Dissipation-driven dynamical topological phase transitions in two-dimensional superconductors

Dissipation driven dynamical topological phase transitions in two-dimensional superconductors

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(Dated: January 23, 2024)

We induce and study a topological dynamical phase transition between two planar superconducting phases. Using the Lindblad equation to account for the interactions of Bogoliubov quasiparticles among themselves and with the fluctuations of the superconducting order parameter, we derive the relaxation dynamics of the order parameter. To characterize the phase transition, we compute the fidelity and the spin-Hall conductance of the open system. Our approach provides crucial informations for experimental implementations, such as the dependence of the critical time on the system-bath coupling.

arXiv:2308.08265v2 [cond-mat.str-el] 20 Jan 2024

Introduction. Phase transitions (PTs) emerge as an effect of fluctuations, both thermal [1] or quantum [2], involving collective degrees of freedom in many-body systems. Typically, a PT is characterized by the divergence of the correlation length as the temperature and/or another control parameter is tuned from the outside and with the corresponding power-law scaling of the physical quantities, with "universal" critical exponents. The usual approach to PTs in systems at equilibrium involves methods, such as looking for singularities and for critical scaling in the free-energy functional describing the system, as the control parameters are tuned close to their critical values.

A continuously increasing interest, both on the theoretical as well as on the experimental side, has been recently gained by "dynamical" PTs in closed many-particle quantum systems, prepared in a nonequlibrium state and then evolving in time with a pertinent Hamiltonian [3–5]. In a DPT, it is the time t that drives the system across criticality and the PT is evidenced by singularities in the matrix elements of the time evolution operator. To date, DPTs have been theoretically predicted and experimentally seen in various isolated quantum systems, in which nonequilibrium is induced by quenching some parameter(s) of the system Hamiltonian [6–15]. In a closed system, the DPT is analyzed by looking at the singularities in the Loschmidt echo (LE) $\mathcal{L}(t) = |\langle\psi(0)|\psi(t)\rangle|^2$, with $|\psi(t)\rangle$ being the state of the system at time t and $|\psi(0)\rangle$ the pre-quench initial state [3–6, 16]. While this approach has been also extended to the case in which the system has not been prepared in a pure state [17–19], it does not apply to DPTs in open systems. The latter are described by a time-dependent density matrix $\rho(t)$, whose dynamics is determined by solving the pertinent evolution equation.

In the context of superconducting electronic systems, nonequilibrium dynamics can be induced, for instance, by suddenly quenching the interaction parameter in a BCS Hamiltonian, and by encoding the following dynamical evolution of the system into an explicit dependence of the superconducting order parameter on t, by means of a time-dependent generalization of the self-consistent BCS mean-field (MF) approach [20].

In this Letter we substantially extend and generalize the approach of Ref.[20], so as to induce a DPT between different superconducting phases realized in a planar, interacting fermionic model with an attractive interaction. In particular, we allow for the coexistence of $x^2 - y^2$ (d-wave) and xy (id-wave) superconducting gaps. Then, after quenching the interaction strength(s) at time $t = 0$, we let the superconductor behave as an open system by exchanging Bogoliubov quasiparticles with the bath. In this way, we account for the dissipative dynamics induced by the interactions between quasiparticles not captured by the BCS approximation and/or by the coupling between the fluctuations of the order parameter and the quasiparticle continuum [9, 21–23] and/or for the coupling to an external metallic contact [24] [a detailed discussion is provided in the Supplemental Material (SM), as well as Refs.[21, 23–25]]. In particular, following Refs. [25, 26], we do so within the Lindblad master equation (LME) approach to the time evolution of the superconductor density matrix $\rho(t)$. Our systematic approach naturally emerges from the microscopic model of Ref.[24]. Moreover, it is perfectly consistent with the one introduced in Ref.[23] on phenomenological grounds, as discussed in Ref.[27], where we also estimate typical experimental values of the coupling between the system and the bath.

Here, we focus onto the DPT between a (topologically trivial) id and a $d+id$ planar superconducting phase, the latter of which is known to describe the class C of topological planar superconductors, characterized by particle-hole conjugation and broken time-reversal symmetry [28–33]. Therefore, we realize a topological DPT (TDPT), to characterize which we first of all look at the transition in time of the superconducting order parameter, between the asymptotic values (at $t=0$ and $t \to \infty$), respectively corresponding to the id and to the $d+id$ phase. Then, rather than the LE, we approach the DPT by using the fidelity $\mathcal{F}(t)$ between the initial pure state and the one described by $\rho(t)$, which is more suitable for an open system [27]. Finally, to make a rigorous statement on the topological properties of the phases separated by the DPT, we compute the spin-Hall conductance of the system as a function of t, $\sigma(t)$. In a stationary state, $\sigma(t)$ is proportional to the topological invariant whose nonzero value is a signal of a nontrivial topological phase [31, 34]. Furthermore, it is well defined even when the system goes through the DPT [34]. In addition to defining a protocol to monitor a DPT in an open system, a topic that has recently become of the utmost relevance [35], our approach provides remarkable results of practical interest in a possible experimental realization of the system that we study (e.g., in ultracold atom lattices), such as the variation of the "critical time" t_* as a function of the system-bath coupling.

Model Hamiltonian. Our main reference Hamiltonian $H_{\rm MF}$ stems from the self-consistent mean-field (SCMF) approximation of the Hamiltonian describing interacting spinful fermions on a two-dimensional (2D) square lattice introduced in Ref.[27], with a nearest-neighbor and a next-to-nearest neighbor density-density interaction, both attractive in the spin singlet channel, respectively, with interaction strengths V and Z (both > 0). We therefore set [20, 27]

$$H_{\rm MF} = \sum_{\mathbf{k}} [c^\dagger_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}] \begin{bmatrix} \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}} \\ -[\Delta_{\mathbf{k}}]^* & -\xi_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{\mathbf{k},\uparrow} \\ c^\dagger_{-\mathbf{k},\downarrow} \end{bmatrix}, \quad (1)$$

with $\mathbf{k}$ summed over the full Brillouin zone and with $c_{\mathbf{k},\sigma}$ and $c^\dagger_{\mathbf{k},\sigma}$ being the fermion operators in $\mathbf{k}$ space. The $\mathbf{k}$-dependent gap in Eq.(1) takes, in general, a nonzero imaginary part. The two parameters take the form

$$\begin{aligned} \xi_{\mathbf{k}} &= -2[\cos(k_x) + \cos(k_y)] - \mu \,, \quad (2) \\ \Delta_{\mathbf{k}} &= 2\Delta_{x^2-y^2}\{\cos(k_x) - \cos(k_y)\} - 4i\Delta_{xy}\sin(k_x)\sin(k_y)\,, \end{aligned}$$

with the chemical potential $\mu = 0$ (half filling) and $\Delta_{x^2-y^2}$ and Δ_{xy} determined by the SCMF equations [27]

$$\begin{aligned} \Delta_{x^2-y^2} &= \frac{V}{4\mathcal{N}} \sum_{\mathbf{k}} \frac{[\cos(k_x) - \cos(k_y)]\Delta_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}\,, \\ \Delta_{xy} &= \frac{iZ}{2\mathcal{N}} \sum_{\mathbf{k}} \frac{\sin(k_x)\sin(k_y)\Delta_{\mathbf{k}}}{\epsilon_{\mathbf{k}}}\,, \quad (3) \end{aligned}$$

$\mathcal{N}$ being the number of lattice sites, $\epsilon_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}$ and the lattice constant $=1$. Our model calculation encompasses all the relevant features that should characterize a TDPT, both in solid-state [24], as well as in quantum-optical open systems [35–37]. In Fig.1**a)** we plot the phase diagram of our system in terms of V and Z (which we regard as our physically tunable parameters) as determined by Eqs.(3). For small, though finite, values of V and Z the system lies within a normal phase (N). Keeping Z (V) small and increasing V (Z), our system undergoes a phase transition, with a gap $\Delta_{x^2-y^2}$ (Δ_{xy}) continuously developing a nonzero value for $V > V_c$ ($Z > Z_c$), with $V_c \approx 0.35$ ($Z_c \approx 0.7$), and with the gaps increasing with V and Z, according to Eqs.(3). At large enough values of both V and Z, the system undergoes a topological phase transition, at which a $d+id$ phase opens, where both $\Delta_{x^2-y^2}$ and Δ_{xy} are $\neq 0$. The latter phase exhibits nontrivial topological properties. We now show how to realize a TDPT between the id and the $d+id$ phase, along the dissipative dynamics of the nonequilibrium superconductor.

Dynamical phase transition: To induce a DPT in our system, we prepare it in the ground state of $H_{\rm MF}$ with $\Delta^{(0)}_{x^2-y^2} = 0$ and $\Delta^{(0)}_{xy} \neq 0$. At time $t=0$, we quench the interaction strengths to $(V^{(1)}, Z^{(1)})$, corresponding to both $\Delta^{(1)}_{x^2-y^2}$ and $\Delta^{(1)}_{xy}$ being $\neq 0$. The induced nonequilibrium dynamics makes $\Delta_{\mathbf{k}}$ to explicitly depend on t. To describe the time evolution of the open system, we extend the time-dependent SCMF approach of Ref.[20] by allowing the system to exchange Bogoliubov quasiparticles with an external bath, thus resorting to the LME approach to the density matrix dynamics (see Sec. III of SM). Following Refs.[9, 21–23, 25, 26], we therefore write the LME for $\rho(t)$ as

$$\begin{aligned} \frac{d\rho(t)}{dt} &= -i[H_{\rm MF}(t), \rho(t)] + g\sum_{\lambda=\pm}\sum_{\mathbf{k}} \quad (4) \\ &\times [f(-\lambda\epsilon_{\mathbf{k}}(t))(2\Gamma_{\mathbf{k},\lambda}(t)\rho(t)\Gamma^\dagger_{\mathbf{k},\lambda}(t) - \{\Gamma^\dagger_{\mathbf{k},\lambda}(t)\Gamma_{\mathbf{k},\lambda}(t), \rho(t)\}) \\ &+ f(\lambda\epsilon_{\mathbf{k}}(t))(2\Gamma^\dagger_{\mathbf{k},\lambda}(t)\rho(t)\Gamma_{\mathbf{k},\lambda}(t) - \{\Gamma_{\mathbf{k},\lambda}(t)\Gamma^\dagger_{\mathbf{k},\lambda}(t), \rho(t)\})]\,. \end{aligned}$$

In Eq.(4) g is the strength of the system-bath coupling and, consistently with the detailed balance principle, recovering the Boltzmann distribution as a stationary so-

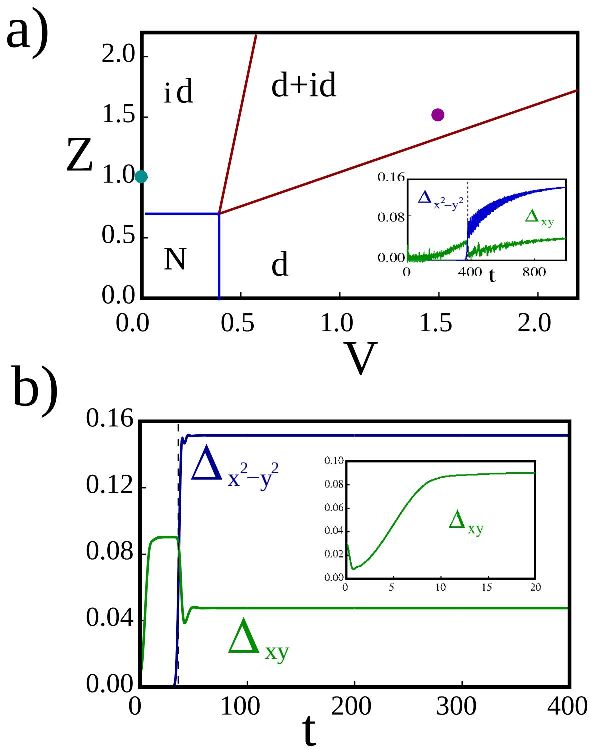

FIG. 1. **a):** Equilibrium phase diagram in the $V - Z$ plane. The cyan (magenta) dot corresponds to $(V^{(0)}, Z^{(0)})$ $(V^{(1)}, Z^{(1)})$ (see text). [**Inset:** Same as in panel **b)** but with $g = 0.002$]. **b):** Time dependent gap $\Delta_{x^2-y^2}(t)$ (blue curve) and $\Delta_{xy}(t)$ (green curve), for $V^{(1)} = Z^{(1)} = 1.5$, $g = 0.2$, and prepared, at $t = 0$, in a state with $\Delta_{xy}^{(0)} \approx 0.03$. [**Inset:** Zoom of the plot of $\Delta_{xy}(t)$ for $0 \leq t \leq 20$].

lution of the LME is assured by our setting of the coupling strength corresponding to the Bogoliubov quasiparticle annihilation and creation operators, $\Gamma_{\mathbf{k},\lambda}$ and $\Gamma^{\dagger}_{\mathbf{k},\lambda}$, to be proportional to $f(-\lambda\epsilon_{\mathbf{k}})$ and to $f(\lambda\epsilon_{\mathbf{k}})$, respectively, with $f(\epsilon)$ being the Fermi distribution function (note that we employ particle-hole symmetry to set $1 - f(\epsilon) = f(-\epsilon)$) [38, 39]. The time dependence of $\Delta_{\mathbf{k}}(t)$ makes $H_{\rm MF}(t)$ in Eq.(4) as well as its eigenvalues $(\pm\epsilon_{\mathbf{k}}(t) = \pm\sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}(t)|^2})$ and eigenmodes $(\Gamma_{\mathbf{k},\pm}(t))$, to acquire an explicit time dependence, as well (see SM and Refs.[20, 25] for further details).

To complete the SCMF approach, we need the relation between $\Delta_{\mathbf{k}}(t)$ and $\rho(t)$. To recover it, we follow the derivation of Ref.[20] by generalizing Eqs.(3) to self-consistent relations between $\Delta_{\mathbf{k}}(t)$ and $f_{\mathbf{k}}(t) = {\rm Tr}[\rho(t)c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\uparrow}]$, as we discuss in detail in the SM.

In Fig.1**b)** we plot $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ in a system prepared in the ground state $|\psi(0)\rangle$ of $H_{\rm MF}$ with $(V^{(0)}, Z^{(0)}) = (0, 1.0)$, corresponding to $\Delta^{(0)}_{x^2-y^2} = 0, \Delta^{(0)}_{xy} = 0.03$. At $t > 0$ we quench the interaction strengths to $(V^{(1)}, Z^{(1)}) = (1.5, 1.5)$, corresponding to $\Delta^{(1)}_{x^2-y^2} = 0.15, \Delta^{(1)}_{xy} = 0.05$, and let the system evolve according to Eqs.(4), with $g = 0.2$ (main figure) and $g = 0.002$ (inset of Fig.1**a)**). Given $(V^{(0)}, Z^{(0)})$ and $(V^{(1)}, Z^{(1)})$, the time evolution of the superconducting gaps is directly determined by Eqs.(4) and by the time-dependent generalizations of Eqs.(3) (see Eq.(8) of SM). We see that there is a finite interval of time $[0, t_*]$, with $t_* \approx 40$, within which $\Delta_{xy}(t)$ stays finite, and basically constant, while $\Delta_{x^2-y^2}(t)$ remains pinned at 0. As t goes across t_* (vertical, dashed line), $\Delta_{xy}(t)$ almost suddenly lowers its value, while $\Delta_{x^2-y^2}(t)$ switches from zero to a finite value, which keeps roughly constant for any $t > t_*$. t_* is determined by the finite time required for the $f_{\mathbf{k}}(t)$'s to take the appropriate threshold value to trigger the onset of the two-component order parameter. The sharp change in $\Delta_{\mathbf{k}}(t)$ across $t = t_*$ corresponds to a transition, in real time, of the system between two different phases characterized by different values of the superconducting order parameter, that is, to a DPT [3–5]. The relatively high value of g (although still quite smaller than any other energy scale in the system) induces a sharp switch in the values of the two different symmetry components of the order parameter at the DPT. To highlight the effects of varying g, in the inset of Fig.1**a)** we show the same plots as in the main figure, but with $g = 0.002$. In this case, t_* becomes much larger than before and $\Delta_{xy}(t)$ oscillates and monotonically increases, starting from $\Delta^{(0)}_{xy}$, as long as $t < t_*$. At the same time, $\Delta_{x^2-y^2}(t) = 0$. At $t \geq t_*$, both $\Delta_{xy}(t)$ and $\Delta_{x^2-y^2}(t)$ undergo a discontinuous jump, after which they start to oscillate around the values they take in the asymptotic $(t \to \infty)$ state. Again, we conclude that, at $t = t_*$, our system goes across a DPT, triggered by the mismatch between the initial and the asymptotic state of the system.

We evidence the onset of the DPT by looking at nonanalyticities in the fidelity $\mathcal{F}(t)$ between the initial state $|\psi(0)\rangle$ and the state described by $\rho(t)$, $\mathcal{F}(t) = \langle\psi(0)|\rho(t)|\psi(0)\rangle$. Indeed, as pointed out in Refs.[40, 41], at a DPT $\mathcal{F}(t)$ is expected to show nonanalyticities similar to what one would obtain in the LE, computed in a closed system. In fact, while the LE at time t is defined if the system lies in a pure state at any t, when the system state is described in terms of a density matrix $\rho(t)$, $\mathcal{F}(t)$ shows to be the pertinent quantity to evidence the DPT. In particular, we look for nonanalyticities in the rate function $\omega(t) = -\frac{1}{N}\log[\mathcal{F}(t)]$ [3–5, 7] (see [27] for the mathematical derivation). In Fig.2 we plot $\omega(t)$ as a function of t in both cases corresponding to the plots in Fig.1. Aside from the different scale t_* at different values of g, we note that, for $0 \leq t < t_*$, $\omega(t)$ takes a mild dependence on t, with $\omega(t) \sim 0.1 - 0.2$, denoting an appreciable overlap between $|\psi(0)\rangle$ and the state at time t. This basically evidences the persistence of the system within the same phase [3–5, 42, 43]. At $t = t_*$, the sudden change in the slope of $\omega(t)$ demonstrates how $t = t_*$ corresponds to a nonanalyticity tied to the DPT. The subsequent rapid increase in $\omega(t)$ for $t > t_*$ corresponds to a drastic reduction in $\mathcal{F}(t)$ (by orders of magnitude), which is a clear signal that, moving across $t = t_*$, the system has gone through a DPT.

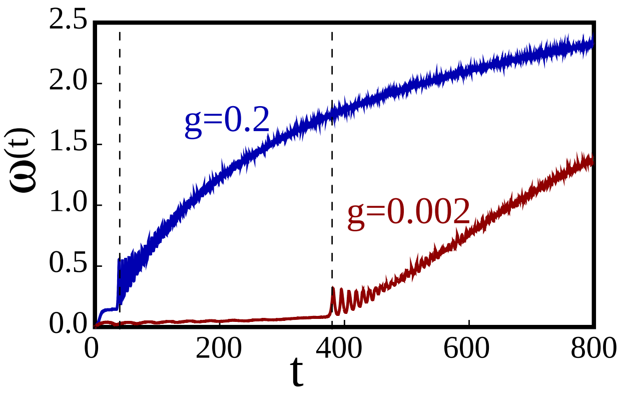

FIG. 2. $\omega(t)$ as a function of time t computed with the time-dependent MF Hamiltonian with parameters $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$, as in Fig.1**b)**, for $g = 0.2$ (blue curve) and $g = 0.002$ (red curve). The dashed vertical lines mark the DPT.

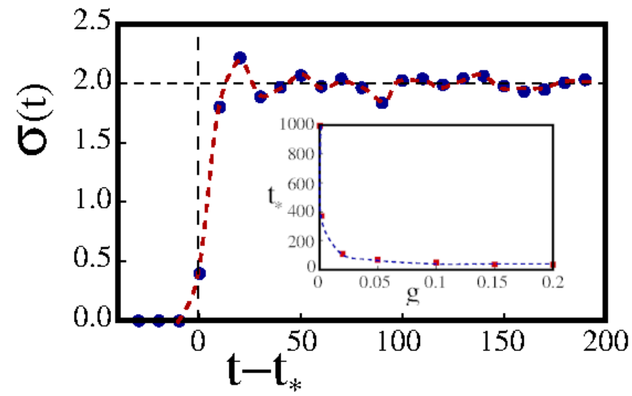

FIG. 3. $\sigma(t)$ (in units of $(2\pi)^{-1}$) computed in the sudden jump approximation (see SM), with $\Delta_{xy}^{(0)} = 0.09$, $\Delta_{x^2-y^2}^{(1)} = 0.15$, $\Delta_{xy}^{(1)} = 0.05$, and $g = 0.2$. The dashed vertical line marks the DPT at $t = t_*$, the dashed horizonal line marks the value of σ_∞. [**Inset**: t_* computed in the same system for different values of g (red squares) between $g = 0.0005$ and $g = 0.2$ (the dashed lines are a guide to the eye).]

Topological phase transition: To physically ground the topological nature of the DPT, we now review the calculation of the spin-Hall conductance $\sigma(t)$ across the DPT of our system (for details, see SM as well as Refs.[15, 28, 30, 31, 34, 36, 37, 44]). In an equilibrium state $\sigma(t)$ is proportional to the Chern number $\mathcal{C}$: in the trivial phase, $\mathcal{C} = 0$, while in a topologically phase, $\mathcal{C} = \pm 2$ [28, 29]. Apparently, $\mathcal{C}$ is ill defined across the DPT. Instead, $\sigma(t)$ is perfectly well defined and can be computed at any finite t within linear (in the applied voltage bias) response theory. Following the derivation of the SM, we note that Fig.1**b)** suggests that, for $g = 0.2$, $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ can be well approximated as $\Delta_{x^2-y^2}(t) = \theta(t - t_*)\Delta_{x^2-y^2}^{(1)}$, and $\Delta_{xy}(t) = \theta(t_* - t)\Delta_{xy}^{(0)} + \theta(t - t_*)\Delta_{xy}^{(1)}$, with $\theta(t)$ being Heaviside's step function, $\Delta_{x^2-y^2}^{(0)} = 0$, $\Delta_{xy}^{(0)} = 0.03$, $\Delta_{x^2-y^2}^{(1)} = 0.15$, and $\Delta_{xy}^{(1)} = 0.05$. In Fig.6 we plot $\sigma(t)$ computed accordingly. As expected, for $t < t_*$ $\sigma(t) = 0$. Passing across the DPT at $t = t_*$, $\sigma(t)$ jumps to a finite value, and then, for $t > t_*$, it shows damped oscillations ($\sim (t - t_*)^{-1}$) toward the asymptotic value $\sigma_\infty = \lim_{t\to\infty} \sigma(t) = 2(2\pi)^{-1}$. This is exactly what is expected for a TDPT. In addition, we have also verified that σ_∞ does not change on varying $\Delta_{x^2-y^2}^{(1)}$ and $\Delta_{xy}^{(1)}$, provided we stay within the $d + id$ phase in Fig.1**a)**.

While our sudden jump approximation only applies for g as large as 0.2, from the inset of Fig.1**a)** we see that, even for $g = 0.002$, the main features of Fig.1**b)** still persist, that is, a sharp reduction in $\Delta_{xy}(t)$ and a corresponding jump in $\Delta_{x^2-y^2}(t)$ from 0 to a finite value at $t = t_*$. From the qualitative point of view, we can still get some hints by enforcing the approximation of $\Delta_{\mathbf{k}}(t)$ with a piecewise function, although along small time intervals (depending, of course, on the frequency of the superimposed oscillations). This would allow us to map out the full time dependence of $\sigma(t)$ on t. Eventually, we expect that $\sigma(t)$ will asymptotically converge to the value dictated by the asymptotic value of $\Delta_{x^2-y^2}(t)$ and of $\Delta_{xy}(t)$ as $t \to \infty$. Again, these correspond to a $d + id$ superconducting state and, therefore, we find that $\sigma(t) \to_{t\to\infty} 2(2\pi)^{-1}$, even at values of g smaller than 0.2 by two orders of magnitude.

Conclusions: In this Letter we have shown that a DPT can take place in an open nonequilibrium planar superconducting system, described by a LME that is determined by the tunnel coupling to an external metallic lead [24], or mimics the residual quasiparticle interaction beyond BCS theory [23]. To monitor the system across the DPT, we synoptically looked at the self-consistently computed superconducting gap, at the fidelity, and at the spin-Hall conductance. In particular, this last quantity is crucial in evidencing the topological nature of the DPT, as it crosses over from being zero for $t < t_*$ to an asymptotic value σ_∞, as $t \to \infty$, which corresponds to a topologically nontrivial $d + id$ phase.

Within our derivation, we explicitly show for the first time a clear evidence that a DPT can take place in an open solid-state system, in particular between two phases with different topological properties. In doing so, we also highlight the importance of the system-bath coupling in stabilizing a DPT transition between two selected phases with given properties. We do so by means of a combined use of the LME and of the time-dependent SCMF approach, thus defining a systematic framework to discuss the peculiar properties of a DPT in an open system. Due to our minimal set of assumptions, we believe that the range of applicability of our approach is much wider than just the one that we discuss here. For instance, it would be important to study whether the topological DPT takes place at any finite g, or whether there is a critical value g_c such that, for $g < g_c$, it is washed out by the uncontrolled oscillations in the superconducting order parameter that arise at small values of g [20]. Along this direction, in the inset of Fig.6 we report the results of a preliminary calculation of t_* at selected values of g between $g = 0.0005$ and $g = 0.2$. We check that a power-law fit of the dependence of t_* on g such as $t_* = Ag^{-B}$, with $A \approx 7.2$ and $B \approx 0.65$ fits the data reasonably well. In addition, the strong increase of t_* as $g \to 0$ is consistent with the absence of any DPT in the closed quenched superconductor studied in Ref.[20]. However, providing a certain answer on those issues requires developing a model calculation of t_* at a given g, which is an interesting topic for future research.

SUPPLEMENTAL MATERIAL

In the following Supplemental Material we provide the details of our calculations that, although being technically important to recover our results, are not essential to follow our main derivation. In particular, in Subsection A we discuss the main steps leading to our time-dependent self-consistent mean field approximation, in Subsection B we present our derivation of the time-dependent spin-Hall conductance, in Subsection C we derive the Lindblad Master Equation from a specific microscopic model of a two-dimensional superconductor coupled to a metallic lead and also discuss how the same equation can arise from the quasiparticle dynamics in the superconductor beyond BCS approximation, in Subsection D we show our results for the rate function and for the spin-Hall conductance across a dynamical phase transition from a planar d+id to a planar s-wave superconducting phase.

A. Time-dependent self-consistent mean field approximation

In order to summarize the main steps behind Eq.(4), here we review the time-dependent SCMF approximation, as introduced in Ref.[20] for an open, superconducting system, together with our specific implementation in the context of the LME, which we derive in detail in Ref.[27].

In order to induce the nonequilibrium dynamics of our system, we prepare it in some initial state, corresponding to the groundstate of a superconducting Hamiltonian such as the one in Eq.(1) with given interaction strengths and then, at $t = 0$, we quench the interaction strengths themselves. At the same time, we couple the system to the external bath. As a result, on one hand, the following time evolution of the density matrix of the resulting open system is described by the LME in Eq.(4), on the other hand, the superconducting gap parameter entering Eq.(4), $\Delta_{\mathbf{k}}(t)$, acquires an explicit dependence on t, which has to be self-consistently determined, in response to the quench in the interaction strengths.

As discussed in Ref.[27], to set up the time-dependent SCMF framework we assume that Eq.(4) holds and define $\nu_{\mathbf{k}}(t)$ and $f_{\mathbf{k}}(t)$ as

$$\begin{aligned} \nu_{\mathbf{k}}(t) &= \mathrm{Tr}\left[\rho(t)\left(c^\dagger_{\mathbf{k},\uparrow}c_{\mathbf{k},\uparrow} - \frac{1}{2}\right)\right] \\ f_{\mathbf{k}}(t) &= \mathrm{Tr}\left[\rho(t)c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\uparrow}\right] \quad . \end{aligned} \tag{5}$$

From Eq.(4), we derive the closed set of differential equations for $\nu_{\mathbf{k}}(t)$ and $f_{\mathbf{k}}(t)$ given by

$$\begin{aligned} \frac{d\nu_{\mathbf{k}}(t)}{dt} &= -\frac{g\xi_{\mathbf{k}}}{\epsilon_{\mathbf{k}}(t)} - 2g\nu_{\mathbf{k}}(t) + \Im m\{[\Delta_{\mathbf{k}}(t)]^* f_{\mathbf{k}}(t)\} \\ \frac{df_{\mathbf{k}}(t)}{dt} &= -(2i\xi_{\mathbf{k}} + 2g)f_{\mathbf{k}}(t) - 2i\Delta_{\mathbf{k}}(t)\nu_{\mathbf{k}}(t) + \frac{g\Delta_{\mathbf{k}}(t)}{\epsilon_{\mathbf{k}}(t)} , \end{aligned} \tag{6}$$

with $\epsilon_{\mathbf{k}}(t) = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}(t)|^2}$ and $\Im m$ denoting the imaginary part. For a superconductor at equilibrium, $\Delta_{\mathbf{k}}$, $\epsilon_{\mathbf{k}}$ and $f_{\mathbf{k}}$, all independent of time, are related to each other by the condition $f_{\mathbf{k}} = \Delta_{\mathbf{k}}/\epsilon_{\mathbf{k}}$. In the nonequilibrium case, following Refs.[20, 27], we assume that $\Delta_{\mathbf{k}}(t)$ takes an explicit dependence on t according to

$$\begin{aligned} \Delta_{\mathbf{k}}(t) = &\, 2\Delta_{x^2-y^2}(t)\{\cos(k_x) - \cos(k_y)\} \\ &- 4i\Delta_{xy}(t)\sin(k_x)\sin(k_y) \quad , \end{aligned} \tag{7}$$

with $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ determined by generalizing Eqs.(3), to account for the explicit dependence on t, as

$$\begin{aligned} \Delta_{x^2-y^2}(t) &= \frac{V}{4\mathcal{N}}\sum_{\mathbf{k}}[\cos(k_x) - \cos(k_y)]f_{\mathbf{k}}(t) \\ \Delta_{xy}(t) &= \frac{iZ}{2\mathcal{N}}\sum_{\mathbf{k}}\sin(k_x)\sin(k_y)f_{\mathbf{k}}(t) \quad , \end{aligned} \tag{8}$$

with $\mathcal{N}$ being the number of lattice sites. Having determined $\Delta_{\mathbf{k}}(t)$ from Eqs.(7,8), we insert the corresponding result in H_{MF} in Eq.(1), thus defining the time-dependent mean-field Hamiltonian $H_{\mathrm{MF}}(t)$. Therefore, regarding t as over-all parameter, we determine the eigenvalues and the corresponding quasiparticle eigenmodes of $H_{\mathrm{MF}}(t)$, respectively given by $E_{\mathbf{k},\pm}(t) = \pm\sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}(t)|^2}$ and $\Gamma_{\mathbf{k},\pm}(t)$, both carrying an explicit dependence on t. In particular, the operators $\Gamma_{\mathbf{k},\lambda}(t)$ are recovered by means of a pertinent, time-dependent generalization of the Bogoliubov-Valatin transformations discussed in detail in Ref.[27]. Inserting the time-dependent parameters and eigenmode operators in the LME for the density matrix $\rho(t)$, we recover Eq.(4). Of course, to preserve self-consistency at any time t, Eq.(4) has to be solved together with the self-consistent Eqs.(8). Due to the apparent complexity of the whole procedure, we resorted to a numerical approach to the solution of the coupled equations, to recover the results for the time-dependent quantities that we discuss in the main paper.

B. Derivation of the time-dependent spin-Hall conductance $\sigma(t)$

To discuss our derivation of the spin-Hall conductance $\sigma(t)$, we refer to Fig.1**b)**, where we plot $\Delta_{x^2-y^2}(t)$ and

$\Delta_{xy}(t)$ for $g = 0.2$. From the figure, we note that the time evolution of $\Delta_{\mathbf{k}}(t)$ can be regarded as made out of two regions ($0 \le t < t_*$ and $t_* < t$), in which both $\Delta_{x^2-y^2}$ and Δ_{xy} are constant, separated by a sudden change in the superconducting gaps at $t = t_*$. Therefore, we conclude that, at least for $g = 0.2$, our approach explicitly realizes, in the case of a two-dimensional superconductor, the protocol outlined in Refs.[36, 37], in a chiral quantum walk of twisted photons. In this specific case, taking into account the sudden change in the parameters of $H_{\rm MF}$ that takes place at $t = t_*$, we describe the time evolution of our system by means of the time-dependent MF Hamiltonian $\hat{H}_{\rm MF}(t)$, given by

$$\hat{H}_{\rm MF}(t) = \sum_{\mathbf{k}} [c^\dagger_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}] \begin{bmatrix} \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}}(t) \\ -[\Delta_{\mathbf{k}}(t)]^* & -\xi_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{\mathbf{k},\uparrow} \\ c^\dagger_{-\mathbf{k},\downarrow} \end{bmatrix} , \tag{9}$$

with $\Delta_{\mathbf{k}}(t) = \theta(t_* - t)\Delta^{(0)}_{\mathbf{k}} + \theta(t - t_*)\Delta^{(1)}_{\mathbf{k}}$. Of course, resorting to the sudden change approximation for $\Delta_{\mathbf{k}}(t)$ remarkably simplifies our analytical calculations of the spin-Hall conductance, without substantially affecting the final result for $\sigma(t)$. In fact, we find only minimal discrepancy between the spin Hall conductance computed with this approximation compared to the fully self-consistent result.

From Eq.(9) we obtain that the spin current operators now explicitly depend on t as well. They are given by ($a = x, y, z$)

$$j_{{\rm sp},a}(t) = \sum_{\mathbf{k}} [c^\dagger_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}] \times \frac{\partial}{\partial k_a} \begin{bmatrix} \xi_{\mathbf{k}} & -\Delta_{\mathbf{k}}(t) \\ -[\Delta_{\mathbf{k}}(t)]^* & -\xi_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{\mathbf{k},\uparrow} \\ c^\dagger_{-\mathbf{k},\downarrow} \end{bmatrix} . \tag{10}$$

The change in $\hat{H}_{\rm MF}$ at $t = t_*$ comes along with a similar change in the expressions for its eigenmodes, which we respectively refer to as $\Gamma^{(0)}_{\mathbf{k},\lambda}$ and as $\Gamma^{(1)}_{\mathbf{k},\lambda}$. Specifically, we obtain

$$\Gamma^{(u)}_{\mathbf{k},+} = \cos\left(\frac{\theta^{(u)}_{\mathbf{k}}}{2}\right) c_{\mathbf{k},\uparrow} - \sin\left(\frac{\theta^{(u)}_{\mathbf{k}}}{2}\right) e^{i\phi^{(u)}_{\mathbf{k}}} c^\dagger_{-\mathbf{k},\downarrow} \tag{11}$$
$$\Gamma^{(u)}_{\mathbf{k},-} = \sin\left(\frac{\theta^{(u)}_{\mathbf{k}}}{2}\right) e^{-i\phi^{(u)}_{\mathbf{k}}} c_{\mathbf{k},\uparrow} + \cos\left(\frac{\theta^{(u)}_{\mathbf{k}}}{2}\right) c^\dagger_{-\mathbf{k},\downarrow} ,$$

with $u = 0, 1$, and

$$\cos(\theta^{(u)}_{\mathbf{k}}) = \frac{\xi_{\mathbf{k}}}{\epsilon^{(u)}_{\mathbf{k}}} ,$$
$$\sin(\theta^{(u)}_{\mathbf{k}}) e^{-i\phi^{(u)}_{\mathbf{k}}} = -\frac{\Delta^{(u)}_{\mathbf{k}}}{\epsilon^{(u)}_{\mathbf{k}}} ,$$
$$\epsilon^{(u)}_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta^{(u)}_{\mathbf{k}}|^2} . \tag{12}$$

To describe the time evolution of the system coupled to the bath, in the following we use a "mixed" representation, in which we let the current operators $j^M_{{\rm sp},a}(t)$ evolve with $\hat{H}_{\rm MF}(t)$, and the density matrix $\bar{\rho}(t)$ depend on time as an effect of the coupling to the bath only. This implies that, for $t > t_*$ and in the zero-temperature limit, $\bar{\rho}(t)$ satisfies the differential equation

$$\frac{d\bar{\rho}(t)}{dt} = g \sum_{\mathbf{k}} \{ [\Gamma^{(1)}_{\mathbf{k},+}, \bar{\rho}(t)[\Gamma^{(1)}_{\mathbf{k},+}]^\dagger] - [[\Gamma^{(1)}_{\mathbf{k},+}]^\dagger, \Gamma^{(1)}_{\mathbf{k},+}\bar{\rho}(t)] \} + g \sum_{\mathbf{k}} \{ [[\Gamma^{(1)}_{\mathbf{k},-}]^\dagger, \bar{\rho}(t)\Gamma^{(1)}_{\mathbf{k},-}] - [\Gamma^{(1)}_{\mathbf{k},-}, [\Gamma^{(1)}_{\mathbf{k},-}]^\dagger \bar{\rho}(t)] \} , \tag{13}$$

which we will employ throughout this section to derive the time evolution of our system, consistently with the approximate expression for the time-dependent mean field Hamiltonian, $\hat{H}_{\rm MF}(t)$, which we are using here. On the other hand, for $t < t_*$, we find $\frac{d\bar{\rho}(t)}{dt} = 0$ and hence $\bar{\rho}(t) = \bar{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|$, with $\Gamma^{(0)}_{\mathbf{k},+}|\psi(0)\rangle = [\Gamma^{(0)}_{\mathbf{k},-}]^\dagger|\psi(0)\rangle = 0 \ \forall \mathbf{k}$. From Eq.(13) one readily derives the time evolution of the covariance matrix element in the energy basis, $\bar{c}_{\mathbf{k};(\alpha,\beta)}(t) = {\rm Tr}\{[\Gamma^{(1)}_{\mathbf{k},\alpha}]^\dagger \Gamma^{(1)}_{\mathbf{k},\beta}\bar{\rho}(t)\}$ in the form

$$\bar{c}_{\mathbf{k};(\alpha,\beta)}(t) = \bar{c}_{\mathbf{k};(\alpha,\beta)}(0)\theta(t_* - t) + \theta(t - t_*) \times \{ e^{-2g(t-t_*)}\bar{c}_{\mathbf{k};(\alpha,\beta)}(0) + [1 - e^{-2g(t-t_*)}]\delta_{\alpha,-}\delta_{\beta,-} \} . \tag{14}$$

We now compute $\sigma(t)$ by resorting to the standard linear response theory of charge transport. In particular, we apply a uniform external field in the y direction at frequency ω_0, compute the corresponding current response in the x direction, and eventually recover $\sigma(t)$, which is given by

$$\sigma(t) = -i \lim_{\omega_0 \to 0} \int_{-\infty}^t d\tau \, {\rm Tr}\{[j^M_{{\rm sp},x}(t), j^M_{{\rm sp},y}(\tau)]\bar{\rho}(t)\} \times \frac{\sin(\omega_0(\tau - t_*))}{\omega_0} e^{\eta(\tau - t_*)} , \tag{15}$$

with $\eta = 0^+$. The spin current operators $j^M_{{\rm sp},a}(t)$ taken in the mixed representation, as discussed above, are given by

$$j^M_{{\rm sp},a}(t) = \theta(t_* - t) \sum_{\mathbf{k}} \sum_{\lambda,\lambda'} [j^{(0)}_{{\rm sp},a}(\mathbf{k})]_{\lambda,\lambda'} [\Gamma^{(0)}_{\mathbf{k},\lambda}]^\dagger \Gamma^{(0)}_{\mathbf{k},\lambda'} e^{i\epsilon^{(0)}_{\mathbf{k}}(\lambda-\lambda')t} + \theta(t - t_*) \sum_{\mathbf{k}} \sum_{\lambda,\lambda'} [j^{(1)}_{{\rm sp},a}(\mathbf{k})]_{\lambda,\lambda'} [\Gamma^{(1)}_{\mathbf{k},\lambda}]^\dagger \Gamma^{(1)}_{\mathbf{k},\lambda'} e^{i\epsilon^{(1)}_{\mathbf{k}}(\lambda-\lambda')t} , \tag{16}$$

with $[j^{(0)}_{\mathrm{sp},a}(\mathbf{k})]_{\lambda,\lambda'}$ and $[j^{(1)}_{\mathrm{sp},a}(\mathbf{k})]_{\lambda,\lambda'}$ denoting the matrix elements of the spin current operators for $t < t_*$ and for $t > t_*$, respectively, in the basis of the eigenmodes of $\hat{H}_{\mathrm{MF}}(t)$.

As long as $t < t_*$, the explicit calculation of $\sigma(t)$ is straightforward: in this case, $\bar{\rho}(t)$ is independent of t and the standard linear response theory approach to equilibrium systems yields

$$\sigma(t < t_*) = -\frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \frac{1}{4(\epsilon^{(0)}_{\mathbf{k}})^2} \Bigg[[j^{(0)}_{\mathrm{sp},x}(\mathbf{k})]_{-,+} [j^{(0)}_{\mathrm{sp},y}(\mathbf{k})]_{+,-} - [j^{(0)}_{\mathrm{sp},y}(\mathbf{k})]_{-,+} [j^{(0)}_{\mathrm{sp},x}(\mathbf{k})]_{+,-} \Bigg] \quad , \tag{17}$$

with the sum over $\mathbf{k}$ taken over the full Brillouin zone. Defining $\hat{d}^{(0)}_{\mathbf{k}}$ as

$$\hat{d}^{z,(0)}_{\mathbf{k}} = \frac{\xi_{\mathbf{k}}}{\epsilon^{(0)}_{\mathbf{k}}} \; , \qquad \hat{d}^{x,(0)}_{\mathbf{k}} = \frac{\Re e[-\Delta^{(0)}_{\mathbf{k}}]}{\epsilon^{(0)}_{\mathbf{k}}} \; , \qquad \hat{d}^{y,(0)}_{\mathbf{k}} = \frac{\Im m[\Delta^{(0)}_{\mathbf{k}}]}{\epsilon^{(0)}_{\mathbf{k}}} \quad , \tag{18}$$

and going through standard formal manipulations, we eventually obtain

$$\sigma(t < t_*) = \frac{1}{16\pi^2} \int_{\mathrm{B.Z.}} d^2k \, \left\{ \hat{d}^{(0)}_{\mathbf{k}} \cdot \left[\frac{\partial \hat{d}^{(0)}_{\mathbf{k}}}{\partial k_x} \times \frac{\partial \hat{d}^{(0)}_{\mathbf{k}}}{\partial k_y} \right] \right\} = \frac{\mathcal{C}_0}{2\pi} \quad . \tag{19}$$

In Eq.(19) we have denoted with $\mathcal{C}_0$ the first Chern number of our specific model [34]. In general, our model Hamiltonian $\hat{H}_{\mathrm{MF}}$ is characterized by broken time-reversal symmetry but particle-hole symmetry [28, 29]. In particular, the latter symmetry is realized as

$$\hat{H}^*_{\mathrm{MF}}(-\mathbf{k}) = U_c \hat{H}_{\mathrm{MF}}(\mathbf{k}) U_c^{-1} \quad , \tag{20}$$

with $U_c = \sigma_2$, $U_c^* U_c = -\mathbf{I}_{2\times 2}$, σ_2 denoting the second Pauli matrix and $\mathbf{I}_{2\times2}$ the two-by-two identity. When both $\Delta_{x^2-y^2}$ and Δ_{xy} are $\neq 0$, our model can be obtained by continuously deforming the Hamiltonian in Eqs.(1,2) of Ref. [31], which is topologically nontrivial. Therefore, we infer that, in this case, our system lies within a topologically superconducting phase, with Chern number $\mathcal{C}_0 = \pm 2$, depending on the relative sign of $\Delta^{(0)}_{x^2-y^2}$ and of $\Delta^{(0)}_{xy}$. At variance, if $\Delta^{(0)}_{x^2-y^2} = 0$, the system is within a topologically trivial case, and $\mathcal{C}_0 = 0$, regardless of whether $\Delta_{xy} = 0$ or not.

To compute $\sigma(t)$ for $t > t_*$ it is operationally useful to split the integral in Eq.(15) according to

$$\sigma(t) = -i \lim_{\omega_0 \to 0} \{ \int_{-\infty}^{t_*} + \int_{t_*}^{t} \} \, d\tau \; \times \mathrm{Tr}\{[j^M_{\mathrm{sp},x}(t), j^M_{\mathrm{sp},y}(\tau)]\rho(t)\} \, \frac{\sin(\omega_0(\tau - t_*))}{\omega_0} e^{\eta(\tau - t_*)} \equiv \sigma^A(t) + \sigma^B(t) \quad . \tag{21}$$

In particular, for $\sigma^A(t)$ one gets

$$\sigma^A(t) = \frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \sum_{\lambda,\lambda'} \sum_{\mu,\nu} \frac{1}{4[\epsilon^{(0)}_{\mathbf{k}}]^2} e^{i\epsilon^{(1)}_{\mathbf{k}}(\lambda-\lambda')(t-t_*)} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{\lambda,\lambda'} [j^{(0)}_{\mathrm{sp},y}(\mathbf{k})]_{\mu,\bar{\mu}} \times [[\mathcal{S}^{(0,1)}_{\mathbf{k};(\mu,\lambda')}]^* \mathcal{S}^{(0,1)}_{\mathbf{k};(\bar{\mu},\nu)} \bar{c}_{\mathbf{k};(\lambda,\nu)}(t) - [\mathcal{S}^{(0,1)}_{\mathbf{k};(\mu,\nu)}]^* \mathcal{S}^{(0,1)}_{\mathbf{k};(\bar{\mu},\lambda)} \bar{c}_{\mathbf{k};(\nu,\lambda')}(t)] \quad , \tag{22}$$

with $[j^{(0)}_{\mathrm{sp},y}]_{\mu,\mu'}$ being the matrix element of $j^{(0)}_{\mathrm{sp},y}$ in the basis of the eigenmodes of $\hat{H}_{\mathrm{MF}}(t)$ for $t < t_*$ and $\mathcal{S}^{(0,1)}_{\mathbf{k}}$ being the unitary transformation matrix defined by

$$\Gamma^{(0)}_{\mathbf{k},\lambda} = \sum_{\mu} \mathcal{S}^{(0,1)}_{\mathbf{k};(\lambda,\mu)} \Gamma^{(1)}_{\mathbf{k},\mu} \quad . \tag{23}$$

By considering that $\bar{c}_{\mathbf{k};(\alpha,\beta)}(t) \to_{t\to\infty} \delta_{\alpha,-}\delta_{\beta,-}$, we readily obtain that the asymptotic form of $\sigma^A(t)$ for $2g(t - t_*) \gg 1$ is given by

$$\sigma^A(t) \to \frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \sum_{\mu} \frac{1}{4[\epsilon^{(0)}_{\mathbf{k}}]^2} \Bigg[e^{-2i\epsilon^{(1)}_{\mathbf{k}}(t-t_*)} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{-,+} [j^{(0)}_{\mathrm{sp},y}(\mathbf{k})]_{\mu,\bar{\mu}} [\mathcal{S}^{(0,1)}_{\mathbf{k};(\mu,+)}]^* S^{(0,1)}_{\mathbf{k};(\bar{\mu},-)} - e^{2i\epsilon^{(1)}_{\mathbf{k}}(t-t_*)} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{+,-} [j^{(0)}_{\mathrm{sp},y}(\mathbf{k})]_{\mu,\bar{\mu}} [\mathcal{S}^{(0,1)}_{\mathbf{k};(\mu,-)}]^* S^{(0,1)}_{\mathbf{k};(\bar{\mu},+)} \Bigg] \quad . \tag{24}$$

As $t - t_*$ gets large, on employing the stationary phase approximation similarly to what has been done in Ref.[37]

for a 1D lattice model, and assuming that $\epsilon_{\mathbf{k}}$ is fully gapped, as in the case of a d+id order parameter, the right-hand side of Eq.(24) can be readily shown to decay like $(t-t_*)^{-1}$. At variance, for $\sigma^B(t)$ we obtain

$$\sigma^B(t) = -\frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \frac{1}{4[\epsilon^{(1)}_{\mathbf{k}}]^2} \sum_\lambda \bar{c}_{\mathbf{k},(\lambda,\lambda)}(t) \Bigg[(1-e^{2i\lambda\epsilon^{(1)}_{\mathbf{k}}(t-t_*)}) [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{\lambda,\bar{\lambda}} [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{\bar{\lambda},\lambda} - (1-e^{-2i\lambda\epsilon^{(1)}_{\mathbf{k}}(t-t_*)}) [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{\lambda,\bar{\lambda}} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{\bar{\lambda},\lambda} \Bigg] + \frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \frac{1}{4[\epsilon^{(1)}_{\mathbf{k}}]^2} \sum_\lambda (1-e^{2i\lambda\epsilon^{(1)}_{\mathbf{k}}(t-t_*)}) \bar{c}_{\mathbf{k},(\lambda,\bar{\lambda})}(t) \Bigg[[j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{\lambda,\lambda} [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{\lambda,\bar{\lambda}} - [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{\lambda,\bar{\lambda}} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{\bar{\lambda},\bar{\lambda}} \Bigg] \quad . \tag{25}$$

Apparently, for $2g(t-t_*) \gg 1$, we get

$$\sigma^B(t) \to -\frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \frac{1}{4[\epsilon^{(1)}_{\mathbf{k}}]^2} \Bigg[(1-e^{-2i\epsilon^{(1)}_{\mathbf{k}}(t-t_*)}) [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{-,+} [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{+,-} - (1-e^{2i\epsilon^{(1)}_{\mathbf{k}}(t-t_*)}) [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{-,+} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{+,-} \Bigg] \equiv \sigma^B_I + \sigma^B_{II}(t) \quad . \tag{26}$$

The former contribution, σ^B_I, is independent of t and is given by

$$\sigma^B_I = -\frac{i}{\mathcal{N}} \sum_{\mathbf{k}} \frac{1}{4(\epsilon^{(1)}_{\mathbf{k}})^2} \Bigg[[j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{-,+} [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{+,-} - [j^{(1)}_{\mathrm{sp},y}(\mathbf{k})]_{-,+} [j^{(1)}_{\mathrm{sp},x}(\mathbf{k})]_{+,-} \Bigg] \quad , \tag{27}$$

which is the same form as the spin-Hall conductance for $t < t_*$ in Eq.(17) and, accordingly, can be rewritten in the same form as in Eq.(19) as

$$\sigma(t\to\infty) = \frac{1}{16\pi^2} \int_{\mathrm{B.Z.}} d^2k \, \left\{ \hat{d}^{(1)}_{\mathbf{k}} \cdot \left[\frac{\partial \hat{d}^{(1)}_{\mathbf{k}}}{\partial k_x} \times \frac{\partial \hat{d}^{(1)}_{\mathbf{k}}}{\partial k_y} \right] \right\} = \frac{\mathcal{C}_1}{2\pi} \quad , \tag{28}$$

with $\hat{d}^{a,(1)}_{\mathbf{k}}$ $(a=x,y,z)$ defined as $\hat{d}^{a,(0)}_{\mathbf{k}}$ in Eq.(18), with $0 \to 1$, and $\mathcal{C}_1$ being the Chern number of the system in the $t\to\infty$ limit. In addition, there is the oscillating, time-dependent contribution which, again, employing the stationary phase approximation can be readily shown to decay as $(t-t_*)^{-1}$, just as $\sigma^A(t)$, for large values of $t-t_*$.

C. Lindblad Master Equation approach to the nonequilibrium dynamics of an open, two-dimensional superconducting system

Our derivation of the time-dependent density matrix of the nonequilibrium two-dimensional superconducting system coupled to the external bath is based on Eq.(4) for $\rho(t)$. Here, we ground our LME approach by sketching its derivation from a general model for a two-dimensional superconductor coupled to a metallic lead working as a dissipative bath [24]. Moreover, following the discussion of Refs.[21, 23], we evidence how one recovers our Eq.(4) form a completely different perspective, that is, within a semiphenomenological formalism that accounts for the damping effects arising from the superconductor dynamics beyond BCS mean-field theory, such as interactions among the quasiparticles within the superconductor, or between the quasiparticles and the fluctuations of the superconducting order parameter [23, 27]. That the LME in Eq.(4), pertinently complemented within the time-dependent SCMF approximation (which we discuss above), accounts for two completely different realizations of the external bath, evidences the wide applicability of our approach to solid-state systems such as the one we consider in our main paper. In addition, it is also worth stressing that LME in many cases provides a good approximation to describe the dissipative dynamics in the case of cold atoms loaded over optical lattices, coupled to a continuum of radiation modes [30]. Thus, it apparently applies equally well to open solid-state, as well as optical, systems.

A minimal microscopic model for a two-dimensional superconductor coupled to a metallic lead can be recovered by properly adapting the formalism of Ref.[24]. Prior to coupling the superconductor to the bath, we describe the former, within mean field approximation, by means of an Hamiltonian $\hat{H}_{\mathrm{MF}}$ in the same form as in

Eq.(9) but, of course, without an explicit dependence on time t. At variance, denoting with $d_{\mathbf{q},\sigma}, d^\dagger_{\mathbf{q},\sigma}$ the single-fermion annihilation/creation operators in the normal lead, we describe it by means of the normal Hamiltonian $\hat{H}_N$, given by

$$\hat{H}_N = \sum_{\mathbf{q}} \sum_{\sigma} \xi_{\mathbf{q}} d^\dagger_{\mathbf{q},\sigma} d_{\mathbf{q},\sigma} \quad , \qquad (29)$$

with $\xi_{\mathbf{q}}$ being the dispersion relations of quasiparticles in the normal contact. Finally, we encode the coupling between the superconductor and the contact in the single-particle tunneling Hamiltonian $\hat{H}_T$, given by

$$\hat{H}_T = \sum_{\mathbf{k},\mathbf{q}} \sum_{\sigma} t_{\mathbf{k},\mathbf{q}} \{ c^\dagger_{\mathbf{k},\sigma} d_{\mathbf{q},\sigma} + d^\dagger_{\mathbf{q},\sigma} c_{\mathbf{k},\sigma} \} \ . \qquad (30)$$

Regarding $\hat{H}_T$ in Eq.(30) we point out that, although it is linear in the single-fermion operators of the superconductor and of the lead, when going along the derivation of the LME, composite operators of the basic jump operators can also account for multi-particle dissipative processes, such as two-particle losses [15, 44]. Yet, those processes enter to higher order in the expansion in the coupling between the superconducting system and the bath and, therefore, being subleading with respect to the single quasiparticle exchange, we do not explicitly consider them here.

To derive the LME for the superconductor density matrix operator $\rho(t)$, we begin with considering the time evolution operator for the density matrix operator of the whole system, $\rho_w(t)$, to second order in $\hat{H}_T$. Assuming that $\hat{H}_T$ is turned on at $t = 0$, this reads [38]

$$\frac{d\rho_w(t)}{dt} = -\int_0^t dt' \, [\hat{H}_T(t), [\hat{H}_T(t'), \rho_w(t)]] \ , \qquad (31)$$

with the operator $\hat{H}_T(t)$ and $\rho_w(t)$ taken in the interaction representation corresponding to the decoupled system. Within weak coupling hypothesis between the system and the lead, we assume that, consistently with the standard Markov approximation, the bath stays at equilibrium at any time t. We therefore set $\rho_w(t) = \rho(t) \otimes \rho_B$, with the bath density matrix $\rho_B = e^{-\beta \hat{H}_N}/\mathrm{Tr}[e^{-\beta \hat{H}_N}]$. Tracing over the bath degrees of freedom and assuming translational invariance in real space, which implies $t_{\mathbf{k},\mathbf{q}} = t_{\mathbf{k}} \delta_{\mathbf{k},\mathbf{q}}$, Eq.(31) yields, after integrating over dt' and resorting to the Schrödinger representation

$$\frac{d\rho(t)}{dt} = -i[H_{\mathrm{MF}}, \rho(t)] + \sum_{\lambda=\pm} \sum_{\mathbf{k}} T_{\mathbf{k}} \qquad (32)$$
$$[f(-\lambda\epsilon_{\mathbf{k}})(2\Gamma_{\mathbf{k},\lambda}\rho(t)\Gamma^\dagger_{\mathbf{k},\lambda} - \{\Gamma^\dagger_{\mathbf{k},\lambda}\Gamma_{\mathbf{k},\lambda}, \rho(t)\})$$
$$+ f(\lambda\epsilon_{\mathbf{k}}(t))(2\Gamma^\dagger_{\mathbf{k},\lambda}\rho(t)\Gamma_{\mathbf{k},\lambda} - \{\Gamma_{\mathbf{k},\lambda}\Gamma^\dagger_{\mathbf{k},\lambda}, \rho(t)\})] \ .$$

with $T_{\mathbf{k}} = 2\pi t^2_{\mathbf{k}}$ and the quasiparticle operators and the other parameters defined according to Eqs.(11,12) above.

On neglecting, just for the sake of simplicity, the dependence of $T_{\mathbf{k}}$ on $\mathbf{k}$ and setting $T_{\mathbf{k}} = g$ and on employing the time-dependent self-consistent approximation described above, we just recover Eq.(4).

A different route also leading to Eq.(4) goes through phenomenologically accounting for damping effects that go beyond mean-field BCS approaximation. Specifically, one begins with the Bloch-like equation for the Nambu pseudospin at a given $\mathbf{k}$, $\mathbf{S}_{\mathbf{k}}$, defined as

$$\mathbf{S}_{\mathbf{k}} = \frac{1}{2}[c^\dagger_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}]\vec{\sigma} \begin{bmatrix} c_{\mathbf{k},\uparrow} \\ c^\dagger_{-\mathbf{k},\downarrow} \end{bmatrix} \quad . \qquad (33)$$

For a system described by $\hat{H}_{\mathrm{MF}}(t)$ in Eq.(9), the equation of motion for the time dependent average, $\langle \mathbf{S}_{\mathbf{k}}(t) \rangle$, is given by [23, 27]

$$\frac{d\langle \mathbf{S}_{\mathbf{k}}(t) \rangle}{dt} = \mathbf{B}_{\mathbf{k}}(t) \times \langle \mathbf{S}_{\mathbf{k}}(t) \rangle \ , \qquad (34)$$

with $\mathbf{B}_{\mathbf{k}}(t) = [-\Re e[\Delta_{\mathbf{k}}(t)], \Im m[\Delta_{\mathbf{k}}(t)], \xi_{\mathbf{k}}]^T$. In order to phenomenologically account for damping effects, in analogy to the spin precession problem, one adds to the right-hand side of Eq.(34) the typical terms that encode the longitudinal (T_1^{-1}) and the transverse (T_2^{-1}) relaxation rates. On setting $T_1^{-1} = T_2^{-1} = 2g$, Eq.(34) is modified into

$$\frac{d\langle \mathbf{S}_{\mathbf{k}}(t) \rangle}{dt} = \mathbf{B}_{\mathbf{k}}(t) \times \langle \mathbf{S}_{\mathbf{k}}(t) \rangle - 2g\langle \mathbf{S}_{\mathbf{k}}(t) \rangle + 2g\langle \mathbf{S}_{\mathbf{k},*}(t) \rangle \ , \qquad (35)$$

with $\langle \mathbf{S}_{\mathbf{k},*}(t) \rangle$ being the thermalized spin configuration at time t. Eq.(35) is exactly recovered within our LME framework by noting that one can express $\langle \mathbf{S}_{\mathbf{k}}(t) \rangle$ as

$$\langle \mathbf{S}_{\mathbf{k}}(t) \rangle = \begin{bmatrix} f^*_{\mathbf{k}}(t) \\ f_{\mathbf{k}}(t) \\ \nu_{\mathbf{k}}(t) \end{bmatrix} \quad . \qquad (36)$$

Given Eq.(36), we use Eqs.(6) to write the equation of motion for $\langle \mathbf{S}_{\mathbf{k}}(t) \rangle$.

It is now easy to check that one exactly recovers Eq.(35), with $\langle \mathbf{S}_{\mathbf{k}}(t) \rangle = \mathbf{B}_{\mathbf{k}}(t)/2\epsilon_{\mathbf{k}}(t)$. We therefore conclude that, as stated above, also the phenomenological approach of Ref.[23] gives us back Eq.(4).

D. Rate function and spin-Hall conductance across the d+id→s topological phase transition

In the main text, we analyze the id→d+id DPT as an explicit example of a TDPT between a topologically trivial and a topologically nontrivial state. As an additional example, we now discuss the d+id→s DPT as an alternative example of a TDPT between a topologically nontrivial and a topologically trivial phase. Specifically, we choose a pure s-wave phase as our topologically trivial one. To trigger a TDPT toward the s phase, at $t = 0$ we turn on an additional, onsite attractive interaction

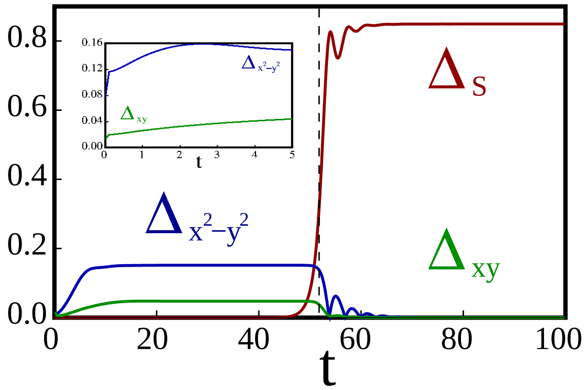

FIG. 4. Time dependent gap $\Delta_S(t)$ (red curve), $\Delta_{x^2-y^2}(t)$ (blue curve), and $\Delta_{xy}(t)$ (green curve), computed in a system with $\mu=0$, $U^{(1)}=30$ and $V^{(1)}=Z^{(1)}=1.5$, with $g=0.2$. For $t<0$ the system is assumed to be prepared in a state with $\Delta_S^{(0)}=0$, $\Delta_{x^2-y^2}^{(0)}=0.077$, and $\Delta_{xy}^{(0)}=0.013$. The dashed vertical line marks the DPT at $t=t_*\approx 50$. [**Inset:** The same plot (for $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$) restricted to $0\le t<5$].

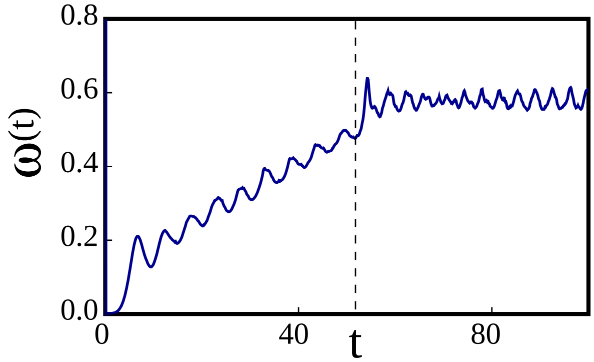

FIG. 5. $\omega(t)$ as a function of t with the time-dependent MF Hamiltonian with parameters $\Delta_{x^2-y^2}(t)$, $\Delta_S(t)$ and $\Delta_{xy}(t)$, as in Fig.4, computed for $g=0.2$. The dashed vertical line marks the DFT.

in the spin singlet channel, with strength equal to $U^{(1)}$. In general, for U,V and Z all different from 0, Eqs.(3) generalizes to [27]

$$\Delta_{\mathbf{k}}=\Delta_S+2\Delta_{x^2-y^2}\{\cos(k_x)-\cos(k_y)\} - 4i\Delta_{xy}\sin(k_x)\sin(k_y) \quad , \tag{37}$$

with Δ_S determined by the self-consistent equation

$$\Delta_S=\frac{U}{2\mathcal{N}}\sum_{\mathbf{k}}\frac{\Delta_{\mathbf{k}}}{\epsilon_{\mathbf{k}}} \quad . \tag{38}$$

To define the nonequilibrium protocol corresponding to the TDPT described above, we prepare our system in a d+id superconducting state with $\Delta_S^{(0)}=0$, $\Delta_{x^2-y^2}^{(0)}=0.077$, and $\Delta_{xy}^{(0)}=0.013$. For $t>0$, we let the system evolve with the Hamiltonian $H_{\rm MF}$ with $Z^{(1)}=V^{(1)}=1.5$ and $U^{(1)}=3.0$ and with the system-bath coupling $g=0.2$. To evidence the DPT, we first of all employ the approach we discuss in the main text, to compute $\Delta_S(t),\Delta_{x^2-y^2}(t)$, and $\Delta_{xy}(t)$. In Fig.4 we plot the three time-dependent gaps as functions of t. We see that apparently the system keeps within the d+id phase, with both $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ finite and with $\Delta_S(t)=0$, as long as $t<t_*$, with $t_*\approx 50$. At $t=t_*$, $\Delta_S(t)$ jumps to a finite value, goes through a small oscillating transient and eventually stabilizes over its constant, asymptotic value $\Delta_{S,\infty}=0.85$. At the same time, after a similar transient, $\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ take their asymptotic values, $\Delta_{x^2-y^2,\infty}=\Delta_{xy,\infty}=0$. The phase with $\Delta_{S,\infty}\neq 0$ and $\Delta_{x^2-y^2,\infty}=\Delta_{xy,\infty}=0$ is topologically trivial. This is, in fact, the first piece of evidence that, at $t=t_*$, our system goes through a TDPT.

As we have done in addressing the id→d+id TDPT in the main text, we proceed by studying the rate function $\omega(t)$ for the d+id→s PT. Computing $\omega(t)$ with the procedure detailed in Ref.[27], we derive the plot of Fig.5, where we draw $\omega(t)$ as a function of t. We clearly see the nonanalyticity at the DPT, corresponding to a change in the slope of $\omega(t)$ at $t=t_*$, and an over-all increase of the rate function at larger values of t, associated to the expected reduction of $\mathcal{F}(t)$ for $t>t_*$.

Finally, we now compute $\sigma(t)$ across the DPT. Again, we note that, since around $t=t_*$ the superimposed oscillations in $\Delta_S(t),\Delta_{x^2-y^2}(t)$ and $\Delta_{xy}(t)$ have no substantial qualitative effects, the behavior of the superconducting gaps can be well approximated by a sudden jump in their values at $t=t_*$. Accordingly, we set $\Delta_{\mathbf{k}}(t)=\theta(t_*-t)\Delta_{\mathbf{k}}^{(<)}+\theta(t-t_*)\Delta_{\mathbf{k}}^{(>)}$, with

$$\begin{aligned}\Delta_{\mathbf{k}}^{(<)} &= 2\Delta_{x^2-y^2}^{(<)}\{\cos(k_x)-\cos(k_y)\} \\ &\quad - 4i\Delta_{xy}^{(<)}\sin(k_x)\sin(k_y) \quad , \\ \Delta_{\mathbf{k}}^{(>)} &= \Delta_S^{(>)} \quad ,\end{aligned} \tag{39}$$

and all the other system parameters chosen consistently with the numerical values of the parameters corresponding to Fig.4, that is, $\mu=0$, $g=0.2$, $\Delta_{x^2-y^2}^{(<)}=0.152$, $\Delta_{xy}^{(<)}=0.048$, and $\Delta_S^{(>)}=0.849$. After the sudden change in the superconducting gaps, $\Delta_{\mathbf{k}}(t)$ is purely s-wave and, therefore, the corresponding superconducting phase is topologically trivial. Thus, we expect that $\sigma(t)\to 0$ as $t\to\infty$. Therefore, we conclude that the time evolution reported in Fig.4 corresponds to a TDPT.

In Fig.6 we plot our result for $\sigma(t)$, computed as discussed above. With a dashed horizontal line we highlight the asymptotic value of $\sigma(t)$. As expected, we see that, for $t-t_*<0$, $\sigma(t)$ is constantly equal to 2 (in the units of the plot), as it must be, in the stationary topological phase, with both $\Delta_{x^2-y^2}$ and Δ_{xy} being $\neq 0$ [27, 31]. At $t-t_*=0$ (which, in Fig.6, we mark with a dashed vertical line), $\sigma(t)$ shows a sudden jump to lower values and, for $t-t_*>0$, it starts to oscillate around the asymptotic value $\sigma^{(1)}=\sigma(t\to\infty)=0$. This is appropriate for the topologically trivial phase

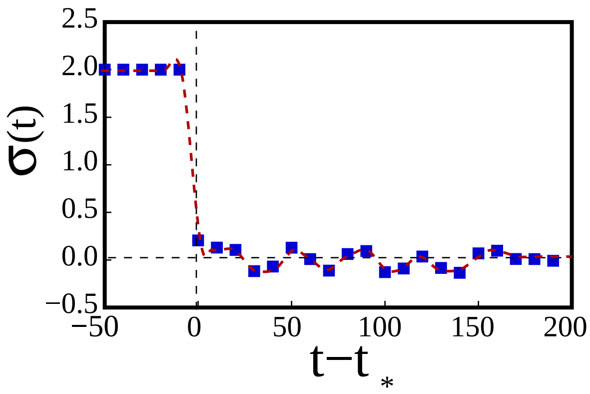

FIG. 6. Spin-Hall conductance $\sigma(t)$ (in units of $(2\pi)^{-1}$) computed with the parameters set as specified in Eq.(39) and in the related discussion. For $t = t_*$ we have used the approximated value $t_* = 50$. The results are shown as a function of $t - t_* \geq 0$ for $-50 \leq t - t_* \leq 200$. The dashed horizontal line marks the asymptotic value $\sigma(t \to \infty)$, the dashed vertical line marks the DPT at $t - t_* = 0$, and the dashed line connecting the dots is a guide to the eye only.

that asymptotically sets in, with $\Delta_S(t \to \infty) \neq 0$ and $\Delta_{x^2-y^2}(t \to \infty) = \Delta_{xy}(t \to \infty) = 0$. On increasing $t - t_*$, the oscillations of $\sigma(t)$ around σ_∞ are apparently suppressed and $\sigma(t)$ flows toward its asymptotic value $\sigma^{(1)} = 0$.

Therefore, we conclude that the time evolution of the superconducting gaps in Fig.4 does, in fact, correspond to a TDPT between a topologically nontrivial and a topologically trivial case. Aside for being relevant per se, our result has also potentially important practical consequences. Indeed, we infer that, although, for $t > 0$, the system evolves with an Hamiltonian whose parameters, at equilibrium, would not correspond to a topological phase, having prepared the system at $t = 0$ in the groundstate of a topologically nontrivial Hamiltonian, the system keeps showing topological properties at finite $t > 0$ until it unavoidably switches to a trivial phase, at $t = t_*$. By analogy with the plots of Fig.4, we infer that the time of persistence of the topological phase (t_*) is determined by the strength of the system-bath coupling (g). Thus, t_* should be, in principle, tunable, by pertinently engineering the system parameters, which, for practical purposes, would allow the persistence, for a long time, of a topological phase in a system whose dynamics is described by a nontopological Hamiltonian. In such an approach, there is no need of tuning and stabilizing the Hamiltonian parameters, but only of pertinently setting the initial state of the system.

ACKNOWLEDGMENTS

We thank N. Lo Gullo and F. Plastina for insightful discussions.

A.N., C.A.P., L.L., and D.G. acknowledge financial support from Italy's MIUR PRIN project TOP-SPIN (Grant No. PRIN 20177SL7HC).

L.L. acknowledges financial support by a project funded under the National Recovery and Resilience Plan (NRRP), Mission 4 Component 2 Investment 1.3 - Call for tender No. 341 of 15/03/2022 of Italian Ministry of University and Research funded by the European Union – NextGenerationEU, award number PE0000023, Concession Decree No. 1564 of 11/10/2022 adopted by the Italian Ministry of University and Research, CUP D93C22000940001, Project title "National Quantum Science and Technology Institute" (NQSTI).

A.N. and R.E. acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grant No. 277101999, TRR 183 (project C01), under Germany's Excellence Strategy - Cluster of Excellence Matter and Light for Quantum Computing (ML4Q) EXC 2004/1 - 390534769, and under Grant No. EG 96/13-1.

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