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# Ornstein-Uhlenbeck operator in convex domains of Banach spaces

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# Introduction

In this work we present some new results about the Ornstein-Uhlenbeck operator $L^\Omega$, and of the semigroup $(T^\Omega(t))_{t\geq 0}$ generated by it, in $L^2(\Omega,\gamma)$. Here $\Omega$ is an open convex subset of an infinite dimensional separable Banach space $X$ endowed with a centered non-degenerate Gaussian measure. The first chapter contains basic notions of measure theory and properties of Gaussian measures, first in finite dimension and then in infinite dimension. In this chapter, we also define the Cameron-Martin space $H$, the $H-$gradient and the Sobolev spaces $W^{1,p}(\Omega,\gamma)$. Moreover we define the Ornstein-Uhlenbeck operator, $L^\Omega$, as the operator associated to the quadratic form

$$\mathcal{E}_{\Omega,\gamma}(u,v) := \int_\Omega \langle \nabla_H u, \nabla_H v\rangle_H d\gamma \quad \text{for} \quad u,v\in W^{1,2}(\Omega,\gamma).$$

Precisely, we set

$$D(L^\Omega) := \Big\{ u\in W^{1,2}(\Omega,\gamma) : \exists f\in L^2(\Omega,\gamma) \text{ s.t. } \mathcal{E}_{\Omega,\gamma}(u,v) = -\int_\Omega fv d\gamma, \forall v\in W^{1,2}(\Omega,\gamma)\Big\}$$

and we put $L^\Omega u := f$.

In the second Chapter we describe some properties of the Ornstein-Uhlenbeck operator. To this aim we approximate $L^\Omega$ by finite-dimensional Ornstein-Uhlenbeck operators, by using the cylindrical approximation of $\Omega$ made in [20]. For finite dimensional Ornstein-Uhlenbeck operators we use the results of [2] and some properties that we show here. In particular we prove that

$$[T^\Omega(t)(fg)]^2 \leq T^\Omega(t)(f^2)T^\Omega(t)(g^2), \text{ a.e. in } \Omega, \ \forall f,g\in L^2(\Omega,\gamma), \ \forall t\geq 0,$$

and

$$|\nabla_H T^\Omega(t)(f)|_H \leq e^{-t}T^\Omega(t)|\nabla_H f|_H, \text{ a.e. in } \Omega, \ \forall f\in W^{1,2}(\Omega,\gamma), \ \forall t\geq 0.$$

Moreover we prove that $(T^{\Omega}(t))_{t\geq 0}$ is a positive semigroup of contraction that satisfies the Beurling-Deny conditions. These properties are used to show the *Poincaré inequality*

$$\int_{\Omega}|f-m_{\Omega}(f)|^2 d\gamma \leq \int_{\Omega}|\nabla_H f|_H^2 d\gamma, \text{ where } m_{\Omega}=\frac{1}{\gamma(\Omega)}\int_{\Omega} f\ d\gamma,$$

and the *Logarithmic-Sobolev inequality*

$$\int_{\Omega} f^2\log(f^2)d\gamma \leq \int_{\Omega}|\nabla_H f|_H^2 d\gamma + \|f\|^2_{L^2(\Omega,\gamma)}\log(\|f\|^2_{L^2(\Omega,\gamma)}),$$

that hold for every $f\in W^{1,2}(\Omega,\gamma)$.

Such inequalities can also be deduced from the theorems shown in [18, Section 6] where the proofs make heavy use of Malliavin calculus and Stochastic Analysis. Our proof is much simpler and relies on analytic tools and on a classical method that goes back to [12].

Infinite dimensional Poincaré and Log-Sobolev inequalities are proved in [3] for $\Omega = X$ through the Wiener chaos decomposition (see Appendix B). In our case we don't have an explicit representation formula for the semigroup neither any sort of Wiener chaos decomposition or explicit expression of the eigenfunctions. As expected, thanks to the Poincaré inequality we prove spectral properties of $L^{\Omega}$.

In the third chapter we consider the equation

$$\lambda u - L^{\Omega}u = f \quad \text{in } \Omega, \tag{1}$$

where $\lambda>0$ and $f\in L^2(\Omega,\gamma)$ are given, and $\Omega$ is an open convex set of $X$. It is not hard to see that for every $\lambda>0$ and $f\in L^2(\Omega,\gamma)$, problem (1) has a unique weak solution $u$, that is

$$\int_{\Omega}\lambda u\varphi\ d\gamma + \int_{\Omega}\langle\nabla_H u,\nabla_H\varphi\rangle_H d\gamma = \int_{\Omega} f\varphi\ d\gamma$$

for all $\varphi\in W^{1,2}(\Omega,\gamma)$.

Here we prove a maximal regularity result for the weak solution $u$ of (1), that is for every $f\in L^2(\Omega,\gamma)$ the weak solution $u$ belongs to $W^{2,2}(\Omega,\gamma)$ and there exists $C>0$ independent of $f$ such that

$$\|u\|_{W^{2,2}(\Omega,\gamma)} \leq C\|f\|_{L^2(\Omega,\gamma)}. \tag{2}$$

It is sufficient to prove that (2) holds if $f$ is a cylindrical smooth bounded function (see Section 1.8), because the space of such functions is dense in $L^2(\Omega,\gamma)$. In this case, we define a sequence of cylindrical functions $\{u_n\}_{n\in\mathbb{N}}$, by using the cylindrical approximation $\{\Omega_n\}_{n\in\mathbb{N}}$ of $\Omega$ made in [20]. In particular,

$$u_n = \varphi_n \circ \pi_n$$

where $\pi_n(X)$ is a finite dimensional subspace of $H$, identified in an obvious way with $\mathbb{R}^q$ with $q = q(n, f)$. So $\pi_n(\Omega_n)$ is identified with an open subset $\mathcal{O}_n$ of $\mathbb{R}^q$, and $\varphi_n : \mathcal{O}_n \subset \mathbb{R}^q \to \mathbb{R}$ solves

$$\begin{cases} \lambda\psi - L^{\mathcal{O}_n}\psi = \widetilde{f} & \text{in } \mathcal{O}_n \subset \mathbb{R}^q, \\ \dfrac{\partial \psi}{\partial \nu} = 0 & \text{on } \partial\mathcal{O}_n \end{cases} \tag{3}$$

where $\widetilde{f}$ is a suitable smooth bounded function. Here, the reference measure is the standard Gaussian measure $N(0, I)$, and $\nabla_H$ is the usual gradient. For the finite dimensional problems (3) we prove dimension free $W^{2,2}$ estimates. Therefore the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{2,2}(\Omega, \gamma)$, and a subsequence weakly converges to $u \in W^{2,2}(\Omega, \gamma)$. Eventually we prove that $u$ is a weak solution of (1).

Moreover, under some regularity assumption on the boundary of $\Omega$, we prove that the weak solution of (1) satisfies

$$\langle \nabla_H u, \nabla_H g\rangle_H = 0 \tag{4}$$

on $\partial\Omega$, in the sense of traces, and $g : X \to \mathbb{R}$ is a suitable convex function such that $g^{-1}(0) = \partial\Omega$. This identity plays the role of the Neumann boundary condition. We use the same sequence $\{u_n\}_{n\in\mathbb{N}}$ defined above, and we show that

$$\int_\Omega (\lambda u_n - L^{\Omega_n} u_n)\varphi \, d\gamma = \int_\Omega f\varphi \, d\gamma,$$

for all smooth cylindrical functions $\varphi$, where $L^{\Omega_n}$ is the Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\Omega_n,\gamma}$, see (1.21). Applying the integration by parts formula (1.20) we get

$$\int_\Omega \lambda\varphi u_n \, d\gamma + \int_\Omega \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma = \int_\Omega f\varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H}\rangle_H \varphi \, d\rho,$$

where $\rho$ is the surface measure associated to the Gaussian measure, see [17]. Taking the limit along a weakly convergent subsequence, we obtain

$$\int_\Omega \lambda\varphi u \, d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H \varphi\rangle_H d\gamma = \int_\Omega f\varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H}\rangle_H \varphi \, d\rho,$$

for all smooth cylindrical functions $\varphi$. Since $u$ is the weak solution of (1) then we can conclude that

$$\int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H}\rangle_H \varphi \, d\rho = 0$$

for all smooth cylindrical functions $\varphi$, that is equivalent to (4).

The maximal $L^p$ regularity for Ornstein-Uhlenbeck equations was established in [24] by Meyer when $\Omega$ is the whole space $X$ for $1 < p < \infty$.

When $\Omega$ is a bounded smooth convex set of an Hilbert space, and $p = 2$ the maximal regularity problem was studied by Barbu, Da Prato and Tubaro in [1] for a more general class Ornstein-Uhlenbeck operators. They approach the weak solution of (1) by a sequence of solutions of penalized problems, in the whole space, that involves the Yosida approximations. They showed, for each of these functions, a maximal regularity estimate, and that the sequence strongly converges to the solution of (1). This allowed them to prove that the weak solution of (1) belongs to $W^{2,2}(\Omega, \gamma)$ and satisfies (4). In our work we consider a simpler Ornstein-Uhlenbeck operator but we study (1) in a separable Banach space, and $\Omega$ can possibly be unbounded.

Maximal $L^2-$regularity is also studied in Hilbert spaces by Da Prato and Lunardi in [9] with Dirichlet boundary condition and in [10] with Neumann boundary condition for a different class of differential operators that doesn't contain the classical Ornstein-Uhlenbeck operator. Also, the proof in [10] is different from ours because it uses a penalization method approaching the weak solution by a sequence of solutions of problems on whole $X$.

In finite dimension more results are available. Maximal $L^p$ regularity, for $p \in (1, \infty)$, was studied by Metafune, Pruess, Rhandi, and Schnaubelt in [23] when $\Omega = \mathbb{R}^n$ for a class of second order differential operators with unbounded coefficients that contains symmetric Ornstein-Uhlenbeck operators. Maximal $L^2$ regularity in open convex sets of $\mathbb{R}^n$, with Neumann boundary condition, was established in [8] again by penalization methods.

The second and third chapter contain new results that have been the subject of two articles.

The properties of the Ornstein-Uhlenbeck semigroup in an open convex set of a separable Banach space, the Logarithmic-Sobolev inequality, the Poincaré inequality and the spectral properties of the Ornstein-Uhlenbeck operator were proved in [5].

The maximal $L^2-$regularity for the Ornstein-Uhlenbeck problem (1) of Chapter 3 was proved in [6].

# Chapter 1

# Preliminaries

This chapter is devoted to introduce the main tools that will be used later, with the aim of fixing the notation. In the first section we present finite real measures and the related notions of $L^p$ space, absolute continuous and singular measure, the Radon-Nikodym theorem, weak convergence and we introduce the Gaussian measures in $\mathbb{R}^d$. In the second section we define the working environment, that is an infinite dimensional separable Banach space endowed with a centered non-degenerate Gaussian measure. The following sections contain definitions and properties of the main tools of this work: the Cameron-Martin space, Sobolev spaces, Ornstein-Uhlenbeck operators, and cylindrical approximations.

## 1.1 Basic notions of measure theory

We introduce sets equipped with a $\sigma-$algebra, that is measurable spaces.

**Definition 1.** Let $\mathcal{F}$ be a collection of subsets of a nonempty set $X$. We say that $\mathcal{F}$ is a $\sigma-$algebra if:

- $\emptyset \in \mathcal{F}$;
- $\forall E_1 \in \mathcal{F} \Rightarrow E_1 \setminus X \in \mathcal{F}$
- $\forall E_1, E_2 \in \mathcal{F} \Rightarrow E_1 \cap E_2 \in \mathcal{F}$
- $\forall \{E_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \Rightarrow \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{F}$.

If $\mathcal{F}$ is a $\sigma-$algebra in $X$, we call the pair $(X, \mathcal{F})$ a measurable space.

**Definition 2.** For any collection $\mathcal{G}$ of subsets of a nonempty set $X$, the $\sigma-$algebra generated by $\mathcal{G}$ is the smallest $\sigma-$algebra containing $\mathcal{G}$. If $(X, \tau)$

is a topological space, we denote by $\mathcal{B}(X)$ the $\sigma-$algebra of Borel subsets of $X$, i.e., the $\sigma-$algebra generated by the open subsets of $X$.

By using the De Morgan laws we get that $\sigma-$algebras are closed under countable intersections. Since the intersection of any family of $\sigma-$algebras is a $\sigma-$algebra and the powerset of $X$ (i.e. the set of all subsets of $X$) is a $\sigma-$algebra, the definition of generated $\sigma-$algebra is well posed. Once we fixed a $\sigma-$algebra, we can introduce positive measures.

**Definition 3.** Let $(X, \mathcal{F})$ be a measurable space and $\mu : \mathcal{F} \to [0, +\infty)$. We say that $\mu$ is a positive finite measure if $\mu(\emptyset) = 0$, $\mu(X) < +\infty$ and $\mu$ is $\sigma-$additive on $\mathcal{F}$, that is, for any sequence $\{E_n\}_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{F}$ the equality

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$$

holds. We say that $\mu$ is a probability measure if $\mu(X) = 1$.

**Definition 4.** A positive finite measure $\mu$ on the Borel sets of a topological space $X$ is called a real Radon measure if for every $B \in \mathcal{B}(X)$ and $\varepsilon > 0$ there is a compact set $K \subset B$ such that $\mu(B \setminus K) < \varepsilon$.

A measure is tight if the same property holds with $B = X$.

**Proposition 1.** *If $(X, d)$ is a separable complete metric space then every positive finite measure on $(X, \mathcal{B}(X))$ is Radon.*

**Definition 5.** Let $(X, \mathcal{F})$, $(Y, \mathcal{G})$ be two measurable spaces. We say that a function $f : X \to Y$ is measurable if $f^{-1}(A) \in \mathcal{F}$ for every $A \in \mathcal{G}$.

Now we denote by $\mathbb{1}_E$ the characteristic function of $E \subset X$ defined below

$$\mathbb{1}_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \neq E. \end{cases}$$

We say that $f : X \to \mathbb{R}$ is a simple function if

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{E_i}$$

where $\alpha_i \in \mathbb{R}$ for all $i = 1, \dots, n$, and $\{E_i\}_{i=1}^n$ is a finite partition of $X$.

We introduce the $L^p$ norms and spaces as follows,

$$\|u\|_{L^p(X,\mu)} := \left(\int_X |u(x)|^p \mu(dx)\right)^{1/p}$$

for $1 \leq p < \infty$, and

$$\|u\|_{L^\infty(X,\mu)} := \inf\{C \in [0, +\infty] : \ |u(x)| \leq C \text{ for } \mu\text{-a.e. } x \in X\}.$$

The space $L^p(X, \mu)$ is the space of equivalence classes of measurable functions agreeing $\mu$-a.e. such that $\|u\|_{L^p(X,\mu)} < \infty$. It is well known that $\|u\|_{L^p(X,\mu)}$ is a norm and $L^p(X, \mu)$ is a Banach space, see e.g. [14, Theorem 5.2.1].

We remark that the continuous and bounded functions belong to $L^1(X, \mu)$ and we define the weak convergence of measures by

$$\mu_j \to \mu \Leftrightarrow \int_X f \, d\mu_j \to \int_X f \, d\mu, \quad \forall f \in C_b(X). \tag{1.1}$$

Let $\mu$, $\nu$ be two positive finite measures on the measurable space $(X, \mathcal{F})$. We say that $\nu$ is absolutely continuous with respect to $\mu$, and we write $\nu \ll \mu$, if $\mu(E) = 0$ implies $\nu(E) = 0$ for all $E \in \mathcal{F}$. We say that they are mutually singular, and write $\nu \perp \mu$, if there exists $E \in \mathcal{F}$ such that $\mu(E) = 0$ and $\nu(X \setminus E) = 0$. If $\mu \ll \nu$ and $\nu \ll \mu$ we say that $\mu$ and $\nu$ are equivalent and write $\mu \sim \nu$. If $\mu$ is a positive measure and $f \in L^1(X, \mu)$, then the measure $\nu$ defined as

$$\nu(E) = \int_E f \, d\mu$$

is absolutely continuous with respect to $\mu$.

**Theorem 1** (Radon-Nikodym theorem)**.** *Let $\mu$, $\nu$ be two positive finite measures on the measurable space $(X, \mathcal{F})$. If $\nu$ is absolutely continuous with respect to $\mu$ then exists a measurable function $f : X \to [0, +\infty)$ , such that*

$$\nu(E) = \int_E f \, d\mu$$

*for all $E \in \mathcal{F}$. The function $f$ is called the Radon-Nikodym derivative and denoted by $\frac{d\nu}{d\mu}$.*

The following theorem is a useful criterion of mutual singularity.

**Theorem 2.** *Let $\mu$, $\nu$ be two probability measures on a measurable space $(X, \mathcal{F})$, and let $\lambda$ be a positive measure such that $\mu \ll \lambda$ and $\nu \ll \lambda$. Then the integral*

$$H(\mu, \nu) := \int_X \sqrt{\frac{d\mu}{d\lambda} \frac{d\nu}{d\lambda}} d\lambda$$

*is independent of $\lambda$ and*

$$2(1 - H(\mu, \nu)) \leq \|\mu - \nu\| \leq 2\sqrt{1 - H(\mu, \nu)^2}$$

*where $\|\mu - \nu\| = \|\frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda}\|_{L^1(X,\lambda)}$ is called variation distance.*

**Corollary 1.** *If $\mu$ and $\nu$ are probability measures, then $\mu \perp \nu$ if and only if $H(\mu, \nu) = 0$.*

Now we recall the notions of image measure that generalizes the classical change of variable formula.

**Definition 6** (Image measure)**.** Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, and let $f : X \to Y$ be a measurable function. For any positive finite measure $\mu$ on $(X, \mathcal{F})$ we define the image law of $\mu$ under $f$, $\mu \circ f^{-1}$, in $(Y, \mathcal{G})$ by

$$(\mu \circ f^{-1})(B) := \mu(f^{-1}(B)), \text{for all } B \in \mathcal{G}.$$

The change of variables formula follows from the previous definition. If $u \in L^1(Y, \mu \circ f^{-1})$, then $u \circ f \in L^1(X, \mu)$ and

$$\int_Y u \; d(\mu \circ f^{-1}) = \int_X (u \circ f) \; d\mu$$

We introduce the cartesian product between two measure spaces.

**Definition 7.** Let $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$ be measure spaces. The product $\sigma-$algebra of $\mathcal{F}_1$ and $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \times \mathcal{F}_2$, is the $\sigma-$algebra generated in $X_1 \times X_2$ by

$$\{E_1 \times E_2 : \; E_1 \in \mathcal{F}_1, \; E_2 \in \mathcal{F}_2\}.$$

**Theorem 3** (Fubini)**.** *Let $(X_1, \mathcal{F}_1, \mu_1)$, $(X_2, \mathcal{F}_2, \mu_2)$ be measure spaces with $\mu_1$, $\mu_2$ positive finite measures. Then, there is a unique positive finite measure $\mu$ on $(X_1 \times X_2, \mathcal{F}_1 \times \mathcal{F}_2)$, denoted also by $\mu_1 \otimes \mu_2$, such that*

$$\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) \quad \forall E_1 \in \mathcal{F}_1, \; \forall E_2 \in \mathcal{F}_2.$$

*Moreover, for any measurable function $f : X_1 \times X_2 \to [0, \infty)$ the functions*

$$x \mapsto \int_{X_2} f(x, y) \; \mu_2(dy), \quad y \mapsto \int_{X_1} f(x, y) \; \mu_1(dx)$$

*are measurable and*

$$\begin{aligned} \int_{X_1 \times X_2} f \; d\mu &= \int_{X_1} \left( \int_{X_2} f(x, y) \; \mu_2(dy) \right) \mu_1(dx) \\ &= \int_{X_2} \left( \int_{X_1} f(x, y) \; \mu_1(dx) \right) \mu_2(dy) \end{aligned}$$

*Remark* 1. It is possible to construct a product measure on infinite cartesian products. If $I$ is a set of indices, typically $I = [0,1]$ or $I = \mathbb{N}$, and $(X_t, \mathcal{F}_t, \mu_t)$, $t \in I$, is a family of measure spaces, the product $\sigma-$algebra is that generated by the family of sets of the form

$$B = B_1 \times \ldots \times B_n \times \bigtimes_{t \in I \setminus \{t_1,\ldots,t_n\}} X_t, \quad B_k \in \mathcal{F}_{t_k},$$

whose measure is $\mu(B) = \mu_{t_1}(B_1) \cdots \mu_{t_n}(B_n)$.

Now we introduce the Fourier transform of measures. Let $\mu$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. We define its Fourier transform by setting

$$\widehat{\mu}(\xi) := \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} \mu(dx) \text{ for } \xi \in \mathbb{R}^d.$$

**Proposition 2.** *Let $\mu$ be a probability measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, then*

- *$\widehat{\mu}$ is uniformly continuous on $\mathbb{R}^d$;*
- *$\widehat{\mu}(0) = 1$;*
- *if $\mu_1$, $\mu_2$ are two probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\widehat{\mu}_1 = \widehat{\mu}_2$, then $\mu_1 = \mu_2$;*
- *if $\{\mu_j\}_{j\in\mathbb{N}}$ is a sequence of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\mu_j \to \mu$ in the sense of (1.1), then $\widehat{\mu}_j \to \widehat{\mu}$ uniformly on compacts;*
- *if $\{\mu_j\}_{j\in\mathbb{N}}$ is a sequence of probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and there is $\varphi : \mathbb{R}^d \to \mathbb{C}$ continuous at $\xi = 0$ such that $\widehat{\mu}_j \to \varphi$ pointwise, then there is a probability measure $\mu$ such that $\widehat{\mu} = \varphi$.*

Now we define the Gaussian measures. They are among the most important finite measures. In this section we consider only the finite dimensional case.

**Definition 8.** A probability measure $\gamma$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called Gaussian if it is either a Dirac measure $\delta_a$ at a point a or if it is a measure absolutely continuous with respect to the Lebesgue measure $\lambda_1$ with density

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}$$

In this case $a$ is called the mean, $\sigma$ the mean-square deviation and $\sigma^2$ the variance of $\gamma$ and we say that $\gamma$ is centered if $a = 0$ and standard if $\sigma = 1$.

It is easy to verify that

$$a = \int_{\mathbb{R}} x \ \gamma(dx), \quad \sigma^2 = \int_{\mathbb{R}} (x-a)^2 \ \gamma(dx),$$

and

$$\widehat{\gamma}(\xi) = \exp\left\{ia\xi - \frac{1}{2}\sigma^2\xi^2\right\}.$$

Moreover by Proposition 2 a probability measure on $\mathbb{R}$ is Gaussian if and only if its Fourier transform has this form. Now we give the definition of Gaussian measure on $\mathbb{R}^d$.

**Definition 9.** A probability measure $\gamma$ on $\mathbb{R}^d$ is said to be Gaussian if for every linear functional $l$ on $\mathbb{R}^d$ the measure $\gamma \circ l^{-1}$ is Gaussian on $\mathbb{R}$.

**Proposition 3.** *A measure $\gamma$ on $\mathbb{R}^d$ is Gaussian if and only if its Fourier transform is*

$$\widehat{\gamma}(\xi) = \exp\left\{i\langle a, \xi\rangle - \frac{1}{2}\langle Q\xi, \xi\rangle\right\}, \quad \xi \in \mathbb{R}^d,$$

*for some $a \in \mathbb{R}^d$ and $Q$ nonnegative $d \times d$ symmetric matrix. Moreover, $\gamma$ is absolutely continuous with respect to the Lebesgue measure $\lambda_d$ if and only if* $\det Q \neq 0$.

## 1.2 Infinite dimensional spaces

In this section we introduce our framework. Let $X$ be an infinite dimensional separable real Banach space, with norm $\|\cdot\|$, and let $X^*$ be its topological dual.

**Definition 10.** We denote by $\mathcal{E}(X)$ the $\sigma-$algebra generated by the cylindrical sets, i.e, the sets of the form

$$C = \{x \in X : (l_1(x), \ldots, l_n(x)) \in C_0\}, \quad l_i \in X^*, \ C_0 \in \mathcal{B}(\mathbb{R}^n),$$

and $C_0$ is called base of $C$.

**Theorem 4.** *If $X$ is a separable Banach space, then $\mathcal{E}(X) = \mathcal{B}(X)$.*

Now we define, as in $\mathbb{R}^d$, the Fourier transform of a finite measure $\mu$ on $\mathcal{E}(X)$, by

$$\widehat{\mu}(f) := \int_X \exp\{if(x)\}\mu(dx), \quad f \in X^*.$$

We extend Proposition 2 to the present context. In particular we recall the injectivity of Fourier transform.

**Proposition 4.** *Let $\mu_1$, $\mu_2$ be two probability measures on $(X, \mathcal{B}(X))$. If $\widehat{\mu}_1 = \widehat{\mu}_2$ then $\mu_1 = \mu_2$.*

Now we introduce Gaussian measures.

**Definition 11.** A probability measure $\gamma$ defined on the $\sigma-$algebra $\mathcal{E}(X)$, is called Gaussian if for all $f \in X^*$ the induced measure $\gamma \circ f^{-1}$ is Gaussian on $\mathbb{R}$. The measure $\gamma$ is called centered if all the measures $\gamma \circ f^{-1}$, with $f \in X^*$, are centered and it is called non-degenerate if for any $f \neq 0$ the measure $\gamma \circ f^{-1}$ is non-degenerate.

First we remark that if $f \in X^*$ then $f \in L^p(X, \gamma)$ for all $p \geq 1$, indeed

$$\int_X |f(x)|^p \gamma(dx) = \int_{\mathbb{R}} |t|^p (\gamma \circ f^{-1})(dt) < +\infty$$

since $\gamma \circ f^{-1}$ is Gaussian on $\mathbb{R}$.

**Definition 12.** Let $\gamma$ be a Gaussian measure on $\mathcal{E}(X)$. The element $a_\gamma$ in the algebraic dual $(X^*)'$ to $X^*$, defined as

$$a_\gamma(f) = \int_X f(x)\ \gamma(dx)$$

is called mean of $\gamma$.

The operator $R_\gamma : X^* \to (X^*)'$ defined by the formula

$$R_\gamma(f)(g) := \int_X [f(x) - a_\gamma(f)]\,[g(x) - a_\gamma(g)]\,\gamma(dx)$$

is called covariance operator of $\gamma$; the corresponding quadratic form on $X^*$ is called covariance of $\gamma$.

**Theorem 5.** *A probability measure $\gamma$ on $X$ is Gaussian if and only if its Fourier transform is given by*

$$\widehat{\gamma}(f) = \exp\left\{ ia(f) - \frac{1}{2}B(f,f) \right\} \quad f \in X^*,$$

*where $a$ is a linear functional on $X^*$ and $B$ is a nonnegative symmetric bilinear form on $X^*$.*

In the following proposition we show some important results about Gaussian measures.

**Proposition 5.** *Let $\gamma$ be a centered Gaussian measure on $X$.*

- *For any $\theta \in \mathbb{R}$ the image measure $(\gamma \otimes \gamma) \circ \varphi_\theta^{-1}$ in $X \times X$ under the map*

$$\begin{aligned} \varphi_\theta : \quad & X \times X \to X \times X \\ & (x, y) \mapsto (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta) \end{aligned}$$

  *coincides with $\gamma$.*

- *For any $\theta \in \mathbb{R}$ the image measures $(\gamma \otimes \gamma) \circ \varphi_i^{-1}$, $i = 1, 2$ in $X$ under the map $\varphi_i : X \times X \to X$,*

$$\varphi_1(x, y) := x \cos\theta + y \sin\theta, \ \varphi_2(x, y) := -x \sin\theta + y \cos\theta$$

  *coincide with $\gamma$.*

We show the useful Fernique Theorem proved in [16].

**Theorem 6** (Fernique Theorem)**.** *Let $\gamma$ be a centered Gaussian measure on a separable Banach space $X$. Then there exists $\alpha > 0$ such that*

$$\int_X \exp\{\alpha \|x\|^2\} \gamma(dx) < +\infty.$$

As a first application of the Fernique theorem, we notice that for every $1 < p < \infty$ we have

$$\int_X \|x\|^p \gamma(dx) < \infty \tag{1.2}$$

since $\|x\|^p \leq c_{\alpha,p} \exp\{\alpha \|x\|^2\}$ for all $x \in X$ and for some constant $c_{\alpha,p} > 0$ depending on $\alpha$ and $p$ only. We already know, through the definition of Gaussian measure, that the functions $f \in X^*$ belong to all $L^p(X, \gamma)$ spaces, for $1 < p < \infty$. The Fernique Theorem tells us much more, since it gives a rather precise description of the allowed growth of the functions in $L^p(X, \gamma)$. Moreover, estimate (1.2) has important consequences on the functions $a_\gamma$ and $B_\gamma$.

**Proposition 6.** *If $\gamma$ is a Gaussian measure on a separable Banach space $X$, then $a_\gamma : X^* \to \mathbb{R}$ and $R_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous. Moreover, there exists $a \in X$ such that*

$$a_\gamma(f) = f(a), \ \textit{for all } f \in X^*.$$

Now we define an important space that will be used several times.

**Definition 13.** We denote by $X^*_\gamma$ the reproducing kernel of the measure $\gamma$, defined as the closure of the set

$$\{f - a_\gamma(f),\ f \in X^*\}$$

in $L^2(X, \gamma)$.

The mapping $R_\gamma$ is extended to $X^*_\gamma$, setting

$$R_\gamma : X^*_\gamma \to (X^*)'$$

$$R_\gamma(f)(g) = \int_X f(x)\,[g(x) - a_\gamma(g)]\,\gamma(dx), \quad f \in X^*_\gamma,\ g \in X^*.$$

We put

$$\sigma(f) := \sqrt{R_\gamma(f)(f)} = \sqrt{\int_X (f(x) - a_\gamma(f))^2 \gamma(dx)}, \quad f \in X^*,$$

$$\sigma(g) := \sqrt{\int_X g(x)^2 \gamma(dx)} = \|g\|_{L^2(X,\gamma)}, \quad g \in X^*_\gamma.$$

The following proposition shows that $R_\gamma$ maps $X^*_\gamma$ into $X$.

**Proposition 7.** *For every $f \in X^*_\gamma$ there exists $y \in X$ such that $R_\gamma(f)(g) = g(y)$ for all $g \in X^*$. Therefore $R_\gamma(X^*_\gamma) \subset (X^*)^*$.*

*Remark* 2. We set $R_\gamma(f) = y$. In fact, thanks to Proposition 7, we can identify $R_\gamma(f)$ with the element $y \in X$ representing it, i.e. we shall write

$$R_\gamma(f)(g) = g\left(R_\gamma(f)\right), \forall g \in X^*.$$

## 1.3 The Cameron-Martin space

This section is devoted to introduce one of the most important tools for the study of infinite dimensional Gaussian analysis, the Cameron-Martin space. Proofs and more details can be found in [3].

**Definition 14.** Let $h \in X$ we define

$$|h|_H := \sup\{l(h) :\ l \in X^*,\ R_\gamma(l)(l) \leq 1\}$$

and

$$H := \{h \in X :\ |h|_{H(\gamma)} < +\infty\}.$$

The space $H = H(\gamma)$ is called the Cameron-Martin space.

*Remark* 3. The following identity holds

$$|h|_H = \sup_{\substack{l\in X^*,\\ \sigma(l)\neq 0}} \frac{l(h)}{\sigma(l)}. \tag{1.3}$$

Indeed

$$\begin{aligned}|h|_H &= \sup_{\substack{l\in X^*,\\ 0<\sigma(l)\leq 1}} l(h) \leq \sup_{\substack{l\in X^*,\\ 0<\sigma(l)\leq 1}} \frac{l(h)}{\sigma(l)} \leq \sup_{\substack{l\in X^*,\\ \sigma(l)\neq 0}} \frac{l(h)}{\sigma(l)}\\ &= \sup_{\substack{0\neq l\in X^*,\\ \widetilde{l}=l/\sigma(l)}} \widetilde{l}(h) \leq \sup_{\substack{l\in X^*,\\ 0<\sigma(l)\leq 1}} l(h) = |h|_H.\end{aligned}$$

**Lemma 1.** *A vector $h \in X$ belongs to the Cameron Martin space $H$ if and only if there exists $g \in X_\gamma^*$ with $h = R_\gamma(g)$. In this case,*

$$|h|_H = \|g\|_{L^2(X,\gamma)}.$$

If $h = R_\gamma(g)$, we put $\widehat{h} := g$. Moreover, the relationship determining $\widehat{h}$ is

$$f(h) = \int_X \widehat{h}(x)[f(x) - a_\gamma(f)]d\gamma(x), \quad f \in X^*.$$

On the Cameron-Martin space we define the inner product

$$\langle h, k\rangle_H := \langle \widehat{h}, \widehat{k}\rangle_{L^2(X,\gamma)} = \int_X \widehat{h}(x)\widehat{k}(x)\ \gamma(dx), \quad h, k \in H.$$

From Lemma 1 it follows that $R_\gamma(X_\gamma^*) = H$. Therefore the Cameron-Martin space $H$, equipped with the norm $|R_\gamma(f)|_H = \sqrt{R_\gamma(f)(f)}$ is a Hilbert space, and $R_\gamma$ is an isomorphism between $X_\gamma^*$ and $H$.

**Proposition 8.** *Let $g \in X_\gamma^*$, then the measure $\nu$ on $X$ given by the density*

$$\varrho(x) = \exp\left(g(x) - \frac{1}{2}\sigma(g)^2\right)$$

*with respect to the measure $\gamma$ is a Gaussian measure.*

Now we get a characterization of the Cameron-Martin space.

**Theorem 7.** *Let $\gamma$ be a Gaussian measure on $X$:*

1. *let $h \in X$ be a vector such that*

$$|h|_H = \sup\{f(h) : \ f \in X^*, \ R_\gamma(f)(f) \le 1\} = \infty$$

*then $\gamma_h := \gamma(\cdot - h)$ and $\gamma$ are mutually singular;*

2. *If $|h|_H < \infty$ then the measures $\gamma_h$ and $\gamma$ are equivalent and the corresponding Radon-Nikodym density is given by the expression*

$$\varrho_h(x) = \exp\left(\widehat{h}(x) - \frac{1}{2}|h|_H^2\right).$$

*In particular*

$$H = \{h \in X : \ \gamma_h \sim \gamma\} = \{h \in X : \ |h|_H < \infty\}.$$

**Proposition 9.** *Let $\gamma$ be a centered Gaussian measure on a separable Banach space $X$. The Cameron-Martin space $H$ coincides with the intersection of all linear subspaces of full $\gamma-$measure (and also with the intersection of all linear subspaces from $\mathcal{E}(X)$ of full $\gamma-$measure). In addition, $\gamma(H) = 0$ if $X^*_\gamma$ is infinite dimensional.*

**Lemma 2.** *There exists an orthonormal basis of $X^*_\gamma$ contained in $X^*$.*

*Proof.* Let $\{f_k\}_{k\in\mathbb{N}}$ be an orthonormal basis of $X^*_\gamma$. Each $f_k$ is the limit, in $L^2(X,\gamma)$ of a sequence $\{g_n^{(k)}\}_{k\in\mathbb{N}} \subset X^*$. On $\operatorname{span}\{g_n^{(k)}, \ n,k \in \mathbb{N}\}$ we construct an orthonormal basis $V$, by the Gram-Schmidt procedure. The linear combinations of the elements of such a basis approach every $g_n^{(k)}$ and hence every $f_k$ in $L^2(X,\gamma)$. Therefore, the linear space spanned by $V$ is dense in $X^*_\gamma$. □

## 1.4 Examples

Now we present two basic examples of infinite dimensional spaces: $\mathbb{R}^\infty$, and a Hilbert space. For both cases we describe the relevant Gaussian measures,the Reproducing Kernel $X^*_\gamma$ and the Cameron-Martin space $H$.

### 1.4.1 The product $\mathbb{R}^\infty$

We define $\mathbb{R}^\infty$ as the space of all the real sequences $x = \{x_n\}_{n\in\mathbb{N}}$. Let $\mathbb{R}^\infty_c$ be the subspace of finite sequences, that is the sequences $\{\xi_n\}_{n\in\mathbb{N}}$ that vanish eventually. Under the map

$$\begin{aligned} \mathbb{R}^\infty_c &\to \mathbb{R} \\ \{\xi_n\}_{n\in\mathbb{N}} &\mapsto \sum_{n=1}^{\infty} \xi_n x_n, \end{aligned}$$

where the sum is finite, $\mathbb{R}^\infty_c$ is isomorphic to the topological dual of $\mathbb{R}^\infty$.

$\mathbb{R}^\infty$ is separable, indeed the elements of $\mathbb{R}^\infty_c$ with rational entries are a countable dense set. The cylindrical $\sigma-$algebra $\mathcal{E}(\mathbb{R}^\infty)$ is generated by the sets of the form

$$\{x \in \mathbb{R}^\infty : \ (x_1, \dots, x_n) \in U, \ U \in \mathcal{B}(\mathbb{R}^n)\}.$$

We equip $\mathbb{R}^\infty$ with the product measure

$$\gamma := \bigotimes_{n\in\mathbb{N}} \gamma_1$$

where $\gamma_1$ is the standard Gaussian measure. As in the Banach space case, we say that a probability measure $\mu$ in $\mathbb{R}^\infty$ is Gaussian if for every $\xi \in \mathbb{R}^\infty_c$ the measure $\mu \circ \xi^{-1}$ is Gaussian on $\mathbb{R}$.

Theorem 5 is also true in $\mathbb{R}^\infty$ , see for instance [3, Theorem 2.2.4].

**Proposition 10.** *The countable product measure $\gamma$ on $\mathbb{R}^\infty$ is a centered Gaussian measure. Its Fourier transform is*

$$\widehat{\gamma}(\xi) = \exp\left\{-\frac{1}{2}\sum_{n=1}^{\infty} |\xi_n|^2\right\} = \exp\left\{-\frac{1}{2}\|\xi\|^2_{l^2}\right\}, \quad \xi \in \mathbb{R}^\infty_c.$$

*The Reproducing Kernel is*

$$X^*_\gamma = \left\{ f \in L^2(\mathbb{R}^\infty, \gamma) : \ f(x) = \sum_{n=1}^{\infty} \xi_n x_n, \ \{\xi_n\}_{n\in\mathbb{N}} \in l^2 \right\}$$

*and the Cameron-Martin space $H$ is $l^2$.*

*Proof.* First we compute the Fourier transform of $\gamma$. For $f(x) = \sum_{n=1}^{\infty} \xi_n x_n$, $x \in \mathbb{R}^\infty$ and $\xi \mathbb{R}^\infty_c$, we have

$$\begin{aligned} \widehat{\gamma}(f) &= \int_{\mathbb{R}^\infty} \exp\{if(x)\}\gamma(dx) = \int_{\mathbb{R}^\infty} \exp\left\{ i\sum_{n=1}^{\infty} \xi_n x_n \right\} \bigotimes_{n\in\mathbb{N}} \gamma_1(dx) \\ &= \prod_{n=1}^{\infty} \int_{\mathbb{R}} \exp\{i\xi_n x_n\}\,\gamma_1(dx_n) = \prod_{n=1}^{\infty} \exp\left\{-\frac{1}{2}|\xi_n|^2\right\} = \exp\left\{-\frac{1}{2}\|\xi\|^2_{l^2}\right\}. \end{aligned}$$

By Theorem 5, $\gamma$ is a Gaussian measure with mean $a_\gamma = 0$ and covariance $B_\gamma(\xi,\xi) = \|\xi\|_{l^2}^2$. Now we consider the Cameron-Martin space. Since $a_\gamma = 0$, for every $f = \{\xi_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty_c$ we have

$$\begin{aligned}
\|f\|^2_{L^2(X,\gamma)} &= \int_X \left|\sum_{n=1}^\infty \xi_n x_n\right|^2 \gamma(dx) = \int_X \left(\sum_{n=1}^\infty |\xi_n|^2|x_n|^2 + \sum_{i\neq j} \xi_i x_j\right)\gamma(dx) \\
&= \sum_{n=1}^\infty |\xi_n|^2 \int_{\mathbb{R}} |x_n|^2 \gamma_1(dx_n) + \sum_{i\neq j} \xi_j \int_{\mathbb{R}} x_j \gamma(dx_j)\ \xi_i \int_{\mathbb{R}} x_i\gamma(dx_i) \\
&= \sum_{n=1}^\infty |\xi_n|^2 = \|\xi\|^2_{l^2}.
\end{aligned}$$

Therefore $X^*_\gamma$ consists of all the functions $f(x) = \sum_{n=1}^\infty \xi_n x_n$ for $\{\xi_n\}_{n\in\mathbb{N}} \in l^2$.

If $h = \{h_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty$, then we have

$$\begin{aligned}
|h|_H &= \sup\left\{f(h) :\ f\in X^*,\ \|f\|_{L^2(X,\gamma)} \le 1\right\} \\
&= \sup\left\{\sum_{n=1}^\infty \xi_n h_n :\ \xi \in \mathbb{R}^\infty_c,\ \sum_{n=1}^\infty |\xi_n|^2 \le 1\right\} = \|h\|_{l^2},
\end{aligned}$$

Therefore $H$ and $l^2$ coincide. □

*Remark* 4. More generally

$$\mu = \bigotimes_{n=1}^\infty N(a_n,\lambda_n), \tag{1.4}$$

with $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, is a Gaussian measures on $\mathbb{R}^\infty$. Then

$$\widehat{\mu}(\xi) = \exp\left\{\sum_{n=1}^\infty \xi_n a_n - \frac{1}{2}\sum_{n=1}^\infty \lambda_n|\xi_n|^2\right\}, \quad \xi = \{\xi_n\}_{n\in\mathbb{N}} \in \mathbb{R}^\infty_c,$$

where all the sums contain a finite number of nonzero elements. Moreover if $\{a_n\}_{n\in\mathbb{N}} \in l^2$ and $\sum_{n=1}^\infty \lambda_n < \infty$ then $\mu$ is concentrated on $l^2$, that is, $\mu(l^2) = 1$. Indeed

$$\int_{\mathbb{R}^\infty} \sum_{n=1}^\infty |x_n|^2 \mu(dx) = \sum_{n=1}^\infty \int_{\mathbb{R}} |x_n|^2 N(a_n,\lambda_n)(dx_n) = \sum_{n=1}^\infty \lambda_n + \sum_{n=1}^\infty |a_n|^2 < \infty.$$

Then $\|x\|_{l^2} < \infty$ $\mu-$a.e. in $\mathbb{R}^\infty$, therefore $\mu(l^2) = 1$.

### 1.4.2 The Hilbert space

In this section we consider an infinite dimensional separable Hilbert space $X$. We denote by $\langle \cdot, \cdot \rangle_X$ its inner product and by $\| \cdot \|_X$ its norm. Moreover we identify $X^*$ with $X$ via the Riesz representation. We recall that the spectrum of a compact operator $A$ on $X$, $\sigma(A)$, is at most countable. If $\sigma(A)$ is infinite it consists of a sequence of eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ that can cluster only at 0.

Moreover, for a compact and self-adjoint operator $A$, there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $X$ consisting of eigenvectors, that is $Ae_n = \lambda_n e_n$ for all $n \in \mathbb{N}$ (see [4, Theorem 6.11]), and $A$ has the representation

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, e_n \rangle_X e_n, \quad x \in X,$$

Furthermore if $A$ is nonnegative, that is $\langle Ax, x \rangle_X \geq 0$ for every $x \in X$, then its eigenvalues are nonnegative and we may define the square root of $A$ by

$$A^{1/2}x = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle x, e_n \rangle_X e_n.$$

We recall that if a linear operator $B$ is such that

$$Bx = \sum_{n=1}^{\infty} a_n \langle x, e_n \rangle_X e_n, \ \{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R},$$

for some orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ with $\lim_{k\to\infty} a_n = 0$, then $B$ is compact. Indeed, if we set

$$B_k x := \sum_{n=1}^{k} a_n \langle x, e_n \rangle_X e_n$$

then

$$\|Bx - B_k x\|_X = \left\| \sum_{n=k+1}^{\infty} a_n \langle x, e_n \rangle_X e_n \right\|_X \leq \sup_{n>k} |a_n| \cdot \|x\|_X,$$

hence

$$\|B - B_k\|_{\mathcal{L}(X)} \leq \sup_{n>k} |a_n| \to 0, \ \text{as} n \to \infty.$$

Therefore $B$ is the limit in the operator norm of the sequence of finite rank operators, that is $B$ is compact. This shows that the operator $A^{1/2}$ is self-adjoint, and compact.

Now we define an important class of nonnegative self-adjoint operators.

**Definition 15.** A nonnegative self-adjoint operator $A \in \mathcal{L}(X)$ is of trace-class if there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $X$ such that

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_X < \infty.$$

Now we show that the sum of the series in Definition 15 does not dependent of the basis. Let $\{f_k\}_{k\in\mathbb{N}}$ be another orthonormal basis of $X$, then we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_X &= \sum_{n=1}^{\infty} \left\langle A\left(\sum_{k=1}^{\infty} \langle Af_k, f_k \rangle_X\right), \sum_{k=1}^{\infty} \langle Af_k, e_k \rangle_X \right\rangle_X \\
&= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \left\langle \langle e_n, f_k \rangle_X Af_k, \langle e_n, f_m \rangle_X f_m \right\rangle_X \\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \langle e_n, f_k \rangle_X \langle e_n, f_m \rangle_X\right) \langle Af_k, f_m \rangle_X \\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} \langle f_k, f_m \rangle_X \langle Af_k, f_m \rangle_X = \sum_{k=1}^{\infty} \langle Af_k, f_k \rangle_X
\end{aligned}$$

we recall that since $A$ is a nonnegative operator, then we can exchange the order of summation. We define the trace of $A$ as

$$\mathrm{Tr}(A) := \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_X$$

for any orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $X$.

**Proposition 11.** *If $A$ is a nonnegative self-adjoint trace-class operator then it is compact.*

Let $\gamma$ be a Gaussian measure in X. By Theorem 5 we have

$$\widehat{\gamma}(f) = \exp\left\{ia_\gamma(f) - \frac{1}{2}B_\gamma(f,f)\right\}, \quad f \in X^*,$$

where the linear function $a_\gamma : X^* \to \mathbb{R}$ and the bilinear symmetric function $B_\gamma : X^* \times X^* \to \mathbb{R}$ are continuous by Proposition 6. Then, there exists $a \in X$ and a self-adjoint $Q \in \mathcal{L}(X)$ such that $a_\gamma(f) = \langle f, a \rangle_X$ and $B_\gamma(f,g) = \langle Qf, g \rangle_X$ for every $f, g \in X^* = X$. Therefore,

$$\langle Qf, g \rangle_X = \int_X \langle f, x-a \rangle_X \langle g, x-a \rangle_X \gamma(dx), \quad f, g \in X, \tag{1.5}$$

and

$$\widehat{\gamma}(f) = \exp\left\{ i\langle f, a\rangle_X - \frac{1}{2}\langle Qf, f\rangle_X \right\}, \quad f \in X. \tag{1.6}$$

We denote by $N(a, Q)$ the Gaussian measure $\gamma$ whose Fourier transform is given by (1.6), $a$ is called the mean and $Q$ is called the covariance of $\gamma$.

With the following theorem we can characterize all Gaussian measures in $X$.

**Theorem 8.** *If $\gamma$ is a Gaussian measure on $X$ then its Fourier transform is given by* (1.6)*, where $a \in X$ and $Q$ is a self-adjoint nonnegative trace-class operator. Conversely, for every $a \in X$ and for every nonnegative self-adjoint trace-class operator $Q$, the function $\widehat{\gamma}$ in* (1.6) *is the Fourier transform of a Gaussian measure with mean $a$ and covariance operator $Q$.*

*Proof.* Let $\widehat{\gamma}$ be the Fourier transform of the Gaussian measure $\gamma$, given by (1.6). Then $a$ is the mean of $\gamma$ by definition. Since the bilinear form $B_\gamma$ is symmetric, then $Q$ is symmetric too. By it follows that $Q$ is nonnegative. Let $\{e_n\}_{n\in\mathbb{N}}$ an orthonormal basis of $X$, then we have

$$\sum_{n=1}^{\infty} \langle Qe_k, e_k\rangle_X = \sum_{n=1}^{\infty} \int_X \langle x - a, e_n\rangle_X^2 \gamma(dx) = \int_X \|x - a\|_X^2 \gamma(dx)$$

which is finite by estimate (1.2). This shows that $Q$ is a trace-class operator.

On the other hand, if $Q$ is a self-adjoint nonnegative trace-class operator, then $Q$ is given by

$$Qx = \sum_{n=1}^{\infty} \lambda_n \langle Qe_k, e_k\rangle_X e_n$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of $Q$, that is, $Qe_n = \lambda_n e_n$ for all $n \in \mathbb{N}$.

Let $\mu$ be the measure on $\mathbb{R}^\infty$ defined by (1.4) and its Fourier transform

$$\widehat{\mu}(\xi) = \exp\left\{ i\sum_{n=1}^{\infty} \xi_n a_n - \frac{1}{2}\lambda_n |\xi_n|^2 \right\}, \quad \xi \in \mathbb{R}_c^\infty,$$

where the series contains only a finite number of nonzero terms. We define $u : l^2 \to X$ by

$$u(y) := \sum_{n=1}^{\infty} y_n e_n.$$

By Remark 4 we can extend arbitrarily in $\mathbb{R}^\infty \setminus l^2$, since $\mu(\mathbb{R}^\infty \setminus l^2) = 0$.

If we define $\gamma := \mu \circ u^{-1}$ then $\gamma = N(a, Q)$. Indeed for $x \in \mathbb{R}^\infty_c$, setting $z = u(y)$ we have

$$\begin{aligned}
\widehat{\mu \circ u^{-1}}(x) &= \int_X \exp\left\{ i \left\langle z, \sum_{n=1}^\infty x_n e_n \right\rangle_X \right\} (\mu \circ u^{-1})(dx) \\
&= \int_{\mathbb{R}^\infty} \exp\left\{ i \sum_{n=1}^\infty y_n x_n \right\} \mu(dy) \\
&= \int_{l^2} \exp\left\{ i \sum_{n=1}^\infty y_n x_n \right\} \bigotimes_{n=1}^\infty N(a_n, \lambda_n)(dy) \\
&= \exp\left\{ i \sum_{n=1}^\infty x_n a_n - \frac{1}{2} \sum_{n=1}^\infty \lambda_n x_n^2 \right\} \\
&= \exp\left\{ i \langle x, a \rangle_X - \frac{1}{2} \langle Qx, x \rangle_X \right\}
\end{aligned}$$

By Theorem 5, $(\widehat{\mu \circ u^{-1}})$ is the Fourier transform of a unique Gaussian measure with mean $a$ and covariance $Q$. $\square$

*Remark* 5. In infinite dimensions the identity is not a trace-class operator, therefore the map $x \mapsto \exp\left\{-\frac{1}{2}\|x\|_X^2\right\}$ is not the characteristic function of a Gaussian measure on $X$.

Thanks to Theorem 8 we can find the best constant in Fernique Theorem.

**Proposition 12.** *Let $\gamma = N(a, Q)$ be a Gaussian measure on $X$ and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be the sequence of the eigenvalues of $Q$. If $\gamma$ is not a Dirac measure, then the integral*

$$\int_X \exp\left\{\alpha \|x\|_X^2\right\} \gamma(dx)$$

*is finite if and only if*

$$\alpha < \inf\left\{ \frac{1}{2\lambda_n} : \lambda_n > 0 \right\}.$$

*Proof.* Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $X$ such that $Qe_n = \lambda_n e_n$, for all $n \in \mathbb{N}$. Then, for $\alpha > 0$, we have

$$\int_X \exp\left\{\alpha\|x\|_X^2\right\}\gamma(dx) = \int_{\mathbb{R}^\infty} \exp\left\{\alpha\sum_{n=1}^{\infty} x_n^2\right\}\bigotimes_{n=1}^{\infty} N(a_n,\lambda_n)(dx)$$

$$= \prod_{n=1}^{\infty}\int_{\mathbb{R}} \exp\left\{\alpha x_n^2\right\} N(a_n,\lambda_n)(dx_n)$$

$$= \prod_{n\colon \lambda_n=0} \exp\left\{\alpha a_n^2\right\}\prod_{n\colon \lambda_n>0}\frac{1}{\sqrt{2\pi\lambda_n}}\int_{\mathbb{R}} \exp\left\{\alpha x_n^2\right\}\exp\left\{-\frac{1}{2\lambda_n}(x_n-a_n)^2\right\}dx_n.$$

If $\alpha \geq \frac{1}{2\lambda_n}$ for some $n \in \mathbb{N}$ then the integral with respect to $dx_n$ is infinite, and the function $x \mapsto \exp\left\{-\frac{1}{2}\|x\|_X^2\right\}$ does not belong to $L^1(X,\gamma)$. If $\alpha < \inf\left\{\frac{1}{2\lambda_n}:\ \lambda_n > 0\right\}$ then each integral is finite and we have

$$\int_{\mathbb{R}} \exp\left\{\alpha x_n^2\right\}\exp\left\{-\frac{1}{2\lambda_n}(x_n-a_n)^2\right\}dx_n = \exp\left\{\frac{\alpha a_n^2}{1-2\alpha\lambda_n}\right\}\frac{1}{\sqrt{1-2\alpha\lambda_n}}\}$$

for all $n \in \mathbb{N}$. Then

$$\int_X \exp\left\{\alpha\|x\|_X^2\right\}\gamma(dx)$$

$$= \exp\left\{\alpha\sum_{n\colon \lambda_n=0} a_n^2\right\}\exp\left\{\alpha\sum_{n\colon \lambda_n>0}\frac{a_n^2}{1-2\alpha\lambda_n}\right\}$$

$$\exp\left\{\sum_{n\colon \lambda_n>0}\log\left(\frac{1}{\sqrt{1-2\alpha\lambda_n}}\right)\right\} < \infty,$$

The convergence of last two series follows from $\sum_{n=1}^{\infty}\lambda_n < \infty$. $\square$

Now we characterize $X_\gamma^*$ and the Cameron-Martin space $H$. Hereafter, if $\gamma = N(a,Q)$ we fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of eigenvectors of $Q$, that is $Qe_n = \lambda_n e_n$ for all $n \in \mathbb{N}$. Moreover we set $x_n := \langle x, e_n\rangle_X$ for all $x \in X$ and $n \in \mathbb{N}$.

**Theorem 9.** *Let $\gamma = N(a,Q)$ be a non-degenerate Gaussian measure on $X$. The space $X_\gamma^*$ is*

$$X_\gamma^* = \left\{f : X \to \mathbb{R} :\ f(x) = \sum_{n=1}^{\infty}(x_n-a_n)z_n\lambda_n^{-1/2},\ z \in X\right\}$$

*and the Cameron-Martin space is the range of $Q^{1/2}$, that is,*

$$H = \left\{ x \in X : \sum_{n=1}^{\infty} x_n^2 \lambda_n^{-1} < \infty \right\}.$$

*For $h = Q^{1/2} z \in H$, we have*

$$\widehat{h}(x) = \sum_{n=1}^{\infty} (x_n - a_n) z_n \lambda_n^{-1/2}$$

*Proof.* Let $z \in X$. We set

$$f_k(z); = \sum_{n=1}^{k} (x_n - a_n) z_n \lambda_n^{-1/2}.$$

Then for every $k \in N$, $f_k \in X^*$. Moreover $\{f_k\}_{k \in \mathbb{N}}$ converges in $L^2(X, \gamma)$, since for $i > j$,

$$\|f_i - f_j\|_{L^2(X,\gamma)}^2 = \sum_{n=j+1}^{i} \lambda_n^{-1} z_n^2 \int_{\mathbb{R}} (x_n - a_n)^2 N(a_n, \lambda_n)(dx_n) = \sum_{n=j+1}^{i} z_n^2$$

and the limit function

$$f(x) = \sum_{n=1}^{\infty} (x_n - a_n) z_n \lambda_n^{-1/2}$$

satisfies

$$\|f\|_{L^2(X,\gamma)}^2 = \sum_{n=1}^{\infty} \lambda_n^{-1} z_n^2 \int_{\mathbb{R}} (x_n - a_n)^2 N(a_n, \lambda_n)(dx_n) = \|z\|_X^2. \tag{1.7}$$

We define

$$V := \left\{ f : X \to \mathbb{R} : \; f(x) = \sum_{n=1}^{\infty} (x_n - a_n) z_n \lambda_n^{-1/2}, \; z \in X \right\}.$$

Then $V$ is contained in the closure of $X^*$ in $L^2(X, \gamma)$, that is $V \subseteq X_\gamma^*$.

Now we show that $X_\gamma^* \subseteq V$. Let $g \in X_\gamma^*$ and let $\{w^{(k)}\}_{k \in \mathbb{N}} \subset X$ be a sequence such that

$$g_k(x) := \langle x - a, w^{(k)} \rangle_X$$

converges to $g$ in $L^2(X,\gamma)$. If we set $z^{(k)} := Q^{1/2}w^{(k)}$, then we have

$$g_k(x) = \sum_{n=1}^{\infty}(x_n - a_n)z_n^{(k)}\lambda_n^{-1/2},$$

and by (1.7),

$$\|z^{(i)} - z^{(j)}\|_X = \|g_i - g_j\|_{L^2(X,\gamma)}, \quad i,j \in \mathbb{N},$$

then $\{z^{(k)}\}_{k\in\mathbb{N}}$ ) is a Cauchy sequence, and it converges to some $z \in X$. By (1.7),

$$\begin{aligned}
&\int_X \left(g(x) - \sum_{n=1}^{\infty}(x_n - a_n)z_n\lambda_n^{-1/2}\right)^2 \gamma(dx) \\
&= \lim_{k\to\infty}\int_X \left(\sum_{n=1}^{\infty}(x_n - a_n)(z_n^{(k)} - z_n)\lambda_n^{-1/2}\right)^2 \gamma(dx) \\
&= \lim_{k\to\infty}\|z^{(k)} - z\|_X^2 = 0,
\end{aligned}$$

that is $g \in V$.

Now we consider the Cameron-Martin space. We recall that $H$ is the range of $R_\gamma : X_\gamma^* \to X$, moreover $R_\gamma f = h$ if and only if $\langle f, g - a_\gamma(g)\rangle_{L^2(X,\gamma)} = g(h)$ for all $g \in X^*$.

Let $f$ be an element of $X_\gamma^*$, that is, $f(x) = \sum_{n=1}^{\infty}(x_n - a_n)z_n\lambda_n^{-1/2}$, for some $z \in X$, and let $g \in X^*$, $g(x) = \sum_{k=1}^{\infty} g_k x_k$. Recalling that $a_\gamma(g) = g(a)$ we have

$$\begin{aligned}
\int_X f(x)(g(x) - a_\gamma(g))\gamma(dx) &= \int_X \sum_{n=1}^{\infty}(x_n - a_n)z_n\lambda_n^{-1/2} \cdot \sum_{k=1}^{\infty}(x_k - a_k)g_k\gamma(dx) \\
&= \int_X \sum_{n=1}^{\infty}(x_n - a_n)^2 z_n\lambda_n^{-1/2}g_n\gamma(dx) = \sum_{n=1}^{\infty} z_n\lambda_n^{-1/2}g_n.
\end{aligned}$$

This is equal to $g(h)$ for $h = \sum_{n=1}^{\infty} z_n\lambda_n^{-1/2}e_n$, that is $h = Q^{1/2}z$. By definition $R_\gamma f = Q^{1/2}z$, therefore $H = Q^{1/2}(X)$ and for $h = Q^{1/2}z$ we have $\widehat{h}(x) = \sum_{n=1}^{\infty}(x_n - a_n)z_n\lambda_n^{-1/2}$. $\square$

## 1.5 Basic properties of linear operators

This section is devoted to show some definition and properties of linear operators. As a rule, we refer to the textbooks like [4], [15], [19] .

Let $X$ be a Banach space on $\mathbb{C}$. Let $\mathcal{L}(X)$ be the set of linear continuous operator $A : X \to X$. We recall that a linear operator $A$ is continuous if and only if

$$\|A\|_{\mathcal{L}(X)} = \sup_{x\in X:\ \|x\|_X=1} \|Ax\|_X = \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_X}{\|x\|_X} < \infty.$$

Let $D(L)$ be a subspace of $X$ and let $L : D(L) \to X$ be a linear operator.

**Definition 16.** A linear operator $L : D(L) \subset X \to X$, is closed if it satisfies one of the following equivalent properties.

1. $\{x_n\}_{n\in\mathbb{N}} \in D(L),\ x_n \to x,\ Lx_n \to y \Rightarrow x \in D(L),\ y = Lx.$
2. The graph $\mathcal{G}(L) = \{(x, Lx) :\ x \in D(L)\}$ is closed in $X \times X$.
3. $(D(L), \|\cdot\|_{D(L)})$ is a Banach space for the graph norm

$$\|x\|_{D(L)} := \|x\|_X + \|Lx\|_X,\ x \in D(L).$$

**Definition 17.** Any subspace $D \subset D(L)$ dense in $(D(L), \|\cdot\|_{D(L)})$ is called core of $L$.

**Definition 18.** We call

$$\rho(L) := \{\lambda \in \mathbb{C} :\ (\lambda - L) : D(L) \to X \text{ is bijective}\}$$

the resolvent set and its complement, $\sigma(L) := \mathbb{C} \setminus \rho(L)$ the spectrum of $L$. For $\lambda \in \rho(L)$, the inverse

$$R(\lambda, L) := (\lambda I - L)^{-1}$$

is, by the closed graph theorem, a bounded operator on $X$ and will be called the resolvent (of $L$ at the point $\lambda$). Each $\lambda \in \mathbb{C}$ such that $\lambda I - A$ is not injective is called an eigenvalue, and each $0 \neq x \in D(A)$ satisfying $(\lambda I - A)x = 0$ is an eigenvector of $L$ (corresponding to $\lambda$).

It follows immediately from the definition that the identity

$$LR(\lambda, L) = \lambda R(\lambda, L) - I$$

holds for every $\lambda \in \rho(L)$. The next identity is the reason for many of the nice properties of the resolvent set $\rho(L)$ and the resolvent map

$$\rho(L) \ni \lambda \mapsto R(\lambda, L) \in \mathcal{L}(X).$$

**Proposition 13** (Resolvent Identity)**.** *For $\lambda, \mu \in \rho(L)$, one has*

$$R(\lambda, L) - R(\mu, L) = (\mu - \lambda) R(\lambda, L) R(\mu, L). \tag{1.8}$$

*Proof.* The definition of the resolvent implies

$$[\lambda R(\lambda, L) - L R(\lambda, L)] R(\mu, L) = R(\mu, L),$$

and

$$R(\lambda, L)[\mu R(\mu, L) - L R(\mu, L)] = R(\lambda, L).$$

If we subtract these equations and use the fact that $R(\lambda, L)$ and $R(\mu, L)$ commute, we get (1.8). □

We say that a family of operators in $\mathcal{L}(X)$, $(T(t))_{t\geq 0}$ is a semigroup if

1. $T(0) = I$,
2. $T(t+s) = T(t)T(s), \ \forall t, s \geq 0.$

A semigroup is called strongly continuous (or $C_0$) if

$$T(\cdot)x : [0, +\infty) \to X$$

is continuous for all $x \in X$.

**Definition 19.** We say that the semigroup $(T(t))_{t\geq 0}$ belongs to $\mathcal{G}(M, \omega)$ if and only if

$$\|T(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t} \ \forall t \geq 0.$$

If $M = 1$ and $\omega = 0$ we say that $(T(t))_{t\geq 0}$ is a contraction semigroup.

Let $(T(t))_{t\geq 0}$ a $C_0-$semigroup on $X$. We define the infinitesimal generator $L$ of $(T(t))_{t\geq 0}$ by setting

$$D(L) := \left\{ x \in X : \ \exists \lim_{t \to 0^+} \frac{T(t)x - x}{t} \in X \right\},$$

and

$$Lx := \lim_{t \to 0^+} \frac{T(t)x - x}{t}.$$

**Proposition 14.** *If $L$ is the infinitesimal generator of a $C_0-$semigroup, $(T(t))_{t\geq 0}$, then*

- *$L$ is a closed linear operator;*
- *$D(L)$ is dense in $X$;*

- *if* $x \in D(L)$ *then* $T(\cdot)x$ *is differentiable for all* $t \geq 0$ *and we have*

$$\frac{d}{dt}T(t)x = LT(t)x = T(t)Lx, \ t \geq 0;$$

- *if* $(T(t))_{t\geq 0}$ *belongs to* $\mathcal{G}(M,\omega)$ *then*

$$\{\lambda \in \mathbb{C} : Re\lambda > \omega\} \subset \rho(L),$$

*and*

$$\|R(\lambda, L)^n\|_{\mathcal{L}(X)} \leq \frac{M}{(Re\lambda - \omega)^n}, \ \forall n \in \mathbb{N}, \ \forall Re\lambda > \omega.$$

**Theorem 10** (Hille-Yosida)**.** *A linear operator* $L : D(L) \subset X \to X$ *is the infinitesimal generator of a* $C_0-$*semigroup,* $(T(t))_{t\geq 0} \in \mathcal{G}(M,\omega)$ *if and only if*

1. $\overline{D(L)} = X$
2. $(\omega, +\infty) \subset \rho(L)$ *and*

$$\|R(\lambda, L)^n\|_{\mathcal{L}(X)} \leq \frac{M}{(\lambda - \omega)^n}, \ \forall n \in \mathbb{N}, \ \forall \lambda > \omega.$$

Now we want to solve a differential equation. More precisely, we look at abstract (i.e., Banach-space-valued) linear initial value problems of the form

$$\begin{cases} u'(t) = Lu(t), & t > 0 \\ u(0) = x \end{cases} \tag{1.9}$$

where the independent variable t represents time, $u(\cdot)$ is a function with values in a Banach space $X$, $L : D(L) \subset X \to X$ is a linear operator, and $x \in X$ is the initial value.

**Definition 20.** A function $u : \mathbb{R}^+ \to X$ is called a strict solution of (1.9) if $u$ is continuously differentiable, $u(t) \in D(L)$ for all $t \geq 0$, and (1.9) holds.

**Proposition 15.** *Let* $(L, D(L))$ *be the generator of the strongly continuous semigroup* $(T(t))_{t\geq 0}$*. Then, for every* $x \in D(L)$*, the function*

$$\begin{aligned} u :& [0, +\infty) \to D(L) \\ & t \mapsto u(t) := T(t)x \end{aligned}$$

*is the unique strict solution of* (1.9)*.*

### 1.5.1 Sectorial operators and analytic semigroups

Now we introduce a class of semigroups enjoying a rather strong regularity property.

**Definition 21.** A closed linear operator $(L, D(L))$ with dense domain $D(L)$ in a Banach space $X$ is called sectorial if there exists $M > 0$, $\omega \in \mathbb{R}$ and $\theta \in (\pi/2, \pi)$ such that

1. $S_{\theta,\omega} := \{\lambda \in \mathbb{C} : \ |\arg \lambda - \omega| < \theta\} \subset \rho(L)$;
2. $\|R(\lambda, L)\|_{\mathcal{L}(X)} \leq \dfrac{M}{|\lambda - \omega|}, \ \forall \lambda \in S_{\theta,\omega}$.

Let $(L, D(L))$ be a sectorial operator. We define $T(0) := I$ and

$$T(t) := \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}+\omega} e^{\lambda t} R(\lambda, L) d\lambda, \ t \geq 0, \tag{1.10}$$

where $r > 0$, $\eta \in (\pi/2, \theta)$, and

$$\begin{aligned}\Gamma_{r,\eta} =& \{\lambda \in \mathbb{C} : \ \lambda = s e^{i\eta}, \ s \geq r\} \cup \{\lambda \in \mathbb{C} : \ \lambda = s e^{-i\eta}, \ s \geq r\} \\ &\cup \{\lambda \in \mathbb{C} : \ \lambda = r e^{i\phi}, \ |\phi| \leq \eta\}.\end{aligned}$$

**Proposition 16.** *Let $(L, D(L))$ be a sectorial operator (with $\omega$ and $\theta$ as in the previous definition). Then the maps $T(t)$, defined in* (1.10)*, satisfy the following properties.*

- *$T(t)x \in D(L^k)$ for all $t > 0$, $x \in X$ and $k \in \mathbb{N}$. Moreover if $x \in D(L^k)$ then $L^k T(t)x = T(t)L^k x$.*
- *$(T(t))_{t \geq 0}$ is a semigroup.*
- *For all $k \in \mathbb{N} \cup \{0\}$ there exists $M_k > 0$ such that*

$$\|(L - \omega I)^k T(t)\|_{\mathcal{L}(X)} \leq \frac{M_k}{t^k} e^{\omega t}, \ \forall t > 0$$

- *the function $t \mapsto T(t)$ belongs to $C^\infty((0, +\infty); \mathcal{L}(X))$ and*

$$\frac{d^k}{dt^k} T(t) = L^k T(t), \ \forall t > 0, \ \forall k \in \mathbb{N}.$$

- *the function $t \mapsto T(t)$ admits an analytic extension in the sector*

$$\{z \in \mathbb{C} : \ \arg(z) < \theta - \pi/2\},$$

*and*

$$T(z) := \frac{1}{2\pi i} \int_{\Gamma_{r,\eta}+\omega} e^{\lambda z} R(\lambda, L) d\lambda.$$

- *$L$ is the infinitesimal generator of $(T(t))_{t\geq 0}$.*

**Definition 22.** We say that $(T(t))_{t\geq 0}$, defined in (1.10), is an analytic semi-group.

**Theorem 11.** *Let $X = H$ be a Hilbert space. Let $L : D(L) \subset H \to H$ be a linear operator such that*

1. *$L$ is self-adjoint;*
2. *$L$ is dissipative.*

*Then $L$ is sectorial, $\sigma(L) \subset (-\infty, 0]$ and, for all $\theta \in (\pi/2, \pi)$, there exists $M_\theta > 0$ such that*

$$\|R(\lambda, L)\|_{\mathcal{L}(H)} \leq \frac{M_\theta}{|\lambda|}, \ \forall \lambda \in S_{0,\theta}.$$

Let $L : D(L) \subset X \to X$ be a sectorial operator. We consider the Cauchy problem

$$\begin{cases} u'(t) = Lu(t), & t \in (0, T] \\ u(0) = x \end{cases} \tag{1.11}$$

**Definition 23.** For $x \in X$, we say that a function $u \in C^1((0, T]; X) \cap C([0, T]; D(L)) \cap C([0, T], X)$ is a classical solution of (1.11) in $[0, T]$ if $u'(t) = Lu(t)$ and $u(0) = x$.

**Theorem 12.** *Let $x \in X$. A function $u$ is a classical solution of* (1.11) *if and only if*

$$u(t) = T(t)x.$$

We recall the Lax-Milgram lemma.

**Proposition 17** (Lax-Milgram)**.** *Let $H$ be a Hilbert real space, and let $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form, that is there exist two constants $C_1, C_2 > 0$ such that*

$$|a(u, v)| \leq C_1 \|u\|_H \|v\|_H, \ \forall u, v \in H,$$
$$a(u, u) \geq C_2 \|u\|_H^2, \ \forall u \in H.$$

*Then for all linear continuous functional $F : H \to \mathbb{R}$ there exists a unique $u \in H$ such that*

$$a(u, v) = F(v), \ \forall v \in H,$$

*and there exists a constant $C > 0$ such that*

$$\|u\|_H \leq C\|F\|_{\mathcal{L}(X;\mathbb{R})}.$$

*By using the representation theorem of continuous linear functional, we have that for every $\varphi \in H$ there exists a unique $u \in H$ such that*

$$a(u, v) = \langle \varphi, v \rangle_H, \ \forall v \in H,$$

*moreover*

$$\|u\|_H \leq C\|\varphi\|_H.$$

## 1.6 Sobolev Spaces

In this section we introduce the notions of gradient and Sobolev spaces. First we define the cylindrical functions, and we give two definitions of differentiability. We recall that, for $X$ and $Y$ Banach spaces, $\mathcal{L}(X, Y)$ is the set of all linear continuous operator from $X$ to $Y$. In particular $\mathcal{L}(X, \mathbb{R}) = X^*$ and we put $\mathcal{L}(X) := \mathcal{L}(X, X)$.

**Definition 24.** We denote by $\mathcal{F}C_b^k(X)$, $k \in \mathbb{N}$, the set of cylindrical bounded $k$ times differentiable function, that is

$$\mathcal{F}C_b^k(X) := \{f : X \to \mathbb{R} : f(x) = \varphi(l_1(x), \ldots, l_d(x)), \ l_i \in X^*, \ \varphi \in C_b^k(\mathbb{R}^d)\}. \tag{1.12}$$

**Definition 25.** Let $X$ be a separable Banach space endowed with a centered non-degenerate Gaussian measure $\gamma$, and let $H$ be its Cameron-Martin space.

**(i)** We say that a function $f : X \to \mathbb{R}$ is Fréchet differentiable at $x \in X$ if there exists $A \in \mathcal{L}(X, \mathbb{R})$ such that

$$f(x + y) - f(x) - Ay = o(\|y\|), \quad y \to 0 \text{ in } X. \tag{1.13}$$

In this case, $A$ is unique, and we set $f'(x) := A$.

**(ii)** We say that a function $f : X \to \mathbb{R}$ is $H-$Fréchet differentiable at $x \in X$ if there exists $A_0 \in \mathcal{L}(H, \mathbb{R})$ such that

$$f(x + h) - f(h) - A_0 h = o(|h|_H), \quad h \to 0 \text{ in } H. \tag{1.14}$$

Most of the properties of differentiable functions can be proved as in the case $X = \mathbb{R}^d$. For instance, if $f$ is Fréchet differentiable at $x$ it is continuous at $x$. Moreover, for all $v \in X$ the directional derivatives

$$\frac{\partial f}{\partial v}(x) := \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t}$$

exists and is equal to $f'(x)(v)$. If $f \in \mathcal{F}C_b^1(X)$, then $f$ is differentiable at every $x$ and

$$f'(x)(v) = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial \xi_i}((l_1(x), \dots, l_d(x)))l_i(v), \quad x, v \in X.$$

If $f : X \to \mathbb{R}$ is $H-$differentiable at $x$, the operator $A_0$ in the definition is called $H-$derivative of $f$ at $x$, and there exists a unique $y \in H$ such that $A_0(h) = \langle h, y \rangle_H$ for every $h \in H$. We put

$$\nabla_H f(x) := y.$$

**Proposition 18.** *If $f : X \to \mathbb{R}$ is Fréchet differentiable at $x \in X$, then is $H-$Fréchet differentiable at $x$, with $H-$derivative given by $h \mapsto f'(x)(h)$ for all $h \in H$. In particular, we have*

$$\nabla_H f(x) = R_\gamma f'(x).$$

If $f$ is $H-$differentiable at $x$, the directional derivative $\frac{\partial f}{\partial v}(x)$ exists for every $v \in H$, and

$$\frac{\partial f}{\partial v}(x) = \langle \nabla_H f(x), v \rangle_H$$

holds. Let $\{h_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $H$, we set

$$\partial_i f(x) := \frac{\partial f}{\partial h_i}(x), \quad i \in \mathbb{N}.$$

Therefore we get

$$\nabla_H f(x) = \sum_{n=1}^{\infty} \partial_i f(x) h_i$$

where the series converges in $H$.

*Remark* 6. Let $X$ be a Hilbert space. In general gradient and $H-$gradient do not coincide. Let $f$ be differentiable function at $x$, if $\gamma = N(0, Q)$, then we have $\nabla_H f(x) = Q \nabla f(x)$. Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $X$ such that is $Qe_n = \lambda_n e_n$, then a canonical orthonormal basis of $H$ is $\{h_n\}_{n \in \mathbb{N}}$, with $h_n = \sqrt{\lambda_n} e_n$, and

$$\partial_n f(x) = \sqrt{\lambda_n} \frac{\partial f}{\partial e_n}(x)$$

holds for $n \in \mathbb{N}$.

### 1.6.1 Sobolev spaces of order 1

Let $\gamma$ be a centered Gaussian measure on a separable Banach space $X$. First we define integrals and $L^p$ spaces of $Y-$valued functions where $Y$ is a Hilbert space. We refer to the books [13] for a detailed treatment. We set

$$L^p(X,\gamma;Y) = \Big\{ f : X \to Y, \text{ measurable and such that}$$
$$\|f\|^p_{L^p(X,\gamma;Y)} = \int_X \|f(x)\|^p_Y \gamma(dx) < \infty \Big\} \Big/ \sim$$

where $f \sim g$ if and only if

$$\gamma(x \in X:\ f(x) \neq g(x)) = 0.$$

Now we show an integration formula for $\mathcal{F}C^1_b$ functions.

**Proposition 19.** *For any $f \in \mathcal{F}C^1_b(X)$ and $h \in H$ we have*

$$\int_X \frac{\partial f}{\partial h}(x)\gamma(dx) = \int_X f(x)\widehat{h}(x)\gamma(dx). \tag{1.15}$$

*Moreover for any $f, g \in \mathcal{F}C^1_b(X)$ and $h \in H$ we have*

$$\int_X \frac{\partial f}{\partial h}(x)\, g(x)\, \gamma(dx) = -\int_X f(x)\frac{\partial g}{\partial h}(x)\, \gamma(dx) + \int_X f(x)\, g(x)\, \widehat{h}(x)\, \gamma(dx).$$

We recall the definition of a closable operator. Let $E$, $F$ be Banach spaces and let $A : D(A) \subset E \to F$ be a linear operator. $A$ is called closable (in $E$) if there exists a linear operator $\bar{A} : D(\bar{A}) \subset E \to F$ whose graph is the closure of the graph of $A$ in $E \times F$. Equivalently, $A$ is closable if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ such that

$$\lim_{n\to\infty} x_n = 0 \text{ in } E, \text{ and } \lim_{n\to\infty} Ax_n = 0 \text{ in } F$$

it follows that $z = 0$. If $A$ is closable, the domain of the closure $A$ of $\bar{A}$ is the set

$$D(\bar{A}) = \Big\{ x \in E :\ \exists \{x_n\}_{n\in\mathbb{N}} \subset D(A),\ \lim_{n\to\infty} x_n = x,\ Ax_n \text{ converges in } F \Big\}$$

and for $x \in D(\bar{A})$ we have

$$\bar{A}x = \lim_{n\to\infty} Ax_n.$$

**Proposition 20.** *For every $1 \le p < \infty$, the operator $\nabla_H : \mathcal{F}C_b^1(X) \to L^p(X,\gamma;H)$ is closable in $L^p(X,\gamma)$.*

**Definition 26.** For each $1 \le p < \infty$, we denote by $W^{1,p}(X,\gamma)$ the domain of the closure of $\nabla_H : \mathcal{F}C_b^1(X) \to L^p(X,\gamma;H)$ in $L^p(X,\gamma)$, still denoted by $\nabla_H$. Hence $f \in L^p(X,\gamma)$ belongs to $W^{1,p}(X,\gamma)$ if and only if there exists a sequence of functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}C_b^1(X)$ such that $f_n \to f$ in $L^p(X,\gamma)$ and $\nabla_H f_n$ converges in $L^p(X,\gamma;H)$. In this case we set $\nabla_H f = \lim_{n\to\infty} \nabla_H f_n$.

The space $W^{1,p}(X,\gamma)$ is complete with the graph norm

$$\begin{aligned}\|f\|_{W^{1,p}(X,\gamma)} &:= \|f\|_{L^p(X,\gamma)} + \|\nabla_H f\|_{L^p(X,\gamma;H)} \\ &= \left(\int_X |f|^p d\gamma\right)^{1/p} + \left(\int_X |\nabla_H f|_H^p d\gamma\right)^{1/p}.\end{aligned}$$

$W^{1,2}(X,\gamma)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{W^{1,2}(X,\gamma)} := \int_X f\, g\, d\gamma + \int_X \langle \nabla_H f, \nabla_H g\rangle_H \, d\gamma,$$

which gives an equivalent norm.

For any orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $H$ and any $f \in W^{1,p}(X,\gamma)$ we have

$$\partial_n f(x) = \langle \nabla_H f(x), h_n\rangle_H, \quad n \in \mathbb{N}.$$

In general, for $h \in H$ we have

$$\partial_h f(x) = \langle \nabla_H f(x), h\rangle_H.$$

Therefore

$$\int_X |\nabla_H f|_H^p = \int_X \left(\sum_{n=1}^\infty \langle \nabla_H f, h_n\rangle^2\right)^{p/2} d\gamma = \int_X \left(\sum_{n=1}^\infty (\partial_n f)^2\right)^{p/2} d\gamma.$$

We give some properties of the spaces $W^{1,p}(X,\gamma)$.

**Proposition 21.** *Let $1 < p < \infty$. Then*

1. *the integration formula* (1.15) *holds for every $f \in W^{1,p}(X,\gamma)$ and $h \in H$;*
2. *if $f \in W^{1,p}(X,\gamma)$ and $g \in W^{1,q}(X,\gamma)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{s} \le 1$, then $fg \in W^{1,s}(X,\gamma)$ and*

$$\nabla_H(fg) = \nabla_H f\, g + f\, \nabla_H g;$$

*3. the space $W^{1,p}(X,\gamma)$ is reflexive;*

*4. if $f_n \to f$ in $L^p(X,\gamma)$ and $\sup_{n\in\mathbb{N}} \|f\|_{W^{1,p}(X,\gamma)} < \infty$ then $f \in W^{1,p}(X,\gamma)$.*

*Remark* 7. Let $X$ be a Hilbert space and let $\gamma = N(0,Q)$ with $Q$ positive definite. For every $f \in \mathcal{F}C_b^1(X)$ we have $\nabla_H f(x) = Qf'(x)$, then

$$|\nabla_H f(x)|_H^2 = \langle Q^{-1/2}Qf'(x), Q^{-1/2}Qf'(x)\rangle_X = \|Q^{1/2}f'(x)\|_X^2,$$

and

$$\|f\|_{W^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \left(\int_X \|Q^{1/2}f'(x)\|_X^2\,\gamma(dx)\right)^{1/p}.$$

Let us fix any orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of X consisting of eigenvectors of $Q$, $Qe_n = \lambda_n e_n$. Then a canonical basis of $H$ is $\{h_n\}_{n\in\mathbb{N}}$ with $h_n = \sqrt{\lambda_n}e_n$, hence

$$\partial_i f(x) = \sqrt{\lambda_n}\frac{\partial f}{\partial e_n},$$

and

$$\|f\|_{W^{1,p}(X,\gamma)} = \|f\|_{L^p(X,\gamma)} + \left(\int_X \left(\sum_{n=1}^{\infty} \lambda_n \left(\frac{\partial f}{\partial e_n}(x)\right)^2\right)^{p/2} \gamma(dx)\right)^{1/p}.$$

### 1.6.2 Sobolev spaces of order 2

First we recall some basic definition and properties of Hilbert-Schmidt operators. We refer to the book [3, A.2] for a detailed treatment. Hereafter $X$ is a separable Banach space endowed with a non-degenerate centered Gaussian measure and $H$ is the Cameron-Martin space associated to $\gamma$.

**Definition 27.** We say that $A \in \mathcal{L}(H)$ is a Hilbert-Schmidt operator if there exists an orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $H$ such that

$$\sum_{n=1}^{\infty} |Ah_n|_H^2 < \infty. \tag{1.16}$$

**Proposition 22.** *If the series in* (1.16) *converges for some orthonormal basis of $H$, then it also converges for every orthonormal basis $H$ and its sum does not depend on the chosen basis.*

We denote by $\mathcal{H} = \mathcal{H}(H)$ the class of all Hilbert-Schmidt operators from $H$ to $H$. Moreover, for $A \in \mathcal{H}$ we put

$$\begin{aligned} \|A\|_{\mathcal{H}}^2 &= \sum_{n=1}^{\infty} |Ah_n|_H^2 = \sum_{n=1}^{\infty} \left\| \sum_{i=1}^{\infty} \langle A(h_n), h_i\rangle_H \, h_i \right\|_H^2 \\ &= \sum_{n,i=1}^{\infty} \left(\langle A(h_n), h_i\rangle_H\right)^2 \end{aligned}$$

*Remark* 8. If $X = \mathbb{R}^d$, then $A$ is associated to the matrix $a_{ij} = \langle A(\xi_j), \xi_i\rangle_{\mathbb{R}^d}$, $\xi \in \mathbb{R}^d$ and

$$\|A\|_{\mathcal{H}(\mathbb{R}^d)}^2 = \sum_{i,j=1}^{\infty} (a_{ij})^2$$

Let $h \in H$. Then we have

$$\begin{aligned} \|A(h)\|_H^2 &= \left\| \sum_{n=1}^{\infty} A(\langle h, h_n\rangle_H \, h_n) \right\|_H^2 = \left\| \sum_{n=1}^{\infty} \langle h, h_n\rangle_H \, A(h_n) \right\|_H^2 \\ &\leq \left( \sum_{n=1}^{\infty} \langle h, h_n\rangle_H \|A(h_n)\|_H \right)^2 \leq \sum_{n=1}^{\infty} \langle h, h_n\rangle_H^2 \cdot \sum_{n=1}^{\infty} \|A(h_n)\|_H^2 \\ &= \|h\|_H^2 \cdot \|A\|_{\mathcal{H}}. \end{aligned}$$

Therefore

$$\|A(h)\|_H^2 \leq \|h\|_H^2 \cdot \|A\|_{\mathcal{H}}$$

for all $h \in H$.

*Remark* 9. 1. It is obvious that the identity on $H$ is not a Hilbert-Schmidt operator.

2. If $w, v \in H^n$, $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, then the operator defined by

$$A(h) := \sum_{i=1}^{n} \langle w_i, h\rangle v_i$$

belongs to $\mathcal{H}$, and

$$\|A\|_{\mathcal{H}} \leq \|v\|_{H^n} \|w\|_{H^n}.$$

3. $\mathcal{H}$ is a Hilbert space equipped with the inner product

$$\langle A_1, A_2\rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \langle A_1(h_n), A_2(h_n)\rangle_H.$$

Let $f : X \to \mathbb{R}$ be Fréchet differentiable in a neighborhood of $x$. We suppose that the function

$$\begin{aligned} X &\to X^* \\ y &\mapsto f'(y) \end{aligned}$$

is Fréchet differentiable at $x$. Then there exist $L \in \mathcal{L}(X, X^*)$ such that:

$$\|f'(x+v) - f'(x) - Lv\|_{X^*} = o(\|v\|_X), \quad \text{as } v \to 0.$$

In this case we say that $f$ is twice Fréchet differentiable at $x$ and we put

$$f''(x) := L.$$

Let $f : X \to \mathbb{R}$ be $H-$differentiable at each point of $X$. We say that $f$ is twice $H-$differentiable at $x$ if there exists $L_H \in \mathcal{L}(H)$ such that

$$\|\nabla_H f(x+h) - \nabla_H f(x) - L_H h\|_H = o(|h|_H), \quad \text{as } h \to 0.$$

If $f$ is twice Fréchet differentiable at $x$, then it is twice $H-$differentiable at $x$ and

$$\nabla_H f(x) = R_\gamma f'(x), \quad L_H = R_\gamma L.$$

Let $f \in \mathcal{F}C_b^2(X)$. Then $f(x) = \varphi(l_1(x), \dots, l_d(x))$, $l_i \in X^*$ for $i = 1, \dots, d$, and $\varphi \in C_b^2(\mathbb{R}^d)$. Therefore $f$ is twice Fréchet differentiable at any point $x$, and

$$L(v)(w) = \sum_{i,j=1}^{d} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}(l_1(x), \dots, l_d(x))\, l_i(v)\, l_j(w)$$

holds for every $v, w \in X$.

Moreover, for all $h, k \in H$, we have

$$\langle L_H(h), k\rangle_H = \sum_{i,j=1}^{d} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}(l_1(x), \dots, l_d(x)) \langle R_\gamma l_i, h\rangle_H \langle R_\gamma l_j, k\rangle_H$$

Now we show that $L_H$ is an Hilbert-Schmidt operator. Let $\{h_n\}_{h\in\mathbb{N}}$ be an orthonormal basis of $H$. Then

$$\sum_{l,k=1}^{\infty} \langle L_H(h_l), h_k \rangle_H^2 = \sum_{l,k=1}^{\infty} \left[ \sum_{i,j=1}^{d} \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}(l_1(x), \dots, l_d(x)) \langle R_\gamma l_i, h_l \rangle_H \langle R_\gamma l_j, h_k \rangle_H \right]^2$$

$$\le \sum_{l,k=1}^{\infty} \left[ \sum_{i,j=1}^{d} \left( \frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j}(l_1(x), \dots, l_d(x)) \right)^2 \cdot \sum_{i=1}^{d} \langle R_\gamma l_i, h_l \rangle_H^2 \cdot \sum_{j=1}^{d} \langle R_\gamma l_j, h_k \rangle_H^2 \right]$$

$$= \|D^2 \varphi(l_1(x), \dots, l_d(x))\|_{\mathcal{H}(\mathbb{R}^d)} \cdot \sum_{l=1}^{\infty} \sum_{i=1}^{d} \langle R_\gamma l_i, h_l \rangle_H^2 \cdot \sum_{k=1}^{\infty} \sum_{j=1}^{d} \langle R_\gamma l_j, h_k \rangle_H^2$$

$$= \|D^2 \varphi(l_1(x), \dots, l_d(x))\|_{\mathcal{H}(\mathbb{R}^d)} \cdot \sum_{i=1}^{d} |R_\gamma l_i|_H^2 \cdot \sum_{j=1}^{d} |R_\gamma l_j|_H^2 < \infty.$$

**Proposition 23.** *For every $1 \le p < \infty$, the operator $(\nabla_H, L_H) : \mathcal{F}C_b^2(X) \to L^p(X,\gamma;H) \times L^p(X,\gamma;\mathcal{H})$ is closable in $L^p(X,\gamma)$.*

**Definition 28.** For each $1 \le p < \infty$, we denote by $W^{2,p}(X,\gamma)$ the domain of the closure of $(\nabla_H, L_H)$ in $L^p(X,\gamma)$, and we denote the closure of $(\nabla_H, L_H)$ by $(\nabla_H, D_H^2)$.

$W^{2,p}(X,\gamma)$ is a Banach space with the graph norm

$$\|f\|_{W^{2,p}(X,\gamma)}^p = \int_X \left[ |f(x)|^p + |\nabla_H f(x)|_H^p + \|D_H^2 f(x)\|_{\mathcal{H}}^p \right] \gamma(dx)$$

$W^{2,2}(X,\gamma)$ is a Hilbert space with the inner product

$$\langle f, g \rangle_{W^{2,2}(X,\gamma)} := \int_X f\, g\, d\gamma + \int_X \langle \nabla_H f, \nabla_H g \rangle_H \, d\gamma + \int_X \langle D_H^2 f, D_H^2 g \rangle_{\mathcal{H}} \, d\gamma,$$

which gives an equivalent norm.

For any orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $H$ and any $f \in W^{2,p}(X,\gamma)$ we have

$$\frac{\partial^2 f}{\partial h_l \partial h_k}(x) = \langle D_H^2 f(x) h_l, h_k \rangle_H.$$

*Remark* 10. Let $X$ be a Hilbert space and let $\gamma = N(0,Q)$ with $Q$ positive definite. As in Remark 7, we fix an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ of $X$ consisting of eigenvectors of $Q$, $Qe_n = \lambda_n e_n$, and $\{h_n\}_{n\in\mathbb{N}}$ with $h_n = \sqrt{\lambda_n} e_n$ is a basis of $H$. Then for all $f \in \mathcal{F}C_b^2(X)$ we have

$$\|D_H^2 f(x)\|_{\mathcal{H}}^2 = \sum_{l,k=1}^{\infty} \left( \frac{\partial^2 f}{\partial h_l \partial h_k}(x) \right)^2 = \sum_{l,k=1}^{\infty} \lambda_l\, \lambda_k \left( \frac{\partial^2 f}{\partial e_l \partial e_k}(x) \right)^2.$$

Therefore

$$\|f\|_{W^{2,p}(X,\gamma)} = \|f\|_{W^{1,p}(X,\gamma)} + \left( \int_X \left( \sum_{l,k=1}^{\infty} \lambda_l \, \lambda_k \left( \frac{\partial^2 f}{\partial e_l \partial e_k}(x) \right)^2 \right)^{p/2} \gamma(dx) \right)^{1/p}.$$

### 1.6.3 Sobolev space on convex set and traces

Let $\gamma$ be a non-degenerate centered Gaussian measure on a separable Banach space $X$, and let $H$ be the Cameron-Martin space related to $\gamma$. Let $\Omega$ be an open convex set of $X$, with $\gamma(\Omega) > 0$. In this section we fix any orthonormal basis $\{h_n\}_{n\in\mathbb{N}}$ of $H$. First we define the Sobolev space on $\Omega$, then, under some assumption of regularity of $\Omega$, we introduce the trace operator. We refer to [7] for a detailed treatment and proofs.

Since $\Omega$ is convex, we fix any $x_0 \in \Omega$ and we define the Minkowski function

$$\mathfrak{m}(x) := \inf \{\lambda \geq 0 : x - x_0 \in \lambda(\Omega - x_0)\}$$

which is Lipschitz continuous. Then $\Omega = \{x \in X : \ g(x) < 0\}$, where

$$g(x) := \mathfrak{m}(x) - 1 \tag{1.17}$$

belongs $W^{1,p}(X,\gamma)$ for every $p > 1$. In particular, since $g$ is continuous, we have $\partial\Omega = g^{-1}(0)$.

Let $\mathcal{F}C_b^k(X)_{|\Omega}$ be the set of the restrictions to $\Omega$ of the elements of $\mathcal{F}C_b^k(X)$.

**Proposition 24.** *For every $p > 1$, the operator $\nabla_H : \mathcal{F}C_b^1(X)_{|\Omega} \to L^p(\Omega,\gamma;H)$ is closable in $L^p(\Omega,\gamma)$. Moreover the operator*

$$\begin{aligned} (\nabla_H, D_H^2) : \mathcal{F}C_b^2(X)_{|\Omega} &\to L^p(\Omega,\gamma;H) \times L^p(\Omega,\gamma;\mathcal{H}) \\ \varphi &\mapsto (\nabla_H \varphi_{|\Omega}, D_H^2 \varphi_{|\Omega}) \end{aligned}$$

*is closable in $L^p(\Omega,\gamma)$. The closure is still denoted by $(\nabla_H, D_H^2)$.*

*Proof.* Let $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{F}C_b^1(X)$ be such that $\varphi_{k|\Omega} \to 0$ in $L^p(\Omega,\gamma)$ and $\nabla_H \varphi_{k|\Omega} \to F$ in $L^p(\Omega,\gamma;H)$, $k \to \infty$. We claim that

$$\int_\Omega \langle F, h_i \rangle_H \psi \, d\gamma = 0$$

for all $\psi \in L^{p'}(\Omega,\gamma)$ and $i \in \mathbb{N}$. We remark that the restrictions to $\Omega$ of the elements of $\mathcal{F}C_b^1(X)$ are dense in $L^{p'}(\Omega,\gamma)$ (this is a consequence Corollary 2). Then it is sufficient to prove that

$$\int_\Omega \langle F, h_i \rangle_H \psi \, d\gamma = 0, \quad \forall \psi \in \mathcal{F}C_b^1(X). \tag{1.18}$$

Let $\psi \in \mathcal{F}C_b^1(X)$. Let $\theta \in C_c^\infty(\mathbb{R})$ be such that $\theta(r) = 0$ for $r \geq -1$, $\theta(r) = 1$ for $r \leq -2$, and we set

$$\theta_n(r) := \theta(nr), \quad \psi_n(x) = \psi(x)\theta_n(g(x)),\ n \in \mathbb{N},\ x \in X.$$

By dominated convergence the sequence $\{\psi_{n|\Omega}\}_{n\in\mathbb{N}}$ goes to $\psi_{|\Omega}$ in $L^{p'}(\Omega, \gamma)$ as $n \to \infty$. Then

$$\int_\Omega \langle F, h_i\rangle_H \psi \, d\gamma = \lim_{n\to\infty} \int_\Omega \langle F, h_i\rangle_H \psi_n \, d\gamma.$$

We remark that $\psi_n$ vanishes in $g^{-1}([-1/n, +\infty])$ and

$$\nabla_H \psi_n = \nabla_H \psi \ \theta_n \circ g + \psi \theta_n' \circ g \ \nabla_H g,$$

then $\psi \in W^{1,r}(X, \gamma)$ for all $r > 1$. Therefore $\varphi_k \psi_n \in W^{1,r}(X, \gamma)$ for all $r > 1$, and $k \in \mathbb{N}$. By applying the integration by parts formula (1.15) we have

$$\int_\Omega \partial_i(\varphi_k \psi_n) d\gamma = \int_\Omega \widehat{h}_i \, \varphi_k \, \psi_n \, d\gamma,$$

then

$$\int_\Omega \psi_n \, \partial_i \varphi_k \, d\gamma = -\int_\Omega \partial_i \psi_n \, \varphi_k \, d\gamma + \int_\Omega \widehat{h}_i \, \varphi_k \, \psi_n \, d\gamma. \tag{1.19}$$

We recall that

$$\lim_{k\to\infty} \int_\Omega \partial_i \psi_n \, \varphi_k \, d\gamma = 0, \quad \lim_{k\to\infty} \int_\Omega \widehat{h}_i \, \varphi_k \, \psi_n \, d\gamma = 0,$$

and

$$\lim_{k\to\infty} \int_\Omega \psi_n \, \partial_i \varphi_k \, d\gamma = \int_\Omega \psi_n \langle F, h_i\rangle_H d\gamma.$$

Then $\int_\Omega \psi_n \langle F, h_i\rangle_H d\gamma = 0$ for all $n \in \mathbb{N}$ and (1.18) holds.

Now let $\{\varphi_k\}_{k\in\mathbb{N}} \subset \mathcal{F}C_b^2(X)$ be such that $\varphi_{k|\Omega} \to 0$ in $L^p(\Omega, \gamma)$, $\nabla_H \varphi_{k|\Omega} \to F$ in $L^p(\Omega, \gamma; H)$, and $D_H^2 \varphi_{k|\Omega} \to \Phi$ in $L^p(\Omega, \gamma; \mathcal{H})$ as $k \to \infty$. By the first part of the proof, $F = 0$; we apply formula (1.19) to $\partial_j \varphi_k$ instead of $\varphi_k$ and we get

$$\int_\Omega \psi_n \, \partial_{ij} \varphi_k \, d\gamma = -\int_\Omega \partial_j \psi_n \, \partial_i \varphi_k \, d\gamma + \int_\Omega \widehat{h}_i \, \partial_j \varphi_k \, \psi_n \, d\gamma.$$

We remark that

$$\lim_{k\to\infty} \int_\Omega \psi_n \, \partial_{ij} \varphi_k \, d\gamma = \int_\Omega \psi_n \, \langle \Phi h_i, h_j\rangle_H \, d\gamma,$$

and
$$\lim_{k\to\infty}\int_\Omega \partial_j\psi_n\,\partial_i\varphi_k\,d\gamma=0,\quad \lim_{k\to\infty}\int_\Omega \widehat{h}_i\,\partial_j\varphi_k\,\psi_n\,d\gamma=0.$$
Then $\int_\Omega \psi_n\,\langle \Phi h_i,h_j\rangle_H\,d\gamma=0$ for all $n\in\mathbb{N}$, and for every $\psi\in\mathcal{F}C_b^1(X)$ we have
$$\int_\Omega \psi\,\langle \Phi h_i,h_j\rangle_H\,d\gamma=\lim_{n\to\infty}\int_\Omega \psi_n\,\langle \Phi h_i,h_j\rangle_H\,d\gamma=0,$$
which implies that $\Phi=0$. $\square$

**Definition 29.** For $1<p<\infty$, we denote by $W^{1,p}(\Omega,\gamma)$ the domain of the closure of $\nabla_H:\mathcal{F}C_b^1(X)_{|\Omega}\to L^p(\Omega,\gamma;H)$ in $L^p(\Omega,\gamma)$, still denoted by $\nabla_H$. Similarly, we denote by $W^{2,p}(\Omega,\gamma)$ the domain of the closure of $(\nabla_H,D_H^2)$ in $L^p(\Omega,\gamma)$, still denoted by $(\nabla_H,D_H^2)$.

We remark that $W^{1,p}(\Omega,\gamma)$ and $W^{2,p}(\Omega,\gamma)$ are Banach spaces with their graphs norm respectively:

$$\begin{aligned}\|f\|^p_{W^{1,p}(\Omega,\gamma)}&:=\|f\|^p_{L^p(\Omega,\gamma)}+\int_\Omega|\nabla_H f(x)|_H^p\gamma(dx),\\ \|f\|^p_{W^{2,p}(\Omega,\gamma)}&:=\|f\|^p_{W^{1,p}(\Omega,\gamma)}+\int_\Omega\|D_H^2 f(x)\|_{\mathcal{H}}^p\gamma(dx).\end{aligned}$$

Moreover
$$\partial_n f(x):=\langle\nabla_H f(x),h_n\rangle_H,$$
and
$$\partial^2_{nm}f(x):=\langle D_H^2 f(x)h_n,h_m\rangle_H.$$
Finally, for $p=2$, $W^{1,2}(\Omega,\gamma)$ and $W^{2,2}(\Omega,\gamma)$ are Hilbert spaces with the inner products:

$$\begin{aligned}\langle f,g\rangle_{W^{1,2}(\Omega,\gamma)}&:=\int_\Omega f\,g\,d\gamma+\int_\Omega\langle\nabla_H f,\nabla_H g\rangle_H\,d\gamma,\\ \langle f,g\rangle_{W^{2,2}(\Omega,\gamma)}&:=\int_\Omega f\,g\,d\gamma+\int_\Omega\langle\nabla_H f,\nabla_H g\rangle_H\,d\gamma+\int_\Omega\langle D_H^2 f,D_H^2 g\rangle_{\mathcal{H}}\,d\gamma.\end{aligned}$$

Our leading assumptions will be the following.

**Assumption 1.** We suppose that there exists a continuous and convex function $g:X\to\mathbb{R}$, such that

- $\Omega=g^{-1}((-\infty,0))$;
- $g\in W^{2,p}(X,\gamma)$ for all $p>1$;

- there exists $\delta > 0$ such that $\dfrac{1}{|\nabla_H g|_H} \in L^p(g^{-1}(-\delta,\delta),\gamma)$ for all $p > 1$.

These conditions allow the definition of traces of Sobolev functions at $g^{-1}(0)$. Now we show some properties that are proved in [7]. We recall that $Lip(X)$ is the space of Lipschitz functions.

**Proposition 25.** *For each $p > 1$ and $\varphi \in W^{1,p}(\Omega,\gamma)$ there exists $\psi \in \cap_{q<p} L^q(\partial\Omega,\rho)$ with the following property: for every $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}C_b^1(X)$ such that $\{\varphi_{n|\Omega}\}_{n\in\mathbb{N}}$ converges to $\varphi$ in $\in W^{1,p}(\Omega,\gamma)$, then $\{\varphi_{n|\partial\Omega}\}_{n\in\mathbb{N}}$ converges to $\psi$ in $L^q(\partial\Omega,\rho)$, for every $q < p$. Here $\rho$ is the the Hausdorff-Gauss surface measure on $\partial\Omega$ studied in [17].*

**Definition 30.** For any $\varphi \in W^{1,p}(\Omega,\gamma)$, $p > 1$, we define the trace $\mathrm{Tr}\,\varphi$ of $\varphi$ at $\partial\Omega$ as the function $\psi$ given by Proposition 25.

In [7] was proved that the trace operator is bounded from $W^{1,p}(\Omega,\gamma)$ to $L^q(\partial\Omega,\rho)$ for all $q \in [1,p)$.

**Theorem 13.** *Let $p > 1$. For each $\varphi \in W^{1,p}(\Omega,\gamma)$, and $n \in \mathbb{N}$ we have*

$$\int_\Omega \partial_n\varphi\, d\gamma = \int_\Omega \widehat{h}_n\varphi\, d\gamma + \int_{\partial\Omega} \frac{\partial_n g}{|\nabla_H g|_H}\,\mathrm{Tr}\,\varphi\, d\rho.$$

Now we show a version of the integration by parts formula.

**Proposition 26.** *Let $\varphi \in W^{1,p}(\Omega,\gamma)$ and $\psi \in W^{1,q}(\Omega,\gamma)$ with $pq/(p+q) > 1$. Then for $k \in \mathbb{N}$*

$$\int_\Omega \partial_k\varphi\ \psi\ d\gamma = -\int_\Omega \varphi\partial_k\psi\ d\gamma + \int_\Omega \varphi\psi\ \widehat{h}_k d\gamma + \int_{\partial\Omega} \mathrm{Tr}\varphi\mathrm{Tr}\psi\frac{\partial_k g}{|\nabla_H g|_H}\ d\rho.$$

The following proposition describes the kernel of the trace operator.

**Proposition 27.** *For every $p > 1$ the kernel of the trace operator in $W^{1,p}(\Omega,\gamma)$ consist of the elements $\varphi \in W^{1,p}(\Omega,\gamma)$ whose null extension $\varphi_0$ to the whole $X$ belongs to $W^{1,p}(X,\gamma)$.*

## 1.7 Construction of Ornstein-Uhlenbeck operator

In this section we describe the construction of the Ornstein-Uhlenbeck operator. Let $\Omega \subset X$ be an open convex set, and let $\gamma$ be a centered non-degenerate Gaussian measure on $X$.

First we recall the integration by parts formula. Let $\varphi \in W^{1,2}(\Omega,\gamma)$ and $\psi \in \mathcal{F}C^1_b(X)$. Then

$$\int_\Omega \partial_k\varphi\, \psi\, d\gamma = -\int_\Omega \varphi\partial_k\psi\, d\gamma + \int_\Omega \varphi\psi\, \widehat{h}_k d\gamma + \int_{\partial\Omega} \mathrm{Tr}\varphi \mathrm{Tr}\psi \frac{\partial_k g}{|\nabla_H g|_H}\, d\rho, \quad (1.20)$$

where in the last integral $\rho$ is the surface measure associated to the Gaussian measure and $\mathrm{Tr}\,\varphi$, $\mathrm{Tr}\,\psi = \psi_{|\partial\Omega}$ are the traces of the function $\varphi, \psi$ (see Section 1.6.3).

The quadratic form

$$\mathcal{E}_{\Omega,\gamma}(u,v) := \int_\Omega \langle \nabla_H u, \nabla_H v\rangle_H \gamma(dx) \quad \text{with } u,v \in W^{1,2}(\Omega,\gamma), \qquad (1.21)$$

is used to define the Ornstein-Uhlenbeck operator $L^\Omega : D(L^\Omega) \subset L^2(\Omega,\gamma) \to L^2(\Omega,\gamma)$ by setting

$$D(L^\Omega) := \Big\{ u \in W^{1,2}(\Omega,\gamma) : \exists f \in L^2(\Omega,\gamma) \text{ s.t.} \\ \mathcal{E}_{\Omega,\gamma}(u,v) = -\int_\Omega f v d\gamma, \forall v \in W^{1,2}(\Omega,\gamma) \Big\} \qquad (1.22)$$

and $L^\Omega u = f$.

The proof of the following properties can be found in [21].

**Proposition 28.** *The operator $(L^\Omega, D(L^\Omega))$ satisfies the following properties:*

1. *it is self-adjoint in $L^2(\Omega,\gamma)$;*
2. *it is dissipative (that is $\langle L^\Omega u, u\rangle_{L^2(\Omega,\gamma)} \leq 0$ for every $u \in D(L^\Omega)$).*

Proposition 28 implies that $L^\Omega$ is the infinitesimal generator of an analytic contraction semigroup $(T^\Omega(t))_{t\geq 0}$ in $L^2(\Omega,\gamma)$. We also recall that, since $T^\Omega(t)$ is an analytic semigroup, then $T^\Omega(t)f \in D(L^\Omega)$ for every $t > 0$, $f \in L^2(\Omega,\gamma)$, moreover

$$\frac{\partial}{\partial t} T^\Omega(t)(f) = L^\Omega T^\Omega(t)(f) \quad \forall f \in L^2(\Omega,\gamma). \qquad (1.23)$$

The following lemma holds also for more general linear operators in spaces $L^p$ space.

**Lemma 3.** *The following conditions are equivalent:*

1. $\int_\Omega T^\Omega(t)f(x)\,\gamma(dx) = \int_\Omega f(x)\,\gamma(dx)$ *for all* $f \in L^2(\Omega,\gamma)$;
2. $\int_\Omega L^\Omega u(x)\,\gamma(dx) = 0$ *for all* $u \in D(L^\Omega)$.

*Proof.* We show the implications separately.

1. $\Rightarrow$ 2. For $u \in D(L^\Omega)$ we have

$$L^\Omega u = \lim_{t \to 0^+} \frac{T^\Omega(t)u - u}{t},$$

where the limit is in $L^2(\Omega, \gamma)$. Then

$$\frac{T^\Omega(t)u - u}{t} - L^\Omega u = \frac{1}{t} \int_0^t (T^\Omega(s) L^\Omega u - L^\Omega u)\, ds.$$

Integrating over $\Omega$, we get

$$\int_\Omega \left( \frac{T^\Omega(t)u - u}{t} - L^\Omega u \right) d\gamma = - \int_\Omega L^\Omega u \, d\gamma,$$

and

$$\int_\Omega \left( \frac{1}{t} \int_0^t (T^\Omega(s) L^\Omega u - L^\Omega u)\, ds \right) d\gamma = \frac{1}{t} \int_0^t \int_\Omega (T^\Omega(s) L^\Omega u - L^\Omega u)\, d\gamma\, ds = 0$$

by hypothesis.

2. $\Rightarrow$ 1. First we take $f \in D(L^\Omega)$. Then

$$\frac{d}{dt} \int_\Omega T^\Omega(t) f \, d\gamma = \int_\Omega L^\Omega T^\Omega(t) f \, d\gamma = 0,$$

for all $t \geq 0$. Therefore, the function $t \mapsto \int_\Omega T^\Omega(t) f \, d\gamma$ is constant, that is

$$\int_\Omega T^\Omega(t) f \, d\gamma = \int_\Omega T^\Omega(0) f \, d\gamma = \int_\Omega f \, d\gamma.$$

Now, by using the density of $D(L^\Omega)$ in $L^2(\Omega, \gamma)$, we get our claim.

□

**Proposition 29.** *The measure $\gamma$ is invariant for the semigroup $(T^\Omega(t))_{t \geq 0}$, that is*

$$\int_\Omega T^\Omega(t) f(x)\, \gamma(dx) = \int_\Omega f(x)\, \gamma(dx), \quad \forall f \in L^2(\Omega, \gamma).$$

*Proof.* For all $u \in D(L^\Omega)$, by construction of $L^\Omega$,we have

$$\int_\Omega L^\Omega u \, d\gamma = \int_\Omega L^\Omega u \cdot 1 \, d\gamma = - \int_\Omega \langle \nabla_H u, \nabla_H 1 \rangle_H \, d\gamma = 0.$$

By applying Lemma 3 we obtain our claim. □

*Remark* 11. We recall that $D((I - L^\Omega)^{1/2})$ is the completion of $D(L^\Omega)$ with respect to the norm

$$\left(\int_\Omega (I - L^\Omega) u\, u\, d\gamma\right)^{1/2}. \tag{1.24}$$

We remark that

$$\begin{aligned}\int_\Omega (I - L^\Omega) u\, u\, d\gamma &= \int_\Omega (|u|^2 - L^\Omega u\, u\, d)\gamma = \int_\Omega (|u|^2 + \langle \nabla_H u, \nabla_H u\rangle_H\,) d\gamma \\ &= \int_\Omega (|u|^2 + |\nabla_H u|_H^2\,) d\gamma.\end{aligned}$$

Then (1.24) is an equivalent norm of $W^{1,2}(\Omega, \gamma)$.

Since $W^{1,2}(\Omega, \gamma) = D((I - L^\Omega)^{1/2})$, the restriction of $T^\Omega(t)$ to $W^{1,2}(\Omega, \gamma)$ is an analytic semigroup, and $D(L^\Omega)$ is also dense in $W^{1,2}(\Omega, \gamma)$.

Now we consider the differential equation (1)

$$\lambda u - L^\Omega u = f$$

where $\lambda > 0$ and $f \in L^2(\Omega, \gamma)$ are given.

**Definition 31** (Weak solution)**.** The function $u \in W^{1,2}(\Omega, \gamma)$ is a weak solution of (1) if

$$\int_\Omega \lambda u \varphi\, d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H \varphi\rangle_H d\gamma = \int_\Omega f \varphi\, d\gamma \quad \forall \varphi \in \mathcal{F}C_b^1(X) \tag{1.25}$$

or equivalently, (1.25) holds for all $\varphi \in W^{1,2}(\Omega, \gamma)$.

By applying Proposition 17, with

$$a(u, v) := \lambda \int_\Omega u\, v\, d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H v\rangle_H\, d\gamma$$

we have that there exists a unique weak solution of (1).

### 1.7.1 Ornstein-Uhlenbeck operators in finite dimension

Now we recall some properties of Ornstein-Uhlenbeck operators in finite dimension. Let $\mathcal{O}$ be a smooth convex set of $\mathbb{R}^n$ and let $\mu$ be the standard Gaussian measure in $\mathbb{R}^n$. Let $L^{\mathcal{O}}$ be the Ornstein-Uhlenbeck operator associated to the quadratic form

$$\mathcal{E}_{\mathcal{O},\mu}(u, v) := \int_{\mathcal{O}} \langle \nabla u, \nabla v\rangle d\mu \quad \text{for} \quad u, v \in W^{1,2}(\mathcal{O}, \mu),$$

since in this case we have $\nabla_H = \nabla$.

It is known, see [8], that

$$D(L^{\mathcal{O}}) = \left\{ f \in W^{2,2}(\mathcal{O},\mu) : \langle x, \nabla f\rangle \in L^2(\mathcal{O},\mu) \text{ and } \frac{\partial f}{\partial \nu} = 0 \right\} \tag{1.26}$$

where $\nu(x)$ is the exterior normal vector to $\partial\mathcal{O}$ at $x$. Moreover

$$L^{\mathcal{O}} f(x) = \Delta f(x) - \langle x, \nabla f(x)\rangle \text{ for every } f \in D(L^{\mathcal{O}}). \tag{1.27}$$

We recall the finite dimensional logarithmic Sobolev inequality

$$\int_{\mathcal{O}} f^2 \log(f^2) d\mu \le \int_{\mathcal{O}} |\nabla_H f|_H^2 d\mu + \|f\|^2_{L^2(\mathcal{O},\mu)} \log(\|f\|^2_{L^2(\mathcal{O},\mu)}), \tag{1.28}$$

that holds for each $f \in W^{1,2}(\mathcal{O},\mu)$, see [8].

## 1.8 Cylindrical approximations

In this section we present an important tool that will be used several times in the following. Let $\Omega \subset X$ an open convex set. First we recall Proposition A.5 in [20].

**Proposition 30.** *Let $\Omega \subset X$ be an open convex set. Then there exists a sequence of open convex cylindrical sets $\Omega_n \supset \Omega_{n+1} \supset \Omega$ with smooth boundaries, such that*

$$\lim_{n\to\infty} \gamma(\Omega_n \setminus \Omega) = 0.$$

In [20], for each $n \in \mathbb{N}$, the sets $\Omega_n$ are given by

$$\Omega_n = \pi_n^{-1}(\mathcal{O}_n)$$

where $\mathcal{O}_n$ is an open smooth convex subset of a finite dimensional subspace $F_n \subset H$, of dimension $j = j(n)$, moreover $F_n \subset F_{n+1}$ for all $n \in \mathbb{N}$. Let $\{h_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of the Cameron-Martin space $H$, , $h_n = R_\gamma \widehat{h}_n$ with $\widehat{h}_n \in X^*$, such that

$$F_n = \text{span}\{h_1, \dots, h_{j(n)}\}.$$

Finally $\pi_n : X \to F_n$ is the finite dimensional projection defined as

$$\pi_n(x) = \sum_{i=1}^{j(n)} \widehat{h}_i(x) h_i.$$

Moreover $\Omega_{n+1} \subset \Omega_n$, $\partial\mathcal{O}_n$ is smooth, $\Omega \subset \Omega_n$ and we have

$$\overline{\Omega} = \bigcap_{n\in\mathbb{N}} \overline{\Omega}_n, \quad \gamma(\partial\Omega) = \gamma(\partial\Omega_n) = 0, \text{ and } \gamma\left(\bigcap_{n\in\mathbb{N}} \Omega_n \backslash \Omega\right) = 0.$$

*Remark* 12. If $x \in H$ then $\widehat{h}_i(x) = \langle x, h_i \rangle_H$ and $\pi_n(x)$ is a orthogonal projection of $X$ on $\text{span}\{h_1, \ldots, h_{j(n)}\}$. Since $\widehat{h}_i \in X^*$ for all $i \in \mathbb{N}$, we have that $\pi_n : X \to X$ is a linear bounded operator moreover $\pi_n(\pi_n(x)) = \pi_n(x)$.

A proof of the following two results can be found in [3].

**Lemma 4.** *Let $X$ be a separable Banach space. Then*

$$\lim_{n\to\infty} \pi_n(x) = x \ \ a.e. \ x \in X.$$

Now we show a useful approximation property.

**Proposition 31.** *Let $f \in L^p(X,\gamma)$ (respectively, $W^{1,p}(X,\gamma)$) then $f_n \to f$ in $L^p(X,\gamma)$, ($W^{1,p}(X,\gamma)$), where*

$$f_n(x) := \int_X f\left(\pi_n x + (I - \pi_n)y\right) \, d\gamma(y), x \in X.$$

*Moreover, $f_n$ is a cylindrical function for all $n \in \mathbb{N}$.*

**Corollary 2.** *The space $\mathcal{F}C_b^k(X)$ is dense in $L^p(X,\gamma)$ with respect to the norm $\|\cdot\|_{L^p(X,\gamma)}$.*

*Proof.* Let $f \in L^p(X,\gamma)$. We fix $\varepsilon > 0$, then, by Proposition 31, there exists a cylindrical function $f_\varepsilon$, such that $\|f - f_\varepsilon\|_{L^p(X,\gamma)} < \varepsilon/2$. Thank to the results of Section 1.9, we have $f_\varepsilon = \widetilde{f} \circ \pi_n$, where $\widetilde{f} \in L^p(\mathbb{R}^n, \gamma_n)$. Therefore, there exists $g \in C_b^k(\mathbb{R}^n)$ such that $\|\widetilde{f}_n - g\|_{L^p(\mathbb{R}^n,\gamma_n)} < \varepsilon/2$. We set $\varphi := g \circ \pi_n$, then

$$\begin{aligned}\|f - \varphi\|_{L^p(X,\gamma)} &\leq \|f - f_\varepsilon\|_{L^p(X,\gamma)} + \|f_\varepsilon - \varphi\|_{L^p(X,\gamma)} \\ &\leq \frac{\varepsilon}{2} + \|\widetilde{f} - g\|_{L^p(\mathbb{R}^n,\gamma_n)} < \varepsilon.\end{aligned}$$

□

Let $L^{(n)} : D(L^{(n)}) \to L^2(\Omega_n, \gamma)$ be the self-adjoint operator associated to the quadratic form (1.21) with $\Omega = \Omega_n$, and $D(L^{(n)})$ is defined by (1.22) with $\Omega = \Omega_n$. Let $(T^{(n)}(t))_{t\geq 0}$ be the semigroup generated by $L^{(n)}$ in $L^2(\Omega_n, \gamma)$.

We recall Proposition 3.3 proved in [20]:

**Proposition 32.** *For any $f \in L^2(X,\gamma)$ and any $\lambda \in \mathbb{C}\backslash(-\infty, 0]$,*

$$\lim_{n\to\infty} \left(R(\lambda, L^{(n)})(f_{|\Omega_n})\right) = R(\lambda, L^{\Omega})(f_{|\Omega}) \quad \text{in } W^{1,2}(\Omega,\gamma),$$

*and*

$$(T^{(n)}(t)u_{|\Omega_n})_{|\Omega} \to T^{\Omega}(t)u_{|\Omega} \quad \text{in } W^{1,2}(\Omega,\gamma)$$

*for any $u \in L^2(X,\gamma)$ and $t > 0$.*

*Remark* 13. We identify $F_n$ with $\mathbb{R}^{j(n)}$ by using function

$$i(h_k) = e_k = (0, \dots, \underbrace{1}_{\text{k-th position}}, \dots, 0),$$

linearly extended to the whole space, that is

$$i\left(\sum_{k=1}^{j(n)} \xi_k h_k\right) = (\xi_1, \dots, \xi_{j(n)}).$$

Then the image measure $\gamma \circ \pi_n^{-1}$ on $\mathbb{R}^{j(n)}$ is the standard Gaussian measure $\gamma_{j(n)}$.

## 1.9 Factorization of the Gaussian measure

Let $\gamma$ be a centered non-degenerate Gaussian measure. We study the decomposition of $\gamma$ as the product of two Gaussian measures on subspaces. We fix an orthonormal basis $\{\widehat{h}_n\}_{n\in\mathbb{N}}$. By Lemma 2, we may assume that each $\widehat{h}_n$ belongs to $X^*$. We recall that the set $\{h_n\}_{n\in\mathbb{N}}$, with $h_n = R_\gamma \widehat{h}_n$, is an orthonormal basis of $H$.

Let $G \subset R_\gamma(X^*)$ be an $q-$dimensional subspace and let us denote by $\pi_G$ the projection given by

$$\pi_G(x) := \sum_{n=1}^{q} \widehat{h}_n(x) h_n, \quad x \in X. \tag{1.29}$$

For $f \in X^*$, we use the notation $\pi_G^* f = f \circ \pi_G$.

We define $\gamma_G = \gamma \circ \pi_G^{-1}$ and we remark that $\gamma_G(G) = 1$ since $\pi_G^{-1}(G) = X$. For $f \in X^*$, we have

$$\begin{aligned}\widehat{\gamma_G}(f) &= \int_X \exp\{if(\pi_G(x))\}\gamma(dx) = \int_X \exp\{i\pi_G^* f(x)\}\gamma(dx) \\ &= \exp\left\{-\frac{1}{2}B_\gamma(\pi_G^* f, \pi_G^* f)\right\}.\end{aligned}$$

By Theorem 5, it follow that $\gamma_G$ is a centered Gaussian measure, with

$$
\begin{aligned}
B_{\gamma_G}(f_1,f_2) &= B_\gamma(\pi_G^* f_1, \pi_G^* f_2) = \int_X \pi_G^* f_1(x)\pi_G^* f_2(x)\gamma(dx) \\
&= \int_X f_1(\pi_G(x))f_2(\pi_G(x))\gamma(dx) = \int_G f_1(\xi)f_2(\xi)\gamma_G(d\xi) \\
&= \langle f_1, f_2\rangle_{L^2(G,\gamma_G)},
\end{aligned}
$$

for all $f_1, f_2 \in X^*$.

Similarly we define the measure $\gamma_{\widetilde{G}} = \gamma \circ (I-\pi_G)^{-1}$ and we set $\widetilde{G} := (I-\pi_G)^{-1}(X)$. Then $\gamma_{\widetilde{G}}(\widetilde{G}) = 1$, and $\gamma_{\widetilde{G}}$ is again a centered Gaussian measure with

$$
\begin{aligned}
B_{\gamma_{\widetilde{G}}}(f_1,f_2) &= B_\gamma((I-\pi_G)^* f_1, (I-\pi_G)^* f_2) \\
&= \int_X (I-\pi_G)^* f_1(x)(I-\pi_G)^* f_2(x)\gamma(dx) \\
&= \int_X f_1((I-\pi_G)(x))f_2((I-\pi_G)(x))\gamma(dx) \\
&= \int_{\widetilde{G}} f_1(y)f_2(y)\gamma_{\widetilde{G}}(dy) \\
&= \langle f_1, f_2\rangle_{L^2(\widetilde{G},\gamma_{\widetilde{G}})},
\end{aligned}
$$

for all $f_1, f_2 \in X^*$. Then the Cameron-Martin spaces of $\gamma_G$ is equal to $G$, while the Cameron-Martin spaces of is $\widetilde{G}$. Moreover $\widetilde{G}$ is the orthogonal complement of $G$ in $H$.

**Lemma 5.** *For any $f \in X^*$, we have*

$$
\pi_G(R_\gamma(f)) = R_\gamma(\pi_G^*(f)), \quad (I-\pi_G)(R_\gamma(f)) = R_\gamma((I-\pi_G)^*(f)).
$$

*As a consequence*

$$
|R_\gamma(f)|_H^2 = \|\pi_G^*(f)\|^2_{L^2(G,\gamma_G)} + \|(I-\pi_G)^*(f)\|^2_{L^2(\widetilde{G},\gamma_{\widetilde{G}})}.
$$

*Proof.* We recall that $G =$span$\{h_1,\dots,h_q\}$. By (1.29) and Remark 2, for all $g \in X^*$ we have

$$
g(\pi_G(R_\gamma(f))) = \sum_{n=1}^q \widehat{h}(R_\gamma(f))g(h_n) = \sum_{n=1}^q \langle f, \widehat{h}_n\rangle_{L^2(X,\gamma)}\, g(h_n).
$$

Moreover

$$
\pi_G^*(f)(x) = f(\pi_G(x)) = \sum_{n=1}^q \widehat{h}(x)f(h_n) = \sum_{n=1}^q \widehat{h}_n(x)\langle f, \widehat{h}_n\rangle_{L^2(X,\gamma)}.
$$

Therefore for any $g \in X^*$, we have

$$\begin{aligned} g(R_\gamma(\pi_G^*(f)) &= g\left(R_\gamma\left(\sum_{n=1}^{q} \langle f, \widehat{h}_n \rangle_{L^2(X,\gamma)} \widehat{h}_n\right)\right) \\ &= g\left(\sum_{n=1}^{q} \langle f, \widehat{h}_n \rangle_{L^2(X,\gamma)} h_n\right) = \sum_{n=1}^{q} \langle f, \widehat{h}_n \rangle_{L^2(X,\gamma)} g(h_n), \end{aligned}$$

and then $\pi_G(R_\gamma(f)) = R_\gamma(\pi_G^*(f))$. Furthermore

$$\begin{aligned} (I - \pi_G)(R_\gamma(f)) &= R_\gamma(f) - \pi_G(R_\gamma(f)) \\ &= R_\gamma(f) - R_\gamma(\pi_G^*(f)) = R_\gamma((I - \pi_G)^*(f)). \end{aligned}$$

We recall that $H = G \oplus \widetilde{G}$, then for $f \in X^*$ we have

$$\begin{aligned} |R_\gamma(f)|_H^2 &= |\pi_G(R_\gamma(f))|_H^2 + |I - \pi_G)(R_\gamma(f))|_H^2 \\ &= \|\pi_G^*(f)\|_{L^2(G,\gamma_G)}^2 + \|(I - \pi_G)^*(f)\|_{L^2(\widetilde{G},\gamma_{\widetilde{G}})}^2 \end{aligned}$$

□

Now we give the main result of this section.

**Theorem 14.** *The equality $\gamma_G \otimes \gamma_{\widetilde{G}} = \gamma$ holds.*

*Proof.* We recall that $X = G \oplus \widetilde{G}$. For any $f \in X^*$ we have

$$\begin{aligned} \widehat{\gamma_G \otimes \gamma_{\widetilde{G}}}(f) &= \int_{G \times \widetilde{G}} \exp\{if(\xi + y)\} \gamma_G \otimes \gamma_{\widetilde{G}}(d(\xi, y)) \\ &= \int_G \exp\{if(\xi)\} \gamma_G(d\xi) \cdot \int_{\widetilde{G}} \exp\{if(y)\} \gamma_{\widetilde{G}}(dy) \\ &= \exp\left\{-\frac{1}{2}\left(B_{\gamma_G}(f,f) + B_{\gamma_{\widetilde{G}}}(f,f)\right)\right\}. \end{aligned}$$

Moreover we have

$$\begin{aligned} B_{\gamma_G}(f,f) + B_{\gamma_{\widetilde{G}}}(f,f) &= \int_G f(\xi)^2 \gamma_G(d\xi) + \int_{\widetilde{G}} f(y)^2 \gamma_{\widetilde{G}}(dy) \\ &= \int_X \left(f(\pi_G(x))^2 + f((I - \pi_G)(x))^2\right) \gamma(dx) \\ &= \int_X \left(f(\pi_G(x))^2 + (f(x) - f(\pi_G(x))^2)^2\right) \gamma(dx) \\ &= \int_X \left(f(\pi_G(x))^2 + f(x)^2 - 2f(x)f(\pi_G(x)) + f(\pi_G(x))^2\right) \gamma(dx) \\ &= \int_X f(x)^2 \gamma(dx) - 2\int_X f(\pi_G(x))\left(f(x) - f(\pi_G(x))\right) \gamma(dx) \\ &= B_\gamma(f,f), \end{aligned}$$

since

$$
\begin{aligned}
\int_X f(\pi_G(x))\,(f(x) - f(\pi_G(x)))\gamma(dx) = \int_X f(\pi_G(x)) f((I-\pi_G)(x))\gamma(dx) \\
= \int_G f(\pi_G(x)) f((I-\pi_G)(x))\gamma(dx) \\
+ \int_{\widetilde{G}} f(\pi_G(x)) f((I-\pi_G)(x))\gamma(dx) = 0
\end{aligned}
$$

This prove that $\widehat{\gamma_G \otimes \gamma_{\widetilde{G}}}(f) = \widehat{\gamma}$, and by Proposition 4 we get our claim. $\square$

# Chapter 2

# The Ornstein-Uhlenbeck operator and the Ornstein-Uhlenbeck semigroup

In this chapter we study the Ornstein-Uhlenbeck operator and the Ornstein-Uhlenbeck semigroup in an open convex subset $\Omega$ of an infinite dimensional separable Banach space $X$. This is done by finite dimensional approximation. In particular we prove Logarithmic-Sobolev and Poincaré inequalities, and thanks to these inequalities we deduce the spectral properties of the Ornstein-Uhlenbeck operator.

## 2.1 The Ornstein-Uhlenbeck semigroups

First we recall Theorem 1.3.3 and Theorem 1.4.1 proved in [11].

**Theorem 15** (Beurling-Deny conditions). *Let $L^{\Omega} \geq 0$ be a self-adjoint operator on $L^2(\Omega, \gamma)$. Then the following are equivalent:*

1. *$T^{\Omega}(t)$ may be extended to a contraction semigroup on $L^{\infty}(\Omega)$.*
2. *$T^{\Omega}(t)$ may be extended to a contraction semigroup on $L^p(\Omega, \gamma)$ for all $1 \leq p \leq \infty$.*
3. *Let $f \in W^{1,2}(\Omega, \gamma)$ and $g \in L^2(\Omega, \gamma)$ satisfy*
$$|g(x)| \leq |f(x)|$$
$$|g(x) - g(y)| \leq |f(x) - f(y)|$$
*for a.e. $x$, $y \in \Omega$. Then $g \in W^{1,2}(\Omega, \gamma)$ and*
$$\mathcal{E}_{\Omega,\gamma}(g, g) \leq \mathcal{E}_{\Omega,\gamma}(f, f).$$

*4. If* $0 \leq f \in W^{1,2}(\Omega,\gamma)$ *then* $f \wedge 1$ *lies in* $W^{1,2}(\Omega,\gamma)$ *and*

$$\mathcal{E}_{\Omega,\gamma}(f \wedge 1, f \wedge 1) \leq \mathcal{E}_{\Omega,\gamma}(f,f).$$

**Theorem 16.** *If* $(T^{\Omega}(t))_{t\geq 0}$ *is a symmetric semigroup on* $L^2(\Omega,\gamma)$ *then* $L^1(\Omega,\gamma) \cap L^{\infty}(\Omega)$ *is invariant under* $(T^{\Omega}(t))_{t\geq 0}$*, and* $(T^{\Omega}(t))_{t\geq 0}$ *may be extended from* $L^1(\Omega,\gamma) \cap L^{\infty}(\Omega)$ *to a positive one-parameter contraction semigroup* $(T_p^{\Omega}(t))_{t\geq 0}$ *on* $L^p(\Omega,\gamma)$ *for all* $1 \leq p \leq \infty$*. These semigroups are strongly continuous if* $1 \leq p < \infty$*, and are consistent in the sense that*

$$T_p^{\Omega}(t)f = T_q^{\Omega}(t)f$$

*if* $f \in L^p(\Omega,\gamma) \cap L^q(\Omega,\gamma)$*. They are self adjoint in the sense that*

$$T_p^{\Omega}(t)^* = T_q^{\Omega}(t)$$

*if* $1 \leq p < \infty$ *and* $p^{-1} + q^{-1} = 1$*.*

Let $L^{\Omega}$ be the Ornstein-Uhlenbeck operator defined in Section 1.7, and let $(T^{\Omega}(t))_{t\geq 0}$ its semigroup. Now we prove that $(T^{\Omega}(t))_{t\geq 0}$ satisfies the Beurling-Deny conditions:

**Proposition 33.** *If* $f \in W^{1,2}(\Omega,\gamma)$ *then* $f \wedge 1 \in W^{1,2}(\Omega,\gamma)$ *and*

$$\int_{\Omega} |\nabla_H (f \wedge 1)|^2 d\gamma \leq \int_{\Omega} |\nabla_H f|^2 d\gamma.$$

*Proof.* We recall that $f \wedge 1 = \min(f,1)$ and

$$\min(a,b) = \frac{|a+b|}{2} - \frac{|a-b|}{2}.$$

Set

$$g_n(\xi) = \frac{1}{2}\left[\sqrt{(\xi+1)^2 + \frac{1}{n}} - \sqrt{(\xi-1)^2 + \frac{1}{n}}\right], \quad \xi \in \mathbb{R}.$$

Let $\{f_n\} \subset \mathcal{F}C_b^1(X)$ be a sequence that approaches $f$ in $W^{1,2}(\Omega,\gamma)$ and pointwise a.e. in $\Omega$. Then the function $g_n \circ f_n$ belongs to $\mathcal{F}C_b^1(X)$ and it approaches $\min(f,1)$ in $W^{1,2}(\Omega,\gamma)$. Indeed this sequence converges to $\min(f,1)$ in $L^2(\Omega,\gamma)$ by the Dominated Convergence Theorem, and $\nabla_H(g_n \circ f_n) = (g_n' \circ f_n)\nabla_H f_n$ converge to

$$\left(\frac{1}{2}(\mathrm{sign}(f+1) - \mathrm{sign}(f-1))\right)\nabla_H f$$

in $L^2(\Omega,\gamma;H)$. Therefore

$$\begin{aligned}\int_\Omega |\nabla_H(f\wedge 1)|_H^2 d\gamma &= \int_\Omega |\nabla_H\left(\frac{|f+1|}{2}-\frac{|f-1|}{2}\right)|_H^2 d\gamma\\ &= \int_\Omega \left|\mathrm{sign}(f+1)\frac{\nabla_H f}{2}-\mathrm{sign}(f-1)\frac{\nabla_H f}{2}\right|^2 d\gamma\\ &= \int_\Omega |\nabla_H f|^2\left|\frac{1}{2}(\mathrm{sign}(f+1)-\mathrm{sign}(f-1))\right|^2 d\gamma\\ &\le \int_\Omega |\nabla_H f|^2 d\gamma.\end{aligned}$$

□

Therefore applying Theorem 15 we have that $(T^\Omega(t))_{t\ge 0}$ can be extended to a semigroup of contractions $(T_p(t))_{t\ge 0}$ on $L^p(\Omega,\gamma)$ for all $1\le p\le\infty$. $(T_p(t))_{t\ge 0}$ is strongly continuous for $1\le p<\infty$ thanks to Theorem 16.

## 2.2 Properties in finite dimension

Let $\mathcal{O}\subset\mathbb{R}^d$ be a smooth open convex set, let $\gamma_d$ be the standard Gaussian measure on $\mathbb{R}^d$ and let $L^{\mathcal{O}}$ be the Ornstein-Uhlenbeck operator defined in Section 1.7 with $\Omega=\mathcal{O}$ and $\gamma=\gamma_d$.

In [8] it is shown that

$$D(L^{\mathcal{O}})=\left\{f\in W^{2,2}(\mathcal{O},\gamma_d):\ \Delta f-\langle x,\nabla f\rangle\in L^2(\mathcal{O},\gamma_d)\text{ and }\frac{\partial f}{\partial\nu}=0\right\}$$

where $\nu(x)$ is the exterior normal vector to $\partial\mathcal{O}$ at $x$. Moreover

$$L^{\mathcal{O}}f(x)=\Delta f(x)-\langle x,\nabla f(x)\rangle\text{ for every }f\in D(L^{\mathcal{O}}). \tag{2.1}$$

We put

$$C^1_\nu(\overline{\mathcal{O}})=\left\{g\in C^1_b(\overline{\mathcal{O}}):\frac{\partial g}{\partial\nu}(x)=0,\quad x\in\partial\mathcal{O}\right\}. \tag{2.2}$$

We show some useful properties proved in [2].

**Proposition 34** (Proposition 2.1 of [2]). *Let $z\in C([0,T]\times\overline{\mathcal{O}})\cap C^1((0,T]\times\overline{\mathcal{O}})\cap C^{1,2}([0,T]\times\mathcal{O})$ be a bounded function satisfying*

$$\begin{cases} z_t(t,x)-L^{\mathcal{O}}z(t,x)\le 0 & 0<t\le T,\ x\in\mathcal{O},\\ \frac{\partial z}{\partial\nu}(t,x)\le 0 & 0<t\le T,\ x\in\partial\mathcal{O},\\ z(0,x)\le 0 & x\in\partial\mathcal{O}.\end{cases}$$

*Then $z(t,x)\le 0$ for all $0\le t\le T$, and $x\in\mathcal{O}$.*

**Proposition 35** (Corollary 3.3 of [2])**.** *For all $\omega > 0$ there exists $C_\omega > 0$ such that*

$$\|\nabla T^{\mathcal{O}}(t)f\|_\infty \leq C_\omega \frac{e^{\omega t}}{\sqrt{t}}\|f\|_\infty,\ t > 0,\ f \in C_b(\overline{\mathcal{O}}).$$

**Proposition 36** (Proposition 4.1 of [2])**.** *For every $p \geq 1$ and $f \in C^1_\nu(\mathcal{O})$ we have*

$$|\nabla T^{\mathcal{O}}(t)f(x)|^p \leq e^{-pt}T^{\mathcal{O}}(t)(|\nabla f|^p)(x),\ t \geq 0,\ x \in \overline{\mathcal{O}}.$$

Then $C^1_\nu(\overline{\mathcal{O}})$ is dense in $W^{1,2}(\mathcal{O},\gamma_d)$, see Lemma 6, and thanks to Proposition 35, we have

$$T^{\mathcal{O}}(t) : C_b(\overline{\mathcal{O}}) \to C^1_\nu(\overline{\mathcal{O}}) \text{ for all } t > 0. \tag{2.3}$$

$L^{\mathcal{O}}$ is one of the operators studied in [2]; thanks to Proposition 36 we can conclude that

$$|\nabla T^{\mathcal{O}}(t)f(x)| \leq e^{-t}T^{\mathcal{O}}(t)|\nabla f(x)| \quad \forall f \in C^1_\nu(\overline{\mathcal{O}}),\ \forall x \in \overline{\mathcal{O}},\ t \geq 0$$

Such estimate may be extended to Sobolev functions, as follows.

**Lemma 6.** *For all $f \in W^{1,2}(\mathcal{O},\gamma_d)$ we have*

$$|\nabla T^{\mathcal{O}}(t)f(x)| \leq e^{-t}T^{\mathcal{O}}(t)|\nabla f(x)|$$

*for a.e. $x \in \mathcal{O}$ and $t \geq 0$.*

*Proof.* If $f \in W^{1,2}(\mathcal{O},\gamma_d)$ there exists $\{f_n\} \subset C^1_b(\overline{\mathcal{O}})$ such that $f_n \to f$ in $W^{1,2}(\mathcal{O},\gamma_d)$ and, up to a subsequence, a.e. in $\mathcal{O}$; now $T^{\mathcal{O}}(t)f_n \to f_n$ for $t \to 0^+$ and thanks to (2.3) we have $T^{\mathcal{O}}(t)f_n \in C^1_\nu(\overline{\mathcal{O}})$. If we set $g_n = T^{\mathcal{O}}(\frac{1}{n})f_n$ then $g_n \in C^1_\nu(\overline{\mathcal{O}})$ for all $n \in \mathbb{N}$ and $g_n \to f$ in $W^{1,2}(\mathcal{O})$. Along a subsequence $\{g_{n_k}\}$ we have

$$|\nabla T^{\mathcal{O}}(t)g_{n_k}(x)| \to |\nabla T^{\mathcal{O}}(t)f(x)| \quad \text{a.e. in } \mathcal{O}$$

and again, up to a subsequence

$$\left(T^{\mathcal{O}}(t)|\nabla g_{n_k}|\right)(x) \to \left(T^{\mathcal{O}}(t)|\nabla f|\right)(x) \quad \text{a.e. in } \mathcal{O}.$$

Therefore

$$\begin{aligned} |\nabla T^{\mathcal{O}}(t)f(x)| &= \lim_{k\to\infty} |\nabla T^{\mathcal{O}}(t)g_{n_k}(x)| \\ &\leq \lim_{k\to\infty} e^{-t}\left(T^{\mathcal{O}}(t)|\nabla g_{n_k}|\right)(x) = e^{-t}\left(T^{\mathcal{O}}(t)|\nabla f|\right)(x) \end{aligned}$$

for a.e. $x \in \mathcal{O}$. □

**Proposition 37.** *For all $f$, $g \in L^2(\mathcal{O}, \gamma_d)$ we have*

$$\left[T^{\mathcal{O}}(t)(fg)\right]^2 \le T^{\mathcal{O}}(t)(f^2)T^{\mathcal{O}}(t)(g^2) \quad \textit{a.e. in } \mathcal{O}$$

*Proof.* First we consider $f, g \in C^1_\nu(\overline{\mathcal{O}})$. We fix $\varepsilon > 0$ and we put

$$v^\varepsilon(t,x) = \sqrt{(T^{\mathcal{O}}(t)(f^2)(x) + \varepsilon) \cdot (T^{\mathcal{O}}(t)(g^2)(x) + \varepsilon)}.$$

For convenience we set

$$\alpha = \alpha(t,x) = T^{\mathcal{O}}(t)(f^2)(x), \quad \beta = \beta(t,x) = T^{\mathcal{O}}(t)(g^2)(x).$$

Then $v^\varepsilon = \sqrt{(\alpha + \varepsilon)(\beta + \varepsilon)}$, and we remark that

$$\begin{aligned} D_i v^\varepsilon &= D_i(\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}) \\ &= \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}}((\beta+\varepsilon)D_i\alpha + (\alpha+\varepsilon)D_i\beta) \end{aligned}$$

moreover

$$\begin{aligned} v^\varepsilon_t &= \frac{\partial v^\varepsilon(t,x)}{\partial t} = \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}}\left(\frac{\partial \alpha}{\partial t}(\beta+\varepsilon) + (\alpha+\varepsilon)\frac{\partial \beta}{\partial t}\right) \\ &= \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}}((\beta+\varepsilon)L^{\mathcal{O}}\alpha + (\alpha+\varepsilon)L^{\mathcal{O}}\beta) \end{aligned}.$$

Then

$$
\begin{aligned}
L^{\mathcal{O}} v^\varepsilon &= \sum_{i=1}^d D_{ii} v^\varepsilon - x_i D_i v^\varepsilon = \sum_{i=1}^d D_i (D_i v^\varepsilon) - x_i D_i v^\varepsilon \\
&= \sum_{i=1}^d D_i \left( \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}} ((\beta+\varepsilon) D_i\alpha + (\alpha+\varepsilon) D_i\beta) \right) \\
&\quad - x_i \left( \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}} ((\beta+\varepsilon) D_i\alpha + (\alpha+\varepsilon) D_i\beta) \right) \\
&= v_t^\varepsilon + \sum_{i=1}^d -\frac{1}{4} \frac{1}{((\alpha+\varepsilon)(\beta+\varepsilon))^{3/2}} \left[ ((\beta+\varepsilon) D_i\alpha)^2 + ((\alpha+\varepsilon) D_i\beta)^2 \right] \\
&\quad + \frac{1}{2\sqrt{(\alpha+\varepsilon)(\beta+\varepsilon)}(\alpha+\varepsilon)(\beta+\varepsilon)} (\beta+\varepsilon) D_i\alpha \cdot (\alpha+\varepsilon) D_i\beta \\
&= v_t^\varepsilon + \sum_{i=1}^d -\frac{1}{4} \frac{1}{((\alpha+\varepsilon)(\beta+\varepsilon))^{3/2}} \left[ ((\beta+\varepsilon) D_i\alpha)^2 + ((\alpha+\varepsilon) D_i\beta)^2 \right. \\
&\quad \left. -2(\beta+\varepsilon) D_i\alpha \cdot (\alpha+\varepsilon) D_i\beta \right] \\
&= v_t^\varepsilon + \sum_{i=1}^d -\frac{1}{4} \frac{1}{((\alpha+\varepsilon)(\beta+\varepsilon))^{3/2}} ((\beta+\varepsilon) D_i\alpha - (\alpha+\varepsilon) D_i\beta)^2 \\
&\le v_t^\varepsilon.
\end{aligned}
$$

Therefore $v_t^\varepsilon \ge L^{\mathcal{O}}(v^\varepsilon)$, $v^\varepsilon(0,x) = \sqrt{(f(x)^2+\varepsilon)(g(x)^2+\varepsilon)}$ and $v^\varepsilon$ satisfies the Neumann boundary condition, thus for the maximum principle (see Proposition 34) we have $v^\varepsilon(t,x) \ge T^{\mathcal{O}}(t)[\sqrt{(f^2+\varepsilon)(g^2+\varepsilon)}](x)$ for all $(t,x)$ and all $\varepsilon > 0$, that is

$$
\begin{aligned}
&T^{\mathcal{O}}(t) \left( \sqrt{(f(x)^2+\varepsilon)(g(x)^2+\varepsilon)} \right)(x) \\
&\qquad \le \sqrt{(T^{\mathcal{O}}(t)(f^2)(x)+\varepsilon) \cdot (T^{\mathcal{O}}(t)(g^2)(x)+\varepsilon)}.
\end{aligned}
$$

Taking the limit as $\varepsilon \to 0^+$ we obtain

$$
T^{\mathcal{O}}(t)(|fg|)(x) \le \sqrt{T^{\mathcal{O}}(t)(f^2)(x) \cdot T^{\mathcal{O}}(t)(g^2)(x)}
$$

for all $(t,x)$, $t \ge 0$ and $f,\ g \in C^1_\nu(\overline{\mathcal{O}})$. Now if $f,\ g \in L^2(\mathcal{O},\gamma_d)$ there exist $\{f_n\}$, $\{g_n\} \subset C^1_\nu(\overline{\mathcal{O}})$ such that $f_n \to f$ and $g_n \to g$ in $L^2(\mathcal{O},\gamma_d)$. For subsequences $\{f_{n_k}\}$, $\{g_{n_k}\}$ we have

$$
T^{\mathcal{O}}(t)(|f_{n_k} g_{n_k}|) \to T^{\mathcal{O}}(t)(|fg|) \quad \text{a.e. in } \mathcal{O}
$$

and again, up to subsequences we have

$$T^{\mathcal{O}}(t)\left(f_{n_k}^2\right) \to T^{\mathcal{O}}(t)(f^2) \quad \text{a.e. in } \mathcal{O}.$$

$$T^{\mathcal{O}}(t)\left(g_{n_k}^2\right) \to T^{\mathcal{O}}(t)(g^2) \quad \text{a.e. in } \mathcal{O}.$$

Therefore

$$\begin{aligned} T^{\mathcal{O}}(t)(|fg|)(x) &= \lim_{k\to\infty} T^{\mathcal{O}}(t)\left(|f_{n_k} g_{n_k}|\right)(x) \\ &\leq \lim_{k\to\infty} \sqrt{T^{\mathcal{O}}(t)\left(f_{n_k}^2\right)(x) \cdot T^{\mathcal{O}}(t)\left(g_{n_k}^2\right)(x)} \\ &= \sqrt{T^{\mathcal{O}}(t)(f^2)(x) \cdot T^{\mathcal{O}}(t)(g^2)(x)} \end{aligned}$$

for a.e. $x \in \mathcal{O}$. □

## 2.3 Properties in infinite dimension

The following lemma will be used several times.

**Lemma 7.** *If* $u \in L^2(\Omega, \gamma)$ *and*

$$\int_\Omega u\varphi_{|\Omega} \, d\gamma \leq 0, \quad \forall \varphi \in \mathcal{F}C_b(X), \ \varphi \geq 0,$$

*then* $u \leq 0$ *a.e. in* $\Omega$.

*Proof.* First we show that

$$\int_\Omega u\psi \, d\gamma \leq 0, \quad \forall \psi \in L^2(\Omega, \gamma), \ \psi \geq 0 \text{ a.e.}$$

We remark that if $\psi \in L^2(\Omega,\gamma)$, $\psi \geq 0$ a.e., then there exists a sequence $\{\varphi_n\} \subset \mathcal{F}C_b(X)$ such that $\varphi_{n|\Omega} \to \psi$ a.e and in $L^2(\Omega,\gamma)$. Now we consider $\varphi_n^+(x) = \max\{0, \varphi_n(x)\} \geq 0$ and we prove that $\varphi^+_{n\,|\Omega} \to \psi$ in $L^2(\Omega,\gamma)$. Indeed,

$$\int_\Omega |\varphi_n^+ - \psi|^2 \, d\gamma = \int_{\{x\in\Omega:\ \varphi_n(x)\geq 0\}} |\varphi_n - \psi|^2 \, d\gamma + \int_{\{x\in\Omega:\ \varphi_n(x)<0\}} \psi^2 \, d\gamma.$$

the first term goes to zero as $n \to \infty$ thanks to the $L^2$-convergence of $\{\varphi_n\}$. Concerning the second one we have $\chi_{\{x\in\Omega:\ \varphi_n(x)<0\}} \to \chi_{\{x\in\Omega:\ \psi(x)\leq 0\}}$ a.e. thanks to the a.e.-convergence of $\{\varphi_n\}$, and $\psi^2\chi_{\{x\in\Omega:\ \varphi_n(x)<0\}} \leq \psi^2 \in L^1(\Omega,\gamma)$. By the Dominated Convergence Theorem,

$$\int_{\{x\in\Omega:\ \varphi_n(x)<0\}} \psi^2 \, d\gamma \xrightarrow{n\to\infty} \int_{\{x\in\Omega:\ \psi(x)\leq 0\}} \psi^2(y)\gamma(dy) = 0,$$

so that

$$\int_\Omega u\psi\, d\gamma = \lim_{n\to\infty}\int_\Omega u\varphi^+_{n\,|\Omega}\, d\gamma \le 0.$$

Now we prove the statement by contradiction: if there exists a measurable set $A$ such that $\gamma(A) > 0$ and $u_{|A} > 0$, then we take $\psi(x) = \chi_A(x) \in L^\infty(\Omega) \subset L^2(\Omega,\gamma)$. Then

$$\int_\Omega u\psi\, d\gamma = \int_A u\, d\gamma > 0$$

that is a contradiction. □

Now we use the cylindrical approximation defined in Section 1.8. Fix $n \in \mathbb{N}$, let $q \in \mathbb{N}$ with $q \ge j(n) = \dim F_n$ and let $G = \operatorname{span}\{h_1,\dots,h_q\}$. Let $\pi_G : X \to G$ be the finite rank projection defined by

$$\pi_G x = \sum_{i=1}^q \widehat{h}_i(x)h_i.$$

We denote by $\gamma_G$ the induced measure $\gamma\circ\pi_G^{-1}$ in $G$. If $G$ is identified with $\mathbb{R}^q$ through the isomorphism $x \mapsto (\widehat{h}_1(x),\dots,\widehat{h}_q(x))$ then $\gamma_G$ is the standard Gaussian measure in $\mathbb{R}^q$. Setting $d = q - \dim F_n$, let $\mathcal{O} := \mathcal{O}_n \times \mathbb{R}^d$ so that $\pi_G(\Omega_n) = \mathcal{O}$; we remark that $\mathcal{O}$ is an open convex subset of $G$. Let $L^G$ be the Ornstein-Uhlenbeck operator defined by (1.21) with $\Omega = \mathcal{O}$ and $\gamma = \gamma_G$ and let $T^G(t)$ be the associated semigroup.

**Lemma 8.** *If $w \in D(L^G)$ then the map $x \mapsto w(\pi_G(x))$ belongs to $D(L^{(n)})$ and*

$$L^{(n)}(w\circ\pi_G) = (L^G w)\circ\pi_G.$$

*Proof.* We show that there exists $f \in L^2(\Omega_n,\gamma)$ such that, given $\varphi \in W^{1,2}(\Omega_n,\gamma)$ we have

$$\int_{\Omega_n}\langle\nabla_H w(\pi_G(x)),\nabla_H\varphi(x)\rangle_H\gamma(dx) = \int_{\Omega_n} f(x)\varphi(x)\gamma(dx). \tag{2.4}$$

We remark that

$$\langle\nabla_H w(\pi_G(x)),\nabla_H\varphi(x)\rangle_H = \sum_{i=1}^q \frac{\partial w}{\partial\xi_i}(\pi_G(x))\frac{\partial\varphi}{\partial h_i}(x)$$

Moreover $\varphi(x) = \varphi(\pi_G(x) + (I-\pi_G)(x))$, the space $X$ can be split $X = G\times\widetilde{G}$ where $\widetilde{G} = (I-\pi_G)(X)$ and also $\gamma = \gamma_G\otimes\gamma_{\widetilde{G}}$, see section 1.9. Let $\xi \in \mathbb{R}^q$ and for every $y \in \widetilde{G}$ let

$$g_y(\xi) := \varphi(\xi_1 h_1 + \ldots + \xi_q h_q + y),$$

then

$$\begin{aligned}
&\int_{\Omega_n} \langle \nabla_H w(\pi_G(x)), \nabla_H \varphi(x)\rangle_H \gamma(dx)\\
&\quad= \int_{\Omega_n} \sum_{i=1}^{q} \frac{\partial w}{\partial \xi_i}(\pi_G(x)) \frac{\partial \varphi}{\partial h_i}(\pi_G(x) + (I-\pi_G)(x))\gamma(dx)\\
&\quad= \int_{\widetilde{G}} \int_{\mathcal{O}} \sum_{i=1}^{q} \frac{\partial w}{\partial \xi_i}(\xi) \frac{\partial g_y}{\partial \xi_i}(\xi)\gamma_G(d\xi)\ \gamma_{\widetilde{G}}(dy)\\
&\quad= \int_{\widetilde{G}} \int_{\mathcal{O}} L^G w(\xi) g_y(\xi)\gamma_G(d\xi)\ \gamma_{\widetilde{G}}(dy) = \int_{\Omega_n} (L^G w)(\pi_G(x))\varphi(x)\gamma(dx).
\end{aligned}$$

Then $w \circ \pi_G \in D(L^{(n)})$ and $L^{(n)}(w \circ \pi_G) = (L^G w) \circ \pi_G$. □

**Lemma 9.** *Let $\widetilde{v} \in C_b^\infty(G)$. Then the function defined by*

$$g(t)(x) := T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)), \quad x \in \Omega_n$$

*belongs to $C((0,\infty); D(L^{(n)}))$.*

*Proof.* By Lemma 8 it follows that $g(t) \in D(L^{(n)})$ for all $t > 0$. Let us prove that $g$ is continuous at $t_0 > 0$. For $t > 0$ we have

$$\begin{aligned}
&\int_{\Omega_n} |g(t)(x) - g(t_0)(x)|^2 \gamma(dx)\\
&\qquad\qquad= \int_{\Omega_n} |T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))|^2\gamma(dx)\\
&\qquad\qquad= \int_{\mathcal{O}} |T^G(t)\widetilde{v}_{|\mathcal{O}}(\xi) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\xi)|^2\gamma_G(d\xi)
\end{aligned}$$

that goes to zero as $t \to t_0$ thanks to the strong continuity of $T^G(t)$ in $L^2(\mathcal{O}, \gamma_G)$. Moreover

$$\begin{aligned}
&\int_{\Omega_n} |L^{(n)}g(t)(x) - L^{(n)}g(t_0)(x)|^2 \gamma(dx)\\
&\qquad\qquad= \int_{\Omega_n} |L^{(n)}T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) - L^{(n)}T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))|^2\gamma(dx)\\
&\qquad\qquad= \int_{\mathcal{O}} |L^G T^G(t)\widetilde{v}_{|\mathcal{O}}(\xi) - L^G T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\xi)|^2\gamma_G(d\xi)
\end{aligned}$$

that goes to zero as $t \to t_0$ since $t \mapsto T^G(t)\widetilde{v}$ belongs to $C((0,\infty), D(L^G))$. Then $g \in C((0,\infty), D(L^{(n)}))$. □

**Theorem 17.** *For all $f \in W^{1,2}(\Omega, \gamma)$ and all $t \geq 0$ we have*

$$|\nabla_H T^\Omega(t) f|_H \leq e^{-t} T^\Omega(t) |\nabla_H f|_H \ a.e. \ on \ \Omega. \tag{2.5}$$

*Proof.* First we prove that (2.5) holds true for the restriction to $\Omega$ of any cylindrical regular function. Let $f \in \mathcal{F}C_b^\infty(X)$ and $\varphi \in \mathcal{F}C_b(X)$ with $\varphi \geq 0$. Then

$$\begin{aligned} f(x) &= v(l_1(x), \ldots, l_k(x)) \\ \varphi(x) &= w(l_{k+1}(x), \ldots, l_{k+m}(x)) \end{aligned}$$

with $l_i \in X^*$, $v \in C_b^\infty(\mathbb{R}^k)$ and $w \in C_b(\mathbb{R}^m)$, $w \geq 0$. We want to estimate the integral

$$\int_{\Omega_n} \varphi(x) |\nabla_H T^{(n)}(t) f_{|\Omega_n}(x)| \gamma(dx).$$

Let $G := \operatorname{span}\{F_n, R_\gamma(l_1), \ldots, R_\gamma(l_{k+m})\}$. Then $G$ is a subspace of $H$ of dimension $q \leq n + k + m$; setting $d = q - \dim F_n$ let $\mathcal{O} := \mathcal{O}_n \times \mathbb{R}^d$. Let $\{h_i\}_{i=1}^q \subset X^*$ be an orthonormal basis of $G$ such that $\{h_1, \ldots, h_j\}$ is an orthonormal basis of $F_n$ if $\dim F_n = j$.

Finally we consider:

$$\begin{aligned} f(x) &= \widetilde{v}(\pi_G(x)) \\ \varphi(x) &= \widetilde{w}(\pi_G(x)) \end{aligned}$$

where $\widetilde{v} \in C_b^\infty(G)$, $\widetilde{w} \in C_b(G)$.

Now we prove that

$$(T^{(n)}(t) f_{|\Omega_n})(x) = T^G(t) \widetilde{v}_{|\mathcal{O}}(\pi_G(x)) := g(t)(x) \quad t > 0, \ x \in \Omega_n. \tag{2.6}$$

We know that the function $t \mapsto T^G(t)\widetilde{v}_{|\mathcal{O}}$ belongs to $C([0,\infty)\,; L^2(\mathcal{O}, \gamma_G)) \cap C^1((0,\infty); L^2(\mathcal{O}, \gamma_G)) \cap C((0,\infty); D(L^G))$ and it is the unique classical solution of

$$\begin{cases} u'(t) = L^G u(t), & t > 0 \\ u(0) = \widetilde{v}_{|\mathcal{O}} \end{cases}$$

in the space $L^2(\mathcal{O}, \gamma_G)$. Then for fixed $t_0 \in [0, +\infty)$ we have

$$\begin{aligned} &\int_{\Omega_n} |g(t)(x) - g(t_0)(x)|^2 \gamma(dx) \\ &\qquad = \int_{\Omega_n} |T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))|^2 \gamma(dx) \\ &\qquad = \int_{\mathcal{O}} |T^G(t)\widetilde{v}_{|\mathcal{O}}(\xi) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\xi)|^2 \gamma_G(d\xi) \end{aligned}$$

that goes to zero for $t \to t_0$ thanks to the strong continuity of $T^G(t)$ in $L^2(\mathcal{O}, \gamma_G)$. Let us prove the differentiability of $g$ at $t_0 \in (0, +\infty)$. For any $t$ such that $t + t_0 > 0$ we have

$$\int_{\Omega_n} \left| \frac{g(t+t_0)(x) - g(t_0)(x)}{t} - L^G T^G(t_0) \widetilde{v}_{|\mathcal{O}}(\pi_G(x)) \right|^2 \gamma(dx)$$
$$= \int_{\Omega_n} \left| \frac{T^G(t+t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))}{t} - L^G T^G(t_0) \widetilde{v}_{|\mathcal{O}}(\pi_G(x)) \right|^2 \gamma(dx)$$
$$= \int_{\mathcal{O}} \left| \frac{T^G(t+t_0)\widetilde{v}_{|\mathcal{O}}(\xi) - T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\xi)}{t} - L^G T^G(t_0) \widetilde{v}_{|\mathcal{O}}(\xi) \right|^2 \gamma_G(d\xi)$$

which tends to zero as $t \to 0$. Then $g'(t_0) = L^G T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(\cdot))$. Moreover $g'$ is continuous at any $t_0 > 0$ with values in $L^2(\Omega_n, \gamma)$ since

$$\begin{aligned} &\int_{\Omega_n} |g'(t)(x) - g'(t_0)(x)|^2 \gamma(dx) \\ &\qquad = \int_{\Omega_n} |L^G T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) - L^G T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))|^2 \gamma(dx) \\ &\qquad = \int_{\mathcal{O}} |L^G T^G(t)\widetilde{v}_{|\mathcal{O}}(\xi) - L^G T^G(t_0)\widetilde{v}_{|\mathcal{O}}(\xi)|^2 \gamma_G(d\xi) \end{aligned}$$

that goes to zero as $t \to t_0$. By Lemma 9, $g \in C((0,\infty); D(L^{(n)}))$ and

$$g'(t) = L^G(T^G(t)\widetilde{v}_{|\mathcal{O}})(\pi_G(x)) = L^{(n)}(T^G(t)\widetilde{v}_{|\mathcal{O}}(\pi_G(x))) = L^{(n)}g(t), \ t > 0.$$

Moreover

$$g(0) = T^G(0)\widetilde{v}_{|\mathcal{O}}(\pi_G(x)) = \widetilde{v}_{|\mathcal{O}}(\pi_G(x)) = f(x)_{|\Omega_n}.$$

Therefore $g$ is a classical solution to

$$\begin{cases} g'(t) = L^{(n)}g(t), & t > 0 \\ g(0) = f_{|\Omega_n}, \end{cases}$$

so that $g(t) = T^{(n)}(t) f_{|\Omega_n}$ and (2.6) is proved.

Now using Lemma 6 we have

$$\begin{aligned}
\int_{\Omega_n} &\varphi(x)|\nabla_H T^{(n)}(t) f_{|\Omega_n}(x)|_H \gamma(dx) \\
&= \int_{\Omega_n} \widetilde{w}(\pi_G(x))|\nabla_H T^G(t)(\widetilde{v}_{|\mathcal{O}})(\pi_G(x))|_H \gamma(dx) \\
&= \int_{\mathcal{O}} \widetilde{w}(\xi)|\nabla T^G(t)(\widetilde{v}_{|\mathcal{O}})(\xi)|\gamma_G(d\xi) \\
&\le \int_{\mathcal{O}} \widetilde{w}(\xi)e^{-t}T^G(t)|\nabla(\widetilde{v}_{|\mathcal{O}})(\xi)|\gamma_G(d\xi) \\
&= \int_{\Omega_n} \widetilde{w}(\pi_G(x))e^{-t}T^G(t)|\nabla_H(\widetilde{v}_{|\mathcal{O}})(\pi_G(x))|_H \gamma(dx) \\
&= \int_{\Omega_n} \varphi(x)e^{-t}T^{(n)}(t)|\nabla_H f_{|\Omega_n}(x)|_H \gamma(dx)
\end{aligned}$$

Therefore for every $f \in \mathcal{F}C_b^\infty(X)$ and $\varphi \in \mathcal{F}C_b(X)$, $\varphi \ge 0$ and every $n$ we have

$$\int_{\Omega_n} \varphi(x)|\nabla_H T^{(n)}(t) f_{|\Omega_n}(x)|\gamma(dx) \le \int_{\Omega_n} \varphi(x)e^{-t}T^{(n)}(t)|\nabla_H f_{|\Omega_n}(x)|\gamma(dx).$$

Now thanks to Proposition 32 we can take the limit as $n \to \infty$ obtaining

$$\int_{\Omega} \varphi(x)|\nabla_H T^{\Omega}(t) f_{|\Omega}(x)|\gamma(dx) \le \int_{\Omega} \varphi(x)e^{-t}T^{\Omega}(t)|\nabla_H f_{|\Omega}(x)|\gamma(dx)$$

and by Lemma 7

$$|\nabla_H T^{\Omega}(t) f_{|\Omega}(x)| \le e^{-t}T^{\Omega}(t)|\nabla_H f_{|\Omega}(x)| \quad \text{for a.e. } x \in \Omega.$$

If $f \in W^{1,2}(\Omega,\gamma)$ then there exists $\{f_n\} \in \mathcal{F}C_b^\infty(X)$ such that $f_{n|\Omega} \to f$ in $W^{1,2}(\Omega,\gamma)$; along a subsequence $\{f_{n_k}\}$ we have

$$|\nabla_H T^{\Omega}(t) f_{n_k}(x)|_H \to |\nabla_H T^{\Omega}(t) f(x)|_H \quad \text{a.e. in } \Omega$$

and again up to a subsequence we have

$$T^{\Omega}(t)|\nabla_H f_{n_k}(x)|_H \to T^{\Omega}(t)|\nabla_H f(x)|_H \quad \text{a.e. in } \Omega.$$

Therefore

$$\begin{aligned}
|\nabla_H T^{\Omega}(t) f(x)|_H &= \lim_{k\to\infty} |\nabla_H T^{\Omega}(t) f_{n_k}(x)|_H \\
&\le \lim_{k\to\infty} e^{-t}T^{\Omega}(t)|\nabla_H f_{n_k}(x)|_H = e^{-t}T^{\Omega}(t)|\nabla_H f(x)|_H
\end{aligned}$$

for a.e. $x \in \Omega$. □

**Proposition 38.** *For all $f$, $g \in L^2(\Omega,\gamma)$ we have*

$$\big[T^\Omega(t)(fg)\big]^2 \le T^\Omega(t)(f^2)T^\Omega(t)(g^2) \quad a.e.\ in\ \Omega$$

*Proof.* As before we consider $f, g, \varphi \in \mathcal{F}C_b(X)$ and using Proposition 37 we prove that

$$\int_{\Omega_n} \varphi(x)[T^{(n)}(t)(fg)(x)]^2\gamma(dx)$$
$$\le \int_{\Omega_n} \varphi(x)\sqrt{T^{(n)}(t)(f^2)(x)\cdot T^{(n)}(t)(g^2)(x)}\gamma(dx).$$

The proof is similar to the previous one with the only difference that now there are more functions but there aren't gradients. Then taking the limit for $n \to \infty$ we get

$$\int_{\Omega} \varphi(x)[T^{\Omega}(t)(fg)(x)]^2\gamma(dx) \le \int_{\Omega} \varphi(x)\sqrt{T^{\Omega}(t)(f^2)(x)\cdot T^{\Omega}(t)(g^2)(x)}\gamma(dx).$$

Since the restrictions to $\Omega$ of the functions of $\mathcal{F}C_b(X)$ are dense in $L^2(\Omega,\gamma)$ we obtain our claim. □

Now we turn to the Poincaré inequality. We set

$$m_\Omega(\varphi) = \frac{1}{\gamma(\Omega)}\int_\Omega \varphi\ d\gamma, \quad \varphi \in L^1(\Omega,\gamma).$$

We recall that if $\mathcal{O} \subset \mathbb{R}^N$ is an open convex set the then the Poincarè inequality holds (see Proposition 4.2 in [8]), that is

$$\int_{\mathcal{O}} |\psi - m_{\mathcal{O}}(\psi)|^2 d\gamma_N \le \int_{\mathcal{O}} |\nabla\psi|^2 d\gamma_N, \quad \forall \psi \in W^{1,2}(\mathcal{O},\gamma_N). \tag{2.7}$$

**Proposition 39.** *For each $\varphi \in W^{1,2}(\Omega,\gamma)$ we have*

$$\int_\Omega |\varphi - m_\Omega(\varphi)|^2 d\gamma \le \int_\Omega |\nabla_H\varphi|_H^2 d\gamma \tag{2.8}$$

*Proof.* First we prove that (2.8) holds for every $f \in \mathcal{F}C_b^1(X)$. Therefore

$$f(x) = v(l_1(x),\dots,l_k(x))$$

with $v \in C_b^1(\mathbb{R}^k)$ and $l_1,\dots,l_k \in X^*$. Let $G := \mathrm{span}\{F_n, R_\gamma(l_1),\dots,R_\gamma(l_k)\}$. Let $\{h_i\}_{i=1}^q \subset X^*$ be an orthonormal basis of $G$ such that $\{h_1,\dots,h_j\}$ is an orthonormal basis of $F_n$ if $\dim F_n = j$. Then $G$ is a subspace of $H$ of dimension $q \le n+k$; setting $d = q - \dim F_n$ let $\mathcal{O} := \mathcal{O}_n \times \mathbb{R}^d$. Let $\{\widehat{h}_i\}_{i=1}^q$ be an orthonormal basis of $G$ such that $\{\widehat{h}_1, \dots, \widehat{h}_j\}$ is an orthonormal basis of $F_n$ if $\dim F_n = j$. Then we have

$$\begin{aligned}
&f(x) = \varphi(\pi_G(x)), \quad \varphi \in C_b^1(G),\\
&m_{\Omega_n}(f) = \frac{1}{\gamma(\Omega_n)} \int_{\Omega_n} f(x)\ \gamma(dx) = \frac{1}{\gamma_G(\mathcal{O})} \int_{\mathcal{O}} \varphi(\xi)\ \gamma_G(d\xi) =: m_{\mathcal{O},\gamma_G}(\varphi),\\
&\lim_{n\to\infty} m_{\Omega_n}(f) = m_\Omega(f).
\end{aligned}$$

Applying (2.7) we obtain

$$\begin{aligned}
&\int_{\Omega_n} |f(x) - m_{\Omega_n}(f)|^2 \gamma(dx)\\
&\qquad = \int_{\Omega_n} |f(x) - m_{\mathcal{O},\gamma_G}(\varphi)|^2 \gamma(dx) = \int_{\mathcal{O}} |\varphi(\xi) - m_{\mathcal{O},\gamma_G}(\varphi)|^2 \gamma_G(d\xi)\\
&\qquad \le \int_{\mathcal{O}} |\nabla \varphi(\xi)|^2 \gamma_G(d\xi) = \int_{\Omega_n} |\nabla_H f(x)|_H^2 \gamma(dx).
\end{aligned}$$

Taking the limit as $n \to \infty$ we get

$$\int_\Omega |f(x) - m_\Omega(f)|^2 \gamma(dx) \le \int_\Omega |\nabla_H f(x)|_H^2 \gamma(dx).$$

Let now $f \in W^{1,2}(\Omega, \gamma)$. There exists a sequence $\{f_n\} \subset \mathcal{F}C_b^1(X)$ such that $f_{n|\Omega} \to f$ in $W^{1,2}(\Omega, \gamma)$. It follows that

$$\begin{aligned}
\int_\Omega |f - m_\Omega(f)|^2 d\gamma &= \lim_{n\to\infty} \int_\Omega |f_n - m_\Omega(f)|^2 d\gamma\\
&\le \lim_{n\to\infty} \int_\Omega |\nabla_H f_n|_H^2 d\gamma = \int_\Omega |\nabla_H f|_H^2 d\gamma.
\end{aligned}$$

□

The Poincaré inequality (2.8) implies that if $\nabla_H \varphi$ vanishes a.e. in $\Omega$, then $\varphi$ is constant a.e. in $\Omega$. By the definition of $L$, this implies that the kernel of $L$ consists of constant functions. As a consequence of the Poincaré inequality other spectral properties of $L$ follow.

**Proposition 40.** *For all $\varphi \in L^2(\Omega, \gamma)$ we have*

$$\|T^\Omega(t)\varphi - m_\Omega(\varphi)\|_{L^2(\Omega,\gamma)} \le e^{-t} \|\varphi\|_{L^2(\Omega,\gamma)} \tag{2.9}$$

*and consequently*

$$\sigma(L) \backslash \{0\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \le -1\}. \tag{2.10}$$

*Proof.* Let $f \in D(L^\Omega)$ be such that $m_\Omega(f) = 0$. By using (2.8) we have

$$\int_\Omega L^\Omega f \cdot f d\gamma = -\int_\Omega \langle \nabla_H f, \nabla_H f\rangle_H d\gamma = -\int_\Omega |\nabla_H f|_H^2 d\gamma \le -\|f\|^2_{L^2(\Omega,\gamma)}.$$

For every $\varphi \in L^2(\Omega,\gamma)$, $m_\Omega(\varphi) = 0$, $T^\Omega(t)\varphi \in D(L^\Omega)$ for $t > 0$. Therefore

$$\frac{d}{dt}\|T^\Omega(t)\varphi\|^2_{L^2(\Omega,\gamma)} = 2\int_\Omega L^\Omega T^\Omega(t)\varphi\, T^\Omega(t)\varphi\, d\gamma \le -2\|T^\Omega(t)\varphi\|^2_{L^2(\Omega,\gamma)}, \quad t > 0.$$

It follows that

$$\|T^\Omega(t)\varphi\|^2_{L^2(\Omega,\gamma)} \le e^{-2t}\|\varphi\|^2_{L^2(\Omega,\gamma)}, \quad t > 0. \tag{2.11}$$

Let now $\varphi \in L^2(\Omega,\gamma)$. Then

$$\begin{aligned}\int_\Omega |T^\Omega(t)\varphi - m_\Omega(\varphi)|^2 d\gamma &= \int_\Omega |T^\Omega(t)(\varphi - m_\Omega(\varphi))|^2 d\gamma \le e^{-2t}\int_\Omega |\varphi - m_\Omega(\varphi)|^2 \\ &= e^{-2t}\left(\int_\Omega |\varphi|^2 d\gamma - \gamma(\Omega)[m_\Omega(\varphi)]^2\right) \le e^{-2t}\int_\Omega |\varphi|^2 d\gamma.\end{aligned}$$

Note that $\varphi \mapsto m_\Omega(\varphi)$ is the orthogonal projection on $\mathrm{Ker}(L^\Omega)$. Splitting $L^2(\Omega,\gamma) = \mathrm{Ker}(L^\Omega) \oplus (\mathrm{Ker}(L^\Omega))^\perp$, $T^\Omega(t)$ maps $(\mathrm{Ker}(L^\Omega))^\perp = \mathrm{Ker}(m_\Omega)$ into itself and the infinitesimal generator of the restriction of $T^\Omega(t)$ to $(\mathrm{Ker}(L^\Omega))^\perp$ is the part $L_0$ of $L^\Omega$ in $(\mathrm{Ker}(L^\Omega))^\perp$. By (2.11), the spectrum of $L_0$ is contained in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \le -1\}$. Since the spectrum of $L^\Omega$ consists of the spectrum of $L_0$ plus the eigenvalue 0, (2.10) follows. □

**Lemma 10.** *If $\varphi \in \mathcal{F}C_b^1(X)$, $\varphi \ge 0$, then $T^\Omega(t)\varphi_{|\Omega} \ge 0$ a.e. in $\Omega$.*

*Proof.* Let $\varphi \in \mathcal{F}C_b^1(X)$, $\varphi \ge 0$, then

$$\varphi(x) = v(l_1(x), \dots, l_k(x))$$

with $l_1, \dots, l_k \in X^*$ and $v \in C_b^1(\mathbb{R}^k)$ with $v \ge 0$. First we prove that $T^{(n)}(t)\varphi \ge 0$ a.e. in $\Omega_n$. Let $G := \mathrm{span}\{F_n, R_\gamma(l_1), \dots, R_\gamma(l_k)\}$. Then $G$ is a subspace of $H$ of dimension $q \le n + k$; setting $d = q - \dim F_n$ let $\mathcal{O} := \mathcal{O}_n \times \mathbb{R}^d$. Let $\{h_i\}_{i=1}^q \subset X^*$ be an orthonormal basis of $G$ such that $\{h_1, \dots, h_j\}$ is an orthonormal basis of $F_n$ if $\dim F_n = j$. Then we have

$$\varphi(x) = \widetilde{v}(\pi_G(x))$$

where $\widetilde{v} \in C_b^1(\mathbb{R}^q)$ and $\widetilde{v} \ge 0$ in $\mathbb{R}^q$. Therefore, by

$$(T^{(n)}(t)\varphi_{|\Omega_n})(x) = T^G(t)\widetilde{v}_{\mathcal{O}}(\pi_G(x)), \quad \text{for } t > 0, \text{ and } x \in \Omega_n.$$

By Proposition 34 we have

$$(T^{(n)}(t)\varphi_{|\Omega_n})(x) \geq 0 \text{ for all } x \in \Omega_n.$$

Thanks to Proposition 32, up to a subsequence,

$$(T^{\Omega}(t)\varphi_{|\Omega})(x) = \lim_{n\to\infty} (T^{(n)}(t)\varphi_{|\Omega_n})(x) \geq 0 \text{ a.e. in } \Omega.$$

$\square$

Now we have all the ingredients to prove the logarithmic Sobolev inequality by the Deuschel-Strook's method (see [12]).

**Proposition 41.** *For all $f \in W^{1,2}(\Omega,\gamma)$ we have*

$$\int_{\Omega} f^2 \log(f) d\gamma \leq \int_{\Omega} |\nabla_H f|_H^2 d\gamma + \|f\|^2_{L^2(\Omega,\gamma)} \log(\|f\|_{L^2(\Omega,\gamma)}). \tag{2.12}$$

*Proof.* First we suppose that $f \in \mathcal{F}C_b^1(\Omega)$ and $f \geq c > 0$ in $\Omega$. Setting $\varphi = f^2$ we have

$$\nabla_H f = \frac{1}{2}\frac{\nabla_H \varphi}{\sqrt{\varphi}}$$

so (2.12) is equivalent to

$$\int_{\Omega} \varphi \log(\varphi) d\gamma - \int_{\Omega} \varphi d\gamma \log\left(\int_{\Omega} \varphi d\gamma\right) \leq \frac{1}{2}\int_{\Omega} \frac{1}{\varphi} |\nabla_H \varphi|_H^2 d\gamma.$$

We remark that

$$\begin{aligned}\frac{d}{dt}\int_{\Omega} T^{\Omega}(t)(\varphi)\log(T^{\Omega}(t)(\varphi))d\gamma &= \int_{\Omega} L^{\Omega}T^{\Omega}(t)(\varphi)\log(T^{\Omega}(t)(\varphi))d\gamma \\ &\quad + \int_{\Omega} L^{\Omega}T^{\Omega}(t)(\varphi)d\gamma.\end{aligned}$$

The second term vanishes for the invariance of $\gamma$ while for the first we recall that

$$\int_{\Omega} (L^{\Omega}\psi) g(\psi) d\gamma = -\int_{\Omega} g'(\psi)|\nabla_H \psi|^2 d\gamma$$

with $g(\xi) = \log \xi$ and $\psi = T^{\Omega}(t)\varphi$. Hence

$$\frac{d}{dt}\int_{\Omega} T^{\Omega}(t)(\varphi)\log(T^{\Omega}(t)(\varphi))d\gamma = -\int_{\Omega} \frac{1}{T^{\Omega}(t)(\varphi)} |\nabla_H T^{\Omega}(t)(\varphi)|^2 d\gamma.$$

Since

$$(|\nabla_H T^{\Omega}(t)(\varphi)|_H)^2 \leq \left(e^{-t} T^{\Omega}(t)|\nabla_H(\varphi)|_H\right)^2$$

and

$$\left(T^\Omega(t)|\nabla_H(\varphi)|_H\right)^2 = \left[T^\Omega(t)\left(\sqrt{\varphi}\frac{|\nabla_H\varphi|_H}{\sqrt{\varphi}}\right)\right]^2 \le T^\Omega(t)(\varphi)T^\Omega(t)\left(\frac{|\nabla_H\varphi|_H^2}{\varphi}\right),$$

we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega T^\Omega(t)(\varphi)\log(T^\Omega(t)(\varphi))d\gamma &\ge -e^{-2t}\int_\Omega T^\Omega(t)\left(\frac{|\nabla_H\varphi|_H^2}{\varphi}\right)d\gamma \\ &= -e^{-2t}\int_\Omega \frac{|\nabla_H\varphi|_H^2}{\varphi}d\gamma.\end{aligned} \tag{2.13}$$

We recall that, by (2.9),

$$T^\Omega(t)\varphi \xrightarrow{t\to\infty} m_\Omega(\varphi), \quad \text{in } L^2(\Omega,\gamma).$$

Moreover

$$\lim_{t\to\infty}\int_\Omega |\log(T^\Omega(t)\varphi) - \log(m_\Omega\varphi)|^2 d\gamma = 0$$

indeed by assumption, we have $\varphi \ge c^2$ and, by Lemma 10, it follows that $T^\Omega(t)\varphi \ge c^2$. Moreover $m_\Omega(\varphi) \ge c^2$, and

$$\begin{aligned}\left|\log(T^\Omega(t)\varphi) - \log(m_\Omega(\varphi))\right| &= \left|\int_{T^\Omega(t)\varphi}^{m_\Omega(\varphi)} \frac{1}{t}\, dt\right| \\ &\le \frac{1}{\min\{T^\Omega(t)\varphi, m_\Omega(\varphi)\}}|T^\Omega(t)\varphi - m_\Omega(\varphi)| \\ &\le \frac{1}{c^2}|T^\Omega(t)\varphi - m_\Omega(\varphi)|.\end{aligned}$$

Then

$$\int_\Omega T^\Omega(t)(\varphi)\log(T^\Omega(t)(\varphi))d\gamma \xrightarrow{t\to\infty} m_\Omega(\varphi)\log\left(m_\Omega(\varphi)\right)$$

Then integrating (2.13) with respect to $t$ between 0 and $\infty$ we get

$$m_\Omega(\varphi)\log\left(m_\Omega(\varphi)\right) - \int_\Omega \varphi\log|\varphi|d\gamma \ge -\frac{1}{2}\int_\Omega \frac{|\nabla_H\varphi|_H^2}{\varphi}d\gamma.$$

Now let $f \in W^{1,2}(\Omega,\gamma)$ and $f \ge 0$ a.e. in $\Omega$. Then there exists a sequence $\{f_n\} \subset \mathcal{F}C_b^1(\Omega)$ such that $f_n \ge 1/n$ and $f_n \to f$ in $W^{1,2}(\Omega,\gamma)$ and almost everywhere. Since $t^2\log(t) > -1$ for all $t > 0$, we can apply the Fatou's lemma and obtain

$$
\begin{aligned}
\int_\Omega f^2\log(f)d\gamma &\le \lim_{n\to\infty}\int_\Omega f_n^2\log(f_n)d\gamma\\
&\le \lim_{n\to\infty}\left[\int_\Omega |\nabla_H f_n|_H^2 d\gamma + \|f_n\|^2_{L^2(\Omega,\gamma)}\log(\|f_n\|_{L^2(\Omega,\gamma)})\right]\\
&= \int_\Omega |\nabla_H f|_H^2 d\gamma + \|f\|^2_{L^2(\Omega,\gamma)}\log(\|f\|_{L^2(\Omega,\gamma)}).
\end{aligned}
$$

For a general $f\in W^{1,2}(\Omega,\gamma)$, (2.12) follows from the fact that $|f|\in W^{1,2}(\Omega,\gamma)$ and $|\nabla|f||_H = |\nabla f|_H$ almost everywhere. $\square$

*Remark* 14. The Log-Sob inequality allows to prove an interesting property of the space $W^{1,2}(\Omega,\gamma)$, namely if $f\in W^{1,2}(\Omega,\gamma)$ and $v$ is a measurable function with $|v(x)|\le k(\|x\|_X+1)$ for some $k>0$ and for a.e. $x\in\Omega$, then $fv\in L^2(\Omega,\gamma)$. To this aim we recall that, by Theorem 6, there exists a constant $\alpha>0$ such that

$$
\int_X e^{\alpha\|x\|_X^2}d\gamma < \infty.
$$

Fix $c<\alpha/4$. Then we have

$$
\begin{aligned}
&\int_\Omega (f(x)v(x))^2 d\gamma\\
&\quad\le k^2\int_\Omega f(x)^2(\|x\|_X+1)^2 d\gamma \le 2k^2\int_\Omega f(x)^2\|x\|_X^2 d\gamma + 2k^2\int_\Omega f(x)^2 d\gamma\\
&\quad= k^2\int_{\{x\in\Omega:\ c\|x\|_X^2>\log|f(x)|\}} f(x)^2\|x\|_X^2 d\gamma\\
&\qquad+ k^2\int_{\{x\in\Omega:\ c\|x\|_X^2\le\log|f(x)|\}} f(x)^2\|x\|_X^2 d\gamma + 2k^2\|f\|^2_{L^2(\Omega,\gamma)}\\
&\quad\le k^2\int_X \|x\|_X^2\ e^{2c\|x\|_X^2}d\gamma + \frac{k^2}{c}\int_\Omega |f(x)|^2\log|f(x)|d\gamma + 2k^2\|f\|^2_{L^2(\Omega,\gamma)}\\
&\quad\le k^2\left(\int_X \|x\|_X^4 d\gamma\right)^{1/2}\left(\int_X e^{4c\|x\|_X^2}d\gamma\right)^{1/2}\\
&\qquad+\frac{k^2}{c}\int_\Omega |f(x)|^2\log|f(x)|d\gamma + 2k^2\|f\|^2_{L^2(\Omega,\gamma)}\\
&\quad\le C+\frac{k^2}{c}\left(\int_\Omega |\nabla_H f|_H^2 d\gamma + \|f\|^2_{L^2(\Omega,\gamma)}\log(\|f\|_{L^2(\Omega,\gamma)})\right) + 2k^2\|f\|^2_{L^2(\Omega,\gamma)},
\end{aligned}
$$

that is $fv\in L^2(\Omega,\gamma)$. The previous estimates shows that the linear function $\Lambda_v: f\mapsto fv$, maps bounded subsets of $W^{1,2}(\Omega,\gamma)$ into bounded subsets of $L^2(\Omega,\gamma)$; this implies that $\Lambda_v$ is continuous.

# Chapter 3

# Maximal Sobolev Regularity for Ornstein-Uhlenbeck equation

In this chapter we study the elliptic equation $\lambda u - L^{\Omega}u = f$ in an open convex subset $\Omega$ of an infinite dimensional separable Banach space $X$ endowed with a centered non-degenerate Gaussian measure $\gamma$, where $L^{\Omega}$ is the Ornstein-Uhlenbeck operator. We prove that for $\lambda > 0$ and $f \in L^2(\Omega, \gamma)$ the weak solution $u$ belongs to the Sobolev space $W^{2,2}(\Omega, \gamma)$. Moreover, under some regularity assumption on the boundary of $\Omega$, we prove that $u$ satisfies a generalized Neumann boundary condition in the sense of traces at the boundary of $\Omega$.

## 3.1 Finite-dimensional estimates

Let $\mathcal{O}$ be an open smooth convex subset of $\mathbb{R}^n$, with fixed $n$. We assume that

$$\mathcal{O} = \{x \in \mathbb{R}^n : \ g(x) < 0\}$$

where $g$ is a smooth convex function whose gradient does not vanish at the boundary $\partial\mathcal{O}$. We denote by $\nu(x)$ the exterior normal vector to $\partial\mathcal{O}$ at $x$, $\nu(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$. Let $\mu$ be the standard Gaussian measure in $\mathbb{R}^n$ and let $L^{\mathcal{O}}$ be the associated Ornstein-Uhlenbeck operator, that is

$$L^{\mathcal{O}}\psi(x) = \sum_{i=1}^{n} D_{ii}\psi(x) - \sum_{i=1}^{n} x_i D_i\psi(x) \text{ for } \psi \in D(L^{\mathcal{O}}).$$

In this section we consider the following problem

$$\begin{cases} \lambda\psi - L^{\mathcal{O}}\psi = f & \text{in } \mathcal{O} \subset \mathbb{R}^n, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{O} \end{cases} \tag{3.1}$$

where $f \in L^2(\mathcal{O}, \mu)$ and $\lambda > 0$.

Let us introduce a weighted surface measure on $\partial\mathcal{O}$:

$$d\sigma = N(x) dH^{n-1}$$

where $N(x) = (2\pi)^{-n/2} \exp(-|x|^2/2)$ is the Gaussian weight and $dH^{n-1}$ is the usual Hausdorff $(n-1)$ dimensional measure. Using the surface measure $d\sigma$ the integration by parts formula reads as:

$$\int_{\mathcal{O}} D_k\varphi \ \psi \ d\mu = -\int_{\mathcal{O}} \varphi D_k\psi \ d\mu + \int_{\mathcal{O}} x_k\varphi\psi \ d\mu + \int_{\partial\mathcal{O}} \frac{D_k g}{|\nabla g|}\varphi\psi \ d\sigma, \qquad (3.2)$$

for each $\varphi, \psi \in W^{1,2}(\mathcal{O}, \mu)$ one of which with bounded support, so the boundary integral is meaningful. Indeed $W^{1,2}(\mathcal{O}, \mu) \subset W^{1,2}_{loc}(\mathcal{O}, dx)$ and the trace at the boundary of any function in $W^{1,2}(\mathcal{O}, \mu)$ belongs to $L^2_{loc}(\partial\mathcal{O}, dH^{n-1}) = L^2_{loc}(\partial\mathcal{O}, d\sigma)$.

Applying (3.2) with $\varphi$ replaced by $D_k\varphi$ and summing up, we find

$$\int_{\mathcal{O}} L^{\mathcal{O}}\varphi\psi \ d\mu = -\int_{\mathcal{O}} \langle\nabla\varphi, \nabla\psi\rangle d\mu + \int_{\partial\mathcal{O}} \frac{\langle\nabla\varphi, \nabla g\rangle}{|\nabla g|}\psi \ d\sigma \qquad (3.3)$$

for every $\varphi \in W^{2,2}(\mathcal{O}, \mu)$ , $\psi \in W^{1,2}(\mathcal{O}, \mu)$ one of which with bounded support.

Now we give dimension free estimates for the weak solution $u \in W^{1,2}(\mathcal{O}, \mu)$ to (3.1) with $\lambda > 0$ and $f \in L^2(\mathcal{O}, \mu)$. We can suppose $f \in C_c^\infty(\mathcal{O})$ because $C_c^\infty(\mathcal{O})$ is dense in $L^2(\mathcal{O}, \mu)$. In this case, thanks to classical results about elliptic equations with smooth coefficients we know that the weak solution $u$ of (3.1) belongs to $C^\infty(\overline{\mathcal{O}}) \subset W^{2,2}_{loc}(\mathcal{O}, \mu)$. Since $\mathcal{O}$ can be unbounded, we introduce a smooth cutoff function $\theta : \mathbb{R}^n \to \mathbb{R}$ such that

$$0 \leq \theta(x) \leq 1, \ |\nabla\theta(x)| \leq 2 \ \forall x \in \mathbb{R}^n, \quad \theta \equiv 1 \text{ in } B(0,1), \quad \theta \equiv 0 \text{ outside } B(0,2)$$

and we set, for $R > 0$

$$\theta_R(x) = \theta(x/R), \quad x \in \mathbb{R}^n.$$

For the $W^{1,2}$ estimates we take $u$ as a test function in the definition of weak solution and we get

$$\lambda \int_{\mathcal{O}} u^2 d\mu + \int_{\mathcal{O}} |\nabla u|^2 d\mu = \int_{\mathcal{O}} fu \ d\mu, \qquad (3.4)$$

then

$$\int_{\mathcal{O}} u^2 d\mu \leq \frac{1}{\lambda^2}\|f\|^2_{L^2(\mathcal{O},\mu)}, \quad \int_{\mathcal{O}} |\nabla u|^2 d\mu \leq \frac{1}{\lambda}\|f\|^2_{L^2(\mathcal{O},\mu)}. \qquad (3.5)$$

The following lemma takes into the account the convexity of $\mathcal{O}$.

**Lemma 11.** *If $u \in C^2(\overline{\mathcal{O}})$ satisfies $\langle \nabla u, \nu\rangle = 0$ on $\partial\mathcal{O}$ then*

$$\langle D^2 u \cdot \nabla u, \nu\rangle \le 0 \text{ on } \partial\mathcal{O}.$$

*Proof.* We recall that $\partial\mathcal{O} = g^{-1}(0)$ where $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth convex function. Let $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu(x)\rangle = 0$ for $x \in \partial\mathcal{O}$, then we have

$$\langle \frac{\partial \nu}{\partial \tau}(x), \tau\rangle \ge 0 \quad \forall x \in \partial\mathcal{O}. \tag{3.6}$$

Indeed

$$\begin{aligned}\frac{\partial \nu}{\partial \tau} &= \frac{\partial}{\partial \tau}\left(\frac{\nabla g}{|\nabla g|}\right) = \frac{1}{|\nabla g|}\frac{\partial}{\partial \tau}(\nabla g) + \frac{\partial}{\partial \tau}\left(\frac{1}{|\nabla g|}\right)\nabla g \\ &= \frac{1}{|\nabla g|} D^2 g \cdot \tau + \langle \nabla\left(\frac{1}{|\nabla g|}\right), \tau\rangle \nabla g,\end{aligned}$$

then for $x \in \partial\mathcal{O}$ we have

$$\begin{aligned}\langle \frac{\partial \nu}{\partial \tau}(x), \tau\rangle &= \frac{1}{|\nabla g(x)|}\langle D^2 g(x)\cdot \tau, \tau\rangle + \langle \nabla\left(\frac{1}{|\nabla g(x)|}\right), \tau\rangle\langle \nabla g(x), \tau\rangle \\ &= \frac{1}{|\nabla g(x)|}\langle D^2 g(x)\cdot \tau, \tau\rangle \ge 0\end{aligned}$$

since $D^2 g$ is a positive semi-definite symmetric matrix. Now we recall that $\langle \nabla u, \nu\rangle = 0$ on $\partial\mathcal{O}$ therefore

$$\frac{\partial}{\partial \tau}(\langle \nabla u(x), \nu(x)\rangle) = \langle D^2 u(x)\ \tau, \nu(x)\rangle + \langle \nabla u(x), \frac{\partial \nu}{\partial \tau}(x)\rangle = 0, \quad x \in \partial\mathcal{O}$$

for each $\tau \in \mathbb{R}^n$ such that $\langle \tau, \nu\rangle = 0$ on $\partial\mathcal{O}$. If we take $\tau = \nabla u(x)$ then we get

$$\langle D^2 u(x)\cdot \nabla u(x), \nu(x)\rangle = -\langle \tau, \frac{\partial \nu}{\partial \tau}(x)\rangle \le 0, \quad x \in \partial\mathcal{O}.$$

□

Now we can give an estimate of the second order derivatives of $u$.

**Proposition 42.** *For every $f \in C_c^\infty(\mathcal{O})$ and $\varepsilon > 0$ there exists $R_0 > 0$ such that for $R > R_0$ the solution $u$ to (3.1) satisfies*

$$(1-\varepsilon)\int_{\mathcal{O}} \theta_R^2 \mathrm{Tr}[(D^2 u)^2] d\mu \le \left(2 + \frac{\varepsilon}{\lambda}\right)\|f\|^2_{L^2(\mathcal{O},\mu)}. \tag{3.7}$$

*Proof.* Recall that $u \in C^\infty(\overline{\mathcal{O}})$; differentiating (3.1) with respect to $x_h$ yields

$$\lambda D_h u - \Delta D_h u - \langle x, \nabla(D_h u)\rangle + D_h u = D_h f.$$

Multiplying by $D_h u \theta_R^2$ we obtain

$$(\lambda + 1)\,(D_h u)^2\theta_R^2 - \Delta D_h u \cdot D_h u\theta_R^2 - \langle x, \nabla(D_h u)\rangle D_h u\theta_R^2 = D_h f D_h u\theta_R^2.$$

Integrating over $\mathcal{O}$ and using (3.3) yields

$$\int_{\mathcal{O}}(\lambda+1)(D_h u)^2\theta_R^2 d\mu + \int_{\mathcal{O}} |\nabla(D_h u)|^2\theta_R^2 d\mu + 2\int_{\mathcal{O}} \theta_R\langle\nabla(D_h u), \nabla\theta_R\rangle D_h u\; d\mu$$
$$= \int_{\partial\mathcal{O}} \frac{\langle\nabla(D_h u), \nabla g\rangle D_h u}{|\nabla g|}\theta_R^2 d\sigma + \int_{\mathcal{O}} D_h f D_h u\theta_R^2 d\mu.$$

Summing over $h$ we obtain

$$\int_{\mathcal{O}}(\lambda+1)|\nabla u|^2\theta_R^2 d\mu + \int_{\mathcal{O}} \mathrm{Tr}[(D^2u)^2]\theta_R^2 d\mu + 2\int_{\mathcal{O}}\langle D^2u\cdot\nabla u, \nabla\theta_R\rangle\theta_R\; d\mu$$
$$= \int_{\partial\mathcal{O}} \frac{\langle D^2u\cdot\nabla u, \nabla g\rangle}{|\nabla g|}\theta_R^2 d\sigma + \int_{\mathcal{O}}\langle\nabla f, \nabla u\rangle\theta_R^2 d\mu.$$

Since $f$ has compact support, for $R$ large enough $\theta_R \equiv 1$ on the support of $f$. For such $R$ we obtain

$$\left|\int_{\mathcal{O}}\langle\nabla f, \nabla u\rangle\theta_R^2 d\mu\right| = \left|-\int_{\mathcal{O}} L^{\mathcal{O}} u f d\mu\right| = \left|\int_{\mathcal{O}}(\lambda u - f) f d\mu\right| \le 2\|f\|^2_{L^2(\mathcal{O},\mu)}.$$

Moreover

$$\left|\int_{\mathcal{O}}\langle D^2u\nabla u, \nabla\theta_R\rangle\theta_R\; d\mu\right| \le \int_{\mathcal{O}}\sum_{i,j=1}^n |D_{ij}u\; D_j u\; D_i\theta_R|\theta_R\; d\mu$$
$$\le \frac{1}{2}\int_{\mathcal{O}}\sum_{i,j=1}^n |D_{ij}u|^2|D_i\theta_R|\theta_R^2\; d\mu + \frac{1}{2}\int_{\mathcal{O}}\sum_{i,j=1}^n |D_j u|^2|D_i\theta_R|\; d\mu$$
$$\le \frac{1}{2}\frac{\||\nabla\theta|\|_\infty}{R}\int_{\mathcal{O}}\theta_R^2\mathrm{Tr}[(D^2u)^2]d\mu + \frac{1}{2}\frac{\||\nabla\theta|\|_\infty}{R}\int_{\mathcal{O}}|\nabla u|^2 d\mu$$
$$\le \left(\frac{1}{2}\int_{\mathcal{O}}\mathrm{Tr}[(D^2u)^2]\theta_R^2 d\mu + \frac{1}{2\lambda}\|f\|^2_{L^2(\mathcal{O},\mu)}\right)\frac{\||\nabla\theta|\|_\infty}{R}.$$

Taking $R$ large enough, such that $\||\nabla\theta|\|_\infty/R \le \varepsilon$, we get

$$(1-\varepsilon)\int_{\mathcal{O}}\theta_R^2\mathrm{Tr}[(D^2u)^2]d\mu \le \left(2 + \frac{\varepsilon}{\lambda}\right)\|f\|^2_{L^2(\mathcal{O},\mu)} + \int_{\partial\mathcal{O}}\theta_R^2\frac{\langle D^2u\cdot\nabla u, \nabla g\rangle}{|\nabla g|}d\sigma.$$

By using Lemma 11, the statement follows. $\square$

**Theorem 18.** *For each $\lambda > 0$ there exists $C = C(\lambda) > 0$, independent of $n$ and $\mathcal{O}$, such that for each $f \in L^2(\mathcal{O}, \mu)$ the weak solution $u$ of problem* (3.1) *belongs to $W^{2,2}(\mathcal{O}, \mu)$, and satisfies*

$$\|u\|_{W^{2,2}(\mathcal{O},\mu)} \leq C\|f\|_{L^2(\mathcal{O},\mu)}. \tag{3.8}$$

*Proof.* Let $f \in C_c^\infty(\mathcal{O})$. Taking the limit as $R \to \infty$ in (3.7) and using the monotone convergence theorem, we get

$$(1-\varepsilon)\int_{\mathcal{O}} \mathrm{Tr}[(D^2u)^2]d\mu \leq \left(2 + \frac{\varepsilon}{\lambda}\right)\|f\|^2_{L^2(\mathcal{O},\mu)}.$$

Now taking the limit as $\varepsilon \to 0$ we get

$$\int_{\mathcal{O}} \mathrm{Tr}[(D^2u)^2]d\mu \leq 2\|f\|^2_{L^2(\mathcal{O},\mu)}. \tag{3.9}$$

Taking into account (3.4), (3.5), and (3.9) we obtain

$$\begin{aligned}\|u\|^2_{W^{2,2}(\mathcal{O},\mu)} &= \|u\|_{L^2(\mathcal{O},\mu)} + \||\nabla u|\|_{L^2(\mathcal{O},\mu)} + \|\mathrm{Tr}[(D^2u)^2]\|_{L^2(\mathcal{O},\mu)} \\ &\leq \left(\frac{1}{\lambda^2} + \frac{1}{\lambda} + 2\right)\|f\|^2_{L^2(\mathcal{O},\mu)}\end{aligned}$$

which is (3.8) with $C(\lambda) = \frac{1}{\lambda^2} + \frac{1}{\lambda} + 2$. For $f \in L^2(\mathcal{O}, \mu)$ the statement follows approaching it by a sequence of functions belonging to $C_c^\infty(\mathcal{O})$. $\square$

We recall that (1.26) holds, and we prove that the condition $\langle \cdot, \nabla f(\cdot)\rangle \in L^2(\mathcal{O}, \mu)$ can be omitted.

**Proposition 43.** *If $f \in W^{2,2}(\mathcal{O}, \mu)$ then the map $x \mapsto \langle x, \nabla f(x)\rangle$ belongs to $L^2(\mathcal{O}, \mu)$, moreover the map*

$$f \mapsto \langle \cdot, \nabla f(\cdot)\rangle$$

*is continuous from $W^{2,2}(\mathcal{O}, \mu)$ to $L^2(\mathcal{O}, \mu)$.*

*Proof.* Let $f \in W^{2,2}(\mathcal{O}, \mu)$, then

$$\int_{\mathcal{O}} |\langle \nabla f, x\rangle|^2 d\mu = \int_{\mathcal{O}} \sum_{i=1}^n (D_i f x_i)^2 d\mu$$

by assumption $D_i f \in W^{1,2}(\mathcal{O},\mu)$ and if $c < 1/4$, by using (1.28), we have

$$\begin{aligned}
&\int_{\mathcal{O}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\
&\quad = \int_{\{x\in\mathcal{O}:\ cx_i^2 > \log|D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\
&\qquad + \int_{\{x\in\mathcal{O}:\ cx_i^2 \le \log|D_i f(x)|\}} (D_i f(x))^2 x_i^2 e^{-|x|^2/2} dx \\
&\quad \le \int_{\mathcal{O}} e^{2cx_i^2} x_i^2 e^{-|x|^2/2} dx + \int_{\mathcal{O}} \frac{1}{c} |D_i f|^2 \log|D_i f| e^{-|x|^2/2} dx \\
&\quad \le C_1 + \frac{1}{c}\left( \int_{\mathcal{O}} |\nabla D_i f| d\mu + \frac{1}{2}\int_{\mathcal{O}} (D_i f)^2 d\mu \log\left( \int_{\mathcal{O}} (D_i f)^2 d\mu \right)\right).
\end{aligned}$$

Summing over $i$ from 1 to $n$ we have $\langle \nabla f, x\rangle \in L^2(\mathcal{O},\mu)$. $\square$

## 3.2 Maximal Sobolev regularity for infinite dimensional problem

Now we consider the infinite dimensional problem. We use the cylindrical approximation defined in Section 1.8. First we show that the restriction to $\Omega$ of cylindrical continuous smooth functions is dense in $L^2(\Omega,\gamma)$. Let $\psi \in L^2(\Omega,\gamma)$, then the zero extension outside $\Omega$, $\overline{\psi}$, belongs to $L^2(X,\gamma)$. By Proposition 31, there exists a sequence of $L^2$ cylindrical functions $\psi_n$ that converges to $\overline{\psi}$ in $L^2(X,\gamma)$. In its turn, each $\psi_n$ may be approached by a sequence of $C_b^\infty$ functions.

Therefore we suppose $f \in \mathcal{F}C_b^\infty(X)$. Then, for some $k \in \mathbb{N}$,

$$f(x) = w(l_1(x),\dots,l_k(x))$$

where $w \in C_b^\infty(\mathbb{R}^k)$, $l_i \in X^*$ for $i = 1,\dots,k$.

Let $G = G(n,f) := \operatorname{span}\{F_n, R_\gamma(l_1),\dots,R_\gamma(l_k)\}$. Then $G$ is a subspace of $H$ of dimension $q = q(n,f) \le j(n) + k$; setting $d := q - j(n)$ let $\mathcal{O} = \mathcal{O}(n,f) := \mathcal{O}_n \times \mathbb{R}^d$. If we denote by

$$\pi_G(x) = \sum_{i=1}^{q} \widehat{h}_i(x) h_i$$

then

$$f(x) = \widetilde{f}(\pi_G(x))$$

where $\widetilde{f} \in C_b^\infty(G)$. Let $\gamma_G$ be the induced measure $\gamma \circ \pi_G^{-1}$ in $G$; if $G$ is identified with $\mathbb{R}^q$ through the isomorphism $x \mapsto (\widehat{h}_1(x), \dots, \widehat{h}_q(x))$ then $\gamma_G$ is the standard Gaussian measure in $\mathbb{R}^q$.

We recall that $L^{\Omega_n}$ is the Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\Omega_n,\gamma}$ while $L^{\mathcal{O}}$ is Ornstein-Uhlenbeck operator associated to the quadratic form $\mathcal{E}_{\mathcal{O},\gamma_G}$.

**Proposition 44.** *Let $v$ be the weak solution of the finite dimensional problem*

$$\lambda v - L^{\mathcal{O}} v = \widetilde{f}_{|\mathcal{O}} \quad \textit{in } \mathcal{O}$$

*Then $u(x) := v(\pi_G(x))$ is the weak solution of*

$$\lambda u - L^{\Omega_n} u = f_{|\Omega_n} \textit{in } \Omega_n$$

*Proof.* We remark that the space $X$ can be split as $X = G \times \widetilde{X}$ where $\widetilde{X} = (I - \pi_G)(X)$, and $\gamma = \gamma_G \otimes \widetilde{\gamma}_G$ where $\widetilde{\gamma}_G = \gamma \circ (I - \pi_G)^{-1}$ is the measure induced on $\widetilde{X}$ by the projection $I - \pi_G$. Let $\varphi \in W^{1,2}(\Omega_n, \gamma)$, then

$$\begin{aligned}
&\int_{\Omega_n} (\lambda u(x)\varphi(x) + \langle \nabla_H u(x), \nabla_H \varphi(x)\rangle_H)\, \gamma(dx) \\
&\quad = \int_{\Omega_n} \lambda v(\pi_G(x))\varphi(\pi_G(x) + (I - \pi_G)(x)) \\
&\qquad + \langle \nabla_H v(\pi_G(x)), \nabla_H \varphi(\pi_G(x) + (I - \pi_G)(x))\rangle_H \gamma(dx) \\
&\quad = \int_{\mathcal{O}\times\widetilde{X}} \lambda v(\xi)\widetilde{\varphi}(\xi + y) \\
&\qquad + \langle \nabla v(\xi), \nabla\widetilde{\varphi}(\xi + y)\rangle \gamma_G(d\xi)\widetilde{\gamma}_G(dy) \quad (\text{where } \widetilde{\varphi}(\cdot + y) \in W^{1,2}(\mathcal{O}, \gamma_G)) \\
&\quad = \int_{\mathcal{O}\times\widetilde{X}} \widetilde{f}(\xi)\widetilde{\varphi}(\xi + y)\gamma_G(d\xi)\widetilde{\gamma}_G(dy) \\
&\quad = \int_{\Omega_n} \widetilde{f}(\pi_G(x))\varphi(x)\gamma(dx) \\
&\quad = \int_{\Omega_n} f(x)\varphi(x)\gamma(dx),
\end{aligned}$$

and the statement follows. □

**Proposition 45.** *The function $u$ satisfies*

$$\|u\|_{W^{2,2}(\Omega,\gamma)} \le K$$

*where*

$$K := C\|f\|_{L^2(\Omega_1,\gamma)}$$

*and $C$ is the constant of Theorem 18.*

*Proof.* We recall that $u(x) = v(\pi_G(x))$. Then

$$\begin{aligned}
\|u\|^2{}_{W^{2,2}(\Omega,\gamma)} &\leq \|u\|^2_{W^{2,2}(\Omega_n,\gamma)} \\
&= \int_{\Omega_n} |u(x)|^2 + \sum_{i=1}^{\infty} |D_i u(x)|^2 + \sum_{i,j=1}^{\infty} |D_{ij} u(x)|^2 \; \gamma(dx) \\
&= \int_{\mathcal{O}} \left( |v(\xi)|^2 + \sum_{i=1}^{q} |\partial_i v(\xi)|^2 \right. \\
&\quad \left. + \sum_{i,j=1}^{q} |\partial_{ij} v(\xi)|^2 \right) \mu(d\xi) \text{ (By using Theorem 18)} \\
&\leq C^2 \|\widetilde{f}\|^2_{L^2(\mathcal{O},\mu)} = C^2 \|f\|^2_{L^2(\Omega_n,\gamma)} \leq C^2 \|f\|^2_{L^2(\Omega_1,\gamma)}
\end{aligned}$$

□

If we consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ of weak solutions of the problems

$$\lambda\psi - L^{\Omega_n}\psi = f_{|\Omega_n} \quad \text{in } \Omega_n.$$

By Proposition 45 it follows

$$\|u_n\|_{W^{2,2}(\Omega,\gamma)} \leq K.$$

Possibly replacing $u_n$ by a subsequence, there exists $u \in W^{2,2}(\Omega,\gamma)$ such that $u_n \rightharpoonup u$ in $W^{2,2}(\Omega,\gamma)$.

**Proposition 46.** *The function $u$ is the weak solution of* (1).

*Proof.* We know that for all $\varphi \in \mathcal{F}C_b^1(X)$

$$\int_{\Omega_n} \lambda u_n \varphi \, d\gamma + \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_{\Omega_n} f\varphi \, d\gamma.$$

We claim that

$$\lim_{n\to\infty} \int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma.$$

Indeed,

$$\int_{\Omega_n} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u_n \varphi \, d\gamma + \int_{\Omega \setminus \Omega_n} \lambda u_n \varphi \, d\gamma; \tag{3.10}$$

by the weak convergence

$$\lim_{n\to\infty} \int_{\Omega} \lambda u_n \varphi \, d\gamma = \int_{\Omega} \lambda u \varphi \, d\gamma$$

while

$$\left|\int_{\Omega\setminus\Omega_n} \lambda u_n\varphi \, d\gamma\right| \le \lambda \left(\int_{\Omega\setminus\Omega_n} |u_n|^2 d\gamma\right)^{1/2} \left(\int_{\Omega\setminus\Omega_n} |\varphi|^2 d\gamma\right)^{1/2} \le \lambda K \left(\int_{\Omega\setminus\Omega_n} |\varphi|^2 d\gamma\right)^{1/2}$$

that goes to zero as $n \to \infty$ by the absolute continuity of the integral. Now we claim that

$$\lim_{n\to\infty} \int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi\rangle_H d\gamma.$$

In fact,

$$\int_{\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma + \int_{\Omega\setminus\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma.$$

By the weak convergence in $W^{1,2}(\Omega, \gamma)$

$$\lim_{n\to\infty} \int_{\Omega} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma = \int_{\Omega} \langle \nabla_H u, \nabla_H \varphi\rangle_H d\gamma$$

while

$$\begin{aligned}\left|\int_{\Omega\setminus\Omega_n} \langle \nabla_H u_n, \nabla_H \varphi\rangle_H d\gamma\right| &\le \lambda \left(\int_{\Omega\setminus\Omega_n} |\nabla_H u_n|_H^2 d\gamma\right)^{1/2} \left(\int_{\Omega\setminus\Omega_n} |\nabla_H \varphi|_H^2 d\gamma\right)^{1/2} \\ &\le \lambda K \left(\int_{\Omega\setminus\Omega_n} |\nabla_H \varphi|_H^2 d\gamma\right)^{1/2}\end{aligned}$$

that goes to zero as $n \to \infty$.

Moreover,

$$\lim_{n\to\infty} \int_{\Omega_n} f\varphi \, d\gamma = \int_{\Omega} f\varphi \, d\gamma.$$

Therefore letting $n \to \infty$ in (3.10) we get that $u$ satisfies (1.25). □

Finally we give the maximal regularity result.

**Theorem 19.** *If $u$ is the weak solution of $\lambda u - L^{\Omega} u = f$ on $\Omega$ then $u \in W^{2,2}(\Omega, \gamma)$ and*

$$\|u\|_{W^{2,2}(\Omega,\gamma)} \le C \|f\|_{L^2(\Omega,\gamma)}$$

*Proof.* By Proposition 45 it follows

$$\|u_n\|_{W^{2,2}(\Omega,\gamma)} \le C \|f\|_{L^2(\Omega_n,\gamma)} \tag{3.11}$$

where $C = C(\lambda)$ is the constant of the Theorem 18.

We remark that

$$\lim_{n\to\infty} \|f\|_{L^2(\Omega_n,\gamma)} = \|f\|_{L^2(\Omega,\gamma)}$$

since $\gamma(\Omega_n \backslash \Omega) \to 0$.

By the weak convergence of $u_n$ to $u$ we have

$$\|u\|_{W^{2,2}(\Omega,\gamma)} \leq \limsup_{n\to\infty} \|u_n\|_{W^{2,2}(\Omega,\gamma)}.$$

Letting $n \to \infty$ in (3.11) we get our claim. □

## 3.3 The Neumann boundary condition

In this section we put under Assumption 1 and we prove that the weak solution $u$ of (1) satisfies a Neumann type boundary condition.

First we prove a useful lemma.

**Proposition 47.** *If $u \in L^p(\partial\Omega, \rho)$ and*

$$\int_{\partial\Omega} u\varphi \, d\rho = 0 \quad \forall \varphi \in \mathcal{F}C_b^1(X),$$

*then $u = 0$ $\rho-$a.e. in $\partial\Omega$.*

*Proof.* Since the map

$$v \mapsto \int_{\partial\Omega} uv \, d\rho$$

is continuous from $W^{1,q}(\Omega, \gamma)$ to $\mathbb{R}$ for all $q > p'$, and $\mathcal{F}C_b^1(X)$ is dense in $W^{1,q}(\Omega, \gamma)$, it follows that

$$\int_{\partial\Omega} u\psi \, d\rho = 0 \quad \forall \psi \in W^{1,q}(\Omega, \gamma).$$

In particular, since the restrictions to $\Omega$ of the Lipschitz continuous and bounded functions $\psi : X \to \mathbb{R}$ belong to $W^{1,q}(\Omega, \gamma)$, we have

$$\int_{\partial\Omega} u\psi \, d\rho = 0 \quad \forall \psi \in Lip_b(X).$$

Lemma 13 yields

$$\int_{\partial\Omega} u\psi \, d\rho = 0 \quad \forall \psi \in L^q(\partial\Omega, \rho)$$

and this implies that $u = 0$ $\rho-$a.e.. □

Now we are ready to prove that the weak solution of (1) satisfies a boundary condition similar to the Neumann boundary condition.

**Proposition 48.** *If $u$ is the weak solution of $\lambda u - Lu = f$ on $\Omega$ then*

$$\langle \nabla_H u(x), \frac{\nabla_H g(x)}{|\nabla_H g(x)|_H} \rangle_H = 0 \quad \rho - a.e\ x \in \partial\Omega. \tag{3.12}$$

*Proof.* We fix $\varphi \in \mathcal{F}C_b^1(X)$. We denote by $u_n$ the solution to

$$\lambda\psi - L^{\Omega_n}\psi = f_{|\Omega_n} \quad \text{in } \Omega_n. \tag{3.13}$$

We recall that $u_n$ is a cylindrical function and, thanks to the result of Section 1.8, we have $u_n \in W^{2,2}(\Omega_n, \gamma)$. We multiply the differential equation (3.13) by $\varphi$ and we integrate on $\Omega$, getting

$$\int_\Omega (\lambda u_n - L^{\Omega_n} u_n)\varphi \, d\gamma = \int_\Omega f\varphi \, d\gamma.$$

We recall that $L^{\Omega_n} u_n$ is cylindrical, then

$$L^{\Omega_n} u_n(x) = \sum_{i=1}^q \partial_{ii} u_n(x) - \widehat{h}_i(x)\partial_i u_n(x).$$

Therefore, by using (1.20), we obtain

$$\int_\Omega \lambda\varphi u_n \, d\gamma + \int_\Omega \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_\Omega f\varphi \, d\gamma + \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \, d\rho,$$

where

$$\langle \nabla_H u_n, \nabla_H \varphi \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i \varphi,$$

and

$$\langle \nabla_H u_n, \nabla_H g \rangle_H = \sum_{i=1}^q \partial_i u_n \partial_i g.$$

As in the previous section we have

$$\lim_{n\to\infty} \int_\Omega \lambda\varphi u_n \, d\gamma = \int_\Omega \lambda\varphi u \, d\gamma,$$

and

$$\lim_{n\to\infty} \int_\Omega \langle \nabla_H u_n, \nabla_H \varphi \rangle_H d\gamma = \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma,$$

We claim that the map

$$v \mapsto \int_{\partial\Omega} \langle \nabla_H v, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \; d\rho$$

from $W^{2,2}(\Omega, \gamma)$ to $\mathbb{R}$ belongs to $(W^{2,2}(\Omega, \gamma))'$. Indeed, the function

$$x \mapsto \langle \nabla_H v(x), \frac{\nabla_H g(x)}{|\nabla_H g(x)|_H} \rangle_H \varphi(x) =: F(x)$$

belongs to $W^{1,q}(\Omega, \gamma)$ for all $q \in (1, 2)$. Moreover $\|F\|_{W^{1,q}(\Omega,\gamma)} \leq \widetilde{C}\|v\|_{W^{2,2}(\Omega,\gamma)}$, and the trace operator is linear and continuous from $W^{1,q}(\Omega, \gamma)$ to $L^1(\partial\Omega, \rho)$. Therefore, since $u_n \rightharpoonup u$ in $W^{2,2}(\Omega, \gamma)$,

$$\lim_{n\to\infty} \int_{\partial\Omega} \langle \nabla_H u_n, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \; d\rho = \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \; d\rho.$$

Then we have

$$\int_\Omega \lambda u \varphi \; d\gamma + \int_\Omega \langle \nabla_H u, \nabla_H \varphi \rangle_H d\gamma = \int_\Omega f \varphi \; d\gamma + \int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \; d\rho$$

and since $u$ is a weak solution of (1) we get

$$\int_{\partial\Omega} \langle \nabla_H u, \frac{\nabla_H g}{|\nabla_H g|_H} \rangle_H \varphi \; d\rho = 0$$

for all $\varphi \in \mathcal{F}C_b^1(X)$. By using Proposition 47 we obtain (3.12). $\square$

Therefore, if $u \in D(L)$ then $u \in W^{2,2}(\Omega, \gamma)$ and $u$ satisfies the Neumann boundary condition (3.12).

# Appendix A

# Density properties

In this appendix we show some density results for which we thank Simone Ferrari. Let $(Y, d)$ be a complete metric space and let $\rho$ be a finite Radon measure defined on the Borel sets of $Y$. Let $BUC(Y)$ be the set of real value uniformly bounded continuous functions and let $Lip_b(Y)$ be the set of Lipschitz bounded functions.

**Lemma 12.** *Let $f : Y \to \mathbb{R}$ be a bounded $\rho-$measurable function. Then for all $\varepsilon > 0$ there exists $g \in BUC(Y)$ such that*

$$\rho(\{x \in Y : \ f(x) \neq f_\varepsilon(x)\}) < \varepsilon$$

*and*

$$\sup_{x \in Y} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|.$$

*Proof.* We fix $\varepsilon > 0$. Since $\rho$ is a Radon measure then there exists $K_0$, compact subset of $Y$, such that $\rho(Y \setminus K_0) < \varepsilon$. By the Lusin theorem there exists a function $f_0 \in C(K_0) = BUC(K_0)$ such that:

$$\rho(\{x \in K_0 : \ f_0(x) \neq f_{|K_0}(x)\}) < \varepsilon$$

and

$$\sup_{x \in K_0} |f_0(x)| \leq \sup_{x \in K_0} |f(x)| \leq \sup_{x \in Y} |f(x)|.$$

We define the following function, studied in [22],

$$g(x) = \begin{cases} f(x) & \text{if } x \in K_0 \\ \inf\limits_{y \in K_0} f_0(y) \dfrac{d(x, y)}{d(x, K_0)} & \text{if } x \notin K_0 \end{cases}$$

then $g$ is a $BUC$ extension of $f_0$ to the whole $Y$. We remark that for $x \notin K_0$ there exists $y_\varepsilon \in K_0$ such that

$$d(x, K_0) = \inf_{y \in K_0} d(x, y) \geq d(x, y_\varepsilon) - \varepsilon,$$

therefore for $x \notin K_0$ we have

$$|g(x)| = \left| \inf_{y \in K_0} f_0(y) \frac{d(x,y)}{d(x,K_0)} \right| \leq \sup_{x \in Y} |f(x)| \frac{d(x, y_\varepsilon)}{d(x, K_0)} \leq \sup_{x \in Y} |f(x)| \frac{d(x, K_0) + \varepsilon}{d(x, K_0)}$$

for all $\varepsilon$. Then for all $x \notin K_0$ we have

$$g(x) \leq \sup_{y \in Y} |f(y)|.$$

Finally

$$\begin{aligned}\sup_{x \in Y} |g(x)| &= \sup_{x \in Y} |g_{|K_0}(x) + g_{|Y \setminus K_0}(x)| \leq \sup_{x \in K_0} |g(x)| + \sup_{x \in Y \setminus K_0} |g(x)| \\ &= \sup_{x \in K_0} |f_0(x)| + \sup_{x \in Y \setminus K_0} |g(x)| \leq 2 \sup_{x \in Y} |f(x)|.\end{aligned}$$

Moreover

$$\begin{aligned}\rho(\{x \in Y : \ g(x) \neq f(x)\}) &\leq \rho(\{x \in K_0 : \ g(x) \neq f(x)\}) \\ &\quad + \rho(\{x \in Y \setminus K_0 : \ g(x) \neq f(x)\}) \\ &\leq \rho(\{x \in K_0 : \ f_0(x) \neq f(x)\}) + \rho(Y \setminus K_0) < 2\varepsilon.\end{aligned}$$

□

**Lemma 13.** *The subspace $Lip_b(Y)$ is dense in $L^p(Y, \rho)$ with respect the norm $\| \cdot \|_{L^p(Y,\rho)}$.*

*Proof.* Let $f \in L^p(Y, \rho)$. For $k \in \mathbb{N}$ we put

$$f_k(x) = \begin{cases} k & \text{if } f(x) > k \\ f(x) & \text{if } f(x) \in [-k, k] \\ -k & \text{if } f(x) < -k \end{cases}$$

so that $f_k(x)$ is bounded and measurable. Then by Lemma 12 there exists $\widetilde{f}_k \in BUC(Y)$ such that

$$\rho(\{x \in Y : \ \widetilde{f}_k(x) \neq f_k(x)\}) \leq \frac{1}{2^k}$$

Then by [25] there exists $g_k \in Lip_b(Y)$ such that

$$\|g_k - \widetilde{f}_k\|_{L^\infty(Y)} \le \frac{1}{2^k}.$$

Now we estimate

$$\|g_k - f\|_{L^p(Y,\rho)} \le \|g_k - \widetilde{f}_k\|_{L^p(Y,\rho)} + \|\widetilde{f}_k - f_k\|_{L^p(Y,\rho)} + \|f_k - f\|_{L^p(Y,\rho)},$$

where

$$\begin{aligned}\|g_k - \widetilde{f}_k\|_{L^p(Y,\rho)} &= \left(\int_Y |g_k(x) - \widetilde{f}_k(x)|^p \rho(dx)\right)^{1/p} \\ &\le \|g_k - \widetilde{f}_k\|_{L^\infty(Y)} \rho(Y)^{1/p} \le \frac{\rho(Y)^{1/p}}{2^k}.\end{aligned}$$

Concerning the second term we recall that

$$\sup_{x \in Y} |\widetilde{f}_k(x)| \le 2 \sup_{x \in Y} |f_k(x)| = 2k$$

then

$$\begin{aligned}\|\widetilde{f}_k - f_k\|_{L^p(Y,\rho)} &= \left(\int_Y |\widetilde{f}_k(x) - f_k(x)|^p \rho(dx)\right)^{1/p} \\ &= \left(\int_{\{x \in Y:\ \widetilde{f}_k(x) \neq f_k(x)\}} |\widetilde{f}_k(x) - f_k(x)|^p \rho(dx)\right)^{1/p} \\ &\le 3k\ \rho(\{x \in Y:\ \widetilde{f}_k(x) \neq f_k(x)\})^{1/p} \le \frac{3k}{2^{k/p}}.\end{aligned}$$

Finally we remark that since $f_k \to f$ $\rho$-a.e. for $k \to \infty$, and $|f_k(x)| \le |f(x)| \in L^p(Y,\rho)$, then the Lebesgue theorem yields

$$\|f_k - f\|_{L^p(Y,\rho)} \to 0,\ k \to \infty.$$

$\square$

# Appendix B

# Hermite polynomials and Wiener chaos decomposition

In this chapter we introduce the Hermite polynomials and their link with the Ornstein-Uhlenbeck operator in the whole space $X$. We refer to the books [3] for a detailed treatment.

First we define the Hermite polynomials in finite dimension.

**Definition 32.** For $k \in \mathbb{N}$, we define the Hermite polynomials of degree $k$ by

$$H_k(x) := \frac{(-1)^k}{\sqrt{k!}} \exp\left\{\frac{x^2}{2}\right\} \frac{d^k}{dx^k}\left(\exp\left\{-\frac{x^2}{2}\right\}\right). \tag{B.1}$$

We set $H_0(t) = 1$.

The first Hermite polynomials are

$$\begin{aligned} &H_1(x) = x, \quad H_2(x) = \frac{1}{\sqrt{2}}(x^2-1), \\ &H_3(x) = \frac{1}{\sqrt{6}}(x^3-3x), \quad H_4(x) = \frac{1}{2\sqrt{6}}(x^4-6x^2+3). \end{aligned}$$

Now give some properties of Hermite polynomials.

**Proposition 49.** *The Hermite polynomials on $\mathbb{R}$ satisfy the following properties:*

1. $\{H_k\}_{k\in\mathbb{N}\cup\{0\}}$ *is an orthonormal basis of* $L^2(\mathbb{R}, \gamma_1)$*;*
2. $H_k'(x) = \sqrt{k}H_{k-1}(x)$*, for all* $k \geq 1$*;*
3. $H_k'(x) = xH_k(x) - \sqrt{k+1}H_{k+1}(x)$*, for all* $k \geq 1$*;*

*4.* $H_k''(x) = -H_k'(x) - kH_k(x)$, *for all* $k \geq 1$.

**Definition 33.** Let $\alpha = (k_1, \dots, k_n)$ be a multi-index. We define the Hermite polynomials on $\mathbb{R}^n$ as

$$H_\alpha(x) = H_{k_1}(x_1) \cdots H_{k_n}(x_n),$$

where $H_{k_i}$ is the Hermite polynomial on $\mathbb{R}$ of degree $k_i$ .

We set

$$\mathcal{X}_k := \overline{\mathrm{span}\{H_\alpha : \ |\alpha| = k\}} \text{ in } L^2(\mathbb{R}^n, \gamma_n).$$

**Proposition 50.** *For all* $h, k \in \mathbb{N} \cup \{0\}$ *with* $h \neq k$*, we have* $\mathcal{X}_h \perp \mathcal{X}_k$*.*

*Proof.* It is sufficient to prove that for $|\alpha| = k$ and $|\beta| = h$, with $h \neq k$, we have

$$\int_{\mathbb{R}^n} H_\alpha(x) H_\beta(x) \gamma_n(dx) = 0.$$

Recalling Definition 33, we have

$$\int_{\mathbb{R}^n} H_\alpha(x) H_\beta(x) \gamma_n(dx) = \prod_{i=1}^n \int_{\mathbb{R}} H_{\alpha_i}(x_i) H_{\beta_i}(x_i) \gamma_1(dx_i) = 0,$$

since $h \neq k$ implies that there exists $j \in \{1, \dots, n\}$ such that $\alpha_j \neq \beta_j$. $\square$

**Proposition 51.** $\{H_\alpha\}_{\alpha \in (\mathbb{N} \cup \{0\})^n}$ *is an orthonormal basis of* $L^2(\mathbb{R}^n, \gamma_n)$*. Moreover*

$$L^2(\mathbb{R}^n, \gamma_n) = \bigoplus_{k=0}^\infty \mathcal{X}_k.$$

## B.1 Hermite polynomials in infinite dimension

Let $X$ be a separable Banach space endowed with a non-degenerate centered Gaussian measure $\gamma$. Let $\{h_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $H$ such that $\{\widehat{h}_n\}_{n \in \mathbb{N}} \subset X^*$ is a basis of $X^*_\gamma$.

**Definition 34.** The Hermite polynomials are function of the type

$$x \mapsto H_{k_1}(\widehat{h}_1(x)) \cdots H_{k_m}(\widehat{h}_m(x)), \ x \in X,$$

with $m \in \mathbb{N}$ and $k_1, \dots, k_m \in \mathbb{N} \cup \{0\}$.

**Proposition 52.** *The space generated by the Hermite polynomials is dense in* $L^2(X, \gamma)$*.*

**Proposition 53.** *The Hermite polynomials are orthonormal in $L^2(X,\gamma)$.*

*Proof.* We fix $n, m \in \mathbb{N}$, $n \geq m$. Let $\alpha, \beta$ be two multi-indices, with $|\alpha| = m$ and $|\beta| = n$. Then

$$\begin{aligned}
&\int_X H_{\alpha_1}(\widehat{h}_1(x)) \cdots H_{\alpha_m}(\widehat{h}_m(x)) \cdot H_{\beta_1}(\widehat{h}_1(x)) \cdots H_{\beta_n}(\widehat{h}_n(x))\, \gamma(dx) \\
&\quad = \int_{\mathbb{R}^n} H_{\alpha_1}(\xi_1) \cdots H_{\alpha_m}(\xi_m) \cdot H_{\beta_1}(\xi_1) \cdots H_{\beta_n}(\xi_n)\, \gamma_n(d\xi) \\
&\quad = \langle H_{(\alpha_1,\ldots,\alpha_m,0,\ldots,0)}, H_\beta \rangle_{L^2(\mathbb{R}^n,\gamma_n)} = \begin{cases} 1 & \text{if } (\alpha, 0) = \beta \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

□

Now we consider $\alpha \in (\mathbb{N} \cup \{0\})^{\mathbb{N}}$, $\alpha = (\alpha_1, \ldots, \alpha_n, \ldots)$, and

$$|\alpha| = \sum_{n=1}^{\infty} \alpha_n.$$

Then $|\alpha| < \infty$ if and only if $\alpha_n \neq 0$ for a finite number if $n$. In this case we put

$$H_\alpha(x) = \prod_{n=1}^{\infty} H_{\alpha_n}(\widehat{h}_n(x)).$$

For $k \in \mathbb{N} \cup \{0\}$ we set

$$\mathcal{X}_k := \overline{\mathrm{span}\{H_\alpha : \ |\alpha| = k\} \text{ in } L^2(X,\gamma)}.$$

Then

$$L^2(X,\gamma) = \bigoplus_{k \in \mathbb{N} \cup \{0\}} \mathcal{X}_k. \tag{B.2}$$

The identity (B.2) is called Wiener chaos decomposition. For $f \in L^2(X,\gamma)$, the orthogonal projection of $f$ on $\mathcal{X}_k$ is given by

$$I_k(f) = \sum_{|\alpha|=k} \langle f, H_\alpha \rangle_{L^2(X,\gamma)} \cdot H_\alpha.$$

We define a family of operators $(T(t))_{t\geq 0} \subset \mathcal{L}(L^2(X,\gamma))$ as

$$T(t)f(x) = \int_X f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right) \gamma(dy).$$

One can prove that $(T(t))_{t\geq 0}$ is a $C_0-$semigroup named Ornstein-Uhlenbeck semigroup (see [3]).

Let $L$ be the infinitesimal generator of $(T(t))_{t\geq 0}$ in $L^2(X,\gamma)$, with

$$D(L) := \left\{ f \in L^2(X,\gamma) : \ \exists \lim_{t\to 0^+} \frac{T(t)f - f}{t} \text{ in } L^2(X,\gamma) \right\}.$$

We call $L$ the Ornstein-Uhlenbeck operator.

**Proposition 54.** *If $L$ is the infinitesimal generator of $(T(t))_{t\geq 0}$, then*

$$D(L) = \Big\{ u \in W^{1,2}(X,\gamma) : \ \exists g \in L^2(X,\gamma) : \\ \int_X \langle \nabla_H u, \nabla_H \varphi \rangle_H \, d\gamma = \int_X g\, \varphi \, d\gamma \ \forall \varphi \in W^{1,2}(X,\gamma) \Big\},$$

*moreover*

$$D(L) = W^{2,2}(X,\gamma),$$

*with equivalence of the norms.*

The following property links the Ornstein-Uhlenbeck semigroup and the Hermite polynomials.

**Proposition 55.** *For all $t \geq 0$ and $f \in L^2(X,\gamma)$, we have*

$$T(t)f = \sum_{k=0}^{\infty} e^{-kt} I_k(f).$$

As a consequence of Proposition 55, we have

$$D(L) = \left\{ f \in L^2(X,\gamma) : \ \sum_{k=0}^{\infty} k^2 \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty \right\},$$

and

$$Lf = \sum_{k=0}^{\infty} -k I_k(f), \quad \forall f \in D(L).$$

Therefore $-k$ is eigenvalue of $L$, and the corresponding eigenspace is $\mathcal{X}_k \cap D(L)$.

**Lemma 14.** *For all $k \in \mathbb{N}$, we have $\mathcal{X}_k \subset D(L)$.*

*Proof.* If $f \in \mathcal{X}_k$, then $I_k(f) = f$ while $I_n(f) = 0$ for all $n \neq k$. Moreover the function $T(t)f = e^{-kt} f$ is differentiable at $t = 0$. □

**Proposition 56.** *The following identity*

$$W^{1,2}(X,\gamma) = \left\{ f \in L^2(X,\gamma) : \ \sum_{k=0}^{\infty} k \|I_k(f)\|^2_{L^2(X,\gamma)} < \infty \right\},$$

*holds.*

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