A mixed Boltzmann–BGK model for inert gas mixtures

Original
A mixed Boltzmann–BGK model for inert gas mixtures / Bisi, Marzia; Groppi, Maria; Lucchin, Enrico; Martalo, Giorgio. - In: KINETIC AND RELATED MODELS. - ISSN 1937-5093. - 17:5(2024), pp. 674-696. [10.3934/krm.2023037]

Publisher:
AMER INST MATHEMATICAL SCIENCES-AIMS

Published
DOI:10.3934/krm.2023037

Manuscript submitted to AIMS Journal

doi:10.3934/xx.xxxxxxx

A MIXED BOLTZMANN–BGK MODEL FOR INERT GAS MIXTURES

Marzia Bisi[1], Maria Groppi[1], Enrico Lucchin[1] and Giorgio Martalò*[1]

[1]Department of Mathematical, Physical and Computer Sciences, University of Parma,
Parco Area delle Scienze 53/A, 43124 Parma, Italy

(Communicated by Handling Editor)

Abstract. We propose a mixed Boltzmann–BGK model for mixtures of monatomic gases, where some kinds of collisions are described by bi–species Boltzmann operators and the others by the binary BGK terms given in [Bobylev et al., *Kinetic and Related Models* **11** (2018)], that is the relaxation model for mixtures with the closest structure to the Boltzmann one. At first we assume that collisions occurring within the same species (intra-species) are modelled by Boltzmann operators, while interactions between different constituents (inter-species) are described by BGK terms. This option allows to rigorously derive hydrodynamic equations not only in the classical collision dominated regime, but also in situations with intra–species collisions playing the dominant role (as in mixtures with very disparate particle masses). Then, we present a general form of this mixed Boltzmann–BGK model, characterized by further parameters allowing to select which binary interactions have to be described by Boltzmann integrals or by BGK operators. We prove that this model preserves conservations of global momentum and energy, positivity of all temperatures, as well as the validity of Boltzmann H-theorem, allowing to conclude that the unique admissible equilibrium state is the expected Maxwellian distribution with all species sharing a common mean velocity and a common temperature.

1. **Introduction.** The kinetic theory approach to gas mixtures has gained much interest in last decades, and the Boltzmann description has been generalized also to reacting mixtures [46] or to polyatomic constituents [22, 25, 27]. As concerns inert mixtures of monatomic gases, Boltzmann equations for species distribution functions are well known in the literature [18, 19], but the investigation of their mathematical properties is still in progress (see for instance some recent results in [1, 15, 24]), as well as the construction of effective numerical schemes for them [12, 20, 51, 53]. Since these equations are quite awkward to deal with, several approximations have been proposed, mainly of BGK type, generalizing the relaxation model proposed in [3] for a single gas. The first mathematically rigorous BGK model built up in [2] assumes that the relaxation towards the equilibrium state is governed by a unique relaxation operator for each species, where the auxiliary Maxwellian attractor is able to take into account the effects due to interactions with all other constituents. The effectiveness of this model, in spite of its simple structure, has motivated various generalizations to reacting mixtures [5, 7, 28], even of

---

2020 *Mathematics Subject Classification.* Primary: 76P05, 82C40.

Key words and phrases. Kinetic theory, Gas mixtures, BGK approximation, Mixed Boltzmann–BGK model, Euler equations.

*Corresponding author: Giorgio Martalò.

monatomic and polyatomic gases [10]. On the other hand, consistent BGK models involving suitable sums of relaxation operators have been also built up, taking into account more details of the binary collision processes occurring in inert gas mixtures [30, 34, 37], and some models in this respect are available also in reactive frames [16, 35].

In the recent paper [11], analogies and differences between existing BGK models for inert mixtures have been commented on, and a new BGK approximation has been built up, with collision operators defined by proper sums of relaxation terms, each one reproducing the exchange rates for momentum and energy of the corresponding binary Boltzmann operator. This correspondence between BGK and Boltzmann rates is exact in case of Maxwell molecule interactions, but a suitable approximation allows to obtain consistent auxiliary parameters (characterizing Maxwellian attractors of BGK operators) even for general intermolecular potentials, widening thus the applicability of the BGK approximation.

In this paper we aim at combining in one model the BGK and the Boltzmann descriptions, and we propose a mixed Boltzmann–BGK model for inert mixtures of monatomic gases, where some kinds of interactions are described by means of classical bi-species Boltzmann operators, and the others by means of the binary BGK operators introduced in [11]. Some preliminary attempts to combine collisional models of Boltzmann and BGK–type, for numerical purposes, can be found in [36, 41], with the aim of balancing computational cost and accuracy. It is well–known indeed that the numerical methods for multi–species Boltzmann equations are computationally expensive (see for instance [30] and the references therein). On the other hand, BGK approximations are not capable to reproduce peculiar nonlinear effects, especially in the evolution of high order moments in mixtures with heavy and light components [29]. This is also due to the fact that BGK operators do not include all the details of the different intermolecular potentials [50]. Taking into account pros and cons of Boltzmann and BGK approaches, we would like to preserve as much as possible the details of collision parameters and intermolecular potentials as in the Boltzmann description, but keeping the kinetic model manageable from the analytical and the numerical points of view. For this reason, we try to replace some of the integral Boltzmann operators by their simpler BGK approximation, which takes into account the collision process only through the Maxwellian attractor and the collision frequency. Among the many BGK models available for inert mixtures, we use the one of Ref. [11] since it has the same structure as the Boltzmann model and simply replaces each bi–species Boltzmann operator by a proper relaxation term. In addition, contrary to other BGK approaches, this model allows to relate its parameters to each binary interaction potential, preserving a more physically realistic description of the collisions. It seems thus well suited for a hybrid kinetic model where only some bi–species interactions are described by relaxation–type operators.

We present at first a model with intra–species collisions (involving particles of the same species) described by means of Boltzmann operators, and inter–species interactions (involving particles of different constituents) by means of BGK terms. We show the consistency of the model for general intermolecular potentials, proving in particular the validity of the Boltzmann H–theorem, allowing us to recover also the correct Maxwellian equilibria of the classical Boltzmann description. The proof of the entropy dissipation estimate is based on the fact that it holds separately for each single–species Boltzmann operator and for each pair of BGK operators

$Q_{sr}(f_s) + Q_{rs}(f_r)$ relevant to interactions (s, r) with $r \neq s$ (where $Q_{sr}(f_s)$ appears in the equation for species s, and $Q_{rs}(f_r)$ in the equation for species r). The choice of the Boltzmann model for intra–species interactions and a proper BGK model for inter–species ones is motivated by the fact that in some physical problems particles of the same species exchange energy faster than particles with disparate masses. This occurs for instance in ε-mixtures of heavy and light gases [23] or in astrophysics problems [49], but also some phenomena in plasma physics are studied neglecting inter–species interactions (as the so–called Z–pinch dynamics, meaning anomalous resistivity and plasma sheath formation [38]). Our description is well suited for these problems since it keeps track of all the interaction details for the dominant single–species collisions and it approximates with relaxation operators only the slower inter–species interactions (loosing thus some microscopic features of such collisions). This structure allows us thus to consider different hydrodynamic closures for practical purposes. In this respect, we show the hydrodynamic closure at Euler level derived from our kinetic model, both in the classical collision dominated regime (leading thus to Euler equations for species number densities, global velocity and temperature of the mixture), and in a situation where intra–species collisions are faster than the others (obtaining Euler equations for species densities, velocities, and temperatures). Details on macroscopic equations in such different asymptotic regimes may be found also in [6] for the full BGK model proposed in [11], and in the references therein. This mixed model could be managed also numerically, since various effective schemes are available for single-species Boltzmann collision operator [14, 40, 45, 48] and for BGK relaxation operators [13, 26, 42]. Some preliminary numerical tests for a hybrid Boltzmann–BGK model for a binary mixture have been recently performed owing to an IMEX finite volume scheme on unstructured meshes [4].

Then, in the second part of the paper we generalize our Boltzmann–BGK model. Specifically, for any pair of species (s, r) with $s, r = 1, \dots, N$, the pertinent collisions may be described by a Boltzmann or a BGK operator: we fix a set of coefficients $\chi_{sr} \in \{0, 1\}$ (with the reasonable symmetry assumption $\chi_{sr} = \chi_{rs}$), and if $\chi_{sr} = 1$ then the (s, r) interactions are modelled by a Boltzmann operator, while if $\chi_{sr} = 0$ by a BGK one. We prove the consistency properties of the model (including the H-theorem) for any choice of coefficients χ_{sr}. The option $\chi_{sr} = 1$, $\forall (s, r)$ reproduces the full Boltzmann model, while the option $\chi_{sr} = 0$, $\forall (s, r)$ yields the BGK model built up in [11]. The general structure of this hybrid kinetic model allows modelling the more significant part of the collision operator by Boltzmann terms (for instance the dominant interactions, or the ones giving rise to specific features of the considered physical problem), and the other ones by means of simpler relaxation terms. For instance, in plasmas the physically admissible scalings are strongly affected by the very large mass ratio between ions and electrons, so that the ions–ions interactions play the dominant role (and should be properly described by Boltzmann operators), whereas ions are almost no affected by electrons [33, 34, 43]. Analogously, when a gaseous mixture is assumed diffusing in a much denser host medium [9, 52], Boltzmann description could be used only for the dominant interactions with the background, leaving other intra- and inter–species interactions modelled by BGK terms.

In more detail, the article is organized as follows. In Section 2 we summarize the main features of the Boltzmann description for gas mixtures and of the BGK–type approximation. Then, in Section 3 we propose a mixed Boltzmann–BGK model with intra–species collisions described by Boltzmann operators and inter–species interactions modelled through binary BGK operators of the type used in [11]; we prove the preservation of Boltzmann collision invariants and of Maxwellian equilibria, as well as the validity of the H-theorem. In Section 4 we show the Euler closure of this kinetic model in two different hydrodynamic regimes: a situation where all collisions are dominant, and a regime where the dominant role is played by intra-species collisions only (described by Boltzmann operators). Finally, in Section 5 we present a general form of our Boltzmann-BGK model, where each different kind of binary interactions may be described by a Boltzmann or by a BGK operator, and we prove its consistency. Section 6 contains some concluding remarks and perspectives.

2. Introduction.

In kinetic theory, the description of a mixture of N monatomic gases is based on evolution equations for the distribution functions of each constituent, here denoted by $f_s = f_s(\mathbf{x}, \mathbf{v}, t)$ $(s = 1, \dots, N)$, depending on position $\mathbf{x} \in \mathbb{R}^3$, velocity $\mathbf{v} \in \mathbb{R}^3$ and time $t \in \mathbb{R}_+$. The most common approach is the Boltzmann one, that will be rapidly summarized in Subsection 2.1, giving rise to a set of N coupled nonlinear integro–differential evolution equations. Simpler approximations are given by the BGK models, and in Subsection 2.2 we describe the main features of a recent model proposed and investigated in [11] for the same inert mixture.

2.1. The Boltzmann formulation.

In the Boltzmann description of the dynamics of rarefied gas mixtures, each distribution function f_s is governed by the following integro-differential equation [18]

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s = \hat{Q}_s(\mathbf{f}) = \sum_{r=1}^{N} \hat{Q}_{sr}(f_s, f_r) \,, \qquad s = 1, \dots, N \,; \tag{1}$$

on the right hand side, $\mathbf{f} = (f_1, f_2, \dots, f_N)$ and the collision term is sum of binary contributions given by

$$\hat{Q}_{sr}(f_s, f_r) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} g_{sr}(|\mathbf{y}|, \hat{\mathbf{y}} \cdot \boldsymbol{\omega}) \left[f_s(\mathbf{v}'_{sr}) f_r(\mathbf{w}'_{sr}) - f_s(\mathbf{v}) f_r(\mathbf{w}) \right] d\mathbf{w} d\boldsymbol{\omega}, \tag{2}$$

where g_{sr} is the collision cross section depending on the pre-collision relative speed $|\mathbf{y}| = |\mathbf{v} - \mathbf{w}|$ and on the angle between its direction $\hat{\mathbf{y}} = \mathbf{y}/|\mathbf{y}|$ and the direction of post-collision relative velocity $\boldsymbol{\omega} \in \mathbb{S}^2$. The cross section takes into account the interaction potential of molecules: for instance, to inverse–power intermolecular potentials of the kind $V_{sr}(d) = d^{-p}$ (with d denoting the intermolecular distance and $p > 1$) there correspond cross sections of the form $g_{sr} = |\mathbf{y}|^{1-4/p}\, \bar{g}_{sr}(\hat{\mathbf{y}} \cdot \boldsymbol{\omega})$; the particular option $p = 4$, in which the dependence on the relative speed disappears, gives the so–called Maxwell molecules. Owing to conservation of momentum and kinetic energy in each binary (elastic) collision, it is easy to check that post-collision velocities $(\mathbf{v}'_{sr}, \mathbf{w}'_{sr})$ are related to the pre-collision ones $(\mathbf{v}, \mathbf{w})$ as follows

$$\begin{aligned} \mathbf{v}'_{sr} &= \frac{m_s}{m_s + m_r} \mathbf{v} + \frac{m_r}{m_s + m_r} \mathbf{w} + \frac{m_r}{m_s + m_r} |\mathbf{y}| \boldsymbol{\omega} \\ \mathbf{w}'_{sr} &= \frac{m_s}{m_s + m_r} \mathbf{v} + \frac{m_r}{m_s + m_r} \mathbf{w} - \frac{m_s}{m_s + m_r} |\mathbf{y}| \boldsymbol{\omega} \,. \end{aligned} \tag{3}$$

We can observe that the Boltzmann term reduces to the classical integral operator for a single gas, when $r = s$. If we denote by $< \cdot, \cdot >$ the usual L^2–product

$$< h, k > = \int_{\mathbb{R}^3} h(\mathbf{v})\, k(\mathbf{v})\, d\mathbf{v} \,, \tag{4}$$

it can be easily checked, by exchanging pre- and post–collision velocities, that the weak form of each Boltzmann operator (2) may be cast as

$$< \hat{Q}_{sr}(f_s, f_r), \varphi_s(\mathbf{v}) > = -\frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} g_{sr}(|\mathbf{y}|, \hat{\mathbf{y}} \cdot \boldsymbol{\omega}) \left[\varphi_s(\mathbf{v}'_{sr}) - \varphi_s(\mathbf{v})\right] \times \left[f_s(\mathbf{v}'_{sr}) f_r(\mathbf{w}'_{sr}) - f_s(\mathbf{v}) f_r(\mathbf{w})\right] d\mathbf{v} d\mathbf{w} d\boldsymbol{\omega}\,, \tag{5}$$

where $\varphi_s(\mathbf{v})$ denotes a general molecular property.

Species number densities n_s, mean velocities $\mathbf{u}_s$, and absolute temperatures T_s (incorporating the Boltzmann constant K_B) may be recovered as suitable moments of s–th distribution function as

$$n_s = \int_{\mathbb{R}^3} f_s(\mathbf{v})\, d\mathbf{v}\,, \quad \mathbf{u}_s = \frac{1}{n_s} \int_{\mathbb{R}^3} \mathbf{v}\, f_s(\mathbf{v})\, d\mathbf{v}\,, \quad T_s = \frac{m_s}{3\, n_s} \int_{\mathbb{R}^3} |\mathbf{v} - \mathbf{u}_s|^2\, f_s(\mathbf{v})\, d\mathbf{v}\,, \tag{6}$$

where m_s is the species particle mass; species mass density will be denoted by $\rho_s = m_s\, n_s$. Total number density n, mass density ρ, global mass velocity $\mathbf{u}$, and global temperature T are provided by the following combinations of species fields

$$n = \sum_{s=1}^{N} n_s\,, \qquad \rho = \sum_{s=1}^{N} m_s\, n_s\,, \qquad \mathbf{u} = \sum_{s=1}^{N} m_s\, n_s\, \mathbf{u}_s\,,$$
$$T = \frac{1}{n} \sum_{s=1}^{N} n_s\, T_s + \frac{1}{3\, n} \sum_{s=1}^{N} m_s\, n_s\, |\mathbf{u}_s - \mathbf{u}|^2\,. \tag{7}$$

Equilibrium states for the set of Boltzmann equations (1) are given by Maxwellian distributions with all species sharing a common mean velocity $\mathbf{u}$ and a common temperature T:

$$\mathcal{M}_s = \mathcal{M}_s(\mathbf{v}; n_s, \mathbf{u}, T) = n_s \left(\frac{m_s}{2\pi T}\right)^{3/2} \exp\left[-\frac{m_s}{2T} |\mathbf{v} - \mathbf{u}|^2\right]. \tag{8}$$

Asymptotic stability of such equilibrium configurations in space homogeneous conditions may be deduced from the Boltzmann H–theorem. Indeed, setting the H–functional

$$\mathcal{H}(\mathbf{f}) = \sum_{s=1}^{N} \int_{\mathbb{R}^3} f_s(\mathbf{v}) \log\left(f_s(\mathbf{v})\right) d\mathbf{v}\,, \tag{9}$$

it can be proved that $\frac{d\mathcal{H}}{dt} \leq 0$ along each space homogeneous solution of the Boltzmann system (1), and $\mathcal{H}(\mathbf{f}) > \mathcal{H}(\mathbf{M})$ for any $\mathbf{f} \neq \mathbf{M}$, where $\mathbf{M} = (\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_N)$ denotes the (unique, in space homogeneous setting) equilibrium state corresponding to the set of distributions $\mathbf{f}$.

2.2. **A BGK–type approximation.** Due to the presence of integral nonlinear operators in the Boltzmann equations, some alternative simpler formulations can be used for the description of gas mixtures. The most famous one is the BGK approximation, built up in the lines established in [3] for a single gas, where the Boltzmann terms are replaced by suitable operators prescribing the relaxation of the distribution functions towards Maxwellian states. The more general structure of the BGK evolution equation for the s−th component is

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s = Q_s(f_s) = \sum_{r=1}^{N} Q_{sr}(f_s)\,, \quad s = 1, \ldots, N\,, \tag{10}$$

where each binary relaxation term takes the form

$$Q_{sr}(f_s) = \nu_{sr}(\mathcal{M}_{sr} - f_s)\,. \tag{11}$$

Here, ν_{sr} stands for a relaxation frequency to be properly determined according to the original Boltzmann interaction rates. The attractor $\mathcal{M}_{sr}$ is a suitable Maxwellian function

$$\mathcal{M}_{sr} = \mathcal{M}_{sr}(\mathbf{v}; n_{sr}, \mathbf{u}_{sr}, T_{sr}) = n_{sr}\left(\frac{m_s}{2\pi T_{sr}}\right)^{3/2} \exp\left[-\frac{m_s}{2T_{sr}}\left|\mathbf{v} - \mathbf{u}_{sr}\right|^2\right], \tag{12}$$

depending on fictitious quantities n_{sr}, $\mathbf{u}_{sr}$ and T_{sr}.

Such auxiliary fields are $5\,N^2$ disposable parameters that can be expressed in terms of species densities, velocities and temperatures in order to make the BGK model (10) a good approximation of the Boltzmann model (1). A requirement to be fulfilled by any kinetic model for inert gas mixtures is the conservation of species number densities, global momentum and kinetic energy; this provides $N + 4$ constraints on the fictitious quantities. Since the number of constraints is lower than the number of free parameters, the generalization of the classical BGK model to a gas mixture is not unique, and several consistent BGK models have been built up in the literature. The simpler approach, proposed in [2] for inert mixtures, consists in assuming that $\mathbf{u}_{sr} = \tilde{\mathbf{u}}$ and $T_{sr} = \tilde{T}$ (while conservation of species densities, in any case, implies $n_{sr} = n_s$), so that a unique relaxation operator appears in each equation (10), and explicit expressions for $\tilde{\mathbf{u}}$ and $\tilde{T}$ are then recovered imposing preservation of total momentum and energy. This way of modelling has been successfully generalized to reacting monatomic species [7] and to mixtures of monatomic and polyatomic gases [10]. However, even models with a sum of BGK operators as in (10) available in the literature [30, 34] can be put in this setting, and are able to take into account not only global conservations, but also some more details of bi–species exchanges.

The BGK description [11] also fits into this line, preserving the same structure of the Boltzmann operators as sum of binary contributions. It assumes at first that when $r = s$ the usual BGK operator for a single gas is reproduced, namely

$$n_{ss} = n_s\,, \qquad \mathbf{u}_{ss} = \mathbf{u}_s\,, \qquad T_{ss} = T_s\,. \tag{13}$$

Then, the other auxiliary fields $\mathbf{u}_{sr}$ and T_{sr} (for $s \neq r$) are determined imposing that each bi–species BGK operator preserves the corresponding bi–species Boltzmann exchange rates of momentum and energy, for general intermolecular potentials. This BGK model will be used in the construction of our mixed Boltzmann–BGK description, and further details about coefficients will be given below in Section 3. Various strategies have been proposed in the literature in order to fix proper relaxation frequencies ν_{sr} for various BGK models [4, 8, 28]. A recent review of existing BGK models for gas mixtures of monatomic and polyatomic particles may be found in [44].

3. **A mixed Boltzmann–BGK model.** We propose a hybrid Boltzmann–BGK description that combines the positive features of the two formulations, useful to describe gas mixtures governed by a two–scale collision process. The main idea is to model the dominant phenomenon by the more accurate Boltzmann term, and the remaining process by means of a more manageable relaxation operator. In this section, we present in detail a model in which collisions among particles of the same species (intra–species collisions) are described by a one–species Boltzmann operator,

while collisions among particles of different constituents (inter–species collisions) are modelled by binary BGK operators as the ones built up in [11]. We will prove the well–posedness of this hybrid model, namely fulfillment of conservation laws and of Boltzmann H-theorem, as well as the preservation of correct Maxwellian equilibria. This model with different kinetic operators for intra–species and inter–species collisions will allow us to investigate, in Section 4, not only the classical collision dominated regime but also a situation in which the dominant process is constituted by intra–species collisions only. This regime could be meaningful in problems involving constituents with very disparate masses (for instance in plasma physics, with ions and electrons), since energy exchanges among particles with very different masses may occur more slowly [23]. In Section 5 we will present a general form of this hybrid kinetic model, with different choices for pair interactions to be described by Boltzmann or by BGK operators, allowing thus to study various hydrodynamic regimes.

We consider here the following integro-differential equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s = \tilde{Q}_s(f_s) = \hat{Q}_{ss}(f_s, f_s) + \sum_{\substack{r=1\\ r\neq s}}^{N} Q_{sr}(f_s)\,, \qquad s = 1,\dots,N\,, \tag{14}$$

where the collisions between molecules of the same component are modeled by Boltzmann terms $\hat{Q}_{ss}(f_s, f_s)$ (see equation (2)), and the interactions between different species are described by BGK terms $Q_{sr}(f_s)$ (see equation (11)). In order to explicitly express the auxiliary parameters in the relaxation contributions, for $r \neq s$, we impose, as done in [11], the conservation of species densities in each collision, and that momentum and energy exchanges for BGK and Boltzmann formulations coincide, i.e.

$$\begin{gathered} < Q_{sr}, 1 > = < \hat{Q}_{sr}, 1 > = 0\,, \qquad < Q_{sr} - \hat{Q}_{sr}, \mathbf{v} > = \mathbf{0}\,, \\ < Q_{sr} - \hat{Q}_{sr}, |\mathbf{v}|^2 > = 0\,, \qquad r \neq s\,. \end{gathered} \tag{15}$$

From the conservation of number densities (first condition in (15)) we easily obtain that for each $s = 1,\dots,N$

$$n_{sr} = n_s\,, \qquad r = 1,\dots,N\,, \quad r \neq s\,. \tag{16}$$

The constraints on momentum and energy exchange rates (second and third conditions in (15)) are much more complicated. Indeed, while BGK contributions may be explicitly computed, providing

$$\begin{aligned} < Q_{sr}, \mathbf{v} > &= \nu_{sr}\, n_s \left(\mathbf{u}_{sr} - \mathbf{u}_s\right), \\ < Q_{sr}, |\mathbf{v}|^2 > &= \nu_{sr}\, n_s \left(3\,\frac{T_{sr} - T_s}{m_s} + |\mathbf{u}_{sr}|^2 - |\mathbf{u}_s|^2\right), \end{aligned} \tag{17}$$

the corresponding Boltzmann moments involve angular integrals of the kind

$$\lambda_{sr}(|\mathbf{y}|)\,\hat{\mathbf{y}} = \int_{\mathbb{S}^2} (\hat{\mathbf{y}} - \boldsymbol{\omega})\, g_{sr}(|\mathbf{y}|, \hat{\mathbf{y}}\cdot\boldsymbol{\omega})\, d\boldsymbol{\omega}\,,$$

that cannot be explicitly calculated for general intermolecular potentials, except for Maxwell molecules, for which λ_{sr} are constant. For other potentials, the authors in [11] propose to approximate each $\lambda_{sr}(|\mathbf{y}|)$ by its value in some typical point, namely $\lambda_{sr}(|\mathbf{y}|) \cong \lambda_{sr}(\bar{y})$, where $\bar{y}$ is an average of the relative speed defined as

$$\begin{aligned}\bar{y} &= \left(\frac{1}{n_s\, n_r} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{w}|^2\, f_s(\mathbf{v})\, f_r(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w} \right)^{1/2} \\ &= \left[3 \left(\frac{T_s}{m_s} + \frac{T_r}{m_r} \right) + |\mathbf{u}_s - \mathbf{u}_r|^2 \right]^{1/2} .\end{aligned}$$

With this approximation, the computation of the Boltzmann exchange rates needed in (15) is similar to that performed in [2] (see also [11] for further details), and it leads to

$$\begin{aligned}&< \hat{Q}_{sr}, \mathbf{v} > = -\frac{m_r}{m_s+m_r}\, \lambda_{sr}\, n_s\, n_r (\mathbf{u}_s - \mathbf{u}_r) \\ &< \hat{Q}_{sr}, |\mathbf{v}|^2 > = -\frac{2\, m_r}{(m_s+m_r)^2}\, \lambda_{sr}\, n_s\, n_r \Big[3(T_s - T_r) + (m_s\, \mathbf{u}_s + m_r\, \mathbf{u}_r) \cdot (\mathbf{u}_s - \mathbf{u}_r) \Big].\end{aligned} \tag{18}$$

By equating these results with the corresponding BGK exchange rates in (17) we get the following explicit expressions for auxiliary parameters:

$$\mathbf{u}_{sr} = (1 - a_{sr})\mathbf{u}_s + a_{sr}\mathbf{u}_r \,, \tag{19}$$

where

$$a_{sr} = \frac{\lambda_{sr}\, m_r\, n_r}{\nu_{sr}\, (m_s + m_r)} \,,$$

and

$$T_{sr} = (1 - b_{sr})T_s + b_{sr}T_r + \gamma_{sr}|\mathbf{u}_s - \mathbf{u}_r|^2 \,, \tag{20}$$

where

$$b_{sr} = 2\, a_{sr} \frac{m_s}{m_s + m_r} \qquad \text{and} \qquad \gamma_{sr} = \frac{1}{3} m_s a_{sr} \left(\frac{2\, m_r}{m_s + m_r} - a_{sr} \right) .$$

The factors λ_{sr} are related to the Boltzmann cross sections as described above, and they are symmetric with respect to their indices.

It can be proved that the auxiliary temperatures defined in (20) are positive if relaxation frequencies ν_{sr} are large enough.

Proposition 3.1. *The fictitious temperatures T_{sr} provided in (20) are well defined, i.e. $T_{sr} > 0$, if relaxation frequencies fulfill the constraint*

$$\nu_{sr} \geq \frac{1}{2} \lambda_{sr} n_r \tag{21}$$

for each $s, r = 1, \ldots, N$, $r \neq s$.

Proof. By definition, it follows that

$$a_{sr} > 0 \,, \qquad b_{sr} > 0 \,; \tag{22}$$

moreover, under the condition (21), one has

$$\begin{aligned} b_{sr} &= 2\, \frac{\lambda_{sr}}{\nu_{sr}}\, n_r\, \frac{m_s\, m_r}{(m_s+m_r)^2} \leq 4\, \frac{m_s\, m_r}{(m_s+m_r)^2} = 4\, \frac{m_s}{m_s+m_r} \left(1 - \frac{m_s}{m_s+m_r} \right) \\ &\leq 4 \cdot \frac{1}{4} = 1 \,, \end{aligned} \tag{23}$$

and

$$\gamma_{sr} = \frac{1}{3} m_s a_{sr} \left(\frac{2\, m_r}{m_s + m_r} - \frac{\lambda_{sr}\, m_r\, n_r}{\nu_{sr}\, (m_s + m_r)} \right) \geq 0 \,. \tag{24}$$

Hence, the positiveness of the fictitious temperatures trivially follows. ☐

3.1. **Consistency of the mixed model.** We show now that the mixed description introduced above satisfies the main properties of consistency; more precisely, we prove

A) conservation of species masses, global momentum and energy, i.e.

$$
\begin{aligned}
&< \hat{Q}_{ss}, 1 >= 0\,, \quad s = 1, \ldots, N\,, \qquad < Q_{sr}, 1 >= 0\,, \quad s, r = 1, \ldots, N\,, \quad r \neq s\,,\\
&\sum_{s=1}^{N} m_s < \tilde{Q}_s, \mathbf{v} >= \mathbf{0}\,, \qquad \sum_{s=1}^{N} m_s < \tilde{Q}_s, |\mathbf{v}|^2 >= 0\,;
\end{aligned}
\tag{25}
$$

B) the validity of Boltzmann H-theorem, and in particular of the entropy dissipation, i.e.

$$\sum_{s=1}^{N} < \tilde{Q}_s, \log f_s >\leq 0\,; \tag{26}$$

C) preservation of Boltzmann equilibrium distribution functions, i.e.

$$\tilde{Q}_s = 0\,, \quad s = 1, \ldots, N \quad \Longleftrightarrow \quad f_s = \mathcal{M}_s = n_s \left(\frac{m_s}{2\pi T} \right)^{3/2} \exp\left[-\frac{m_s}{2T} |\mathbf{v} - \mathbf{u}|^2 \right], \tag{27}$$

where $\mathbf{u}$ and T are the global mean velocity and temperature, respectively.

In the following, Proposition 3.2 will be devoted to the proof of item A), Proposition 3.3 to the proof of item B), and Proposition 3.5 to the proof of item C).

Proposition 3.2. *The conservation laws (25) hold for the mixed model (14).*

Proof. The conservation of species number densities trivially follows from (15).

As concerns the conservation of momentum, one has to prove that

$$\sum_{s=1}^{N} < \hat{Q}_{ss}, m_s \mathbf{v} > + \sum_{s=1}^{N} \sum_{\substack{r=1\\ r\neq s}}^{N} < Q_{sr}, m_s \mathbf{v} >= \mathbf{0}\,; \tag{28}$$

we can observe that each contribution in the first sum vanishes, since $\varphi_s(\mathbf{v}) = m_s \mathbf{v}$ is a collision invariant for the single gas Boltzmann operator. The remaining terms can be rewritten as follows

$$
\begin{aligned}
&\sum_{s=1}^{N} \sum_{\substack{r=1\\ r>s}}^{N} (< Q_{sr}, m_s \mathbf{v} > + < Q_{rs}, m_r \mathbf{v} >)\\
&= \sum_{s=1}^{N} \sum_{\substack{r=1\\ r>s}}^{N} \left(-\lambda_{sr} \frac{m_s m_r}{m_s + m_r} n_s n_r (\mathbf{u}_s - \mathbf{u}_r) + \lambda_{rs} \frac{m_s m_r}{m_s + m_r} n_s n_r (\mathbf{u}_s - \mathbf{u}_r) \right) = \mathbf{0}\,,
\end{aligned}
\tag{29}
$$

where the first equality follows from (18) and we use the symmetry property $\lambda_{rs} = \lambda_{sr}$ to conclude.

Similar arguments are used to prove the conservation of total energy, leading to

$$\begin{aligned}
&\sum_{s=1}^{N}\sum_{\substack{r=1\\ r>s}}^{N}(< Q_{sr}, m_s|\mathbf{v}|^2 > + < Q_{rs}, m_r|\mathbf{v}|^2 >)\\
&= \sum_{s=1}^{N}\sum_{\substack{r=1\\ r>s}}^{N}\Big[-\lambda_{sr}\,\frac{2\,m_s\,m_r}{(m_s+m_r)^2}\,n_s\,n_r\Big[3(T_s-T_r)+(m_s\,\mathbf{u}_s+m_r\,\mathbf{u}_r)\cdot(\mathbf{u}_s-\mathbf{u}_r)\Big]\\
&+\lambda_{rs}\,\frac{2\,m_s\,m_r}{(m_s+m_r)^2}\,n_s\,n_r\Big[3(T_s-T_r)+(m_s\,\mathbf{u}_s+m_r\,\mathbf{u}_r)\cdot(\mathbf{u}_s-\mathbf{u}_r)\Big]\Big] = 0\,.
\end{aligned} \tag{30}$$

□

We prove now the validity of the Boltzmann H-theorem for our mixed Boltzmann–BGK model.

Proposition 3.3. *In the space homogeneous case, the functional*

$$\mathcal{H}[\mathbf{f}] = \sum_{s=1}^{N} < f_s, \log f_s > = \sum_{s=1}^{N}\int_{\mathbb{R}^3} f_s(\mathbf{v})\,\log\big(f_s(\mathbf{v})\big)\,d\mathbf{v} \tag{31}$$

is a Lyapunov functional, namely $\frac{d\mathcal{H}}{dt} \leq 0$ *along each solution of the mixed Boltzmann–BGK system (14), and* $\mathcal{H}(\mathbf{f}) > \mathcal{H}(\mathbf{M})$ *for any* $\mathbf{f} \neq \mathbf{M}$.

Proof. The Gibbs inequality $\mathcal{H}(\mathbf{f}) > \mathcal{H}(\mathbf{M})$ is a standard result of kinetic theory for gas mixtures [18]. As concerns the entropy dissipation, at first we note that, from (25), in the space homogeneous case we have

$$\frac{d}{dt} < f_s, 1 > = < \frac{df_s}{dt}, 1 > = < \tilde{Q}_s, 1 > = < \hat{Q}_{ss}, 1 > + \sum_{\substack{r=1\\ r\neq s}}^{N} < Q_{sr}, 1 > = 0\,, \tag{32}$$

hence

$$\frac{dn_s}{dt} = 0 \qquad \Longrightarrow \qquad n_s = n_s^{(0)}\,, \tag{33}$$

where $n_s^{(0)}$ is the given initial density of the $s-$th component. We want to prove that

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \frac{d}{dt}\left(\sum_{s=1}^{N} < f_s, \log f_s >\right) = \sum_{s=1}^{N}\left(< \frac{df_s}{dt}, \log f_s > + < f_s, \frac{1}{f_s}\frac{df_s}{dt} >\right)\\
&= \sum_{s=1}^{N}\left(< \frac{df_s}{dt}, \log f_s > + < 1, \frac{df_s}{dt} >\right) = \sum_{s=1}^{N} < \frac{df_s}{dt}, \log f_s >\\
&= \sum_{s=1}^{N} < \hat{Q}_{ss}, \log f_s > + \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} < Q_{sr}, \log f_s >
\end{aligned} \tag{34}$$

is non-positive. As classically done for a single gas, it is easy to observe that

$$\begin{aligned}
< \hat{Q}_{ss}, \log f_s > = -\frac{1}{4}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} g_{ss}(\mathbf{y}, \hat{\mathbf{y}}\cdot\boldsymbol{\omega})\log\left(\frac{f_s(\mathbf{v}'_{ss})f_s(\mathbf{w}'_{ss})}{f_s(\mathbf{v})f_s(\mathbf{w})}\right)\\
\times\left[\frac{f_s(\mathbf{v}'_{ss})f_s(\mathbf{w}'_{ss})}{f_s(\mathbf{v})f_s(\mathbf{w})} - 1\right] f_s(\mathbf{v})f_s(\mathbf{w})d\mathbf{v}d\mathbf{w}d\boldsymbol{\omega} \leq 0\,;
\end{aligned} \tag{35}$$

therefore

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &\le \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} < Q_{sr}, \log f_s >= \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \nu_{sr} < \mathcal{M}_{sr} - f_s, \log f_s > \\ &\le \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \nu_{sr} \left[< \mathcal{M}_{sr}, \log \mathcal{M}_{sr} - 1 > - < f_s, \log f_s - 1 >\right] \\ &= \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \nu_{sr} \left[< \mathcal{M}_{sr}, \log \mathcal{M}_{sr} > - < f_s, \log f_s >\right],\end{aligned} \tag{36}$$

where the second inequality follows from

$$(y-x)\log x \le y(\log y - 1) - x(\log x - 1), \qquad \text{for any } x, y > 0 \tag{37}$$

and in last equality we have taken into account that $< \mathcal{M}_{sr}, 1 >=< f_s, 1 >= n_s$. Now, we can use the classical inequality

$$< f_s, \log f_s > \ge < \mathcal{M}_{ss}, \log \mathcal{M}_{ss} >, \tag{38}$$

where $\mathcal{M}_{ss}$, defined according to (12), is a Maxwellian function having the same moments as f_s; it follows that

$$\frac{d\mathcal{H}}{dt} \le \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \nu_{sr} \left[< \mathcal{M}_{sr}, \log \mathcal{M}_{sr} > - < \mathcal{M}_{ss}, \log \mathcal{M}_{ss} >\right]. \tag{39}$$

We compute explicitly

$$\begin{aligned}< \mathcal{M}_{sr}, \log \mathcal{M}_{sr} > &=< \mathcal{M}_{sr}, \log n_s + \frac{3}{2}\log\left(\frac{m_s}{2\pi T_{sr}}\right) - \frac{m_s}{2T_{sr}}|\mathbf{v} - \mathbf{u}_{sr}|^2 > \\ &= \left[\log n_s + \frac{3}{2}\log\left(\frac{m_s}{2\pi T_{sr}}\right)\right] < \mathcal{M}_{sr}, 1 > - \frac{m_s}{2T_{sr}} < \mathcal{M}_{sr}, |\mathbf{v} - \mathbf{u}_{sr}|^2 > \\ &= n_s\left[\log n_s + \frac{3}{2}\log\left(\frac{m_s}{2\pi T_{sr}}\right)\right] - \frac{m_s}{2T_{sr}} n_s \left(\frac{m_s}{2\pi T_{sr}}\right)^{3/2} \frac{3}{2}\pi^{3/2}\left(\frac{m_s}{2T_{sr}}\right)^{-5/2} \\ &= n_s\left[\log n_s + \frac{3}{2}\log\left(\frac{m_s}{2\pi T_{sr}}\right) - \frac{3}{2}\right],\end{aligned} \tag{40}$$

hence

$$\frac{d\mathcal{H}}{dt} \le -\frac{3}{2}\sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \nu_{sr} n_s \log\frac{T_{sr}}{T_s}. \tag{41}$$

From the expressions of auxiliary temperatures given in (20), we have

$$\begin{aligned}&T_{sr} \ge (1-b_{sr})T_s + b_{sr}T_r \\ \Longrightarrow &\log T_{sr} \ge \log[(1-b_{sr})T_s + b_{sr}T_r] \ge (1-b_{sr})\log T_s + b_{sr}\log T_r,\end{aligned} \tag{42}$$

where last inequality follows from concavity arguments, recalling that $b_{sr} < 1$; by such result, we can rewrite

$$\log\frac{T_{sr}}{T_s} = \log T_{sr} - \log T_s \ge (1-b_{sr})\log T_s + b_{sr}\log T_r - \log T_s = b_{sr}\log\frac{T_r}{T_s}, \tag{43}$$

and

$$\frac{d\mathcal{H}}{dt} \leq -\frac{3}{2}\sum_{s=1}^{N}\sum_{\substack{r=1\\r\neq s}}^{N}\nu_{sr}n_s b_{sr}\log\frac{T_r}{T_s} = -\frac{3}{2}\sum_{s=1}^{N}\sum_{\substack{r=1\\r>s}}^{N}\nu_{sr}n_s b_{sr}\left(\log\frac{T_r}{T_s}+\log\frac{T_s}{T_r}\right) = 0\,, \tag{44}$$

where we have noted that $\nu_{rs}n_r b_{rs} = \nu_{sr}n_s b_{sr}$. This concludes the proof of the entropy dissipation of our mixed Boltzmann–BGK model. □

From the H-theorem we can deduce that our mixed model preserves the same equilibrium distributions as the original Boltzmann model for inert gas mixtures; before entering into the proof, we state the following auxiliary lemma.

Lemma 3.4. *Let $\{\kappa_{ij}\,, i,j = 1,\ldots,N\,, i\neq j\}$ be a set of $N(N-1)$ positive numbers such that $\kappa_{ij} = \kappa_{ji}$ for each admissible pair. Then, the linear system*

$$\sum_{\substack{j=1\\j\neq i}}^{N}\kappa_{ij}(\xi_j-\xi_i) = 0\,,\quad i = 1,\ldots,N \tag{45}$$

admits only the trivial solution $\xi_1 = \ldots = \xi_N = const.$

Proof. We multiply each equation by ξ_i and we sum over i; we obtain

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\kappa_{ij}(\xi_i^2-\xi_i\xi_j) = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\kappa_{ij}(\xi_i-\xi_j)^2 = 0\,. \tag{46}$$

Since all coefficients κ_{ij} are positive, the only admissible solution is $\xi_i - \xi_j = 0$ for each pair of indices. □

Proposition 3.5. *The equilibrium solutions of the Boltzmann–BGK model (14) are provided by Maxwellian functions depending on global mean velocity and temperature, i.e.*

$$\tilde{Q}_s = 0\,,\quad s = 1,\ldots,N \quad\Longleftrightarrow\quad f_s = \mathcal{M}_s = n_s\left(\frac{m_s}{2\pi T}\right)^{3/2}\exp\left[-\frac{m_s}{2T}\left|\mathbf{v}-\mathbf{u}\right|^2\right]. \tag{47}$$

Proof. Firstly we observe that one of the two implications is trivial: if the Maxwellian distribution functions share the same mean velocity and temperature then each term in $\tilde{Q}_s$ vanishes. For the reverse implication, we remark that in the proof of Proposition 3.3 we have shown that $\frac{d\mathcal{H}}{dt}$ is a sum of non–negative terms

$$\frac{d\mathcal{H}}{dt} = \sum_{s=1}^{N}\underbrace{<\hat{Q}_{ss},\log f_s>}_{\leq 0} + \underbrace{\sum_{s=1}^{N}\sum_{\substack{r=1\\r\neq s}}^{N}<Q_{sr},\log f_s>}_{\leq 0}\,. \tag{48}$$

If $\mathbf{f} = (f_1,\ldots,f_N)$ is a stationary state, then $\frac{d\mathcal{H}}{dt}$ must vanish hence, in particular, each term coming from Boltzmann operators must be zero. Then, by using the classical arguments for a single gas, it can be proved that each distribution at equilibrium is a Maxwellian function depending on its own velocity and temperature

$$f_s^{\mathrm{eq}} = n_s \left(\frac{m_s}{2\pi T_s} \right)^{3/2} \exp \left[-\frac{m_s}{2T_s} \left| \mathbf{v} - \mathbf{u}_s \right|^2 \right] . \tag{49}$$

We multiply now $\tilde{Q}_s = 0$ by the test function $m_s \mathbf{v}$, and we obtain

$$< \tilde{Q}_s, m_s \mathbf{v} >=< \hat{Q}_{ss}, m_s \mathbf{v} > + \sum_{\substack{r=1 \\ r\neq s}}^{N} < Q_{sr}, m_s \mathbf{v} >= \sum_{\substack{r=1 \\ r\neq s}}^{N} \mathcal{C}_{sr} (\mathbf{u}_r - \mathbf{u}_s) = \mathbf{0} , \tag{50}$$

where $\mathcal{C}_{sr} = \lambda_{sr} \dfrac{m_s m_r}{m_s + m_r} n_s n_r > 0$ (see (29)). By Lemma 3.4, we can conclude that $\mathbf{u}_1 = \ldots = \mathbf{u}_N = \mathbf{u}$. Using this last result, one can rewrite

$$< \tilde{Q}_s, m_s |\mathbf{v}|^2 >=< \hat{Q}_{ss}, m_s |\mathbf{v}|^2 > + \sum_{\substack{r=1 \\ r\neq s}}^{N} < Q_{sr}, m_s |\mathbf{v}|^2 >= 3 \sum_{\substack{r=1 \\ r\neq s}}^{N} \mathcal{D}_{sr} (T_r - T_s) = 0 , \tag{51}$$

where $\mathcal{D}_{sr} = 2\mathcal{C}_{sr}/(m_s + m_r) > 0$ (see (30)), and the equalization of species temperatures trivially follows from Lemma 3.4. $\square$

4. **Euler hydrodynamic limits.** In this section we derive from our Boltzmann–BGK model a fluid–dynamic description of the evolution of a gas mixture in two different hydrodynamic regimes. To this aim, we first present the dimensionless form of the kinetic equations (14). We fix some reference values m_0, n_0, T_0, ν_0 for particle mass, number density, temperature and collision frequency, respectively, and we introduce the following scaling for time, space, velocity variables and for distribution functions

$$\tilde{t} = \nu_0 t , \qquad \tilde{\mathbf{x}} = \frac{\nu_0 \mathbf{x}}{\sqrt{T_0/m_0}} , \qquad \tilde{\mathbf{v}} = \frac{\mathbf{v}}{\sqrt{T_0/m_0}} , \qquad \tilde{f}_s = \left(\frac{T_0}{m_0} \right)^{3/2} \frac{f_s}{n_0} , \tag{52}$$

and for macroscopic fields

$$\tilde{n}_s = \frac{n_s}{n_0} , \qquad \tilde{\mathbf{u}}_s = \frac{\mathbf{u}_s}{\sqrt{T_0/m_0}} , \qquad \tilde{T}_s = \frac{T_s}{T_0} . \tag{53}$$

As concerns collision frequencies and cross sections, we scale such quantities by means of two parameters: α, useful to distinguish different hydrodynamic regimes, and the classical Knudsen number ε, giving the ratio between microscopic and macroscopic length scales; more precisely, we introduce

$$\tilde{\nu}_{sr} = \frac{\varepsilon}{\alpha \nu_0} \nu_{sr} , \qquad \tilde{g}_{sr} = \frac{\varepsilon n_0}{\nu_0} g_{sr} , \qquad \tilde{\lambda}_{sr} = \frac{\varepsilon n_0}{\alpha \nu_0} \lambda_{sr} . \tag{54}$$

Substituting these quantities in (14) and omitting all tildes for sake of simplicity, we obtain the following dimensionless kinetic equations

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s = \frac{1}{\varepsilon} \hat{Q}_{ss}(f_s, f_s) + \frac{\alpha}{\varepsilon} \sum_{\substack{r=1 \\ r\neq s}}^{N} Q_{sr}(f_s) , \qquad s = 1, \ldots, N . \tag{55}$$

We are interested in discussing two different asymptotic regimes: the classical hydrodynamic limit with all dominant collisions (corresponding to $\alpha = 1$), that will be dealt with in Subsection 4.1, and the case with dominant intra-species interactions (corresponding to $\alpha = \varepsilon$), meaningful for instance in gas mixtures with very disparate masses [23, 31, 39], that will be investigated in Subsection 4.2. In both cases, use will be made of classical Chapman-Enskog expansion of the distribution functions

$$f_s = f_s^0 + \varepsilon f_s^1 + \dots\,, \tag{56}$$

leading to similar expansions for macroscopic quantities but, as typical in Chapman-Enskog procedure, with the constraints that the macroscopic fields related to the collision invariants of the dominant operator remain unexpanded [19].

4.1. **Collision dominated regime.** When $\alpha = 1$, at the leading order the set of re-scaled equations (55) reduces to

$$\tilde{Q}_s(f_s^0) = \hat{Q}_{ss}(f_s^0, f_s^0) + \sum_{\substack{r=1\\ r\neq s}}^{N} Q_{sr}(f_s^0) = 0\,, \quad s = 1, \dots, N\,, \tag{57}$$

giving that the zero-th order approximation coincides with the global Maxwellian equilibrium

$$f_s^0 = n_s \left(\frac{m_s}{2\pi T}\right)^{3/2} \exp\left[-\frac{m_s}{2T}\,|\mathbf{v}-\mathbf{u}|^2\right]\,, \quad s = 1, \dots, N\,, \tag{58}$$

with all species sharing a common mean velocity and temperature. Collision invariants for the dominant part of the collision operator (in this regime, the whole operator) are $N+4$, and correspond to species densities, global mean velocity and global temperature. By standard procedure, we get the Euler level of accuracy using the approximation (58) into the set of macroscopic equations that can be deduced taking the weak form of (14) relevant to collision invariants. Then, the evolution of these macroscopic fields turns out to be governed by

$$\begin{aligned}
&\frac{\partial n_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (n_s \mathbf{u}) = 0\,, \qquad s = 1, \dots, N\\
&\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u} + nT\mathbf{I}) = \mathbf{0}\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{3}{2}nT\right) + \nabla_{\mathbf{x}} \cdot \left[\left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{5}{2}nT\right)\mathbf{u}\right] = 0\,,
\end{aligned} \tag{59}$$

with $\mathbf{I}$ denoting the identity matrix. These are the classical Euler equations for inert gas mixtures, describing conservations of mass of each component, of global momentum and of total energy.

4.2. **Dominant intra–species collisions.** When $\alpha = \varepsilon$, at the leading order equations (55) yield

$$\hat{Q}_{ss}(f_s^0, f_s^0) = 0\,, \quad s = 1, \dots, N\,, \tag{60}$$

and the zero-th order approximation is thus constituted by equilibria of single-species Boltzmann operators, namely by Maxwellian functions depending on species macroscopic fields n_s, $\mathbf{u}_s$ and T_s as

$$f_s^0 = n_s \left(\frac{m_s}{2\pi T_s}\right)^{3/2} \exp\left[-\frac{m_s}{2T_s}\,|\mathbf{v}-\mathbf{u}_s|^2\right]\,, \quad s = 1, \dots, N\,. \tag{61}$$

In this regime, the collision invariants of the dominant operators are thus $5\,N$, and correspond to species macroscopic fields, i.e. density, velocity and temperature of each constituent. Their evolution equations, at Euler level, contain not only the streaming terms but also source terms coming from the slow interaction process (inter–species BGK contributions) and taking into account momentum and energy exchanges between different components. The computation is essentially the same already performed for the full BGK model (10)–(11) in [6], and leads to the same results. Precisely, we obtain

$$\begin{aligned}
&\frac{\partial n_s}{\partial t} + \nabla_{\mathbf{x}} \cdot (n_s \mathbf{u}) = 0\,, \qquad s = 1, \dots, N\\
&\frac{\partial(\rho_s \mathbf{u}_s)}{\partial t} + \nabla_{\mathbf{x}} \cdot (\rho_s \mathbf{u}_s \otimes \mathbf{u}_s + n_s T_s \mathbf{I}) = \sum_{\substack{r=1\\ r\neq s}}^{N} \mathbf{R}_{sr}\,,\\
&\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_s |\mathbf{u}_s|^2 + \frac{3}{2} n_s T_s\right) + \nabla_{\mathbf{x}} \cdot \left[\left(\frac{1}{2}\rho_s |\mathbf{u}_s|^2 + \frac{5}{2} n_s T_s\right)\mathbf{u}_s\right] = \sum_{\substack{r=1\\ r\neq s}}^{N} S_{sr}\,,
\end{aligned} \tag{62}$$

where exchange terms are provided by

$$\begin{aligned}
\mathbf{R}_{sr} &= \nu_{sr}^0 \int_{\mathbb{R}^3} m_s\, \mathbf{v} \Big[\mathcal{M}_{sr}^0(\mathbf{v}) - f_s^0(\mathbf{v})\Big] d\mathbf{v}\,,\\
S_{sr} &= \nu_{sr}^0 \int_{\mathbb{R}^3} \frac{1}{2} m_s\, |\mathbf{v}|^2 \Big[\mathcal{M}_{sr}^0(\mathbf{v}) - f_s^0(\mathbf{v})\Big] d\mathbf{v}\,.
\end{aligned}$$

Here, since species fields n_s, $\mathbf{u}_s$ and T_s have to remain unexpanded, the attractors $\mathcal{M}_{sr}$ may depend on the Knudsen parameter ε only through collision frequencies ν_{sr}. Leading order auxiliary parameters read then as

$$\mathbf{u}_{sr}^0 = \left(1 - a_{sr}^0\right) \mathbf{u}_s + a_{sr}^0\, \mathbf{u}_r\,, \qquad T_{sr}^0 = \left(1 - b_{sr}^0\right) T_s + b_{sr}^0\, T_r + \gamma_{sr}^0\, |\mathbf{u}_s - \mathbf{u}_r|^2\,,$$

with coefficients a_{sr}^0, b_{sr}^0, γ_{sr}^0 depending thus on ν_{sr}^0. Simple computations provide the following explicit expressions for the source terms (see [6] for details):

$$\begin{aligned}
\mathbf{R}_{sr} &= -\lambda_{sr} \frac{m_s m_r}{m_s + m_r} n_s n_r (\mathbf{u}_s - \mathbf{u}_r)\,,\\
S_{sr} &= -\,\lambda_{sr} \frac{m_s\, m_r}{(m_s + m_r)^2} n_s n_r \left[3(T_s - T_r) + (m_s \mathbf{u}_s + m_r \mathbf{u}_r) \cdot (\mathbf{u}_s - \mathbf{u}_r)\right]\,.
\end{aligned} \tag{63}$$

Note that when $\mathbf{u}_s = \mathbf{u}_r$ and $T_s = T_r$, $\forall s, r = 1, \dots, N$, namely when the collision equilibrium of the whole Boltzmann–BGK model is reached, the collision contributions $\mathbf{R}_{sr}$ and S_{sr} correctly vanish.

5. **General form of the hybrid kinetic model.** In the previous sections we have proposed and investigated a Boltzmann–BGK model where intra–species collisions are described by Boltzmann operators, and inter–species collisions by BGK operators like the ones built up in [11]. This model has allowed us to study also the asymptotic regime where the dominant role in the evolution is played by intra–species interactions only. In this section we present a more general form of such hybrid kinetic model, where each kind of binary interaction may be alternatively described by means of Boltzmann or BGK terms. The general Boltzmann–BGK model for an inert mixture of N gaseous species reads as

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s = \sum_{r=1}^{N} \Big[\chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1 - \chi_{sr})\, Q_{sr}(f_s)\Big]\,, \quad s = 1, \dots, N\,, \tag{64}$$

where $\hat{Q}_{sr}(f_s, f_r)$ is the bi–species Boltzmann operator defined in (2), and $Q_{sr}(f_s)$ is the BGK operator (11) constructed as described in Section 3; moreover, coefficients $\chi_{sr} \in \{0, 1\}$ and are such that $\chi_{sr} = \chi_{rs}$, $\forall s, r = 1, \dots, N$.

Notice that the option $\chi_{sr} = 1$, $\forall (s,r)$ provides the pure Boltzmann model for inert mixtures, while the option $\chi_{sr} = 0$, $\forall (s,r)$ allows to recover the full BGK model proposed in [11]. The general collision operator in (64) takes into account that, for any pair (s,r), with $s,r = 1,\dots,N$, we can choose to describe the interactions between particles of species s and r by means of a classical Boltzmann operator (keeping all the details of the intermolecular interaction potential), or by means of a simpler BGK operator that, as described in Section 3, introduces some proper approximations. A possible reasonable choice could be to model the dominant collisions (for instance the ones involving particles with much larger mass) with Boltzmann terms, and the others, providing momentum and energy exchanging on a slower time scale, with BGK terms. The hybrid model described in Section 3, distinguishing intra–species and inter–species collisions, may be recovered with the option $\chi_{ss} = 1$, $s = 1,\dots,N$, and $\chi_{sr} = 0$ for any $r \neq s$. In addition, many other choices can be put in this frame, including also less symmetric scenarios, like for instance $\chi_{11} = 1$ and $\chi_{sr} = 0$ for any pair $(s,r) \neq (1,1)$, suitable to describe a mixture in which the component labelled by 1 is much heavier than the others. If two gas components s and r are similar, namely with particle masses of the same order of magnitude and with analogous interaction potentials, one should choose $\chi_{sj} = \chi_{rj}$, $\forall j = 1,\dots,N$; consequently, the particular limiting case where all species become equal should be described only by the full Boltzmann model (with all coefficients equal to 1) or by the full BGK approximation (with all coefficients equal to 0).

We prove now the consistency properties of the general kinetic system (64), namely the fulfillment of conservation laws, the entropy dissipation estimate, and the preservation of Maxwellian equilibria. In order to check the conservations of species number densities, global momentum, and total kinetic energy, we have to show that

$$\sum_{r=1}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), 1 >= 0\,, \quad \forall\, s = 1,\dots,N\,, \tag{65}$$

and

$$\sum_{s=1}^{N}\sum_{r=1}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), \varphi_s(\mathbf{v}) >= 0\,, \tag{66}$$

for $\varphi_s(\mathbf{v}) = m_s\,\mathbf{v}$ and $\varphi_s(\mathbf{v}) = m_s\,|\mathbf{v}|^2$. Since all binary Boltzmann and BGK operators separately preserve mass, equations (65) trivially hold. Equation (66) may be rewritten as

$$\begin{aligned} &\sum_{s=1}^{N} < \chi_{ss}\, \hat{Q}_{ss}(f_s, f_s) + (1-\chi_{ss})\, Q_{ss}(f_s), \varphi_s(\mathbf{v}) > \\ &+ \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), \varphi_s(\mathbf{v}) >= 0\,. \end{aligned} \tag{67}$$

The first sum vanishes, since $\varphi_s(\mathbf{v}) = m_s\,\mathbf{v}$ and $\varphi_s(\mathbf{v}) = m_s\,|\mathbf{v}|^2$ are collision invariants for the single–species Boltzmann operators, and BGK contributions (17) also vanish when $r = s$ (since $\mathbf{u}_{ss} = \mathbf{u}_s$, and $T_{ss} = T_s$). As concerns the second sum, involving interactions between different species, by exchanging indices r and

s and recalling that $\chi_{rs} = \chi_{sr}$ we get

$$\begin{aligned}
&\sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), \varphi_s(\mathbf{v}) > \\
&= \frac{1}{2}\sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} \Big[\chi_{sr}\Big(< \hat{Q}_{sr}(f_s, f_r), \varphi_s(\mathbf{v}) > + < \hat{Q}_{rs}(f_r, f_s), \varphi_r(\mathbf{v}) > \Big) \\
&+ (1-\chi_{sr})\Big(< Q_{sr}(f_s), \varphi_s(\mathbf{v}) > + < Q_{rs}(f_r), \varphi_r(\mathbf{v}) > \Big)\Big].
\end{aligned}$$

Owing to the weak form (5) of the Boltzmann operators, we see that Boltzmann contributions vanish due to conservations of momentum and energy in each collision; even BGK contributions vanish as already proved in (29)–(30), therefore also the second sum in (67) is equal to zero, concluding thus the proof of conservation laws.

In order to verify the validity of Boltzmann H-theorem as stated in Proposition 3.3, we have to prove that along any solution of the mixed model (64) in space homogeneous conditions we have $\dfrac{d\mathcal{H}}{dt} \leq 0$. Similarly to before, it can be checked that

$$\begin{aligned}
\frac{d\mathcal{H}}{dt} &= \sum_{s=1}^{N}\sum_{r=1}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), \log f_s > \\
&= \sum_{s=1}^{N} < \chi_{ss}\, \hat{Q}_{ss}(f_s, f_s) + (1-\chi_{ss})\, Q_{ss}(f_s), \log f_s > \\
&+ \sum_{s=1}^{N}\sum_{\substack{r=1\\ r\neq s}}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s, f_r) + (1-\chi_{sr})\, Q_{sr}(f_s), \log f_s > .
\end{aligned}$$

Since $\hat{Q}_{ss}(f_s, f_s)$ is the one–species Boltzmann operator [17] and $Q_{ss}(f_s)$ is the standard BGK operator for a single gas [3], it is well known that $< \hat{Q}_{ss}(f_s, f_s), \log f_s > \leq 0$ and $< Q_{ss}(f_s), \log f_s > \leq 0$. As concerns contributions involving operators with $r \neq s$, for the BGK ones we have already proved in Subsection 3.1 that

$$< Q_{sr}(f_s), \log f_s > + < Q_{rs}(f_r), \log f_r > \leq 0\,,$$

while for the Boltzmann terms, recalling the weak form (5), we have

$$\begin{aligned}
&< \hat{Q}_{sr}(f_s, f_r), \log f_s > + < \hat{Q}_{rs}(f_r, f_s), \log f_r > \\
&\qquad = -\frac{1}{2}\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{S}^2} g_{sr}(\mathbf{y}, \hat{\mathbf{y}}\cdot\boldsymbol{\omega}) \log\left(\frac{f_s(\mathbf{v}'_{sr}) f_r(\mathbf{w}'_{sr})}{f_s(\mathbf{v}) f_r(\mathbf{w})}\right) \\
&\qquad \times \left[\frac{f_s(\mathbf{v}'_{sr}) f_r(\mathbf{w}'_{sr})}{f_s(\mathbf{v}) f_r(\mathbf{w})} - 1\right] f_s(\mathbf{v}) f_r(\mathbf{w})\, d\mathbf{v}\, d\mathbf{w}\, d\boldsymbol{\omega} \leq 0\,,
\end{aligned}$$

where the inequality holds since $(y-1)\log y \geq 0$, $\forall y > 0$. In conclusion, the inequality $\dfrac{d\mathcal{H}}{dt} \leq 0$ is fulfilled and, more specifically, $\dfrac{d\mathcal{H}}{dt}$ may be seen as a sum of non–positive contributions as follows:

$$\begin{aligned}\frac{d\mathcal{H}}{dt} &= \sum_{s=1}^{N}\chi_{ss}\underbrace{<\hat{Q}_{ss}(f_s,f_s),\log f_s>}_{\leq 0}+\sum_{s=1}^{N}(1-\chi_{ss})\underbrace{<Q_{ss}(f_s),\log f_s>}_{\leq 0}\\ &+\sum_{s=1}^{N}\sum_{\substack{r=1\\ r>s}}^{N}\chi_{sr}\Big(\underbrace{<\hat{Q}_{sr}(f_s,f_r),\log f_s>+<\hat{Q}_{rs}(f_r,f_s),\log f_r>}_{\leq 0}\Big)\\ &+\sum_{s=1}^{N}\sum_{\substack{r=1\\ r>s}}^{N}(1-\chi_{sr})\Big(\underbrace{<Q_{sr}(f_s),\log f_s>+<Q_{rs}(f_r),\log f_r>}_{\leq 0}\Big).\end{aligned} \tag{68}$$

This will allow us to prove that the collision equilibria of the general mixed model (64) are only the standard Maxwellians (8), with all species sharing a common velocity and a common temperature. Indeed, in any equilibrium state it holds $\frac{d\mathcal{H}}{dt}=0$, therefore each non–positive term in the expression above has to vanish. By considering at first the interactions between particles of the same species we get that, for any $s=1,...,N$, if $\chi_{ss}=1$ then $<\hat{Q}_{ss}(f_s,f_s),\log f_s>=0$, while if $\chi_{ss}=0$ then $<Q_{ss}(f_s),\log f_s>=0$; in both cases this obviously yields (recalling classical results for one–species Boltzmann and BGK H-theorem) that $f_s=f_s^{\rm eq}$ given in (49), namely it is a Maxwellian distribution with species macroscopic fields n_s, $\mathbf{u}_s$ and T_s.

Taking into account this result, let us consider the interactions between different constituents. For any (s,r) with $r\neq s$ such that $\chi_{rs}=1$, from the computations above and from the well known detailed balance principle for bi–species Boltzmann operators [18] we have that the pertinent contribution to the entropy dissipation estimate (68) vanishes only if

$$\log\big(f_s(\mathbf{v}'_{sr})\big)+\log\big(f_r(\mathbf{w}'_{sr})\big)=\log\big(f_s(\mathbf{v})\big)+\log\big(f_r(\mathbf{w})\big)\,,\qquad \forall\mathbf{v},\mathbf{w}\in\mathbb{R}^3,\boldsymbol{\omega}\in\mathbb{S}^2\,.$$

By substituting into this equality the expressions of the local Maxwellians $f_s^{\rm eq}$ and $f_r^{\rm eq}$ given in (49) we get

$$\frac{m_s}{T_s}|\mathbf{v}'_{sr}-\mathbf{u}_s|^2+\frac{m_r}{T_r}|\mathbf{w}'_{sr}-\mathbf{u}_r|^2=\frac{m_s}{T_s}|\mathbf{v}-\mathbf{u}_s|^2+\frac{m_r}{T_r}|\mathbf{w}-\mathbf{u}_r|^2,$$

hence

$$\frac{m_s}{T_s}\Big(|\mathbf{v}'_{sr}|^2-|\mathbf{v}|^2\Big)+\frac{m_r}{T_r}\Big(|\mathbf{w}'_{sr}|^2-|\mathbf{w}|^2\Big)-2\frac{m_s}{T_s}\left(\mathbf{v}'_{sr}-\mathbf{v}\right)\cdot\mathbf{u}_s-2\frac{m_r}{T_r}\left(\mathbf{w}'_{sr}-\mathbf{w}\right)\cdot\mathbf{u}_r=0\,;$$

choosing $\mathbf{w}=-\mathbf{v}$ and $\boldsymbol{\omega}=-\mathbf{v}/|\mathbf{v}|$ we get $\mathbf{v}'_{sr}=\left(1-4\,\frac{m_r}{m_s+m_r}\right)\mathbf{v}$ and $\mathbf{w}'_{sr}=\left(4\,\frac{m_s}{m_r+m_s}-1\right)\mathbf{v}$, and consequently

$$8\,\frac{m_s\,m_r}{m_s+m_r}\left(2\,\frac{m_r}{m_s+m_r}-1\right)\left(\frac{1}{T_s}-\frac{1}{T_r}\right)|\mathbf{v}|^2+8\,\frac{m_s\,m_r}{m_s+m_r}\,\mathbf{v}\cdot\left(\frac{\mathbf{u}_s}{T_s}-\frac{\mathbf{u}_r}{T_r}\right)=0\,.$$

Since such equality must hold $\forall\mathbf{v}\in\mathbb{R}^3$, we obtain $\mathbf{u}_r=\mathbf{u}_s$ and $T_r=T_s$ for all pairs (s,r) with $\chi_{sr}=1$. Under these constraints, distributions (49) are such that $\hat{Q}_{sr}(f_s^{\rm eq},f_r^{\rm eq})=\hat{Q}_{rs}(f_r^{\rm eq},f_s^{\rm eq})=0$, for any (s,r) with $\chi_{rs}=1$. Therefore, by multiplying the s-th collision operator in (64) by $m_s\,\mathbf{v}$, only BGK contributions remain:

$$\sum_{r=1}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s^{\mathrm{eq}}, f_r^{\mathrm{eq}}) + (1-\chi_{sr})\, Q_{sr}(f_s^{\mathrm{eq}}), m_s\, \mathbf{v} >$$

$$= \sum_{\substack{r=1\\ r\neq s}}^{N} (1-\chi_{sr}) < Q_{sr}(f_s^{\mathrm{eq}}), m_s\, \mathbf{v} >= \sum_{\substack{r=1\\ r\neq s}}^{N} (1-\chi_{sr}) \mathcal{C}_{sr}(\mathbf{u}_r - \mathbf{u}_s)\,,$$

with coefficients $\mathcal{C}_{sr} > 0$ already detailed in the proof of Proposition 3.5. At the equilibrium we have thus

$$\sum_{r\in\,\Xi_s} \mathcal{C}_{sr}(\mathbf{u}_r - \mathbf{u}_s) = \mathbf{0}\,, \qquad s = 1, \ldots, N,$$

where Ξ_s denotes the set of indices $r = 1, \ldots, N$ such that $\chi_{sr} = 0$. According to Lemma 1, these equalities hold if and only if $\mathbf{u}_r = \mathbf{u}_s$ for any (s, r) such that $\chi_{rs} = 0$. Using this result in the weak form of the s-th collision operator in (64) with weight $m_s\, |\mathbf{v}|^2$ we obtain

$$\sum_{r=1}^{N} < \chi_{sr}\, \hat{Q}_{sr}(f_s^{\mathrm{eq}}, f_r^{\mathrm{eq}}) + (1-\chi_{sr})\, Q_{sr}(f_s^{\mathrm{eq}}), m_s\, |\mathbf{v}|^2 >$$

$$= \sum_{\substack{r=1\\ r\neq s}}^{N} (1-\chi_{sr}) < Q_{sr}(f_s^{\mathrm{eq}}), m_s\, |\mathbf{v}|^2 >= \sum_{\substack{r=1\\ r\neq s}}^{N} (1-\chi_{sr}) \mathcal{D}_{sr}(T_r - T_s)\,,$$

with coefficients $\mathcal{D}_{sr} > 0$ given explicitly in the proof of Proposition 3.5, and the same arguments as above imply $T_r = T_s$ for any (s, r) such that $\chi_{rs} = 0$. This concludes the proof of equalization of mean velocities and temperatures of all species in any equilibrium configuration.

6. Concluding remarks.

We have proposed and investigated a mixed Boltzmann–BGK kinetic model for gas mixtures of N constituents, where some kinds of interactions are modelled by Boltzmann operators and the remaining ones by binary BGK relaxation terms built up in [11]. As well known, the Boltzmann description is able to take into account the intermolecular potential and the microscopic impact parameters of each collision; on the other hand, a BGK approximation neglects some microscopic details of single interactions, preserving only some basic features as conservation laws and entropy dissipation, but it turns out to be more manageable from the numerical point of view and also in the analytical derivation of hydrodynamic closures [4, 6]. Some possible combined uses of nonlinear terms coming from Boltzmann descriptions and of linear contributions from BGK operators have already been proposed in view of numerical approximations [36, 41], but for the first time in this paper we have proved that a proper Boltzmann-BGK description is consistent even from the analytical point of view. Specifically, we have shown that our general model correctly reproduces conservation laws and Maxwellian equilibria of the Boltzmann kinetic equations, and even the entropy dissipation property is fulfilled. The proof of the H-theorem is based on the fact that entropy dissipation holds separately for the single–species Boltzmann or BGK operators, and for the two symmetric operators relevant to each pair of interacting species (s, r), with $r \neq s$, both of Boltzmann or of BGK type.

In several problems for mixtures of rarefied gases, differences between the results corresponding to various interaction potentials may be more appreciated in presence of nonlinear terms [47]. On the other hand, it is known that the evolution is driven by heaviest molecules, leading to unexpected behaviors in high order moments, as heat flux [29]. In this respect, our hybrid model is devised to describe the evolution of gas mixtures whose constituents show different features, and allows to treat the most meaningful binary interactions, as the ones involving heavy particles or higher collision frequencies, by accurate Boltzmann operators, and to approximate by simpler BGK terms the interactions playing a minor role in the global evolution of the mixture.

We have investigated in more detail the particular case with intra–species collisions modelled by Boltzmann operators and inter–species interactions by BGK terms. This option is suitable to investigate also situations where intra–species interactions are much more frequent than the others, as it may occur in mixtures of gases with very different masses [23, 39]. In this respect, we have performed the Chapman–Enskog asymptotic procedure at Euler accuracy, leading to classical $N+4$ Euler equations in the collision dominated regime, and to $5N$ Euler equations for species densities, velocities and temperatures when only intra–species collisions are dominant, showing proper source terms due to the slower inter–species interactions described by BGK terms. The hydrodynamic limit of this mixed kinetic model up to the Navier–Stokes accuracy is in progress in both asymptotic regimes, and some additional difficulties appear with respect to analogous fluid–dynamic limits of kinetic equations [6, 7], due to the simultaneous presence of the linearized single–species Boltzmann operators and of the first order correction of BGK relaxation terms, involving a proper linearization of Maxwellian attractors.

Of course different options for this hybrid Boltzmann–BGK model are worth to be investigated in the future. For instance, in plasmas the interactions between (heavy) ions usually play the dominant role (and should be thus described by Boltzmann terms), while the influence of (light) electrons on the evolution of the mixture is almost negligible (and could be modelled by simple BGK terms). Another physical problem that could be suitably described by our kinetic model is the evolution of a rarefied gas mixture in a much denser host medium, as a polluting powder in the atmosphere: collisions between particles of the rarefied powder could be modelled by BGK operators, while interactions with the background could be described by bi–species Boltzmann operators, improving thus the commonly used kinetic models based on linear Boltzmann operators, that assume the background distribution fixed and not influenced by the gaseous powder diffusing in it. Hydrodynamic limits of these models should involve a rigorous scaling of kinetic equations in terms of both masses and collision frequencies, leading thus to a multi–scale setting in case of very disparate masses and different interaction rates. Suitable numerical methods should be also arranged for these hybrid kinetic equations, in the spirit of the asymptotic–preserving schemes available for multi–scale Boltzmann equations or fluid–kinetic models [21, 31, 32].

Another long-term goal could be the generalization of this Boltzmann-BGK description to mixtures involving monatomic and polyatomic gases, even in presence of chemical reactions. To this aim, the BGK model built up in [11], where each binary relaxation operator reproduces the exchange rates of momentum and energy of the corresponding Boltzmann operator, should be suitably established for polyatomic particles and for bimolecular reactions. Some preliminary computations show that additional technicalities appear in the construction of auxiliary parameters and in the proof of the consistency of the BGK model, since exchange rates for reactive Boltzmann operators may be explicitly computed only under suitable simplifying assumptions, introducing thus additional approximations in the kinetic model, whose order of accuracy has to be carefully estimated.

Acknowledgments. This work has been performed in the frame of activities sponsored by INdAM–GNFM and by the University of Parma. Specifically, University of Parma granted the research of EL and GM through the action Bando di Ateneo 2021 per la ricerca (project *Hybrid Boltzmann-BGK models for rarefied gas mixtures*) and the research of MB and MG through the action Bando di Ateneo 2022 per la ricerca (project *Collective and self-organised dynamics: kinetic and network approaches*). Both actions were cofunded by MUR-Italian Ministry of Universities and Research - D.M. 737/2021 - PNR - PNRR - NextGenerationEU. The authors also thank the support by the Italian PRIN Research Project *Multiscale phenomena in Continuum Mechanics: singular limits, off-equilibrium and transitions* (Prin 2017YBKNCE).

REFERENCES

[1] R. Alonso and H. Orf, Statistical moments and integrability properties of monatomic gas mixtures with long range interactions, preprint, 2022, `arXiv:2204.09160`.

[2] P. Andries, K. Aoki and B. Perthame, A consistent BGK-type model for gas mixtures, *J. Stat. Phys.*, **106** (2002), 993-1018.

[3] P.L. Bhatnagar, E.P. Gross and K. Krook, A model for collision processes in gases, *Phys. Rev.*, **94** (1954), 511-524.

[4] M. Bisi, W. Boscheri, G. Dimarco, M. Groppi and G. Martalò, A new mixed Boltzmann–BGK model for mixtures solved with an IMEX finite volume scheme on unstructured meshes, *Appl. Math. Comput.*, **433** (2022), 127416.

[5] M. Bisi and M.J. Cáceres, A BGK relaxation model for polyatomic gas mixtures, *Commun. Math. Sci.*, **14** (2016), 297-325.

[6] M. Bisi, M. Groppi and G. Martalò, Macroscopic equations for inert gas mixtures in different hydrodynamic regimes, *J. Phys. A: Math. Theor.*, **54** (2021), 085201.

[7] M. Bisi, M. Groppi and G. Spiga, Kinetic Bhatnagar–Gross–Krook model for fast reactive mixtures and its hydrodynamic limit, *Phys. Rev. E*, **81** (2010), 036327.

[8] M. Bisi and S. Lorenzani, High–frequency sound wave propagation in binary gas mixtures flowing through microchannels, *Phys. Fluids*, **28** (2016), 052003.

[9] M. Bisi and G. Spiga, Diatomic gas diffusing in a background medium: kinetic approach and reaction–diffusion equations, *Commun. Math. Sci.*, **4** (2006), 779-798.

[10] M. Bisi and R. Travaglini, A BGK model for mixtures of monoatomic and polyatomic gases with discrete internal energy, *Physica A*, **547** (2020), 124441.

[11] A.V. Bobylev, M. Bisi, M. Groppi, G. Spiga and I.F. Potapenko, A general consistent BGK model for gas mixtures, *Kinet. Relat. Models*, **11** (2018), 1377-1393.

[12] A. Bondesan, L. Boudin and B. Grec, A numerical scheme for a kinetic model for mixtures in the diffusive limit using the moment method, *Numer. Methods Partial Differ. Equ.*, **35** (2019), 1184-1205.

[13] W. Boscheri and G. Dimarco, High order central WENO-implicit-explicit Runge-Kutta schemes for the BGK model on general polygonal meshes, *J. Comput. Phys.*, **422** (2020), 109766.

[14] W. Boscheri and G. Dimarco, High order finite volume schemes with IMEX time stepping for the Boltzmann model on unstructured meshes, *Comput. Methods Appl. Mech. Engrg.*, **387** (2021), 114180.

[15] M. Briant and E.S. Daus, The Boltzmann equation for a multi–species mixture close to global equilibrium, *Arch. Ration. Mech. Anal.*, **222** (2016), 1367-1443.

[16] S. Brull and J. Schneider, Derivation of a BGK model for reacting gas mixtures, *Commun. Math. Sci.*, **12** (2014), 1199-1223.

[17] C. Cercignani, *The Boltzmann Equation and its Applications*, Springer, New York, 1988.

[18] C. Cercignani, *Rarefied Gas Dynamics: From Basic Concepts to Actual Calculations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000.

[19] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non–Uniform Gases*, Cambridge University Press, Cambridge, 1970.

[20] A. Crestetto, C. Klingenberg and M. Pirner, Kinetic/fluid micro–macro numerical scheme for a two component gas mixture, *Multiscale Model. Simul.*, **18** (2020), 970-998.

[21] P. Degond and F. Deluzet, Asymptotic-preserving methods and multiscale models for plasma physics, *J. Comput. Phys.*, **336** (2017), 429-457.

[22] L. Desvillettes, R. Monaco and F. Salvarani, A kinetic model allowing to obtain the energy law of polytropic gases in the presence of chemical reactions, *Eur. J. Mech. B Fluids*, **24** (2005), 219-236.

[23] V.S. Galkin and N.K. Makashev, Kinetic derivation of the gas-dynamic equation for multicomponent mixtures of light and heavy particles, *Fluid Dynam.*, **29** (1994), 140-155.

[24] I.M. Gamba and M. Pavic–Colic, On existence and uniqueness to homogeneous Boltzmann flows of monatomic gas mixtures, *Arch. Ration. Mech. Anal.*, **235** (2020), 723-781.

[25] V. Giovangigli, *Multicomponent Flow Modeling*, Birkhäuser, Boston, 1999.

[26] M. Groppi, G. Russo and G. Stracquadanio, High order semi–Lagrangian methods for the BGK equation, *Commun. Math. Sci.*, **14** (2016), 389-414.

[27] M. Groppi and G. Spiga, Kinetic approach to chemical reactions and inelastic transitions in a rarefied gas, *J. Math. Chem.*, **26** (1999), 197-219.

[28] M. Groppi and G. Spiga, A Bhatnagar–Gross–Krook–type approach for chemically reacting gas mixtures, *Phys. Fluids*, **16** (2004), 4273-4284.

[29] V.K. Gupta and M. Torrilhon, Comparison of relaxation phenomena in binary gas-mixtures of Maxwell molecules and hard spheres, *Comput. Math. Appl.*, **70** (2015), 73-88.

[30] J.R. Haack, C.D. Hauck and M.S. Murillo, A conservative, entropic multispecies BGK model, *J. Stat. Phys.*, **168** (2017), 826-856.

[31] J. Hu, S. Jin and L. Wang, An asymptotic–preserving scheme for the semiconductor Boltzmann equation with two-scale collisions: a splitting approach, *Kinet. Relat. Mod.*, **8** (2015), 707-723.

[32] S. Jin, Asymptotic preserving (AP) schemes for multiscale kinetic and hyperbolic equations: a review, *Riv. Math. Univ. Parma (N.S.)*, **3**(2) (2012), 177-216.

[33] A. Jüngel and Y.J. Peng, A hierarchy of hydrodynamic models for plasmas. Zero-electron-mass limits in the drift-diffusion equations, *Ann. Inst. Henri Poincaré, Analyse non linéare*, **17** (2000), 83-118.

[34] C. Klingenberg, M. Pirner and G. Puppo, A consistent kinetic model for a two–component mixture with an application to plasma, *Kinet. Relat. Mod.*, **10** (2017), 445-465.

[35] G.M. Kremer, M. Pandolfi Bianchi and A.J. Soares, A relaxation kinetic model for transport phenomena in a reactive flow, *Phys. Fluids*, **18** (2006), 037104.

[36] R. Li, Y. Lu, Y. Wang and H. Xu, Hermite spectral method for multi-species Boltzmann equation, *J. Comp. Phys.*, **471** (2022), 111650.

[37] F.J. McCormack, Construction of linearized kinetic models for gaseous mixtures and molecular gases, *Phys. Fluids*, **16** (1973), 2095-2105.

[38] E. Meier and U. Shumlak, Development of five–moment two–fluid modeling for Z–pinch physics, *Phys. Plasmas*, **28** (2021), 092512.

[39] M. Mezhericher, A. Levy and I. Borde, Multi–scale multiphase modeling of transport phenomena in spray–drying processes, *Dry. Technol.*, **33** (2015), 2-23.

[40] L. Pareschi and G. Russo, Numerical solution of the Boltzmann equation I: Spectrally accurate approximation of the collision operator, *SIAM J. Numer. Anal.*, **37** (2000), 1217-1245.

[41] M. Pfeiffer, A. Mirza and P. Nizenkov, Multi–species modeling in the particle–based ellipsoidal statistical Bhatnagar-Gross-Krook method for monatomic gas species, *Phys. Fluids*, **33** (2021), 036106.

[42] S. Pieraccini and G. Puppo, Implicit–explicit schemes for BGK kinetic equations, *J. Sci. Comput.*, **32** (2007), 1-28.

[43] M. Pirner, *Kinetic modelling of gas mixtures*, Ph.D. thesis, Würzburg University, Germany, 2018.

[44] M. Pirner, A review on BGK models for gas mixtures of mono and polyatomic molecules, *Fluids*, **6** (2021), 393.

[45] J.M. Reese, L. Wu, J. Zhang and Y. Zhang, A fast spectral method for the Boltzmann equation for monoatomic gas mixtures, *J. Comput. Phys.*, **298** (2015), 602-621.

[46] A. Rossani and G. Spiga, A note on the kinetic theory of chemically reacting gases, *Physica A*, **272** (1999), 563-573.

[47] F. Sharipov and J.L. Strapasson, Benchmark problems for mixtures of rarefied gases. I. Couette flow, *Phys. Fluids*, **25** (2013), 027101.

[48] W. Su, P. Wang, Y. Zhang and L. Wu, Implicit discontinuous Galerkin method for the Boltzmann equation, *J. Sci. Comput.*, **82** (2020), 35.

[49] J. Vranjes and P.S. Krstic, Collisions, magnetization, and transport coefficients in the lower solar atmosphere, *Astron. Astrophys.*, **554** (2013), A22.

[50] L. Wu, H. Liu, Y. Zhang and J.M. Reese, Influence of intermolecular potentials on rarefied gas flows: Fast spectral solutions of the Boltzmann equation, *Phys. Fluids*, **27** (2015), 082002.

[51] L. Wu, J. Zhang, J.M. Reese and Y. Zhang, A fast spectral method for the Boltzmann equation for monatomic gas mixtures, *J. Comp. Phys.*, **298** (2015), 602-621.

[52] D.H. Zanette and R.O. Barrachina, Nonlinear particle diffusion in a time–dependent host medium, *Phys. Fluids*, **31** (1988), 502-505.

[53] Y. Zhang, L. Zhu, R. Wang and Z. Guo, Discrete unified gas kinetic scheme for all Knudsen number flows. III. Binary gas mixtures of Maxwell molecules, *Phys. Rev. E*, **97** (2018), 053306.

Received xxxx 20xx; revised xxxx 20xx; early access xxxx 20xx.