0. \tag{4.10}$$

We then set for $j\in\mathbb{N}_0$

$$C_j := \fint_{B_j}|Du|\,dx+\sigma^{-2n}\fint_{B_j}\left|Du-(Du)_{B_j}\right|dx,\qquad \mathcal{J}:=\left\{j\in\mathbb{N}_0: C_j<\frac{\lambda}{40}\right\};$$

note that $\mathcal{J}\neq\emptyset$ (since $0\in\mathcal{J}$ due to the choice of $\bar{c}$) and that if $\mathcal{J}$ is infinite we are done, since along some subsequence $\fint_{B_{j_m}}|Du|\,dx<\lambda$ and therefore the (existing) limit of the averages would be less or equal than λ. Therefore to complete the boundedness proof we can suppose $\mathcal{J}$ to be non-empty and finite and accordingly set $j_e=\max\mathcal{J}$; in particular we have

$$C_{j_e}=\fint_{B_{j_e}}|Du|\,dx+\sigma^{-2n}\fint_{B_{j_e}}\left|Du-(Du)_{B_{j_e}}\right|dx\le\frac{\lambda}{40},$$

$$\fint_{B_j}|Du|\,dx+\sigma^{-2n}\fint_{B_j}\left|Du-(Du)_{B_j}\right|dx\ge\frac{\lambda}{40}\quad\text{for } j\ge j_e+1. \tag{4.11}$$

To prove (4.8) we want to show by induction that

$$a_j+E_j\le\lambda\qquad\text{for all } j\ge j_e.$$

Note that the base of induction is true (notice that $a_{j_e}+E_{j_e}\le C_{j_e}\le\lambda$ by the definition of C_j and (4.11)) and we are thus left to prove that if for some $k\ge j_e$ $a_j+E_j\le\lambda$ for all $j\in\{j_e,\dots,k\}$, then $a_{k+1}+E_{k+1}\le\lambda$.

For the inductive step, we notice that for $j\in\{j_e,\dots,k\}$

$$\fint_{B_j}|Du|\,dx\le\left|\fint_{B_j}Du\,dx\right|+\fint_{B_j}\left|Du-(Du)_{B_j}\right|dx=a_j+E_j\le\lambda;$$

on the other hand by (4.11) and (4.4) for $j\in\{j_e,\dots,k\}$ we also have

$$\frac{\lambda}{40}\le\fint_{B_{j+1}}|Du|\,dx+\sigma^{-2n}\fint_{B_{j+1}}\left|Du-(Du)_{B_{j+1}}\right|dx\le\fint_{B_{j+1}}|Du|\,dx+\frac{\lambda}{80};$$

therefore

$$\fint_{B_{j+1}} |Du|\,dx \geq \frac{\lambda}{80} \qquad \text{for the same indexes.}$$

We are thus in position to use (4.6) with $A = 80$ uniformly for $j \in \{j_e + 1, \dots, k\}$; in particular

$$\begin{aligned} E_{k+1} &= \fint_{B_{k+1}} \left|Du - (Du)_{B_{k+1}}\right| dx \\ &\leq \frac{1}{4}\fint_{B_k} \left|Du - (Du)_{B_k}\right| dx + c_{ld}\left[\mathfrak{A}_{k-1} + \mathfrak{P}_{k-1}\right]\lambda \leq \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2} \end{aligned}$$

thanks to our inductive assumption and (4.10). On the other hand summing up for $j \in \{j_e + 1, \dots, k\}$ (and performing some simple algebraic manipulations) gives

$$\begin{aligned} \sum_{j=j_e+1}^{k+1} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx &\leq \frac{1}{4} \sum_{j=j_e+1}^{k+1} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx \\ &\quad + \fint_{B_{j_e+1}} \left|Du - (Du)_{B_{j_e+1}}\right| dx + c_{ld}\,\lambda \sum_{j=j_e}^{k-1} \left[\mathfrak{A}_j + \mathfrak{P}_j\right] \end{aligned}$$

from which

$$\sum_{j=j_e+1}^{k+1} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx \leq 2 \fint_{B_{j_e+1}} \left|Du - (Du)_{B_{j_e+1}}\right| dx + 2c_{ld}\,\lambda \sum_{j=j_e}^{k-1} \left[\mathfrak{A}_j + \mathfrak{P}_j\right]$$

and manipulating the integrals defining the excesses over B_{j_e} and B_{j_e+1}

$$\sum_{j=j_e}^{k+1} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx \leq 5\sigma^{-n} \fint_{B_{j_e}} \left|Du - (Du)_{B_{j_e}}\right| dx + 2c_{ld}\lambda \sum_{j=0}^{\infty} \left[\mathfrak{A}_j + \mathfrak{P}_j\right].$$

Now we see that (4.11) together with (4.10) guarantees

$$\sum_{j=j_e}^{k+1} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx \leq \left[\frac{\lambda}{8} + \frac{\lambda}{8}\right]\sigma^n.$$

Finally we can estimate by telescopic summation and triangle's inequality

$$a_{k+1} \le a_{j_e} + \sigma^{-n} \sum_{j=j_e}^{k} \fint_{B_j} \left| Du - (Du)_{B_j} \right| dx \le \frac{\lambda}{4} + \frac{\lambda}{4} = \frac{\lambda}{2} \tag{4.12}$$

and the boundedness proof is concluded.

4.2. Gradient continuity

Now that we have proved that the gradient is locally bounded, we proceed with its continuity, with a proof similar to that of the previous section; in particular, we are going to prove that the gradient is continuous being the local uniform limit of continuous functions - namely, its averages on small balls. Therefore, for a compactly supported subset $K \Subset \Omega$, we here take an intermediate subset $K \Subset \tilde{K} \Subset \Omega$ such that $d = \mathrm{dist}(K, \partial\tilde{K}) = \mathrm{dist}(\tilde{K}, \partial\Omega)$; we then set

$$\lambda = \|Du\|_{L^\infty(\tilde{K})} + 1$$

and we take balls with center in K and radius smaller than d.

We start fixing $\varepsilon > 0$ and a radius $R_1 = \min\{d/2, R_0\}$, being R_0 the threshold from Section 3; we are going to further reduce its value. Then we choose $\varepsilon_3 = 1/2$, $A = 48/\varepsilon$ in (4.6), we define the corresponding small constant $\sigma_2(\mathtt{data}, \varepsilon)$, set $\sigma = \sigma_2$ and finally take the corresponding $M_3 = M_3(\mathtt{data}, \varepsilon)$.

We begin the proof noticing that for every $\varepsilon_4 > 0$, it is possible to find a threshold R_2, depending on $\mathtt{data}, a(\cdot), p(\cdot), M, d$ and ε_4 such that

$$\sup_{R \le R_2} \sup_{x_0 \in K} \fint_{B_R(x_0)} \left| Du - (Du)_{B_R(x_0)} \right| dx \le \varepsilon_4 \lambda : \tag{4.13}$$

we simply take $R_2 = \sigma^2 \min\{R_1, \tilde{R}\}$ where $\tilde{R} \equiv \tilde{R}(\mathtt{data}, p(\cdot), a(\cdot), \varepsilon_4)$ is such that

$$S_{q,0}(Da)(x_0, R) + S_{q,1}(Dp)(x_0, R) = \sum_{j=0}^{\infty} \mathfrak{A}_j + \sum_{j=0}^{\infty} \mathfrak{P}_j \le \frac{1}{M_2} \qquad \text{if } R \le \tilde{R},$$

being M_2 the constant provided by Lemma 4.1 for the value $\varepsilon_2 = \varepsilon_4$ ($\mathfrak{A}_j, \mathfrak{P}_j$ built starting from a generic radius $R \le \tilde{R}$ as in (4.2)). We have $B_R(x_0) = B_{j+1}$ with $j \ge 1$ for an appropriate choice of the starting radius $r \in (0, \min\{R_1, \tilde{R}\}]$ and this proves (4.13); note that the first condition in (4.3) is satisfied by our choice of λ.

Now we take a generic but fixed radius $R \le \sigma R_1$ and a point $x_0 \in K$ and we build the sequence of balls B_j as the quantities $\mathfrak{A}_j, \mathfrak{P}_j$ as in (4.1)-(4.2). For $\varepsilon > 0$, we reduce R_1, again thanks to Corollary 2.2 and the previous result, so that

$$\sum_{j=0}^{\infty} \left[\mathfrak{A}_j + \mathfrak{P}_j \right] \le \sigma^n \frac{\varepsilon}{48 c_{ld}} \qquad \text{for every starting radius } R \le R_1$$

and (4.13) holds with $\varepsilon_4 = \sigma^{2n}\varepsilon/48$ for $R \le R_1$; in particular

$$\fint_{B_j} \left|Du-(Du)_{B_j}\right| dx \le \sigma^{2n}\frac{\varepsilon}{48}\lambda \qquad \text{for all } j\in\mathbb{N}_0. \tag{4.14}$$

This makes R_1 depend on $\texttt{data}$, $p(\cdot), a(\cdot), M, d$ and ε. We also define

$$\mathcal{J}_\varepsilon := \Big\{ j\in\mathbb{N}_0 : \fint_{B_j} |Du|\,dx \le \frac{\varepsilon}{48}\lambda\Big\}, \qquad j_e = \begin{cases} \min\mathcal{J}_\varepsilon & \text{if } \mathcal{J}_\varepsilon\neq\emptyset \\ +\infty & \text{if } \mathcal{J}_\varepsilon=\emptyset \end{cases}$$

and we write $\mathbb{N}\setminus\mathcal{J}_\varepsilon$ as disjoint union of nonempty (possibly infinite) maximal intervals $\mathcal{C}_i$ ($i\in\mathcal{I}$ for some set of indexes $\mathcal{I}\subset\mathbb{N}_0$) so that $\mathcal{C}_i\subset\mathbb{N}\setminus\mathcal{J}_\varepsilon$, $i=\min\mathcal{C}_i$ and $\mathcal{C}_i$ is maximal with respect to the inclusion. In particular, for $i\in\mathcal{I}\setminus\{0\}$, $i-1\in\mathcal{J}_\varepsilon$. What we want to prove here is that

$$\left|(Du)_{B_\ell}-(Du)_{B_k}\right| \le \varepsilon\lambda \qquad \text{for } 1\le k<\ell; \tag{4.15}$$

later we will show how does this lead to the conclusion of the proof.

We prove the estimate in the display above distinguishing three cases. The first one is when $k<\ell<j_e$: in this case we aim at linearizing the system due to the fact that, by definition,

$$\fint_{B_{j+1}} |Du|\,dx > \frac{\varepsilon}{48}\lambda \qquad \text{for all } j\in\{k-1,\dots,\ell-1\}$$

and, obviously

$$\fint_{B_{j-1}} |Du|\,dx \le \lambda \qquad \text{for } j\in\mathbb{N}.$$

Summing (4.6) for $j\in\{k,\dots,\ell-1\}$ and reabsorbing yields

$$\begin{aligned}\sum_{j=k}^{\ell}\fint_{B_j}\left|Du-(Du)_{B_j}\right|dx &\le 2\fint_{B_k}\left|Du-(Du)_{B_k}\right|dx + 2c_{ld}\sum_{j=0}^{\infty}\left[\mathfrak{A}_j+\mathfrak{P}_j\right]\lambda \\ &\le \sigma^n\frac{\varepsilon}{24}\lambda+\sigma^n\frac{\varepsilon}{24}\lambda \le \sigma^n\frac{\varepsilon}{12}\lambda\end{aligned}$$

using also the smallness information given by our choice of R_0; hence, by telescopic summation, as in (4.12)

$$\left|(Du)_{B_\ell}-(Du)_{B_k}\right| \le \sigma^{-n}\sum_{j=k}^{\ell-1}\fint_{B_j}\left|Du-(Du)_{B_j}\right|dx \le \frac{\varepsilon}{12}\lambda\le\varepsilon\lambda.$$

The second case we consider is when $j_e\le k<\ell$ (if $j_e<+\infty$); in this case we prove

$$\left|(Du)_{B_\ell}\right|\le\frac{\varepsilon}{2}\lambda, \qquad \left|(Du)_{B_k}\right|\le\frac{\varepsilon}{2}\lambda \tag{4.16}$$

and (4.15) will follow by triangle's inequality. The proof of the first of the previous inequality (being the second one the same) is anyway simple: if $\ell \in \mathcal{J}_\varepsilon$, then simply

$$\left|(Du)_{B_\ell}\right| \le \fint_{B_\ell} |Du|\,dx \le \frac{\varepsilon}{48}\lambda;$$

if $\ell \notin \mathcal{J}_\varepsilon$, on the other hand, we can consider the maximal interval $\mathcal{C}_{\bar{\imath}}$ containing ℓ and we redo the argument of the previous case, replacing k with $\bar{\imath}-1$ (notice that $\bar{\imath} > j_e \ge 0$ since $\ell \ge \bar{\imath}$ and $\ell > j_e$), to get

$$\begin{aligned}
\left|(Du)_{B_\ell} - (Du)_{B_{\bar{\imath}-1}}\right| &\le \sigma^{-n} \sum_{j=\bar{\imath}-1}^{\ell} \fint_{B_j} \left|Du - (Du)_{B_j}\right| dx \\
&\le 2\sigma^{-n} \fint_{B_{\bar{\imath}-1}} \left|Du - (Du)_{B_{\bar{\imath}-1}}\right| dx + 2\sigma^{-n} c_{ld} \sum_{j=0}^{\infty} [\mathfrak{A}_j + \mathfrak{P}_j]\lambda \\
&\le \frac{\varepsilon}{8}\lambda + \frac{\varepsilon}{24}\lambda
\end{aligned}$$

and therefore, since $\bar{\imath}-1 \in \mathcal{J}_\varepsilon$ by definition of $\mathcal{C}_{\bar{\imath}}$,

$$\left|(Du)_{B_\ell}\right| \le \left|(Du)_{B_\ell} - (Du)_{B_{\bar{\imath}-1}}\right| + \left|(Du)_{B_{\bar{\imath}-1}}\right| \le \frac{\varepsilon}{4}\lambda + \frac{\varepsilon}{48}\lambda \le \frac{\varepsilon}{2}\lambda.$$

We complete the proof of (4.15) by showing that also in the case $k < j_e \le \ell$ (4.16) holds. Indeed we can proceed as in the first case replacing ℓ with $j_e - 1 \ge 1$ getting

$$\begin{aligned}
&\left|(Du)_{B_{j_e-1}} - (Du)_{B_k}\right| \le \frac{\varepsilon}{12}\lambda \quad \Longrightarrow \\
&\qquad \left|(Du)_{B_{j_e}} - (Du)_{B_k}\right| \le \sigma^{-n} \fint_{B_{j_e}} \left|Du - (Du)_{B_{j_e}}\right| dx + \frac{\varepsilon}{12}\lambda \quad \Longrightarrow \\
&\qquad\qquad \left|(Du)_{B_k}\right| \le \left|(Du)_{B_{j_e}} - (Du)_{B_k}\right| + \left|(Du)_{B_{j_e}}\right| \le \frac{\varepsilon}{48}\lambda + \frac{\varepsilon}{12}\lambda + \frac{\varepsilon}{48}\lambda \le \frac{\varepsilon}{2}\lambda
\end{aligned}$$

using (4.14) too and recalling that $j_e \in \mathcal{J}_\varepsilon$; on the other hand, if we proceed as in the second one we directly have

$$\left|(Du)_{B_\ell}\right| \le \frac{\varepsilon}{2}$$

and also in this last case we have (4.15).

To conclude, we notice that the previous result implies that for every $\varepsilon > 0$ there exists a constant $R_\varepsilon > 0$ depending on $\texttt{data}$, $p(\cdot)$, $a(\cdot)$, M, d and ε but uniform with respect to $x_0 \in K$ such that

$$\left|(Du)_{B_{r_1}(x_0)} - (Du)_{B_{r_2}(x_0)}\right| \le \varepsilon\lambda \tag{4.17}$$

for all $0 < r_1 < r_2 \leq R_\varepsilon$; as a direct consequence Du is continuous for every $x_0 \in K$ and thus in Ω. Let indeed be $R_\varepsilon = \sigma R_1$: there exist two indexes $1 \leq k \leq \ell$ such that

$$\sigma^{\ell+1} R_2 < r_1 \leq \sigma^{\ell} R_2, \qquad \sigma^{k+1} R_2 < r_2 \leq \sigma^{k} R_2;$$

we have by (4.13)

$$\left|(Du)_{B_{r_1}(x_0)} - (Du)_{B_{\ell+1}}\right| \leq \sigma^{-n} \fint_{B_{r_1}(x_0)} \left|Du - (Du)_{B_{r_1}(x_0)}\right| dx \leq \frac{\varepsilon}{48},$$

$$\left|(Du)_{B_{r_2}(x_0)} - (Du)_{B_{k+1}}\right| \leq \frac{\varepsilon}{48}$$

and these two estimates, together with (4.15), give (4.17).

Data availability

No data was used for the research described in the article.

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