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Abstract

Nonlinear Fredholm-Hammerstein integral equations with logarithmic kernel are here taken into account and numerically solved by spline quasi-interpolating projectors based collocation and Kulkarni methods, both in their basic and iterated versions. Theoretical analysis of discretization error and convergence order is provided, together with numerical results validating the estimates obtained under the hypothesis of sufficiently smooth solutions. Finally, some results in case of less regular solutions show the robustness of the proposed approach even in a non smooth framework.

Keywords: Fredholm-Hammerstein integral equation; Logarithmic kernel; Spline quasi-interpolating projector; Kulkarni

1. Introduction

Many problems in the applied sciences lead to mathematical models described by nonlinear integral equations. For instance, the Fredholm-Hammerstein integral equations appear in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electromagnetic fluid dynamics, antenna synthesis problem, communication theory, mathematical economics, population genetics, radiation, the particle transport problems of astrophysics and reactor theory, fluid mechanics (see e.g. [13, 22] and references therein). In the nineties, many papers appeared and handled these integral equations. Among them, significant results have been proved in [13, 25, 26, 27, 32] and, in the literature, it is highlighted that the convergence orders of numerical methods are affected by possible singular behaviour of the solutions near the domain boundary.

On the other side, various practical engineering problems, such as for instance the analysis of linear transport equation in slab geometry [31] or the study of steady potential flow past obstacles [29], such as around airfoils by Boundary Element Methods (BEMs) [18], give rise to integral equations with weakly singular kernels of logarithmic type.

Here, the focus is on the numerical solution of nonlinear Fredholm-Hammerstein integral equations with logarithmic kernel, appearing for instance in the integral reformulation of two-dimensional elliptic boundary value problems with a nonlinear boundary condition. For example, in [35], several applications of integral equations are presented and for the considered model the exterior boundary value problem for the two-dimensional Helmholtz equation is proposed. In general, it is difficult to find the exact solution of these type of integral equations and hence to obtain approximate solutions is often mandatory. The problem has recently received increasing attention and several papers in the literature investigate the topic, see e.g. [10, 11, 24, 30].

Classical methods to search an approximate solution of the above kind of problems are the projection ones, such as the Galerkin and collocation; recently, a modified projection method, based on a sequence of orthogonal or interpolatory projectors onto finite dimensional subspaces, usually spaces of piecewise polynomials

of degree d at most continuous, has been proposed by Kulkarni [24, 28].
On the other side, the use of the spline quasi-interpolation has been proved to work well for the approximation of the solution of Fredholm integral equations, also in the bivariate case (see e.g. [5, 6, 7, 8, 16, 19, 20, 21]). Therefore, here the novelty w.r.t. the state of the art lies in the use of spline Quasi-Interpolating Projectors (QIPs) in the space of splines of degree d and smoothness C^{d-1}, as approximation tools in the framework of collocation and Kulkarni discretization techniques for the solution of nonlinear Fredholm-Hammerstein integral equations with logarithmic kernel, under the hypothesis that the data are assigned such that the problems admit sufficiently regular solutions (for the interested reader, some examples of integral equations with logarithmic kernel and regular solutions can be found in [12, 4, 35, 9, 14]).
This in order to theoretically study the convergence order of the proposed numerical approaches. In fact, the regularity of the solution is needed to take advantage of the use of the spline QIPs, since their convergence properties, on which the convergence order of the employed methods relies, are based on the regularity of the function to be approximated, i.e. in this framework, the solution of the integral equation [17].
We underline that this work can be conceived as a prosecution of [21, 19], where spline QIPs were introduced for the discretization of linear and nonlinear Fredholm integral equations, in both cases with smoother kernels. We also remark that the use of smooth splines allows an advantage from the computational point of view with respect to the growth of the dimension of linear systems to be solved in the construction of the approximate solutions, as explained at the end of Section 4.1.
The paper is structured as follows: in the next Section 2 the problem at hand is described, while in Section 3 spline QIPs are briefly recalled. Then in Section 4, spline QIP based collocation and Kulkarni numerical methods, in basic and iterated versions, are illustrated and their discretization error, with subsequent convergence order, is studied. Section 5 contains numerical results that validate the theoretical estimates. Moreover, the reader will find some results in case of less regular solutions, which show the robustness of the proposed approach even in a non smooth framework. Conclusions are summarized, together with future research lines, in Section 6. For readers' convenience, basic quadrature rules for logarithmic kernel are resumed from [2] in Appendix A, together with the description of their iterated use as implemented for the numerical simulations, while an extension of a theoretical result in [19], needed to prove some error estimates, is given in Appendix B.

2. Fredholm-Hammerstein integral equations with logarithmic kernel

In this paper we consider Fredholm-Hammerstein integral equations of the second kind of the form

$$x - K(x) = f, \tag{2.1}$$

where K is the integral operator

$$K(x)(s) := \int_0^1 k(s,t)\psi(t,x(t))dt, \qquad s \in \mathcal{I} := [0,1], \qquad x \in X := C(\mathcal{I}),$$

$k(s,t) = \log|s-t|$ is the logarithmic kernel taken into account, f and ψ are known functions and x is the unknown solution to be determined.

We assume throughout this paper the following conditions on f and ψ:

1. $f \in X$,
2. for $x \in X$, $\psi(\cdot, x(\cdot)) \in X$,
3. $\psi(t,x(t))$ is bounded and continuous over $\mathcal{I} \times \mathbb{R}$ and Lipschitz continuous in x, i.e. there exists a constant $c_1 > 0$ for which $|\psi(t,u) - \psi(t,v)| \le c_1 |u-v|$, $\forall u, v \in \mathbb{R}$,
4. the partial derivative $\frac{\partial \psi}{\partial x}(t,x(t))$ of ψ with respect to the second variable exists and we assume it is Lipschitz continuous with respect to the second variable, i.e. there exists $c_2 > 0$ such that

$$\left| \frac{\partial \psi}{\partial x}(t,u) - \frac{\partial \psi}{\partial x}(t,v) \right| \le c_2 |u-v|, \quad \forall u, v \in \mathbb{R},$$

5. $B = \sup_{t\in\mathcal{I}} \left| \frac{\partial\psi}{\partial x}(t, x(t)) \right| < \infty$.

The Fréchet derivative of K is given by [30]

$$(K'(x)q)(s) = \int_0^1 k(s,t)\frac{\partial\psi}{\partial x}(t, x(t))q(t)dt \tag{2.2}$$

(for a general definition see [15]).
Let us note that, for any $s \in \mathcal{I}$, we have

$$\int_0^1 |k(s,t)|\, dt \le p_1 < \infty, \qquad \text{where } p_1 = 1.7 \tag{2.3}$$

and if $c_2\, p_1 < 1$ equation (2.1) admits a unique solution [25], denoted in the following by φ. Furthermore,

$$\lim_{t\to\tau} \|k(\cdot,t) - k(\cdot,\tau)\|_1 = 0, \qquad \tau \in \mathcal{I}.$$

Finally, we define the operator T by $T(u) = f + K(u)$, so that (2.1) can be written as $x = T(x)$.

3. Spline quasi-interpolating projectors

In order to make the paper self-contained, in this section we recall definition and properties of the spline QIPs proposed in [19].

Let us consider the space of splines of degree d and class $C^{d-1}(\mathcal{I})$ on the uniform knot partition $\mathcal{T}_n := \{t_i = ih, 0 \le i \le n\}$, with $h = 1/n$. We denote such a space by X_n and here we consider QIPs on X_n of the following form

$$\pi_n x := \sum_{i=1}^{N} \lambda_i(x) B_i, \qquad x \in X \tag{3.1}$$

where:

- $N := \dim(X_n) = n + d$;
- $\mathcal{T}_n^e := \mathcal{T}_n \cup \{t_{-d} = \ldots = t_0 = 0; 1 = t_n = \ldots = t_{n+d}\}$ is the usual extended knot sequence associated with $\mathcal{T}_n$;
- $\{B_i\}_{i=1}^N$ are the B-splines with support $\mathrm{supp} B_i = [t_{i-d-1}, t_i]$ on $\mathcal{T}_n^e$, forming a basis for X_n;
- $\{\lambda_i\}_{i=1}^N$ are point coefficient functionals of the form

  $$\lambda_i(x) := \sum_{j=2(i-d-1)}^{2i} \sigma_{i,j} x_j, \qquad x_j := x(\xi_j) \tag{3.2}$$

  based on the quasi-interpolation nodes $\{\xi_j\}_{j=0}^{2n}$ (the QI nodes involved in (3.2) are inside $\mathrm{supp} B_i$) with

  $$\begin{cases} \xi_{2i} := t_i, & 0 \le i \le n \\ \xi_{2i-1} := s_i := \frac{1}{2}(t_{i-1} + t_i) & 1 \le i \le n \end{cases}$$

  and the $\sigma_{i,j}$'s chosen such that $\pi_n x = x$, for all $x \in X_n$.

The QIP π_n can also be written in the quasi-Lagrange form, by means of the so-called fundamental functions, given by linear combination of B-splines, according to (3.2)

$$\pi_n x = \sum_{i=0}^{2n} x_i L_i. \tag{3.3}$$

Here we recall some properties of the QIP π_n:

- π_n is bounded, i.e.

$$\|\pi_n\|_\infty := \sup_{x\in X, x\neq 0} \frac{\|\pi_n x\|_\infty}{\|x\|_\infty} < \infty$$

  because the λ_i are continuous linear functionals;

- from classical results in approximation theory

$$\|x - \pi_n x\|_\infty \leq C\, \mathrm{dist}(x, X_n), \qquad C := 1 + \|\pi_n\|_\infty\,;$$

- for $x \in C^j(\mathcal{I})$, there exists a constant $\overline{C}_j$, depending on C and j, such that,

$$\|x - \pi_n x\|_\infty \leq \overline{C}_j h^j \omega(x^{(j)}, h), \quad \text{with} \quad 0 \leq j \leq d,$$

  where ω is the modulus of continuity of $x^{(j)}$. Moreover, if $x \in C^{d+1}(\mathcal{I})$ we have

$$\|x - \pi_n x\|_\infty = O(h^{d+1}). \tag{3.4}$$

  These results are deduced from Jackson type theorem for splines [17, 34].

Examples of spline QIPs of the form (3.1) can be found in [19]. They are denoted by Q_2 and Q_3 and are used in the numerical tests here proposed. The operators Q_2 is defined in $\mathcal{S}_2^1(\mathcal{I}, \mathcal{T}_n)$ and Q_3 is defined on $\mathcal{S}_3^2(\mathcal{I}, \mathcal{T}_n)$. For both of them, the following inequality holds:

$$\sup_n \|Q_j\| \leq p_j, \quad j = 2, 3,$$

where p_j are suitable positive real constants [21].

The following theorems present some interesting properties of the projectors π_n, in case of even degree d (for details see [19]).

Theorem 1. *Let π_n be a QIP on X_n of kind (3.1) and let the degree d be even. If the functionals λ_i, $i = d+1, \ldots, n$, are such that the values $\sigma_{i,j}$ in (3.2), associated to QI nodes symmetric w.r.t. the center of suppB_i, are equal, then*

$$\int_{t_{i-1}}^{t_i} (\pi_n m_{d+1}(t) - m_{d+1}(t))dt = 0, \qquad i = d+1, \ldots, n-d, \qquad m_{d+1}(t) = t^{d+1}.$$

It is interesting to consider QIPs for which Theorem 1 is valid also in the case of odd degree, as it is the case for the QIP Q_3 (see Appendix B).

Theorem 2. *If Theorem 1 holds, for any function $g \in W^{1,1}$ (i.e. with $\|g'\|_1$ bounded) and any function x such that $\left\|x^{(d+2)}\right\|_\infty$ is bounded, there results*

$$\left|\int_0^1 g(t)(\pi_n x(t) - x(t))dt\right| = O(h^{d+2}).$$

For the considered QIPs Q_2 and Q_3, Theorem 2 holds.

4. Spline projection methods

Given a spline QIP operator $\pi_n : X \to X_n$, defined as in Section 3, we introduce in the following two projection methods based on it. The first one is considered for its good numerical performance, while the second one, which is standard, is taken into account for comparison purposes:

1. *QIP spline Kulkarni's type method.* K is approximated by

$$K_n^k := \pi_n K + K\pi_n - \pi_n K \pi_n \tag{4.1}$$

in (2.1) and the approximate equation is

$$\varphi_n^k - K_n^k(\varphi_n^k) = f. \tag{4.2}$$

Defining T_n^k by $T_n^k(u) = f + K_n^k(u)$, then (4.2) can be written as $\varphi_n^k = T_n^k(\varphi_n^k)$.
We also consider the iterated version of this Kulkarni's type method. By using the approximation φ_n^k and the equation (2.1) we get

$$\tilde{\varphi}_n^k = K(\varphi_n^k) + f,$$

where $\tilde{\varphi}_n^k$ is the approximation of the solution by this iterated method.

2. *QIP spline collocation method.* K is approximated by $K_n^c := \pi_n K \pi_n$ and the right hand side f by $\pi_n f$ in (2.1), obtaining the approximate equation

$$\varphi_n^c - \pi_n K(\varphi_n^c) = \pi_n f. \tag{4.3}$$

Also in this case we consider the iterated version of the method. By using φ_n^c and (2.1) we get

$$\tilde{\varphi}_n^c = K(\varphi_n^c) + f,$$

where $\tilde{\varphi}_n^c$ is the approximation of the solution by this iterated method.

4.1. Construction of the approximate solutions

Starting from equations (4.2) and (4.3), in this section we construct the corresponding approximate solutions.

1. *QIP spline Kulkarni's type method.*
We consider definition (4.1) and equation (4.2), we project them in X_n by using the spline QIP π_n, and we join them, obtaining

$$\pi_n\varphi_n^k - \pi_n K(\varphi_n^k) = \pi_n f. \tag{4.4}$$

Combining (4.4) with (4.2) we obtain

$$\varphi_n^k = K(\pi_n\varphi_n^k) - \pi_n K(\pi_n\varphi_n^k) + \pi_n\varphi_n^k - \pi_n f + f. \tag{4.5}$$

Now we define $\psi_n := \pi_n\varphi_n^k$; from (4.4) we have

$$\psi_n - \pi_n K(\varphi_n^k) = \pi_n f \tag{4.6}$$

and from (4.5) we obtain

$$\varphi_n^k = \psi_n + (I - \pi_n)(K(\psi_n) + f). \tag{4.7}$$

Replacing (4.7) in (4.6) we finally have

$$\psi_n - \pi_n K(\psi_n + (I - \pi_n)(K(\psi_n) + f)) = \pi_n f, \tag{4.8}$$

where the unknown ψ_n by its definition lies in X_n.
In order to find ψ_n, we define the functional

$$F_n(y) = y - \pi_n K(y + (I - \pi_n)(K(y) + f)) - \pi_n f, \qquad y \in X_n, \tag{4.9}$$

with Fréchet derivative

$$F_n'(y)q = q - \pi_n K'(y + (I - \pi_n)(K(y) + f))(I + (I - \pi_n)K'(y))q. \tag{4.10}$$

We notice that (4.8) is equivalent to

$$F_n(\psi_n) = 0,$$

which is iteratively solved by Newton-Kantorovich method, an extension to functional spaces of the classical Newton method for the numerical solutions of nonlinear equations in one variable (see e.g. [33] for details).

Let $\psi_n^{(0)}$ be the initial approximation needed by the method. The iterates $\psi_n^{(r)}, r = 0, 1, 2, \ldots,$ are given by

$$\psi_n^{(r+1)} = \psi_n^{(r)} - [F_n'(\psi_n^{(r)})]^{-1} F_n(\psi_n^{(r)})$$

or, equivalently

$$F_n'(\psi_n^{(r)})\psi_n^{(r+1)} = F_n'(\psi_n^{(r)})\psi_n^{(r)} - F_n(\psi_n^{(r)}). \tag{4.11}$$

By using (4.9) and (4.10), and also (4.7), the equation (4.11) can be written in this way

$$\begin{aligned} &\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r+1)} \\ &= \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\psi_n^{(r)} - \pi_n K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})\psi_n^{(r)}. \end{aligned} \tag{4.12}$$

Recalling that $\psi_n^{(r)} \in X_n$, we can express it as a linear combination of B-splines

$$\psi_n^{(r)} = \sum_{j=1}^{N} x_n^{(r)}(j) B_j \qquad x_n^{(r)} \in \mathbb{R}^N. \tag{4.13}$$

After some algebra, we can write (4.12) in this way:

$$\begin{aligned} &x_n^{(r+1)}(i) - \sum_{j=1}^{N} x_n^{(r+1)}(j)\lambda_i(K'(\varphi_n^{(r)})B_j) - \sum_{j=1}^{N} x_n^{(r+1)}(j)\lambda_i(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j) \\ &= \lambda_i(K(\varphi_n^{(r)})) + \lambda_i(f) - \sum_{j=1}^{N} x_n^{(r)}(j)\lambda_i(K'(\varphi_n^{(r)})B_j) - \sum_{j=1}^{N} x_n^{(r)}(j)\lambda_i(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j), \end{aligned}$$

$i = 1, \ldots, N$. This is a linear system of size N, whose matrix form is

$$(I - \Lambda_n^{(r)} - \Xi_n^{(r)})x_n^{(r+1)} = \delta_n^{(r+1)}, \tag{4.14}$$

where, for $i, j = 1, \ldots, N$

$$\begin{aligned} \Lambda_n^{(r)}(i,j) &:= \lambda_i(K'(\varphi_n^{(r)})B_j) \\ \Xi_n^{(r)}(i,j) &:= \lambda_i(K'(\varphi_n^{(r)})(I - \pi_n)K'(\psi_n^{(r)})B_j) \\ \delta_n^{(r)}(i) &:= \lambda_i(K(\varphi_n^{(r)})) + \lambda_i(f) - (\Lambda_n^{(r)}x_n^{(r)})(i) - (\Xi_n^{(r)}x_n^{(r)})(i). \end{aligned}$$

By solving the system (4.14), due to the non singularity of the related matrix, we get the vector $x_n^{(r+1)}$. Using this we can calculate $\psi_n^{(r+1)}$ from (4.13). The approximate solution at the $(r+1)$ iteration is $\varphi_n^{(r+1)}$, which can be constructed using (4.7).

2. *QIP spline collocation method.*
   We start considering equation (4.3). We recall that in this method $\varphi_n^c \in X_n$. Since the equation is nonlinear, we solve it by Newton-Kantorovich method. In this case the functional needed in order to set the method is given by

   $$F_n(y) = y - \pi_n K(y) - \pi_n f \qquad y \in X_n$$

   and its Fréchet derivative by

   $$F_n'(y)q = q - \pi_n K'(y)q.$$

Consequently, we notice that the equation

$$F_n(\varphi_n^c) = 0$$

is equivalent to (4.3), so the iteration of the Newton-Kantorovich method (4.11) is given by

$$\varphi_n^{(r+1)} - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r+1)} = \pi_n(K(\varphi_n^{(r)}) + f) - \pi_n K'(\varphi_n^{(r)})\varphi_n^{(r)}. \tag{4.15}$$

Since $\varphi_n^{(r)} \in X_n$, we can write

$$\varphi_n^{(r)} = \sum_{j=1}^{N} x_n^{(r)}(j) B_j, \qquad x_n^{(r)} \in \mathbb{R}^N. \tag{4.16}$$

From (4.15), after some algebra, we obtain

$$x_n^{(r+1)}(i) - \sum_{j=1}^{N} x_n^{(r+1)}(j)\lambda_i\big[K'(\varphi_n^{(r)})B_j\big] = \lambda_i\big[K(\varphi_n^{(r)})\big] + \lambda_i(f) - \sum_{j=1}^{N} x_n^{(r)}(j)\lambda_i\big[K'(\varphi_n^{(r)})B_j\big],$$

$i = 1, \ldots, N$, that is a linear system of size N, whose matrix form is given by

$$(I - \Phi_n^{(r)})x_n^{(r+1)} = \omega_n^{(r)}, \tag{4.17}$$

where, for $i, j = 1, \ldots, N$

$$\Phi_n^{(r)}(i,j) = \lambda_i\big(K'(\varphi_n^{(r)})B_j\big)$$
$$\omega_n^{(r)}(i) = \lambda_i\big(K(\varphi_n^{(r)})\big) + \lambda_i(f) - \big(\Phi_n^{(r)} x_n^{(r)}\big)(i).$$

By solving the system (4.17), due to the non singularity of the related matrix, we get the vector $x_n^{(r+1)}$. Using this we can calculate $\varphi_n^{(r+1)}$ from (4.16), which is the approximate solution at the $(r+1)$ iteration.

We remark that classical methods for the solution of the above kind of problems are the projection ones based on a sequence of orthogonal or interpolatory projectors, usually onto spaces of piecewise polynomials of degree d at most continuous. In this case the dimension of the linear systems is related to the product between the number of subintervals n and the degree d. Instead, in our approach the dimension of the linear systems is related to the sum between the number of subintervals n and the degree d and therefore we have an advantage from the computational point of view for increasing values of n.

4.2. Convergence of the methods

In this section we study the convergence of the spline projection methods (4.2) and (4.3) and their iterated version.

Concerning the existence and uniqueness of the approximate solutions φ_n^k and φ_n^c, we can refer to the general results given in [30] and [26], respectively, that also hold for the considered spline QIPs.

First of all we prove the following lemma.

Lemma 3. *Let $\varphi \in C^{d+2}(\mathcal{I})$ be an isolated solution of (2.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$ and let $\pi_n : X \to X_n$ be a spline QIP of kind (3.1) for which Theorem 1 is valid. Then*

$$\|K(\pi_n\varphi) - K(\varphi)\|_\infty = O(h^{d+2}\log(h)).$$

Proof. Consider

$$\begin{aligned}
|(K(\pi_n\varphi) - K(\varphi))(s)| = & \left| \int_0^1 \log|s-t| \left[\psi(t, \pi_n\varphi(t)) - \psi(t, \varphi(t)) \right] dt \right| \\
\leq & \left| \int_0^1 \log|s-t| \left[\frac{\partial\psi}{\partial\varphi}(t, \varphi(t) + \theta_1(\pi_n\varphi - \varphi)(t)) - \frac{\partial\psi}{\partial\varphi}(t, \varphi(t)) \right] (\pi_n\varphi - \varphi)(t) dt \right| \\
& + \left| \int_0^1 \log|s-t| \frac{\partial\psi}{\partial\varphi}(t, \varphi(t))(\pi_n\varphi - \varphi)(t) dt \right| = |\clubsuit| + |\spadesuit|,
\end{aligned}$$

with $0 < \theta_1 < 1$. Then, since $\frac{\partial\psi}{\partial x}(t, x(t))$ is Lipschitz continuous with constant c_2, from (2.3) and (3.4) we have

$$\begin{aligned}
|\clubsuit| & \leq \int_0^1 |\log|s-t|| \left| \frac{\partial\psi}{\partial\varphi}(t, \varphi(t) + \theta_1(\pi_n\varphi - \varphi)(t)) - \frac{\partial\psi}{\partial\varphi}(t, \varphi(t)) \right| |(\pi_n\varphi - \varphi)(t)| \, dt \\
& \leq c_2\theta_1 \int_0^1 |\log|s-t|| \, |(\pi_n\varphi - \varphi)(t)|^2 \, dt \leq c_2\theta_1 p_1 \, \|\pi_n\varphi - \varphi\|_\infty^2 = O(h^{2d+2}).
\end{aligned}$$

Now we consider the second term $|\spadesuit|$. Defining $g_s(t) := \log|s-t| \frac{\partial\psi}{\partial\varphi}(t, \varphi(t))$ and choosing $\phi_s(t)$ the polynomial of degree less than or equal to n as in Lemma 3.7 in [30] (see also [34, p. 92]), such that $\|g_s - \phi_s\|_1 = O(h\log(h))$, from Theorem 2 and again (3.4), we get

$$\begin{aligned}
|\spadesuit| & = \left| \int_0^1 g_s(t)(\pi_n\varphi - \varphi)(t) dt \right| \leq \left| \int_0^1 (g_s - \phi_s)(t)(\pi_n\varphi - \varphi)(t) dt \right| + \left| \int_0^1 \phi_s(t)(\pi_n\varphi - \varphi)(t) dt \right| \\
& \leq \|g_s - \phi_s\|_1 \|\pi_n\varphi - \varphi\|_\infty + O(h^{d+2}) = O(h^{d+2}\log(h)) + O(h^{d+2}) = O(h^{d+2}\log(h))
\end{aligned}$$

and the thesis follows. □

Now, we consider the QIP spline collocation method and its iterated version and we prove the following results.

Theorem 4. *Let $\varphi \in C^{d+1}(\mathcal{I})$ be an isolated solution of (2.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : X \to X_n$ be a spline QIP of kind (3.1). Then (4.3) has a unique solution $\varphi_n^c \in B(\varphi, \delta) = \{x : \|x - \varphi\|_\infty < \delta\}$ for some $\delta > 0$ and for sufficiently large n. Moreover*

$$\|\varphi_n^c - \varphi\|_\infty = O(h^{d+1}).$$

Proof. We follow the lines of the proof of Theorem 2.1 in [25], reaching the inequality

$$\|\varphi_n^c - \varphi\|_\infty \leq c_3\|\pi_n\varphi - \varphi\|_\infty,$$

for a suitable constant c_3. Using (3.4) the thesis holds. □

Theorem 5. *Let $\varphi \in C^{d+2}(\mathcal{I})$ be an isolated solution of (2.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : X \to X_n$ be a spline QIP of kind (3.1) for which Theorem 1 is valid. Let $\tilde{\varphi}_n^c$ be the iterated approximation of the spline collocation method. Then, there holds*

$$\|\tilde{\varphi}_n^c - \varphi\|_\infty = O(h^{d+2}\log(h)).$$

Proof. Following the same path of reasoning used in the proof of Lemma 3, from the definition of $\tilde{\varphi}_n^c$ we have

$$\tilde{\varphi}_n^c - \varphi = K(\varphi_n^c) - K(\varphi) = [K'(\varphi + \theta_2(\varphi_n^c - \varphi)) - K'(\varphi)](\varphi_n^c - \varphi) + K'(\varphi)(\varphi_n^c - \varphi), \tag{4.18}$$

with $0 < \theta_2 < 1$. Taking into account that $\pi_n\tilde{\varphi}_n^c = \varphi_n^c$, we have

$$\tilde{\varphi}_n^c - \varphi = [K'(\varphi + \theta_2(\varphi_n^c - \varphi)) - K'(\varphi)](\varphi_n^c - \varphi) + K'(\varphi)(\pi_n(\tilde{\varphi}_n^c - \varphi)) + K'(\varphi)(\pi_n\varphi - \varphi)$$

and consequently

$$[I - K'(\varphi)\pi_n](\tilde{\varphi}_n^c - \varphi) = [K'(\varphi + \theta_2(\varphi_n^c - \varphi)) - K'(\varphi)](\varphi_n^c - \varphi) + K'(\varphi)(\pi_n\varphi - \varphi).$$

Following the theory of the collectively compact operators [26, 27], which can be applied thanks to the hypothesis reported in Section 2, we have that the inverse of the operator $[I - K'(\varphi)\pi_n]$ exists with bounded infinity norm. Therefore

$$\|\tilde{\varphi}_n^c - \varphi\|_\infty \leq c_4 \|\varphi_n^c - \varphi\|_\infty^2 + c_5 \|K'(\varphi)(\pi_n\varphi - \varphi)\|_\infty,$$

for suitable constants c_4, c_5. Following the proof of Lemma 3 and using the results there obtained, there exixst a constant c_6 such that

$$\|K'(\varphi)(\pi_n\varphi - \varphi)\|_\infty \leq c_6\, h\log(h)\, \|\pi_n\varphi - \varphi\|_\infty + \sup_{s\in\mathcal{I}} \left| \int_0^1 \phi_s(t)(\pi_n\varphi - \varphi)(t)dt \right|$$

where $\phi_s(t)$ is the polynomial of degree less than or equal to n already chosen in Lemma 3. Hence, from (3.4) and Theorem 2, the thesis follows. □

Remark. The rigorous proof of Theorem 5 can be substituted by an alternative one, based on a combination of theoretical steps and an assumption supported by an intensive numerical testing, which can be outlined as follows. Treating the first term in the right-hand side of (4.18) similarly as ♣ and the second term similarly as ♠, we obtain

$$\|\tilde{\varphi}_n^c - \varphi\|_\infty \leq c_7 \|\varphi_n^c - \varphi\|_\infty^2 + c_8\, h\log(h)\, \|\varphi_n^c - \varphi\|_\infty + \sup_{s\in\mathcal{I}} \left| \int_0^1 \phi_s(t)(\varphi_n^c - \varphi)(t)dt \right|, \tag{4.19}$$

for suitable constants c_7, c_8 and $\phi_s(t)$ the polynomial of degree less than or equal to n already chosen in Lemma 3. On the basis of several numerical evidences, we can conjecture that the last term of (4.19) decays as $O(h^{d+2})$, i.e. with an extra order with respect to the error $\|\varphi_n^c - \varphi\|_\infty$. Therefore, from this fact and from Theorem 4, (4.19) gives the thesis.

Considering the QIP spline Kulkarni method, we obtain the following result.

Theorem 6. *Let $\varphi \in C^{d+2}(\mathcal{I})$ be an isolated solution of (2.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : X \to X_n$ be a spline QIP of kind (3.1) for which Theorem 1 is valid. Then (4.2) has a unique solution $\varphi_n^k \in B(\varphi, \delta) = \{x : \|x - \varphi\|_\infty < \delta\}$ for some $\delta > 0$ and for sufficiently large n. Moreover, there exists a constant $0 < q < 1$, independent of n, such that*

$$\frac{\alpha_n}{1+q} \leq \left\|\varphi_n^k - \varphi\right\|_\infty \leq \frac{\alpha_n}{1-q} \tag{4.20}$$

where $\alpha_n := \left\|[I - (T_n^k)'(\varphi)]^{-1}(T_n^k(\varphi) - T(\varphi))\right\|_\infty$. Further

$$\left\|\varphi_n^k - \varphi\right\|_\infty = O(h^{d+2}\log(h)). \tag{4.21}$$

Proof. Following the theory of the collectively compact operators [26, 27], which can be applied thanks to the hypothesis reported in Section 2, we have that $(I - (T_n^k)'(\varphi))$ is invertible and

$$\left\|[I - (T_n^k)'(\varphi)]^{-1}\right\|_\infty \leq c_9.$$

Using this inequality and Lemma 3.4 of [30], after some algebra, we can say that hypothesis of Theorem 3.2 of [30] hold, so (4.20) is proved.
From the last inequality of (4.20) we can write

$$\begin{aligned}\left\|\varphi_n^k - \varphi\right\|_\infty &\leq \frac{\alpha_n}{1-q} \leq \frac{c_9}{1-q}\left\|T_n^k(\varphi) - T(\varphi)\right\|_\infty \leq \frac{c_9}{1-q}\|(I-\pi_n)(K(\pi_n\varphi) - K(\varphi))\|_\infty \\ &\leq c_{10}\|K(\pi_n\varphi) - K(\varphi)\|_\infty,\end{aligned}$$

for a suitable constant c_{10}. Applying Lemma 3, (4.21) holds. $\square$

Finally, for the iterated version of the QIP spline Kulkarni method, we give the following result, whose claim has been proved only by a combination of rigorous theoretical steps and an assumption supported by an intensive numerical testing, following the path of reasoning as in the Remark written at the end of Theorem 5.

Proposition 7. *Let $\varphi \in C^{d+2}(\mathcal{I})$ be an isolated solution of (2.1) and assume that 1 is not an eigenvalue of $K'(\varphi)$. Let $\pi_n : X \to X_n$ be a spline QIP of kind (3.1) for which Theorem 1 is valid. Let $\tilde{\varphi}_n^k$ be the iterated approximation of the Kulkarni's type method. Then, there holds*

$$\left\|\tilde{\varphi}_n^k - \varphi\right\|_\infty = O(h^{d+3}(\log(h))^2). \tag{4.22}$$

Proof. From the definition of $\tilde{\varphi}_n^k$, we have

$$\tilde{\varphi}_n^k - \varphi = K(\varphi_n^k) - K(\varphi) = \left[K'\left(\varphi + \theta_3\left(\varphi_n^k - \varphi\right)\right) - K'\left(\varphi\right)\right]\left(\varphi_n^k - \varphi\right) + K'\left(\varphi\right)\left(\varphi_n^k - \varphi\right),$$

with $0 < \theta_3 < 1$. Following the same path of reasoning used in the proof of Lemma 3, we obtain

$$\left\|\tilde{\varphi}_n^k - \varphi\right\|_\infty \le c_{11}\left\|\varphi_n^k - \varphi\right\|_\infty^2 + c_{12}\, h\log(h)\left\|\varphi_n^k - \varphi\right\|_\infty + \sup_{s\in\mathcal{I}}\left|\int_0^1 \phi_s(t)(\varphi_n^k - \varphi)(t)dt\right|, \tag{4.23}$$

for suitable constants c_{11}, c_{12} and $\phi_s(t)$ the polynomial of degree less than or equal to n chosen in Lemma 3. On the basis of several numerical evidences, we can conjecture that the last term of (4.23) decays as $O(h^{d+3}\log(h))$, i.e. with an extra order with respect to the error $\left\|\varphi_n^k - \varphi\right\|_\infty$. Therefore, from this fact and from Theorem 6, (4.23) gives the thesis. $\square$

From Theorem 6 and Proposition 7, we can notice that the iterated QIP spline Kulkarni's type method improves over the QIP spline Kulkarni's type method.

Moreover, from Theorem 6 and Theorem 5, we can notice that the QIP spline Kulkarni's type method and the iterated QIP spline collocation method have the same order of convergence. However, from Proposition 7, we remark that the iterated QIP spline Kulkarni's type method improves over both iterated QIP spline collocation method and QIP spline Kulkarni's type method.

5. Numerical results

In this section, at first, we present results related to two integral equations of type (2.1), in order to give a numerical counterpart of the theoretical estimates given in the previous section. In fact, we have numerically solved the mentioned equations with QIP spline collocation method and QIP spline Kulkarni's type method in their basic and iterated versions, using both projectors Q_2 and Q_3.
The integrals occurring in the various methods are evaluated by using the quadrature formulas of composite type presented in Appendix A, which are suitable in order to evaluate integrals with logarithmic kernel.
For all the tests, for increasing values of n, we have computed the maximum absolute error by an approximate infinity norm calculated in this way

$$\|\varphi - \varphi_n\|_\infty := \max_{v\in G}|\varphi(v) - \varphi_n(v)|$$

where φ is the exact solution of the equation, φ_n is the approximate solution by one of the mentioned methods and G is a partition of the interval $\mathcal{I}$ with mesh size $h/7$.
We have also computed the corresponding numerical convergence order, obtained applying the base 2 logarithm to the ratio between two consecutive errors.

Test 1

The first considered weakly singular Fredholm-Hammerstein integral equation reads

$$x(s) - \int_0^1 \log|t-s| x^2(t)dt = f(s), \qquad s \in \mathcal{I}$$

where

$$f(s) = -\frac{1}{9}\log(1-s) + \frac{1}{9}\log(1-s)s^9 - \frac{1}{9}\log(s)s^9 + \frac{1}{9}s^8 + \frac{1}{18}s^7 + \frac{1}{27}s^6 + \frac{1}{36}s^5 + \frac{46}{45}s^4 + \frac{1}{54}s^3 + \frac{1}{63}s^2 + \frac{1}{72}s + \frac{1}{81}.$$

We note that

$$\lim_{s\to 0} f(s) = \frac{1}{81}, \qquad \lim_{s\to 1} f(s) = \frac{29809}{22680}.$$

The exact solution of this equation is $\varphi(s) = s^4$.
By using Matlab environment, we have constructed the computational procedure in order to numerically solve this equation with the various methods presented in this paper and we have obtained the results reported in Table 1.
In the first, second and last column it is clear that the convergence order increases when the mesh size decreases. This fact is due to the term $\log(h)$ that is present in the order of convergence of the corresponding methods, stated in Theorem 6, Proposition 7 and Theorem 5, whose effect in decreasing the convergence order is more prevailing when the mesh size h is wider.
As stated theoretically in the previous section, we underline that QIP spline Kulkarni's type method and iterated QIP spline collocation method are equivalent in terms of convergence order, but it is interesting to point out that the second one is easier to construct and it is cheaper in terms of computational cost. On the other side, the computational effort for the QIP spline Kulkarni's type method can be justified by the increased convergence order of the iterated QIP spline Kulkarni's type method. In fact, with a little additional computational effort with respect to its basic version, this last method can achieve a better order of convergence, as stated in Proposition 7 and as confirmed by the numerical results in the second column of Table 1.

Table 1: Numerical results for Test 1 with all presented methods and both projectors.

Test 1								
	Kulkarni		Iterated Kulkarni		Collocation		Iterated collocation	
n	$\Vert\varphi-\varphi_n^k\Vert_\infty$	O_∞^k	$\Vert\varphi-\tilde{\varphi}_n^k\Vert_\infty$	$\tilde{O}_\infty^k$	$\Vert\varphi-\varphi_n^c\Vert_\infty$	O_∞^c	$\Vert\varphi-\tilde{\varphi}_n^c\Vert_\infty$	$\tilde{O}_\infty^c$
Methods based on Q_2								
2	3.68(−03)		2.08(−03)		3.26(−02)		7.18(−03)	
4	4.22(−04)	3.1	1.14(−04)	4.2	3.11(−03)	3.4	6.98(−04)	3.4
8	3.47(−05)	3.6	4.93(−06)	4.5	3.68(−04)	3.1	5.82(−05)	3.6
16	2.42(−06)	3.8	2.05(−07)	4.6	4.48(−05)	3.0	4.53(−06)	3.7
32	1.58(−07)	3.9	8.19(−09)	4.6	5.51(−06)	3.0	3.37(−07)	3.8
Methods based on Q_3								
4	1.38(−05)		4.71(−06)		2.41(−04)		6.57(−05)	
8	1.12(−06)	3.6	1.37(−07)	5.1	1.45(−05)	4.1	2.90(−06)	4.5
16	5.03(−08)	4.5	4.38(−09)	5.0	8.90(−07)	4.0	1.11(−07)	4.7
32	1.85(−09)	4.8	9.79(−11)	5.5	5.57(−08)	4.0	3.88(−09)	4.8

Test 2

The second considered weakly singular Fredholm-Hammerstein integral equation reads

$$x(s) - \int_0^1 \log|s-t|\sqrt{1+x^2(t)}dt = f(s), \qquad s \in \mathcal{I}$$

where

$$f(s) = \mathrm{Chi}(1-s)\sinh(s) + \sinh(s) + \mathrm{Shi}(s)\cosh(s) \\ - \mathrm{Chi}(-s)\sinh(s) - \mathrm{Shi}(s-1)\cosh(s) - \sinh(1)\log|s-1|,$$

with (see [1])

$$\mathrm{Shi}(s) = \int_0^s \frac{\sinh(t)}{t}dt, \qquad \mathrm{Chi}(s) = \gamma + \log(s) + \int_0^s \frac{\cosh(t)-1}{t}dt$$

and γ the Eulero-Mascheroni constant.
We point out that we consider f as a real function. We note also that

$$\lim_{s\to 0} f(s) \simeq 1.05725, \qquad \lim_{s\to 1} f(s) \simeq 2.50031$$

The exact solution of this equation is $\varphi(s) = \sinh(s)$.
In Table 2 we reported the results of numerical simulations related to this equation. Also these numerical tests confirm all the theoretical results and the remarks written previously.

Table 2: Numerical results for Test 2 with all presented methods and both projectors.

Test 2								
	Kulkarni		Iterated Kulkarni		Collocation		Iterated collocation	
n	$\|\varphi-\varphi_n^k\|_\infty$	O_∞^k	$\|\varphi-\tilde{\varphi}_n^k\|_\infty$	$\tilde{O}_\infty^k$	$\|\varphi-\varphi_n^c\|_\infty$	O_∞^c	$\|\varphi-\tilde{\varphi}_n^c\|_\infty$	$\tilde{O}_\infty^c$
Methods based on Q_2								
2	2.47(−04)		6.50(−05)		1.50(−03)		2.86(−04)	
4	1.62(−05)	3.9	1.95(−06)	5.1	1.85(−04)	3.0	2.21(−05)	3.7
8	1.04(−06)	4.0	6.20(−08)	5.0	2.29(−05)	3.0	1.69(−06)	3.7
16	6.50(−08)	4.0	2.39(−09)	4.7	2.83(−06)	3.0	1.26(−07)	3.7
32	4.04(−09)	4.0	8.48(−11)	4.8	3.50(−07)	3.0	9.10(−09)	3.8
Methods based on Q_3								
4	6.87(−07)		1.12(−07)		7.66(−06)		1.24(−06)	
8	3.14(−08)	4.5	3.04(−09)	5.2	5.84(−07)	3.7	5.29(−08)	4.6
16	1.13(−09)	4.8	5.60(−11)	5.8	4.01(−08)	3.9	1.97(−09)	4.7
32	3.80(−11)	4.9	9.21(−13)	5.9	2.62(−09)	3.9	6.85(−11)	4.8

Test 3

Let us conclude this section, presenting some results related to a non smooth solution, as found in [10]. We consider the weakly singular Fredholm-Hammerstein integral equation

$$x(s) - \int_0^1 \log|t-s|x^2(t)dt = f(s), \qquad s \in \mathcal{I}$$

where f is chosen so that the exact solution is $\varphi(s) = s\log(s)$.
Numerical results are collected in Table 3: the presented approach shows an expected decay of its performance, if applied to an integral equation having a less regular solution. Moreover, the use of Q_3 does not

improve the order of convergence of Q_2 as before, even if the errors are smaller. In any case, errors are in line with those presented in [10], where piecewise constant basis functions have been used on suitable graded meshes, instead of splines on classical uniform grids as employed here.
Even if not yet supported by the theory, these last simulations and analogous ones, not reported here, show the robustness of the proposed approach also in a non smooth framework.

Table 3: Numerical results for Test 3 with all presented methods and both projectors.

Test 3								
	Kulkarni		Iterated Kulkarni		Collocation		Iterated collocation	
n	$\|\varphi-\varphi_n^k\|_\infty$	O_∞^k	$\|\varphi-\tilde{\varphi}_n^k\|_\infty$	$\tilde{O}_\infty^k$	$\|\varphi-\varphi_n^c\|_\infty$	O_∞^c	$\|\varphi-\tilde{\varphi}_n^c\|_\infty$	$\tilde{O}_\infty^c$
Methods based on Q_2								
8	6.89(−04)		1.94(−04)		1.46(−02)		3.87(−03)	
16	9.37(−05)	2.9	9.08(−06)	4.4	6.32(−03)	1.2	5.25(−04)	2.9
32	1.37(−05)	2.8	4.22(−07)	4.4	3.29(−03)	0.9	7.76(−05)	2.8
Methods based on Q_3								
8	6.23(−05)		6.48(−06)		1.00(−02)		4.30(−04)	
16	1.01(−05)	2.6	4.29(−07)	3.9	4.92(−03)	1.0	5.82(−05)	2.9
32	1.61(−06)	2.6	2.22(−08)	4.3	2.44(−03)	1.0	8.68(−06)	2.7

6. Conclusions

In this paper, spline quasi-interpolating projectors have been used to efficiently solve nonlinear Fredholm-Hammerstein integral equations with logarithmic kernel by means of collocation and Kulkarni methods, both in their basic and iterated versions. Theoretical analysis of discretization error and convergence order has been provided, and numerical results have been shown validating the estimates, obtained under the hypothesis of sufficiently smooth solutions. The analysis of the proposed approach performance in a non-smooth framework is currently under study, but related numerical results appear promising.
Moreover, the methodologies proposed in this paper can be extended to an integral operator of the form

$$K(x)(s) := \int_0^1 r(s,t) \log|s-t| \, \psi(t, x(t))dt, \qquad s \in \mathcal{I}, \qquad x \in X,$$

where $r(s,t)$ is a smooth function defined in $\mathcal{I} \times \mathcal{I}$.
Future investigations will be devoted to treat kernels with higher order of singularity, such as those giving rise to Cauchy principal value or Hadamard finite part integrals (the reader is referred, for instance, to [2, 3] for some examples of such types of kernels arising in BEMs).

Appendix A. Quadrature formulas

Throughout this work we met several times the following type of integrals

$$\int_a^b \log|t-s| f(t)dt, \quad s \in [a,b].$$

By reader's convenience, in this paragraph we briefly recall suitable quadrature formulas in order to numerically calculate them in an efficient way (see [2, 3]).

We start by considering $a = -1$, $b = -1$ and the application of the following interpolating quadrature formula

$$\int_{-1}^{1} \log|t-s| f(t)dt \approx \sum_{k=1}^{n} \omega_k \tilde{\gamma}_k(s) f(x_k),$$

where x_k and ω_k are respectively the knots and the weights of the classical n points Gauss-Legendre quadrature formula in $[-1, 1]$ and $\tilde{\gamma}_k(s)$ are defined by

$$\tilde{\gamma}_k(s) = \frac{1}{2} \sum_{i=0}^{n-1} (2i+1)\mu_i(s) P_i(x_k).$$

In this equality $P_i(t)$ is the Legendre polynomial of degree i and $\mu_i(s)$ are the *modified moments* of the kernel $\log|t-s|$, defined as

$$\mu_i(s) = \int_{-1}^{1} \log|t-s| P_i(t)dt.$$

They can be computed by the following recursive procedure:

$$\begin{cases} \mu_0(s) = (1+s)\log(1+s) + (1-s)\log(1-s) - 2 \\ \mu_j(s) = \dfrac{1}{2j}\Theta_{j-1}(s), \quad j \geq 1 \end{cases},$$

where:

$$\begin{cases} \Theta_0(s) = (1-s^2)\log\left(\dfrac{1-s}{1+s}\right) - 2s \\ \Theta_1(s) = 2s\Theta_0(s) + \dfrac{8}{3} \\ \Theta_j(s) = \dfrac{j+1}{j(j+2)}[(2j+1)s\Theta_{j-1}(s) - j\Theta_{j-2}(s)], \qquad j \geq 2 \end{cases}.$$

Regarding the degree of accuracy of these quadrature formulas, the reader can refer to [3].

Using standard techniques, such as change of variable or subdivision of the integration domain, we can construct formulas in order to calculate this type of integrals over an interval $[a, b]$, and also composite formulas.

For sake of completeness we briefly construct the composite quadrature formula for this type of integrals over $[a, b]$. We point out that this formula has been used throughout this work, all times where an integral of such type has occurred.

Defining a positive integer m and setting $h = (b-a)/m$, we set a uniform partition of the interval $[a, b]$, made by m subintervals

$$Z = \{z_\eta = a + \eta h, \eta = 0, \ldots, m\}.$$

So we have

$$\int_a^b \log|t-s| f(t)dt = \sum_{\eta=0}^{m-1} \int_{z_\eta}^{z_{\eta+1}} \log|t-s| f(t)dt.$$

Fixing s, we call $\tilde{\eta}$ the index such that $s \in [z_{\tilde{\eta}}, z_{\tilde{\eta}+1}]$. We note that the only singular integral is the one on the domain $[z_{\tilde{\eta}}, z_{\tilde{\eta}+1}]$. So only this integral must be calculated using the formula outlined in this section, while for the other subintervals we can use a classic Gauss-Legendre formula. After a change of variable and some algebra we reach

$$\int_a^b \log|t-s| f(t)dt \approx \frac{h}{2}\Big[\log\Big(\frac{h}{2}\Big)\sum_{k=1}^{n} \omega_k f(x_{k,\tilde{\eta}}) + \sum_{k=1}^{n} \omega_k \tilde{\gamma}_k(\sigma) f(x_{k,\tilde{\eta}}) + \sum_{\eta=0, \eta\neq\tilde{\eta}}^{m-1} \sum_{k=1}^{n} \omega_k f(x_{k,\eta}) \log|x_{k,\eta} - s|\Big],$$

where $x_{k,*} = \dfrac{h}{2}x_k + \dfrac{z_* + z_{*+1}}{2}$ with $* = \eta, \tilde{\eta}$, $\sigma = -\dfrac{2}{h}\Big(\dfrac{z_{\tilde{\eta}} + z_{\tilde{\eta}+1}}{2} - s\Big)$.

Appendix B. Proof of Theorem 1 for Q_3

In this appendix we prove that the QIP Q_3 satisfies Theorem 1 and therefore Theorem 2 holds.

Let $\mathbb{P}_d$ be the space of polynomials of degree at most d. Consider the interval $[t_{i-1}, t_i]$, $i = 4, \ldots, n-4$, the middle point s_i, defined as in Section 3 and $m_4(t) = t^4$. Therefore, we can write $m_4(t) = (t - s_i)^4 + p_3(t) = p_4(t) + p_3(t)$, where $p_3 \in \mathbb{P}_3$. As $Q_3 p_3 = p_3$, we can write

$$\int_{t_{i-1}}^{t_i} (Q_3 m_4(t) - m_4(t))dt = \int_{t_{i-1}}^{t_i} (Q_3 p_4(t) - p_4(t))dt.$$

Now, as $\displaystyle\int_{t_{i-1}}^{t_i} p_4(t)dt = \frac{h^5}{80}$, it is sufficient to prove that also $\displaystyle\int_{t_{i-1}}^{t_i} Q_3 p_4(t)dt$ is equal to $\displaystyle\frac{h^5}{80}$.

From the expression of the coefficient functionals $\lambda_i(x)$, $i = d+1, \ldots, n$ of Q_3 given in [19], it is possible to obtain the quasi-Lagrange form (3.3) of Q_3. Therefore

$$\int_{t_{i-1}}^{t_i} Q_3 p_4(t)dt = \int_{t_{i-1}}^{t_i} \sum_{j=0}^{2n} (\xi_j - s_i)^4 L_j(t)dt = \sum_{j=0}^{2n} (\xi_j - s_i)^4 \int_{t_{i-1}}^{t_i} L_j(t)dt.$$

Taking into account the locality of the B-splines, the symmetry of the data points with respect to s_i and the symmetry properties of the coefficients $\lambda_i(x)$, $i = d+1, \ldots, n$, we can compute $\displaystyle\int_{t_{i-1}}^{t_i} L_j(t)dt$ and after some algebra we deduce $\displaystyle\int_{t_{i-1}}^{t_i} Q_3 p_4(t)(t) = \frac{h^5}{80}$. Therefore, considering the QIP Q_3, Theorem 1 holds also for the odd case $d = 3$.

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