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Escaping Fatou components with disjoint hyperbolic limit sets

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Abstract

We construct automorphisms of $\mathbb{C}^2$ of constant Jacobian with a cycle of escaping Fatou components, on which there are exactly two limit functions, both of rank 1. On each such Fatou component, the limit sets for these limit functions are two disjoint hyperbolic subsets of the line at infinity. In the literature there are currently very few examples of automorphisms of $\mathbb{C}^2$ with rank one limit sets on the boundary of Fatou components. To our knowledge, this is the first example in which such limit sets are hyperbolic, and moreover different limit sets of rank 1 coexist.

Keywords Holomorphic dynamics · Fatou set · Henon maps · Escaping Fatou components · Baker domains · Trascendental automorphisms of $\mathbb{C}^2$

Mathematics Subject Classification Primary 37F80 · 37F10 · 32H50

1 Introduction

Transcendental Hénon maps are automorphisms of $\mathbb{C}^2$ with constant Jacobian of the form

$$F(z, w) := (f(z) - \delta w, z) \text{ with } f : \mathbb{C} \to \mathbb{C} \text{ entire transcendental.}$$

In analogy with classical complex Hénon maps, for which f is assumed to be a polynomial (see e.g. [4–6, 15–17]), the dynamical investigation of transcendental Hénon maps can rely on tools and knowledge from one dimensional complex dynamics, which is better understood than its higher dimensional counterpart. They have been introduced in [13]. General properties of transcendental Hénon maps were established in [1–3] and examples with interesting dynamical features were presented.

Let $\mathbb{P}^2$ be the complex projective space obtained by compactifying $\mathbb{C}^2$ by adding the line at infinity ℓ_∞. We define the Fatou set of F as the set of points in $\mathbb{C}^2$ near which the

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iterates form a normal family with respect to the complex structure induced by $\mathbb{P}^2$ (compare with [1, section 1]). A Fatou component is a connected component of the Fatou set. Given a Fatou component Ω we call a function $h : \Omega \to \mathbb{P}^2$ a *limit function* for Ω if there exists a subsequence n_k such that $F^{n_k} \to h$ uniformly on compact subsets of Ω. The image $h(\Omega)$ of a limit function h is called a *limit set* (for Ω). By Lemma 4.3 and 2.4 in [1], each limit set is either contained in $\mathbb{C}^2$ or contained in ℓ_∞.

In this paper we investigate *escaping Fatou components*, that is Fatou components for which all limit sets lie in the line at infinity.

More precisely we construct a transcendental Hénon map with a cycle of escaping Fatou components satisfying the following properties. Let $\mathbb{H}$ denote the right half plane, $-\mathbb{H}$ denote the left half plane.

Theorem 1.1 *Let*

$$F(z, w) := (e^{-z^2} + e^{\pi i}\delta w, z), \quad \delta > 2.$$

Then F has a cycle of four Fatou components Ω^{ab} with $a, b \in \{+, -\}$, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$. There are exactly two limit functions h_1, h_2, both of rank 1, such that

$$h_1(\Omega^{aa}) = h_2(\Omega^{a(-a)}) = \mathbb{H} \text{ and } h_1(\Omega^{a(-a)}) = h_2(\Omega^{aa}) = -\mathbb{H} \text{ for all } a.$$

Moreover, F is conjugate to its linear part on every Ω^{ab}.

Notice that δ is the Jacobian of F, hence the latter is expansive. Its role is not relevant, as long as $\delta > 1$. For convenience we take $\delta > 2$.

The main points of interest of this result are that the limit functions have rank one, that each Fatou component has two disjoint limit sets (compare [18] for restrictions on the presence of several limit sets), and that the limit sets $\mathbb{H}$, $-\mathbb{H}$ are hyperbolic.

For general automorphisms of $\mathbb{C}^2$ there are very few examples of limit functions of rank 1 [9, 18], and for polynomial Hénon maps, it is not even known whether rank 1 limit functions can exist; in fact, their existence has been excluded provided the Jacobian is small enough [20]. On the other hand, they are abundant for holomorphic *endomorphisms* of $\mathbb{C}^2$ [10, Theorem 4]. For transcendental Hénon maps, rank 1 limit functions seem to appear naturally for escaping Fatou components [8]. To our knowledge there were no previous examples of hyperbolic limit sets for automorphisms of $\mathbb{C}^2$. One possible reason for the natural appearance of these phenomena might be that F is not defined on ℓ_∞, hence there is no natural dynamics on limit sets contained there.

One can see F as a special case of maps of the form

$$F(z, w) := \left(e^{-z^k} + e^{\frac{2\pi i}{k}}\delta w, z\right), \quad \delta > 2, \ k \in \mathbb{N}$$

Analogous results hold for such maps, and are proven in [7] with similar techniques.

2 Proof of Theorem 1.1

From now on let F be as in Theorem 1.1,

$$F(z, w) = (e^{-z^2} - \delta w, z) \quad \text{with } \delta > 2. \tag{2.1}$$

Throughout the paper, given a point $P = (z_0, w_0) \in \mathbb{C}^2$ and $n \in \mathbb{N}$ we denote its iterates by $F^n(P) =: (z_n, w_n)$.

2.1 Computing limit functions

In this section we give an explicit expression for the iterates of F and their formal limit. A direct computation (compare [8]) shows that

$$F^{2n}(z_0, w_0) = (-\delta)^n \left(z_0 + \sum_{j=1}^{n} (-\delta)^{-j} f(z_{2j-1}),\ w_0 + \sum_{j=1}^{n} (-\delta)^{-j} f(z_{2j-2}) \right)$$

$$F^{2n+1}(z_0, w_0) = (-\delta)^n \left(-\delta \left(w_0 + \sum_{j=1}^{n+1} (-\delta)^{-j} f(z_{2j-2}) \right),\ z_0 + \sum_{j=1}^{n} (-\delta)^{-j} f(z_{2j-1}) \right).$$

For $n \in \mathbb{N}$ define the following holomorphic functions from $\mathbb{C}^2$ to $\hat{\mathbb{C}}$

$$\Delta_1^n(z_0, w_0) := \sum_{j=1}^{n} (-\delta)^{-j} f(z_{2j-1})$$

$$\Delta_2^n(z_0, w_0) := \sum_{j=1}^{n} (-\delta)^{-j} f(z_{2j-2})$$

With this notation the iterates of F take the form

$$F^{2n}(z_0, w_0) = (-\delta)^n \left((z_0 + \Delta_1^n(z_0, w_0), w_0 + \Delta_2^n(z_0, w_0) \right) \tag{2.2}$$

$$F^{2n+1}(z_0, w_0) = (-\delta)^n \left(-\delta w_0 - \delta \Delta_2^{n+1}(z_0, w_0), z_0 + \Delta_1^n(z_0, w_0) \right). \tag{2.3}$$

Let

$$\Delta_1(z, w) = \Delta_1^\infty(z, w) := \lim_{n\to\infty} \Delta_1^n(z, w)$$

$$\Delta_2(z, w) = \Delta_2^\infty(z, w) := \lim_{n\to\infty} \Delta_2^n(z, w)$$

$$\Delta(z, w) = \max\left(|\Delta_1(z, w)|, |\Delta_2(z, w)|\right).$$

Notice that Δ_1, Δ_2 are holomorphic functions to $\hat{\mathbb{C}}$ on open sets on which they are well defined.

We can deduce the following formal limits.

$$h_1(z, w) := \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}} = \frac{z + \Delta_1(z, w)}{w + \Delta_2(z, w)} \tag{2.4}$$

$$h_2(z, w) := \lim_{n\to\infty} \frac{z_{2n+1}}{w_{2n+1}} = \frac{-\delta(w + \Delta_2(z, w))}{z + \Delta_1(z, w)} = -\frac{\delta}{h_1(z, w)}. \tag{2.5}$$

We have that h_1, h_2 are holomorphic functions to $\hat{\mathbb{C}}$ on open sets on which Δ_1 and Δ_2 are holomorphic functions to $\hat{\mathbb{C}}$. We will show in Proposition 2.7 that $h_1 \neq h_2$.

2.2 Existence of Fatou components and rank of the limit functions

In this section we construct a forward invariant open set W on which the even and the odd iterates converge, from which we deduce the existence of Fatou components. We then show that the limit functions have rank 1 on such Fatou components.

For $A \subseteq \mathbb{C}^2$ and $a, b \in \{+, -\}$ define

$$A^{ab} := A \cap \{(z, w) \in \mathbb{C}^2 \; a\,\mathrm{Re}(z) > 0,\; b\,\mathrm{Re}(w) > 0\}. \tag{2.6}$$

If $A \cap (\{\mathrm{Re}\, z = 0\} \cup \{\mathrm{Re}\, w = 0\}) = \varnothing$ then $A = \bigcup_{a,b\in\{+,-\}} A^{ab}$.

We start by defining a set on which we have control on the dynamics. Let

$$\begin{aligned} \mathcal{S} &:= \{z \in \mathbb{C} : |\mathrm{Im}(z)| < |\mathrm{Re}(z)|\} \subset \mathbb{C} \\ S &:= \mathcal{S} \times \mathcal{S} \subset \mathbb{C}^2 \end{aligned}$$

A sketch of $\mathcal{S}$ can be found in Fig. 1.

Lemma 2.1 *Let $z \in \mathcal{S}$, then $|f(z)| = |e^{-z^2}| < 1$.*

Proof If $z \in \mathcal{S}$, then $|\arg(z)| < \frac{\pi}{4}$ and hence $\mathrm{Re}(z^2) > 0$ from which we have $|e^{-z^2}| = e^{-\mathrm{Re}\, z^2} < 1$. □

Lemma 2.2 (Orbits contained in S) *For any $P = (z_0, w_0) \in S^{ab}$ such that $F(P) \in S$ and $|\mathrm{Re}\, w_0| > \frac{1}{\delta}$ we have that $F(P) \in S^{(-b)a}$.*

From now on assume that $F^n(P) \in S$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} F^{2n}(z_0, w_0) &\to h_1(z_0, w_0) \\ F^{2n+1}(z_0, w_0) &\to h_2(z_0, w_0). \end{aligned}$$

Fix $\lambda > 0$ and assume also that $|\mathrm{Re}\, z_0|, |\mathrm{Re}\, w_0| > \frac{1+\lambda}{\delta-1}$. Then

$$|\mathrm{Re}\, z_{2n-1}| = |\mathrm{Re}\, w_{2n}| > |\mathrm{Re}\, w_0| + n\lambda \tag{2.7}$$

$$|\mathrm{Re}\, z_{2n}| = |\mathrm{Re}\, w_{2n+1}| > |\mathrm{Re}\, z_0| + n\lambda. \tag{2.8}$$

Proof By hypothesis, $F(P) \in S$ hence $F(P) \in S^{\tilde{a}\tilde{b}}$ for some $\tilde{a}, \tilde{b} \in \{+, -\}$. Since $\mathrm{Re}\, w_1 = \mathrm{Re}\, z_0$ we have that $\tilde{b} = a$. Moreover $\mathrm{Re}\, z_1 = -\delta\,\mathrm{Re}\, w_0 + \mathrm{Re}(e^{-z_0^2})$ and since $P \in S$, $|\mathrm{Re}(e^{-z_0^2})| < 1$ by Lemma 2.1. Hence the sign of $\mathrm{Re}\, z_1$ is opposite to the sign of $\mathrm{Re}\, w_0$ provided $|\mathrm{Re}\, w_0| > \frac{1}{\delta}$, and $\tilde{a} = -b$ as required.

Assume from now on that $F^n(P) \in S$ for all $n \in \mathbb{N}$. It follows that $z_n \in \mathcal{S}$ for all $n \in \mathbb{N}$ and hence by Lemma 2.1 $|f(z_n)| < 1$ for all $n \in \mathbb{N}$. Since $\delta > 2$ this implies

$$\Delta(z_0, w_0) < \sum_{j=1}^{\infty} \delta^{-j} < 1 \text{ whenever } F^n(z_0, w_0) \in S \text{ for all } n \in \mathbb{N}, \tag{2.9}$$

which implies convergence of the even and odd iterates of F according to the expression in (2.2), (2.3).

We now prove (2.7), (2.8). Using the expression of F and since $P \in S$, by Lemma 2.1 we have $|\mathrm{Re}\, z_1| \geq \delta|\mathrm{Re}\, w_0| - |e^{-z_0^2}| \geq \delta|\mathrm{Re}\, w_0| - 1$ which is larger than $|\mathrm{Re}\, w_0| + \lambda$ if $|\mathrm{Re}\, w_0| > \frac{1+\lambda}{\delta-1}$. It follows that

$$|\mathrm{Re}\, z_1| > |\mathrm{Re}\, w_0| + \lambda \tag{2.10}$$

$$|\mathrm{Re}\, z_2| > |\mathrm{Re}\, w_1| + \lambda = |\mathrm{Re}\, z_0| + \lambda, \tag{2.11}$$

where the claim for z_2 follows because $w_1 = z_0$. The more general formula follows by induction, using that $F^n(P) \in S$ for all $n \in \mathbb{N}$. □

Corollary 2.3 *Let $A \subset S$ be forward invariant. If $P = (z_0, w_0) \in A$ such that $|\operatorname{Re} w_0| > \frac{1}{\delta}$ then Lemma 2.2 holds for $\lambda = 1$, in particular, if $P \in A^{ab}$ then $F(P) \in A^{(-b)a}$.*

For $R > 0$ and $0 < k < 1$ define the sets (see Fig. 1.)

$$\begin{aligned} \mathcal{W}_{k,R} &:= \{z \in \mathbb{C} : |\operatorname{Im} z| < k|\operatorname{Re} z|, \ |\operatorname{Re} z| > R\} \subset \mathbb{C} \\ W_{k,R_1,R_2} &:= \mathcal{W}_{k,R_1} \times \mathcal{W}_{k,R_2} \subset \mathbb{C}^2. \end{aligned}$$

Observe that $\mathcal{W}_{k,R} \subset \mathcal{S}$ and that $\mathcal{W}_{1,0} = \mathcal{S}$.

Lemma 2.4 *Let $n \in \mathbb{N}$, and let $(z_0, w_0) \in W_{k,R_1,R_2}$. Let $0 < k < \tilde{k} < 1$. If $R_2 > \frac{2}{\delta(\tilde{k}-k)}$ then*

$$\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < \tilde{k} \quad \textit{and} \quad \left|\frac{\operatorname{Im} w_1}{\operatorname{Re} w_1}\right| < k.$$

Proof Let $(z_0, w_0) \in W_{k,R_1,R_2}$. The claim for w_1 is immediate because $w_1 = z_0$. Using the expression of F, the triangular inequality, the estimate in Lemma 2.1 and the fact that $|\operatorname{Im} w_0| < k|\operatorname{Re} w_0|$ we have

$$\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < \frac{\delta|\operatorname{Im} w_0| + 1}{\delta|\operatorname{Re} w_0| - 1} < \frac{k\delta|\operatorname{Re} w_0| + 1}{\delta|\operatorname{Re} w_0| - 1}.$$

Setting the resulting expression to be less than $\tilde{k}$ we get $|\operatorname{Re} w_0| > \frac{1+\tilde{k}}{\delta(\tilde{k}-k)}$. Since $\tilde{k} < 1$, it is enough to take $|\operatorname{Re} w_0| > \frac{2}{\delta(\tilde{k}-k)}$ as required. □

Let $k_n := 1 - \frac{1}{n+2}$ and $R_n := (\frac{\delta}{2})^{\frac{n}{2}} R_0$ for $R_0 > 2$ sufficiently large depending only on δ (see (2.12)). Let $R_{-1} = R_0$ and set

$$W_n := \begin{cases} W^{++}_{k_n,R_n,R_{n-1}} & \text{if } n = 0 \mod 4 \\ W^{-+}_{k_n,R_n,R_{n-1}} & \text{if } n = 1 \mod 4 \\ W^{--}_{k_n,R_n,R_{n-1}} & \text{if } n = 2 \mod 4 \\ W^{+-}_{k_n,R_n,R_{n-1}} & \text{if } n = 3 \mod 4 \end{cases}.$$

and define

$$W := \bigcup_{n \in \mathbb{N}} W_n.$$

Proposition 2.5 (Invariance of W) *The set W is open and $W \subset S$. For any $n \in \mathbb{N}$ we have that $F(W_n) \subset W_{n+1}$, hence W is forward invariant. The set W consists of four connected components W^{ab} with $a, b \in \{+, -\}$ and $F(W^{ab}) \subset W^{(-b)a}$.*

Proof The fact that W is open and $W \subset S$ follows from the definition. Fix $n \in \mathbb{N}$. Let $(z_0, w_0) \in W_n$ and let (z_1, w_1) be its image. Since $w_1 = z_0$, the signs of $\operatorname{Re} w_1, \operatorname{Re} z_0$ are the same, and we have that $|\operatorname{Re} w_1| = |\operatorname{Re} z_0| > R_n$ and that

$$\left|\frac{\operatorname{Im} w_1}{\operatorname{Re} w_1}\right| = \left|\frac{\operatorname{Im} z_0}{\operatorname{Re} z_0}\right| < k_n < k_{n+1}.$$

Hence to show that $F(W_n) \subset W_{n+1}$ it is enough to see that $|\operatorname{Re} z_1| > R_{n+1}$ and that

$$\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < k_{n+1}.$$

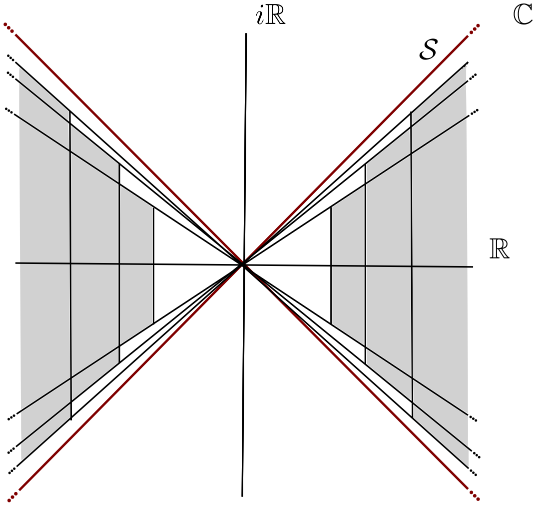

Fig. 1 A sketch in $\mathbb{C}$ of the set $\mathcal{S}$ and three of the sets $\mathcal{W}_{k_n,R_n}$. The set W is obtained by taking appropriate products of (parts of) the sets $\mathcal{W}_{k_n,R_n}$

Let $\lambda_n := R_{n+1} - R_{n-1}$. Since $P \in S$, by (2.10) we have that

$$|\operatorname{Re} z_1| > |\operatorname{Re} w_0| + \lambda_n > R_{n-1} + \lambda_n = R_{n+1}$$

provided $R_{n-1} > \frac{1+\lambda_n}{\delta-1}$. Substituting the expression for λ_n we get $R_{n+1} < \delta R_{n-1} - 1$. Substituting the expression for R_{n+1} and R_{n-1} we get

$$\delta^{\frac{n+1}{2}} R_0 > 2^{\frac{n+1}{2}}$$

which is satisfied because $\delta > 2$, provided $R_0 \geq 1$. This gives $|\operatorname{Re} z_1| > R_{n+1}$.

We now prove $\left|\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1}\right| < k_{n+1}$. By Lemma 2.4, it is enough to check that $R_{n-1} > \frac{2}{\delta(k_{n+1}-k_n)} = \frac{2(n+2)(n+3)}{\delta}$, that is

$$R_0 > 2^{\frac{n+1}{2}} \delta^{-\frac{n+1}{2}} (n+2)(n+3) \quad \text{for all } n \in \mathbb{N}. \tag{2.12}$$

Since the function on the right hand side is bounded in n for any $\delta > 2$ (in fact, it tends to 0 as $n \to \infty$), such R_0 exists and depends only on δ.

Finally, for any $(z, w) \in W$ we have $\operatorname{Re} z, \operatorname{Re} w \neq 0$, so the sets W^{ab} are well defined. By construction, $W_n \cap W_{n+4} \neq \varnothing$ so each W^{ab} is connected. It follows that W consists of 4 connected components W^{ab}. Since W is forward invariant and contained in S, the orbits of points in W are contained in S hence Corollary 2.3 applies. □

Proposition 2.6 (Existence of Fatou components) *On each W^{ab} we have that*

$$F^{2n} \to h_1, \ F^{2n+1} \to h_2 \text{ uniformly on compact subsets of } W^{ab}.$$

It follows that each W^{ab} is contained in a Fatou component that we denote by Ω^{ab}.

Proof Since $W \subset S$ and is forward invariant by Proposition 2.5, (2.9) holds hence F^{2n} and F^{2n+1} converge uniformly on W to h_1, h_2 respectively, hence W is contained in the Fatou set. Since each W^{ab} is open and connected it is contained in a unique Fatou component that we denote by Ω^{ab}. □

We will see in Proposition 2.18 that in fact the components Ω^{ab} are all distinct and that the notation Ω^{ab} matches the definition of A^{ab} given in Sect. 2.1 for a general set A.

Proposition 2.7 *Both h_1 and h_2 have (generic) rank 1 on W, and $h_1 \neq h_2$.*

Proof Recall that $\Delta(z, w) < 1$ on W by (2.9). It follows that by the explicit expression of F^{2n}, F^{2n+1}, the iterates of any point in W converge to the line at infinity. So $h_i(W) \subset \ell_\infty$, and h_1, h_2 either have generic rank 1 or are constants. Suppose by contradiction that $h_1 = c$ is constant. If $|c| \neq \infty$, then one has:

$$|z_0| - \Delta(z_0, w_0) \leq |z_0 + \Delta_1(z_0, w_0)| = |c|\,|w_0 + \Delta_2(z_0, w_0))| \leq |c||w_0| + |c|\Delta(z_0, w_0),$$

hence

$$|z_0| \leq |c||w_0| + (|c| + 1),$$

contradicting the fact that (z_0, w_0) could be any point in W, which is unbounded in the z direction for any choice of w. If $c = \infty$, we have $|w_0| \leq 1$, again a contradiction. It follows that $h_1 \neq h_2$. Indeed, $h_1 \cdot h_2 = -\delta$ is constant, if we had $h_1 = h_2$ it would follow that h_1^2 (and hence h_1) would be constant as well, contradicting the argument above. □

2.3 Construction of an absorbing set

Let Ω^{ab} with $a, b \in \{+, -\}$ be the Fatou components defined in Proposition 2.6 and let

$$\Omega := \bigcup_{ab} \Omega^{ab}.$$

Since each Ω^{ab} is connected, Ω consists of at most 4 Fatou components. This section is devoted to find an absorbing set W_I for Ω under F. Its existence will be used in Sect. 2.5 to show that the Fatou components Ω^{ab} are all distinct and to describe both their limit sets and their geometric structure. We use an argument based on harmonic functions used also in [1, 8, 14].

Definition 2.8 (*Absorbing sets*) A set A is *absorbing* for an open set $\Omega \supset A$ under a map F if for any compact $K \subset \Omega$ there exists $N > 0$ such that

$$F^n(K) \subset A \quad \text{for all } n \geq N.$$

If A is absorbing for Ω, then $\Omega = \bigcup_n F^{-n}(A)$.

Fix $C \geq 1$ and let

$$I = I(C) := \{z \in \mathbb{C} \mid |\operatorname{Im} z|^2 < |\operatorname{Re} z|^2 - C^2\} = \{z \in \mathbb{C} : \operatorname{Re}(z^2) > C^2\} \subset \mathcal{S}.$$

Notice that if $z \in I$, then $|\operatorname{Re} z| > C$.

Define

$$W_I = W_I(C) := \Omega \cap \{(z, w) \in \mathbb{C}^2 : F^n(z, w) \in I \times I \text{ for all } n \geq 0\}.$$

Proposition 2.9 *We have that $W_I^{ab} \neq \varnothing$ for all $a, b \in \{+, -\}$. For every $a, b \in \{+, -\}$,*

$$F(W_I^{ab}) \subset W_I^{(-b)a}.$$

The sets $W_I^{++} \cup W_I^{--}$, $W_I^{-+} \cup W_I^{+-}$ are both forward invariant under F^2. Moreover F^{2n} and F^{2n+1} are convergent on W_I.

Proof Each W_I^{ab} contains the set $\{(z, w) \in \mathbb{C}^2 : a \operatorname{Re} z > M,\ b \operatorname{Re} w > M,\ \operatorname{Im} z = \operatorname{Im} w = 0\}$ for M sufficiently large. The set $W_I \subset S$ is forward invariant hence Corollary 2.3 applies. Convergence of even and odd iterates follows by (2.9). □

It will turn out that W_I is open as well (Proposition 2.16).

The rest of this section is devoted to proving the following proposition.

Proposition 2.10 *The set W_I is absorbing for Ω under F, that is,*

$$\Omega = \bigcup_n F^{-n}(W_I) =: \mathcal{A}_I.$$

Let

$$\mathcal{X} := \{(z, w) \in \Omega : \ h_1(z, w) = 0, \infty\}.$$

Since $\mathcal{X}$ is an analytic set, being the union of the 0-set and the ∞-set of a meromorphic function, it is locally a finite union of 1-complex-dimensional varieties (see [11]).

Let K be a compact subset of $\Omega \backslash \mathcal{X}$, hence $h_i(P) \neq 0, \infty$ for all $P \in K$, and $i = 1, 2$. Define

$$M := \max_{K,i} |h_i| < \infty. \tag{2.13}$$

Note that $M > 1$ because $h_2 = -\frac{\delta}{h_1}$ and $\delta > 1$. By Corollary 2.3 in [8] if $\varepsilon > 0$ is sufficiently small there exists a constant c such that for every $(z_0, w_0) \in K$

$$|z_n| \leq c(M + \varepsilon)^n. \tag{2.14}$$

$$|w_n| = |z_{n-1}| \leq c(M + \varepsilon)^{n-1}. \tag{2.15}$$

The proof of Proposition 2.10 relies on the following technical lemma. Recall that for $P = (z_0, w_0)$, we write $F^n(P) = (z_n, w_n)$.

Lemma 2.11 *Define the sequence of harmonic functions u_n from Ω to $\mathbb{R}$ as*

$$u_n(z_0, w_0) := \frac{-\operatorname{Re}(z_n^2)}{n}. \tag{2.16}$$

Then

1. *Let $K \subset \Omega$ compact. Then there exists $M = M(K)$ and $N \in \mathbb{N}$ such that $u_n \leq \log M$ on K for $n > N$;*
2. *$u_n \to -\infty$ uniformly on compact subsets of W;*
3. *If $P \in \Omega \backslash \mathcal{A}_I$, for every $\varepsilon > 0$ there is a subsequence $n_k \to \infty$ such that $u_{n_k}(P) \geq -\varepsilon$.*

Lemma 2.12 *Let $z \in \mathbb{C}$, $k < 1$. If*

$$\left|\frac{\operatorname{Im} z}{\operatorname{Re} z}\right| \leq k < 1 \textit{ then} \tag{2.17}$$

$$\left|\frac{\operatorname{Im} z^2}{\operatorname{Re} z^2}\right| \leq \frac{2k}{1 - k^2}. \tag{2.18}$$

Proof Let $z = re^{i\theta}$ satisfying (2.17); then $|\tan\theta| \leq k < 1$. Hence since $z^2 = r^2 e^{2i\theta}$,

$$\left|\frac{\operatorname{Im} z^2}{\operatorname{Re} z^2}\right| = |\tan(2\theta)| = \left|\frac{2\tan\theta}{1 - \tan^2\theta}\right| \leq \frac{2k}{1 - k^2}.$$

□

The following fact is certainly known, however we give a proof in the Appendix. Given a set A, let $\mathring{A}$ denote its interior.

Proposition 2.13 *Let L be a compact set and H be an analytic subset of dimension one of $\mathbb{C}^2$. For any compact K s.t. $K \subset \mathring{L}$ there exists $\eta = \eta(K, L, H)$ such that for any u harmonic defined in a neighborhood of L and such that*

$$u \leq \alpha < \infty \text{ on } L \backslash (\eta\text{-neighborhood of } H)$$

we have

$$u \leq \alpha \text{ on } K$$

Proof of Lemma 2.11 (1) Let K be a compact subset of Ω. Let η as obtained by applying Proposition 2.13 to a slightly larger compact set $L \subset \Omega$ and to the analytic set $\mathcal{X}$. Let $U_\eta(\mathcal{X})$ be an η-neighborhood of $\mathcal{X}$. In view of Proposition 2.13 it is enough to prove that there exists $N \in \mathbb{N}$ such that $u_n \leq \log M$ for $n > N$ and for some M on the set

$$\tilde{K} := K \backslash U_\eta(\mathcal{X})$$

which is a compact subset of $\Omega \backslash \mathcal{X}$. Hence it is enough to prove the claim for any K compact subset of $\Omega \backslash \mathcal{X}$.

Fix $\varepsilon > 0$ sufficiently small and let M, c be as in (2.14) and (2.15) for K. Suppose that there exists a subsequence (n_j) and points $(z, w) = (z(j), w(j)) \in K$ such that

$$-\frac{\operatorname{Re}(z_{n_j}^2)}{n_j} > \beta$$

for some β. We will show that $\beta \leq M$.

Using (2.14) and (2.15) we have that

$$\begin{aligned} c(M+\varepsilon)^{n_j+1} &\geq |z_{n_j+1}| = |e^{-z_{n_j}^2} - \delta w_{n_j}| \geq |e^{-z_{n_j}^2}| - \delta |w_{n_j}| \\ &\geq e^{-\operatorname{Re}(z_{n_j}^2)} - \delta c(M+\varepsilon)^{n_j-1} \geq e^{\beta n_j} - \delta c(M+\varepsilon)^{n_j-1}. \end{aligned}$$

Hence, using $M > 1$ and $\varepsilon > 0$ sufficiently small,

$$e^{\beta n_j} \leq \delta c(M+\varepsilon)^{n_j-1} + c(M+\varepsilon)^{n_j+1} \leq c(\delta+1)(M+\varepsilon)^{n_j+1}.$$

Then

$$\beta \leq \frac{\log\big(c(\delta+1)\big)}{n_j} + \frac{n_j+1}{n_j}\log(M+\varepsilon) \to \log M$$

as $n_j \longrightarrow \infty$ and $\varepsilon \longrightarrow 0$.

(2) It is enough to show that $u_n(z_0, w_0) \to -\infty$ for any point $(z_0, w_0) \in W$ and it will follow for any compact subset of W. Since W is forward invariant, $F^n(z_0, w_0) \subset W \subset S$ for all $n \in \mathbb{N}$ and $\Delta(z_0, w_0) < 1$ by (2.9). Using the explicit expression for iterates of F given by (2.2), (2.3) we have

$$|z_n^2| = |z_n|^2 = \begin{cases} \delta^n |z_0 + \Delta_1^{n/2}(z_0, w_0)|^2 \geq \delta^n |z_0 - 1|^2 & \text{if n even;} \\ \delta^{(n+1)} |w_0 + \Delta_2^{(n+1)/2}(z_0, w_0)|^2 \geq \delta^{n+1} |w_0 - 1|^2 & \text{if n odd.} \end{cases}$$

In both cases, since $|z_0|, |w_0| > R_0 > 2$ we obtain $|z_n^2| \geq \delta^n$. Since $W = \bigcup_j W_j$ as defined in Sect. 2.2, $(z_0, w_0) \in W_j$ for some j, hence by Proposition 2.5,

$$F^n(z_0, w_0) \in W_{j+n} \quad \text{for all } n \in \mathbb{N},$$

hence $\left|\frac{\operatorname{Im} z_n}{\operatorname{Re} z_n}\right| \leq k_{j+n} < 1$ and by Lemma 2.12 we obtain

$$\left|\frac{\operatorname{Im} z_n^2}{\operatorname{Re} z_n^2}\right| \leq \frac{2k_{j+n}}{1-k_{j+n}^2} =: \alpha_n \sim n \quad \text{as } n \to \infty,$$

where the estimate $\alpha_n \sim n$ as $n \to \infty$ is computed using the explicit expression for k_{j+n}. It follows that

$$\delta^n \leq |z_n^2| = \sqrt{(\operatorname{Re}(z_n^2))^2 + (\operatorname{Im}(z_n^2))^2} \leq \operatorname{Re}(z_n^2)\sqrt{1+\alpha_n^2}$$

hence $\operatorname{Re}(z_n^2) \geq \frac{\delta^n}{\sqrt{1+\alpha_n^2}} \sim \frac{\delta^n}{n} \geq \delta^{n/2}$ for n large. Finally

$$u_n(z_0, w_0) = -\frac{\operatorname{Re}(z_n^2)}{n} \leq -\frac{\delta^{n/2}}{n} \to -\infty \text{ as } n \to \infty$$

(3) Suppose by contradiction that there exists $P = (z_0, w_0) \in \Omega \backslash \mathcal{A}_I$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$u_n(z_0, w_0) = \frac{-\operatorname{Re} z_n^2}{n} < -\varepsilon \quad \text{for all } n \geq N.$$

Hence there exists $N' > N$ depending on ε, C (where C is the constant used to define W_I) such that

$$\operatorname{Re}(z_n^2) > \varepsilon n > C^2 \text{ for all } n \geq N'.$$

Since $w_n = z_{n-1}$ and since $P \in \Omega$ for hypothesis, we have that $F^n(P) \in I \times I$ for all $n \geq N'$ hence $P \in F^{-n}(W_I) \subset \mathcal{A}_I$, a contradiction.

This concludes the proof. □

Lemma 2.14 (Good holomorphic disks) *Let $P \in \Omega$, W as before. Then there exists $\varphi : \overline{\mathbb{D}} \to \Omega$ holomorphic in a neighborhood of $\overline{\mathbb{D}}$ such that*

- $\varphi(0) = P$
- $\varphi(\mathbb{D}) \Subset \Omega$ *and* $\partial\varphi(\mathbb{D})$ *is analytic*
- *The one-dimensional Lebesgue measure of* $\partial\varphi(\mathbb{D}) \cap W$ *is greater than* 0.

Proof Since W is open it is enough to have $\varphi(\mathbb{D}) \cap W \neq \varnothing$ to ensure that the one-dimensional Lebesgue measure of $\partial\varphi(\mathbb{D}) \cap W$ is greater than 0. Let $a, b \in \{+, -\}$ such that $P \in \Omega^{ab}$. Since $W^{ab} \neq \varnothing$ for all $a, b \in \{+, -\}$ there exists $Q \in W^{ab}$. Since Ω^{ab} is connected and open there exists a simple real analytic curve passing through P and Q in Ω^{ab}. Complexifying this curve we obtain a holomorphic disc passing through P that we can write as $\varphi(\mathbb{D})$ for some φ holomorphic defined in a neighborhood of $\overline{\mathbb{D}}$. Up to precomposing φ with a Moebius transformation we can assume that $P = \varphi(0)$. □

In our proof, we are going to use the mean value property for the harmonic functions u_N.

Lemma 2.15 (Mean value property for holomorphic disks) *Let $\mathbb{D} \subset \mathbb{C}$ be the open unit disk and $\varphi : \overline{\mathbb{D}} \to \Omega$ be a holomorphic map. Let u be harmonic on the holomorphic open disk $D = \varphi(\mathbb{D})$ and continuous up to the boundary of D. Let $P_0 := \varphi(0)$. Then*

$$u(P_0) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)|\varphi'(\zeta)|^{-1} d\zeta$$

Proof Consider the function $u\circ\varphi : \overline{\mathbb{D}} \to \mathbb{R}$. First, note that it is harmonic on $\mathbb{D}$ and continuous up to the boundary. Indeed if $u : D \to \mathbb{R}$ is $\mathcal{C}^2$-smooth, then we can explicitly compute its Laplacian

$$\nabla^2(u\circ\varphi) = \nabla^2(u)|\varphi'|^2 = 0$$

while if u is not $\mathcal{C}^2$-smooth, the result follows by approximating u with harmonic smooth functions.

Hence for $u\circ\varphi$ the classical Mean Value Property holds. By computing $u(P_0)$ we get

$$u(P_0) = u(\varphi(0)) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\varphi(\eta))\,d\eta = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)\,|\varphi'(\zeta)|^{-1}\,d\zeta. \tag{2.19}$$

□

Proof of Proposition 2.10 Let $P\in\Omega\backslash\mathcal{A}_I$ and $D := \varphi(\mathbb{D})$ where φ is given by Lemma 2.14. Let μ be the pushforward under φ of the one-dimensional Lebesgue measure on $\partial\mathbb{D}$. Let K be a compact subset of W such that $\mu(K\cap\partial D) > 0$.

Let $\mu_{\mathrm{good}} = \mu(\partial D\cap K) > 0$ and $\mu_{\mathrm{bad}} = \mu(\partial D\cap(\Omega\backslash K))$. Since Ω contains D, $\partial D = (\partial D\cap K)\cup(\partial D\cap(\Omega\backslash K))$, and since K is compact and Ω is open, the sets in question are measurable.

By Lemma 2.11 for any given $\mathcal{M} > 0$ there exists N such that $u_N \le -\mathcal{M}$ on K, $u_N(P) \ge -\varepsilon$ for some $\varepsilon > 0$ since $P\in\Omega\backslash\mathcal{A}_I$, and $u_N \le \log M$ on $\overline{D}$ (with $M = M(\overline{D})$). By the Mean value property (2.19) for u_N we have

$$\begin{aligned} -\varepsilon \le u_N(P) = \frac{1}{2\pi}\int_{\partial D} u_N(\zeta)|\varphi'(\zeta)|d\zeta &= \frac{1}{2\pi}\int_{\partial D\cap K} u_N(\zeta)|\varphi'(\zeta)|d\zeta \\ &+ \frac{1}{2\pi}\int_{\partial D\cap(\Omega\backslash K)} u_N(\zeta)|\varphi'(\zeta)|d\zeta \le \\ &\le \frac{1}{2\pi}\left(-\mathcal{M}\mu_{\mathrm{good}} + \log M\mu_{\mathrm{bad}}\right)\cdot \sup_{\partial\mathbb{D}}|\varphi'|^{-1}. \end{aligned}$$

Since $\mathcal{M}$ is arbitrarily large, this gives a contradiction. □

Proposition 2.16 *The set W_I is open.*

Proof Let $P\in W_I$. We want to find $V\subset W_I$ neighborhood of P. Since $W_I\subset\Omega\cap(I\times I)$ which is open there is a neighborhood U of P which is compactly contained in $\Omega\cap(I\times I)$. Since W_I is absorbing for Ω under F there exists $N > 0$ such that

$$F^n(\overline{U})\subset W_I \quad \text{for all}\, n\ge N. \tag{2.20}$$

As usual let us define $P_j := F^j(P)$; by definition of W_I, $P_j\subset I\times I$ for all $j\ge 0$, which is an open set. Hence for each $j\ge 0$ there is a neighborhood $U_j\subset I\times I$ of P_j. So up to making the U_j smaller, we can assume that $U_j\subset F^j(U)$.

Let

$$V := \bigcap_{j=0}^{N} F^{-j}(U_j)\subset U.$$

The set V is open since it is a finite intersection of open sets. We only need to check that $V\subset W_I$, or equivalently, that $F^j(V)\subset I\times I$ for all $j\ge 0$. For $j\le N-1$, this is true by definition, since $F^j(V)\subset U_j\subset I\times I$. For $j\ge N$, this is true by (2.20). Since $P\in V$ by construction, V is a neighborhood of P in W_I as required. □

2.4 Geometric structure of Ω

In this section we show that Ω is the union of four disjoint Fatou components Ω^{ab}, $a, b \in \{+, -\}$, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$.

We first show conjugacy of F to its linear part on Ω, and estimate the distance between the conjugacy and the identity map.

Proposition 2.17 (Conjugacy) *F is conjugate to the linear map $L(z, w) = (-\delta w, z)$ on Ω via an injective holomorphic map φ. If P is such that $F^n(P) \in S$ for all $n \in \mathbb{N}$, then $\|(\varphi - Id)(P)\| < \sqrt{2}$. Finally, $\varphi(\Omega) \subset S$.*

Proof We first show that F is conjugate to L on W_I.

For $n \in \mathbb{N}$ let $\varphi_n : \mathbb{C}^2 \to \mathbb{C}^2$ be the automorphisms defined as

$$\varphi_n := L^{-n} \circ F^n.$$

If we show that the φ_n converge to a map φ uniformly on W_I we obtain that φ satisfies the functional equation $\varphi = L^{-1} \circ \varphi \circ F$ and hence is a conjugacy between F and L.

Computing L^{-n} and using the explicit expressions for the iterates of F we obtain

$$\varphi_{2n}(z, w) = \left(z + \Delta_1^n(z, w), w + \Delta_2^n(z, w)\right), \tag{2.21}$$

$$\varphi_{2n+1}(z, w) = \Big(z + \Delta_1^n(z, w), w + \Delta_2^{n+1}(z, w)\Big). \tag{2.22}$$

Both have the same formal limit

$$\varphi(z, w) = (z + \Delta_1(z, w), w + \Delta_2(z, w)).$$

If $P = (z, w) \in W_I$, then $F^n(P) = (z_n, w_n) \subset I \times I \subset S$ for all j, hence, by (2.9), we have that $\Delta(z, w) < 1$; in particular, $\Delta_1(z, w)$ and $\Delta_2(z, w)$ are convergent. Hence φ is a holomorphic map from W_I to $\varphi(W_I)$ (W_I is open by Proposition 2.16). Moreover, for any point (z, w) whose orbit is contained in S,

$$\begin{aligned} \|(\varphi - \mathrm{Id})(z, w)\| &= \|(\Delta_1(z, w), \Delta_2(z, w))\| \\ &< \sqrt{2}\Delta(z, w) < \sqrt{2}. \end{aligned} \tag{2.23}$$

It follows that φ is open because W_I is an unbounded set, hence if φ had rank 0 or 1, $\|(\varphi - \mathrm{Id})\|$ could not be bounded on W_I. Hence the map φ is injective by Hurwitz Theorem (see [19, Exercise 3 on page 310]) because the maps φ_n are injective and their limit has rank 2. It follows that φ is a biholomorphism between W_I and $\varphi(W_I)$.

To extend φ to all of Ω recall that W_I is absorbing for Ω. So if $P \in \Omega$, we have that $F^k(P) \in W_I$ for some $k \in \mathbb{N}$, hence we can define $\varphi(P) = L^{-k} \circ \varphi \circ F^k(P)$. Since F is an automorphism, φ extends as a biholomorphism to Ω.

It remains to show that $\varphi(\Omega) \subseteq S$. By (2.23) we have that $\varphi(W_I)$ is contained in a $\sqrt{2}$ neighborhood U of W_I. Suppose by contradiction that there exists $Q = (z, w) \in \varphi(W_I) \backslash S$. Since W_I is forward invariant under F and φ is a conjugacy we have that $\varphi(W_I)$ is forward invariant under L. Up to considering $L(Q)$ if necessary, and since θ is such that $re^{i\theta} \notin S$, we can assume that $z = re^{i\theta} \notin S$. By forward invariance $L^{2n}(Q) = ((-\delta^n)re^{i\theta}, (-\delta)^n w) \in \varphi(W_I)$.

Since $(-\delta)^n r$ tends to infinity, the distance of $L^{2n}(Q)$ from the boundary of S tends to infinity, hence so does the distance of $L^{2n}(Q)$ from $W_I \subset S$, contradicting $\varphi(W_I) \subset U$.

Hence $\varphi(W_I) \subset S$. Since W_I is an absorbing set for Ω under F, $\varphi \circ F = L \circ \varphi$, and $\varphi(W_I)$ is completely invariant under L, we have that

$$\varphi(\Omega) = \varphi\left(\bigcup_{n \geq 0} F^{-n}(W_I)\right) = \bigcup_{n \geq 0} L^{-n}(\varphi(W_I)) \subset \varphi(W_I) \subset S. \tag{2.24}$$

□

We are now able to understand the geometric structure of Ω.

Proposition 2.18 (Geometry of Ω) *Ω consists of four distinct connected components, each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$, and which form a cycle of period 4.*

We recall the following simple topological lemma. Here ∂ denotes the topological boundary.

Lemma 2.19 *Let $A, B \subset \mathbb{C}^n$ be open, A connected. If $A \cap B \neq \varnothing$ and $\partial B \cap A = \varnothing$ then $A \subseteq B$.*

Proof Since $A \cap \partial B = \varnothing$ we can write

$$A = (A \cap B) \cup (A \backslash \overline{B}).$$

Both $A \cap B$ and $A \backslash \overline{B}$ are open and $A \cap B \neq \varnothing$ by assumption, so since A is connected, $A \backslash \overline{B} = \varnothing$. □

Recall also that if a set A is invariant under a map F, by continuity of the latter we have $F(\overline{A}) \subset \overline{A}$. The following lemma is also known.

Lemma 2.20 *Let Ω_1, Ω_2 be two Fatou components for an automorphism F of $\mathbb{C}^2$. Then if $F(\Omega_1) \cap \Omega_2 \neq \varnothing$, $F(\Omega_1) = \Omega_2$.*

Proof We have that $F(\Omega_1) \subset \Omega_2$, indeed otherwise, $F(\Omega_1)$ would intersect the boundary of Ω_2 which is contained in the forward Julia set, and this is impossible because the Fatou set is completely invariant. On the other hand suppose for a contradiction that there is $P \in \Omega_2 \backslash F(\Omega_1)$. Then since $F(\Omega_1) \cap \Omega_2 \neq \varnothing$ and both Ω_1, $F(\Omega_1)$ are connected there exists $Q \in \Omega_2 \cap F(\partial \Omega_1)$, which is impossible because $\partial \Omega_1$ is contained in the forward Julia set which is forward invariant. □

Observe that we could not simply use the same argument applied to F^{-1}, since the Fatou components for F and F^{-1} are, in general, different sets.

Proof of Proposition 2.18 We prove the claim by showing that Ω is biholomorphic to S. Since S has four connected components S^{ab} each of which is biholomorphic to $\mathbb{H} \times \mathbb{H}$, the same holds for Ω. Since by definition $\Omega = \bigcup_{a,b \in \{+,-\}} \Omega^{ab}$ and each Ω^{ab} is connected, these are exactly the connected components of Ω.

Recall the definition of the set $W \subset S$ from Sect. 2.2 and recall that it is forward invariant and contained in S. Hence (2.23) holds. Also recall that by (2.24) $\varphi(\Omega) \subset S$.

Let U be a $2\sqrt{2}$-neighborhood of ∂W. Fix $a, b \in \{+, -\}$. We want to apply Lemma 2.19 to the sets $A = W^{ab} \backslash \overline{U}$ and $B = \varphi(W^{ab})$. So we need to show that

- $A \cap B = (W^{ab} \backslash \overline{U}) \cap \varphi(W^{ab}) \neq \varnothing$
- $\partial B \cap A = \partial \varphi(W^{ab}) \cap (W^{ab} \backslash \overline{U}) = \varnothing$.

The second item is true because $\partial(\varphi(W^{ab})) \subset \varphi(\partial W^{ab}) \Subset U$ by (2.23). So we now show that $(W^{ab}\backslash\overline{U}) \cap \varphi(W^{ab}) \neq \varnothing$. Let $P \in W^{ab}$ such that the ball of radius $\sqrt{2}$ centered at P is contained in $W^{ab}\backslash\overline{U}$. This is possible because this set contains arbitrarily large balls. By (2.23), $\|P - \varphi(P)\| < \sqrt{2}$ hence $\varphi(P) \in W^{ab}\backslash\overline{U}$.

Hence, applying Lemma 2.19 we obtain that for each $a, b \in \{+, -\}$ we have $\varphi(W^{ab}) \supset (W^{ab}\backslash U)$ hence

$$\varphi(\Omega) \supset \varphi(W) \supset (W\backslash U).$$

We now show that this implies that $\varphi(\Omega) \supset S$. Notice that S can be written as

$$S = \left\{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \in \mathbb{C}^2 : r_1, r_2 > 0, \text{ and for each } i = 1, 2 \text{ either } |\theta_i| < \frac{\pi}{4} \text{ or } |\theta_i - \pi| < \frac{\pi}{4}\right\}.$$

Fix $\alpha < \frac{\pi}{4}$. By definition of W there exists $R = R(\alpha)$ such that $W\backslash U$ contains the set

$$W\backslash U \supset X_{\alpha,R} := \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : r_1, r_2 > R \text{ and for each } i = 1, 2 \text{ either} |\theta_i| < \alpha \text{ or } |\theta_i - \pi| < \alpha\}.$$

Hence $\varphi(\Omega) \supset \varphi(W) \supset (W\backslash U) \supset X_{\alpha,R}$.

By the explicit form of L, $\bigcup_{j\geq 0} L^{-j} X_{\alpha,R} = X_{\alpha,0}$. Hence by backward invariance of $\varphi(\Omega)$ under L we have that

$$\varphi(\Omega) \supset \bigcup_{j\geq 0} L^{-j} X_{\alpha,R} = X_{\alpha,0} \quad \text{for every } \alpha < \frac{\pi}{4}.$$

It follows that

$$\varphi(\Omega) \supset \bigcup_{\alpha<\frac{\pi}{4}} X_{\alpha,0} = S.$$

Hence $\varphi(\Omega) = S$.

It remains to show that the Fatou components Ω^{ab} with $a, b \in \{+, -\}$ form a cycle of period four, more precisely, that

$$F(\Omega^{ab}) = \Omega^{(-b)a} \quad \text{for all } a, b \in \{+, -\}. \tag{2.25}$$

By definition $\Omega^{ab} \supset W^{ab}$ and by Lemma 2.2, $F(W^{ab}) \cap W^{(-b)a} \neq \varnothing$. Hence $F(\Omega^{ab}) \cap \Omega^{(-b)a} \neq \varnothing$. By Lemma 2.20, $F(\Omega^{ab}) = \Omega^{(-b)a}$. □

2.5 Limit sets on Ω^{ab}

Let $\mathbb{H}$ and $-\mathbb{H}$ denote the right and left half plane respectively. In this section we show the following.

Proposition 2.21 (Limit set for Ω) *We have that*

$$h_1(\Omega^{ab}) = \mathbb{H} \quad \textit{and} \quad h_2(\Omega^{ab}) = -\mathbb{H} \textit{ if } a = b$$
$$h_1(\Omega^{ab}) = -\mathbb{H} \quad \textit{and} \quad h_2(\Omega^{ab}) = \mathbb{H} \textit{ if } a \neq b.$$

Let W be as defined in Sect. 2.2 and W_I as defined in Sect. 2.3. Since both are forward invariant and contained in S we have that, for any $a, b \in \{+, -\}$, $F(W^{ab}) \subset W^{(-b)a}$ and $F(W_I^{ab}) \subset W_I^{(-b)a}$. Compare with Lemma 2.2 and Corollary 2.3.

We first study the image of W_I^{ab} under h_1, h_2.

Lemma 2.22

$$\begin{array}{lcl} h_1(W_I^{ab}) \subset \mathbb{H} & \textit{and} & h_2(W_I^{ab}) \subset -\mathbb{H} \ \textit{if}\, a=b \\ h_1(W_I^{ab}) \subset -\mathbb{H} & \textit{and} & h_2(W_I^{ab}) \subset \mathbb{H} \ \textit{if}\, a \neq b \end{array}$$

Proof Recall that $h_1(z_0, w_0) = \lim_{n\to\infty} \frac{z_{2n}}{w_{2n}}$. Let $I_+ := \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $I_- := \left(\frac{3}{4}\pi, \frac{5}{4}\pi\right)$.

For $a, b \in \{+, -\}$ and $(z, w) \in W_I^{ab}$, then $\arg(z) \in I_a$ and $\arg(w) \in I_b$. Hence $\arg\left(\frac{z}{w}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ if $a = b$, and $\arg\left(\frac{z}{w}\right) \in \left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$ if $a \neq b$. Since $F^2(W_I^{++} \cup W_I^{--}) \subset W_I^{++} \cup W_I^{--}$, If $(z, w) \in W_I^{++} \cup W_I^{--}$ then all of its even iterates $(z_{2n}, w_{2n}) \in W_I^{++} \cup W_I^{--}$, hence by taking the limit $h_1(W_I^{++}), h_1(W_I^{--}) \subset \overline{\mathbb{H}}$. Similarly if $(z, w) \in W_I^{+-} \cup W_I^{-+}$ then all of its even iterates $(z_{2n}, w_{2n}) \in W_I^{+-} \cup W_I^{-+}$, hence $h_1(W_I^{+-}), h_1(W_I^{-+}) \subset \overline{-\mathbb{H}}$. The analogous results for h_2 hold by observing that $h_2 = \frac{-\delta}{h_1}$. Since W_I is open by Proposition 2.16, its image under a holomorphic map of maximal rank is open, hence we can replace $\overline{\mathbb{H}}, \overline{-\mathbb{H}}$ by $\mathbb{H}, -\mathbb{H}$. □

Lemma 2.23

$$\begin{array}{lcl} \mathbb{H} \subset h_1(W^{ab}) & \textit{and} & -\mathbb{H} \subset h_2(W^{ab}) \ \textit{if}\, a=b \\ -\mathbb{H} \subset h_1(W^{ab}) & \textit{and} & \mathbb{H} \subset h_2(W^{ab}) \ \textit{if}\, a \neq b. \end{array}$$

Before proving Lemma 2.23 let us see how Lemmas 2.22 and 2.23 imply Proposition 2.21.

Proof of Proposition 2.21 We prove the claims for h_1; for $h_2 = \frac{-\delta}{h_1}$, it follows by symmetry.

Clearly $h_1(\Omega^{ab}) \supset h_1(W^{ab})$ for any $a, b \in \{+, -\}$ since $\Omega^{ab} \supset W^{ab}$. So in view of Lemma 2.23, $h_1(\Omega^{ab}) \supset -\mathbb{H}$ or $h_1(\Omega^{ab}) \supset \mathbb{H}$ depending on whether $a = b$.

We now consider limit sets for Ω^{++} and Ω^{--}; the other cases are analogous. By (2.25),

$$F^2(\Omega^{++}) \subset \Omega^{--} \quad \text{and} \quad F^2(\Omega^{--}) \subset \Omega^{++}.$$

It follows that for any $n > 0$,

$$F^{2n}(\Omega^{++} \cup \Omega^{--}) \subset \Omega^{++} \cup \Omega^{--}.$$

In view of this, and since W_I is absorbing for Ω, we have that for any $P \in (\Omega^{++} \cup \Omega^{--})$

$$F^{2n}(P) \subset W_I \cap (\Omega^{++} \cup \Omega^{--}) = W_I^{++} \cup W_I^{--} \text{ for any } n \text{ large enough.}$$

Hence $h_1(P) \in h_1(W_I^{++} \cup W_I^{--}) \subset \mathbb{H}$ for every $P \in (\Omega^{++} \cup \Omega^{--})$, hence $h_1(\Omega^{++} \cup \Omega^{--}) \subset \mathbb{H}$. It follows that $h_1(\Omega^{++}) = h_1(\Omega^{--}) = \mathbb{H}$. □

We devote the rest of this section to proving Lemma 2.23. We first give a version of Rouché's Theorem which relies on one of the many versions of Rouché's Theorem existing in one variable (compare with Theorem 3.4 in [8]; we will use it with the spherical instead of the Euclidean metric). This is certainly known to experts in the field but we are not aware of a reference. In this section ∂ denotes the topological boundary, and $\mathrm{dist}_{\mathrm{spher}}$ denotes the spherical distance.

Theorem 2.24 (Rouché's Theorem in $\mathbb{C}^2$) *Let $B \subset \mathbb{C}^2$ be a polydisk, F, G be holomorphic maps defined in a neighborhood of $\overline{B}$ which take values in $\hat{\mathbb{C}}$. Let $c \in G(B)$, let $\varepsilon =$* $\mathrm{dist}_{\mathrm{spher}}(c, G(\partial B)) > 0$ *and assume*

$$\mathrm{dist}_{\mathrm{spher}}(F, G) < \varepsilon \quad \textit{on } \partial B.$$

Then $c \in F(B)$.

Notice that the assumptions imply that F, G have generic rank 1: They cannot have rank 2 because the target is $\hat{\mathbb{C}}$, and G cannot be constant otherwise there could not be $c \in G(B)$ with positive distance from $G(\partial B)$. One can check that F cannot be constant either.

Proof Let D be a horizontal disk passing through a point $P_c \in G^{-1}(c) \cap B$, such that $\partial D \subset \partial B$. Let $g := G|_D$, $f := F_D$. They are holomorphic in a slightly larger horizontal disk. Notice that $\mathrm{dist}_{\mathrm{spher}}(g, f) < \varepsilon$ on ∂D, and that $\mathrm{dist}_{\mathrm{spher}}(G(\partial D), c) \geq \varepsilon$ because $\partial D \subset \partial B$. By Rouché's Theorem in one variable, $c \in f(D) \subset F(B)$ as required. □

Remark 2.25 Unless P_c is an isolated point in $G^{-1}(c)$ we obtain a curve of points in $F^{-1}(c)$. Indeed, the proof gives a point in $F^{-1}(c)$ for any Euclidean disk passing through points in $G^{-1}(c)$, for example, a family of disks passing through P_c along different complex directions. The points obtained for $F^{-1}(c)$ are distinct unless they always coincide with P_c. On the other hand, if P_c is an isolated point in $G^{-1}(c)$ then $P_c \in F^{-1}(c)$ is also isolated in $F^{-1}(c)$. Indeed otherwise we could reverse the role of F and G and obtain one point in $G^{-1}(c)$ for any Euclidean disk passing through any point in $F^{-1}(c)$ and obtain a curve of points in $G^{-1}(c)$. The proof as it is works when B is any $\mathbb{C}$-convex set instead of a polydisk, and can certainly be generalized further.

Proof of Lemma 2.23 We show $\mathbb{H} \subset h_1(W_I^{++})$. The other cases are analogous. Recall that orbits of points in W are contained in S, hence (2.9) holds. Since

$$\frac{z_{2n}}{w_{2n}} = \frac{z_0 + \Delta_1^n(z_0, w_0)}{w_0 + \Delta_2^n(z_0, w_0)},$$

dividing the numerator and the denominator by w_0 and using $\frac{1}{1+x} = 1 + \sum_{j=1}^{\infty}(-x)^j$ for $|x| < 1$ we have that

$$\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0} = \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0, w_0)}{w_0}\right)\left(1 + \sum_{j=1}^{\infty}\left(\frac{-\Delta_2^n(z_0, w_0)}{w_0}\right)^j\right) - \frac{z_0}{w_0} \tag{2.26}$$

$$= \frac{\Delta_1^n(z_0, w_0)}{w_0} + \left(\frac{z_0}{w_0} + \frac{\Delta_1^n(z_0, w_0)}{w_0}\right)\sum_{j=1}^{\infty}\left(\frac{-\Delta_2^n(z_0, w_0)}{w_0}\right)^j \ \forall n \geq 0. \tag{2.27}$$

This expression makes sense for $|x| = \left|\frac{-\Delta_2^n(z_0,w_0)}{w_0}\right| < 1$, hence, in view of (2.9), for $|w_0| > 1$. Recall also that $|\sum_{j=1}^{\infty} x^j| = \frac{|x|}{1-x} \leq 2|x|$ if $|x| < \frac{1}{2}$. Let $K \subset \hat{\mathbb{C}}$ be a compact set and suppose that $\frac{z_0}{w_0}$ takes values in K. By (2.26) and using (2.9), for any $\varepsilon > 0$ there exists $M = M(K, \varepsilon)$ such that

$$\left|\frac{z_{2n}}{w_{2n}} - \frac{z_0}{w_0}\right| < \varepsilon \text{ for } |w_0| > M \text{ and } \frac{z_0}{w_0} \in K. \tag{2.28}$$

Consider the function $G(z, w) := \frac{z}{w}$ Observe that

$$G^{-1}(re^{i\theta}) = \left\{(r_1e^{i\theta_1}, r_2e^{i\theta_2}) \in \mathbb{C}^2 : \frac{r_1}{r_2} = r, \theta = \theta_1 - \theta_2\right\}.$$

Let $c \in \mathbb{H}$. By the shape of W we have that $G(W^{++}) = \mathbb{H}$, that $\varepsilon := \frac{1}{2}\,\mathrm{dist}_{\mathrm{spher}}(c, G(\partial W)) > 0$, and that we can choose $Q = (z_0, w_0) \in W^{++} \in G^{-1}(c)$

such that $|w_0|$ is arbitrarily large. By taking a limit in n in equation (2.26) and on a sufficiently small polydisk centered at Q we can ensure that $\mathrm{dist}_{\mathrm{spher}}(h_1, G) < \varepsilon$, hence the claim follows by Rouché's Theorem. □

The main Theorem is a direct consequence of Propositions 2.6, 2.17, 2.18, 2.21.

3 Appendix: Proof of Proposition 2.13

We split the proof of Proposition 2.13 over several lemmas.

Definition 3.1 Let $E \subset \mathbb{C}^n$. A vector $v \in \mathbb{C}^n$ is called tangent to E at a point $P \in \overline{E}$ if there exist a sequence of points $P_J \in E$ and real numbers $t_j > 0$ such that $P_i \to P$ and $t_j(P_j - P) \to v$ as $j \to \infty$. The set of all such tangent vectors is the *tangent cone* to E at P.

The tangent cone is indeed a cone in $\mathbb{C}^n = T_P\mathbb{C}^n$. If the set E is a $\mathcal{C}^1$-smooth manifold, the tangent cone coincides with the tangent space.

For complex analytic sets of dimension one, the following is a well known fact. For a proof, see [11, Corollary on page 80].

Lemma 3.2 *Let $H \subset \mathbb{C}^n$ be an analytic set of dimension one. For all $x \in H$ the tangent cone of H at x consists of a finite union of complex lines (whose number is not greater than the number of irreducible components of H at x).*

Definition 3.3 Let $B \subset \mathbb{C}^n$ be a polidisc. The torus $\mathbb{T}$ with same center and same poliradius as B is called its *Šilov boundary*. We will denote it by $\partial_S B$.

The Šilov boundary is a very general notion, for Banach algebras, but we will not need it here in all generality. For details, we refer to [12, from page 325].

Lemma 3.4 *Let B be a polydisk, $\partial_S B$ be its Šilov boundary, and $u : U \to \mathbb{R}$ be a harmonic function defined on a neighbourhood U of $\overline{B}$. Then*

$$\max_{\overline{B}} u = \max_{\partial_S B} u.$$

Recall that ∂ denotes the topological boundary.

Proof For every $P \in \partial B \backslash \partial_S B$ there is a horizontal Euclidean disc D through P which is contained in ∂B whose boundary is in $\partial_S B$. Being u harmonic in a neighbourhood of $\overline{B}$, u is harmonic on such a closed disc, hence its value at P is less or equal to the maximum of u at its boundary $\partial D \subset \partial_S B$. Hence

$$\max_{\partial B} u = \max_{\partial_S B} u.$$

If $P \in B$, we can find a disc through P with boundary in ∂B and repeat the argument, getting the conclusion

$$\max_{\overline{B}} u = \max_{\partial B} u = \max_{\partial_S B} u.$$

□

Lemma 3.5 *Let $H \subset \mathbb{C}^2$ be an analytic set of dimension one. Then for every $P \in H$ there exists an arbitrarily small torus $\mathbb{T}_P$ centered in P such that $\mathbb{T}_P \cap H = \varnothing$.*

Proof Let $P \in H$. Consider the tangent cone C_P of H at P. By Lemma 3.2, C_P is a finite set of directions $\alpha_1, \ldots, \alpha_k \in \hat{\mathbb{C}}$. Up to a rotation, we can suppose all directions to be in $\mathbb{C}$. Up to choosing $\eta > 0$ small enough, we can ensure that the polidisk B_η of poliradius η centered in P intersects only one connected component of H. Moreover, by the definition of tangent cone, we can choose a small neighbourhood $K \subset \mathbb{C}$ of all α_j such that

$$H \cap U_\eta \ \subset \ \cup_{\alpha \in K} (P + (z, \alpha z)).$$

We can suppose K to be small enough that there is $0 < b < 1$ such that $K \cap \{\beta \in \mathbb{C} \mid |\beta| = b\} = \varnothing$.

For any $0 < \delta < \eta$, defining

$$\mathbb{T}_P = \{|z - z_P| = \delta\} \times \{|w - w_P| = b\delta\}$$

we have that $\mathbb{T}_P \subset U_\delta$ and if $(z, w) \in \mathbb{T}_P$, $\frac{w-w_P}{z-z_P} = \beta$ with $|\beta| = b$. So

$$\mathbb{T}_P \cap H \ = \ T_P \cap U_\eta \cap H \ = \ \varnothing.$$

□

Proof of Proposition 2.13 Let K and L be compact sets as in the statement. For each $P \in H \cap K$, by Lemma 3.5 there exists a torus $\mathbb{T}_P \subset L$ centered in P such that $\mathbb{T}_P \cap H = \varnothing$. Each torus $\mathbb{T}_P$ is the Šilov boundary of a polidisk B_P centered in P. Since $\{B_P\}_{P \in H \cap K}$ is a covering of $H \cap K$, by compactness we can extract a finite covering $\{B_1, \ldots, B_k\}$.

There is a η-neighbourhood U_η of H such that $U_\eta \cap K \subset \cup B_j$. If the harmonic function u satisfies $u \leq \alpha$ on $L \backslash U_\eta$ then it satisfies the same estimate on all tori $\mathbb{T}_j$, and by Lemma 3.4 the same estimate holds on all B_j. Hence $u \leq \alpha$ on K. □

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