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A Cesàro average for an additive problem with an arbitrary number of prime powers and squares

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Abstract

In this paper we extend and improve all the previous results known in literature about weighted average, with Cesàro weight, of representations of an integer as sum of a positive arbitrary number of prime powers and a non-negative arbitrary number of squares. Our result includes all cases dealt with so far and allows us to obtain the best possible outcome using the chosen technique.

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1 introduction

Additive problems with prime variables are a very popular topic in Analytic Number Theory. The most famous open problem is the binary Goldbach conjecture, viz. that every even integer larger than 4 is the sum of two odd primes. The corresponding problem for odd integers has been partially solved by Vinogradov [27] and then finally settled by Helfgott in a series of papers [16–18]. For a historical account of the progress in the binary Goldbach problem see e.g. Bhowmik and Halupczok [2].

Given the inherent difficulty of the problem, some variants have also been studied: in this paper we are concerned with weighted averages of the number of representations of an integer in the desired shape. We will consider as summands both powers of prime numbers and perfect squares. Given a real number $k \geq 0$, we introduce the Cesàro weight w_k defined by

$$w_k(x) = \begin{cases} \dfrac{(1-x)^k}{\Gamma(k+1)} & \text{if } x \in [0,1], \\ 0 & \text{if } x > 1. \end{cases}$$

This approach was used by Languasco and Zaccagnini in [22] for the binary Goldbach problem: they proved the "explicit formula"

$$\sum_{n\ge 1} R_G(n) w_k\Big(\frac{n}{N}\Big) = \sum_{n\le N} R_G(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \frac{N^2}{\Gamma(k+3)} - 2\sum_{\rho}\frac{\Gamma(\rho)}{\Gamma(\rho+k+2)}N^{\rho+1} + \sum_{\rho_1}\sum_{\rho_2}\frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2+k+1)}N^{\rho_1+\rho_2} + O_k(N)$$

for $k>1$ where $R_G(n) = \sum_{m_1+m_2=n} \Lambda(m_1)\Lambda(m_2)$ and $\Lambda(n)$ is the von Mangoldt function. Here ρ ranges over non-trivial zeros of the Riemann zeta-function. We point out that the error term in [22] has been corrected in Languasco's survey paper [19], where the Author also gives a thorough introduction to these problems. This result was extended to all $k>0$, with a more precise error term and a suitable interpretation for the infinite sums when $0<k<1$, by Brüdern, Kaczorowski and Perelli [3], who used a completely different technique.

Given the flexibility of the technique introduced in [22], the latter has been applied to other types of additive problems like the one studied in [23]. Similar averages of arithmetical functions are common in the literature: see, e.g., Berndt [1].

The presence of the Gamma function in the above asymptotic development stems from the use of an integral formula due to Laplace, see (1) below, and of the explicit formula for the infinite exponential sum over power of primes, which is defined in (3); see Lemma 2. We outline the method in Sect. 2 and then give all definitions in Sect. 3.

However, the same technique can also be used in mixed problems, namely when we consider representations of an integer as a sum of powers of primes and of squares of integers; see Cantarini [4,5], Languasco and Zaccagnini [21] and [24]. In this case, Bessel functions naturally arise because the infinite exponential sum over squares satisfies a functional equation, see (4) below. The asymptotic expansion we seek tends to be more complicated than in the case without squares, but nevertheless it can be obtained as in the cases described above.

The presence of a smooth weight allows to obtain an asymptotic formula with terms of decreasing orders of magnitude and depending on the non-trivial zeros of the Riemann zeta function; furthermore, the weight allows to obtain results independent of the Riemann hypothesis. Since the Cesàro weights depend on a non-negative real parameter k and are equal to 1 if $k=0$ (that is, for $k=0$ we have a simple average without weights) it is important to obtain results with k as small as possible. During the last few years there have been some improvements regarding the optimal k in the case of the Goldbach's problem: in [22] results hold for $k>1$, in Goldston and Yang [15] (assuming Riemann hypothesis) and in Cantarini [7] (unconditionally) for $k=1$ and in Brüdern et al. [3] for $k>0$. Unfortunately, for case $k=0$, that is, without the Cesàro weights, it is not yet possible to obtain the same form or the same quantity of terms as in the other cases (see, for example, [6,20] and Pintz [25]).

In this paper, we prove a result which incorporates all the previous results in the case of Cesàro averages and we show how the technique, although very general and applicable to many problems, makes the lower bound of k worse as the number of primes and squares involved increases, so as to confirm what has already been suggested by the lower bound for k obtained in [4] and [5]. Indeed, in these two articles, for the first time, the technique is extended to problems with a number of summands greater than two; this translates into a greater number of series / integral exchanges to be made and, consequently, a more restrictive constraint on k.

2 Outline of the method

We now briefly explain the major ideas behind this theorem and why we use series instead of finite exponential sums: one of the main tools of this technique is Laplace's formula (1) in [12], namely

$$\frac{1}{2\pi i}\int_{(a)} v^{-s}e^{v}\,\mathrm{d}v = \frac{1}{\Gamma(s)} \qquad \text{where} \qquad \int_{(a)} := \int_{a-i\infty}^{a+i\infty} \tag{1}$$

for $\mathrm{Re}(s) > 0$ and $a > 0$; see formula 5.4(1) on page 238 of [13]. Assume that we have a problem with r summands, and call $R(n)$ the number of representation $R(n)$ of the integer n in the desired shape. The Laplace transform allows us to transform a weighted sum with Cesàro weight into a product of generating functions, which we will describe in detail below. A general version of the transformation is the following:

$$\sum_{n\le N} R(n)\frac{(N-n)^k}{\Gamma(k+1)} = \int_{(a)} e^{Nz}z^{-k-1}F_1(z)\cdots F_r(z)\,\mathrm{d}z.$$

Here $F_1, \ldots, F_r$ denote the infinite exponential sums related to either the powers of primes or to the squares, which we will define more precisely in (2), but of course this formula is quite general. We use a different normalization of Cesàro's weight from [22].

The use of series complicates the treatment from the point of view of computation, but has the advantage of providing the desired functional relationships that would not be possible with truncated sums. For example, in Lemma 2 we have the sum over all zeros of the Riemann zeta function without truncation. Furthermore, the series connected with the squares of integers is related to the Jacobi functional equation, see (4) below for the version that we actually use, which in turn gives rise to the Bessel functions.

3 Preliminary definitions and main theorem

We now describe in detail the notations and conventions that we need throughout the paper. Let $d, h, N \in \mathbb{N}$, $d > 0, N \ge 2$, $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, $\boldsymbol{r} := (r_1, \ldots, r_d) \in \left(\mathbb{N}^+\right)^d$, where $1 \le r_1 \le r_2 \le \cdots \le r_d$, $\boldsymbol{t} := (t_1, \ldots, t_h) \in \mathbb{N}^h$ and, in general, with bold letters, for example $\boldsymbol{f}$, we will indicate some vector that belongs to $\mathbb{N}^\alpha$ or $\left(\mathbb{N}^+\right)^\alpha$, for some positive integer α. With the symbol $\|\cdot\|$ we will indicate the usual Euclidean norm, with the symbol ρ, with or without subscripts, we will always indicate the non-trivial zeros of the Riemann zeta function and the series $\sum_\rho$ will always indicates the sum over all non trivial zeros of $\zeta(s)$, with or without subscripts. With $\boldsymbol{\rho} := (\rho_{s_1}, \ldots, \rho_{s_v})$, where s_j, $j = 1, \ldots, v$ belong to some subset of $\mathbb{N}^+$.

For every $\mathfrak{J} \subseteq \mathfrak{D} := \left\{1, \ldots, d\right\}$ we will define the scalar product

$$\tau\left(\boldsymbol{\Psi}, \boldsymbol{r}, \mathfrak{J}\right) := \sum_{j\in\mathfrak{J}} \frac{\Psi_j}{r_j}$$

where $\boldsymbol{\Psi} = (\Psi_1, \ldots, \Psi_d)$. In most cases along the paper we will use $\boldsymbol{\Psi} = \boldsymbol{\rho}$; in addition, we will use the short definition $\tau\left(\mathbf{1}, \boldsymbol{r}, \mathfrak{J}\right) = \tau\left(\boldsymbol{r}, \mathfrak{J}\right) := \sum_{j\in\mathfrak{J}} \frac{1}{r_j}$.

We will also indicate by $\sum_{\mathfrak{J}\subseteq\mathfrak{D}}$ the sums over all the possible subsets of $\mathfrak{D}$. Taking $n \in \mathbb{N}$, we set

$$R_{d,h,\boldsymbol{r}}(n) := \sum_{m_1^{r_1}+\cdots+m_d^{r_d}+t_1^2+\cdots+t_h^2=n} \Lambda(m_1)\cdots\Lambda(m_d)$$

where $\Lambda(m)$ is the usual von Mangoldt-function. We want to find an asymptotic formula, as $N \to +\infty$, for

$$\sum_{n\le N} R_{d,h,\boldsymbol{r}}(n)\frac{(N-n)^k}{\Gamma(k+1)}$$

where $k>0$ is a real parameter and $\Gamma(x)$ is the Euler Gamma-function.

Let $Z := \{s\in\mathbb{C},\, 0\le \mathrm{Re}(s)\le 1 : \zeta(s)=0\}$ be the set of the non-trivial zeros of the Riemann zeta-function and let $\mathfrak{J}\subseteq\mathfrak{D}$. We will use the symbols

$$\sum_{\boldsymbol{\rho}\in Z^{|\mathfrak{J}|}} = \sum_{\rho_{j_1}}\cdots\sum_{\rho_{j_{|\mathfrak{J}|}}}$$

and

$$\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) := \frac{1}{r_{j_1}}\Gamma\left(\frac{\rho_{j_1}}{r_{j_1}}\right)\cdots\frac{1}{r_{j_{|\mathfrak{J}|}}}\Gamma\left(\frac{\rho_{j_{|\mathfrak{J}|}}}{r_{j_{|\mathfrak{J}|}}}\right)$$

where $j_\alpha\in\mathfrak{J}$, $\alpha=1,\dots,|\mathfrak{J}|$, every $\rho_{j_\alpha}\in Z$ and r_{j_α} is the j_α-th coordinate of the fixed vector $\boldsymbol{r}=(r_1,\dots,r_d)$.

In analogy to the previous definition, we will use the following symbol

$$\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right) := \frac{1}{r_1}\Gamma\left(\frac{1}{r_1}\right)\cdots\frac{1}{r_d}\Gamma\left(\frac{1}{r_d}\right).$$

We finally introduce the following definitions for the terms of the development: notice that summands containing the Bessel function, namely M_2, M_4 and M_5 only appear if $h>0$. We set

$$M_1(N,k,d,h,\boldsymbol{r}) := \frac{1}{2^h}\sum_{\ell=0}^{h}\binom{h}{\ell}\frac{\pi^{\frac{\ell}{2}}(-1)^{h-\ell}N^{k+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\ell}{2}}}{\Gamma\left(k+1+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\ell}{2}\right)}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right),$$

$$M_2(N,k,d,h,\boldsymbol{r}) := \frac{N^{\frac{k+\tau(\boldsymbol{r},\mathfrak{D})}{2}}}{\pi^{k+\tau(\boldsymbol{r},\mathfrak{D})}}\sum_{\eta=0}^{h-1}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}(-1)^{\eta-\ell}\mathfrak{B}\big(\tau(\boldsymbol{r},\mathfrak{D})\big)\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right),$$

$$\begin{aligned}M_3(N,k,d,h,\boldsymbol{r}) := {}& \frac{N^k(-1)^d}{2^h}\sum_{\ell=0}^{h}\binom{h}{\ell}(N\pi)^{\frac{\ell}{2}}(-1)^{h-\ell}\\ &\times\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)\frac{N^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}}{\Gamma\left(k+1+\frac{\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})\right)}.\end{aligned}$$

$$\begin{aligned}M_4(N,k,d,h,\boldsymbol{r}) := {}& \frac{N^{k/2}(-1)^d}{\pi^k}\sum_{\eta=0}^{h-1}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}(-1)^{\eta-\ell}\\ &\times\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)\frac{N^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})/2}}{\pi^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}}\mathfrak{B}\big(\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})\big),\end{aligned}$$

$$\begin{aligned}M_5(N,k,d,h,\boldsymbol{r}) := {}& \frac{N^{k/2}}{\pi^k}\sum_{\substack{I\subseteq\mathfrak{D}\\|I|\ge1}}N^{\frac{\tau(\boldsymbol{r},I)}{2}}(-1)^{|\mathfrak{D}\setminus I|}\sum_{\eta=0}^{h}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}(-1)^{\eta-\ell}\\ &\times\sum_{\boldsymbol{\rho}\in Z^{|\mathfrak{D}\setminus I|}}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)\frac{N^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)/2}}{\pi^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)}}\mathfrak{B}\big(\tau(\boldsymbol{r},I)+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)\big),\end{aligned}$$

where

$$\mathfrak{B}(x) = \mathfrak{B}_{k,h,\eta,\ell,N}(x) = N^{\frac{h-\eta+\ell}{4}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \frac{J_{x+k+(h-\eta+\ell)/2}\left(2\pi\sqrt{N}\,\|\boldsymbol{f}\|\right)}{\|\boldsymbol{f}\|^{x+k+(h-\eta+\ell)/2}},$$

with

$$\sum_{\boldsymbol{f}\in(\mathbb{N}^+)^c} := \sum_{f_1\geq 1}\cdots\sum_{f_c\geq 1}.$$

and $J_\nu(u)$ is the Bessel J function of real argument u and complex order ν.

The convergence of the mentioned series will be proved in section Sect. 5. The main result of this article is the following theorem:

Theorem 1 *Let $d, h \in \mathbb{N}$, $d > 0$, let N be a sufficiently large integer. Let $\mathfrak{D} = \{1, \dots, d\}$ and, for every $\mathfrak{J} \subseteq \mathfrak{D}$ (or with the notation $I \subseteq \mathfrak{D}$) let $\tau(\boldsymbol{r}, \mathfrak{J}) := \sum_{j\in\mathfrak{J}} \frac{1}{r_j}$, where $1 \leq r_1 \leq r_2 \leq \cdots \leq r_d$. Then, for $k > \frac{d+h}{2}$, we have that*

$$\sum_{n\leq N} R_{d,h,\boldsymbol{r}}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \sum_{j=1}^{5} M_j(N,k,d,h,\boldsymbol{r}) + O_{\boldsymbol{r},d,h}\left(N^{k+h/2+\tau(\boldsymbol{r},\mathfrak{D})-1/r_d}\right).$$

It is important to underline that in some particular configurations of the parameters some terms of the asymptotic (but not the dominant term) could be incorporated in the error. Despite the apparently complicated form of the terms, it is not difficult to recognize the results obtained in the previous work on this topic, for example setting $d = 2$, $h = 0$ and $\boldsymbol{r} = (1, 1)$ (the Goldbach numbers case [22]) or $\boldsymbol{r} = (\ell_1, \ell_2)$, $1 \leq \ell_1 \leq \ell_2$ integers (the generalized Goldbach numbers case [23]). Furthermore, it is quite natural to conjecture that at least the main term of this asymptotic is valid for $k \geq 0$ instead of $k > \frac{d+h}{2}$ as suggested by similar studies but with other techniques (see, e.g., the papers by the present authors [10] and with Languasco [8]).

From (1) and suitable hypotheses, which we will explain in detail in the next sections, we deduce

$$\sum_{n\leq N} R_{d,h,\boldsymbol{r}}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}_{r_1}(z)\cdots\widetilde{S}_{r_d}(z)\,\omega_2(z)^h\,\mathrm{d}z \tag{2}$$

where $z = a + iy$, $a > 0$, $y \in \mathbb{R}$, where

$$\widetilde{S}_r(z) := \sum_{m\geq 1}\Lambda(m)\,e^{-m^r z}, \qquad \omega_2(z) := \sum_{m\geq 1} e^{-m^2 z}, \tag{3}$$

are the series that embody the prime powers and the squares, respectively. Since, as we will see, it is possible to develop $\widetilde{S}_r(z)$ as an asymptotic formula, the idea is to substitute this formula for $\widetilde{S}_r(z)$, exchange the integral with all the terms which are obtained from the various products and finally bound the error. Another important aspect to emphasize is that we work with squares, and so with $\omega_2(z)$, because this function is linked to the well-known Jacobi theta 3 function

$$\theta_3(z) := \sum_{m\in\mathbb{Z}} e^{-m^2 z} = 1 + 2\omega_2(z)$$

and $\theta_3(z)$ satisfies the functional equation

$$\theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right), \ \mathrm{Re}(z) > 0$$

(see, for example, Proposition VI.4.3, page 340, of [14]) which implies a functional equation for $\omega_2(z)$

$$\omega_2(z) = \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right). \tag{4}$$

This is fundamental for the present technique, because this functional equation allows us to find the terms involving the Bessel J function and, since we do not have a functional equation of this type for other powers than squares, we can only deal with this particular case.

Various constraints on k arise at several places of our proof, the strongest being needed in Sect. 6.2.2 in order to ensure convergence in the series for M_3. See also the comment just before Lemma 3.

4 Settings

For our purposes, we need a general version of the formula (1), so we recall the following relations:

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDu}}{(a+iu)^s} \, \mathrm{d}u = \begin{cases} \dfrac{D^{s-1}e^{-aD}}{\Gamma(s)}, & D > 0 \\ 0, & D < 0 \end{cases} \tag{5}$$

with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(a) > 0$ and

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{(a+iu)^s} \, \mathrm{d}u = \begin{cases} 0, & \mathrm{Re}(s) > 1 \\ 1/2, & \mathrm{Re}(s) = 1 \end{cases} \tag{6}$$

with $\mathrm{Re}(a) > 0$ (see formulas (8) and (9) of [11]). We also need an integral representation of the Bessel J function with real argument u and complex order ν:

$$J_\nu(u) := \frac{(u/2)}{2\pi i} \int_{(a)} s^{-\nu-1} e^s e^{-u^2/(4s)} \mathrm{d}s \tag{7}$$

for $a > 0$, $u, \nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) > -1$ (see, e.g., equation (8) on page 177 of [28]).

Assume that $k > 0$. From the definition of $\widetilde{S}_r(z)$ and $\omega_2(z)$ (3), it is not difficult to note that

$$\widetilde{S}_{r_1}(z) \cdots \widetilde{S}_{r_d}(z)\, \omega_2(z)^h = \sum_{n \geq 1} R_{d,h,\boldsymbol{r}}(n)\, e^{-nz}.$$

Furthermore, from (5) and (6), we have that

$$\sum_{n \leq N} R_{d,h,\boldsymbol{r}}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \sum_{n \geq 1} R_{d,h,\boldsymbol{r}}(n) \left(\frac{1}{2\pi i} \int_{(a)} e^{(N-n)z} z^{-k-1} \, \mathrm{d}z \right). \tag{8}$$

Now we want to show that it is possible to exchange the integral with the series in the right side of (8). By the Prime Number Theorem, we have that

$$\widetilde{S}_{r_j}(a) \sim \frac{\Gamma\left(\frac{1}{r_j}\right)}{r_j a^{1/r_j}} \tag{9}$$

as $a \to 0^+$(see [24]) and

$$|\omega_2(z)| \leq \omega_2(a) \leq \int_0^{+\infty} e^{-au^2}\,\mathrm{d}u \leq a^{-1/2}\int_0^{+\infty} e^{-v^2}\,\mathrm{d}v \ll a^{-1/2} \tag{10}$$

and so

$$\begin{aligned}\sum_{n\geq 1}\left|R_{d,h,\boldsymbol{r}}(n)e^{-nz}\right| &= \sum_{n\geq 1}R_{d,h,\boldsymbol{r}}(n)e^{-na} = \widetilde{S}_{r_1}(a)\cdots\widetilde{S}_{r_d}(a)\,\omega_2(a)^h \\ &\ll_{\boldsymbol{r},d,h} a^{-\tau(\boldsymbol{r},\mathfrak{D})-h/2}.\end{aligned}$$

From the trivial estimate

$$\left|e^{Nz}\right|\left|z^{-k-1}\right| \asymp e^{Na}\begin{cases} a^{-k-1}, & |y| \leq a \\ |y|^{-k-1}, & |y| > a\end{cases} \tag{11}$$

where $f \asymp g$ means $g \ll f \ll g$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}_{r_1}(z)\cdots\widetilde{S}_{r_d}(z)\,\omega_2(z)^h\,\mathrm{d}z \\ &\quad\ll_{\boldsymbol{r},d,h} e^{Na}a^{-\tau(\boldsymbol{r},\mathfrak{D})-h/2}\left(\int_{-a}^{a} a^{-k-1}\,\mathrm{d}y + \int_a^{+\infty} y^{-k-1}\,\mathrm{d}y\right) \\ &\quad\ll_{\boldsymbol{r},d,h} e^{Na}a^{-\tau(\boldsymbol{r},\mathfrak{D})-h/2-k}\end{aligned}$$

for $k > 0$. Then, we can exchange the integral with the series and so we obtain the main formula (2).

5 Lemmas

In this section we present some technical lemmas that will be useful later and some basic facts in complex analysis. First, we recall that if $z = a + iy$, $a > 0$ and $w \in \mathbb{C}$, we have that

$$z^{-w} = |z|^{-\mathrm{Re}(w)-i\mathrm{Im}(w)}\exp\left((-i\mathrm{Re}(w) + \mathrm{Im}(w))\arctan\left(\frac{y}{a}\right)\right)$$

and so

$$\left|z^{-w}\right| = |z|^{-\mathrm{Re}(w)}\exp\left(\mathrm{Im}(w)\arctan\left(\frac{y}{a}\right)\right). \tag{12}$$

We also recall the Stirling formula

$$\left|\Gamma(x+iy)\right| \sim \sqrt{2\pi}e^{-\pi|y|/2}|y|^{x-1/2} \tag{13}$$

which holds uniformly for $x \in [x_1, x_2]$, x_1, x_2 fixed and $|y| \to +\infty$ (see, e.g., [26], section 4.42).

Now we introduce the "explicit formula" of $\widetilde{S}_r(z)$, $r \in \mathbb{N}^+$.

Lemma 2 (Lemma 1 of [23]) *Let $r \geq 1$ be an integer, let $z = a + iy$, $a > 0$, $y \in \mathbb{R}$. Let*

$$T(z,r) := \frac{\Gamma\left(\frac{1}{r}\right)}{rz^{1/r}} - \frac{1}{r}\sum_{\rho} z^{-\rho/r}\Gamma\left(\frac{\rho}{r}\right). \tag{14}$$

Then

$$\widetilde{S}_r(z) = T(z,r) + E(a,y,r). \tag{15}$$

where

$$\left|E(a,y,r)\right| \ll_r 1 + |z|^{1/2}\begin{cases} 1, & |y| \leq a \\ 1 + \log^2\left(\frac{|y|}{a}\right), & |y| > a. \end{cases} \tag{16}$$

Note that in Lemma 1 of [23] $T(z,r)$ is defined as

$$T(z,r) := \frac{\Gamma\left(\frac{1}{r}\right)}{rz^{1/r}} - \frac{1}{r}\sum_{\rho} z^{-\rho/r}\Gamma\left(\frac{\rho}{r}\right) - \log(2\pi)$$

but in our context, to make the main term combinatorically more tractable, it is better to insert $\log(2\pi)$ in the error term $E(a,y,r)$. Furthermore, from (9) and (16) we immediately get the important estimate

$$\left|\sum_{\rho} z^{-\rho/r}\Gamma\left(\frac{\rho}{r}\right)\right| \ll_r a^{-1/r} + 1 + |z|^{1/2}\begin{cases} 1, & |y| \leq a \\ 1 + \log^2\left(\frac{|y|}{a}\right), & |y| > a \end{cases} \tag{17}$$

which can be rewritten, if $0 < a < 1$ and $r \geq 1$, in the more compact form

$$\left|\sum_{\rho} z^{-\rho/r}\Gamma\left(\frac{\rho}{r}\right)\right| \ll_r \begin{cases} a^{-1/r}, & |y| \leq a \\ a^{-1/r} + |z|^{1/2}\log^2\left(\frac{2|y|}{a}\right), & |y| > a. \end{cases} \tag{18}$$

The following two technical lemmas highlight the constraints that are necessary for convergence of the series and integrals and that will subsequently be reflected in the constraint regarding the parameter k.

Lemma 3 *Let $\lambda \in \mathbb{N}^+$, $r_1, \ldots, r_\lambda \in \mathbb{N}^+$ and $\boldsymbol{r} := (r_1, \ldots, r_\lambda) \in \left(\mathbb{N}^+\right)^\lambda$. Let $\rho_j = \beta_j + i\gamma_j$, $j \in \{1, \ldots, \lambda\}$, run over the non trivial zeros of Riemann zeta function and $\alpha > 1$ be a parameter. Then, for any fixed $b > 1$ and $c \geq 0$, the series*

$$\sum_{\rho_1 : \gamma_1 > 0}\left(\frac{\gamma_1}{r_1}\right)^{\beta_1/r_1 - 1/2} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0}\left(\frac{\gamma_\lambda}{r_\lambda}\right)^{\beta_\lambda/r_\lambda - 1/2}$$
$$\times \int_1^{+\infty} \log^c(bu)\exp\left(-\arctan\left(\frac{1}{u}\right)\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)\right)\frac{\mathrm{d}u}{u^{\alpha+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)}}$$

converges if $\alpha > \frac{\lambda}{2} + 1$.

Proof Following the proof of Lemma 2 of [23], we can see that

$$\int_1^{+\infty}\exp\left(-\arctan\left(\frac{1}{u}\right)\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)\right)\frac{\mathrm{d}u}{u^{\alpha+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)}}$$
$$\ll_{\alpha,\boldsymbol{r}} \tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)^{1-\alpha-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)}\int_0^{+\infty} e^{-w}w^{\alpha+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)-2}\,\mathrm{d}w$$

and the integral converges since $0 < \beta_j < 1$, $j = 1, \ldots, \lambda$ and $\alpha > 1$. Hence, we have to consider

$$\sum_{\rho_1 : \gamma_1 > 0} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0} \frac{\left(\frac{\gamma_1}{r_1}\right)^{\beta_1/r_1 - 1/2} \cdots \left(\frac{\gamma_\lambda}{r_\lambda}\right)^{\beta_\lambda/r_\lambda - 1/2}}{\tau\left(\boldsymbol{\gamma}, \boldsymbol{r}, \mathfrak{J}_\lambda\right)^{\alpha + \tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda) - 1}}.$$

Now, it is not difficult to note that

$$\frac{\left(\frac{\gamma_1}{r_1}\right)^{\beta_1/r_1} \cdots \left(\frac{\gamma_\lambda}{r_\lambda}\right)^{\beta_\lambda/r_\lambda}}{\tau\left(\boldsymbol{\gamma}, \boldsymbol{r}, \mathfrak{J}_\lambda\right)^{\tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda)}} \leq 1 \tag{19}$$

so we analyze

$$\begin{aligned} &\sum_{\rho_1 : \gamma_1 > 0} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0} \frac{\left(\frac{\gamma_1}{r_1}\right)^{-1/2} \cdots \left(\frac{\gamma_\lambda}{r_\lambda}\right)^{-1/2}}{\tau\left(\boldsymbol{\gamma}, \boldsymbol{r}, \mathfrak{J}_\lambda\right)^{\alpha - 1}} \\ &\quad \leq \sum_{\rho_1 : \gamma_1 > 0} \left(\frac{\gamma_1}{r_1}\right)^{-\frac{1}{2} - \frac{\alpha-1}{\lambda}} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0} \left(\frac{\gamma_\lambda}{r_\lambda}\right)^{-\frac{1}{2} - \frac{\alpha-1}{\lambda}} \end{aligned}$$

by the inequality of arithmetic and geometric means. From the asymptotic formula of $N(T)$, where $N(T)$ is the number of non-trivial zeros of the Riemann zeta function with imaginary part $0 \leq \gamma \leq T$, it is not difficult to prove, putting $\gamma(k)$ the imaginary part of the k-th non-trivial zeros of $\zeta(s)$, that

$$\gamma(k) \sim \frac{2\pi k}{\log(k)}$$

as $k \to +\infty$. So the series converges if $\alpha > \frac{\lambda}{2} + 1$. The treatment is similar for the case $c > 0$. □

Lemma 4 *Let N, λ, α be positive integers, let $h \in \mathbb{Q}^+$, let $\rho_j = \beta_j + i\gamma_j$, $j \in \{1, \ldots, \lambda\}$, run over the non-trivial zeros of the Riemann zeta function, $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$, $d \in \mathbb{N}^+$ and $k > 0$ a real number. For sake of simplicity we define $\delta := \sum_{j=1}^{\lambda} \gamma_j$. Then, for every fixed integer $b > 1$ and $c > 0$,*

$$\begin{aligned} &\sum_{\rho_1 : \gamma_1 > 0} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0} \frac{\gamma_1^{-\frac{1}{2}} \cdots \gamma_\lambda^{-\frac{1}{2}}}{\delta^{k+h+\alpha}} \sum_{\boldsymbol{f} \in (\mathbb{N}^+)^\alpha} \\ &\quad \times \int_0^\delta v^{k-1+h+\alpha+\tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda)} e^{-\|\boldsymbol{f}\|^2 N v^2/\delta^2 - v} \log^{2c}\left(\frac{b\,\delta}{v}\right) \mathrm{d}v \end{aligned}$$

converges if $k > \frac{\lambda}{2} - h$.

Proof We consider the integral

$$\begin{aligned} &\sum_{\rho_1 : \gamma_1 > 0} \cdots \sum_{\rho_\lambda : \gamma_\lambda > 0} \frac{\gamma_1^{-\frac{1}{2}} \cdots \gamma_\lambda^{-\frac{1}{2}}}{\delta^{k+h+\alpha}} \sum_{\boldsymbol{f} \in (\mathbb{N}^+)^\alpha} \\ &\quad \times \int_0^\delta v^{k-1+h+\alpha+\tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda)} e^{-\|\boldsymbol{f}\|^2 N v^2 \delta^{-2}} \exp(-v)\, \mathrm{d}v. \end{aligned} \tag{20}$$

Now we claim that we can exchange the integral with the multiple series $\sum_{\boldsymbol{f} \in (\mathbb{N}^+)^\alpha}$. To show this we consider

$$\int_0^\delta v^{k-1+h+\alpha+\tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda)} \sum_{f_1 \geq 1} e^{-f_1 N v^2 \delta^{-2}} \omega_2^{\alpha-1}(N v^2 \delta^{-2}) \exp(-v)\, \mathrm{d}v.$$

Now, since for every $M \geq 1$ we have

$$\sum_{f_1 \leq M} e^{-f_1 N v^2 \delta^{-2}} \leq \sum_{f_1 \geq 1} e^{-f_1 N v^2 \delta^{-2}} = \omega_2(N v^2 \delta^{-2}) \ll_N \frac{\delta}{v}$$

from (10) and so we have to deal with

$$\int_0^\delta v^{k-2+h+\alpha+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \omega_2^{\alpha-1}(N v^2 \delta^{-2}) \exp(-v)\, dv$$
$$\ll_{N,\alpha} \delta^{\alpha-1} \int_0^\delta v^{k+h-1+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \exp(-v)\, dv$$

which is convergent since $k > 0$, then we obtain

$$\sum_{f_1 \geq 1} \int_0^\delta v^{k-1+h+\alpha+\frac{\beta_1}{r_1}+\cdots+\frac{\beta_\lambda}{r_\lambda}} e^{-f_1 N v^2 \delta^{-2}} \omega_2^{\alpha-1}(N v^2 \delta^{-2}) \exp(-v)\, dv$$

by the Dominated Convergence Theorem. Clearly, we can repeat the same argument for every factor in the product $\omega_2^{\alpha-1}(N v^2 \delta^{-2})$ and so we can write (20) as

$$\sum_{\rho_1:\, \gamma_1 > 0} \cdots \sum_{\rho_\lambda:\, \gamma_\lambda > 0} \frac{\gamma_1^{-\frac{1}{2}} \cdots \gamma_\lambda^{-\frac{1}{2}}}{\delta^{k+h+\alpha}} \int_0^\delta v^{k-1+h+\alpha+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \omega_2^{\alpha}(N v^2 \delta^{-2}) \exp(-v)\, dv$$

and again using (10) we have to deal with

$$\sum_{\rho_1:\, \gamma_1 > 0} \cdots \sum_{\rho_\lambda:\, \gamma_\lambda > 0} \frac{\gamma_1^{-\frac{1}{2}} \cdots \gamma_\lambda^{-\frac{1}{2}}}{\delta^{k+h+\alpha}} \int_0^\delta v^{k-1+h+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \exp(-v)\, dv.$$

Now, since $k + h + \tau(\boldsymbol{\beta}, \boldsymbol{r}, \mathfrak{J}_\lambda) > 0$, then

$$\int_0^\delta v^{k-1+h+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \exp(-v)\, dv \ll \int_0^{+\infty} v^{k-1+h+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \exp(-v)\, dv < +\infty.$$

Then, from arithmetic mean - geometric mean inequality, we get

$$\sum_{\rho_1:\, \gamma_1 > 0} \cdots \sum_{\rho_\lambda:\, \gamma_\lambda > 0} \frac{\gamma_1^{-\frac{1}{2}} \cdots \gamma_\lambda^{-\frac{1}{2}}}{\delta^{k+h+\alpha}} \ll \sum_{\rho_1:\, \gamma_1 > 0} \gamma_1^{-\frac{k}{\lambda}-\frac{1}{2}-\frac{h}{\lambda}} \cdots \sum_{\rho_\lambda:\, \gamma_\lambda > 0} \gamma_\lambda^{-\frac{k}{\lambda}-\frac{1}{2}-\frac{h}{\lambda}} \tag{21}$$

and the series converges if $k > \frac{\lambda}{2} - h$. Clearly, if we have a log factor into the integral the bound for k is the same. Indeed, we note that

$$\log^{2c}\left(b\frac{\delta}{v}\right) \ll \log^{2c}(\delta) + \log^{2c}(bv)$$

and

$$\log^{2c}(\delta) \leq \log^{2c}\left(\lambda \max_{\gamma_j,\, j=1,\ldots,\lambda} \gamma_j\right) := \log^{2c}(\lambda\gamma_\star)$$

so we have in (21) one series such that

$$\sum_{\rho_\star:\, \gamma_\star > 0} \gamma_\star^{-\frac{k}{\lambda}-\frac{1}{2}-\frac{h}{\lambda}} \log^{2c}(\lambda\gamma_\star)$$

and clearly the log factor does not affect the bound for k; if we have

$$\int_0^{+\infty} v^{k-1+h+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \exp(-v) \log^{2c}(bv)\, dv$$

again, we have the same bounds for k and so the Lemma is proved. $\square$

6 Proof of the main theorem

In this section we prove the main theorem. We first show that the error bound in the main formula is "small", then we prove that all the exchange of symbols is justified and finally we evaluate the integrals.

6.1 Error term

From (14), (15) and following the subdivision in [9], formula (2), we can write

$$\begin{aligned}\widetilde{S}_{r_1}(z)\cdots\widetilde{S}_{r_d}(z) &= T(z,r_1)\cdots T(z,r_d) + \sum_{j=1}^{d} E(a,y,r_j)\left(\prod_{i\neq j}\widetilde{S}_{r_i}(z)\right)\\ &\quad + \sum_{\substack{I\subseteq\mathfrak{D}\\ |I|\geq 2}} c_d(I)\left(\prod_{i\in\mathfrak{D}\setminus I} T(z,r_i)\right)\left(\prod_{\ell\in I} E(a,y,r_\ell)\right)\end{aligned}$$

for some suitable coefficients $c_d(I)$. We multiply by ω_2^h and integrate, getting

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\widetilde{S}_{r_1}(z)\cdots\widetilde{S}_{r_d}(z)\,\omega_2(z)^h\,\mathrm{d}z\\ &= \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}T(z,r_1)\cdots T(z,r_d)\,\omega_2(z)^h\,\mathrm{d}z\\ &+ \frac{1}{2\pi i}\sum_{j=1}^{d}\int_{(a)} e^{Nz}z^{-k-1}E(a,y,r_j)\left(\prod_{i\neq j}\widetilde{S}_{r_i}(z)\right)\omega_2(z)^h\,\mathrm{d}z\\ &+ \sum_{\substack{I\subseteq\mathfrak{D}\\ |I|\geq 2}} c_d(I)\int_{(a)} e^{Nz}z^{-k-1}\left(\prod_{i\in\mathfrak{D}\setminus I} T(z,r_i)\right)\left(\prod_{\ell\in I} E(a,y,r_\ell)\right)\omega_2(z)^h\,\mathrm{d}z\\ &=: A_1 + A_2 + A_3,\end{aligned}$$

say. Now we have to estimate the error term. From (9), (10) and (16) we obtain

$$\begin{aligned}|A_2| &\ll \sum_{j=1}^{d}\int_{(a)}\left|e^{Nz}\right|\left|z^{-k-1}\right|\left|E(a,y,r_j)\right|\prod_{i\neq j}\left|\widetilde{S}_{r_i}(z)\right|\left|\omega_2(z)^h\right|\,\mathrm{d}y\\ &\ll_{\boldsymbol{r},d,h} e^{Na}a^{-h/2}\sum_{j=1}^{d} a^{-\tau(\boldsymbol{r},\mathfrak{D})+\frac{1}{r_j}}\\ &\quad\times\left(\int_0^a a^{-k-1}\left(1+a^{1/2}\right)\mathrm{d}y + \int_a^{+\infty} y^{-k-1}\left(1+y^{1/2}\left(1+\log^2\left(\frac{y}{a}\right)\right)\right)\mathrm{d}y\right)\\ &\ll_{\boldsymbol{r},d,h} e^{Na}a^{-k-h/2}\sum_{j=1}^{d} a^{-\tau(\boldsymbol{r},\mathfrak{D})+\frac{1}{r_j}}\end{aligned}\tag{22}$$

for $k>0$.

For the estimation of A_3 we fix $I\subseteq\mathfrak{D}$ and we consider

$$|A_{3,I}| := \int_{(a)}\left|e^{Nz}\right|\left|z^{-k-1}\right|\prod_{i\in\mathfrak{D}\setminus I}\left|T(z,r_i)\right|\prod_{\ell\in I}\left|E(a,y,r_\ell)\right|\left|\omega_2(z)^h\right|\,\mathrm{d}y.$$

We know from (9) and (15) that

$$|T(z,r)| \ll_r a^{-1/r} + \left|E(a,y,r)\right|$$

hence, using formula (10) it is enough to work with

$$a^{-h/2}\int_{(a)}\left|e^{Nz}\right|\left|z^{-k-1}\right|\prod_{i\in\mathfrak{D}\setminus I}\left(a^{-1/r_i}+\left|E\left(a,y,r_i\right)\right|\right)\prod_{\ell\in I}\left|E\left(a,y,r_\ell\right)\right|\,\mathrm{d}y.$$

Now, observing that

$$\prod_{i\in\mathfrak{D}\setminus I}\left(a^{-1/r_i}+\left|E\left(a,y,r_i\right)\right|\right)=\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}a^{-\tau(\boldsymbol{r},\mathfrak{J})}\prod_{i\in\mathfrak{D}\setminus(I\cup\mathfrak{J})}\left|E\left(a,y,r_i\right)\right|$$

we have by (16),

$$\begin{aligned}
|A_{3,I}| &\ll_{\boldsymbol{r},d,h} e^{Na}a^{-h/2}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}\int_{(a)}a^{-\tau(\boldsymbol{r},\mathfrak{J})}\prod_{i\in\mathfrak{D}\setminus(I\cup\mathfrak{J})}\left|E\left(a,y,r_i\right)\right|\prod_{\ell\in I}\left|E\left(a,y,r_\ell\right)\right|\left|z^{-k-1}\right|\,\mathrm{d}y\\
&\ll_{\boldsymbol{r},d,h} e^{Na}a^{-h/2}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}a^{-\tau(\boldsymbol{r},\mathfrak{J})}\int_0^a a^{-k-1}\left(1+a^{1/2}\right)^{|\mathfrak{D}\setminus\mathfrak{J}|}\,\mathrm{d}y\\
&\quad+e^{Na}a^{-h/2}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}a^{-\tau(\boldsymbol{r},\mathfrak{J})}\int_a^{+\infty}y^{-k-1}\left(1+y^{1/2}\left(1+\log^2\left(\frac{y}{a}\right)\right)\right)^{|\mathfrak{D}\setminus\mathfrak{J}|}\,\mathrm{d}y\\
&\ll_{\boldsymbol{r},d,h} e^{Na}a^{-k-h/2}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}a^{-\tau(\boldsymbol{r},\mathfrak{J})}
\end{aligned}$$

for $k>\frac{|\mathfrak{D}\setminus\mathfrak{J}|}{2}$ and since this inequality must holds for all subsets $\mathfrak{J}\subseteq\mathfrak{D}$, we have to assume $k>\frac{d}{2}$. Hence

$$|A_3|\ll_{\boldsymbol{r},d,h} e^{Na}a^{-k-h/2}\sum_{\substack{I\subseteq\mathfrak{D}\\|I|\ge 2}}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}a^{-\tau(\boldsymbol{r},\mathfrak{J})}.$$

Now we take $a=1/N$ and we observe that

$$\sum_{\substack{I\subseteq\mathfrak{D}\\|I|\ge 2}}\sum_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}N^{\tau(\boldsymbol{r},\mathfrak{J})}\ll_d\max_{\substack{I\subseteq\mathfrak{D}\\|I|\ge 2}}\max_{\mathfrak{J}\subseteq\mathfrak{D}\setminus I}N^{\tau(\boldsymbol{r},\mathfrak{J})}\ll_d N^{\tau(\boldsymbol{r},\mathfrak{D})-\frac{1}{r_{j_1}}-\frac{1}{r_{j_2}}}\ll_d N^{k+h/2+\tau(\boldsymbol{r},\mathfrak{D})-1/r_d}$$

remembering that $1\le r_1\le r_2\le\ldots\le r_d$. This error term is compatible with that of Theorem 1.

6.2 Evaluation of the main term

According to (14) we rewrite A_1 in the following form

$$\begin{aligned}
A_1&=\frac{1}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-1}T\left(z,r_1\right)\cdots T\left(z,r_d\right)\omega_2\left(z\right)^h\,\mathrm{d}z\\
&=\frac{1}{2\pi i}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right)\int_{(1/N)}e^{Nz}z^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})}\omega_2\left(z\right)^h\,\mathrm{d}z\\
&\quad+\frac{(-1)^d}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-1}\left(\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\right)\omega_2\left(z\right)^h\,\mathrm{d}z\\
&\quad+\frac{1}{2\pi i}\sum_{\substack{I\subseteq\mathfrak{D}\\|I|\ge 1}}(-1)^{|\mathfrak{D}\setminus I|}\int_{(1/N)}e^{Nz}z^{-k-1-\tau(\boldsymbol{r},I)}\left(\sum_{\boldsymbol{\rho}\in Z^{|\mathfrak{D}\setminus I|}}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)}\right)\omega_2\left(z\right)^h\,\mathrm{d}z\\
&=:I_1+I_2+I_3.
\end{aligned}$$

I_1 corresponds to the terms M_1 and M_2 of Theorem 1, I_2 corresponds to the terms M_3 and M_4 and finally I_3 corresponds to M_5.

6.2.1 Evaluation of I_1

We study I_1. By (4) and the binomial theorem, we get

$$\begin{aligned}\omega_2(z)^h &= \left(\frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{z}\right)^{1/2}\omega_2\left(\frac{\pi^2}{z}\right)\right)^h \\ &= \sum_{\eta=0}^{h}\frac{\binom{h}{\eta}}{2^\eta}\left(\left(\frac{\pi}{z}\right)^{1/2}-1\right)^\eta \omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta}\left(\frac{\pi}{z}\right)^{\frac{h-\eta}{2}} \\ &= \sum_{\eta=0}^{h}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}(-1)^{\eta-\ell}\left(\frac{\pi}{z}\right)^{\frac{h-\eta+\ell}{2}}\omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta}\end{aligned}$$

and so

$$\begin{aligned}I_1 &= \frac{1}{2\pi i}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right)\int_{(1/N)} e^{Nz}z^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})}\omega_2(z)^h\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\sum_{\eta=0}^{h}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}\pi^{\frac{h-\eta+\ell}{2}}(-1)^{\eta-\ell}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right) \\ &\quad\times\int_{(1/N)} e^{Nz}z^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta+\ell}{2}}\omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta}\mathrm{d}z.\end{aligned}$$

Our main goal is to show that, for a suitable k, we can exchange the integral with the involved series; in this case, with the series related to ω_2. We consider two cases: if $\eta = h$ we get

$$I_{1,1} := \frac{1}{2^{h+1}\pi i}\sum_{\ell=0}^{h}\binom{h}{\ell}\pi^{\frac{\ell}{2}}(-1)^{h-\ell}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right)\int_{(1/N)} e^{Nz}z^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{\ell}{2}}\,\mathrm{d}z$$

which corresponds to the term M_1 in Theorem 1 and, from the substitution $Nz = u$ and (1), we get

$$\begin{aligned}I_{1,1} &= \frac{1}{2^{h+1}\pi i}\sum_{\ell=0}^{h}\binom{h}{\ell}\pi^{\frac{\ell}{2}}(-1)^{h-\ell}N^{k+\tau(\boldsymbol{r},d)+\frac{\ell}{2}}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right)\int_{(1)} e^{u}u^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{\ell}{2}}\,\mathrm{d}u \\ &= \frac{1}{2^h}\sum_{\ell=0}^{h}\binom{h}{\ell}\frac{\pi^{\frac{\ell}{2}}(-1)^{h-\ell}N^{k+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\ell}{2}}}{\Gamma\left(k+1+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\ell}{2}\right)}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{1}{\boldsymbol{r}}\right)\end{aligned}$$

for $k+1+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\ell}{2} > 0$, which is trivially true if $k > 0$. Now, fix $1 \le \lambda \le h-\eta$.

We consider the general case

$$\begin{aligned}I_{1,2,\lambda} := \sum_{f_1\ge1}\cdots\sum_{f_\lambda\ge1}\int_{(1/N)}\left|e^{Nz}\right||z|^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta+\ell}{2}}e^{-\pi^2\mathrm{Re}(1/z)\left(f_1^2+\cdots+f_\lambda^2\right)} \\ \times\left|\omega_2\left(\frac{\pi^2}{z}\right)\right|^{h-\eta-\lambda}|\,\mathrm{d}z|.\end{aligned} \tag{23}$$

Note that if $I_{1,2,\lambda}$ converges for all λ, the exchange between series and integral is justified. By the trivial estimate

$$\mathrm{Re}\left(\frac{1}{z}\right) = \frac{N}{1+y^2N^2} \gg \begin{cases} N, & |y| \le 1/N \\ 1/\left(Ny^2\right), & |y| > 1/N, \end{cases}$$

by (11) and by (10), we obtain

$$I_{1,2,\lambda} \ll_{h,\eta,\lambda} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{\lambda}} \int_0^{1/N} N^{k+1+\tau(\boldsymbol{r},\mathfrak{D})+\frac{\lambda+\ell}{2}} e^{-\pi^2 N\left(\|\boldsymbol{f}\|^2\right)} \, dy$$
$$+ N^{h-\eta-\lambda} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{\lambda}} \int_{1/N}^{+\infty} y^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})+\frac{h-\eta-\ell}{2}-\lambda} e^{-\frac{\pi^2\left(\|\boldsymbol{f}\|^2\right)}{Ny^2}} \, dy.$$

The first integral and the series trivially converge since N is positive, then we can consider only the second integral. Making the substitution $v = \frac{\pi^2\left(\|\boldsymbol{f}\|^2\right)}{Ny^2}$, we get

$$\sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{\lambda}} \int_{1/N}^{+\infty} y^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})+\frac{h-\eta-\ell}{2}-\lambda} e^{-\frac{\pi^2\left(\|\boldsymbol{f}\|^2\right)}{Ny^2}} \, dy$$
$$\ll_{N,h,\eta,\lambda} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{\lambda}} \|\boldsymbol{f}\|^{-\left(k+\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta-\ell}{2}+\lambda\right)} \int_0^{+\infty} v^{\frac{1}{2}\left(k+\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta-\ell}{2}+\lambda-1\right)-1} e^{-v} \, dv.$$

Now, the integral is convergent if $k + \tau(\boldsymbol{r},\mathfrak{D}) - \frac{h-\eta-\ell}{2} + \lambda - 1 > 0$, which means $k > -\tau(\boldsymbol{r},\mathfrak{D}) + \frac{h-\eta-\ell}{2} - \lambda + 1$ and the series is convergent if $k + \tau(\boldsymbol{r},\mathfrak{D}) - \frac{h-\eta-\ell}{2} + \lambda > \lambda$, from the inequality of arithmetic and geometric means, and so $k > -\tau(\boldsymbol{r},\mathfrak{D}) + \frac{h-\eta-\ell}{2}$. Since the inequalities must holds for all $1 \le \lambda \le h-\eta$, for all $0 \le \ell \le \eta$ and for all $0 \le \eta \le h-1$, we can conclude that we can exchange all the series with the integral if $k > -\tau(\boldsymbol{r},\mathfrak{D}) + \frac{h}{2}$. Hence, using (7) we can finally write

$$I_{1,2} = \frac{1}{2\pi i} \sum_{\eta=0}^{h-1} \frac{\binom{h}{\eta}}{2^{\eta}} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} \pi^{\frac{h-\eta+\ell}{2}} (-1)^{\eta-\ell} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{1}{\boldsymbol{r}}\right)$$
$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \int_{(1/N)} e^{Nz} z^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta+\ell}{2}} e^{-\frac{\pi^2\|\boldsymbol{f}\|^2}{z}} \, dz$$
$$= \frac{N^{k+\tau(\boldsymbol{r},\mathfrak{D})}}{2\pi i} \sum_{\eta=0}^{h-1} \frac{\binom{h}{\eta}}{2^{\eta}} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (N\pi)^{\frac{h-\eta+\ell}{2}} (-1)^{\eta-\ell} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{1}{\boldsymbol{r}}\right)$$
$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \int_{(1)} e^{u} u^{-k-1-\tau(\boldsymbol{r},\mathfrak{D})-\frac{h-\eta+\ell}{2}} e^{-\frac{\pi^2\|\boldsymbol{f}\|^2 N}{u}} \, du$$
$$= \frac{N^{\frac{k+\tau(\boldsymbol{r},\mathfrak{D})}{2}}}{\pi^{k+\tau(\boldsymbol{r},\mathfrak{D})}} \sum_{\eta=0}^{h-1} \frac{\binom{h}{\eta}}{2^{\eta}} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} N^{\frac{h-\eta+\ell}{4}} (-1)^{\eta-\ell} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{1}{\boldsymbol{r}}\right)$$
$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \frac{J_{k+\tau(\boldsymbol{r},\mathfrak{D})+\frac{h-\eta+\ell}{2}}\left(2\pi\sqrt{N}\,\|\boldsymbol{f}\|\right)}{\|\boldsymbol{f}\|^{k+\tau(\boldsymbol{r},\mathfrak{D})+\frac{h-\eta+\ell}{2}}}$$

for $k > -\tau(\boldsymbol{r},\mathfrak{D}) + \frac{h}{2}$. This term corresponds to M_2 in Theorem 1.

6.2.2 Evaluation of I_2

As in the previous case, we split the integral into two pieces

$$
\begin{aligned}
I_2 &= \frac{(-1)^d}{2\pi i}\int_{(1/N)} e^{Nz}z^{-k-1}\left(\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\right)\omega_2(z)^h\,\mathrm{d}z\\
&= \frac{(-1)^d}{2\pi i}\sum_{\eta=0}^{h}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}\pi^{\frac{h-\eta+\ell}{2}}(-1)^{\eta-\ell}\int_{(1/N)} e^{Nz}z^{-k-1-\frac{h-\eta+\ell}{2}}\\
&\quad\times\left(\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\right)\omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta}\mathrm{d}z\\
&= \frac{(-1)^d}{2^{h+1}\pi i}\sum_{\ell=0}^{h}\binom{h}{\ell}\pi^{\frac{\ell}{2}}(-1)^{h-\ell}\int_{(1/N)} e^{Nz}z^{-k-1-\frac{\ell}{2}}\left(\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\right)\mathrm{d}z\\
&\quad+\frac{(-1)^d}{2\pi i}\sum_{\eta=0}^{h-1}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}\pi^{\frac{h-\eta+\ell}{2}}(-1)^{\eta-\ell}\int_{(1/N)} e^{Nz}z^{-k-1-\frac{h-\eta+\ell}{2}}\\
&\quad\times\left(\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\right)\omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta}\mathrm{d}z\\
&=: I_{2,1}+I_{2,2}.
\end{aligned}
$$

Let us consider $I_{2,1}$ which corresponds to M_3 in Theorem 1. We want to show that it is possible to exchange the integral with the product of the series involving the non-trivial zeros if the Riemann zeta function. To prove this, we fix an arbitrary $1 \le \lambda \le d$ and we analyze

$$
\sum_{\rho_1}\frac{\left|\Gamma\left(\frac{\rho_1}{r_1}\right)\right|}{r_1}\cdots\sum_{\rho_\lambda}\frac{\left|\Gamma\left(\frac{\rho_\lambda}{r_\lambda}\right)\right|}{r_\lambda}\int_{(1/N)}\left|e^{Nz}\right||z|^{-k-1-\frac{\ell}{2}}\left|z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{I}_\lambda)}\right|\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right||\mathrm{d}z|
$$

with the convention that, if $\lambda = d$, then $\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right| = 1$. From Stirling formula (13) and (17) we have that it is enough to study the convergence of

$$
\begin{aligned}
&\sum_{\rho_1}|\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}}\cdots\sum_{\rho_\lambda}|\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}}\int_{\mathbb{R}}|z|^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{I}_\lambda)}\\
&\quad\times\exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny)-\frac{\pi|\gamma_j|}{2r_j}\right)\right)\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right|\mathrm{d}y.
\end{aligned}
$$

We split the integral in $|y| \le 1/N$ and $|y| > 1/N$. Assume that $|y| \le 1/N$, then, by (17), we have

$$
\begin{aligned}
&\sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \int_{-1/N}^{1/N} |z|^{-k-1-\frac{\ell}{2}} \, |z|^{-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} \\
&\quad \times \exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\left|\gamma_j\right|}{2r_j}\right)\right) N^{d-\lambda}\, \mathrm{d}y \\
&\quad \ll_{N,\lambda,d} \sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \exp\left(-\frac{\pi\,|\gamma_1|}{4r_1}\right) \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \exp\left(-\frac{\pi\,|\gamma_\lambda|}{4r_\lambda}\right)
\end{aligned}
$$

and the series trivially converges, so assume that $\left|y\right| > 1/N$. It is enough considering the case

$$
\begin{aligned}
&\sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \int_{|y|>1/N} |z|^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\alpha}{2}} \\
&\quad \times \exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\left|\gamma_j\right|}{2r_j}\right)\right) \log^{2\alpha}\left(2N\left|y\right|\right)\, \mathrm{d}y
\end{aligned}
$$

for $1 \le \alpha \le d-\lambda$, since the powers of N do not affect the study of the convergence and so can be omitted. Assume $y > 1/N$ and $\gamma_j > 0$, $j = 1, \dots, \lambda$. Putting $Ny = u$ and using the well-known identity $\arctan(x) - \frac{\pi}{2} = -\arctan\left(\frac{1}{x}\right)$ we get

$$
\begin{aligned}
&\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda:\,\gamma_\lambda>0} \gamma_\lambda^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \int_1^{+\infty} u^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\alpha}{2}} \\
&\quad \times \exp\left(-\arctan\left(\frac{1}{u}\right)\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)\right) \log^{2\alpha}(2u)\, \mathrm{d}u
\end{aligned}
$$

and, by Lemma 3, we have the convergence if $k > \frac{\lambda+\alpha-\ell}{2}$, and since this inequality must hold for all $0 \le \ell \le d$ and all $1 \le \alpha \le d-\lambda$ we can conclude that $k > \frac{d}{2}$. Now, fix $1 \le \eta \le \lambda$ and assume that $\gamma_1, \dots, \gamma_\eta > 0$ and $\gamma_{\eta+1}, \dots, \gamma_\lambda < 0$. In this case, recalling that $y > 1/N$ and so $\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi|\gamma_j|}{2r_j} \le -\frac{\pi|\gamma_j|}{2r_j}$ for $j > \eta$, we have to work with

$$
\begin{aligned}
&\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\eta:\,\gamma_\eta>0} \gamma_\eta^{\frac{\beta_\eta}{r_\eta}-\frac{1}{2}} \sum_{\rho_{\eta+1}:\,\gamma_{\eta+1}<0} \left|\gamma_{\eta+1}\right|^{\frac{\beta_{\eta+1}}{r_{\eta+1}}-\frac{1}{2}} \\
&\quad \times \exp\left(-\frac{\pi\left|\gamma_{\eta+1}\right|}{2r_{\eta+1}}\right) \cdots \sum_{\rho_\lambda:\,\gamma_\lambda<0} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \exp\left(-\frac{\pi\,|\gamma_\lambda|}{2r_\lambda}\right) \\
&\quad \times \int_{y>1/N} y^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\alpha}{2}} \exp\left(\sum_{j=1}^{\eta}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\gamma_j}{2r_j}\right)\right) \log^{2\alpha}(2Ny)\, \mathrm{d}y
\end{aligned}
$$

with $1 \le \alpha \le d-\lambda$. Letting $Ny = u$, we note that we have to deal with

$$
\begin{aligned}
&\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\eta:\,\gamma_\eta>0} \gamma_\eta^{\frac{\beta_\eta}{r_\eta}-\frac{1}{2}} \sum_{\rho_{\eta+1}:\,\gamma_{\eta+1}<0} \left|\gamma_{\eta+1}\right|^{\frac{\beta_{\eta+1}}{r_{\eta+1}}-\frac{1}{2}} \\
&\quad \times \exp\left(-\frac{\pi\left|\gamma_{\eta+1}\right|}{2r_{\eta+1}}\right) \cdots \sum_{\rho_\lambda:\,\gamma_\lambda<0} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \exp\left(-\frac{\pi\,|\gamma_\lambda|}{2r_\lambda}\right) \\
&\quad \times \int_1^{+\infty} u^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\alpha}{2}} \exp\left(-\arctan\left(\frac{1}{u}\right)\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\eta\right)\right) \log^{2\alpha}(2u)\, \mathrm{d}u
\end{aligned}
$$

(also in this case we omit the powers of N because they do not affect the convergence) and, since

$$\left(\frac{1}{u}\right)^{\frac{\beta_j}{r_j}} < 1,\ u > 1,\ j = 1, \ldots, \lambda,$$

it is enough to consider

$$\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\eta:\,\gamma_\eta>0} \gamma_\eta^{\frac{\beta_\eta}{r_\eta}-\frac{1}{2}} \int_1^{+\infty} u^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\eta)+\frac{\alpha}{2}}$$
$$\times \exp\left(-\arctan\left(\frac{1}{u}\right)\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\eta\right)\right)\log^{2\alpha}(2u)\,\mathrm{d}u$$

and so, arguing as in the previous case, the convergence if $k > \frac{\eta+d-\lambda}{2}$ and so, since $\eta \le \lambda$, a complete convergence in the case $k > \frac{d}{2}$. If $y < -1/N$ we get the same bounds for k, by symmetry.

Now, since $|\mathfrak{D}| = d$, we have

$$\begin{aligned} I_{2,1} &= \frac{(-1)^d}{2^{h+1}\pi i}\sum_{\ell=0}^{h}\binom{h}{\ell}\pi^{\frac{\ell}{2}}(-1)^{h-\ell}\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)\int_{(1/N)} e^{Nz}z^{-k-1-\frac{\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\,\mathrm{d}z \\ &= \frac{N^k(-1)^d}{2^h}\sum_{\ell=0}^{h}\binom{h}{\ell}(N\pi)^{\frac{\ell}{2}}(-1)^{h-\ell}\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)\frac{N^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}}{\Gamma\left(k+1+\frac{\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})\right)}, \end{aligned}$$

from (1).

Now, we analyze $I_{2,2}$ (which corresponds to M_4 in Theorem 1) and we prove that we can switch the integral with the series involving the non-trivial zeros of the Riemann zeta function and with the powers of $\omega_2\left(\frac{\pi^2}{z}\right)$. As the previous calculations, we fix $1 \le \lambda \le d$ and $1 \le \alpha \le h-\eta$. So we have to consider

$$\sum_{\rho_1}\frac{\left|\Gamma\left(\frac{\rho_1}{r_1}\right)\right|}{r_1}\cdots\sum_{\rho_\lambda}\frac{\left|\Gamma\left(\frac{\rho_\lambda}{r_\lambda}\right)\right|}{r_\lambda}\sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha}\int_{(1/N)}\left|e^{Nz}\right|e^{-\mathrm{Re}\left(\frac{\pi^2}{z}\right)\|\boldsymbol{f}\|^2}|z|^{-k-1-\frac{h-\eta+\ell}{2}}\left|z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{J}_\lambda)}\right|$$
$$\times\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right|\left|\omega_2\left(\frac{\pi^2}{z}\right)\right|^{h-\eta-\alpha}|\,\mathrm{d}z|$$

again with the convention $\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right| = 1$ if $\lambda = d$. From (13) and recalling that the powers of N do not affect the convergence, it is enough to study the convergence of

$$\sum_{\rho_1}|\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}}\cdots\sum_{\rho_\lambda}|\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}}\sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha}\int_{\mathbb{R}}|z|^{-k-1-\frac{h-\eta+\ell}{2}}|z|^{-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)}e^{-\frac{\|\boldsymbol{f}\|^2N}{1+N^2y^2}}$$
$$\times\exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny)-\frac{\pi|\gamma_j|}{2r_j}\right)\right)\prod_{s=\lambda+1}^{d}\left|\sum_{\rho_s}\frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s}z^{-\frac{\rho_s}{r_s}}\right|\left(\frac{1+y^2N^2}{N}\right)^{\frac{h-\eta-\alpha}{2}}\,\mathrm{d}y.$$

If $\left|y\right| \leq 1/N$ we have

$$\sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_{-1/N}^{1/N} |z|^{-k-1-\frac{h-\eta+\ell}{2}} |z|^{-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} e^{-\frac{\|\boldsymbol{f}\|^2 N}{1+N^2y^2}}$$
$$\times \exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\left|\gamma_j\right|}{2r_j}\right)\right) \prod_{s=\lambda+1}^{d} \left|\sum_{\rho_s} \frac{\Gamma\left(\frac{\rho_s}{r_s}\right)}{r_s} z^{-\frac{\rho_s}{r_s}}\right| \left(\frac{1+y^2N^2}{N}\right)^{\frac{h-\eta-\alpha}{2}} \mathrm{d}y$$
$$\ll_{N,k,\alpha,h,\eta,\ell,\boldsymbol{r}} \sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \exp\left(-\frac{\pi\left|\gamma_1\right|}{4r_1}\right) \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \exp\left(-\frac{\pi\left|\gamma_1\right|}{4r_1}\right) \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} e^{-\|\boldsymbol{f}\|^2 N}$$

and trivially the convergence, so we consider now $\left|y\right| > 1/N$. If we fix $1 \leq \mu \leq d-\lambda$, from (17), it is enough to work with

$$\sum_{\rho_1} |\gamma_1|^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda} |\gamma_\lambda|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_{|y|>1/N} |z|^{-k-1-\frac{h-\eta+\ell}{2}+\frac{\mu}{2}} |z|^{-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)} e^{-\frac{\|\boldsymbol{f}\|^2}{Ny^2}}$$
$$\times \exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\left|\gamma_j\right|}{2r_j}\right)\right) \log^{2\mu}\left(2N\left|y\right|\right) \left|y\right|^{h-\eta-\alpha} \mathrm{d}y.$$

Assume $y > 1/N$ and $\gamma_j > 0$, $j = 1, \ldots, \lambda$. Since $\arctan\left(\frac{1}{Ny}\right) \gg \frac{1}{Ny}$, We have

$$\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda:\,\gamma_\lambda>0} \gamma_\lambda^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_{1/N}^{+\infty} y^{-k-1-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\mu}{2}} e^{-\frac{\|\boldsymbol{f}\|^2}{Ny^2}}$$
$$\times \exp\left(-\frac{\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)}{Ny}\right) \log^{2\mu}(2Ny)\, y^{h-\eta-\alpha}\, \mathrm{d}y.$$

Putting $\nu = \dfrac{\sum_{j=1}^{\lambda}\frac{\gamma_j}{r_j}}{Ny}$, we obtain

$$\frac{\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\lambda:\,\gamma_\lambda>0} \gamma_\lambda^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}}}{\left(\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)\right)^{k+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)-h+\eta+\alpha-\frac{\mu}{2}}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_0^{\tau(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda)} \nu^{k-1+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)-\frac{\mu}{2}}$$
$$\times e^{-\frac{\|\boldsymbol{f}\|^2 N\nu^2}{\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\gamma)^2}} \exp(-\nu) \log^{2\mu}\left(2\frac{\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)}{\nu}\right) \nu^{-h+\eta+\alpha}\, \mathrm{d}\nu.$$

Now, from (19) and by elementary manipulations we can study

$$\frac{\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{-\frac{1}{2}} \cdots \sum_{\rho_\lambda:\,\gamma_\lambda>0} \gamma_\lambda^{-\frac{1}{2}}}{\left(\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)\right)^{k-\frac{h-\eta}{2}+\frac{\ell}{2}+\alpha-\frac{\mu}{2}}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_0^{\tau(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda)} \nu^{k-1-\frac{h-\eta}{2}+\frac{\ell}{2}+\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\alpha-\frac{\mu}{2}}$$
$$\times e^{-\frac{\|\boldsymbol{f}\|^2 N\nu^2}{\tau(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda)^2}} \exp(-\nu) \log^{2\nu}\left(2\frac{\tau\left(\boldsymbol{\gamma},\boldsymbol{r},\mathfrak{J}_\lambda\right)}{\nu}\right) \mathrm{d}\nu$$

and so, by Lemma 4, we have the convergence if $k > \frac{\lambda}{2} + \frac{h-\eta-\ell+\mu}{2}$. Since $\mu \leq d-\lambda$ and $0 \leq \ell \leq \eta$ we have the complete convergence for all possible cases if $k > \frac{d+h}{2}$.

Now fix $1 \le \xi \le \lambda$ and assume that $\gamma_1, \ldots, \gamma_\xi > 0$ and $\gamma_{\xi+1}, \ldots, \gamma_\lambda < 0$. In this case, recalling that $y > 1/N$, we have to work with

$$\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\xi:\,\gamma_\xi>0} \gamma_\xi^{\frac{\beta_\xi}{r_\xi}-\frac{1}{2}} \sum_{\rho_{\xi+1}:\,\gamma_{\xi+1}<0} \left|\gamma_{\xi+1}\right|^{\frac{\beta_{\xi+1}}{r_{\xi+1}}-\frac{1}{2}} \cdots \sum_{\rho_\lambda:\,\gamma_\lambda<0} \left|\gamma_\lambda\right|^{\frac{\beta_\lambda}{r_\lambda}-\frac{1}{2}}$$
$$\times \sum_{\boldsymbol{m}\in(\mathbb{N}^+)^\alpha} \int_{1/N}^{+\infty} y^{-k-1-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\lambda)+\frac{\mu}{2}} e^{-\frac{\|\boldsymbol{m}\|^2}{Ny^2}}$$
$$\times \exp\left(\sum_{j=1}^{\lambda}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\left|\gamma_j\right|}{2r_j}\right)\right)\log^{2\mu}(2Ny)\, y^{h-\eta-\alpha}\,\mathrm{d}y.$$

Now, since $y > 1/N$, if $\gamma_j < 0$, we observe that $\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi|\gamma_j|}{2r_j} \le -\frac{\pi|\gamma_j|}{2r_j}$, $y^{-\beta_j/r_j} \le N^{\beta_j/r_j} \le N^{1/r_j}$ and

$$\sum_{\rho_j:\,\gamma_j<0} \left|\gamma_j\right|^{\frac{\beta_j}{r_j}-\frac{1}{2}} \exp\left(-\frac{\pi\left|\gamma_j\right|}{2r_j}\right)$$

trivially converges, so it is enough to consider

$$\sum_{\rho_1:\,\gamma_1>0} \gamma_1^{\frac{\beta_1}{r_1}-\frac{1}{2}} \cdots \sum_{\rho_\xi:\,\gamma_\xi>0} \gamma_\xi^{\frac{\beta_\xi}{r_\xi}-\frac{1}{2}} \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^\alpha} \int_{1/N}^{+\infty} y^{-k-1-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\beta},\boldsymbol{r},\mathfrak{J}_\xi)+\frac{\mu}{2}} e^{-\frac{\|\boldsymbol{f}\|^2}{Ny^2}}$$
$$\times \exp\left(\sum_{j=1}^{\xi}\left(\frac{\gamma_j}{r_j}\arctan(Ny) - \frac{\pi\gamma_j}{2r_j}\right)\right)\log^{2\mu}(2Ny)\, y^{h-\eta-\alpha}\,\mathrm{d}y$$

and so, following the previous case, we have the convergence if $k > \frac{h+d-\lambda+\xi}{2}$ and since $\xi \le \lambda$, we have the complete convergence if $k > \frac{d+h}{2}$. If $y < -1/N$ we have the same bounds, by the symmetry of the non-trivial zeros of the Riemann zeta function, so we can finally exchange the integral with the series and get

$$I_{2,2} = \frac{(-1)^d}{2\pi i}\sum_{\eta=0}^{h-1}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}\pi^{\frac{h-\eta+\ell}{2}}(-1)^{\eta-\ell}\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)$$
$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \int_{(1/N)} e^{Nz} z^{-k-1-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})} e^{-\frac{\pi^2\|\boldsymbol{f}\|^2}{z}}\,\mathrm{d}z$$

which is, taking $Nz = v$,

$$\frac{(-1)^d}{2\pi i}\sum_{\eta=0}^{h-1}\frac{\binom{h}{\eta}}{2^\eta}\sum_{\ell=0}^{\eta}\binom{\eta}{\ell}\pi^{\frac{h-\eta+\ell}{2}}(-1)^{\eta-\ell}\sum_{\boldsymbol{\rho}\in Z^d}\frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right)N^{k+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\times$$
$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^+)^{h-\eta}} \int_{(1)} z^{-k-1-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})} e^{v-\frac{\pi^2 N\|\boldsymbol{f}\|}{v}}\,\mathrm{d}v$$

$$
\begin{aligned}
&= \frac{N^{k/2}(-1)^d}{\pi^k} \sum_{\eta=0}^{h-1} \frac{\binom{h}{\eta}}{2^\eta} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (-1)^{\eta-\ell} \sum_{\boldsymbol{\rho} \in Z^d} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) \frac{N^{\frac{h-\eta+\ell}{4}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})/2}}{\pi^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}} \times \\
&\times \sum_{\boldsymbol{f} \in (\mathbb{N}^+)^{h-\eta}} \frac{J_{k+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}\left(2\pi\sqrt{N}\,\|\boldsymbol{f}\|\right)}{\|\boldsymbol{f}\|^{k+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})}}.
\end{aligned}
$$

7 Evaluation of I_3

In this section we evaluate

$$
I_3 = \frac{1}{2\pi i} \sum_{\substack{I \subseteq \mathfrak{D} \\ |I| \geq 1}} (-1)^{|\mathfrak{D}\setminus I|} \int_{(1/N)} e^{Nz} z^{-k-1-\tau(\boldsymbol{r},I)} \left(\sum_{\boldsymbol{\rho} \in Z^{|\mathfrak{D}\setminus I|}} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) z^{-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)} \right) \omega_2(z)^h \, \mathrm{d}z
$$

which corresponds to M_5 in Theorem 1 and has a similar structure to I_2. If we fix $I \subseteq \mathfrak{D}$ we can repeat the previous argument to justify the exchange the integral with the series only considering $k + \tau(\boldsymbol{r}, I)$ instead of k and $|\mathfrak{D} \setminus I|$ instead of $d = |\mathfrak{D}|$. So, we can conclude that all exchanges are justified if $k > \frac{|\mathfrak{D}\setminus I|+h}{2} - \tau(\boldsymbol{r}, I)$, and so we have

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi i} \sum_{\substack{I \subseteq \mathfrak{D} \\ |I| \geq 1}} (-1)^{|\mathfrak{D}\setminus I|} \sum_{\eta=0}^{h} \frac{\binom{h}{\eta}}{2^\eta} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (-1)^{\eta-\ell} \pi^{\frac{h-\eta+\ell}{2}} \\
&\times \sum_{\boldsymbol{\rho} \in Z^{|\mathfrak{D}\setminus I|}} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) \int_{(1/N)} e^{Nz} z^{-k-1-\tau(\boldsymbol{r},I)-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})} \omega_2\left(\frac{\pi^2}{z}\right)^{h-\eta} \mathrm{d}z \\
&= \frac{1}{2\pi i} \sum_{\substack{I \subseteq \mathfrak{D} \\ |I| \geq 1}} (-1)^{|\mathfrak{D}\setminus I|} \sum_{\eta=0}^{h} \frac{\binom{h}{\eta}}{2^\eta} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (-1)^{\eta-\ell} \pi^{\frac{h-\eta+\ell}{2}} \\
&\times \sum_{\boldsymbol{\rho} \in Z^{|\mathfrak{D}\setminus I|}} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) \sum_{\boldsymbol{f} \in (\mathbb{N}^+)^{h-\eta}} \int_{(1/N)} e^{Nz} z^{-k-1-\tau(\boldsymbol{r},I)-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})} e^{-\frac{\pi^2\|\boldsymbol{f}\|^2}{z}} \mathrm{d}z
\end{aligned}
$$

and so, taking $Nz = \nu$ and using (7), we have that

$$
\begin{aligned}
I_3 &= \frac{N^k}{2\pi i} \sum_{\substack{I \subseteq \mathfrak{D} \\ |I| \geq 1}} N^{\tau(\boldsymbol{r},I)} (-1)^{|\mathfrak{D}\setminus I|} \sum_{\eta=0}^{h} \frac{\binom{h}{\eta}}{2^\eta} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (-1)^{\eta-\ell} \pi^{\frac{h-\eta+\ell}{2}} \\
&\times \sum_{\boldsymbol{\rho} \in Z^{|\mathfrak{D}\setminus I|}} \frac{1}{\boldsymbol{r}} \Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) N^{\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)} \\
&\times \sum_{\boldsymbol{f} \in (\mathbb{N}^+)^{h-\eta}} \int_{(1)} e^{\nu} \nu^{-k-1-\tau(\boldsymbol{r},I)-\frac{h-\eta+\ell}{2}-\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D})} e^{-\frac{\pi^2 N\|\boldsymbol{f}\|^2}{\nu}} \mathrm{d}\nu \\
&= \frac{N^{k/2}}{\pi^k} \sum_{\substack{I \subseteq \mathfrak{D} \\ |I| \geq 1}} N^{\tau(\boldsymbol{r},I)/2} (-1)^{|\mathfrak{D}\setminus I|} \sum_{\eta=0}^{h} \frac{\binom{h}{\eta}}{2^\eta} \sum_{\ell=0}^{\eta} \binom{\eta}{\ell} (-1)^{\eta-\ell}
\end{aligned}
$$

$$\times \sum_{\boldsymbol{\rho}\in Z^{|\mathfrak{D}\setminus I|}} \frac{1}{\boldsymbol{r}}\Gamma\left(\frac{\boldsymbol{\rho}}{\boldsymbol{r}}\right) \frac{N^{\frac{h-\eta+\ell}{4}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)/2}}{\pi^{\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)}}$$

$$\times \sum_{\boldsymbol{f}\in(\mathbb{N}^{+})^{h-\eta}} \frac{J_{k+\tau(\boldsymbol{r},I)+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)}\left(2\pi\sqrt{N}\,\|\boldsymbol{f}\|\right)}{\|\boldsymbol{f}\|^{k+\tau(\boldsymbol{r},I)+\frac{h-\eta+\ell}{2}+\tau(\boldsymbol{\rho},\boldsymbol{r},\mathfrak{D}\setminus I)}}$$

and this completes the proof.

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