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02 May 2026

The relaxed energy of fractional Sobolev maps with values into the circle

Domenico Mucci

Abstract. We deal with the weak sequential density of smooth maps in the fractional Sobolev classes of $W^{s,p}$ maps in high dimension domains and with values into the circle. When s is lower than one, using interpolation theory we introduce a natural energy in terms of optimal extensions on suitable weighted Sobolev spaces. The relaxation problem is then discussed in terms of Cartesian currents. When $sp = 1$, the energy gap in the relaxed functional is always finite and is given by the minimal connection of the singularities times an energy weight, obtained through a minimum problem for one dimensional $W^{1/p,p}$ maps with degree one. When $sp > 1$, instead, concentration on codimension one sets needs unbounded energy. We finally treat the case where s is greater than one, obtaining an almost complete picture.

Keywords: fractional Sobolev spaces; weighted Sobolev spaces; relaxation; singularities; minimal connections; Cartesian currents.

AMS classification codes: 49Q20; 46E35; 28A75.

Introduction

In this paper, we deal with the weak sequential density property of smooth maps in the fractional Sobolev class $W^{s,p}(B^n, \mathbb{S}^1)$, where B^n is the unit ball in \mathbb{R}^n , \mathbb{S}^1 the unit circle in \mathbb{R}^2 , $s > 0$, and $1 < p < \infty$.

In case $0 < s < 1$, the fractional Sobolev space $W^{s,p}(B^n)$ is given by the L^p -functions $u : B^n \rightarrow \mathbb{R}$ with finite Gagliardo seminorm

$$|u|_{s,p}^p := \int_{B^n} \int_{B^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy.$$

It is Banach space when equipped with the norm $\|u\|_{s,p} := \|u\|_{L^p} + |u|_{s,p}$. When $s > 1$ is not integer, denoting by m and σ the integer and fractional part of s , respectively, the space $W^{s,p}(B^n)$ is given by the Sobolev functions $u \in W^{m,p}(B^n)$ such that the Gagliardo seminorm $|D^m u|_{\sigma,p}$ is finite, where $D^m u$ is the tensor of the m -th order weak derivatives of u . It is again a Banach space when normed by $\|u\|_{s,p} := \|u\|_{W^{m,p}} + |D^m u|_{\sigma,p}$. Finally, $W^{s,p}(B^n, \mathbb{R}^2)$ is the space of functions $u : B^n \rightarrow \mathbb{R}^2$ with components in $W^{s,p}(B^n)$.

Denoting by

$$\mathbb{S}^1 := \{y \in \mathbb{R}^2 : |y| = 1\}$$

the unit circle centered at the origin in the target space, we thus let

$$W^{s,p}(B^n, \mathbb{S}^1) := \{u \in W^{s,p}(B^n, \mathbb{R}^2) : |u(x)| = 1 \text{ for a.e. } x \in B^n\}.$$

The problem of *strong $W^{s,p}$ -density of smooth maps* $u : \bar{B}^n \rightarrow \mathbb{S}^1$ in $W^{s,p}(B^n, \mathbb{S}^1)$ is completely settled:

Theorem 0.1 *The class $C^\infty(\bar{B}^n, \mathbb{S}^1)$ is dense in $W^{s,p}(B^n, \mathbb{S}^1)$ in low dimension $n = 1$ for any s and p , and, when $n \geq 2$, if and only if $sp < 1$ or $sp \geq 2$.*

When $sp < 1$, the statement readily follows from the existence of a lifting in $W^{s,p}(B^n)$, see [5], through a density argument, see [6].

When $s \geq 1$ and $sp \geq 2$, a similar argument based on existence of suitable liftings, see [11], and on a convolution argument, see [10], applies.

When $sp > n$, strong density of smooth maps follows from the embedding of $W^{s,p}(B^n)$ in the class $C^0(B^n)$, and when $sp = n$ (or in low dimension $n = 1$ for any s and p) from the embedding in the class VMO of functions with vanishing mean oscillation. In these cases, in fact, one may apply an argument that goes back to [29] and is based on an approximation by convolutions with a smooth kernel, followed by a projection onto \mathbb{S}^1 . Therefore, such an argument works for more general target manifolds \mathcal{Y} through the Nash embedding theorem.

When $0 < s < 1$ and $2 \leq sp < n$, Theorem 0.1 is proved by Brezis-Mironescu in [12], and it is essentially based on two facts. Firstly, the authors introduce the class $\mathcal{R}_{s,p}(B^n, \mathbb{S}^1)$ of maps u in $W^{s,p}(B^n, \mathbb{S}^1)$ which are smooth (continuous) outside a singular set Σ_u given by a finite union of $(n - [sp] - 1)$ -manifolds of B^n , where $[sp]$ denotes the integer part of sp . For example, Σ_u is a finite set of points when $n = 2$ and $sp = 1$. Extending results that go back to [4, 3] for Sobolev maps in $W^{1,p}(B^n, \mathcal{Y})$, they in fact prove:

Theorem 0.2 *Every map in $W^{s,p}(B^n, \mathbb{S}^1)$ is the strong limit of a sequence in $\mathcal{R}_{s,p}(B^n, \mathbb{S}^1)$.*

Secondly, since the high order homotopy groups of \mathbb{S}^1 are all trivial, $\pi_i(\mathbb{S}^1) \simeq 0$ for $i \geq 2$ integer, they are able to remove the singular set Σ_u of maps in $\mathcal{R}_{s,p}$, when $[sp] \geq 2$.

The first homotopy group $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ being non-trivial, counterexamples to the strong density of smooth maps exist when $n \geq 2$ and the integer part of sp is equal to one.

Example 0.3 For $n = 2$, the map $u(x) = x/|x|$ belongs to $W^{s,p}(B^2, \mathbb{S}^1)$ if $1 \leq sp < 2$, and u has degree one around the origin. On the other hand, smooth maps in $C^\infty(B^2, \mathbb{S}^1)$ have degree zero around the origin, and the degree is continuous w.r.t. the strong convergence in $W^{s,p}$, when $1 \leq sp < 2$. Therefore, the map u cannot be the strong $W^{s,p}$ limit of a sequence $\{u_h\} \subset C^\infty(B^2, \mathbb{S}^1)$. If $n \geq 3$, a similar counterexample is given by the map $u(x) = (x_1, x_2)/|(x_1, x_2)|$, see e.g. [12].

Denoting by $W_S^{s,p}(B^n, \mathbb{S}^1)$ the *strong closure of $C^\infty(B^n, \mathbb{S}^1)$ in the $W^{s,p}$ -norm*, and assuming $n \geq 2$, we thus have:

$$W_S^{s,p}(B^n, \mathbb{S}^1) = W^{s,p}(B^n, \mathbb{S}^1) \iff sp < 1 \quad \text{or} \quad sp \geq 2 \quad (0.1)$$

whereas

$$W_S^{s,p}(B^n, \mathbb{S}^1) \subsetneq W^{s,p}(B^n, \mathbb{S}^1) \iff 1 \leq sp < 2 \quad \text{and} \quad n \geq 2.$$

However, in [13] it is defined a *distributional Jacobian* $\mathbb{J}u$ of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ that characterizes the obstruction to the strong approximation by smooth maps: namely, the strong closure of $C^\infty(B^n, \mathbb{S}^1)$ in the $W^{s,p}$ -norm agrees with the class of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ such that $\mathbb{J}u = 0$.

In this paper, we deal with the problem of *weak sequential density of smooth maps* in any dimension $n \geq 2$. More precisely, when strong density fails to hold, we ask if for every $u \in W^{s,p}(B^n, \mathbb{S}^1)$ one can find a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ strongly converging to u in $L^p(B^n, \mathbb{R}^2)$ and such that $\sup_h |u_h|_{s,p} < \infty$.

The answer is negative in case $1 < sp < 2$, the counterexample being given by maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ with non-zero Jacobian $\mathbb{J}u$, see [8].

In the relevant case $sp = 1$, the latter problem is completely settled for $s = 1/2$ and $p = 2$, i.e., in the class $W^{1/2,2}(B^n, \mathbb{S}^1)$. In that framework, in fact, using that functions in $W^{1/2,2}(B^n)$ identify the traces of the Sobolev space $W^{1,2}(B^n \times (0, +\infty))$, the distributional Jacobian $\mathbb{J}u$ was defined by Hang-Lin [25], and actually in any dimension $n \geq 2$ it can be written by means of homological arguments, see [23]. For maps in $W^{1/2,2}(B^n, \mathbb{S}^1)$, in fact, the explicit formula for the relaxed energy was obtained in [23], using tools from Geometric Measure Theory.

In this paper, we solve the problem in case $0 < s < 1$ and $p > 1$. To this purpose, we introduce on the class $W^{s,p}(B^n, \mathbb{S}^1)$ a natural energy $u \mapsto \mathcal{E}_{s,p}(u)$, see (0.5) below. When $s = 1/2$ and $p = 2$, a case already considered in [18, 23], the energy $\mathcal{E}_{1/2,2}(u)$ is given by the Dirichlet integral $\int |DU|^2 dx dt$ of the harmonic extension $U : B^n \times (0, 1) \rightarrow \mathbb{R}^2$ of u . We then analyze the corresponding *relaxed energy* (0.6). Some more notation is in order.

0.1 The energy

For $\gamma \in \mathbb{R}$ and $p > 1$, we denote by $W_\gamma^{1,p}(B^n \times (0, +\infty))$ the weighted Sobolev space given by the functions $U \in L^p(B^n \times (0, +\infty))$ which are approximately differentiable a.e. and with approximate gradient DU a

measurable function satisfying

$$\int_{\Omega \times (0, +\infty)} t^\gamma |DU(x, t)|^p dx dt < \infty, \quad \Omega = B^n. \quad (0.2)$$

Assume now $0 < s < 1$. By interpolation theory, see e.g. [27], it turns out that the fractional Sobolev space $W^{s,p}(B^n)$ agrees with the Besov space $B_{p,p}^s(B^n)$, and hence with the class of traces $u(x) = U(x, 0)$ on $t = 0$ of functions U in $W_\gamma^{1,p}(B^n \times (0, +\infty))$, where

$$\gamma = \gamma(s, p) := p(1 - s) - 1, \quad p > 1, \quad 0 < s < 1. \quad (0.3)$$

Note that when $s = 1 - 1/p$, one has $\gamma = 0$ and $W^{1-1/p,p}(B^n)$ agrees with the class of traces on $t = 0$ of the Sobolev space $W^{1,p}(B^n \times (0, +\infty))$.

A particular case of our interest is when $sp = 1$, so that $\gamma = p - 2$. In fact, since the Gagliardo seminorm becomes

$$|u|_{1/p,p}^p = \int_{B^n} \int_{B^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+1}} dx dy,$$

it turns out that in any dimension n , the class of bounded functions in $W^{1/p,p}(B^n)$ is continuously embedded in $W^{1/q,q}(B^n)$, for each $1 < p < q$. Moreover, in low dimension $n = 1$, when $\Omega = \mathbb{R}$ and $sp = 1$, the energy (0.2) is scale invariant for any $p > 1$.

Denote by $W_\gamma^{1,p}(\mathbf{C}^{n+1}, \mathbb{R}^2)$, where \mathbf{C}^{n+1} the $(n + 1)$ -dimensional cylinder

$$\mathbf{C}^{n+1} := B^n \times (0, 1),$$

the class of functions $U = (U^1, U^2) : \mathbf{C}^{n+1} \rightarrow \mathbb{R}^2$ with components U^j in $W_\gamma^{1,p}(\mathbf{C}^{n+1})$, and consider the energy

$$\mathcal{E}_{\gamma(s,p)}^p(U) := \int_{\mathbf{C}^{n+1}} t^{\gamma(s,p)} |DU(x, t)|^p dx dt, \quad \gamma(s, p) := p(1 - s) - 1. \quad (0.4)$$

For any bounded function $u \in W^{s,p}(B^n, \mathbb{R}^2) \cap L^\infty(B^n, \mathbb{R}^2)$, we let

$$U := \text{Ext}(u)$$

denote a bounded function that *minimizes the energy* $\mathcal{E}_{\gamma(s,p)}^p(U)$ among all $U \in W_\gamma^{1,p}(\mathbf{C}^{n+1}, \mathbb{R}^2) \cap L^\infty(\mathbf{C}^{n+1}, \mathbb{R}^2)$ such that $U(x, 0) = u(x)$ on $B^n \times \{0\}$ in the sense of the traces.

Such a minimizer exists and is smooth inside \mathbf{C}^{n+1} , by the convexity of the functional $U \mapsto \mathcal{E}_{\gamma(s,p)}^p(U)$. Moreover, if $u \in W^{s,p}(B^n, \mathbb{S}^1)$, by a projection argument we may assume $\text{Ext}(u) : \mathbf{C}^{n+1} \rightarrow \mathbb{D}^2$, where

$$\mathbb{D}^2 := \{y \in \mathbb{R}^2 : |y| \leq 1\}$$

is the unit disk in the target space. In addition, see [27], if $\{u_h\}$ is a sequence in $W^{s,p}(B^n, \mathbb{R}^2) \cap L^\infty(B^n, \mathbb{R}^2)$ converging a.e. in B^n to a function $u \in W^{s,p}(B^n, \mathbb{R}^2) \cap L^\infty(B^n, \mathbb{R}^2)$, it turns out that *the strong convergence* $u_h \rightarrow u$ in $W^{s,p}(B^n, \mathbb{R}^2)$ is equivalent to the convergence $u_h \rightarrow u$ in $L^p(B^n, \mathbb{R}^2)$ joined with the energy convergence

$$\lim_{h \rightarrow \infty} \mathcal{E}_{\gamma(s,p)}^p(\text{Ext}(u_h)) = \mathcal{E}_{\gamma(s,p)}^p(\text{Ext}(u)).$$

When $0 < s < 1$, it is then natural to introduce on the class of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ the $W^{s,p}$ -energy

$$\mathcal{E}_{s,p}(u) := \mathcal{E}_{\gamma(s,p)}^p(U), \quad U = \text{Ext } u : \mathbf{C}^{n+1} \rightarrow \mathbb{D}^2 \quad (0.5)$$

where $\mathcal{E}_{\gamma(s,p)}^p(U)$ is given by (0.4). In the same spirit as for Lebesgue's relaxed area, we correspondingly introduce the *relaxed energy*

$$\tilde{\mathcal{E}}_{s,p}(u) := \inf \left\{ \liminf_{h \rightarrow \infty} \mathcal{E}_{s,p}(u_h) \mid \begin{array}{l} \{u_h\} \subset C^\infty(B^n, \mathbb{S}^1), \\ u_h \rightarrow u \text{ strongly in } L^p(B^n, \mathbb{R}^2) \end{array} \right\}. \quad (0.6)$$

In fact, a map $u \in W^{s,p}(B^n, \mathbb{S}^1)$ belong to the $W^{s,p}$ weak sequential closure of smooth maps in $C^\infty(B^n, \mathbb{S}^1)$ if and only if it has finite relaxed energy, $\tilde{\mathcal{E}}_{s,p}(u) < \infty$.

0.2 The current of the singularities

Assume now $sp = 1$. In that case, we rewrite the distributional Jacobian $\mathbb{J}u$ from [13], that carries the relevant information on the *singularities* of u , in terms of an $(n-2)$ -dimensional current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$, see Definition 3.1.

In dimension $n = 2$, the distribution $\mathbb{P}(u)$ agrees with the extension of the distributional determinant $T : W^{1/p,p}(B^2, \mathbb{S}^1) \rightarrow W^{1,\infty}(B^2, \mathbb{R})^*$ by Bourgain-Brezis-Mironescu [7].

We show that for every u in $W^{1/p,p}(B^n, \mathbb{S}^1)$, the current $\mathbb{P}(u)$ is an *integral flat chain*, i.e., that there exists an integer multiplicity (say i.m.) rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ satisfying $(\partial L) \llcorner B^n = \mathbb{P}(u)$.

Such a property holds true when $p = 2$ as a consequence of the *coarea formula* first considered by Almgren-Browder-Lieb [1], see [23]. In fact, extensions of maps in $W^{1/2,2}(B^n, \mathbb{S}^1)$ belong to the Sobolev space $W^{1,2}(\mathbb{C}^{n+1}, \mathbb{R}^2)$.

Therefore, when $1 < p < 2$, by the continuous embedding of the class $W^{1/p,p}(B^n, \mathbb{S}^1)$ into $W^{1/2,2}(B^n, \mathbb{S}^1)$, the integral flat chain $\mathbb{P}(u)$ is automatically defined for all maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$.

If $p > 2$, there exist maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ such that $\|u\|_{1/2,2} = \infty$, and hence the latter direct argument fails to hold. Notwithstanding, we are able to treat the case $p > 2$ by means of a new coarea formula, whose proof is based on some relevant estimates due to Bourgain-Brezis-Mironescu [7].

0.3 The energy gap

On account of property (0.1), it turns out that the *energy gap*

$$\mathcal{G}_{s,p}(u) := \tilde{\mathcal{E}}_{s,p}(u) - \mathcal{E}_{s,p}(u) \quad (0.7)$$

can be non-zero only if $1 \leq sp < 2$ and $n \geq 2$.

In this paper, we show that *the energy gap $\mathcal{G}_{1/p,p}(u)$ in (0.7) is always finite* in the class $W^{1/p,p}(B^n, \mathbb{S}^1)$, for any $p > 1$. More precisely, for any $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ we have:

$$\mathcal{G}_{1/p,p}(u) = 0 \quad \iff \quad \mathbb{P}(u) = 0.$$

In addition, when $\mathbb{P}(u) \neq 0$, we obtain the explicit formula:

$$\mathcal{G}_{1/p,p}(u) = E_p \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty \quad \forall u \in W^{1/p,p}(B^n, \mathbb{S}^1), \quad \forall n \geq 2, \quad p > 1. \quad (0.8)$$

In equation (0.8), the term $m_{i,B^n}(\mathbb{P}(u))$ denotes the *minimal integral connection* of the singularity $\mathbb{P}(u)$, i.e., the mass $\mathbf{M}(L)$ of the minimal i.m. rectifiable current $L \in \mathcal{R}_{n-1}(B^n)$ satisfying $(\partial L) \llcorner B^n = \mathbb{P}(u)$.

If e.g. $n = 2$ and $u \in R_{1/p,p}(B^2, \mathbb{S}^1)$, letting $\Sigma_u = \{x_1, \dots, x_m\}$ and denoting by $d_i \in \mathbb{Z}$ the degree of u at x_i , the minimal connection of the singular set Σ_u is the mass-minimizing current $L \in \mathcal{R}_1(B^2)$ satisfying $(\partial L) \llcorner B^2 = \sum_{i=1}^m d_i \cdot \delta_{x_i}$, where δ_x is the unit Dirac mass at the point $x \in B^2$.

Moreover, the positive constant E_p in formula (0.8) is given by the *energy infimum of degree one maps* in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$. When $p = 2$, in [23, 28] we in fact obtained that $E_2 = 2\pi$, that is equal to the Dirichlet energy $\int |DU|^2$ of the harmonic map from the half-space to \mathbb{R}^2 whose trace agrees with the inverse of the stereographic map from \mathbb{S}^1 onto \mathbb{R} , see Appendix A.

Note that formula (0.8) implies that *every map in $W^{1/p,p}(B^n, \mathbb{S}^1)$ belongs to the $W^{1/p,p}$ -weak sequential closure of smooth maps in $C^\infty(B^n, \mathbb{S}^1)$.*

0.4 Other results and open questions

In case $sp > 1$, the cited results from [13] yield the existence of maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ with non-zero distributional Jacobian $\mathbb{J}u$ and for which $\mathcal{G}_{s,p}(u) = +\infty$.

When $n \geq 2$, $0 < s < 1$, $p > 1$, and $1 < sp < 2$, in this paper we obtain:

$$\tilde{\mathcal{E}}_{s,p}(u) = \begin{cases} \mathcal{E}_{s,p}(u) & \text{if } \mathbb{P}(u) = 0 \\ +\infty & \text{if } \mathbb{P}(u) \neq 0 \end{cases} \quad \forall u \in W^{s,p}(B^n, \mathbb{S}^1). \quad (0.9)$$

Therefore, differently from the case $sp = 1$, where energy concentration only occurs on codimension one sets and with a finite amount of energy, when $sp > 1$, in presence of non-trivial singularities the relaxed energy blows up.

In particular, when $n \geq 2$, $0 < s < 1$, and $p > 1$, with $sp \geq 1$, by our results we conclude that:

$$W_S^{s,p}(B^n, \mathbb{S}^1) = \{u \in W^{s,p}(B^n, \mathbb{S}^1) \mid \mathbb{P}(u) = 0\}. \quad (0.10)$$

It thus remains to consider the ranges of s and p for which the strong density of smooth maps fails to hold, see Theorem 0.1, but $s > 1$. In this case, we have a partial result: if $1 < p < 2$, $1 < s < 2$, $1 < sp < 2$, $n \geq 2$, and $u \in W^{s,p}(B^n, \mathbb{S}^1)$ is such that there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging a.e. to u and with equibounded $W^{s,p}$ -norms, $\sup_h \|u_h\|_{s,p} < \infty$, then one has $\mathbb{P}(u) = 0$.

We expect that the converse implication holds. However, when $s > 1$, our definition of energy (0.5) does not make sense.

Coming back to [27], we recall that when $1 < s < 2$, the class $W^{s,p}(B^n)$ is given by the traces on $t = 0$ of Sobolev functions U in $W^{1,p}(B^n \times (0, +\infty))$ with approximate second gradient D^2U a measurable function satisfying

$$\int_{B^n \times (0, +\infty)} t^\gamma |D^2U(x, t)|^p dx dt < \infty, \quad \gamma = p(2 - s) - 1.$$

Similarly as before, one may thus introduce on maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$ the energy $\mathcal{E}_{s,p}(u)$ as e.g.

$$\mathcal{E}_{s,p}(u) := \int_{\mathbf{C}^{n+1}} t^{p(2-s)-1} |D^2U(x, t)|^p dx dt < \infty, \quad U = \text{Ext}(u). \quad (0.11)$$

A positive answer to the following Open Question would imply the validity of formula (0.10) for any $p > 1$ and $s > 0$, with $1 \leq sp < 2$.

Open Question: Let $1 < p < 2$, $1 < sp < 2$, $1 < s < 2$, and $n \geq 2$. If $u \in W^{s,p}(B^n, \mathbb{S}^1)$ satisfies condition $\mathbb{P}(u) = 0$, then there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging to u a.e. and such that $\mathcal{E}_{s,p}(u_h) \rightarrow \mathcal{E}_{s,p}(u)$, where the energy is given by (0.11) and $\text{Ext}(u)$ is an energy minimizer among the smooth maps U from \mathbf{C}^{n+1} to \mathbb{D}^2 with trace on $t = 0$ equal to u .

For the sake of brevity, the case of maps with *prescribed boundary values* is not treated in this paper: it can be readily discussed by making straightforward modifications as e.g. in [23]. Finally, in the case $s = 1$, the relaxation problem of Sobolev maps in $W^{1,p}(B^n, \mathbb{S}^1)$ is treated in [24].

0.5 Content of the paper

In Sec. 1, we use some estimates taken from [7] to find a suitable extension of maps in $W^{1/p,p}(B^n, \mathbb{S}^1)$ with finite mapping area, Proposition 1.1, which yields to the validity of a suitable coarea formula, Theorem 1.2.

In Sec. 2, we recall some notation from Geometric Measure Theory and introduce the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, for which we establish a closure-compactness property, Theorem 2.5. Roughly speaking, a current T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ takes the form:

$$T = G_u + L \times \llbracket \mathbb{S}^1 \rrbracket \quad (0.12)$$

where G_u is the *current carried by the graph* of some map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $L \in \mathcal{R}_{n-1}(B^n)$ is an i.m. rectifiable current in such a way that $\partial T = 0$ on compactly supported smooth $(n-1)$ -forms in $B^n \times \mathbb{S}^1$. In low dimension $n = 1$, we then discuss a notion of degree in our context, by extending the definition given in [7] for maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$.

In Sec. 3, we introduce the $(n-2)$ -current of the singularities $\mathbb{P}(u)$, showing that it is an integral flat chain, Proposition 3.7. For this purpose, we take advantage of Theorem 0.2 by Brezis-Mironescu [12]. Note that when $n \geq 2$, for a current T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ as in (0.12), the null boundary condition of T is equivalent to the equality $(\partial L) \llcorner B^n = -\mathbb{P}(u)$, whereas it is automatically satisfied when $n = 1$.

In Sec. 4, we deal with the factor E_p in the energy gap formula (0.8), showing that for any $p > 1$ and any fixed integer $d \in \mathbb{Z}$, the energy infimum among all maps in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ with degree d is equal to $|d| \cdot E_p$, Proposition 4.3.

In Sec. 5, we introduce a suitable functional $T \mapsto E_{1/p,p}(T)$ on Cartesian currents, see (5.1), that agrees with the energy $\mathcal{E}_{1/p,p}(u)$ in case of graphs of “smooth” maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, see (0.5). Our functional turns out to be lower semicontinuous along weakly converging sequences of smooth graphs, Theorem 5.1.

In Sec. 6, we provide in low dimension $n = 2$ the approximation of dipoles for $W^{1/p,p}$ -maps with values in \mathbb{S}^1 , Theorem 6.4. Our Proposition 6.5 is in accordance with the case $N = 1$ of [7, Thm. 2.4], where the authors analyzed the dipole problem for maps in $W^{N/p,p}(\mathbb{S}^{N+1}, \mathbb{S}^N)$. We also show how to remove *homologically trivial* point singularities, Proposition 6.1.

In Sec. 7, we prove for any $p > 1$ and $n \geq 2$ a strong density result for our class of Cartesian currents. Namely, in Theorem 7.1 we show that *for every T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ we can find a smooth sequence $\{u_h\}$ in $C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(u_h) \rightarrow E_{1/p,p}(T)$.*

We briefly sketch here its proof. On account of Proposition 3.7 and of Federer’s strong polyhedral approximation theorem [14], we are able to reduce to the case where the current T takes the form (0.12) for some map $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, where the current $L \in \mathcal{R}_{n-1}(B^n)$ is a finite sum of pairwise disjoint oriented polyhedral $(n-1)$ -chains. In order to approximate the $(n-1)$ -dimensional “dipoles”, in high dimension $n \geq 3$ we apply Proposition 7.3, that is proved in Appendix B. Finally, in order to remove the $(n-2)$ -dimensional singular set, in high dimension $n \geq 3$ we apply Proposition 7.4, that is proved in Appendix C. In the easier case $n = 2$, we directly apply Theorem 6.4 and Proposition 6.1.

In Sec. 8, we first collect the closure-compactness properties for the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, Theorem 8.1, extending results proved in [23] when $p = 2$. In Theorem 8.2, we obtain the explicit formula (0.8) for the energy gap (0.7). Its proof is based on our main results previously stated: Proposition 3.7 and Theorems 2.5, 5.1, and 7.1. In Theorem 8.4, we prove formula (0.9). Finally, in Corollary 8.6, we prove the cited partial result for the case $1 < p < 2$, $1 < s < 2$, $1 < sp < 2$, and $n \geq 2$.

1 Coarea formula

In this section, we find a coarea formula for a suitable extension of maps in $W^{1/p,p}(B^n, \mathbb{S}^1)$. To this purpose, we make use of some relevant estimates obtained by Bourgain-Brezis-Mironescu in [7].

1.1 A relevant estimate

Let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, and let $U \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ be the harmonic extension of u to $\mathbf{C}^{n+1} := B^n \times (0, 1)$, where $n \geq 1$ is integer and $p > 1$ real. Following [7, Lemma 1.2], we denote

$$G := \{(x, t) \in \mathbf{C}^{n+1} : |U(x, t)| \leq 1/2\}$$

and we let $d : B^n \rightarrow]0, 1/2]$ be the function such that $d(x) := 1/2$ if $|U(x, t)| \geq 1/2$ for each $t \in (0, 1/2)$, and

$$d(x) := \min\{t \in (0, 1/2) : |U(x, t)| \leq 1/2\}$$

otherwise. Using that $|DU(x, t)| \leq c/t$ for some absolute constant c , for any exponent $\alpha > 1$ one has

$$\int_G |DU(x, t)|^\alpha dx dt \leq c \int_{B^n} \left(\int_{d(x)}^1 t^{-\alpha} dt \right) dx \leq C \int_{B^n} \frac{1}{d(x)^{\alpha-1}} dx.$$

In a similar way to the case $\alpha = 2$, using that $t > d(x)$ if $(x, t) \in G$, for each $p > 1$ we estimate

$$\int_G t^{p-2} |DU(x, t)|^p dx dt \leq \int_G \frac{C}{t^2} dx dt \leq C \int_{B^n} \frac{1}{d(x)} dx \quad (1.1)$$

where $C = C(n, p)$. Moreover, as in [7, Lemma 1.3], since $U \in W^{2/p,p}(B^n \times I, \mathbb{R}^2)$, where $I = (0, 1/2)$, using the embedding of $W^{2/p,p}(I)$ in the Hölder class $C^{0,1/p}(I)$, it turns out that for a.e. $x \in B^n$ the function $\varphi_x(t) := U(x, t)$ belongs to $W^{2/p,p}(I, \mathbb{R}^2)$, whence to $C^{0,1/p}(I, \mathbb{R}^2)$, so that we have:

$$\frac{1}{2} \leq |\varphi_x(d(x)) - \varphi_x(0)| \leq C d(x)^{1/p} \|\varphi_x\|_{C^{0,1/p}(I)} \leq C d(x)^{1/p} \|\varphi_x\|_{W^{2/p,p}(I, \mathbb{R}^2)}$$

and hence

$$\frac{1}{d(x)} \leq C \|\varphi_x\|_{W^{2/p,p}(I)}^p.$$

Therefore, using the inequality on Besov-type spaces

$$\begin{aligned} \int_{B^n} \|\varphi_x\|_{W^{2/p,p}(I,\mathbb{R}^2)}^p dx &= \int_{B^n} \|U(x,\cdot)\|_{W^{2/p,p}(I,\mathbb{R}^2)}^p dx \\ &\leq C \|U\|_{W^{2/p,p}(\mathbf{C}^{n+1},\mathbb{R}^2)}^p \leq C \|u\|_{W^{1/p,p}(B^n,\mathbb{R}^2)}^p \end{aligned}$$

by (1.1) one gets the estimate

$$\int_G t^{p-2} |DU(x,t)|^p dx dt \leq C_1 \int_{B^n} \frac{1}{d(x)} dx \leq C_2 \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dx dt \quad (1.2)$$

for some positive constants C_1, C_2 only depending on n and p .

In the sequel, we choose a smooth function $\Phi : \mathbb{R}^2 \rightarrow \mathbb{D}^2$ such that $\Phi(y) = y/|y|$ if $|y| \geq 1/2$, where $y = (y_1, y_2)$, and Φ is a bi-Lipschitz map from $\{y : |y| \leq 1/2\}$ to \mathbb{D}^2 .

Setting $V := \Phi \circ U$, we clearly have:

$$\begin{aligned} |DV(x,t)| &\leq C_1 |DU(x,t)| \quad \forall (x,t) \in \mathbf{C}^{n+1}, \\ |DU(x,t)| &\leq C_2 |DV(x,t)| \quad \forall (x,t) \in G. \end{aligned} \quad (1.3)$$

Denote now by $V^\#(dy^1 \wedge dy^2)$ the 2-form in \mathbf{C}^{n+1} given by the pull-back by V of the 2-form $dy^1 \wedge dy^2$. One has

$$|V^\#(dy^1 \wedge dy^2)| = J_V \quad (1.4)$$

where J_V is the Jacobian of the map V , so that $J_V(x,t)^2$ is the sum of the square of the 2×2 minors of the gradient matrix $DV(x,t)$. Therefore, by the area formula one has $J_V(x,t) = 0$ if $(x,t) \notin G$, whereas by the parallelogram inequality one gets the general estimate $J_V(x,t) \leq C_n |DV(x,t)|^2$.

These are the main facts that led Bourgain-Brezis-Mironescu [7] to obtain the estimate

$$|\deg g| \leq C_p \|g\|_{1/p,p}^p \quad \forall p > 1$$

on the degree $\deg g$ of maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$. In dimension $n = 2$, they also build up for any $p > 1$ the unique extension $T : W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \rightarrow W^{1,\infty}(\mathbb{S}^2, \mathbb{R})^*$ of the *distributional determinant* $T(g) = \text{Det}(\nabla g)$ of maps $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \cap W^{1,2}$, obtaining the estimate

$$|\langle T(g), \zeta \rangle| \leq C_p \|g\|_{1/p,p}^p \cdot \|\nabla \zeta\|_{L^\infty} \quad \forall \zeta \in W^{1,\infty}(\mathbb{S}^2)$$

for any $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$. Extending previous facts from the easier case $p = 2$, they also prove for every $p > 1$ and $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$ the existence of two sequences $(P_i), (N_i) \subset \mathbb{S}^2$ such that

$$T(g) = \pi \cdot \sum_i (\delta_{P_i} - \delta_{N_i}), \quad \sum_i |P_i - N_i| \leq C_p \|g\|_{1/p,p}^p.$$

1.2 Coarea formula

With the previous notation, we similarly obtain in any dimension $n \geq 1$ the following estimate:

Proposition 1.1 *Let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ for some $p > 1$. Then we have:*

$$\int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge dy^2)| dx dt \leq C \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dx dt \quad (1.5)$$

for some real constant $C > 0$ only depending on n, p .

PROOF: By the previous facts, inequality (1.5) readily follows when $p = 2$, and hence for $1 < p < 2$, by the continuous embedding $W^{1/p,p}(B^n, \mathbb{S}^1) \subset W^{1/2,2}(B^n, \mathbb{S}^1)$. When $p > 2$, letting $\alpha = \alpha(p) = 2(p-2)/p$, by the Hölder inequality with exponents $q = p/2$ and $q' = p/(p-2)$ we get:

$$\begin{aligned} \int_G |DV(x,t)|^2 dx dt &\leq C \int_G (t^\alpha |DU(x,t)|^2) t^{-\alpha} dx dt \\ &\leq C \left(\int_G t^{p-2} |DU(x,t)|^p dx dt \right)^{2/p} \left(\int_G t^{-2} dx dt \right)^{(p-2)/p} \end{aligned}$$

where by (1.1) and (1.2) we can estimate

$$\left(\int_G t^{-2} dx dt \right)^{(p-2)/p} \leq C_2 \left(\int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dx dt \right)^{(p-2)/p}.$$

Since by (1.3) and (1.4)

$$\int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge dy^2)| dx dt = \int_G J_V(x,t) dx dt \leq C \int_G |DV(x,t)|^2 dx dt$$

the assertion readily follows. \square

We thus obtain the validity of the *coarea formula* in the sense of Almgren-Browder-Lieb [1]:

Theorem 1.2 *Let $n \geq 1$ and $p > 1$. Denote by \mathcal{H}^{n-1} the $(n-1)$ -dimensional Hausdorff measure. For every map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, there exists a smooth extension $V \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ and a regular value $y \in \mathbb{D}^2$ for V such that*

$$\mathcal{H}^{n-1}(V^{-1}(\{y\})) \leq C \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dx dt \quad (1.6)$$

for some real constant C only depending on n and p .

PROOF: Choose $V := \Phi \circ U$, where $U \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u . We have

$$\int_{\mathbb{D}^2} \mathcal{H}^{n-1}(V^{-1}(\{y\})) d\mathcal{H}^2(y) = \int_{\mathbf{C}^{n+1}} J_V(x,t) dx dt = \int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge dy^2)| dx dt$$

and hence we can find a regular value $y \in \mathbb{D}^2$ such that

$$\mathcal{H}^{n-1}(V^{-1}(\{y\})) \leq \frac{1}{\pi} \cdot \int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge dy^2)| dx dt.$$

The assertion follows from Proposition 1.1. \square

2 Cartesian currents and degree

In this section, we recall some notation from Geometric Measure Theory and introduce the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, for which we establish a closure-compactness property, Theorem 2.5. In low dimension $n = 1$, we then discuss a notion of degree in our context, by extending the definition given in [7] for maps $g \in W^{1/p,p}(\mathbb{S}^1, \mathbb{S}^1)$.

2.1 Rectifiable currents

Let $0 \leq k \leq N$ be integer and let $\Omega \subset \mathbb{R}^N$ be an open set. The space $\mathcal{D}_k(\Omega)$ of k -currents in Ω is the strong dual of the space $\mathcal{D}^k(\Omega)$ of compactly supported smooth k -forms. The weak convergence $T_h \rightharpoonup T$ in $\mathcal{D}_k(\Omega)$ is defined by duality through the formula

$$\lim_{h \rightarrow \infty} T_h(\omega) = T(\omega), \quad \forall \omega \in \mathcal{D}^k(\Omega).$$

The *mass* of a current $T \in \mathcal{D}_k(\Omega)$ is defined by

$$\mathbf{M}(T) := \sup\{T(\omega) \mid \omega \in \mathcal{D}^k(\Omega), \|\omega\| \leq 1\}$$

where $\|\omega\|$ is the *comass* norm of ω . Therefore, the mass functional is lower semicontinuous w.r.t. the weak convergence. The *boundary* of a current T in $\mathcal{D}_k(\Omega)$, when $k \geq 1$, is defined by duality as

$$\partial T(\eta) := T(d\eta), \quad \forall \eta \in \mathcal{D}^{k-1}(\Omega),$$

yielding to a current ∂T in $\mathcal{D}_{k-1}(\Omega)$.

A current $T \in \mathcal{D}_k(\Omega)$ is called *integer multiplicity* (say i.m.) *rectifiable* in $\mathcal{R}_k(\Omega)$ if one has

$$T(\omega) = \int_{\mathcal{M}} \theta \langle \omega, \xi \rangle d\mathcal{H}^k \quad \forall \omega \in \mathcal{D}^k(\Omega),$$

where \mathcal{H}^k is the k -dimensional Hausdorff measure, \mathcal{M} is a countably k -rectifiable set of Ω , with $\mathcal{H}^k(\mathcal{M}) < \infty$, the $\mathcal{H}^k \llcorner \mathcal{M}$ -measurable function $\xi : \mathcal{M} \rightarrow \Lambda^k \mathbb{R}^N$ gives for \mathcal{H}^k -a.e. $z \in \mathcal{M}$ a unit simple k -vector $\xi(z)$ that provides an orientation to the approximate tangent k -space to \mathcal{M} at z , and θ is an integer valued, $\mathcal{H}^k \llcorner \mathcal{M}$ -summable, and non-negative multiplicity function. Therefore, one has $\mathbf{M}(T) = \int_{\mathcal{M}} \theta d\mathcal{H}^k < \infty$. We also denote by

$$\text{set } T := \{z \in \mathcal{M} \mid \theta(z) \in \mathbb{N}^+\} \quad (2.1)$$

the set of points in \mathcal{M} with positive density.

If e.g. \mathcal{M} is an oriented k -submanifold of Ω with finite k -volume, the linear functional $\omega \mapsto \int_{\mathcal{M}} \omega$ on $\mathcal{D}^k(\Omega)$ defines a current $[\mathcal{M}]$ in $\mathcal{R}_k(\Omega)$ with finite mass equal to $\mathcal{H}^k(\mathcal{M})$. We address to [30] or [?] for further details on GMT tools.

In particular, when $\Omega = A \times \mathbb{R}^m$, where $A \subset \mathbb{R}^k$ is a bounded domain, and $\mathcal{M} = \mathcal{G}_v$ is the graph of a Lipschitz function $v : A \rightarrow \mathbb{R}^m$, the k -current $G_v = [\mathcal{G}_v]$ carried by the graph of v acts on k -forms $\omega \in \mathcal{D}^k(A \times \mathbb{R}^m)$ as

$$G_v(\omega) = (Id \bowtie v)_{\#} [A](\omega) := \int_A (Id \bowtie v)^{\#} \omega$$

where $(Id \bowtie v)^{\#} \omega$ is the pull-back of ω through the graph map $(Id \bowtie v)(x) := (x, v(x))$. For example, if $\omega = \gamma \wedge \psi \in \mathcal{D}^k(A \times \mathbb{R}^m)$, where $\gamma \in \mathcal{D}^{k-h}(A)$, $\psi \in \mathcal{D}^h(\mathbb{R}^m)$, and $0 \leq h \leq \min\{k, m\}$, then

$$G_v(\gamma \wedge \psi) = \int_A (Id \bowtie v)^{\#}(\gamma \wedge \psi) = \int_A \gamma \wedge v^{\#} \psi. \quad (2.2)$$

By the area formula, one then computes

$$\mathbf{M}(G_v) = \int_A J_{Id \bowtie v}(x) dx = \mathcal{H}^k(\mathcal{G}_v)$$

where $J_{Id \bowtie v}$ is the Jacobian of the graph map. If e.g. $k \geq m = 2$, one has

$$J_{Id \bowtie v} = \sqrt{1 + |Dv|^2 + |M_2(Dv)|^2}$$

where $|M_2(Dv)|^2$ is the sum of the square of the 2×2 minors of the gradient matrix Dv , so that $|M_2(Dv)| = J_v$ and in particular $|M_2(Dv)| = |\det Dv|$ if $k = 2$, see e.g. [?].

Example 2.1 If U is a Sobolev map in $W^{1,2}(\mathbf{C}^{n+1}, \mathbb{R}^2)$, the $(n+1)$ -current G_U in $\mathbf{C}^{n+1} \times \mathbb{R}^2$ carried by its graph is defined (in an approximate sense) by $G_U := (Id \bowtie U)_{\#} [\mathbf{C}^{n+1}]$, compare [?]. Actually, G_U has finite mass and is an i.m. rectifiable current in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1} \times \mathbb{R}^2)$. In fact, by the area formula and the parallelogram inequality we get the bound

$$\mathbf{M}(G_U) = \int_{\mathbf{C}^{n+1}} J_{Id \bowtie U} dz \leq c \left(1 + \int_{\mathbf{C}^{n+1}} |DU|^2 dz \right) < \infty$$

for some absolute constant $c > 0$, not depending on U . We also have

$$\partial G_U(\eta) = 0 \quad \forall \eta \in \mathcal{D}^n(\mathbf{C}^{n+1} \times \mathbb{R}^2),$$

a property that reads as the null-boundary condition

$$(\partial G_U) \llcorner \mathbf{C}^{n+1} \times \mathbb{R}^2 = 0. \quad (2.3)$$

In fact, equation (2.3) is readily checked if U is smooth, by Stokes' theorem, and it is preserved by the weak convergence $G_{U_h} \rightharpoonup G_U$ as currents, that holds true by dominated convergence if $U_h \rightarrow U$ strongly in $W^{1,2}(\mathbf{C}^{n+1}, \mathbb{R}^2)$. Then, a standard density argument applies to infer (2.3).

By Proposition 1.1, we are able to extend the previous features to our setting, as follows.

Proposition 2.2 *Let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, where $p > 1$, and let $V := \Phi \circ U$, where $U \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u . Then the current G_V is i.m. rectifiable in $\mathcal{R}_{n+1}(\mathbf{C}^{n+1} \times \mathbb{R}^2)$, with finite mass bounded by*

$$\mathbf{M}(G_V) = \int_{\mathbf{C}^{n+1}} J_{Id \times V} dz \leq c \left(1 + \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dz \right) \quad (2.4)$$

for some constant $c > 0$, not depending on u . Moreover, the null-boundary condition (2.3) holds true, with $U = V$.

PROOF: The continuous embedding $W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{R}^2) \subset W^{1,1}(\mathbf{C}^{n+1}, \mathbb{R}^2)$ holds for any $p > 1$. In fact, letting $\alpha = (p-2)/p$, by the Hölder inequality with exponents p and $p' = p/(p-1)$ we get:

$$\begin{aligned} \int_{\mathbf{C}^{n+1}} |DU(z)| dz &= \int_{\mathbf{C}^{n+1}} (t^\alpha |DU(x,t)|) t^{-\alpha} dx dt \\ &\leq \left(\int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x,t)|^p dx dt \right)^{1/p} \left(\int_{\mathbf{C}^{n+1}} t^{-\alpha p/(p-1)} dx dt \right)^{(p-1)/p} < \infty \end{aligned}$$

as $-\alpha p/(p-1) = (2-p)/(p-1) > -1$. Recalling that $|V^\#(dy^1 \wedge dy^2)| = |M_2(DV)|$ and $|DV| \leq C|DU|$, the mass estimate (2.4) follows from (1.5). Finally, similarly to the $W^{1,2}$ case in Example 2.1, the null boundary condition (2.3) is readily checked by a standard density argument, on account of the mass estimate (2.4) and of the dominated convergence theorem. \square

Due to the previous facts, similarly to the case $p = 2$ analyzed in [18, 22], we are able to introduce a good notion of current G_u carried by the graph of a map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$.

Definition 2.3 *To any map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ we associate an n -current G_u in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ by setting*

$$G_u := (-1)^{n-1} (\partial G_V) \llcorner ((B^n \times \{0\}) \times \mathbb{R}^2) \quad \text{on } \mathcal{D}^n(B^n \times \mathbb{S}^1), \quad (2.5)$$

where $V := \Phi \circ U$ and $U \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u .

In formula (2.5), the boundary ∂G_V is seen by extending the action of the current G_V to forms in $\mathcal{D}^{n+1}(\mathbb{R}^{n+1} \times \mathbb{R}^2)$. By Federer's support theorem [14], we in fact infer that the current G_u actually belongs to the class $\mathcal{D}_n(B^n \times \mathbb{S}^1)$. Note however that in general G_u is not i.m. rectifiable, even in low dimension $n = 2$, and fails to satisfy the null-boundary condition $(\partial G_u) \llcorner B^n \times \mathbb{S}^1 = 0$.

Remark 2.4 In case $n = 1$, property $(\partial G_u) \llcorner B^1 \times \mathbb{S}^1 = 0$ holds true as a consequence of the strong density of smooth maps, see Theorem 0.1.

2.2 Cartesian currents

Following [?, 23], we introduce for any $p > 1$ and $n \geq 1$ the class of Cartesian currents $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. They are given by the currents T in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ that decompose as

$$T = G_{u_T} + L_T \times \llbracket \mathbb{S}^1 \rrbracket \quad (2.6)$$

for some $u_T \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $L_T \in \mathcal{R}_{n-1}(B^n)$, and that satisfy the null-boundary condition

$$(\partial T) \llcorner B^n \times \mathbb{S}^1 = 0. \quad (2.7)$$

Note that equation (2.7) is automatically satisfied when $n = 1$, see Remark 2.4. Moreover, in general a current T in $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ fails to have bounded mass and to be i.m. rectifiable current in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$, if e.g. $u_T \notin W^{1,1}(B^n, \mathbb{R}^2)$. However, the following compactness property holds:

Theorem 2.5 *Let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ be such that $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$ for some $p > 1$. Then, there exists a Cartesian current $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ as in (2.6) such that, possibly passing to a not relabeled subsequence, $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $u_h \rightarrow u_T$ in $L^p(B^n, \mathbb{R}^2)$.*

PROOF: Let $V_h := \Phi \circ U_h$, where $U_h \in W_{p-2}^{1,p}(\mathbb{C}^{n+1}, \mathbb{D}^2)$ is the harmonic extension of u_h . According to Definition 2.3, the current G_{u_h} satisfies the slicing formula in (2.5), with $u = u_h$ and $V = V_h$.

Following [23], we now define a suitable map $W_h : \tilde{\mathbb{C}}^{n+1} \rightarrow \mathbb{S}^2$, where $\tilde{\mathbb{C}}^{n+1} := B^n \times (-1, 1)$ and $\mathbb{S}^2 := \{y \in \mathbb{R}^3 : |y| = 1\}$ is the unit sphere. Denoting by $\mathbb{S}_\pm^2 := \{y \in \mathbb{S}^2 : \pm y_3 \geq 0\}$ the upper and lower half-spheres, we consider a couple of bi-Lipschitz maps $\Phi^\pm : \mathbb{D}^2 \rightarrow \mathbb{S}_\pm^2$ such that $\Phi^\pm|_{\mathbb{S}^1}(z) = (z, 0)$, define

$$W_h(x, t) := \begin{cases} \Phi^+ \circ V_h(x, t) & \text{if } t \geq 0 \\ \Phi^- \circ V_h(x, -t) & \text{if } t \leq 0 \end{cases} \quad x \in B^n,$$

and denote $G_{W_h} \in \mathcal{D}_{n+1}(\tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2)$ the current carried by the graph of W_h . According to (0.4), we shall thus work with the energy $W \mapsto \mathcal{E}_{p-2}^p(W) := \int_{\tilde{\mathbb{C}}^{n+1}} |t|^{p-2} |DW|^p dx dt$.

Since $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$, by the mass estimate (2.4) we infer that

$$\sup_h \mathbf{M}(G_{W_h}) \leq C \cdot \sup_h \mathcal{E}_{p-2}^p(W_h) < \infty,$$

whence $\{G_{W_h}\}$ is a sequence of i.m. rectifiable currents in $\mathcal{R}_{n+1}(\tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2)$ with equibounded masses. Furthermore, by smoothness of W_h , Stokes' theorem implies the validity of the null-boundary condition:

$$(\partial G_{W_h}) \llcorner \tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2 = 0 \quad \forall h.$$

Therefore, by Federer-Fleming's closure theorem [16], a subsequence of $\{G_{W_h}\}$ weakly converges in $\mathcal{D}_{n+1}(\tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2)$ to an i.m. rectifiable current $\tilde{T} \in \mathcal{R}_{n+1}(\tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2)$ satisfying $(\partial \tilde{T}) \llcorner \tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2 = 0$.

Arguing as in [17], by the rectifiable slices theorem [31] it turns out that $\tilde{T} = G_W + L \llcorner \llbracket \mathbb{S}^2 \rrbracket$ for some function $W \in W^{1,1}(\tilde{\mathbb{C}}^{n+1}, \mathbb{S}^2)$, with Jacobian $J_{\text{Id} \times W}$ in $L^1(\tilde{\mathbb{C}}^{n+1})$, and some i.m. rectifiable current $L \in \mathcal{R}_{n-1}(\tilde{\mathbb{C}}^{n+1})$. Therefore, \tilde{T} is a Cartesian current in the class $\text{cart}(\tilde{\mathbb{C}}^{n+1} \times \mathbb{S}^2)$, see [?].

Since moreover $W_h \rightarrow W$ in $L^p(\tilde{\mathbb{C}}^{n+1}, \mathbb{R}^2)$, by lower semicontinuity of the energy $W \mapsto \mathcal{E}_{p-2}^p(W)$ it turns out that, in an obvious sense, $W \in W_{p-2}^{1,p}(\tilde{\mathbb{C}}^{n+1}, \mathbb{S}^2)$. Also, using that $W_h(x, 0) = u_h(x)$, a further subsequence of $\{u_h\}$ strongly converges in $L^p(B^n, \mathbb{R}^2)$ to some map $u \in L^p(B^n, \mathbb{S}^1)$, whence we get $W(x, 0) = u(x)$ in the sense of the traces and definitely $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$.

Furthermore, by the definition of W_h , it turns out that the current L is supported in the closure of $B^n \times \{0\}$. Therefore, on account of equation (2.5), where $u = u_h$ and $V = V_h$, by a slicing argument we infer that the current $T = G_u + L \llcorner \llbracket \mathbb{S}^1 \rrbracket$ satisfies the null-boundary condition (2.7).

In conclusion, T belongs to the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ and actually $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, as required. \square

2.3 Degree

For $p > 1$, denote now by $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ the class of locally summable maps $u : \mathbb{R} \rightarrow \mathbb{S}^1$ such that $u(x) - P_u \in L^p(\mathbb{R}, \mathbb{R}^2)$ for some point $P_u \in \mathbb{S}^1$, and $|u|_{1/p,p} < \infty$, where

$$|u|_{1/p,p}^p := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|u(x) - u(y)|^p}{|x - y|^2} dx dy < \infty.$$

The class $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ is equipped with the norm $\|u - P_u\|_{L^p} + |u|_{1/p,p}$.

We define the *degree* of a map u in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ through the formula

$$\deg u := \frac{1}{\pi} \int_{\mathbb{R}_+^2} V^\#(dy^1 \wedge dy^2) \quad (2.8)$$

where $\mathbb{R}_+^2 := \{(x, t) \in \mathbb{R}^2 \mid t > 0\}$ denotes the upper half plane, $U \in W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{D}^2)$ is the harmonic extension of u , and $V := \Phi \circ U$. We have:

Proposition 2.6 *The degree of maps in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ is strongly continuous. Moreover, $\deg u \in \mathbb{Z}$ for each $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$.*

PROOF: Let $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$. Arguing as in the proof of Proposition 1.1, we can find a real constant $C_p > 0$ depending on p , such that

$$\int_{\mathbb{R}_+^2} |V^\#(dy^1 \wedge dy^2)| dx dt \leq C_p \int_{\mathbb{R}_+^2} t^{p-2} |DU(x, t)|^p dx dt.$$

Let $\{u_h\} \subset W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ be such that $u_h \rightarrow u$ strongly in $W^{1/p,p}(\mathbb{R}, \mathbb{R}^2)$. For each h , denote $V_h := \Phi \circ U_h$, where $U_h \in W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{D}^2)$ is the harmonic extension of u_h . The strong convergence $u_h \rightarrow u$ in $W^{1/p,p}(\mathbb{R}, \mathbb{R}^2)$ implies the strong convergence $V_h \rightarrow V$ in $W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{R}^2)$. Therefore, by the above estimate, the dominated convergence theorem yields

$$\lim_{h \rightarrow \infty} \int_{\mathbb{R}_+^2} V_h^\#(dy^1 \wedge dy^2) = \int_{\mathbb{R}_+^2} V^\#(dy^1 \wedge dy^2)$$

whence $\deg u_h \rightarrow \deg u$. Since moreover $n = 1$, there exists a sequence $\{u_h\} \subset C^1(\mathbb{R}, \mathbb{S}^1)$ such that $u_h \rightarrow u$ strongly in $W^{1/p,p}(\mathbb{R}, \mathbb{R}^2)$. By means of a cut-off argument, we readily find a smooth map $W_h : \mathbb{R}^2 \rightarrow \mathbb{D}^2$ and a point $P_h \in \mathbb{S}^1$ such that $W_h(x, t) - P_h$ has compact support contained in \mathbb{R}_+^2 , and

$$\int_{\mathbb{R}_+^2} |W_h^\#(dy^1 \wedge dy^2) - V_h^\#(dy^1 \wedge dy^2)| dx dt < \frac{1}{2}, \quad \forall h.$$

It is then readily checked that

$$\int_{\mathbb{R}_+^2} W_h^\#(dy^1 \wedge dy^2) = d_h \cdot \pi$$

for some $d_h \in \mathbb{Z}$. Therefore, we get $\deg u_h = d_h$ for each h , whence $\deg u \in \mathbb{Z}$, as $\deg u_h \rightarrow \deg u$. \square

As in Definition 2.3, with $n = 1$, if $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ we can define the 1-current G_u in $\mathcal{D}_1(\mathbb{R} \times \mathbb{S}^1)$ by setting

$$G_u := (\partial G_V)_\#((\mathbb{R} \times \{0\}) \times \mathbb{R}^2) \quad \text{on} \quad \mathcal{D}^1(\mathbb{R} \times \mathbb{S}^1).$$

Actually, using a cut-off argument on the function V , and arguing essentially as in the proof of Proposition 3.2 below, it can be checked that

$$\deg u = \frac{1}{\pi} G_u(\pi_2^\# \omega_{\mathbb{S}^1}), \quad (2.9)$$

where $\omega_{\mathbb{S}^1}$ is the 1-form defined in (3.1), and $\pi_2 : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the orthogonal projection onto the second factor.

Remark 2.7 Due to the *bubbling phenomenon* (cf. [?, p. 324]), the degree fails to be continuous w.r.t. the weak sequential convergence in $W^{1/p,p}$. It suffices to consider a sequence $\{u_h\} \subset C^1(\mathbb{R}, \mathbb{S}^1)$ with $\sup_h \|u_h\|_{1/p,p} < \infty$, $\deg u_h = 1$ for each h , and such that $u_h \rightarrow P$ a.e., but $G_{u_h} \rightarrow G_P + \delta_0 \times \llbracket \mathbb{S}^1 \rrbracket$ in $\mathcal{D}_1(\mathbb{R} \times \mathbb{S}^1)$, where G_P is the graph current of the constant map equal to some $P \in \mathbb{S}^1$.

3 Singularities and minimal connections

In this section, we describe in any dimension $n \geq 2$ the singular set of maps $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ in terms of homological tools. Namely, by means of the coarea formula, Theorem 1.2, we build up an $(n-2)$ -dimensional *integral flat chain* $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ that in dimension $n = 2$ agrees with the extension $T : W^{1/p,p}(B^2, \mathbb{S}^1) \rightarrow W^{1,\infty}(B^2, \mathbb{R})^*$ of the distributional determinant by Bourgain-Brezis-Mironescu [7].

3.1 Singularities

Let $\omega_{\mathbb{S}^1}$ be the *normalized volume 1-form* in \mathbb{S}^1

$$\omega_{\mathbb{S}^1} := \frac{1}{2\pi}(y^1 dy^2 - y^2 dy^1), \quad (3.1)$$

so that $[\mathbb{S}^1](\omega_{\mathbb{S}^1}) = \int_{\mathbb{S}^1} \omega_{\mathbb{S}^1} = 1$. Moreover, let $\pi_1 : A \times \mathbb{R}^2 \rightarrow A$ and $\pi_2 : A \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote the orthogonal projections onto the two factors, where $A = B^n$ or $A = \mathbf{C}^{n+1}$.

Definition 3.1 *The singularities of a map u in $W^{1/p,p}(B^n, \mathbb{S}^1)$, where $p > 1$, are represented by the $(n-2)$ -dimensional current $\mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$*

$$\mathbb{P}(u)(\phi) := \partial G_u(\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}), \quad \phi \in \mathcal{D}^{n-2}(B^n)$$

where G_u is given by Definition 2.3.

We write explicitly the action of $\mathbb{P}(u)$, recovering in the case $p = 2$ the definition of singularities introduced by Hang-Lin [25]. For this purpose, following [23], we choose a smooth decreasing cut-off function $\eta : [0, 1] \rightarrow [0, 1]$ such that $\eta(t) = 1$ for $t \in [0, 1/4]$ and $\eta(t) = 0$ for $t \in [3/4, 1]$, and for any form $\phi \in \mathcal{D}^k(B^n)$ we denote by $\tilde{\phi}$ the k -form in \mathbf{C}^{n+1} given by $\tilde{\phi} := \phi \wedge \eta$.

Proposition 3.2 *For every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $\phi \in \mathcal{D}^{n-2}(B^n)$, we have*

$$\mathbb{P}(u)(\phi) = \frac{1}{\pi} \int_{\mathbf{C}^{n+1}} d\tilde{\phi} \wedge V^\#(dy^1 \wedge dy^2),$$

where the extension V is chosen as in Definition 2.3.

PROOF: Since $d\pi_2^\# \omega_{\mathbb{S}^1} = \pi_2^\# d\omega_{\mathbb{S}^1} = 0$, as $\omega_{\mathbb{S}^1}$ is a closed 1-form in \mathbb{S}^1 , then

$$\mathbb{P}(u)(\phi) = G_u(d\pi_1^\# \phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}) = G_u(\pi_1^\# d\phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}).$$

Denote by $\widehat{\omega}_{\mathbb{S}^1}$ a 1-form in $\mathcal{D}^1(\mathbb{R}^2)$ that agrees with the right-hand side of (3.1) on \mathbb{D}^2 . By the definition (2.5), using that V satisfies the null-boundary condition (2.3) we have

$$\begin{aligned} \partial G_V(\pi_1^\# d\tilde{\phi} \wedge \pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) &= (-1)^{n-1} G_u(\pi_1^\# (d_x \tilde{\phi} + d_t \tilde{\phi})|_{t=0} \wedge \pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) \\ &= (-1)^{n-1} G_u(\pi_1^\# d\phi \wedge \pi_2^\# \omega_{\mathbb{S}^1}). \end{aligned}$$

We thus obtain:

$$\begin{aligned} \mathbb{P}(u)(\phi) &= (-1)^{n-1} \partial G_V(\pi_1^\# d\tilde{\phi} \wedge \pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) \\ &= G_V(\pi_1^\# d\tilde{\phi} \wedge d\pi_2^\# \widehat{\omega}_{\mathbb{S}^1}) = G_V(\pi_1^\# d\tilde{\phi} \wedge \pi_2^\# d\widehat{\omega}_{\mathbb{S}^1}). \end{aligned} \quad (3.2)$$

Therefore, it suffices to observe that since $V(\mathbf{C}^{n+1}) \subset \mathbb{D}^2$, then

$$V^\# d\widehat{\omega}_{\mathbb{S}^1} = \frac{1}{\pi} V^\#(dy^1 \wedge dy^2)$$

and recall the action (2.2) of the current G_V , on account of Proposition 2.2. \square

3.2 Cartesian maps

By the previous notation, it turns out that a map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ has zero homological singularities, that is, it satisfies $\mathbb{P}(u) = 0$, if and only if the current G_u associated to its graph has no inner boundary, i.e.,

$$\partial G_u = 0 \quad \text{on} \quad \mathcal{D}^{n-1}(B^n \times \mathbb{S}^1). \quad (3.3)$$

For that reason, we give the following

Definition 3.3 *Let $n \geq 1$ and $p > 1$. A map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ is said to be a Cartesian map in $\text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$ if the current G_u satisfies the null-boundary condition (3.3).*

Note that the strong convergence $u_h \rightarrow u$ in $W^{1/p,p}(B^n, \mathbb{R}^2)$ yields the weak convergence $G_{u_h} \rightharpoonup G_u$ in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$. Therefore, in low dimension $n = 1$ we get the equality

$$W^{1/p,p}(B^1, \mathbb{S}^1) = \text{cart}^{1/p,p}(B^1, \mathbb{S}^1)$$

by the strong density of smooth maps, whereas in high dimension, clearly

$$W^{1/p,p}(B^n, \mathbb{S}^1) \subsetneq \text{cart}^{1/p,p}(B^n, \mathbb{S}^1) \quad \forall n \geq 2.$$

When $n \geq 2$, condition $\mathbb{P}(u_h) = 0$ clearly holds true if $u_h : B^n \rightarrow \mathbb{S}^1$ is smooth, say Lipschitz, and the null-boundary condition (3.3) is preserved by the weak convergence $G_{u_h} \rightharpoonup G_u$ in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, which implies the weak convergence $\mathbb{P}(u_h) \rightharpoonup \mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$. Therefore, we readily obtain that

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) \subset \text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$$

where, we recall, $W_S^{1/p,p}(B^n, \mathbb{S}^1)$ denotes the strong closure of smooth maps $u \in C^\infty(B^n, \mathbb{S}^1)$ in the $W^{1/p,p}$ -norm.

Example 3.4 Coming back to Example 0.3, the map $u(x) = x/|x|$ belongs to $W^{s,p}(B^2, \mathbb{S}^1)$ if $1 \leq sp < 2$. Following [?, Sec. 4.2.5], we have:

$$(\partial G_u) \llcorner (B^2 \times \mathbb{S}^1) = -\delta_{\mathbf{0}} \times \llbracket \mathbb{S}^1 \rrbracket$$

where $\mathbf{0}$ is the origin in \mathbb{R}^2 . Therefore, on account of Definition 3.1 we infer that $\mathbb{P}(u) = -\delta_{\mathbf{0}}$.

If $n \geq 3$, the map $u(x) = (x_1, x_2)/|(x_1, x_2)|$ belongs to $W^{s,p}(B^n, \mathbb{S}^1)$ if $1 \leq sp < 2$, and this time

$$(\partial G_u) \llcorner (B^n \times \mathbb{S}^1) = -\llbracket \Delta \rrbracket \times \llbracket \mathbb{S}^1 \rrbracket$$

where the $(n-2)$ -disk $\Delta := \{x \in B^n \mid (x_1, x_2) = (0, 0)\}$ is oriented by $e_3 \wedge \cdots \wedge e_n$, $\{e_i\}_{i=1}^n$ being the canonical basis in \mathbb{R}^n . As a consequence, we get $\mathbb{P}(u) = -\llbracket \Delta \rrbracket$.

Instead, an example of Cartesian map according to Definition 3.3 is given e.g. by the content of [12, Lemma 5]. Taking in fact

$$u(x) := (\cos \psi(x), \sin \psi(x)), \quad \psi : B^n \setminus \{0\} \rightarrow \mathbb{R}, \quad \psi(x) := \frac{1}{|x|^\alpha}$$

it turns out that $u \in W^{s,p}(B^n, \mathbb{S}^1)$ for every $0 < s < 1$ and $p > 1$ with $1 \leq sp < n$, provided that $0 < \alpha < (n-sp)/sp$. Therefore, if $0 < \alpha < n-1$, where $n \geq 2$, we have $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ for every $p > 1$. Now, letting $u_h = (\cos \psi_h, \sin \psi_h)$, where $\psi_h(x) := \max\{\psi(x), h\}$ and $h \in \mathbb{N}^+$, we infer that $\{u_h\} \subset W^{1/p,p}(B^n, \mathbb{S}^1)$ is a sequence of Lipschitz maps strongly converging to u in $W^{1/p,p}(B^n, \mathbb{R}^2)$. Using that $\mathbb{P}(u_h) = 0$ for each h , we infer that $\mathbb{P}(u) = 0$, whence $u \in \text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$ for any $p > 1$.

In Corollary 8.3, we shall prove the equality

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) = \text{cart}^{1/p,p}(B^n, \mathbb{S}^1) \quad \forall n \geq 2, \quad \forall p > 1$$

so that for every map $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$,

$$u \in W_S^{1/p,p}(B^n, \mathbb{S}^1) \iff \mathbb{P}(u) = 0.$$

Moreover, we shall see that when $u \notin W_S^{1/p,p}(B^n, \mathbb{S}^1)$, the energy gap $\mathcal{G}_{1/p,p}(u)$ in the relaxation process, see (0.7), is given by the *minimal integral connection* of the singularity $\mathbb{P}(u)$ times a constant weight E_p only depending on the exponent $p > 1$, with $E_2 = 2\pi$ in the easier case $p = 2$.

3.3 Real and integral mass

Let $0 \leq k \leq n - 2$ be integers. Recall from [?]:

Definition 3.5 For any current $\Gamma \in \mathcal{D}_k(B^n)$, we denote by

$$\begin{aligned} m_{r,B^n}(\Gamma) &:= \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{k+1}(B^n), \quad (\partial D) \llcorner B^n = \Gamma\} \\ m_{i,B^n}(\Gamma) &:= \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{k+1}(B^n), \quad (\partial L) \llcorner B^n = \Gamma\} \end{aligned}$$

the real mass and integral mass of Γ relative to B^n , respectively.

In case $m_{i,B^n}(\Gamma) < \infty$, an i.m. rectifiable current $L \in \mathcal{R}_{k+1}(B^n)$ is an *integral minimal connection for the mass of Γ allowing connections to the boundary of B^n* , if $(\partial L) \llcorner B^n = \Gamma$ and $\mathbf{M}(L) = m_{i,B^n}(\Gamma)$. In general, one has $m_{r,B^n}(\Gamma) \leq m_{i,B^n}(\Gamma)$. However, by Federer's theorem [15], for $k = 0$, or by Hardt-Pitts' result [26], when $k = n - 2$, if $m_{i,B^n}(\Gamma) < \infty$ one has

$$m_{r,B^n}(\Gamma) = m_{i,B^n}(\Gamma).$$

Following [23], we now introduce for every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ the current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(B^n)$ given by

$$\mathbb{D}(u)(\gamma) := G_V(\pi_1^\# \tilde{\gamma} \wedge \pi_2^\# d\widehat{\omega}_{\mathbb{S}^1}), \quad \gamma \in \mathcal{D}^{n-1}(B^n), \quad (3.4)$$

where the extension V is chosen as in Definition 2.3 and the $(n-1)$ -form $\tilde{\gamma}$ in \mathbf{C}^{n+1} is given as above by $\tilde{\gamma} := \gamma \wedge \eta$. Since we have

$$\mathbf{M}(\mathbb{D}(u)) \leq \int_{\mathbf{C}^{n+1}} |V^\#(dy^1 \wedge dy^2)| dx dt,$$

by Proposition 1.1 we infer that $\mathbb{D}(u)$ has finite mass, namely:

$$\mathbf{M}(\mathbb{D}(u)) \leq C \int_{\mathbf{C}^{n+1}} t^{p-2} |DU(x, t)|^p dx dt \quad (3.5)$$

for some real constant $C > 0$ only depending on n and p .

Furthermore, on account of formula (3.2), in Proposition 3.2 we have actually obtained that

$$\mathbb{P}(u) = (\partial \mathbb{D}(u)) \llcorner B^n.$$

Therefore, $\mathbb{P}(u)$ is an $(n-2)$ -dimensional *real flat chain*, and

$$m_{r,B^n}(\mathbb{P}(u)) \leq \mathbf{M}(\mathbb{D}(u)) < \infty.$$

3.4 Minimal integral connection

By means of the coarea formula, Theorem 1.2, we now obtain that if $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, the current $\mathbb{P}(u)$ of the singularities is an *integral flat chain*. In dimension $n = 2$, since u has a finite singular set, the latter property implies that $\mathbb{P}(u)$ is a finite sum of Dirac masses.

Proposition 3.6 Let $u \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, where $n \geq 2$ and $p > 1$. Then there exists $L \in \mathcal{R}_{n-1}(B^n)$ with $\mathbf{M}(L) < \infty$ such that $\mathbb{P}(u) = (\partial L) \llcorner B^n$.

PROOF: Choose a regular value $y \in \mathbb{D}^2$ of the extension $V \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{D}^2)$ such that (1.6) holds, so that $\mathcal{M} := V^{-1}(\{y\})$ is a countably $(n-1)$ -rectifiable set of \mathbf{C}^{n+1} . Consider the current $\tilde{L} \in \mathcal{D}_{n-1}(\mathbf{C}^{n+1})$ given by

$$\tilde{L}(\omega) := \int_{\mathcal{M}} \langle \omega, \xi \rangle d\mathcal{H}^{n-1}, \quad \omega \in \mathcal{D}^{n-1}(\mathbf{C}^{n+1})$$

where $\xi := \eta/|\eta|$ and η is the $(n-1)$ -vector $\eta := *V^\#(dy^1 \wedge dy^2)$, where $*$ is the *Hodge operator* in \mathbb{R}^{n+1} . Therefore, when $n = 2$ the 1-vector field η agrees with the D-field introduced by Brezis-Coron-Lieb [9].

Since ξ is an orienting unit $(n-1)$ -vector field of the approximate tangent $(n-1)$ -space to \mathcal{M} at $\mathcal{H}^{n-1} \llcorner \mathcal{M}$ -a.e. $z \in \mathbf{C}^{n+1}$, it turns out that \tilde{L} is i.m. rectifiable in $\mathcal{R}_{n-1}(\mathbf{C}^{n+1})$, with finite mass bounded by the right-hand side of formula (1.6). In addition, by (2.5) and Definition 3.1 it turns out that

$$(\partial\tilde{L}) \llcorner (B^n \times \{0\}) = \mathbb{P}(u).$$

It then suffices to take L equal to the push forward of \tilde{L} through the orthogonal projection of \mathbf{C}^{n+1} onto $B^n \times \{0\}$. \square

We now recall that any map u in $W^{1/p,p}(B^n, \mathbb{S}^1)$ is the strong limit of a sequence $\{u_h\}$ in the class $\mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, see Theorem 0.2. Arguing as e.g. in [?, Sec. 4.2.5], we in fact prove the following:

Proposition 3.7 *For any $n \geq 2$ and $p > 1$, let $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ and $\{u_h\} \subset \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ be such that $u_h \rightarrow u$ strongly in $W^{1/p,p}(B^n, \mathbb{R}^2)$. Then*

- (i) $\mathbf{M}(\mathbb{D}(u) - \mathbb{D}(u_h)) \rightarrow 0$ as $h \rightarrow \infty$;
- (ii) there exists $L \in \mathcal{R}_{n-1}(B^n)$ such that $\mathbb{P}(u) = (\partial L) \llcorner B^n$;
- (iii) if $L_{u_h, u}$ denotes an i.m. rectifiable current of least mass in $\mathcal{R}_{n-1}(B^n)$ such that

$$(\partial L_{u_h, u}) \llcorner B^n = \mathbb{P}(u) - \mathbb{P}(u_h), \quad (3.6)$$

then $\mathbf{M}(L_{u_h, u}) \rightarrow 0$ as $h \rightarrow \infty$.

PROOF: Since $m_{i, B^n}(\mathbb{P}(u_h)) < \infty$, for every h there exists an i.m. rectifiable current $L_h \in \mathcal{R}_{n-1}(B^n)$ such that $\mathbb{P}(u_h) = (\partial L_h) \llcorner B^n$ and $m_{i, B^n}(\mathbb{P}(u_h)) = \mathbf{M}(L_h)$. By the bound (3.5), since the strong convergence of $u_h \rightarrow u$ is equivalent to the strong convergence of $U_h \rightarrow U$ in $W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{R}^2)$, using the dominated convergence theorem we obtain property (i). Therefore, possibly passing to a (not relabeled) subsequence we may and do assume that

$$m_{r, B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) \leq 2^{-h} \quad \forall h \in \mathbb{N}$$

and again that

$$\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h) = (\partial\tilde{L}_h) \llcorner B^n,$$

where \tilde{L}_h is an integral minimal connection of $\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)$. Therefore, by Hardt-Pitts' result [26]

$$\mathbf{M}(\tilde{L}_h) = m_{i, B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) = m_{r, B^n}(\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) \leq 2^{-h}.$$

Therefore, the current $L := L_0 + \sum_{h=0}^{\infty} \tilde{L}_h$ is i.m. rectifiable in $\mathcal{R}_{n-1}(B^n)$, whereas due to the weak convergence $\mathbb{P}(u_h) \rightarrow \mathbb{P}(u)$ in $\mathcal{D}_{n-2}(B^n)$, we get

$$\begin{aligned} (\partial L) \llcorner B^n &= (\partial L_0) \llcorner B^n + \sum_{h=0}^{\infty} (\partial\tilde{L}_h) \llcorner B^n \\ &= \mathbb{P}(u_0) + \sum_{h=0}^{\infty} (\mathbb{P}(u_{h+1}) - \mathbb{P}(u_h)) = \mathbb{P}(u). \end{aligned}$$

We thus obtain property (ii), whereas property (iii) readily follows. \square

Remark 3.8 If $n = 2$, and replacing B^2 with \mathbb{S}^2 , our definition of singular set $\mathbb{P}(g) \in \mathcal{D}_0(\mathbb{S}^2)$ of maps $g \in W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$ agrees essentially with the extension $T : W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1) \rightarrow W^{1,\infty}(\mathbb{S}^2, \mathbb{R})^*$ of the distributional determinant $\text{Det}(\nabla g)$ by Bourgain-Brezis-Mironescu [7]. Therefore, if e.g. $u \in W^{1/p,p}(B^2, \mathbb{S}^1)$ is constant in a neighborhood of ∂B^2 , arguing as in [7] for the case of maps in $W^{1/p,p}(\mathbb{S}^2, \mathbb{S}^1)$, we obtain the existence of two sequences $(P_i), (N_i) \subset B^2$ such that

$$\mathbb{P}(u) = \sum_{i=1}^{\infty} (\delta_{P_i} - \delta_{N_i}), \quad m_i(\mathbb{P}(u)) = \sum_{i=1}^{\infty} |P_i - N_i| < \infty$$

where the integral mass of the integral flat chain $\mathbb{P}(u) \in \mathcal{D}_0(B^2)$ is given by

$$m_i(\mathbb{P}(u)) := \min\{\mathbf{M}(L) \mid L \in \mathcal{R}_1(B^2), \partial L = \mathbb{P}(u)\}.$$

For our purposes, we finally point out the following:

Remark 3.9 Let $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, so that (2.6) holds. It is readily checked that

$$(2.7) \text{ holds } \iff \mathbb{P}(u_T) = -(\partial L_T) \lrcorner B^n. \quad (3.7)$$

Therefore, on account of property (ii) in Proposition 3.7, it turns out that the class of Cartesian currents with underlying map u_T equal to u

$$\mathcal{T}_u^{1/p,p} := \{T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1) \mid u_T = u \text{ in (2.6)}\} \quad (3.8)$$

is non-empty, for every $u \in W^{1/p,p}(B^n \times \mathbb{S}^1)$ and $p > 1$.

4 Energy concentration

In this section, we discuss a minimum problem that turns out to be strictly related to the energy concentration phenomenon in the relaxation process on the class $W^{1/p,p}(B^n, \mathbb{S}^1)$.

4.1 A minimum problem

For any $p > 1$, denote

$$\mathcal{F}_p := \{u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1) \mid \deg u = 1\}$$

see (2.8) and Proposition 2.6, and define

$$E_p := \inf\{\mathcal{E}_{1/p,p}(u, \mathbb{R}) \mid u \in \mathcal{F}_p\} \quad (4.1)$$

where, similarly as above, for any $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ we let

$$\mathcal{E}_{1/p,p}(u, \mathbb{R}) := \mathcal{E}_{p-2}^p(U, \mathbb{R}_+^2) = \int_{\mathbb{R}_+^2} t^{p-2} |DU(x, t)|^p dx dt < \infty, \quad U := \text{Ext}(u),$$

$\text{Ext}(u)$ being the energy minimizer among all functions $U \in W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{D}^2)$ such that $U(x, 0) = u(x)$ on $\mathbb{R} \times \{0\}$ in the sense of the traces.

Remark 4.1 The energy functional $U \mapsto \mathcal{E}_{p-2}^p(U, \mathbb{R}_+^2)$ is scale-invariant for each $p > 1$. More precisely, if $U_{(r)}(x, t) := U(rx, rt)$, where $U = \text{Ext}(u)$ for some map $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$, then the trace $u_{(r)}$ of $U_{(r)}$ on $t = 0$ belongs to $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$, and $\mathcal{E}_{p-2}^p(U_{(r)}, \mathbb{R}_+^2) = \mathcal{E}_{p-2}^p(U, \mathbb{R}_+^2)$ for each $r > 0$, whence $U_{(r)} = \text{Ext}(u_{(r)})$.

By convexity of the integrand, the functional $u \mapsto \mathcal{E}_{1/p,p}(u, \mathbb{R})$ is sequentially weakly lower semicontinuous in \mathcal{F}_p . However, the class \mathcal{F}_p is not closed w.r.t. the weak convergence, see Remark 2.7, so that (a part from the case $p = 2$, see Appendix A) we expect that the minimum in (4.1) fails to be attained. Notwithstanding, we have:

Proposition 4.2 *For every $p > 1$, the energy infimum (4.1) is a real positive constant $E_p > 0$.*

PROOF: Arguing as in the proof of Proposition 1.1 in the case $n = 1$, but this time with \mathbb{R}_+^2 instead of \mathbb{C}^2 , for every $p > 1$ and $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ we get the estimate

$$\int_{\mathbb{R}_+^2} |V^\#(dy^1 \wedge dy^2)| dx dt \leq C_p \int_{\mathbb{R}_+^2} t^{p-2} |DU(x, t)|^p dx dt$$

for some real constant $C_p > 0$ only depending on p . On account of definition (2.8), this yields that

$$\pi \leq \left| \int_{\mathbb{R}_+^2} V^\#(dy^1 \wedge dy^2) \right| \leq C_p \cdot \mathcal{E}_{1/p,p}(u, \mathbb{R})$$

for each $u \in \mathcal{F}_p$, whence $E_p \geq \pi/C_p > 0$, as required. \square

When $p = 2$, in Example A.1 from the first appendix we compute the weight $E_2 = 2\pi$. Therefore, extending the result obtained in [23] in the case $p = 2$, the explicit formula for the relaxed energy (0.6) is expected to be:

$$\tilde{\mathcal{E}}_{1/p,p}(u) = \mathcal{E}_{1/p,p}(u) + E_p \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty$$

for every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, $n \geq 2$, and $p > 1$. This property will be proved in Theorem 8.2 below.

4.2 Energy minimum with fixed degree

In the sequel, for each map $u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ and each open set $A \subset \mathbb{R}$, we define the energy of u on A by means of the restriction $u|_A$, i.e.,

$$\begin{aligned} \mathcal{E}_{1/p,p}(u, A) &:= \mathcal{E}_{p-2}^p(U, A \times (0, 1)) \\ &= \int_{A \times (0, 1)} t^{p-2} |DU(x, t)|^p dx dt, \quad U = \text{Ext}(u|_A), \end{aligned} \quad (4.2)$$

where $\text{Ext}(u|_A)$ is the energy minimizer among all maps $U \in W_{p-2}^{1,p}(A \times (0, 1), \mathbb{R}^2) \cap L^\infty(A \times (0, 1), \mathbb{R}^2)$ such that $U(x, 0) = u(x)$ on $A \times \{0\}$. Note that in general $\mathcal{E}_{1/p,p}(u, A) \leq \mathcal{E}_{p-2}^p(\text{Ext}(u), A \times (0, 1))$, where $\text{Ext}(u)$ is the energy minimizer in $W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{D}^2)$, so that $\mathcal{E}_{1/p,p}(u, \mathbb{R}) := \mathcal{E}_{p-2}^p(\text{Ext}(u), \mathbb{R}_+^2)$.

Recalling the definition (2.8), Proposition 2.6, and (4.1), we now prove:

Proposition 4.3 *For any $p > 1$ and $d \in \mathbb{Z}$, we have*

$$\inf\{\mathcal{E}_{1/p,p}(u, \mathbb{R}) \mid u \in \mathcal{F}_p(d)\} = |d| \cdot E_p \quad (4.3)$$

where $\mathcal{F}_p(d) := \{u \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1) \mid \deg u = d\}$.

PROOF: We prove in two steps the inequalities “ \leq ” and “ \geq ” in (4.3), where the case $d \in \{0, 1, -1\}$ trivially follows, whence it clearly suffices to consider the case $d \in \mathbb{N}^+$, with $d \geq 2$.

Step 1: the inequality “ \leq ”. Let $P_S := (0, -1)$, the “south pole” in \mathbb{S}^1 . We make use of the following

Lemma 4.4 *For each $\varepsilon > 0$, we can find a map $u_\varepsilon \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ such that $u_\varepsilon(x) = P_S$ if $|x| \geq 1$, $\deg u_\varepsilon = 1$, and $\mathcal{E}_{1/p,p}(u_\varepsilon, \mathbb{R}) \leq E_p + \varepsilon$.*

PROOF: Let $u \in \mathcal{F}_p = \mathcal{F}_p(1)$ be such that $\mathcal{E}_{1/p,p}(u, \mathbb{R}) < E_p + \varepsilon/2$. Since $u - P_u \in L^p(\mathbb{R}, \mathbb{R}^2)$ for some $P_u \in \mathbb{S}^1$, by left composition with a rotation in the target space we can choose $P_u = P_S$. Let now $U = \text{Ext}(u)$, and for each $h \in \mathbb{N}^+$ define

$$U_h(x, t) = \phi(|(x, t)| - h) U(x, t) + (1 - \phi(|(x, t)| - h)) \cdot P_S$$

for some smooth decreasing cut-off function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ with $\phi(\rho) = 1$ for $\rho \leq 0$ and $\phi(\rho) = 0$ for $\rho \geq 1$. We have that $U_h \rightarrow U$ strongly in $W_{p-2}^{1,p}(\mathbb{R}_+^2, \mathbb{R}^2)$, whence for h sufficiently large we get

$$|\mathcal{E}_{p-2}^p(U_h, \mathbb{R}_+^2) - \mathcal{E}_{p-2}^p(U, \mathbb{R}_+^2)| < \varepsilon/2.$$

By scale invariance of the energy, Remark 4.1, it suffices to define $u_\varepsilon(x) = u_h(x/(h+1))$ for h large. \square

Now, by gluing together d “copies” of the map u_ε from Lemma 4.4, i.e., by letting $u_{\varepsilon,d}(x) = u_\varepsilon(x+2i)$ if $x \in (2i, 2i+1)$ for $i = 0, \dots, d-1$, and $u_{\varepsilon,d}(x) = P_S$ elsewhere in \mathbb{R} , we find a map $u \in \mathcal{F}_p(d)$ satisfying $\mathcal{E}_{1/p,p}(u_{\varepsilon,d}, \mathbb{R}) \leq d \cdot E_p + d \cdot \varepsilon$. Therefore, inequality “ \leq ” in (4.3) follows by letting $\varepsilon \rightarrow 0$.

Step 2: the inequality “ \geq ”. Assume by contradiction that there exists $u \in \mathcal{F}_p(d)$ such that

$$\mathcal{E}_{1/p,p}(u, \mathbb{R}) = (E_p - \eta) d \quad (4.4)$$

for some constant $\eta > 0$. By a density argument, a truncation procedure as in the previous lemma, by scale invariance of the energy, and by using a left composition with a rotation, we may and do assume that $u \in C^\infty(\mathbb{R}, \mathbb{S}^1)$ with $u(x) \equiv P_S$ for $|x| \geq 1$. Let $C = u^{-1}(\{P_S\})$ and $A = \mathbb{R} \setminus C$, so that A is a bounded open subset in B^1 .

Let $\{A_j\}$ denote the (at most countable) family of connected components (open intervals) of A . For each j , we let u_j denote the function equal to u on A_j and to P_S on $\mathbb{R} \setminus A_j$. We have $u_j \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$, whence the degree $d_j \in \mathbb{Z}$ of u_j is well-defined, and by (2.9) we readily infer that $\sum_j d_j = d$.

We now see that $d_j \in \{0, -1, +1\}$ for each j . Since in fact $u_j \equiv P_S$ in $\mathbb{R} \setminus A_j$ and $u_j(x) \neq P_S$ for each $x \in A_j$, it cannot happen that $|d_j| \geq 2$, otherwise for topological reasons the continuous map u_j should cover the whole circle \mathbb{S}^1 at least once outside A_j , a contradiction.

We thus may assume (after a relabeling) that $d_j = 1$ and hence $u_j \in \mathcal{F}_p$ for each for $j = 1, \dots, d$. Using that $\mathcal{E}_{1/p,p}(u, \mathbb{R}) \geq \sum_{j=1}^d \mathcal{E}_{1/p,p}(u_j, \mathbb{R})$, by (4.4) we find for some $j = 1, \dots, d$

$$\mathcal{E}_{1/p,p}(u_j, \mathbb{R}) \leq \frac{1}{d} \mathcal{E}_{1/p,p}(u, \mathbb{R}) = E_p - \eta, \quad \eta > 0$$

so that by (4.1) we get contradiction in the formula (4.4), as required. \square

5 A lower semicontinuous energy on currents

In this section, we introduce a functional $T \mapsto E_{1/p,p}(T)$ on Cartesian currents that agrees with the energy $\mathcal{E}_{1/p,p}(u)$ if $T = G_u$ for some map $u \in \text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$, see (5.1). Our functional turns out to be lower semicontinuous along weakly converging sequences of smooth graphs, Theorem 5.1.

For any $p > 1$, if $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ as in (2.6), we define

$$E_{1/p,p}(T) := \mathcal{E}_{1/p,p}(u_T) + E_p \cdot \mathbf{M}(L_T) \quad \text{if } T = G_{u_T} + L_T \times \llbracket \mathbb{S}^1 \rrbracket \quad (5.1)$$

where $E_p > 0$ is the real constant given by (4.1). The localized energy on open sets $A \subset B^n$ is

$$E_{1/p,p}(T, A) := \mathcal{E}_{1/p,p}(u, A) + E_p \cdot \mathbf{M}(L \llcorner A), \quad u = u_T, \quad L = L_T$$

where $\mathcal{E}_{1/p,p}(u, A)$ is defined again by (4.2), the function $\text{Ext}(u|_A)$ being the energy minimizer among all maps $U \in W_{p-2}^{1,p}(A \times (0, 1), \mathbb{R}^2) \cap L^\infty(A \times (0, 1), \mathbb{R}^2)$ such that $U(x, 0) = u(x)$ on $A \times \{0\}$. We recall that in general $\mathcal{E}_{p-2}^p(\text{Ext}(u), A \times (0, 1)) \geq \mathcal{E}_{1/p,p}(u, A)$, and that if A_1, A_2 are pairwise disjoint open sets in B^n

$$\mathcal{E}_{1/p,p}(u, A_1 \cup A_2) \geq \mathcal{E}_{1/p,p}(u, A_1) + \mathcal{E}_{1/p,p}(u, A_2), \quad A_1 \cap A_2 = \emptyset. \quad (5.2)$$

The following lower semicontinuity property holds true for any $p > 1$ and in any dimension n .

Theorem 5.1 *Let $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. Let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ be such that $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$, $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, and $u_h \rightarrow u_T$ in $L^p(B^n, \mathbb{R}^2)$. Then we have:*

$$E_{1/p,p}(T) \leq \liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h). \quad (5.3)$$

PROOF: We divide the proof in three steps. In Step 1, we first consider the case of low dimension $n = 1$. In Step 2, we deal with the case $n = 2$, using a dimension reduction argument and Step 1. In Step 3, we treat the case of high dimension $n \geq 3$ by induction.

Step 1: the case $n = 1$. If $T \in \text{cart}^{1/p,p}(B^1 \times \mathbb{S}^1)$, in formula (2.6) we have $L = L_T \in \mathcal{R}_0(B^1)$, hence L is a finite sum of Dirac masses with integer weight, namely

$$L = \sum_{i=1}^m d_i \delta_{x_i}, \quad \mathbf{M}(L) = \sum_{i=1}^m |d_i| \quad (5.4)$$

where the x_i 's are distinct points in B^1 and $d_i \in \mathbb{Z} \setminus \{0\}$ for each i . Choose a family of open intervals $\{A_i \mid i = 1, \dots, m\}$, each A_i centered at the point x_i , with pairwise disjoint closures $\overline{A_i}$ which are also well contained in B^1 . Due to the continuous embedding of $W^{1/p,p}(B^1, \mathbb{S}^1)$ in the class VMO, see e.g. [7], for each $\varepsilon > 0$ we can select each A_i small enough in such a way that:

i) the map u_T has small oscillation on each A_i and small energy, namely:

$$\mathcal{E}_{1/p,p}(u_T, A_i) < \varepsilon/m;$$

ii) for $i = 1, \dots, m$, we can find a map $v_i \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ such that

$$\|v_i\|_{W^{1/p,p}(\mathbb{R} \setminus \overline{A_i}, \mathbb{R}^2)} + \|u_T - v_i\|_{W^{1/p,p}(A_i, \mathbb{R}^2)} < \frac{\varepsilon}{m};$$

iii) setting $\Omega = B^1 \setminus \bigcup\{\overline{A_i} \mid i = 1, \dots, m\}$, we have

$$\mathcal{E}_{1/p,p}(u_T) \leq \mathcal{E}_{1/p,p}(u_T, \Omega) + \sum_{i=1}^m \mathcal{E}_{1/p,p}(u_T, A_i) + \varepsilon;$$

iv) on account of Proposition 4.3, if $\varepsilon > 0$ is small enough, then $\deg v_i = 0$ for each i .

By lower semicontinuity of the energy $u \mapsto \mathcal{E}_{1/p,p}(u, \Omega)$, we thus get

$$\begin{aligned} \liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h, \Omega) &\geq \mathcal{E}_{1/p,p}(u_T, \Omega) \\ &\geq \mathcal{E}_{1/p,p}(u_T) - \sum_{i=1}^m \mathcal{E}_{1/p,p}(u, A_i) - \varepsilon \geq \mathcal{E}_{1/p,p}(u_T) - 2\varepsilon. \end{aligned} \quad (5.5)$$

Now, for each i we have that $G_{u_h} \llcorner (A_i \times \mathbb{S}^1) \rightarrow G_{u_T} \llcorner (A_i \times \mathbb{S}^1) + d_i \delta_{x_i}$ in $\mathcal{D}_1(A_i \times \mathbb{S}^1)$. Moreover, using that $\{u_h\}$ has equibounded energy, for h large enough we can find a smooth function $v_h(i) \in W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ such that

$$\|u_h - v_h(i)\|_{W^{1/p,p}(A_i, \mathbb{R}^2)} \leq a_h, \quad \|v_i - v_h(i)\|_{W^{1/p,p}(\mathbb{R} \setminus \overline{A_i}, \mathbb{R}^2)} \leq a_h$$

where $a_h \rightarrow 0$. This yields that $G_{v_h(i)} \rightarrow G_{v_i} + d_i \delta_{x_i}$ in $\mathcal{D}_1(\mathbb{R} \times \mathbb{S}^1)$.

On account of (2.9), and using that $\deg v_i = 0$, we thus infer that $\deg(v_h(i)) \rightarrow d_i$ as $h \rightarrow \infty$, whence Proposition 4.3 yields that for h large $\mathcal{E}_{1/p,p}(v_h(i), \mathbb{R}) \geq E_p \cdot |d_i|$, and therefore:

$$\mathcal{E}_{1/p,p}(u_h, A_i) \geq E_p \cdot |d_i| - \varepsilon/m \quad \forall i = 1, \dots, m. \quad (5.6)$$

By the estimates (5.5) and (5.6), using that by (5.2)

$$\mathcal{E}_{1/p,p}(u_h) \geq \mathcal{E}_{1/p,p}(u_h, \Omega) + \sum_{i=1}^m \mathcal{E}_{1/p,p}(u_h, A_i) \quad \forall h$$

we thus get

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h) \geq \mathcal{E}_{1/p,p}(u_T) + E_p \cdot \sum_{i=1}^m |d_i| - 3\varepsilon$$

and hence, letting $\varepsilon \rightarrow 0$, the lower semicontinuity inequality (5.3) follows on account of (5.1) and (5.4). Finally, the inequality

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h, A) \geq E_{1/p,p}(T, A)$$

is similarly obtained for any open subset $A \subset B^1$.

Step 2: the case $n = 2$. Denote for simplicity $u_T := u_\infty$, $T_h := G_{u_h}$, $T_\infty := G_{u_\infty} + L_T \times \mathbb{S}^1$, and let $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Choose a direction $\nu \in S_+^1 := \{x \in \mathbb{R}^2 : |x| = 1, x_1 \geq 0\}$, denote by π_ν the 1-D space spanned by ν , and fix an orienting unit vector $\tau(\nu)$ of the 1-D subspace of \mathbb{R}^2 orthogonal to π_ν . For any non-empty and open subset A of B^2 , denote by A_ν the orthogonal projection of A onto π_ν , and for any $y \in A_\nu$

$$A_y^\nu := \{z \in \mathbb{R} \mid y\nu + z\tau(\nu) \in A\}$$

the (non-empty) section of A corresponding to y . Accordingly, for any function $u_h : A \rightarrow \mathbb{S}^1$ and any $y \in A_\nu$, the sliced function $(u_h)_y^\nu : A_y^\nu \rightarrow \mathbb{S}^1$ is defined by

$$(u_h)_y^\nu(z) := u_h(y\nu + z\tau(\nu)), \quad h \in \overline{\mathbb{N}}.$$

Taking $\Omega := B^2$, for any $h \in \overline{\mathbb{N}}$ the 1-dimensional slice (cf. [?, Sec. 2.2.5])

$$(T_h)_y^\nu := T_h \llcorner (\Omega_y^\nu \times \mathbb{S}^1)$$

defines a Cartesian current in $\text{cart}^{1/p,p}(\Omega_y^\nu \times \mathbb{S}^1)$ for \mathcal{H}^1 -a.e. $y \in \Omega_\nu$, and

$$(T_h)_y^\nu = G_{(u_h)_y^\nu}, \quad (T_\infty)_y^\nu = G_{(u_\infty)_y^\nu} + (L_T \llcorner \Omega_y^\nu) \times \llbracket \mathbb{S}^1 \rrbracket.$$

Moreover, by the definition (5.1), the energy of the sliced current $(T_h)_y^\nu$ is given for \mathcal{L}^1 -a.e. $y \in A_\nu$ by

$$\begin{aligned} E_{1/p,p}((T_h)_y^\nu, A_y^\nu) &:= \mathcal{E}_{1/p,p}((u_h)_y^\nu, A_y^\nu) \quad \forall h \in \overline{\mathbb{N}} \\ E_{1/p,p}((T_\infty)_y^\nu, A_y^\nu) &:= \mathcal{E}_{1/p,p}((u_\infty)_y^\nu, A_y^\nu) + E_p \cdot \mathbf{M}(L_T \llcorner A_y^\nu). \end{aligned}$$

Therefore, setting

$$E_{1/p,p}(T_h, A; \nu) := \int_{\pi_\nu} E_{1/p,p}((T_h)_y^\nu, A_y^\nu) dy, \quad h \in \overline{\mathbb{N}} \quad (5.7)$$

by the inequalities

$$\begin{aligned} \mathcal{E}_{1/p,p}(u_h, A) &\geq \int_{A_\nu} \mathcal{E}_{1/p,p}((u_h)_y^\nu, A_y^\nu) dy \quad \forall h \in \overline{\mathbb{N}}, \\ \mathbf{M}(L_T \llcorner A) &\geq \int_{A_\nu} \mathbf{M}(L_T \llcorner A_y^\nu) dy, \end{aligned}$$

we infer that

$$E_{1/p,p}(T_h, A) \geq E_{1/p,p}(T_h, A; \nu) \quad \forall h \in \overline{\mathbb{N}}. \quad (5.8)$$

Moreover, using that

$$\lim_{h \rightarrow \infty} \int_{\pi_\nu} \int_{A_y^\nu} |(u_h)_y^\nu - (u_\infty)_y^\nu|^p dz dy = \lim_{h \rightarrow \infty} \int_A |u_h - u_\infty|^p dx = 0$$

we can find a strictly increasing sequence $\{h(k)\} \subset \overline{\mathbb{N}}$ such that

$$\liminf_{h \rightarrow \infty} E_{1/p,p}(T_h, A; \nu) = \lim_{k \rightarrow \infty} E_{1/p,p}(T_{h(k)}, A; \nu)$$

and the sliced currents $(T_{h(k)})_y^\nu$ converge to $(T_\infty)_y^\nu$ weakly in $\mathcal{D}_1(A_y^\nu \times \mathbb{S}^1)$ as $k \rightarrow \infty$, for \mathcal{H}^1 -a.e. $y \in \pi_\nu$. By Step 1, we thus have for any such y

$$\liminf_{k \rightarrow \infty} E_{1/p,p}((T_{h(k)})_y^\nu, A_y^\nu) \geq E_{1/p,p}((T_\infty)_y^\nu, A_y^\nu). \quad (5.9)$$

Integrating both sides of (5.9) on π_ν , using Fatou's lemma, (5.7), and (5.8) we thus get for any $\nu \in S_+^1$

$$\liminf_{h \rightarrow \infty} E_{1/p,p}(T_h, A) \geq \liminf_{k \rightarrow \infty} E_{1/p,p}(T_{h(k)}, A; \nu) \geq E_{1/p,p}(T_\infty, A; \nu). \quad (5.10)$$

Consider now the Radon measure

$$\lambda := \mathcal{L}^3 \llcorner \mathbf{C}^3 + \theta \mathcal{H}^1 \llcorner (\text{set } L_T \times \{0\})$$

where θ is the density function of the i.m. rectifiable current $L_T \in \mathcal{R}_1(B^2)$ corresponding to the weak limit current T , and set L_T is the 1-rectifiable subset of B^2 given by the points with positive density θ , see (2.1).

Let $\{\nu^{(i)}\}_i \subset S_+^1$ be a countable dense sequence. Setting for $(x, t) \in B^n \times J$, where $J := [0, 1]$,

$$\varphi_i(x, t) := \begin{cases} t^{p-2} |D \operatorname{Ext}(u_\infty)_{y}^{\nu^{(i)}}(x, t)|^p & \text{if } (x, t) \in \mathbf{C}^3 \setminus (\operatorname{set} L_T \times \{0\}), \\ & x = y \nu^{(i)} + z \tau(\nu^{(i)}) \\ E_p & \text{if } x \in \operatorname{set} L_T, \quad t = 0 \end{cases}$$

we obtain for each i and each open set $A \subset B^2$:

$$E_{1/p,p}(T_\infty, A; \nu^{(i)}) = \int_{A \times J} \varphi_i d\lambda. \quad (5.11)$$

By (5.2) and by the superadditivity of the \liminf operator, using (5.10) and (5.11) we thus get

$$\begin{aligned} \liminf_{h \rightarrow \infty} E_{1/p,p}(T_h) &\geq \sum_i \liminf_{h \rightarrow \infty} E_{1/p,p}(T_h, A_i) \\ &\geq \sum_i E_{1/p,p}(T_\infty, A; \nu^{(i)}) = \sum_i \int_{A_i \times J} \varphi_i d\lambda \end{aligned} \quad (5.12)$$

for any finite family of pairwise disjoint open sets $A_i \subset B^2$. We now recall that by [2, Lemma 2.35]

$$\int_{B^2 \times J} \sup_{i \in \mathbb{N}} \varphi_i d\lambda = \sup \left\{ \sum_{i \in I} \int_{A_i \times J} \varphi_i d\lambda \right\}$$

where the supremum is taken over all finite sets of indices $I \subset \mathbb{N}$ and all families $\{A_i\}_{i \in I}$ of pairwise disjoint open sets with compact closure in B^2 . By (5.12), we then conclude that

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h) = \liminf_{h \rightarrow \infty} E_{1/p,p}(T_h) \geq \int_{B^2 \times J} \sup_{i \in \mathbb{N}} \varphi_i d\lambda = E_{1/p,p}(T_\infty).$$

Finally, the inequality

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h, A) \geq E_{1/p,p}(T_\infty, A)$$

is similarly obtained for any open subset $A \subset B^2$, as required.

Step 3: the case $n \geq 3$. Arguing as e.g. in the proof of [2, Thm. 5.4], we apply a reduction argument to the case $n - 1$ in a similar way to Step 2, and an induction argument on the dimension n . We omit any further detail. \square

6 Approximate dipoles

In this section, we provide in low dimension $n = 2$ the approximation of dipoles for $W^{1/p,p}$ -maps with values in \mathbb{S}^1 , see [9, 19], and [?, Sec. 4.2.3]. Using an argument similar to [23], we first show how to remove *homologically trivial* point singularities in dimension $n = 2$.

6.1 Removing point singularities

If $p > 1$, $0 < s < 1$, and $1 \leq sp < 2$, by the continuous embedding $W^{s,p}(B^n, \mathbb{S}^1) \subset W^{1/p,p}(B^n, \mathbb{S}^1)$, it turns out that Definitions 2.3 and 3.1 concerning the graph current G_u and the singularity $\mathbb{P}(u)$ extend to maps $u \in W^{s,p}(B^n, \mathbb{S}^1)$. We have:

Proposition 6.1 *Let $p > 1$, $0 < s < 1$, and $1 \leq sp < 2$. Let $u \in \mathcal{R}_{s,p}(B^2, \mathbb{S}^1)$ be such that $\mathbb{P}(u) = 0$, so that (3.3) holds, with $n = 2$. Then there exists a sequence of smooth maps $\{u_h\} \subset C^\infty(B^2, \mathbb{S}^1)$ which converges to u strongly in $W^{s,p}$.*

PROOF: Since we use a local argument, we may assume that u has only one singularity at the origin, i.e., $u \in C^\infty(B^2 \setminus \{\mathbf{0}\}, \mathbb{S}^1)$. For $0 < r < 1$, we denote

$$\begin{aligned} Q_r &:= B_r^3 \cap \mathbf{C}^3, & F_r &:= Q_r \cap (B^2 \times \{\mathbf{0}\}), \\ \partial^+ Q_r &:= \partial B_r^3 \cap \{z = (x, t) \in \mathbf{C}^3 \mid t > 0\}. \end{aligned}$$

Let $U = \text{Ext}(u) \in W_{\gamma(s,p)}^{1,p}(\mathbf{C}^3, \mathbb{R}^2)$, where $\gamma(s,p)$ is given by (0.3). According to (0.4), for any Borel set $\Omega \subset \mathbf{C}^3$ we let

$$\mathcal{E}_{\gamma(s,p)}^p(U, \Omega) := \int_{\Omega} t^{\gamma(s,p)} |DU(x, t)|^p dx dt, \quad \gamma = \gamma(s, p) := p(1-s) - 1.$$

Given $\varepsilon > 0$, let $0 < R = R(\varepsilon) \ll 1$ be such that $\mathcal{E}_{\gamma(s,p)}^p(U, Q_R) \leq \varepsilon$. Since

$$\mathcal{E}_{\gamma(s,p)}^p(U, Q_R \setminus Q_{R/2}) = \int_{R/2}^R dr \int_{\partial^+ Q_r} t^{\gamma(s,p)} |DU|^p d\mathcal{H}^2,$$

there exists $r = r_\varepsilon \in [R/2, R]$ such that

$$\mathcal{E}_{\gamma(s,p)}^p(U, \partial^+ Q_r) := \int_{\partial^+ Q_r} t^{\gamma(s,p)} |DU|^p d\mathcal{H}^2 \leq \frac{4}{R} \mathcal{E}_{\gamma(s,p)}^p(U, Q_R \setminus Q_{R/2}) \leq \frac{4\varepsilon}{R}. \quad (6.1)$$

In order to remove the singularity of u , we have to show that

$$\{w \in W^{s,p}(B_r^2, \mathbb{R}^2) \cap C^0(\bar{B}_r^2, \mathbb{S}^1) \mid w|_{\partial B_r^2} = u|_{\partial B_r^2}\} \neq \emptyset, \quad (6.2)$$

i.e., that $u|_{\partial B_r^2}$ is homotopic to a constant map in \mathbb{S}^1 . Since the first homotopy group $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$ is commutative, it suffices to show that $d u|_{\partial B_r^2} \# \omega_{\mathbb{S}^1} = 0$. This property holds true since by condition (3.3) we get:

$$\begin{aligned} \int_{\partial B_r^2} u|_{\partial B_r^2} \# \omega_{\mathbb{S}^1} &= G_{u|_{\partial B_r^2}}(\pi_2 \# \omega_{\mathbb{S}^1}) = \partial G_{u|_{B_r^2}}(\pi_2 \# \omega_{\mathbb{S}^1}) \\ &= G_{u|_{B_r^2}}(d\pi_2 \# \omega_{\mathbb{S}^1}) = G_{u|_{B_r^2}}(\pi_2 \# d\omega_{\mathbb{S}^1}) = 0. \end{aligned} \quad (6.3)$$

As a consequence, there exists a smooth extension $u_r : B_r^2 \rightarrow \mathbb{S}^1$ of $u|_{\partial B_r^2}$ with finite $W^{s,p}$ -norm.

Let $V_r : Q_r \rightarrow \mathbb{D}^2$ be a minimizer of the energy $V \mapsto \mathcal{E}_{\gamma(s,p)}^p(V, Q_r)$ among the maps in $W_{\gamma(s,p)}^{1,p}(Q_r, \mathbb{R}^2)$ satisfying the boundary conditions

$$\begin{cases} V = U & \text{on } \partial^+ Q_r \\ V = u_r & \text{on } F_r. \end{cases}$$

Let $0 < \delta < r$ to be fixed later. Define $U_r : \mathbf{C}^3 \rightarrow \mathbb{D}^2$ by

$$U_r(z) := \begin{cases} V_r\left(\frac{r}{\delta}z\right) & \text{if } |z| \leq \delta \\ U\left(r\frac{z}{|z|}\right) & \text{if } \delta \leq |z| \leq r \\ U(z) & \text{if } |z| \geq r \end{cases}$$

so that $U_r \in W_{\gamma(s,p)}^{1,p}(\mathbf{C}^3, \mathbb{R}^2)$ is continuous and with trace $u_r(x) := U_r(x, 0)$ in $W^{s,p}(B^2, \mathbb{S}^1)$. We have

$$\mathcal{E}_{\gamma(s,p)}^p(U_r, Q_\delta) = \left(\frac{\delta}{r}\right)^{2-sp} \mathcal{E}_{\gamma(s,p)}^p(V_r, Q_r).$$

Moreover, as in the case $s = 1/2$ and $p = 2$, we can estimate

$$\mathcal{E}_{\gamma(s,p)}^p(U_r, Q_r \setminus Q_\delta) \leq cr \mathcal{E}_{\gamma(s,p)}^p(U, \partial^+ Q_r) \quad (6.4)$$

for some absolute constant $c > 0$. Therefore, by (6.1), using that $r < R$, and taking $\delta = \delta(\varepsilon)$ sufficiently small, since $2 - sp > 0$ we get:

$$\mathcal{E}_{\gamma(s,p)}^p(U_r) \leq \mathcal{E}_{\gamma(s,p)}^p(U) + 4c\varepsilon + \left(\frac{\delta}{r}\right)^{2-sp} \mathcal{E}_{\gamma(s,p)}^p(V_r, Q_r) \leq \mathcal{E}_{\gamma(s,p)}^p(U) + (4c + 1)\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we infer that $U_{r_\varepsilon} \rightarrow U$ in $W_{\gamma(s,p)}^{1,p}(\mathcal{C}^3, \mathbb{R}^2)$ and finally that $u_{r_\varepsilon} \rightarrow u$ in $W^{s,p}(B^2, \mathbb{R}^2)$, with $u_{r_\varepsilon} \in W^{s,p}(B^2, \mathbb{S}^1)$ continuous. By a standard argument as e.g. in [29], we finally approximate u_{r_ε} by smooth functions in $C^\infty(B^2, \mathbb{S}^1)$, as required. \square

6.2 The dipole construction

We now restrict to the case $sp = 1$, where we adapt some results from [21], to which we refer for further details. To fix the notation, let $a_+, a_- \in B^2$ and let $L \in \mathcal{R}_1(B^2)$ be the 1-current integration over the segment joining a_- to a_+ , with mass $\mathbf{M}(L) = l := |a_+ - a_-| \in (0, 1)$, oriented so that the boundary $\partial L = \delta_{a_+} - \delta_{a_-}$. We assume for simplicity

$$a_+ := (l, 0), \quad a_- := (0, 0).$$

Also, if $P \in \mathbb{S}^1$, we let G_P denote the current carried by the graph of the map equal to P on B^2 .

In the sequel we also denote $D^2 := \{(x, t) \in \mathbb{R}^2 \mid x^2 + t^2 < 1\}$ and

$$\begin{aligned} B^+ &:= \bar{D}^2 \cap \mathcal{C}^2, & \partial^+ B &:= \partial D^2 \cap \{(x, t) \in \mathcal{C}^2 \mid t > 0\}, \\ J &:= \partial B^+ \setminus \partial^+ B = [-1, 1] \times \{0\}. \end{aligned}$$

We first notice that by Lemma 4.4, taking a left composition with a rotation, we readily obtain:

Proposition 6.2 *For every $P \in \mathbb{S}^1$, there exists a family $\{f_\varepsilon^P\}_{\varepsilon > 0}$ of Lipschitz functions $f_\varepsilon^P : B^+ \rightarrow \mathbb{D}^2$ such that $f_{\varepsilon|\partial^+ B}^P \equiv P$, $f_\varepsilon^P(J) \subset \mathbb{S}^1$, $f_{\varepsilon\#}^P[B^+] = [\mathbb{D}^2]$, $f_{\varepsilon\#}^P[J] = [\mathbb{S}^1]$, and*

$$\mathcal{E}_{1/p,p}(f_\varepsilon^P, B^+) \leq E_p + \frac{\varepsilon}{2}.$$

In Lemma 6.3, we extend a result proved in [21] when $p = 2$. For $0 < \delta < 1$ and $0 < m \leq 1$, let

$$\varphi_\delta^m(y) := \min\{my, m(l - y), \delta\}, \quad 0 \leq y \leq l.$$

Consider the map $\phi_\delta^m : (0, l) \times B^+ \rightarrow \mathbf{C}^3$

$$\phi_\delta^m(z) := (x_1, \varphi_\delta^m(x_1)x_2, \varphi_\delta^m(x_1)t), \quad z = (x_1, x_2, t)$$

and define

$$\begin{aligned} \Omega_\delta^m &:= \phi_\delta^m((0, l) \times B^+) \\ &= \{(x_1, x_2, t) \in \mathbf{C}^3 \mid (x_2)^2 + t^2 < \varphi_\delta^m(x_1)^2, \quad 0 < x_1 < l\}. \end{aligned}$$

Lemma 6.3 *Let $V : (0, l) \times B^+ \rightarrow \mathbb{R}^2$ be a $W_{p-2}^{1,p}$ function and let*

$$V_\delta^m(z) := V \circ (\phi_\delta^m)^{-1}(z), \quad z \in \Omega_\delta^m.$$

Then there exists an absolute constant $c > 0$ such that

$$\begin{aligned} \int_{\Omega_\delta^m} t^{p-2} |DV_\delta^m|^p dz &\leq \int_{(0,l) \times B^+} t^{p-2} |D_{(x_2,t)} V|^p dz + c\delta^p \int_{(0,l) \times B^+} t^{p-2} |D_{x_1} V|^2 dz \\ &+ cm^p \int_{((0,\delta/m) \cup (l-\delta/m,l)) \times B^+} t^{p-2} |D_{(x_2,t)} V|^p dz. \end{aligned} \tag{6.5}$$

PROOF: It suffices to note that with $s = \varphi_\delta^m(x_1)t$, then

$$s^{p-2} |\det D\phi_\delta^m(\phi_\delta^m(x, t))| = t^{p-2} \varphi_\delta^m(x_1)^p.$$

We then argue by a change of variables as in the case $p = 2$ considered in [?, Sec. 4.2.3]. \square

6.3 Approximate dipoles

Theorem 6.4 *For every $P \in \mathbb{S}^1$, there exists a sequence of maps $\{u_h\} \subset C^1(B^2 \setminus \{a_-, a_+\}, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup G_P + L \times \mathbb{S}^1$ weakly in $\mathcal{D}_2(B^2 \times \mathbb{S}^1)$, and*

$$\mathcal{E}_{1/p,p}(u_h) \rightarrow l \cdot E_p, \quad l := |a_+ - a_-|.$$

PROOF: For $\varepsilon > 0$ small, first define $W_\varepsilon : (0, l) \times B^+ \rightarrow \mathbb{R}^2$ by $W_\varepsilon(x_1, x_2, t) := f_\varepsilon^P(x_2, t)$, where f_ε^P is given by Proposition 6.2, so that W_ε is Lipschitz-continuous. Then apply Lemma 6.3 with $V := W_\varepsilon$ and $m = 1$, to obtain a map $W_{\varepsilon, \delta} := W_\varepsilon \circ (\phi_\delta^1)^{-1}$. Finally set

$$U_\varepsilon(z) := \begin{cases} W_{\varepsilon, \delta}(z) & \text{if } z \in \Omega_\delta^1 \\ P & \text{otherwise,} \end{cases}$$

so that U_ε belongs to $\text{Lip}_{\text{loc}}(\mathbb{C}^3 \setminus \{a_-, a_+\}, \mathbb{R}^2)$, the trace $u_\varepsilon(x) := U_\varepsilon(x, 0)$ belongs to $W^{1/p,p}(B^2, \mathbb{S}^1)$, and by (6.5)

$$l \cdot E_p \leq \mathcal{E}_{p-2}^p(U_\varepsilon) \leq l \cdot \mathcal{E}_{p-2}^p(f_\varepsilon^P, B^+) + \frac{\varepsilon}{2}$$

if δ is sufficiently small in dependence of ε and of the Lipschitz constant of W_ε . Letting $u_h = u_{\varepsilon_h}$ with $\varepsilon_h \searrow 0$, the assertion follows from Proposition 6.2, on account of a convolution argument. \square

6.4 The dipole problem

The function u_ε in Theorem 6.4 has degree $\deg(u_\varepsilon, a_\pm) = \pm 1$, where the degree is defined in the classical sense by taking the restriction of u_ε to small circles around the singularities a_\pm . Equivalently, according to Example 3.4 we have: $\mathbb{P}(u_\varepsilon) = -\delta_{a_+} + \delta_{a_-}$. As a consequence, we readily obtain:

Proposition 6.5 *For $p > 1$, let \mathcal{G}_p be the class of maps $u \in W^{1/p,p}(\mathbb{R}^2, \mathbb{S}^1)$ which are smooth outside the points a_\pm and such that $\deg(u, a_\pm) = \pm 1$. Then*

$$\inf\{\mathcal{E}_{1/p,p}(u, \mathbb{R}^2) \mid u \in \mathcal{G}_p\} = l \cdot E_p, \quad l = |a_+ - a_-|.$$

PROOF: Inequality “ \leq ” follows from Theorem 6.4, extending u_ε as the constant function P on $\mathbb{R}^2 \setminus B^2$. On the other hand, for every $u \in \mathcal{G}_p$ and for a.e. $x_1 \in (0, l)$, the restriction $u_{x_1}(y) := u(x_1, y)$ is a map in $W^{1/p,p}(\mathbb{R}, \mathbb{S}^1)$ with degree one, whence $\mathcal{E}_{1/p,p}(u_{x_1}, \mathbb{R}) \geq E_p$, by (4.1). Integrating on $y \in (0, l)$, we get

$$\mathcal{E}_{1/p,p}(u, \mathbb{R}^2) \geq \int_0^l \mathcal{E}_{1/p,p}(u_{x_1}, \mathbb{R}) dx_1 \geq l \cdot E_p$$

which yields the inequality “ \geq ”. \square

Proposition 6.5 is in accordance with the case $N = 1$ of [7, Thm. 2.4], where the authors analyzed the dipole problem for maps in $W^{N/p,p}(\mathbb{S}^{N+1}, \mathbb{S}^N)$.

7 Approximation by smooth graphs

In this section, and in the appendices B and C, we prove the following strong density result:

Theorem 7.1 *Let $n \geq 2$ and $p > 1$. For every $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, there exists a sequence of smooth maps $\{u_h\}$ in $C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and*

$$\lim_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h) = E_{1/p,p}(T).$$

PROOF: We make use of a readaptation of the proof for the case $p = 2$ taken from [23], in the simpler case where the target space is the circle \mathbb{S}^1 . We divide the proof in four steps.

Step 1: Reduction to finite mass singularities. Let $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, so that (2.6) holds and hence $\mathbb{P}(u_T) = -(\partial L_T) \llcorner B^n$, see (3.7). By Proposition 3.7, we readily infer:

Lemma 7.2 *There exists a sequence $\{u_h\}$ in $\mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ strongly converging to $u = u_T$ in $W^{1/p,p}(B^n, \mathbb{R}^2)$ and such that, if $L_{u_h, u}$ is given by (3.6), then the current*

$$T_h := G_{u_h} + (L_{u_h, u} + L_T) \times \mathbb{S}^1$$

belongs to $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, all the boundary masses $\mathbf{M}(\partial(L_{u_h, u} + L_T))$ are finite, $T_h \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, and $E_{1/p,p}(T_h) \rightarrow E_{1/p,p}(T)$ as $h \rightarrow \infty$.

As a consequence, we can assume that $u_T \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$ and that $\mathbb{P}(u_T)$ has finite mass, whence $T = G_{u_T} + L \llbracket \mathbb{S}^1 \rrbracket$, where $L \in \mathcal{R}_{n-1}(B^n)$ satisfies $(\partial L) \llcorner B^n = -\mathbb{P}(u_T)$. Therefore, by Federer's boundary rectifiability theorem [14], we infer that the boundary current ∂L is i.m. rectifiable in $\mathcal{R}_{n-2}(B^n)$.

Step 2: Approximation by polyhedral chains. We can thus write $L = \sum_{q=1}^m L_q$ where the L_q 's are integral $(n-1)$ -currents in B^n with pairwise disjoint supports. Using Federer's strong polyhedral approximation theorem [14], for every $\varepsilon > 0$ and $q = 1, \dots, m$ we find an integral polyhedral $(n-1)$ -chain P_q^ε with support contained in a small neighborhood of radius $c\varepsilon$ of the support of the L_q 's, and a function $U_\varepsilon \in C^1(\mathbf{C}^{n+1}, \mathbb{R}^2)$, with trace $u_\varepsilon(x) := U_\varepsilon(x, 0) \in \mathcal{R}_{1/p,p}(B^n, \mathbb{S}^1)$, such that if

$$T_\varepsilon := G_{u_\varepsilon} + \sum_{q=1}^m P_q^\varepsilon \times \mathbb{S}^1,$$

then $T_\varepsilon \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, T_ε converges weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to T as $\varepsilon \rightarrow 0$, and

$$E_{1/p,p}(T_\varepsilon) = \mathcal{E}_{1/p,p}(u_\varepsilon) + E_p \cdot \sum_{q=1}^m \mathbf{M}(P_q^\varepsilon) \rightarrow \mathcal{E}_{1/p,p}(u_T) + E_p \cdot \mathbf{M}(L) = E_{1/p,p}(T).$$

Moreover, since the L_q 's have disjoint supports, we may and do choose the polyhedral chains P_q^ε in such a way that for every small $\varepsilon > 0$ they have pairwise disjoint supports contained in B^n , and u_ε is locally Lipschitz on $B^n \setminus \bigcup_{q=1}^m \text{spt } \partial P_q^\varepsilon$, i.e., outside the $(n-2)$ -skeleton of each P_q^ε . Also, possibly dividing the simplices of a triangulation of P_q^ε , we may and do assume that every polyhedral $(n-1)$ -chain P_q^ε is the union of a finite number of oriented $(n-1)$ -simplices Δ which only intersect at the boundary points.

Step 3: Approximate dipoles. In dimension $n = 2$, we apply Proposition 6.2. In high dimension $n \geq 3$, we apply Proposition 7.3, that is proved in Appendix B. For this purpose, we fix some notation.

For $n \geq 3$, let Δ be the $(n-1)$ -simplex in B^n given by the convex hull

$$\Delta := \text{coh}(\{0_{\mathbb{R}^n}, l e_1, l e_2, \dots, l e_{n-1}\}), \quad 0 < l < 1, \quad (7.1)$$

(e_1, \dots, e_n) being the standard basis in \mathbb{R}^n . Denote by

$$z = (x, t) = (\tilde{x}, x_n, t), \quad \tilde{x} = (x_1, \dots, x_{n-1})$$

a generic point z in \mathbf{C}^{n+1} . Also, for $\delta > 0$ and $0 < m \ll 1$, we let

$$\varphi_\delta^m(y) := \min\{my, \delta\}, \quad y \geq 0,$$

we denote by

$$y(\tilde{x}) := \text{dist}(\tilde{x}, \partial\Delta)$$

the distance of \tilde{x} from the boundary of the $(n-1)$ -simplex Δ , and we set

$$\phi_\delta^m(z) := (\tilde{x}, \varphi_\delta^m(y(\tilde{x}))x_n, \varphi_\delta^m(y(\tilde{x}))t)$$

so that if

$$\Omega_\delta^m := \phi_\delta^m(\Delta \times B^+), \quad B^+ := \{(x_n, t) \in B^2 \mid t > 0\},$$

then Ω_δ^m is a small ‘‘neighbor’’ of the $(n-1)$ -simplex $\Delta \times \{0\}$ in \mathbf{C}^{n+1} .

Proposition 7.3 *Let $U \in W_{p-2}^{1,p}(\mathbf{C}^{n+1}, \mathbb{R}^2)$ be a map which is smooth in the interior of $\Omega_{\delta_0}^{m_0}$, for some fixed small $m_0, \delta_0 > 0$, and such that the trace $u(x) := U(x, 0)$ belongs to $W^{1/p,p}(B^n, \mathbb{S}^1)$. Then for every $\varepsilon > 0$, $0 < \delta < \delta_0$, and $0 < m < m_0$, there exists a map $U_\varepsilon : \mathbf{C}^{n+1} \rightarrow \mathbb{R}^2$ with trace $u_\varepsilon(x) := U_\varepsilon(x, 0)$ in $W^{1/p,p}(B^n, \mathbb{S}^1)$ such that U_ε is smooth in the closure of Ω_δ^m , except for the $(n-2)$ -skeleton of a triangulation of Δ . Moreover, $G_{u_\varepsilon} \rightharpoonup G_u + \llbracket \Delta \rrbracket \times \llbracket \mathbb{S}^1 \rrbracket$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ as $\varepsilon \rightarrow 0$, and*

$$\mathcal{E}_{1/p,p}(u_\varepsilon) \leq \mathcal{E}_{1/p,p}(u) + E_p \cdot \mathcal{H}^{n-1}(\Delta) + \varepsilon. \quad (7.2)$$

Once we have applied Proposition 6.2, when $n = 2$, or Proposition 7.3, when $n \geq 3$, in order to approximate the dipoles $P_q \times \mathbb{S}^1$, by a diagonal argument we find a sequence $\{u_\varepsilon\}$ in $R_{1/p,p}(B^n \times \mathbb{S}^1)$ whose graphs G_{u_ε} weakly converge to T in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, with $E_{1/p,p}(G_{u_\varepsilon}) \rightarrow E_{1/p,p}(T)$. Moreover, by the construction it turns out that u_ε is smooth except on a singular set Σ_ε of B^n given by the $(n-2)$ -skeleton of a triangulation of the union of the polyhedral $(n-1)$ -chains P_q , and that $\mathbb{P}(u_\varepsilon) = 0$, i.e., u_ε is a Cartesian map in $\text{cart}^{1/p,p}(B^n, \mathbb{S}^1)$.

Step 4: Removing the singularities. We remove the $(n-2)$ -dimensional singular set Σ_ε . In low dimension $n = 2$ we apply Proposition 6.1, whereas in high dimension $n \geq 3$ we make use of the following variant of a result from [23], that is proved in Appendix C.

Proposition 7.4 *Under the previous hypotheses, for $\varepsilon > 0$ small enough there exists a sequence of smooth maps $\{u_h^{(\varepsilon)}\} \subset C^\infty(B^n, \mathbb{S}^1)$ which converges to u_ε strongly in $W^{1/p,p}(B^n, \mathbb{R}^2)$ as $h \rightarrow \infty$.*

The proof of Theorem 7.1 is then completed by a diagonal argument. \square

8 Relaxed energy

In this section, we provide in any dimension $n \geq 2$ the explicit formula for the relaxed energy (0.6) in the class $W^{s,p}(B^n, \mathbb{S}^1)$ for any $0 < s < 1$ and $p > 1$. We then give a partial result concerning for the case $1 < s < 2$. Recalling that in dimension $n \geq 2$ we have $W_S^{s,p}(B^n, \mathbb{S}^1) = W^{s,p}(B^n, \mathbb{S}^1)$ if and only if $sp < 1$ or $sp \geq 2$, see Theorem 0.1, in the sequel we assume $1 \leq sp < 2$ and $p > 1$, by firstly considering the case $s = 1/p$, Theorem 8.2, where we apply previous results proved in this paper.

For the sake of completeness, we first collect some properties concerning the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ from Definition 3.3, which were already proved in [23] when $p = 2$

8.1 Cartesian currents

Using Theorems 2.5, 5.1, and 7.1, we obtain:

Theorem 8.1 *Let $n \geq 2$ and $p > 1$. Then:*

- i) *for every $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ there exists a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $u_h \rightarrow u_T$ strongly in $L^p(B^n, \mathbb{R}^2)$, $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$, and $\mathcal{E}_{1/p,p}(u_h) \rightarrow E_{1/p,p}(T)$;*
- ii) *the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ is closed along weakly converging sequences of currents with equibounded energies;*
- iii) *the functional $T \mapsto E_{1/p,p}(T)$ is sequentially lower semicontinuous in the class $\text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$.*

PROOF: Property i) is Theorem 7.1. As to property ii), assume that $\{T_h\} \subset \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ is such that $\sup_h E_{1/p,p}(T_h) < \infty$. By applying Theorem 7.1, for each h we find a sequence $\{v_k^{(h)}\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{v_k^{(h)}} \rightharpoonup T_h$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(v_k^{(h)}) \rightarrow E_{1/p,p}(T_h)$ as $k \rightarrow \infty$. Letting $v_h := v_h^{(h)}$, we have $\sup_h \mathcal{E}_{1/p,p}(v_h) < \infty$, whence by Theorem 2.5 a (not relabeled) subsequence is such that $G_{v_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to some $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. We now recall that the weak convergence restricted to currents in $\mathcal{R}_n(B^n \times \mathbb{S}^1)$ with no inner boundary is metrizable, being equivalent to the flat metric convergence, see [30, Thm. 31.2]. Therefore, by a diagonal argument we find a subsequence of $\{T_h\}$ weakly converging to T .

As to property iii), assume now that $\{T_h\} \subset \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$ satisfies $\sup_h E_{1/p,p}(T_h) < \infty$ and $T_h \rightharpoonup T$ for some $T \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$. As before, we find a sequence $\{v_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{v_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(v_h) \leq \liminf_{h \rightarrow \infty} E_{1/p,p}(T_h).$$

On account of Theorem 5.1, we thus get

$$E_{1/p,p}(T) \leq \liminf_{h \rightarrow \infty} E_{1/p,p}(T_h).$$

We omit any further detail. \square

8.2 An explicit formula

Assume now $sp = 1$ and $p > 1$. The following theorem implies that *every map in $W^{1/p,p}(B^n, \mathbb{S}^1)$ belongs to the $W^{1/p,p}$ -weak sequential closure of smooth maps in $C^\infty(B^n, \mathbb{S}^1)$.*

Theorem 8.2 *Let $n \geq 2$ and $p > 1$. For every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$ the relaxed energy $\tilde{\mathcal{E}}_{1/p,p}(u)$ is finite, and we have:*

$$\tilde{\mathcal{E}}_{1/p,p}(u) = \mathcal{E}_{1/p,p}(u) + E_p \cdot m_{i,B^n}(\mathbb{P}(u)) < \infty$$

where $E_p > 0$ is given by the minimum problem (4.1), and $m_{i,B^n}(\mathbb{P}(u))$ is the integral mass relative to B^n of the current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(B^n)$ of the singularities of u , see Definitions 3.1 and 3.5.

PROOF: We claim that for every $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$

$$\tilde{\mathcal{E}}_{1/p,p}(u) = \inf\{E_{1/p,p}(T) \mid T \in \mathcal{T}_u^{1/p,p}\} < \infty \quad (8.1)$$

where the class $\mathcal{T}_u^{1/p,p}$ of Cartesian currents with underlying function u is defined in (3.8).

In fact, we already know that the class $\mathcal{T}_u^{1/p,p}$ is non-empty, see Remark 3.9, whereas by Theorem 7.1, for any $T \in \mathcal{T}_u^{1/p,p}$ we can find a smooth sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and $\mathcal{E}_{1/p,p}(u_h) \rightarrow E_{1/p,p}(T)$. Since $u_h \rightarrow u_T$ in $L^p(B^n, \mathbb{R}^2)$ and $u_T = u$, we infer that the inequality “ \leq ” holds in (8.1), and hence that $\tilde{\mathcal{E}}_{1/p,p}(u) < \infty$.

On the other hand, if $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ satisfies $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$, and $u_h \rightarrow u$ in $L^p(B^n, \mathbb{R}^2)$, by Theorem 2.5 we find a (not relabeled) subsequence such that $G_{u_h} \rightharpoonup T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to some $T \in \mathcal{T}_u^{1/p,p}$. By Theorem 5.1, we have that $E_{1/p,p}(T) \leq \liminf_h \mathcal{E}_{1/p,p}(u_h)$, whence the inequality “ \geq ” holds too in (8.1).

Now, using again Proposition 3.7, we know that $\mathbb{P}(u)$ is an integral flat chain in $\mathcal{D}_{n-2}(B^n)$, whence there exists $L_u \in \mathcal{R}_{n-2}(B^n)$ such that

$$(\partial L_u) \llcorner B^n = -\mathbb{P}(u) \quad \text{and} \quad \mathbf{M}(L_u) = m_{i,B^n}(\mathbb{P}(u)).$$

Setting then $T_u := G_u + L_u \times \llbracket \mathbb{S}^1 \rrbracket$, by (3.7) we infer that the null-boundary condition (2.7) holds, whence by (2.6) it turns out that $T_u \in \mathcal{T}_u^{1/p,p}$. By the definition in (5.1) we thus get

$$\inf\{E_{1/p,p}(T) \mid T \in \mathcal{T}_u^{1/p,p}\} = E_{1/p,p}(T_u) = \mathcal{E}_{1/p,p}(u) + E_p \cdot \mathbf{M}(L_u)$$

which implies the explicit formula for $\tilde{\mathcal{E}}_{1/p,p}(u)$, on account of (8.1). \square

Coming back to Definition 3.3, we thus readily obtain:

Corollary 8.3 *For any $p > 1$, we have:*

$$W_S^{1/p,p}(B^n, \mathbb{S}^1) = \text{cart}^{1/p,p}(B^n, \mathbb{S}^1) \quad \forall n \geq 2.$$

8.3 The case $0 < s < 1$, $1 < sp < 2$

In these ranges of s and p , the strong density of smooth maps fails to hold, see Theorem 0.1. Since $sp > 1$, by the continuous embedding $W^{s,p}(B^n, \mathbb{S}^1) \subset W^{s,1/s}(B^n, \mathbb{S}^1)$, both the graph current G_u in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ and the singular set $\mathbb{P}(u)$, an integral flat chain in $\mathcal{D}_{n-2}(B^n)$, are well defined for each map $u \in W^{s,p}(B^n, \mathbb{S}^1)$.

However, differently to the scale invariance property for the case $sp = 1$, see Remark 4.1, letting $\gamma(s,p)$ as in (0.3), if $U \in W_{\gamma(s,p)}^{1,p}(\mathbb{R}_+^2, \mathbb{R}^2)$ and $U_{(r)}(x,t) := U(rx, rt)$ we get

$$\mathcal{E}_{\gamma(s,p)}^p(U_{(r)}, \mathbb{R}_+^2) = r^{sp-1} \mathcal{E}_{\gamma(s,p)}^p(U, \mathbb{R}_+^2) \quad \forall r > 0.$$

Therefore, since $sp > 1$ we have $\mathcal{E}_{\gamma(s,p)}^p(U_{(r)}, \mathbb{R}_+^2) \rightarrow 0$ as $r \rightarrow 0^+$.

On account of Theorem 8.2, by the above facts we infer that the relaxed energy $\tilde{\mathcal{E}}_{s,p}(u)$ is finite if and only if $\mathbb{P}(u) = 0$, in which case it agrees with the energy $\mathcal{E}_{s,p}(u)$. More precisely, we have:

Theorem 8.4 *Let $n \geq 2$, $0 < s < 1$, and $p > 1$, with $1 < sp < 2$. Then for every $u \in W^{s,p}(B^n, \mathbb{S}^1)$*

$$\tilde{\mathcal{E}}_{s,p}(u) = \begin{cases} \mathcal{E}_{s,p}(u) & \text{if } \mathbb{P}(u) = 0 \\ +\infty & \text{if } \mathbb{P}(u) \neq 0. \end{cases}$$

PROOF: The implication

$$\mathbb{P}(u) = 0 \implies \tilde{\mathcal{E}}_{s,p}(u) = \mathcal{E}_{s,p}(u) < \infty$$

is a consequence of the following approximation result:

Proposition 8.5 *With the values of s , p , and n as in Theorem 8.4, if $u \in W^{s,p}(B^n, \mathbb{S}^1)$ satisfies $\mathbb{P}(u) = 0$, there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ such that $u_h \rightarrow u$ in $L^p(B^n, \mathbb{R}^2)$ and $\mathcal{E}_{s,p}(u_h) \rightarrow \mathcal{E}_{s,p}(u)$ as $h \rightarrow \infty$.*

PROOF: Since $G_u \in \text{cart}^{1/p,p}(B^n \times \mathbb{S}^1)$, arguing as in Steps 1–2 of the proof of Theorem 7.1 we find a sequence of maps u_ε satisfying $\mathbb{P}(u_\varepsilon) = 0$, which are smooth except on a singular set Σ_ε of B^n given by the $(n-2)$ -skeleton of a finite triangulation in B^n , and with $u_\varepsilon \rightarrow u$ strongly in $W^{s,p}(B^n, \mathbb{R}^2)$ as $\varepsilon \rightarrow 0$. We then apply Proposition C.1 from Appendix C, in high dimension $n \geq 3$, or argue in a way similar to Proposition 6.2, when $n = 2$, in order to remove the homologically trivial singularities. Further details are omitted. \square

Conversely, we now show that

$$\tilde{\mathcal{E}}_{s,p}(u) < \infty \implies \mathbb{P}(u) = 0.$$

Let $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ be such that $\sup_h \mathcal{E}_{s,p}(u_h) < \infty$ and $u_h \rightarrow u$ in $L^p(B^n, \mathbb{R}^2)$. Since by the continuous embedding $\sup_h \mathcal{E}_{1/p,p}(u_h) < \infty$ and $u \in W^{1/p,p}(B^n, \mathbb{S}^1)$, by Theorem 2.5 we find a (not relabeled) subsequence such that $G_{u_h} \rightarrow T$ weakly in $\mathcal{D}_n(B^n \times \mathbb{S}^1)$ to some $T \in \mathcal{T}_u^{1/p,p}$.

Assume by contradiction that $\mathbb{P}(u) \neq 0$. Then, $T = G_u + L \times \llbracket \mathbb{S}^1 \rrbracket$ for some $L \in \mathcal{R}_{n-1}(B^n)$ with positive mass, $\mathbf{M}(L) > 0$. Therefore, if $\mathcal{L} := \text{set } L$ is the set of points of positive density for L , see (2.1), we have $\mathcal{H}^{n-1}(\mathcal{L}) > 0$.

For \mathcal{H}^{n-1} -a.e. $x \in \mathcal{L}$, we denote by I_x the line segment given by the intersection of B^n with the straight line of \mathbb{R}^n containing x and orthogonal to the approximate tangent $(n-1)$ -space to \mathcal{L} at x . Then, by a slicing argument, the 1-dimensional restriction of G_{u_h} to $I_x \times \mathbb{S}^1$ yields (possibly passing to a subsequence) a sequence of graphs of smooth maps $u_{h|I_x} : I_x \rightarrow \mathbb{S}^1$ with equibounded $\mathcal{E}_{s,p}$ -energies. Moreover, denoting by

$$U_h^{(x)} := \text{Ext}(u_{h|I_x}) : I_x \times (0, 1) \rightarrow \mathbb{R}^2$$

the extension of $u_{h|I_x}$ in $W_{\gamma(s,p)}^{1,p}(I_x \times (0, 1), \mathbb{R}^2)$, where $\gamma(s,p)$ is given by (0.3), we have

$$\sup_h \int_{I_x \times (0,1)} t^{\gamma(s,p)} |DU_h^{(x)}(y,t)|^p dy dt = C < \infty.$$

Therefore, using that $t^{p-2} = t^{sp-1} \cdot t^{\gamma(s,p)}$, for $0 < r < 1$ we estimate

$$\int_{I_x \times (0,r)} t^{p-2} |DU_h^{(x)}(y,t)|^p dy dt \leq r^{sp-1} \int_{I_x \times (0,r)} t^{\gamma(s,p)} |DU_h^{(x)}(y,t)|^p dy dt \leq C r^{sp-1} \quad (8.2)$$

for any h , where $C r^{sp-1} \rightarrow 0$ as $r \rightarrow 0^+$, since $sp > 1$.

On the other hand, by a slicing argument, we can find a neighborhood J_x of x in I_x such that the 1-currents $G_{u_h|_{J_x}} \llcorner (J_x \times \mathbb{S}^1)$ have to converge near the point x to the current $G_{u|_{J_x}} \llcorner (J_x \times \mathbb{S}^1) + \sigma \delta_x \times [\mathbb{S}^1]$, where $\sigma \in \mathbb{Z} \setminus \{0\}$ agrees (up to the sign) with the density of the current L at x . Therefore, by lower semicontinuity, Theorem 5.1, we have

$$\liminf_{h \rightarrow \infty} \mathcal{E}_{1/p,p}(u_h|_{I_x}, I_x) \geq E_p > 0 \quad (8.3)$$

where E_p is given by (4.1). Since (8.3) is in contradiction with (8.2), we must have $\mathbf{M}(L) = 0$, which yields $\mathbb{P}(u) = 0$, by (3.7), as required. \square

8.4 The case $1 < s < 2$, $1 < sp < 2$

These are the ranges of s and p for which the strong density of smooth maps fails to hold, see Theorem 0.1, but $s > 1$. Even if our definition of energy (0.5), with $\mathcal{E}_{\gamma(s,p)}^p(U)$ given by (0.4), does not make sense, we obtain a partial result:

Corollary 8.6 *Let $u \in W^{s,p}(B^n, \mathbb{S}^1)$, where $1 < p < 2$, $1 < s < 2$, $1 < sp < 2$, and $n \geq 2$. Assume that there exists a sequence $\{u_h\} \subset C^\infty(B^n, \mathbb{S}^1)$ converging a.e. to u and such that $\sup_h \|u_h\|_{s,p} < \infty$. Then, $\mathbb{P}(u) = 0$.*

PROOF: Since $sp > 1$, we have already seen that the current G_u and the singular set $\mathbb{P}(u)$ are well defined. This time, by means of a Gagliardo-Nirenberg type inequality, see [5, Appendix D], the continuous embedding $W^{s,p}(B^n, \mathbb{S}^1) \subset W^{r,q}(B^n, \mathbb{S}^1)$ holds for any $0 < r < 1$ and $q > p$ such that $rq = sp$, with $\|u\|_{r,q} \leq C \|u\|_{s,p}^{r/s}$. This yields that $\sup_h \mathcal{E}_{r,q}(u_h) < \infty$, whence u has finite relaxed energy, $\tilde{\mathcal{E}}_{r,q}(u) < \infty$. Since $rq > 1$ and $0 < r < 1$, by Theorem 8.4 we conclude that $\mathbb{P}(u) = 0$. \square

Appendix A Harmonic maps

The Euler-Lagrange system associated to the minimum problem (4.1) reads as

$$\operatorname{div}(|t DU(x,t)|^{p-2} DU) = 0, \quad (x,t) \in \mathbb{R}_+^2$$

and finding the energy minimum E_p in (4.1) is a difficult task. However, when $p = 2$ we reduce to the harmonic map equation $\Delta U(x,t) = 0$ and one has $E_2 = 2\pi$, an energy minimizer $u_2 \in W^{1/2,2}(\mathbb{R}, \mathbb{S}^1)$ being given by the trace on $t = 0$ of the function $U_2 : \mathbb{R}_+^2 \rightarrow \mathbb{D}^2$

$$U_2(x,t) := \left(\frac{2}{x^2 + (t+1)^2} x, \frac{1 - (x^2 + t^2)}{x^2 + (t+1)^2} \right),$$

see (A.1), that is by the inverse u_2 of the stereographic map $\sigma : \mathbb{S}^1 \rightarrow \mathbb{R}$ from the south pole $P_S = (0, -1)$.

Example A.1 Let $p = 2$. By the parallelogram inequality, $2|\det G| \leq |G|^2$ for each $G \in \mathbb{R}^{2 \times 2}$ and hence, using that $U(\Omega) = \mathbb{D}^2$ if $u \in \mathcal{F}_2$ is smooth, by the area formula we get the energy lower bound

$$\mathcal{E}_2(U, \mathbb{R}_+^2) = \int_{\mathbb{R}_+^2} |DU(x,t)|^2 dx dt \geq 2 \int_{\mathbb{R}_+^2} |\det DU(x,t)|^2 dx dt \geq 2\pi.$$

Therefore, by a density argument we infer that $E_2 \geq 2\pi$. Following [28], consider now the complex map

$$h(z) := \frac{1 - i\bar{z}}{\bar{z} - i}, \quad z \in \mathbb{C} \setminus \{-i\}.$$

It is readily checked that h is a biholomorphic map between the half-space $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ and the unit disc $\mathbb{E} := \{z \in \mathbb{C} : |z| < 1\}$, and that $h(z) \rightarrow -i$ as $|z| \rightarrow +\infty$. Setting $z = x + it$ for $x \in \mathbb{R}$ and $t > 0$, we have $h(z) = f(x, t) + i g(x, t)$, where

$$f(x, t) := \frac{2x}{\Delta(x, t)}, \quad g(x, t) := \frac{1 - (x^2 + t^2)}{\Delta(x, t)}, \quad \Delta = \Delta(x, t) := x^2 + (t + 1)^2.$$

Therefore, the following Cauchy-Riemann equations are satisfied:

$$f_{,x} = -g_{,t} = \frac{2}{\Delta(x, t)^2} ((t + 1)^2 - x^2), \quad f_{,t} = g_{,x} = \frac{4x(t + 1)}{\Delta(x, t)^2}.$$

In particular, the mapping $U_2(x, t) := (f, g)(x, t)$ is conformal, i.e.,

$$\langle D_i U_2, D_j U_2 \rangle_{\mathbb{R}^2} = \delta_{ij} \frac{4}{\Delta^2} \quad \forall (x, t) \in \mathbb{R}_+^2, \quad 1 \leq i \leq j \leq 2.$$

Therefore, one has $|DU_2|^2 = 2|\det DU_2|$ for each $(x, t) \in \mathbb{R}_+^2$, and since $U_2 : \mathbb{R}_+^2 \rightarrow \mathbb{D}$ is bijective

$$\mathcal{E}_0^2(U_2, \mathbb{R}_+^2) = \int_{\mathbb{R}_+^2} |DU_2(x, t)|^2 dx dt = 2 \int_{\mathbb{R}_+^2} |\det DU_2(x, t)| dx dt = 2\pi.$$

Finally, the trace $u_2 : \mathbb{R} \rightarrow \mathbb{S}^1$

$$u_2(x) := U_2(x, 0) = \left(\frac{2x}{1 + x^2}, \frac{1 - x^2}{1 + x^2} \right), \quad x \in \mathbb{R} \tag{A.1}$$

satisfies $\deg u_2 = 1$, whence $u_2 \in \mathcal{F}_2$ and definitely $E_2 = 2\pi$.

Appendix B Approximate dipoles

In this appendix we give the proof of Proposition 7.3. It is a readaptation of the proof of [23, Prop. 7.3] for the case $p = 2$, in the simpler situation where the target manifold is the unit circle \mathbb{S}^1 .

PROOF: [Proof of Proposition 7.3] Recall that $n \geq 3$. Let ψ be a bi-Lipschitz homeomorphism of B^n which takes the $(n - 1)$ -simplex Δ given by (7.1) onto the $(n - 1)$ -disk $D \times \{0\}$ of diameter l , where

$$D := \{\tilde{x} \in \mathbb{R}^{n-1} : |\tilde{x}| \leq l/2\},$$

with Lipschitz constants

$$\text{Lip } \psi \leq K, \quad \text{Lip } \psi^{-1} \leq K, \tag{B.1}$$

where $K = K(n)$ does not depend on l , but possibly on the distance of Δ from ∂B^n . Also, let $V : \mathbb{C}^{n+1} \rightarrow \mathbb{R}^2$ be given by

$$V(z) := U \circ \Psi^{-1}(z), \quad \Psi(z) = \Psi(x, t) := (\psi(x), t).$$

Denoting $\mathbf{0} = (0, 0) \in \mathbb{R}^2$, we finally set

$$\begin{aligned} \Omega_\rho &:= \{z \in \mathbb{C}^{n+1} \mid \text{dist}(z, \partial D \times \{\mathbf{0}\}) < \rho\}, \\ \partial^+ \Omega_\rho &:= \{z \in \mathbb{C}^{n+1} \mid \text{dist}(z, \partial D \times \{\mathbf{0}\}) = \rho\}, \end{aligned}$$

where $\rho > 0$, fix $0 < R < l/2$, and denote by $p : \Omega_R \rightarrow \partial D \times \{\mathbf{0}\}$ the nearest point projection, so that for every $z \in \Omega_R$

$$|z - p(z)| = \text{dist}(z, \partial D \times \{\mathbf{0}\}).$$

By applying the coarea formula w.r.t. “cylindrical type” coordinates defined around the $(n-2)$ -sphere $\partial D \times \{\mathbf{0}\}$, since property

$$\int_{\Omega_R} t^{p-2} |DV|^p dz = \int_0^R d\rho \int_{\partial^+ \Omega_\rho} t^{p-2} |DV|^p d\mathcal{H}^n < \infty$$

yields

$$\liminf_{\rho \rightarrow 0^+} \rho \int_{\partial^+ \Omega_\rho} t^{p-2} |DV|^2 d\mathcal{H}^n = 0,$$

we can choose a small radius $r > 0$ and replace V on Ω_r by the map

$$V_r(z) := V\left(p(z) + r \frac{z - p(z)}{|z - p(z)|}\right) \quad (\text{B.2})$$

so that as in (6.4) we estimate

$$\mathcal{E}_{p-2}^p(V_r, \Omega_r) \leq c_n \cdot r \int_{\partial^+ \Omega_r} t^{p-2} |DV|^p d\mathcal{H}^n = O(r) \quad (\text{B.3})$$

where $O(r_j) \rightarrow 0$ along a sequence $r_j \searrow 0$.

We now define $\tilde{\Omega}_\delta^m := \tilde{\phi}_\delta^m(D \times B^+)$, where for $z = (\tilde{x}, x_n, t)$ we let

$$\tilde{\phi}_\delta^m(z) := (\tilde{x}, \varphi_\delta^m(\tilde{y}(\tilde{x}))x_n, \varphi_\delta^m(\tilde{y}(\tilde{x}))t), \quad \tilde{y}(\tilde{x}) := \text{dist}(\tilde{x}, \partial D).$$

Moreover, setting $r_{\delta,m} := \delta \frac{\sqrt{1+m^2}}{m}$ we define

$$K_\delta^m := \left\{ z \in \mathbf{C}^{n+1} \mid \begin{array}{l} 0 < \text{dist}(z, \partial D \times \{\mathbf{0}\}) < r_{\delta,m}, \\ 0 < \tilde{y}(\tilde{x}) < \delta/m, \sqrt{x_n^2 + t^2} < m \cdot \tilde{y}(\tilde{x}) \end{array} \right\}$$

and note that if $r_{\delta,m} < r$, by (B.2) it turns out that the restriction of V to K_δ^m does not depend on the distance of z from $\partial D \times \{\mathbf{0}\}$.

We now wish that the following conditions hold true:

- (i) V maps K_δ^m into a set of diameter ε ;
- (ii) V maps $\tilde{\Omega}_\delta^m$ into a set of diameter ε .

If it is not the case, we let $\{\Delta_i\}_{i=1}^{c(n)}$ be a barycentric-type subdivision of Δ into smaller simplices of side $l/2$, whose number $c(n)$ only depends on n , starting from the 1-faces of Δ . Moreover, possibly slightly moving the centers of the 1-faces of Δ , without loss of generality we can assume that the restriction of V to each k -face of Δ_i has finite $\mathcal{E}_{1/p,p}$ -energy, for every $k = 2, \dots, n$ and every i . We then apply the previous construction to each Δ_i , where K , see (B.1), is an upper bound for the Lipschitz constants of the homeomorphisms of B^n which map Δ_i onto D_i , the $(n-1)$ -disk of diameter $l/2$, for every i .

If V does not satisfy conditions (i) and (ii) on the sets $K_{\delta,i}^m$ and $\Omega_{\delta,i}^m$ corresponding to D_i , we start again with the previous procedure, by taking a barycentric subdivision of Δ_i as above.

Note that V is smooth on the interior of Ω_δ^m , for δ and m sufficiently small, and, by paying a small amount of energy, we can assume that V does not depend on the distance of z from ∂D_i on $K_{\delta,i}^m$. We then infer that the conditions (i) and (ii) above are obtained after a finite number of barycentric subdivisions, by first taking $0 < m = m(\varepsilon) \ll 1$ and then $\delta = \delta(m, r) > 0$ small. Therefore, in the sequel we omit to write the index i corresponding to the simplex Δ_i of the given (finite) subdivision of Δ .

Let now $W_\varepsilon : D \times B^+ \rightarrow \mathbb{R}^N$ be given by

$$W_\varepsilon(\tilde{x}, x_n, t) := f_\varepsilon^P(x_n, t) \quad (\text{B.4})$$

where f_ε^P is given by Proposition 6.2 in correspondence to the point $P := U(x)$ for some given $x \in \text{int}(\Delta)$. Setting

$$\Phi_\varepsilon(z) := W_\varepsilon \circ (\tilde{\phi}_\delta^m)^{-1}(z), \quad z \in \tilde{\Omega}_\delta^m$$

arguing as in Theorem 6.4 and (B.3) we estimate

$$\mathcal{E}_{p-2}^p(\Phi_\varepsilon, \tilde{\Omega}_\delta^m) \leq \mathcal{H}^{n-1}(\Delta) \cdot \mathcal{E}_{p-2}^p(f_\varepsilon^P, B^+) + \frac{\varepsilon}{2\mu K^2}$$

if we choose $\delta = \delta(W_\varepsilon, \varepsilon, m, K, \mu)$ sufficiently small. Here, μ is the finite number of the Δ_i 's obtained in the previous subdivision of Δ . We now introduce the cylindrical coordinates

$$z = (\tilde{x}, x_n, t) = F(\rho, \theta, \tilde{x}) := (\tilde{x}, \rho \cos \theta, \rho \sin \theta), \quad \rho > 0, \quad \theta \in [0, \pi]$$

so that $\rho = \sqrt{x_n^2 + t^2}$, denote

$$\widehat{W}(\rho, \theta, \tilde{x}) := W(F(\rho, \theta, \tilde{x}))$$

and define $V_\varepsilon : \tilde{\Omega}_\delta^m \rightarrow \mathbb{R}^2$ by

$$\widehat{V}_\varepsilon(\rho, \theta, \tilde{x}) := \begin{cases} \widehat{\Phi}_\varepsilon(2\rho, \theta, \tilde{y}) & \text{if } 0 \leq \rho < \varphi_\delta^m(\tilde{y})/2 \\ \widehat{\Psi}_\delta^m(\rho, \theta, \tilde{y}) & \text{if } \varphi_\delta^m(\tilde{y})/2 \leq \rho < \varphi_\delta^m(\tilde{y}) \end{cases}$$

for all $\theta \in [0, \pi]$ and $\tilde{x} \in \text{int}(\Delta)$, where $\tilde{y} = \tilde{y}(\tilde{x}) := \text{dist}(\tilde{x}, \partial D)$ and

$$\widehat{\Psi}_\delta^m(\rho, \theta, \tilde{y}) := \left(\frac{2\rho}{\varphi_\delta^m(\tilde{y})} - 1 \right) \cdot \widehat{V}(\varphi_\delta^m(\tilde{y}), \theta, \tilde{y}) + \left(2 - \frac{2\rho}{\varphi_\delta^m(\tilde{y})} \right) \cdot P.$$

We also extend $V_\varepsilon \equiv V$ outside $\tilde{\Omega}_\delta^m$. By conditions (i) and (ii) above we thus estimate

$$\mathcal{E}_{p-2}^p(V_\varepsilon, \tilde{\Omega}_\delta^m) \leq \mathcal{H}^{n-1}(\Delta) \cdot (E_p + 4\varepsilon^2) + \frac{\varepsilon}{2\mu K^2}. \quad (\text{B.5})$$

We finally define

$$U_\varepsilon(z) := V_\varepsilon \circ \Psi(z).$$

Possibly repeating the argument for each simplex Δ_i of the given subdivision of Δ , by (B.5) and (B.1) we estimate

$$\mathcal{E}_{p-2}^p(U_\varepsilon) \leq \mathcal{E}_{p-2}^p(U) + \mathcal{H}^{n-1}(\Delta) \cdot (E_p + 4K^2\varepsilon^2) + \frac{\varepsilon}{2}$$

so that (7.2) follows for $\varepsilon > 0$ small. \square

Appendix C Removing homologically trivial singularities

In this appendix, we give the proof of Proposition 7.4 by taking $s = 1/p$ in Proposition C.1 below. As before, it is a readaptation of the proof of [23, Prop. 7.4] for the case $s = 1/2$ and $p = 2$, in the simpler case where the target manifold is the unit circle \mathbb{S}^1 .

Proposition C.1 *Let $n \geq 3$, $p > 1$, and $0 < s < 1$ be such that $1 \leq sp < 2$, and let $u_\varepsilon \in \mathcal{R}_{s,p}(B^n \times \mathbb{S}^1)$ be smooth except on a singular set Σ_ε of B^n given by the $(n-2)$ -skeleton of a triangulation of the union of polyhedral $(n-1)$ -chains P_q , $q = 1, \dots, m$. If $\mathbb{P}(u_\varepsilon) = 0$, there exists a sequence of smooth maps $\{u_h^{(\varepsilon)}\} \subset C^\infty(B^n, \mathbb{S}^1)$ which converges to u_ε strongly in $W^{s,p}(B^n, \mathbb{R}^2)$.*

PROOF: Let $U_\varepsilon : B^n \times]-1, 1[\rightarrow \mathbb{D}^2$ be given by $U_\varepsilon(x, t) := (\text{Ext } u_\varepsilon)(x, t)$ if $t > 0$, and $U_\varepsilon(x, t) := (\text{Ext } u_\varepsilon)(x, -t)$ if $t < 0$. In the proof, we shall then work with the energy

$$U \mapsto \mathcal{E}_{\gamma(s,p)}^p(U) := \int_{B^n \times]-1, 1[} |t|^{\gamma(s,p)} |DU(x, t)|^p dx dt$$

where $\gamma(s, p)$ is given by (0.3). For $m \in \mathbb{N}^*$ and $a \in [1/4m, 3/4m]^{n+1}$, we denote by $\mathcal{L}_m = \mathcal{L}_m(a)$ the grid of \mathbb{R}^{n+1}

$$\mathcal{L}_m := a + \bigcup_{z \in \mathbb{Z}^{n+1}} \frac{1}{m} \cdot z$$

and by $\mathcal{L}_m^{(k+1)}$ the family of all the $(k+1)$ -faces Q of the $(n+1)$ -cubes of \mathcal{L}_m which intersect the n -disk $B^n \times \{0\}$, for $k = 1, \dots, n$. Moreover, we let $\mathcal{F}_m^{(k)}$ denote the set of k -faces F obtained by intersecting the $(k+1)$ -faces Q of $\mathcal{L}_m^{(k+1)}$ with the n -disk $B^n \times \{0\}$, i.e.,

$$F = Q \cap (B^n \times \{0\}). \quad (\text{C.1})$$

We finally set

$$G_m := B^n \times]-10m^{-1}, 10m^{-1}[.$$

Similarly to [22], we may and do choose $a = a(m, U_\varepsilon)$ so that the following conditions hold:

- (i) for every $k = 1, \dots, n-1$, the restriction of U_ε to every $(k+1)$ -face of $\mathcal{L}_m^{(k+1)}$ is a $W_{\gamma(s,p)}^{1,p}$ function;
- (ii) there exists an absolute constant $c > 0$ such that

$$\mathcal{E}_{\gamma(s,p)}^p(U_\varepsilon, \cup \mathcal{L}_m^{(k+1)}) \leq c m^{n-k} \mathcal{E}_{\gamma(s,p)}^p(U_\varepsilon, G_m) \quad \forall k = 1, \dots, n-1. \quad (\text{C.2})$$

Moreover, recalling that $n \geq 3$, since the singular set Σ_ε is given by the $(n-2)$ -skeleton of some fixed triangulation of the P_q 's, by a slicing argument, for m sufficiently large we can also require that

- (iii) Σ_ε does not intersect the 1-faces of $\mathcal{F}_m^{(1)}$;
- (iv) every 2-face F of $\mathcal{F}_m^{(2)}$ intersects Σ_ε at almost one interior point $p_F \in \text{int}(F)$, which does not belong to the $(n-3)$ -skeleton of the triangulation of the polyhedral $(n-1)$ -chains P_q ;
- (v) the restriction $u_{\varepsilon|_F}$ of u_ε to any 2-face F of $\mathcal{F}_m^{(2)}$ is continuous, possibly except at the point p_F ;
- (vi) in this case, if $p_F \in \text{spt } P_q$, we have

$$\partial G_{u_{\varepsilon|_F}} \lrcorner F \times \mathbb{S}^1 = 0 \quad \text{on } \mathcal{D}^1(F \times \mathbb{S}^1). \quad (\text{C.3})$$

As a consequence, arguing as in (6.3), by (C.3) we infer that

$$\{w \in W^{s,p}(F, \mathbb{R}^2) \cap C^0(F, \mathbb{S}^1) \mid w|_{\partial F} = u_{\varepsilon|_{\partial F}}\} \neq \emptyset \quad (\text{C.4})$$

holds true for every 2-face F of $\mathcal{F}_m^{(2)}$.

In order to remove the singular set Σ_ε of u_ε , we make use of an argument taken from [22]. To this aim, at the 1st step we set $U_m^{(\varepsilon)} \equiv U_\varepsilon$ on $\cup \mathcal{L}_m^{(2)}$. We then argue by induction on the dimension $k = 2, \dots, n$ and, at the k^{th} step, we set $U_m^{(\varepsilon)} \equiv U_\varepsilon$ on every $Q \in \mathcal{L}_m^{(k+1)}$ which does not intersect the n -disk $B^n \times \{0\}$. Moreover, we define $U_m^{(\varepsilon)}$ on every $Q \in \mathcal{L}_m^{(k+1)}$ which intersects $B^n \times \{0\}$ by means of a ‘‘cone’’ construction starting from the restriction $U_m^{(\varepsilon)}|_{\partial Q}$ of $U_m^{(\varepsilon)}$ to the boundary ∂Q . To do this, if $F \in \mathcal{F}_m^{(k)}$ is given by (C.1), it suffices to require that the trace φ_F of $U_m^{(\varepsilon)}|_{\partial Q}$ on the boundary of F has a continuous extension $\Phi_F \in W^{s,p}(F, \mathbb{S}^1)$.

Note that at the 2nd step, this last condition is given by (C.4). In order to extend this condition to the case $k \geq 3$, for every $k \geq 2$ at the k^{th} step we first modify the definition of $u_m^{(\varepsilon)}(x) := U_m^{(\varepsilon)}(x, 0)$ on $\mathcal{F}_m^{(k)}$ in a suitable way (see the k^{th} step of the proof of Theorem 2 on p. 457 of [22] for further details).

We secondly extend $U_m^{(\varepsilon)}$ to every $Q \in \mathcal{L}_m^{(k+1)}$ in a continuous way, so that its trace $u_m^{(\varepsilon)}$ belongs to $W^{s,p}(F, \mathbb{S}^1)$ and

$$\mathcal{E}_{\gamma(s,p)}^p(U_m^{(\varepsilon)}, Q) \leq \frac{c}{m} \mathcal{E}_{\gamma(s,p)}^p(U_\varepsilon, \partial Q). \quad (\text{C.5})$$

More precisely, let $v_Q : Q \rightarrow \mathbb{R}^2$ be defined by $v_Q(z) = v_Q^\pm(z)$ if $z \in Q^\pm$, where

$$Q^\pm := \{z = (x, t) \in Q \mid \pm t \geq 0\}$$

and $v_Q^\pm : Q^\pm \rightarrow \mathbb{R}^2$ is the solution of the minimum problem for the energy $\int_{Q^\pm} |t|^{\gamma(s,p)} |DV(x, t)|^p dx dt$ with boundary condition

$$\begin{cases} v_Q^\pm = U_m^{(\varepsilon)} & \text{on } \partial Q^\pm \cap \{(x, t) \mid \pm t > 0\} \\ v_Q^\pm = \Phi_F & \text{on } F \end{cases}$$

where $\Phi_F : F \rightarrow \mathbb{S}^1$ is a continuous $W^{s,p}$ -extension of the boundary datum $\varphi_F(x) := U_m^{(\varepsilon)}|_{\partial Q}(x, 0)$. Assuming e.g. that the center of Q is the origin $0_{\mathbb{R}^{n+1}}$, we define $U_m^{(\varepsilon)}$ on Q by setting, for $0 < \delta \ll 1/2m$,

$$U_m^{(\varepsilon)}(z) := \begin{cases} v_Q\left(\frac{z}{2m\delta}\right) & \text{if } \|z\| \leq \delta \\ U_m^{(\varepsilon)}\left(\frac{z}{2m\|z\|}\right) & \text{if } \delta \leq \|z\| \leq \frac{1}{2m} \end{cases} \quad z \in Q$$

where $\|z\| := \sup_i |z_i|$ if $z = (z_1, \dots, z_{n+1})$, so that $\|z\| = 1/2m$ if $z \in \partial Q$. Therefore, we have:

$$\mathcal{E}_{\gamma(s,p)}^p(U_m^{(\varepsilon)}, \{\|z\| < \delta\}) \leq (2m\delta)^{n-sp} \mathcal{E}_{\gamma(s,p)}^p(v_Q, \{\|z\| < 1/2m\})$$

where $n - sp > 0$. A similar definition works in the general case, so that (C.5) holds true and $u_m^{(\varepsilon)}(x) := U_m^{(\varepsilon)}(x, 0)$ belongs to $W^{s,p}(F, \mathbb{S}^1)$.

Repeating the argument for $k = 2, \dots, n$, using (C.5) we estimate

$$\mathcal{E}_{\gamma(s,p)}^p(U_m^{(\varepsilon)}, \cup \mathcal{L}_m^{(n+1)}) \leq C(n) \sum_{k=1}^{n-1} \frac{1}{m^{n-k}} \mathcal{E}_{\gamma(s,p)}^p(U_\varepsilon, \mathcal{L}_m^{(k+1)})$$

and hence, by (C.2), we obtain

$$\mathcal{E}_{\gamma(s,p)}^p(U_m^{(\varepsilon)}, \cup \mathcal{L}_m^{(n+1)}) \leq C(n) \mathcal{E}_{\gamma(s,p)}^p(U_\varepsilon, G_m) \rightarrow 0$$

as $m \rightarrow +\infty$, since $|G_m| \rightarrow 0$. We finally set $U_m^{(\varepsilon)} = U_\varepsilon$ on $\mathbf{C}^{n+1} \setminus \cup \mathcal{L}_m^{(n+1)}$, as required. \square

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