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Sharp and Fast Bounds for the Celis-Dennis-Tapia Problem

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**Author queries:**

Q1: Should this be defined?

Q2: Should this be defined?

Q3: Should this be defined?

Q4: Should “being” and “H” be transposed?

Q5: Should this be defined?

Q6: Delete [17] since it is a duplicate of [16]?

1           **SHARP AND FAST BOUNDS FOR THE CELIS-DENNIS-TAPIA**  
2           **PROBLEM\***

                  LUCA CONSOLINI<sup>†</sup> AND MARCO LOCATELLI<sup>†</sup>

3           **Abstract.** In the Celis–Dennis–Tapia (CDT) problem a quadratic function is minimized over  
4 a region defined by two strictly convex quadratic constraints. In this paper we rederive a necessary  
5 and sufficient optimality condition for the exactness of the dual Lagrangian bound (equivalent to the  
6 Shor relaxation bound in this case). Starting from such a condition, we propose strengthening the  
7 dual Lagrangian bound by adding one or two linear cuts to the Lagrangian relaxation. Such cuts are  
8 obtained from supporting hyperplanes of one of the two constraints. Thus, they are redundant for  
9 the original problem, but they are not for the Lagrangian relaxation. The computational experiments  
10 show that the new bounds are effective and require limited computing times. In particular, one of the  
11 proposed bounds is able to solve all but one of the 212 hard instances of the CDT problem presented  
12 in [S. Burer and K. M. Anstreicher, *{\it SIAM J. Optim.}*, 23 (2013), pp. 432–451].

13           **Key words.** CDT problem, dual Lagrangian bound, linear cuts

14           **MSC codes.** 90C20, 90C22, 90C26

**DOI.** 10.1137/21M144548X

15           **1. Introduction.** The Celis–Dennis–Tapia problem (CDT problem in what fol-  
16 lows) is defined as follows:

$$(1.1) \quad \begin{aligned} p^* = \min \quad & \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{q}^\top \mathbf{x} \\ & \mathbf{x}^\top \mathbf{x} \leq 1 \\ & \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} \leq a_0, \end{aligned}$$

17 where  $\mathbf{Q}, \mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{q}, \mathbf{a} \in \mathbb{R}^n$ ,  $a_0 \in \mathbb{R}$ , while  $\mathbf{A}$  is assumed to be positive definite. We  
18 will denote by

$$H = \{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} \leq a_0 \}$$

19 the ellipsoid defined by the second constraint, by  $\partial H$  its border, and by  $\text{int}(H)$  its  
20 interior. The CDT problem was originally proposed in [13] and has attracted a lot  
21 of attention in the last two decades. For some special cases a convex reformulation  
22 is available. For instance, in [26] it is shown that a semidefinite reformulation is  
23 possible when no linear terms are present, i.e., when  $\mathbf{q} = \mathbf{a} = \mathbf{0}$ . However, up to now  
24 no tractable convex reformulation of general CDT problems has been proposed in the  
25 literature. In spite of that, recently three different works [9, 14, 22] independently  
26 proved that the CDT problem is solvable in polynomial time. More precisely, in [14,  
27 22] polynomial solvability is proved by identifying all KKT points through the solution  
28 of a bivariate polynomial system with polynomials of degree at most  $2n$ . The two  
29 unknowns are the Lagrange multipliers of the two quadratic constraints. Instead, in  
30 [9] an approach based on the solution of a sequence of feasibility problems for systems  
31 of quadratic inequalities is proposed. The systems are solved by a polynomial-time

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32 algorithm based on Barvinok’s construction [6]. Though polynomial, all of these  
 33 approaches are computationally demanding since the degree of the polynomial is quite  
 34 large. Conditions guaranteeing that the classical Shor SDP relaxation or, equivalently  
 35 in this case, the dual Lagrangian bound is exact, are discussed in [2, 7]. In particular,  
 36 in [2] a necessary and sufficient condition is presented. It is shown that the lack of  
 37 exactness is related to the existence of KKT points with the same Lagrange multipliers  
 38 but two distinct primal solutions, both active at one of the two constraints but one  
 39 violating and the other one fulfilling the other constraint. In [10] necessary and  
 40 sufficient conditions for local and global optimality are discussed based on copositivity.  
 41 In [11] an exactness condition is given for a copositive relaxation, also for the case  
 42 with additional linear constraints. A trajectory following method to solve the CDT  
 43 problem has been discussed in [26], while different branch-and-bound solvers are tested  
 44 in [19].

45 Recently, different papers proposed valid bounds for the CDT problem. In [12]  
 46 the Shor relaxation bound is strengthened by adding all RLT constraints obtained by **AQ1**  
 47 supporting hyperplanes of the two ellipsoids. By fixing the supporting hyperplane for  
 48 one ellipsoid, the RLT constraints obtained with all the supporting hyperplanes of the  
 49 other can be condensed into a single **SOC-RLT** constraint. Varying the supporting hy- **AQ2**  
 50 perplane of the first ellipsoid gives rise to an infinite number of **SOC-RLT** constraints  
 51 which, however, can be separated in polynomial time. The addition of these con-  
 52 straints does not allow one to close the duality gap, but it is computationally shown  
 53 that many instances which are not solved via the SDP bound, are solved with the  
 54 addition of these **SOC-RLT** cuts. The authors generate 1000 random test instances  
 55 for each  $n = 5, 10, 20$ , following a procedure described in [18] to generate trust-region  
 56 problems with one local and nonglobal minimizer. The proposed bound based on  
 57 **SOC-RLT** cuts allows for solving most instances except for 212 (38 for  $n = 5$ , 70 for  
 58  $n = 70$ , and 104 for  $n = 20$ ). Such unsolved instances are considered as hard ones  
 59 in subsequent works. In [25] lifted-RLT cuts are introduced and it is shown that the  
 60 new constraints allow one to derive an exact bound for  $n = 2$  but also to improve the  
 61 bounds of [12] over the hard instances for  $n > 2$ . In [27] it is proved that the duality  
 62 gap can be reduced to 0 by solving two subproblems with SOC constraints when the  
 63 second constraint is the product of two linear functions and an exactness result is also  
 64 provided for the case of problems with two variables. Due to its relations with the  
 65 approach proposed in this work, we will further discuss the approach proposed in [27]  
 66 at the end of section 3. In [3] cuts are introduced. These are Kronecker product **AQ3**  
 67 constraints which generalize both the classical RLT constraints obtained from two linear  
 68 inequality constraints, and the **SOC-RLT** constraints obtained from one linear inequal-  
 69 ity constraint and a SOC constraint. Further hard instances from [12] are solved with  
 70 the addition of these cuts. In the very recent paper [4] a branch and bound approach is  
 71 proposed. The main feature of this approach is eigenvector branching, i.e., a branch-  
 72 ing rule based on the use of the eigenvector corresponding to the maximal eigenvalue  
 73 of  $\mathbf{X}^* - \mathbf{x}^* \mathbf{x}^{*\top}$ , where  $(\mathbf{X}^*, \mathbf{x}^*)$  is the optimal solution of an SDP relaxation.

74 In this paper we investigate ways to strengthen the dual Lagrangian bound  
 75 through the addition of one or two linear cuts. In particular, the paper is structured  
 76 as follows. In section 2 we derive some theoretical results for a class of problems with  
 77 two constraints which includes the CDT problem as a special case. We develop a bi-  
 78 section technique to solve the dual Lagrangian relaxation for such class of problems.  
 79 In the following sections we apply the results of section 2 to the CDT problem. In  
 80 particular, in section 3 we introduce some results through which it will be possible  
 81 to rederive the necessary and sufficient exactness condition discussed in [2] and we

82 discuss how to improve the dual bound for the CDT problem by the addition of a  
 83 linear cut. Next, in sections 4 and 5 we discuss techniques to further improve the  
 84 bound. More precisely, in section 4 we still present a bound based on the addition of  
 85 a linear cut, but we develop a technique to locally adjust a given linear cut, while in  
 86 section 5 we consider a bound based on the addition of two linear cuts. Finally, in  
 87 section 6 we present some computational experiments which show that the newly pro-  
 88 posed bounds, in particular those based on two linear cuts, are both computationally  
 89 cheap and effective. In particular, one of the bounds will be able to solve all but one  
 90 of the hard instances from [12]. We also investigate which are the most challenging  
 91 instances for the proposed bounds and, as we will see, the difficulties are related to  
 92 the existence of multiple solutions of Lagrangian relaxations.

93 It is also worthwhile to highlight the contribution of this paper under another  
 94 perspective. As previously discussed, while there is no known convex relaxation of  
 95 the CDT problem, there are several problems, related to CDT, which do have exact  
 96 SDP relaxations. These include the trust-region subproblem (TRS), the TRS with a  
 97 single linear constraint (TRS1, see [12, 23]), and the TRS with two linear constraints,  
 98 at least one of which is tight (TRS2eq, see [26]). This paper shows that such special  
 99 cases, for which an exact convex relaxation exists, can be used to help solve the general  
 100 CDT problem. Indeed, we first observe that the subproblems to be solved in section 3  
 101 to improve the dual Lagrangian bound, obtained by adding a linear cut corresponding  
 102 to a supporting hyperplane for the second ellipsoidal constraint, turn out to be TRS1  
 103 problems. In section 4 we also discuss how to pick a "good" supporting hyperplane,  
 104 i.e., one which leads to a good SDP relaxation and, in fact, we also provide a necessary  
 105 and sufficient condition under which we can guarantee that the supporting hyperplane  
 106 is the best one. Next, in section 5 we observe that the bound can be further improved  
 107 by adding two linear cuts, one of which must be active, so that the subproblems to  
 108 be solved in this case are TRS2eq problems.

109 **2. Lower bounds obtained from the Lagrangian relaxation.** The CDT  
 110 problem (1.1) is a specific instance of the following, more general, one:

$$(2.1) \quad p^* = \min_{\mathbf{x} \in \mathbb{R}^n}, \quad \begin{aligned} & f(\mathbf{x}), \\ & g(\mathbf{x}) \leq 0, \\ & h(\mathbf{x}) \leq 0. \end{aligned}$$

111 In this section, we discuss a class of lower bounds on the solution of problem (2.1)  
 112 that can be obtained from its Lagrangian relaxation. In the next sections, we will  
 113 apply these bounds to the specific case of the CDT problem (1.1). Throughout this  
 114 and the following sections, we make the following assumptions.

115 *Assumption 2.1.* In problem (2.1)

- 117 (a)  $g, h$  are continuous;
- 118 (b) the set  $\{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$  is bounded;
- 119 (c) it holds that

$$(2.2) \quad h_0 = \min_{\mathbf{x} : g(\mathbf{x}) \leq 0} h(\mathbf{x}) < 0;$$

- 120 (d) the solution set of problem (2.1) without the last constraint, that is

$$\bar{P} = \arg \min_{\mathbf{x} \in \mathbb{R}^n}, \quad \begin{aligned} & f(\mathbf{x}), \\ & g(\mathbf{x}) \leq 0, \end{aligned}$$

121 is such that  $(\forall \mathbf{x} \in \bar{P}) h(\mathbf{x}) > 0$ .

122 Note that if the last condition in Assumption 2.1 is violated, we can find the  
 123 solution of problem (2.1) by removing the last constraint and the relaxation discussed  
 124 in this section is useless. Now, let  $G = \{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) \leq 0\}$  and  $H = \{\mathbf{x} \in \mathbb{R}^n :$   
 125  $h(\mathbf{x}) \leq 0\}$ . Let  $X \supset H$  be a closed subset of  $\mathbb{R}^n$  and for  $\lambda \in \mathbb{R}$ , with  $\lambda \geq 0$ , define the  
 126 Lagrangian relaxation

$$(2.3) \quad p_X(\lambda) = \min_{\mathbf{x} \in X \cap G} f(\mathbf{x}) + \lambda h(\mathbf{x}),$$

127 and the corresponding solution set

$$P_X(\lambda) = \arg \min_{\mathbf{x} \in X \cap G} f(\mathbf{x}) + \lambda h(\mathbf{x}).$$

128 Note that  $P_X(\lambda)$  is compact, since  $G \cap X$  is nonempty (in view of part (c))  
 129 of Assumption 2.1) and compact (in view of the compactness of  $G$  which follows  
 130 from parts (a) and (b) of Assumption 2.1), and  $f + \lambda h$  is continuous. Due to well-  
 131 known properties of the Lagrangian relaxation, we have that function  $p_X$  is such that  
 132  $(\forall \lambda \geq 0) p_X(\lambda) \leq p^*$ , and is concave (it is the pointwise minimum of a set of functions  
 133 linear in  $\lambda$ ). The best bound that can be obtained as the solution of (2.3) is given by

$$(2.4) \quad \bar{p}_X = \max_{\lambda \geq 0} p_X(\lambda),$$

134 and corresponds to the solution of the dual Lagrangian problem. Note that function  
 135  $p_X$  depends on the choice of set  $X$ .

136 Now, we recall that the supergradient of a function  $q : \mathbb{R} \rightarrow \mathbb{R}$  at  $x \in \mathbb{R}$  is defined  
 137 as

$$\partial q(x) = \{z \in \mathbb{R} : (\forall y \in \mathbb{R}) q(y) - q(x) \leq z(y - x)\}.$$

138 Since  $p_X$  is concave, for any  $\lambda \in \mathbb{R}$ , the supergradient  $\partial p_X(\lambda)$  is nonempty.

139 For  $A \subset \mathbb{R}^n$  define the following subset of  $\mathbb{R}$ :

$$h(A) = \{h(\mathbf{x}) : \mathbf{x} \in A\}.$$

140 For  $X \subset \mathbb{R}^n$ , define a (set-valued) function  $Q_X : \mathbb{R}_+ \rightarrow \mathcal{P}(\mathbb{R})$ ,

$$(2.5) \quad Q_X(\lambda) = h(P_X(\lambda))$$

141 ( $\mathbb{R}_+$  denotes the set of nonnegative reals and  $\mathcal{P}(\mathbb{R})$  is the power set of the set of real  
 142 numbers). Also set  $h_X^{\min}(\lambda) = \min Q_X(\lambda)$  and  $h_X^{\max}(\lambda) = \max Q_X(\lambda)$ . The following  
 143 proposition shows that function  $Q_X$  is monotone nonincreasing (see Definition 3.5.1  
 144 of [5]) and upper semicontinuous (see Definition 1.4.1 of [5]). These two properties  
 145 will play an important role in the computation of a lower bound for problem (2.1).  
 146 Moreover, this proposition characterizes the supergradient of  $p_X$  at each  $\lambda \geq 0$ . In the  
 147 proof of the proposition we will make use of Berge's maximum theorem (see [8]). In  
 148 particular, we will consider the slightly different formulation presented as the corollary  
 149 to Theorem 3 on page 30 of [15].

150 **COROLLARY 2.1.** *Let the correspondence (i.e., the set-valued function)  $\beta$  of  $S$*   
 151 *into  $T$  be compact-valued and continuous, and let  $\phi : S \times T \rightarrow \mathbb{R}$  be a continuous*  
 152 *function. Then, we have the following:*

153 (a) *The function  $z \mapsto m(z) := \max\{\phi(z, y) | y \in \beta(z)\}$  is continuous.*

154 (b) *The correspondence  $z \mapsto \{y \in \beta(z) | \phi(z, y) = m(z)\}$  is nonempty and compact-*  
 155 *valued and upper semicontinuous.*

156 PROPOSITION 2.2. For any  $X \subset \mathbb{R}^n$ ,

157 (i)  $Q_X$  is monotone non-increasing, that is if  $\lambda_1 \geq \lambda_2$ ,  $y_1 \in Q_X(\lambda_1)$ ,  $y_2 \in Q_X(\lambda_2)$ ,  
 158 then  $y_1 \leq y_2$ .

159 (ii)  $Q_X$  is upper semicontinuous, that is, if  $Q_X(\lambda) \subset U$ , where  $U$  is an open subset  
 160 of  $\mathbb{R}$ , then there exists a neighborhood  $V$  of  $\lambda$  such that  $(\forall z \in V) Q_X(z) \subset U$ .

161 (iii)  $\partial p_X(\lambda) = [\min Q_X(\lambda), \max Q_X(\lambda)]$ .

162 *Proof.* (i) Let  $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$  be such that  $y_1 = h(\mathbf{x}_1)$  and  $y_2 = h(\mathbf{x}_2)$ , then  $f(\mathbf{x}_1) +$   
 163  $\lambda_1 h(\mathbf{x}_1) \leq f(\mathbf{x}_2) + \lambda_1 h(\mathbf{x}_2)$  and  $f(\mathbf{x}_2) + \lambda_2 h(\mathbf{x}_2) \leq f(\mathbf{x}_1) + \lambda_2 h(\mathbf{x}_1)$ . By adding up the  
 164 previous inequalities, it follows that  $(\lambda_1 - \lambda_2)(h(\mathbf{x}_1) - h(\mathbf{x}_2)) \leq 0$ .

165 (ii) Apply Corollary 2.1 with  $T = G \cap X$ ,  $S = \mathbb{R}_+$ , constant function  $(\forall \lambda \in$   
 166  $S) \beta(\lambda) = G \cap X$ ,  $\phi(\lambda, \mathbf{x}) = -f(\mathbf{x}) - \lambda \cdot h(\mathbf{x})$ . Since  $\phi$  is continuous, set-valued function  
 167  $P_X(\lambda) = \{\mathbf{x} \in G \cap X : \phi(\lambda, \mathbf{x}) = \max_{\mathbf{y} \in G \cap X} \phi(\lambda, \mathbf{y})\}$  is upper semicontinuous. Hence,  
 168 also  $Q_X$  is upper semicontinuous, since it is obtained as the composition of  $P_X$  with  
 169  $h$ , which is continuous (see Theorem 1' on page 113 of [8]).

170 (iii) It is a consequence of Theorem 4.4.2 in [16], being  $G$  compact.  $\square$

171 The next proposition characterizes the optimal solution of the dual Lagrangian prob-  
 172 lem (2.4).

173 PROPOSITION 2.3. Under Assumption 2.1, the optimal value of the dual La-  
 174 grangian problem (2.4) is: (i) either attained at  $\lambda_X = 0$  in case  $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$ ,  
 175 where  $\mathbb{R}_-$  denotes the set of nonpositive real numbers; (ii) or is attained at  $\lambda_X > 0$   
 176 such that  $0 \in \partial p_X(\lambda_X)$ . In the former case,  $\bar{p}_X = p^*$  holds.

177 *Proof.* The proposition, apart from the last statement, is a direct consequence of  
 178 the optimality conditions for the maximum of concave functions (see Theorem 1.1.1  
 179 in Chapter 7 of [16]). To prove the last statement, namely that if (i) holds  $\bar{p}_X = p^*$ ,  
 180 note that, if  $\partial p_X(0) \cap \mathbb{R}_- \neq \emptyset$ , then, in view of part (iii) of Proposition 2.2,  $h_X^{\min}(0) \leq 0$   
 181 and, thus, there exists an optimal solution  $\mathbf{x}^*$  of  $\min_{\mathbf{x} \in X \cap G} f(\mathbf{x})$  such that  $h(\mathbf{x}^*) \leq 0$ .  
 182 This implies that  $\mathbf{x}^*$  is also an optimal solution of the original problem (2.1), so that  
 183  $\bar{p}_X = p^*$  holds.  $\square$

184 The following property shows that it is always possible to find a sufficiently high  
 185 value of  $\lambda$  such that  $P_X(\lambda) \subset H$ , that is, the elements of  $P_X(\lambda)$  are feasible solutions  
 186 of problem (2.1).

187 LEMMA 2.4. If

$$(2.6) \quad \lambda \geq \hat{\lambda} = \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h_0|},$$

188 where  $h_0$  is defined in (2.2), then  $P_X(\lambda) \subset H$ .

189 *Proof.* By contradiction, assume that there exists  $\mathbf{x} \in P_X(\lambda)$  such that  $h(\mathbf{x}) > 0$ ,  
 190 and let  $\mathbf{x}_0 \in G \cap H$  be such that  $h(\mathbf{x}_0) = h_0 < 0$ ; then  $f(\mathbf{x}) + \lambda h(\mathbf{x}) \leq f(\mathbf{x}_0) + \lambda h(\mathbf{x}_0)$ .  
 191 Since  $h(\mathbf{x}) > 0$ , it follows that  $\lambda \leq \frac{f(\mathbf{x}_0) - f(\mathbf{x})}{|h(\mathbf{x}_0)| + h(\mathbf{x})} < \frac{\max_{\mathbf{x} \in G \cap X} f(\mathbf{x}) - \min_{\mathbf{x} \in G \cap X} f(\mathbf{x})}{|h(\mathbf{x}_0)|}$ , which  
 192 contradicts the assumption on  $\lambda$ .  $\square$

193 The following proposition shows that if  $0 \in Q_X(\lambda)$ , then  $p_X(\lambda)$  is equal to the  
 194 optimal value of problem (2.1).

195 PROPOSITION 2.5. Under Assumption 2.1, the following statements are equiva-  
 196 lent for  $\lambda > 0$ :

197 (i)  $0 \in Q_X(\lambda)$ ,

198 (ii)  $p^* = p_X(\lambda)$  and there exists  $\bar{\mathbf{x}} \in \arg \min_{\mathbf{x} \in G \cap H} f(\mathbf{x})$  such that  $h(\bar{\mathbf{x}}) = 0$ .

199 *Proof.* (i)  $\Rightarrow$  (ii). Let  $\bar{\mathbf{x}}$  be such that  $h(\bar{\mathbf{x}}) \in Q_X(\lambda)$  and  $h(\bar{\mathbf{x}}) = 0$ . Let  $\mathbf{x}^*$  be a  
 200 solution of (2.1). Then,  $p_X(\lambda) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$ , hence  
 201  $p_X(\lambda) \leq p^*$ . Moreover,  $p_X(\lambda) = f(\bar{\mathbf{x}}) \geq \min_{\mathbf{x} \in G \cap H} f(\mathbf{x}) = p^*$ .

202 (ii)  $\Rightarrow$  (i). Assume that  $p_X(\lambda) = p^*$ , and let  $\mathbf{x} \in P_X(\lambda)$ . Then, by (ii),  $f(\mathbf{x}) +$   
 203  $\lambda h(\mathbf{x}) = f(\bar{\mathbf{x}}) = f(\bar{\mathbf{x}}) + \lambda h(\bar{\mathbf{x}})$ . It follows that  $\bar{\mathbf{x}} \in P_X(\lambda)$  and  $Q_X(\lambda) \ni h(\bar{\mathbf{x}}) = 0$ .  $\square$

204 *Remark 2.6.* If  $0 \in Q_X(\lambda)$ , by point (iii) of Proposition 2.2,  $\partial p_X(\lambda) \ni 0$ , so that  
 205  $\lambda$  corresponds to a maximizer of the dual Lagrangian. Note that equation  $0 \in Q_X(\lambda)$   
 206 always admits a solution if  $Q_X$  is continuous. However, in the general case,  $Q_X$  is  
 207 only upper semicontinuous. In this case, the value of  $\lambda$  for which  $\partial p_X(\lambda) \ni 0$  may not  
 208 satisfy  $0 \in Q_X(\lambda)$ . Thus, the optimal value of the dual Lagrangian (2.4) is not equal  
 209 to the optimal value of (2.1) but it represents a lower bound of it.

210 In order to evaluate a numerical solution algorithm, we define the following weak  
 211 solution of (2.1).

212 **DEFINITION 2.7.**  $\mathbf{x}$  is an  $\eta$ -solution of (2.1) if  $\mathbf{x} \in G \cap H$  and  $f(\mathbf{x}) - p^* \leq \eta$ .

213 The following proposition presents a bound on the error committed on the esti-  
 214 mation of  $p^*$ .

215 **PROPOSITION 2.8.** For any  $\lambda \geq 0$  such that  $P_X(\lambda) \cap H \neq \emptyset$ , and for any  $\mathbf{x} \in$   
 216  $P_X(\lambda) \cap H$ , it holds that  $f(\mathbf{x}) - p^* \leq \lambda |h(\mathbf{x})|$ , i.e.,  $\mathbf{x}$  is an  $\eta$ -solution of problem of  
 217 (2.1) with  $\eta = \lambda |h(\mathbf{x})|$ .

218 *Proof.* Since  $\mathbf{x} \in P_X(\lambda)$  and observing that  $\mathbf{x}^* \in G \cap X$  for any  $X \supset H$ ,  $f(\mathbf{x}) +$   
 219  $\lambda h(\mathbf{x}) \leq f(\mathbf{x}^*) + \lambda h(\mathbf{x}^*) \leq f(\mathbf{x}^*)$ , from which  $f(\mathbf{x}) - f(\mathbf{x}^*) \leq \lambda |h(\mathbf{x})|$ .  $\square$

220 Now we introduce Algorithm 2.1 which is based on a binary search through differ-  
 221 ent  $\lambda$  values and is able to return the solution of the dual Lagrangian problem, i.e., the  
 222 maximum of function  $p_X(\lambda)$  and, in some cases, even the solution of problem (2.1).  
 223 The algorithm also returns a point  $\mathbf{z}_1(\lambda^{\max}) \in H$  and (possibly) a point  $\mathbf{z}_2(\lambda^{\min}) \notin H$ .  
 224 Note that according to Proposition 2.8, point  $\mathbf{z}_1(\lambda^{\max})$  is an  $\eta$ -solution of problem  
 225 (2.1) with  $\eta = \lambda |h(\mathbf{z}_1(\lambda^{\max}))|$ .

226 The algorithm starts with an initial interval of  $\lambda$  values  $[\lambda^{\min}, \lambda^{\max}] = [0, \lambda^{\text{init}}]$ ,  
 227 where  $\lambda^{\text{init}}$  is a suitably large value and can be set equal to  $\hat{\lambda}$  as defined in Lemma  
 228 2.4. At each iteration the algorithm halves such interval by evaluating the set  $Q_X^\lambda$   
 229 at  $\lambda = (\lambda^{\max} + \lambda^{\min})/2$ . Then, the algorithm sets- are  $\lambda^{\min} = \lambda$ , if  $h_X^{\min}(\lambda) > 0$ ;  
 230  $\lambda^{\max} = \lambda$  if  $h_X^{\max}(\lambda) < 0$ . Instead, if  $0 \in \partial p_X(\lambda) = [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$ , the algorithm  
 231 sets  $\lambda^{\max} = \lambda^{\min} = \lambda$  and exits the loop.

232 The following proposition characterizes Algorithm 2.1.

233 **PROPOSITION 2.9.** (i) *Algorithm 2.1 terminates in a finite number of iterations:*

234 (ii) *at each iteration*  $\lambda^{\min} \leq \lambda_X \leq \lambda^{\max}$ ,

235 (iii) *at termination*  $|\lambda^{\max} - \lambda_X| \leq \epsilon$ ,

236 (iv) *at each iteration, if*  $\lambda^{\min} < \lambda_X < \lambda^{\max}$ , *then*  $[h_X^{\max}(\lambda^{\max}), h_X^{\min}(\lambda^{\min})] \supset$   
 237  $\partial p_X(\lambda_X)$ ,

238 (v) *point*  $\mathbf{z}_1(\lambda^{\max}) \in P_X(\lambda^{\max}) \cap H$  *is an*  $\eta$ -*solution of (2.1) with*  $\eta = \lambda_{\max} |$   
 239  $h(\mathbf{z}_1(\lambda^{\max}))|$ .

240 *Proof.* (i) At each iteration the length of the interval  $[\lambda^{\min}, \lambda^{\max}]$  is halved. Hence,  
 241 in a sufficient large number of iterations, the termination condition of the main loop  
 242 will be satisfied.

243 (ii) At the beginning of the algorithm we have that  $\lambda^{\min} \leq \lambda_X \leq \lambda^{\max}$ . Every time  
 244  $\lambda^{\min}$  is updated, we set  $\lambda^{\min} = \lambda$  if condition  $h_X^{\min}(\lambda) > 0$  holds. Since  $h_X^{\min}(\lambda_X) \leq 0$ ,

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**Algorithm 2.1** Binary search algorithm for the solution of the dual Lagrangian problem for (1.1).

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**DualLagrangian**( $X, \lambda_{\text{init}}$ )

Set  $\lambda_{\min} = 0, \lambda_{\max} = \lambda_{\text{init}}$

**while**  $\lambda_{\max} - \lambda_{\min} > \varepsilon$  **do**

Set  $\lambda = (\lambda_{\max} + \lambda_{\min})/2$

Solve problem (2.3), and let  $P_X(\lambda)$  be its set of optimal solutions

Compute the set  $Q_X(\lambda)$  and the values  $h_X^{\min}(\lambda), h_X^{\max}(\lambda)$

**if**  $h_X^{\min}(\lambda) > 0$  **then**

Set  $\lambda_{\min} = \lambda$

**else if**  $h_X^{\max}(\lambda) < 0$  **then**

Set  $\lambda_{\max} = \lambda$

**else**

Set  $\lambda_{\max} = \lambda_{\min} = \lambda$

**end if**

**end while**

Set  $Lb = p_X(\lambda_{\max})$ , and let  $\mathbf{z}_1(\lambda_{\max})$  be some point in  $P_X(\lambda_{\max}) \cap H$  and  $\mathbf{z}_2(\lambda_{\min})$  be some point (if any) in  $P_X(\lambda_{\min}) \setminus H$

**return**  $Lb, \lambda_{\max}, \mathbf{z}_1(\lambda_{\max}), \mathbf{z}_2(\lambda_{\min})$

---

245 by the monotonicity of function  $h_X^{\min}$ , which is a consequence of the monotonicity of  
 246 function  $Q_X$ , condition  $\lambda^{\min} \leq \lambda_X$  is maintained. The same reasoning can be used to  
 247 prove that  $\lambda^{\max} \geq \lambda_X$ .

248 (iii) It is a consequence of (ii) and the termination condition.

249 (iv)  $\partial p_X(\lambda_X) = [h_X^{\min}(\lambda_X), h_X^{\max}(\lambda_X)] \subset [h_X^{\max}(\lambda_{\max}), h_X^{\min}(\lambda_{\min})]$ , due to point  
 250 (ii) and the monotonicity of functions  $h_X^{\max}$  and  $h_X^{\min}$ , which is a consequence of the  
 251 monotonicity of function  $Q_X$ .

252 (v) It is a consequence of Proposition 2.8.  $\square$

253 The following property is a direct consequence of the upper semicontinuity of  $Q_X$ .

255 PROPOSITION 2.10. *Let  $X \supset H$  be such that  $\sup Q_X(\lambda) < 0$ ; then there exists a*  
 256 *neighborhood  $U$  of  $\lambda$  such that  $(\forall \eta \in U) \max Q_X(\eta) < 0$ .*

257 As a consequence of the previous proposition, it is possible to improve the lower  
 258 bound on problem (2.1), obtained as the solution of (2.3), by replacing set  $X$  with a  
 259 different set  $Y \supset H$  fulfilling a given condition.

260 PROPOSITION 2.11. *Let  $Y \supset H$  be such that  $\max Q_Y(\lambda_X) \leq 0$  or, equivalently,*  
 261  *$P_Y(\lambda_X) \setminus H = \emptyset$ , and assume that  $\bar{p}_X = p_X(\lambda_X) < p^*$ . Then  $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$ .*

262 *Proof.* Note that, by Proposition 2.3,  $\bar{p}_X = p_X(\lambda_X) < p^*$  implies  $\lambda_X > 0$ . Now,  
 263 in case  $\max Q_Y(\lambda_X) = 0$ , then  $0 \in Q_Y(\lambda_X)$  and, by Proposition 2.5,  $\bar{p}_Y = p^* > \bar{p}_X$ .  
 264 Thus, we only consider the case  $\max Q_Y(\lambda_X) < 0$ . In such case, by Proposition 2.10,  
 265  $\lambda_Y < \lambda_X$ . If  $\lambda_Y = 0$ , by Proposition 2.3 we have that  $\bar{p}_Y = p^* > \bar{p}_X$  and we are done.  
 266 Otherwise, if  $\lambda_Y > 0$ , again by Proposition 2.3 we have that  $0 \in [h_Y^{\min}(\lambda_Y), h_Y^{\max}(\lambda_Y)]$ ,  
 267 and, consequently, there exists  $\mathbf{y} \in P_Y(\lambda_Y)$  such that  $h(\mathbf{y}) \leq 0$ . Note that  $\bar{p}_Y =$   
 268  $f(\mathbf{y}) + \lambda_Y h(\mathbf{y})$ . If  $h(\mathbf{y}) = 0$ , then, by Proposition 2.5,  $p_Y(\lambda_Y) = p^*$ , so that the thesis  
 269 is satisfied in view of  $\bar{p}_X < p^*$ . Otherwise, if  $h(\mathbf{y}) < 0$ , let  $\mathbf{x} \in \mathbb{R}^n$  be such that  $\bar{p}_X =$   
 270  $f(\mathbf{x}) + \lambda_X h(\mathbf{x})$ . Then  $\bar{p}_X = f(\mathbf{x}) + \lambda_X h(\mathbf{x}) \leq f(\mathbf{y}) + \lambda_X h(\mathbf{y}) < f(\mathbf{y}) + \lambda_Y h(\mathbf{y}) = \bar{p}_Y$ ,  
 271 where we used the facts that  $h(\mathbf{y}) < 0$  and that  $\lambda_Y < \lambda_X$ .  $\square$

272 The following proposition deals with the special case of the previous result when  
 273  $Y \subset X$ .

274 **PROPOSITION 2.12.** *Let  $X \supset Y \supset H$  be such that  $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$ , and*  
 275 *assume that  $\bar{p}_X = p_X(\lambda_X) < p^*$ . Then  $\bar{p}_Y = p_Y(\lambda_Y) > \bar{p}_X$ .*

276 *Proof.* Since  $h_X^{\min}(\lambda_X) < 0$  we have that  $P_X(\lambda_X) \cap H \neq \emptyset$  and, consequently, since  
 277  $Y \supset H$ , also  $Y \cap P_X(\lambda_X) \neq \emptyset$ . Then,  $Y \subset X$  implies  $P_Y(\lambda_X) = Y \cap P_X(\lambda_X)$ . Moreover,  
 278 if  $Y \cap (P_X(\lambda_X) \setminus H) = \emptyset$ , then the condition  $\max Q_Y(\lambda_X) \leq 0$  is satisfied and the result  
 279 follows from Proposition 2.11.  $\square$

280 Stated in another way, the previous propositions show that, in case the lower  
 281 bound  $\bar{p}_X$  is not exact, we are able to improve (increase) it if we are able to replace  
 282 set  $X$  with a new set  $Y$  which cuts away all members of  $P_X(\lambda_X)$  outside  $H$ .

283 *Remark 2.13.* Up to now we have not discussed the difficulty of computing the  
 284 values of function  $p_X$  or, equivalently, the difficulty of solving problem (2.3). Such  
 285 difficulty is strictly related to the specific problem (i.e., to the specific functions  $f, g, h$ ),  
 286 and also to the specific set  $X$ . In the next sections we apply the general theory  
 287 developed in this section to the CDT problem. We show that for suitably defined sets  
 288  $X$  (defined by one or two linear cuts), the computation of function  $p_X$  can be done  
 289 efficiently, and, moreover, the corresponding lower bounds  $\bar{p}_X$  improve the standard  
 290 dual Lagrangian bound, corresponding to the case  $X = \mathbb{R}^n$ .

291 *Remark 2.14.* In principle one could also define a cutting algorithm where a  
 292 sequence of sets  $\{X_k\}$  is generated such that (i)  $X_k \supset X_{k+1} \supset H$  for all  $k$ ; (ii)  
 293  $X_{k+1} \cap (P_{X_k}(\lambda_{X_k}) \setminus H) = \emptyset$ ; (iii)  $\bigcap_{k=1}^{\infty} X_k = H$ . The corresponding sequence of lower  
 294 bounds  $\{\bar{p}_{X_k}\}$  is strictly increasing in view of Proposition 2.12, and converges to  $p^*$ .  
 295 However, the difficulty related to such an algorithm is that forcing (ii) may not be  
 296 trivial and, moreover, as already commented in Remark 2.13, computing  $p_{X_k}$  may be  
 297 computationally demanding.

298 The following algorithm, Algorithm 2.2, in principle, is able to always find an  
 299 approximate solution of (2.1). The algorithm is based on an iterative reduction of  
 300 set  $X$ , in order to eliminate its elements in which function  $h$  is positive. In practice,  
 301 Algorithm 2.2 could be unimplementable. Indeed, it may require a large number of  
 302 cuts on set  $X$  and each added cut may increase the complexity of the optimization  
 303 problem that we need to solve to evaluate **DualLagrangian**. In section 4, we will see  
 304 that, to refine the lower bound on the solution of the CDT problem, it is computa-  
 305 tionally more convenient to adjust existing cuts instead of adding new ones. We stress  
 306 that we will not actually use Algorithm 2.2 for the solution of the CDT problem. We  
 307 present this algorithm just as a theoretical contribution.

308 **PROPOSITION 2.15.** *Algorithm 2.2 terminates and  $\bar{\mathbf{x}}$  is such that  $h(\bar{\mathbf{x}}) \leq \frac{\eta}{\lambda^{\max}}$*   
 309 *and  $|\bar{f} - f^*| \leq \eta$ .*

310 *Proof.* By contradiction, assume that the algorithm does not terminate. Let  $l_i$   
 311 be the value of  $\lambda^{\min}$  returned by the  $i$ th call to **DualLagrangian**. Sequence  $l_i$  is  
 312 monotone nonincreasing; moreover, the domain of the sequence is a subset of finite  
 313 cardinality of interval  $[0, \lambda^{\text{init}}]$  (its maximum cardinality depends on  $\lambda^{\text{init}}$  and  $\epsilon$ ).  
 314 Indeed, the termination condition of function **DualLagrangian** allows only for a  
 315 finite number of divisions of the interval  $[0, \lambda^{\text{init}}]$ . Hence, sequence  $l_i$  converges in  
 316 a finite number of iterations to its limit  $l_\infty = \lim_{i \rightarrow \infty} l_i$  and there exists  $\bar{i} \in \mathbb{N}$  such  
 317 that  $(\forall i \geq \bar{i}) l_i = l_\infty$ . By (iv) of Proposition 2.9,  $h^{\max}(l_\infty) \geq 0$  and, since the

---

**Algorithm 2.2** Bound improvement through redefinition of set  $X$ .

---

- 1: Set  $X = \mathbb{R}^n$
  - 2: Set  $\lambda^{\max} = \lambda^{\text{init}}$
  - 3: **repeat**
  - 4:   Let  
 $[Lb, \lambda^{\min}, \lambda^{\max}, \mathbf{z}_1(\lambda^{\max}), \mathbf{z}_2(\lambda^{\min}), h_X^{\min}, h_X^{\max}] = \mathbf{DualLagrangian}(X, \lambda^{\text{init}})$
  - 5:   Set  $Z = \{\mathbf{x} \in P_X(\lambda^{\min}) : h(\mathbf{x}) > 0\}$
  - 6:   Redefine  $X = Y$ , where  $Y$  is such that  $X \supset Y \supset H$  and  $Z \cap Y = \emptyset$ .
  - 7: **until**  $\min\{h_X^{\max}(\lambda^{\min}), -h_X^{\min}(\lambda^{\max})\}\lambda^{\max} \leq \eta$
  - 8: **return**  $\bar{\mathbf{x}} \in P_X(\lambda_X^{\min}) \cup P_X(\lambda_X^{\max})$  with  $|h(\bar{\mathbf{x}})| \leq \eta$ ,  $\bar{f} = f(\bar{\mathbf{x}})$ .
- 

318 algorithm does not terminate,  $h^{\max}(l_\infty) \geq \eta$ . At the  $\bar{i} + 1$ -iteration, the algorithm  
 319 calls  $\mathbf{DualLagrangian}(X, l_\infty)$ , which returns the value  $\lambda^{\min} = l_\infty$ . Anyway, at the  
 320 previous iteration  $\bar{i}$ , the elements  $P_X(\lambda^{\min})$  at which function  $h$  is positive had already  
 321 been removed from  $X$ . This implies that  $\mathbf{DualLagrangian}(X, l_\infty)$  cannot return the  
 322 strictly positive value  $\lambda^{\min} = l_\infty$ , leading to a contradiction. Hence, the algorithm  
 323 terminates and the stated bounds hold because of the termination condition and by  
 324 Proposition 2.8.  $\square$

325 **3. Dual Lagrangian bound and a possible improvement.** In this section,  
 326 we apply the general properties presented in section 2 to the CDT problem (1.1). In  
 327 fact, the CDT problem is a specific instance of (2.1) in which  $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{q}^\top \mathbf{x}$ ,  
 328  $g(\mathbf{x}) = \mathbf{x}^\top \mathbf{x} - 1$ ,  $h(\mathbf{x}) = \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0$ .

329 Note that the first two requirements of Assumption 2.1 are satisfied; in order to  
 330 satisfy the third one we assume that

$$(3.1) \quad h_0 = \min_{\mathbf{x} : \mathbf{x}^\top \mathbf{x} \leq 1} \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0 < 0,$$

331 i.e., the feasible region of (1.1) has a nonempty interior. Note that the assumption can  
 332 be checked in polynomial time by the solution of a trust region problem. As before,  
 333 we denote by  $X \subseteq \mathbb{R}^n$  a closed set such that  $X \supset H$ , i.e., it contains the ellipsoid  
 334 defined by the second constraint. For each  $\lambda \geq 0$ , the Lagrangian relaxation (2.3)  
 335 takes on the form

$$(3.2) \quad p_X(\lambda) = \min_{\mathbf{x} \in X} \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0$$

$$\mathbf{x}^\top \mathbf{x} \leq 1.$$

336 If  $X = \mathbb{R}^n$ , this is the standard Lagrangian relaxation of problem (1.1) and it can be  
 337 solved efficiently since it is a trust region problem. Following the notation of section  
 338 2, let

$$P_X(\lambda) = \arg \min_{\mathbf{x} \in X : \mathbf{x}^\top \mathbf{x} \leq 1} \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x}$$

339 be the set of optimal solutions of (3.2). To apply Algorithm 2.1 to the CDT problem  
 340 with  $X = \mathbb{R}^n$ , we need to characterize the set of optimal solutions  $P_{\mathbb{R}^n}(\lambda)$  of problem  
 341 (3.2) with  $X = \mathbb{R}^n$ , which is a trust region problem. The set of optimal solutions of a  
 342 trust region problem has been derived, e.g., in [1, 20, 21]. Here we briefly recall the  
 343 different cases. For simplicity, let  $\mathbf{S}_\lambda = \mathbf{Q} + \lambda \mathbf{A}$  and  $\mathbf{s}_\lambda = \mathbf{q} + \lambda \mathbf{a}$ . We distinguish the  
 344 following cases:

346 **Case 1** If  $\mathbf{S}_\lambda > \mathbf{O}$  and  $\|-\frac{1}{2}\mathbf{S}_\lambda^{-1}\mathbf{s}_\lambda\| \leq 1$ , then  $-\frac{1}{2}\mathbf{S}_\lambda^{-1}\mathbf{s}_\lambda$  is the unique optimal solution  
347 of (3.2);

348 **Case 2** Let  $\mathbf{u}_j$  be the orthonormal eigenvectors of matrix  $\mathbf{S}_\lambda$ , and let  $\gamma_j$  be the cor-  
349 responding eigenvalues. Let  $\gamma_{\min} = \min_j \gamma_j$  and  $J_\lambda = \arg \min_j \gamma_j$ . For each  $\gamma$   
350 such that  $(\forall j) \gamma \neq \gamma_j$ , let

$$\mathbf{y}(\gamma) = \mathbf{y}_1(\gamma) + \mathbf{y}_2(\gamma),$$

351 where

$$\mathbf{y}_1(\gamma) = - \sum_{j \notin J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma} \mathbf{u}_j, \quad \mathbf{y}_2(\gamma) = - \sum_{j \in J_\lambda} \frac{\mathbf{s}_\lambda^\top \mathbf{u}_j}{\gamma_j - \gamma} \mathbf{u}_j.$$

352 Then, we have the following subcases.

353 **Case 2.1** It holds that  $\mathbf{s}_\lambda^\top \mathbf{u}_j \neq 0$  for some  $j \in J_\lambda$ . Then, there exists a unique  
354  $\gamma^* \in (-\gamma_{\min}, +\infty)$  such that  $\|\mathbf{y}(\gamma^*)\| = 1$  and  $\mathbf{y}(\gamma^*)$  is the unique  
355 optimal solution of (3.2).

356 **Case 2.2** It holds that  $\mathbf{s}_\lambda^\top \mathbf{u}_j = 0$  for all  $j \in J_\lambda$  but  $\|\mathbf{y}_1(\gamma_{\min})\| \geq 1$ . In this  
357 case there exists a unique  $\gamma^* \in [-\gamma_{\min}, +\infty)$  such that  $\|\mathbf{y}_1(\gamma^*)\| = 1$   
358 and  $\mathbf{y}_1(\gamma^*)$  is the unique optimal solution of (3.2).

359 **Case 2.3** It holds that  $\mathbf{s}_\lambda^\top \mathbf{u}_j = 0$  for all  $j \in J_\lambda$  and  $\|\mathbf{y}_1(\gamma_{\min})\| < 1$ . In this  
360 case we have that  $P_{\mathbb{R}^n}(\lambda)$  is not a singleton and is made up by the  
361 following points:  
362

$$(3.3) \quad P_{\mathbb{R}^n}(\lambda) = \left\{ \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_\lambda} \xi_j \mathbf{u}_j : \sum_{j \in J_\lambda} \xi_j^2 = 1 - \|\mathbf{y}_1(\gamma_{\min})\|^2 \right\}.$$

363 Thus, we recognize two further subcases.

364 **Case 2.3.1**  $|J_\lambda| = 1$ , in which case  $P_{\mathbb{R}^n}(\lambda)$  contains exactly two distinct  
365 points.

366 **Case 2.3.2**  $|J_\lambda| \geq 2$ , in which case the set  $P_{\mathbb{R}^n}(\lambda)$  contains an infinite num-  
367 ber of points and is a connected set.

368 Note that in Cases 2.3.1 and 2.3.2 we can compute the two values  $h_{\mathbb{R}^n}^{\min}(\lambda), h_{\mathbb{R}^n}^{\max}(\lambda)$   
369 by solving a trust region problem over the border of a  $|J_\lambda|$ -dimensional ball. More  
370 precisely, we need to solve the following problems:  
371

$$(3.4) \quad \min / \max_{\boldsymbol{\xi}} \quad \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0 \\ \|\mathbf{w}(\boldsymbol{\xi})\|^2 = 1,$$

372 where  $\mathbf{w}(\boldsymbol{\xi}) = \mathbf{y}_1(\gamma_{\min}) + \sum_{j \in J_\lambda} \xi_j \mathbf{u}_j$ . In these cases, where  $P_{\mathbb{R}^n}(\lambda)$  is not a singleton,  
373 we also set

$$(3.5) \quad \mathbf{z}_1(\lambda) = \mathbf{w}(\boldsymbol{\xi}_1), \quad \boldsymbol{\xi}_1 \in \arg \min_{\boldsymbol{\xi} : \|\mathbf{w}(\boldsymbol{\xi})\|=1} \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0, \\ \mathbf{z}_2(\lambda) = \mathbf{w}(\boldsymbol{\xi}_2), \quad \boldsymbol{\xi}_2 \in \arg \max_{\boldsymbol{\xi} : \|\mathbf{w}(\boldsymbol{\xi})\|=1} \mathbf{w}(\boldsymbol{\xi})^\top \mathbf{A} \mathbf{w}(\boldsymbol{\xi}) + \mathbf{a}^\top \mathbf{w}(\boldsymbol{\xi}) - a_0,$$

374 while in all other cases, when  $P_{\mathbb{R}^n}(\lambda) = \{\mathbf{z}^*(\lambda)\}$  is a singleton, we set

$$(3.6) \quad \mathbf{z}_1(\lambda) = \mathbf{z}_2(\lambda) = \mathbf{z}^*(\lambda).$$

375 The following statement is a direct consequence of Proposition 2.5.

376 **PROPOSITION 3.1.** *In the CDT problem (1.1), if  $\lambda > 0$  and*

- 377 •  $h_{\mathbb{R}^n}^{\min}(\lambda) = 0$ ;
- 378 • or if  $h_{\mathbb{R}^n}^{\max}(\lambda) = 0$ ;
- 379 • or  $h_{\mathbb{R}^n}^{\min}(\lambda) < 0 < h_{\mathbb{R}^n}^{\max}(\lambda)$  and  $|J_\lambda| \geq 2$  (i.e., we are in Case 2.3.2);

380 then  $p_{\mathbb{R}^n}(\lambda) = p^*$ .

382 *Proof.* Since  $\{h_{\mathbb{R}^n}^{\min}(\lambda), h_{\mathbb{R}^n}^{\max}(\lambda)\} \in Q_X(\lambda)$ , in the first two cases  $0 \in Q_X(\lambda)$  and  
 383 the thesis is a consequence of Proposition 2.5. If  $h_{\mathbb{R}^n}^{\min}(\lambda) < 0 < h_{\mathbb{R}^n}^{\max}(\lambda)$  and  $|J_\lambda| \geq 2$ ,  
 384 we observed that  $P_{\mathbb{R}^n}(\lambda)$  is a connected set. Then, there exists  $\mathbf{x}^* \in P_{\mathbb{R}^n}(\lambda)$  such  
 385 that  $\mathbf{x}^* \in \partial H$ . More precisely,  $\mathbf{x}^*$  is a point along the curve in  $P_{\mathbb{R}^n}(\lambda)$  connecting  
 386 points  $\mathbf{z}_1(\lambda)$  and  $\mathbf{z}_2(\lambda)$ , defined in (3.5). Thus, the lower bound  $p_{\mathbb{R}^n}(\lambda)$  is equal to the  
 387 optimal value of problem (1.1).  $\square$

388 Note that the first two conditions of Proposition 3.1 imply exactness of the bound  
 389 also for generic regions  $X \supset H$ , while the last condition is specific to the case  $X = \mathbb{R}^n$ .  
 390 The following result is related to the necessary and sufficient condition under which  
 391 the dual Lagrangian bound is not exact discussed in [2].

392 **PROPOSITION 3.2.** *In the CDT problem (1.1),  $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$  if and only if  $\lambda_{\mathbb{R}^n} >$   
 393  $0$ ,  $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n})$  contains exactly two points (Case 2.3.1), and  $0 \in (h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n}))$ .*

394 *Proof.* First note that, in view of Proposition 2.3, the dual Lagrangian bound is  
 395 always exact when  $\lambda_{\mathbb{R}^n} = 0$ . When  $\lambda_{\mathbb{R}^n} > 0$ , the result is a consequence of Proposition  
 396 3.1 and the fact that for  $|J_{\lambda_{\mathbb{R}^n}}| = 1$  it holds that  $Q_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) = \{h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n})\} \not\subseteq$   
 397  $0$ .  $\square$

398 Now, we introduce an example where  $p_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \neq p^*$ , that is the dual Lagrangian  
 399 bound is not exact, which will also be helpful in the following sections.

400 *Example 3.3.* Let us consider the following example taken from [12]:

$$\mathbf{Q} = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{q} = (1 \ 1) \quad \mathbf{a} = (0 \ 0), \quad a_0 = 2.$$

401 Such an instance has optimal value  $-4$  attained at points  $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$  and  $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ .  
 402 The maximizer of  $p_{\mathbb{R}^2}(\lambda)$  is  $\lambda_{\mathbb{R}^2} = 1$  for which we have

$$h_{\mathbb{R}^2}^{\min} \approx -0.66 < 0 < 0.66 \approx h_{\mathbb{R}^2}^{\max},$$

403 and, moreover,  $|J_{\lambda_{\mathbb{R}^2}}| = 1$ , so that we have exactly two optimal solutions of (3.2),  
 404 one violating the second constraint, namely  $\mathbf{z}_2(\lambda_{\mathbb{R}^2}) = (-0.911, 0.4114)$ , point  $x_1$  in  
 405 Figure 1, displayed as  $\circ$ , the other in  $\text{int}(H)$ , point  $z_1$  in Figure 1, displayed as  $\times$ .  
 406 The lower bound is  $p_{\mathbb{R}^2}(1) = -4.25$ , which is not exact.

407 Now, let us assume that the dual Lagrangian bound is not exact, i.e., as previously  
 408 stated in Proposition 3.2

$$0 \in (h_{\mathbb{R}^n}^{\min}(\lambda_{\mathbb{R}^n}), h_{\mathbb{R}^n}^{\max}(\lambda_{\mathbb{R}^n})), \quad |J_{\lambda_{\mathbb{R}^n}}| = 1.$$

409 Recall that, by Proposition 3.2, in this case, there exists a single point  $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in$   
 410  $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \cap H$  (actually  $\mathbf{z}_1(\lambda_{\mathbb{R}^n}) \in \text{int}(H)$ ), and a single point  $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \in P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$ .  
 411 Now we show that the dual Lagrangian bound can be strictly improved through the  
 412 addition of a linear cut. We first observe that the optimal value of problem (1.1) does  
 413 not change if we add constraints which are implied by the second one.

414 In the following proposition, we define a projection  $\Pi_{\mathbf{A}, \mathbf{a}} : \mathbb{R}^n \setminus H \rightarrow \partial H$  that  
 415 maps  $\mathbf{x} \notin H$  to the element of  $\partial H$  located on the segment that joins  $\mathbf{x}$  to the center  
 416 of the ellipsoid  $H$  (given by  $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$ ).

417 **PROPOSITION 3.4.** *For  $\mathbf{x} \notin H$ , set  $\Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{x}) = \sqrt{\frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x})-h(\boldsymbol{\alpha})}}(\mathbf{x} - \boldsymbol{\alpha}) + \boldsymbol{\alpha}$ , where  
 418  $\boldsymbol{\alpha} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{a}$  is the center of the ellipsoid. Then  $h(\Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{x})) = 0$ .*

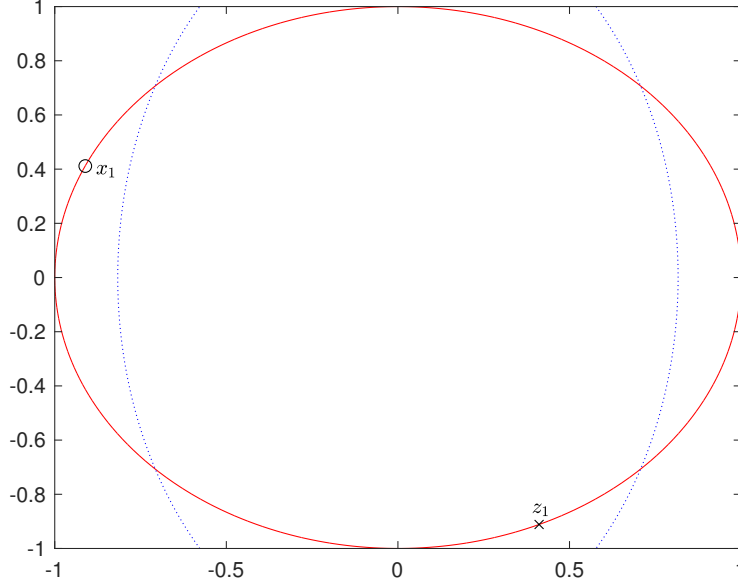


FIG. 1. Optimal solutions of the dual Lagrangian bound outside  $H$  ( $x_1$ ) and in  $\text{int}(H)$  ( $z_1$ ), denoted by  $\circ$  and  $\times$ , respectively. The continuous red curve is the border of the unit ball, while the dotted blue curve is the border of the ellipsoid  $H$ . (Figure in color online.)

419 *Proof.* Note that  $(\forall \beta \in \mathbb{R}) h(\beta(\mathbf{x} - \boldsymbol{\alpha}) + \boldsymbol{\alpha}) - h(\boldsymbol{\alpha}) = \beta^2(h(\mathbf{x}) - h(\boldsymbol{\alpha}))$  (it is a  
 420 consequence of the fact that function  $h$  is quadratic and it can be verified by direct  
 421 substitution). Then  $h(\Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{x})) = h\left(\sqrt{\frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x}) - h(\boldsymbol{\alpha})}}(\mathbf{x} - \boldsymbol{\alpha}) + \boldsymbol{\alpha}\right) = \frac{-h(\boldsymbol{\alpha})}{h(\mathbf{x}) - h(\boldsymbol{\alpha})}(h(\mathbf{x}) -$   
 422  $h(\boldsymbol{\alpha})) + h(\boldsymbol{\alpha}) = 0$ .  $\square$

423 Given any  $\bar{\mathbf{x}} \in \mathbb{R}^n$ , it holds, by convexity, that

$$\mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} \geq \bar{\mathbf{x}}^\top \mathbf{A} \bar{\mathbf{x}} + \mathbf{a}^\top \bar{\mathbf{x}} + (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}).$$

424 Thus, the following linear constraint is implied by the second constraint in (1.1):

$$(3.7) \quad (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top \mathbf{x} - \bar{\mathbf{x}}^\top \mathbf{A} \bar{\mathbf{x}} \leq a_0,$$

425 and, consequently, it can be added to problem (1.1) without modifying its feasible  
 426 region. In particular, if  $\bar{\mathbf{x}} \in \partial H$ , being  $\bar{\mathbf{x}}^\top \mathbf{A} \bar{\mathbf{x}} + \mathbf{a}^\top \bar{\mathbf{x}} = a_0$ , the linear constraint is

$$(3.8) \quad (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0.$$

427 Due to the redundancy of the linear constraint for problem (1.1), we can define, for a  
 428 given  $\bar{\mathbf{x}} \in \partial H$ , the new Lagrangian problem

$$(3.9) \quad \begin{aligned} p_X(\lambda) = \min_{\mathbf{x}} \quad & \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top \mathbf{x} \leq 1 \\ & (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0, \end{aligned}$$

429 where

$$(3.10) \quad X = \Omega_{\bar{\mathbf{x}}} = \{\mathbf{x} : (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0\} \supset H.$$

430 If we set  $\bar{\mathbf{x}} = \Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$ , i.e.,  $\bar{\mathbf{x}}$  is the projection over  $\partial H$  of the single point in  
 431  $P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H$ , then  $\mathbb{R}^n \supset X \supset H$  and, moreover,  $X \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$ , so that,  
 432 by Proposition 2.12,  $\bar{p}_X > \bar{p}_{\mathbb{R}^n}$ . Then, if we run again Algorithm 2.1 with input  
 433  $X = \Omega_{\bar{\mathbf{x}}}$  defined in (3.10) and  $\lambda_{\text{init}} = \lambda_{\mathbb{R}^n}$  (or  $\lambda_{\text{init}} = \lambda_{\mathbb{R}^n}^{\text{max}}$ ), we are able to improve  
 434 strictly the dual Lagrangian bound. Note that problem (3.9), needed to compute  
 435 function  $p_{\Omega_{\bar{\mathbf{x}}}}$ , can be solved in polynomial time according to the results proved in  
 436 [12, 23]. But we also discuss an alternative way to solve problem (3.9), based on the  
 437 solution of a trust region problem. For  $\lambda = \lambda_{\mathbb{R}^n}$ , after the addition of the linear cut,  
 438 a unique optimal solution exists, lying in  $\text{int}(H)$  and, consequently, in  $\text{int}(\Omega_{\bar{\mathbf{x}}})$ , since  
 439 also the linear constraint in (3.9) is not active at it, being  $H$  a subset of the region **AQ4**  
 440 defined by the linear cut. By continuity, for  $\lambda$  values smaller than but close to  $\lambda_{\mathbb{R}^n}$ ,  
 441 the unique optimal solution of (3.9) also lies in  $\text{int}(H)$ , i.e.,  $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$  with  
 442  $\mathbf{z}_1(\lambda) \in \text{int}(H)$ . Thus, such optimal solution must be a local and nonglobal optimal  
 443 solution of the trust region problem (3.2) with  $X = \mathbb{R}^n$ . Indeed, the globally optimal  
 444 solutions of this trust region problem always violate the second constraint in (1.1) for  
 445 all  $\lambda < \lambda_{\mathbb{R}^n}$ . Now, for all  $\lambda \in [0, \lambda_{\mathbb{R}^n})$ , we first check whether a local and nonglobal  
 446 optimal solution of problem (3.2) with  $X = \mathbb{R}^n$  exists by exploiting the necessary and  
 447 sufficient condition stated in [24]. Also recall that, if it exists, the local and nonglobal  
 448 minimizer is unique. If it does not exist, then we set  $f_1 = +\infty$ . Otherwise, if it exists,  
 449 we denote it by  $\mathbf{z}_1(\lambda)$ . If  $\mathbf{z}_1(\lambda) \notin \Omega_{\bar{\mathbf{x}}}$ , then we set again  $f_1 = +\infty$ , otherwise we denote  
 450 by  $f_1$  the value of the objective function of (3.9) evaluated at  $\mathbf{z}_1(\lambda)$ . If some globally  
 451 optimal solution of the trust region problem (3.2) with  $X = \mathbb{R}^n$  belongs to  $\Omega_{\bar{\mathbf{x}}} \setminus H$ , then  
 452 it is also a solution of (3.9) and we set  $f_2$  equal to the optimal value of this problem.  
 453 Note that in this case  $f_2 < f_1$ , since  $f_1$  is the function value at a local and nonglobal  
 454 solution of the trust region problem. Then, Algorithm 2.1 sets  $\lambda^{\text{min}} = \lambda$ . Instead,  
 455 if all globally optimal solutions of the trust region problem do not belong to  $\Omega_{\bar{\mathbf{x}}}$ , we  
 456 proceed as follows. We consider the best feasible solutions of problem (3.9) for which  
 457 the linear constraint is imposed to be active. The resulting problem is converted into  
 458 a trust region problem, after the change of variable  $\mathbf{x} = \bar{\mathbf{x}} + \mathbf{V}\mathbf{z}$ , where  $\mathbf{V} \in \mathbb{R}^{n \times (n-1)}$   
 459 is a matrix whose columns form a basis for the null space of vector  $2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a}$ . The  
 460 resulting (trust region) problem is

$$(3.11) \quad \min_{\mathbf{w} \in \mathbb{R}^{n-1}} \quad \mathbf{w}^\top \mathbf{V}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{V} \mathbf{w} + [2\bar{\mathbf{x}}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{V} + (\mathbf{q} + \lambda \mathbf{a})^\top] \mathbf{w} + \ell(\bar{\mathbf{x}}, \lambda) \\ \|\bar{\mathbf{x}} + \mathbf{V} \mathbf{w}\|^2 \leq 1,$$

461 where  $\ell(\bar{\mathbf{x}}, \lambda) = \bar{\mathbf{x}}^\top (\mathbf{Q} + \lambda \mathbf{A}) \bar{\mathbf{x}} + (\mathbf{q} + \lambda \mathbf{a})^\top \bar{\mathbf{x}} - \lambda a_0$  is constant with respect to the  
 462 vector of variables  $\mathbf{w}$ . Let  $W^*(\lambda)$  be the set of optimal solutions of problem (3.11)  
 463 and

$$P_1^*(\lambda) = \{\bar{\mathbf{x}} + \mathbf{V} \mathbf{w}^* : \mathbf{w}^* \in W^*(\lambda)\}.$$

464 Note that the set  $W^*(\lambda)$  can be computed through the procedure presented in section  
 465 3 with the different cases (namely, Cases 1, 2.1, 2.2, 2.3.1, 2.3.2) after rewriting it as a  
 466 classical trust region problem. Moreover, let  $f_2 < +\infty$  be the optimal value of problem  
 467 (3.11). Now, after comparing  $f_1$  and  $f_2$ , we are able to define the set  $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda)$  of optimal  
 468 solutions for problem (3.9). More precisely, if  $f_2 > f_1$ , then  $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = \{\mathbf{z}_1(\lambda)\}$ , i.e.,  
 469  $\mathbf{z}_1(\lambda)$  is the unique optimal solution of problem (3.9). In this case

$$h_{\Omega_{\bar{\mathbf{x}}}}^{\text{min}}(\lambda) = h_{\Omega_{\bar{\mathbf{x}}}}^{\text{max}}(\lambda) = \mathbf{z}_1(\lambda)^\top \mathbf{A} \mathbf{z}_1(\lambda) + \mathbf{a}^\top \mathbf{z}_1(\lambda) - a_0.$$

470 Instead, if  $f_2 < f_1$ , which always holds, e.g., if  $f_1 = +\infty$ , then  $P_{\Omega_{\bar{\mathbf{x}}}}(\lambda) = P_1^*(\lambda)$ . Since  
 471 all points in  $P_1^*(\lambda)$  lie over a supporting hyperplane of  $H$ , we must have that

$$h_{\Omega_{\bar{\mathbf{x}}}}^{\min}(\lambda) = \min_{\mathbf{x} \in P_1^*(\lambda)} \mathbf{x}^\top \mathbf{A} \mathbf{x} + \mathbf{a}^\top \mathbf{x} - a_0 \geq 0,$$

472 and equality holds only if  $\bar{\mathbf{x}} \in P_1^*(\lambda)$ . In the latter case, the bound is exact, otherwise  
 473 Algorithm 2.1 sets  $\lambda^{\min} = \lambda$ . Finally, if  $f_1 = f_2$ , then  $P_{\Omega_{\bar{\mathbf{x}}}(\lambda)} = P_1^*(\lambda) \cup \{\mathbf{z}_1(\lambda)\}$  and  
 474 in this case  $0 \in [h_{\Omega_{\bar{\mathbf{x}}}(\lambda)}^{\min}, h_{\Omega_{\bar{\mathbf{x}}}(\lambda)}^{\max}]$  and the algorithm exits the loop. The following  
 475 result is a straightforward consequence of Proposition 2.12.

476 **PROPOSITION 3.5.** *Algorithm 2.1 with  $\varepsilon = 0$  will stop after a finite number of*  
 477 *iterations or will converge to some  $\lambda_{\Omega_{\bar{\mathbf{x}}}} < \lambda_{\mathbb{R}^n}$  with a new lower bound  $\bar{p}_{\Omega_{\bar{\mathbf{x}}}} > \bar{p}_{\mathbb{R}^n}$ .*

478 *Proof.* Strict inequalities hold in view of Proposition 2.12 with  $X = \mathbb{R}^n$  and  
 479  $Y = \Omega_{\bar{\mathbf{x}}}$ , since, as already observed,  $\Omega_{\bar{\mathbf{x}}} \cap (P_{\mathbb{R}^n}(\lambda_{\mathbb{R}^n}) \setminus H) = \emptyset$ .  $\square$

480 If the final bound is not exact, i.e.,  $\bar{p}_{\Omega_{\bar{\mathbf{x}}}} = p_{\Omega_{\bar{\mathbf{x}}}(\lambda_{\Omega_{\bar{\mathbf{x}}})} < p^*$ , at  $\lambda_{\Omega_{\bar{\mathbf{x}}}}$  we have  $f_1 = f_2$  and  
 481  $P_{\Omega_{\bar{\mathbf{x}}}(\lambda_{\Omega_{\bar{\mathbf{x}}})}$  contains multiple optimal solutions, in particular, one in  $\text{int}(H)$  and the  
 482 other(s) outside  $H$ , more precisely on  $\partial\Omega_{\bar{\mathbf{x}}} \setminus H$ . We illustrate all this on Example 3.3.  
 483

484 *Example 3.6.* The optimal solution of (3.2) with  $X = \mathbb{R}^n$  for  $\lambda_{\mathbb{R}^n} = 1$  which  
 485 violates the second constraint is  $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) = (-0.911, 0.4114)$ . The lower bound is  
 486  $p_{\mathbb{R}^n}(1) = -4.25$ . After the addition of the linear inequality (3.7) obtained with  $\bar{\mathbf{x}} =$   
 487  $\Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{z}_2(\lambda_{\mathbb{R}^n}))$ , equal to the projection of  $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$  over the boundary of the second  
 488 constraint, we can run again Algorithm 2.1 with  $X = \Omega_{\bar{\mathbf{x}}}$  and we get to  $\lambda_{\Omega_{\bar{\mathbf{x}}}} \approx 0.726$   
 489 and  $p_{\Omega_{\bar{\mathbf{x}}}(\lambda_{\Omega_{\bar{\mathbf{x}}})} \approx -4.097$ , which improves the previous lower bound. In Figure 2 we  
 490 show the linear cut and the two new optimal solutions outside  $H$  and in  $\text{int}(H)$  ( $x_2$   
 491 and  $z_2$ , respectively) obtained at  $\lambda_{\Omega_{\bar{\mathbf{x}}}}$ . In the same figure we also display the previous  
 492 pair of optimal solutions in order to show the progress of the algorithm.

493 It is worthwhile to discuss at this point the relations between the approach pro-  
 494 posed in this work and the one proposed in [27], where the classical SDP relaxation

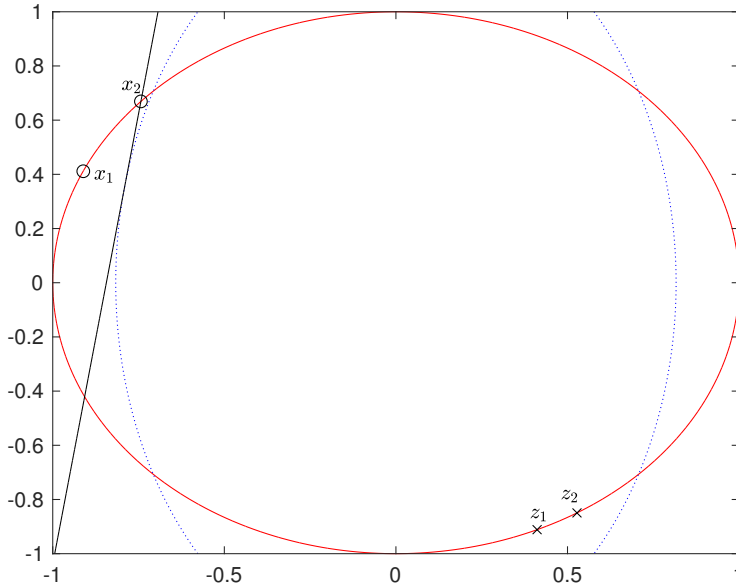


FIG. 2. First linear cut and the two optimal solutions lying outside  $H$  ( $x_2$ ) and in  $\text{int}(H)$  ( $z_2$ ), denoted by  $\circ$  and  $\times$ , respectively.

495 of problem (1.1) is considered. Both approaches stem from the necessary and suf-  
 496 ficient condition under which the dual Lagrangian bound is not exact discussed in  
 497 [2], namely the existence of two distinct optimal solutions, one belonging to  $\text{int}(H)$   
 498 and the other outside  $H$ . In both cases it is observed that, in order to improve the  
 499 bound, it is necessary to separate such optimal solutions. But the way the separation  
 500 is carried on in the two approaches is different. Following the terminology employed  
 501 in Integer Programming, in [27] the separation is performed through a branching oper-  
 502 ation, while in this work it is performed through the addition of a cutting plane.  
 503 Indeed, in [27] first, a hyperplane  $\mathbf{w}^T \mathbf{x} = v$  separating the two optimal solutions is  
 504 introduced; then, two distinct subproblems are solved, one by adding the inequality  
 505  $\mathbf{w}^T \mathbf{x} \leq v$  (converted into an SOCP constraint) to the SDP relaxation, the other by  
 506 adding the inequality  $\mathbf{w}^T \mathbf{x} \geq v$  to the SDP relaxation; finally, the new bound is set  
 507 equal to the minimum of the bounds over the two subregions into which the original  
 508 feasible region has been split. Note that one of the two subregions may be empty, in  
 509 which case its corresponding lower bound is set equal to  $+\infty$  and the linear inequality  
 510 is a separating hyperplane between  $H$  and the optimal solution outside  $H$ . In this  
 511 paper the separation is performed through the addition of a linear cut and a single  
 512 subproblem is solved. Moreover, in [27] it is observed that one could search for an  
 513 ‘‘optimal’’ hyperplane separating the two optimal solutions, namely one which leads  
 514 to the best possible bound. In that paper such a hyperplane is derived in the special  
 515 case when the function  $h$  is the product of two affine functions and an exactness result  
 516 is also provided for the case of problems with two variables, but the question about  
 517 how to characterize an ‘‘optimal’’ affine function is left open in the general case. In  
 518 the next section we will be able to provide a necessary and sufficient condition for a  
 519 linear cut to be the one delivering the best bound (Proposition 4.2). Based on this  
 520 condition, we will also be able to propose a procedure to improve the bound by local  
 521 adjustments of the linear cut. Finally, in this paper we will also show in section 5  
 522 that the bound can be further improved through the addition of a second linear cut,  
 523 possibly followed by a local adjustment of the two linear cuts. The experiments in  
 524 section 6 will show that the bound obtained by the addition of two linear cuts is quite  
 525 a good one, allowing one to solve all except one of the 212 hardest instances intro-  
 526 duced in [12]. As a final remark, we observe that the approach presented in [27] and  
 527 the one discussed in this paper could actually be combined by performing a branching  
 528 operation (as in [27]) followed by the addition of a linear cut (as in this work) in each  
 529 branch. Borrowing again from the terminology of Integer Programming, this can be  
 530 viewed as a branch-and-cut approach.

531 **4. Improving the bound by local adjustments of the linear cut.** In the  
 532 previous section we proposed to set  $\bar{\mathbf{x}}$  equal to the projection over  $\partial H$  of  $\mathbf{z}_2(\lambda_{\mathbb{R}^n})$ , the  
 533 optimal solution of problem (3.2) with  $X = \mathbb{R}^n$  lying outside  $H$ . However, this point  
 534 can be improved by some local adjustment. We first give a necessary and sufficient  
 535 condition under which the current point  $\bar{\mathbf{x}}$  cannot be improved. The proof will also  
 536 suggest how to improve the point (and the bound) when the condition is not fulfilled.  
 537 Let

$$(4.1) \quad r(\mathbf{w}, \mathbf{x}) = (2\mathbf{A}\mathbf{w} + \mathbf{a})^\top (\mathbf{x} - \mathbf{w}) + \mathbf{w}^\top \mathbf{A}\mathbf{w} + \mathbf{a}^\top \mathbf{w} - a_0 = (2\mathbf{A}\mathbf{w} + \mathbf{a})^\top \mathbf{x} - \mathbf{w}^\top \mathbf{A}\mathbf{w} - a_0$$

538 be the linearization of the ellipsoid constraint at  $\mathbf{w}$ . Note that constraint (3.7) can be  
 539 written as  $r(\bar{\mathbf{x}}, \mathbf{x}) \leq 0$ . Also note that for each  $\mathbf{x}$ ,  $r$  is a concave function with respect  
 540 to  $\mathbf{w}$ . Next, we set

$$p(\lambda, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\lambda),$$

541 in order to highlight the dependency of the bound not only on  $\lambda$  but also on  $\mathbf{w}$ . Then,  
542 in order to maximize the lower bound, we need to solve the following problem:

$$\max_{\lambda \geq 0, \mathbf{w}} p(\lambda, \mathbf{w}).$$

543 As before, we denote by  $P_{\Omega_{\mathbf{w}}}(\lambda)$  the optimal set of problem (3.9) with  $\bar{\mathbf{x}} = \mathbf{w}$ , while we  
544 denote by  $P_{\Omega_{\mathbf{w}}}^1(\lambda) = P_{\Omega_{\mathbf{w}}}(\lambda) \setminus \text{int}(H)$  the set of optimal solutions of the same problem  
545 lying outside the interior of the ellipsoid  $H$ . We will need the following lemma.

546 LEMMA 4.1. *Set-valued functions  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$  and  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}^1(\lambda)$  are upper semi-*  
547 *continuous for any  $\lambda \geq 0$ .*

548 *Proof.* Upper semicontinuity of  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$  follows from the maximum theorem  
549 (see, for instance, Theorem 1.4.16 of [5]), while upper semicontinuity of  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}^1(\lambda)$   
550 follows from the fact that  $P_{\Omega_{\mathbf{w}}}^1(\lambda)$  is obtained by intersecting the upper semicontinuous  
551 function  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}(\lambda)$  with the compact set  $\{\mathbf{x} : \|\mathbf{x}\| \leq 1\} \setminus \text{int}(H)$  (see, for instance,  
552 Proposition 1.4.9 of [5]).  $\square$

553 Now, the following proposition characterizes the maxima of  $p$ .

554 PROPOSITION 4.2. *Let  $(\lambda^*, \mathbf{w}^*)$  be such that  $\mathbf{w}^* \in \partial H$ ,  $\lambda^* > 0$ ,  $0 \in (h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*),$   
555  $h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$ , and  $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ . Assume also that  $(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$ . Then,*  
556 *the following statements are equivalent:*

- 557 (i)  $(\lambda^*, \mathbf{w}^*) = \operatorname{argmax}_{(\lambda \geq 0, \mathbf{w})} p(\lambda, \mathbf{w})$ .  
558 (ii)  $(\forall \mathbf{w} \in \mathbb{R}^n) P_{\Omega_{\mathbf{w}}}^1(\lambda^*) \neq \emptyset$ .  
559 (iii)

$$(4.2) \quad (\forall \mathbf{d} \in \mathbb{R}^n) (\exists \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) : -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) \leq 0.$$

560 *Proof.* Before proving the result we make some remarks. First, note that  $0 \in$   
561  $(h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*), h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$  implies that  $\lambda^* = \lambda_{\Omega_{\mathbf{w}^*}}$ . Moreover, since  $0 \notin Q_{\Omega_{\mathbf{w}^*}}(\lambda^*)$  and  
562  $\lambda^* > 0$ , by Propositions 2.3 and 2.5,  $p_{\Omega_{\mathbf{w}^*}}(\lambda^*) < p^*$  (i.e., the bound is not exact). If  
563 the bound were exact, the current pair  $(\mathbf{w}^*, \lambda^*)$  would obviously be optimal. Also  
564 note that  $h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*) < 0 < h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*)$  implies that  $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \neq \emptyset$ . Finally, condition  
565  $(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$  means that  $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \subset \partial \Omega_{\mathbf{w}^*}$ , which, according to  
566 the discussion about the optimal solutions of problem (3.9), holds true provided that  
567  $\mathbf{z}_2(\lambda_{\mathbb{R}^n}) \notin \Omega_{\mathbf{w}^*}$ .

568 (i)  $\rightarrow$  (ii) By contradiction, let  $\mathbf{w}$  be such that  $P_{\Omega_{\mathbf{w}}}^1(\lambda^*) = \emptyset$ . Then,  $P_{\Omega_{\mathbf{w}}}(\lambda^*) \subset \text{int}(H)$ .  
569 Therefore, function  $\lambda \rightsquigarrow p_{\Omega_{\mathbf{w}}}(\lambda)$  is strictly decreasing at  $\lambda^*$ . As a consequence,  
570 there exists  $0 \leq \bar{\lambda} < \lambda^*$  such that  $p(\bar{\lambda}, \mathbf{w}) = p_{\Omega_{\mathbf{w}}}(\bar{\lambda}) > p_{\Omega_{\mathbf{w}}}(\lambda^*) = p(\lambda^*, \mathbf{w})$ . More-  
571 over,  $p(\lambda^*, \mathbf{w}^*) = p(\lambda^*, \mathbf{w})$ . Indeed, since  $0 \in (h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*), h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$ , we have that  
572  $h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*) < 0$ , so that  $P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap \text{int}(H)$  is not empty and is equal to  $P_{\Omega_{\mathbf{w}^*}}(\lambda^*)$ .  
573 Hence,  $p(\bar{\lambda}, \mathbf{w}) > p(\lambda^*, \mathbf{w}^*)$ , which contradicts (i).

574 (ii)  $\rightarrow$  (iii) By contradiction, let us assume that there exists  $\mathbf{d} \in \mathbb{R}^n$  such that

$$(4.3) \quad (\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0.$$

575 By continuity of the left-hand side of the inequality in (4.3) with respect to  $\mathbf{v}$ , there  
576 exists a neighborhood  $B_1$  of  $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)$  such that

$$(\forall \mathbf{v} \in B_1) -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0,$$

577 which implies that

$$(4.4) \quad (\forall \mathbf{v} \in B_1) 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) > 0.$$

578 By upper semicontinuity of set-valued function  $\mathbf{w} \rightsquigarrow P_{\Omega_{\mathbf{w}}}^1(\lambda^*)$  (see Lemma 4.1),  
 579 there exists a neighborhood  $B_2$  of  $\mathbf{w}^*$  such that  $(\forall \mathbf{w} \in B_2) P_{\Omega_{\mathbf{w}}}^1(\lambda^*) \subset B_1$ . Let  $\bar{\eta} > 0$   
 580 be such that  $\mathbf{w}^* + \bar{\eta}\mathbf{d} \in B_2$  and consider function  $\rho : [0, \bar{\eta}] \times \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\rho(\eta, \mathbf{v}) =$   
 581  $r(\mathbf{w}^* + \eta\mathbf{d}, \mathbf{v})$ . Then, by definition of  $r$  in (4.1),

$$\rho(\eta, \mathbf{v}) = -\eta^2 \mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\eta \mathbf{d}^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) + \rho(0, \mathbf{v}).$$

582 By (4.4),  $(\forall \mathbf{v} \in B_1) \partial_\eta \rho(0, \mathbf{v}) > 0$ , where  $\partial_\eta$  denotes the partial derivative with respect  
 583 to  $\eta$ . By continuity of  $\rho$ , there exists a continuous function  $\hat{\eta} : B_1 \rightarrow (0, \bar{\eta}]$  such that

$$(4.5) \quad (\forall \mathbf{v} \in B_1, \eta \in (0, \hat{\eta}(\mathbf{v}))) \rho(\eta, \mathbf{v}) > \rho(0, \mathbf{v}).$$

584 Hence, since  $B_1$  is a compact set and  $\hat{\eta}$  is continuous and strictly positive, setting  
 585  $\tilde{\eta} = \min_{\mathbf{v} \in B_1} \hat{\eta}(\mathbf{v})$ , it follows that  $(\forall \eta \in [0, \tilde{\eta}]) P_{\Omega_{\mathbf{w}^* + \eta\mathbf{d}}}^1(\lambda^*) \subseteq P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)$ . Moreover,  
 586 since, by assumption,  $(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\mathbf{w}^*, \mathbf{v}) = 0$ ,

$$(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) \rho(0, \mathbf{v}) = 0.$$

587 Hence, (4.5) implies that

$$(4.6) \quad (\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) \rho(\tilde{\eta}, \mathbf{v}) > 0.$$

588 Being  $P_{\Omega_{\mathbf{w}^* + \tilde{\eta}\mathbf{d}}}^1(\lambda^*) \subseteq P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)$ , (4.6) implies that  $P_{\Omega_{\mathbf{w}^* + \tilde{\eta}\mathbf{d}}}^1(\lambda^*) = \emptyset$ , which contradicts  
 589 (ii).

590 (iii)  $\rightarrow$  (i) By contradiction, there exists a couple  $(\bar{\lambda}, \bar{\mathbf{w}})$  such that  $p(\bar{\lambda}, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*)$ .  
 591 In particular, we can take  $\bar{\lambda} = \lambda_{\Omega_{\bar{\mathbf{w}}}}$ . In case  $\bar{\lambda} = 0$ , by assumption  $\bar{\lambda} < \lambda^*$ . Now we  
 592 show that the same inequality holds true also when  $\bar{\lambda} > 0$ . If  $\bar{\lambda} > 0$ , then  $\bar{\lambda} = \lambda_{\Omega_{\bar{\mathbf{w}}}}$   
 593 implies that  $0 \in [h_{\Omega_{\bar{\mathbf{w}}}}^{\min}(\bar{\lambda}), h_{\Omega_{\bar{\mathbf{w}}}}^{\max}(\bar{\lambda})]$ . Since, by assumption,  $0 \in (h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*), h_{\Omega_{\mathbf{w}^*}}^{\max}(\lambda^*))$ ,  
 594 we have that both  $P_{\Omega_{\bar{\mathbf{w}}}}(\bar{\lambda}) \cap H \neq \emptyset$  and  $P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap H \neq \emptyset$  (i.e., the minimum values  
 595  $p(\lambda^*, \mathbf{w}^*)$  and  $p(\bar{\lambda}, \bar{\mathbf{w}})$  are both attained in  $H$ ). Hence,  $p(\bar{\lambda}, \bar{\mathbf{w}}) > p(\lambda^*, \mathbf{w}^*)$  implies  
 596 that  $\bar{\lambda} < \lambda^*$ . Indeed, let us assume that  $\bar{\lambda} \geq \lambda^*$  and let  $\mathbf{z} \in P_{\Omega_{\mathbf{w}^*}}(\lambda^*) \cap H$ . Then,

$$\begin{aligned} p(\lambda^*, \mathbf{w}^*) &= \mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} + \lambda^* (\mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0) \\ &\geq \mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} + \bar{\lambda} (\mathbf{z}^\top \mathbf{Q} \mathbf{z} + \mathbf{q}^\top \mathbf{z} - a_0) \geq p(\bar{\lambda}, \bar{\mathbf{w}}), \end{aligned}$$

597 which is a contradiction. Then, function  $p(\lambda^*, \bar{\mathbf{w}})$  must be decreasing at  $\lambda^*$  or, equiv-  
 598 alently,  $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset \text{int}(H)$  and  $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \neq \emptyset$ . Since  $h_{\Omega_{\mathbf{w}^*}}^{\min}(\lambda^*) < 0$ , then  $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$   
 599 and  $P_{\Omega_{\mathbf{w}^*}}(\lambda^*)$  have a common nonempty intersection within  $H$  and, consequently,  
 600  $p(\lambda^*, \bar{\mathbf{w}}) = p(\lambda^*, \mathbf{w}^*)$  holds. This implies that  $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$ . Indeed, assume  
 601 there exists  $\mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}}$ . Note that  $\mathbf{v} \notin \text{int}(H)$  and, since  $P_{\Omega_{\bar{\mathbf{w}}}}^1(\lambda^*) \subset P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$ ,  
 602  $\mathbf{v}$  would also belong to  $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*)$  which, however, contradicts  $P_{\Omega_{\bar{\mathbf{w}}}}(\lambda^*) \subset \text{int}(H)$ .

603 Condition  $P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*) \cap \Omega_{\bar{\mathbf{w}}} = \emptyset$  is equivalent to

$$(\forall \mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)) r(\bar{\mathbf{w}}, \mathbf{v}) > 0.$$

604 Note that, by assumption,  $\mathbf{v} \in P_{\Omega_{\mathbf{w}^*}}^1(\lambda^*)$  implies  $\mathbf{v} \in \partial\Omega_{\mathbf{w}^*}$ . Moreover,

$$\begin{aligned} r(\bar{\mathbf{w}}, \mathbf{v}) &= r((\bar{\mathbf{w}} - \mathbf{w}^*) + \mathbf{w}^*, \mathbf{v}) = -(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\bar{\mathbf{w}} - \mathbf{w}^*) \\ &\quad + 2(\bar{\mathbf{w}} - \mathbf{w}^*)^\top \mathbf{A}(\mathbf{v} - \mathbf{w}^*) + r(\mathbf{w}^*, \mathbf{v}) > 0. \end{aligned}$$

605 Being  $\mathbf{w}^* \in \partial H$  and  $\mathbf{v} \in \partial \Omega_{\mathbf{w}^*}$ , we have that  $r(\mathbf{w}^*, \mathbf{v}) = 0$ . Then, by taking  $\mathbf{d} = \bar{\mathbf{w}} - \mathbf{w}^*$ ,  
 606 (iii) is contradicted.  $\square$

607 Given the current point  $\bar{\mathbf{x}}$  with  $\lambda_{\Omega_{\bar{\mathbf{x}}}} > 0$ , the question now is either to find a direction  
 608  $\mathbf{d}$  fulfilling

$$(4.7) \quad (\forall \mathbf{v} \in P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})) - \mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}) > 0$$

609 or to establish that it does not exist. In case it does not exist,

$$p(\bar{\mathbf{x}}, \lambda_{\Omega_{\bar{\mathbf{x}}}}) = \max_{\lambda \geq 0, \mathbf{w}} p(\lambda, \mathbf{w}).$$

610 Otherwise, direction  $(\mathbf{d}, -1)$  is an increasing direction for function  $p$ . We discuss  
 611 different cases depending on the cardinality of  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})$  (see the cases discussed in  
 612 section 3 for the trust region problem).

613 **4.1.**  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})| = 1$ . In this case, let  $\mathbf{v}$  be the unique point in  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ . Then  
 614 we need to solve the following convex optimization problem:

$$\max_{\mathbf{d} \in \mathbb{R}^n} -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}),$$

615 whose optimal solution is  $\mathbf{d} = \mathbf{v} - \bar{\mathbf{x}}$  and its optimal value is  $(\mathbf{v} - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}) > 0$ .  
 616 Therefore, if  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})| = 1$ , we are always able to locally adjust the current point  $\bar{\mathbf{x}}$   
 617 in such a way that the bound can be improved.

618 **4.2.**  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})| = 2$ . In this case, let  $\mathbf{v}_1$  and  $\mathbf{v}_2$  be the two optimal points in  
 619  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ . Then, we need to solve the following optimization problem:

$$(4.8) \quad \max_{\mathbf{d} \in \mathbb{R}^n} \min \{-\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v}_1 - \bar{\mathbf{x}}), -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}})\},$$

620 or, equivalently

$$\begin{aligned} \max \quad & v \\ & v \leq -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v}_1 - \bar{\mathbf{x}}) \\ & v \leq -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}). \end{aligned}$$

621 This is a convex optimization problem, whose solution can be obtained in closed form.  
 622 Indeed, by imposing the KKT conditions, it can be seen that the optimal solution has  
 623 the following form:

$$(4.9) \quad \mathbf{d} = \beta(\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta)(\mathbf{v}_2 - \bar{\mathbf{x}}), \quad \beta \in [0, 1].$$

624 Now, let

$$\begin{aligned} a &= (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_1 - \bar{\mathbf{x}}) > 0, \\ b &= (\mathbf{v}_2 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}) > 0, \\ c &= (\mathbf{v}_1 - \bar{\mathbf{x}})^\top \mathbf{A}(\mathbf{v}_2 - \bar{\mathbf{x}}). \end{aligned}$$

625 By replacing (4.9) in the objective function of (4.8), we have that (4.8) can be rewritten  
 626 as

$$\max_{\beta \in [0, 1]} \min \{(-\beta^2 + 2\beta)a - (1 - \beta)^2 b + 2(1 - \beta)^2 c, -\beta^2 a + (1 - \beta^2)b + 2\beta^2 c\}.$$

627 The optimal solution of this problem is

$$\beta^* = \begin{cases} 0 & \text{if } b \leq c, \\ 1 & \text{if } a \leq c, \\ \frac{b-c}{a+b-2c} & \text{otherwise.} \end{cases}$$

628 Then, the optimal value is

$$\begin{cases} b & \text{if } b \leq c, \\ a & \text{if } a \leq c, \\ \frac{ab-c^2}{a+b-2c} & \text{otherwise.} \end{cases}$$

629 We notice that  $a, b > 0$ ,

$$a + b - 2c = (\mathbf{v}_1 - \mathbf{v}_2)^\top \mathbf{A}(\mathbf{v}_1 - \mathbf{v}_2) > 0,$$

630 and, by the Cauchy–Schwarz inequality,

$$ab - 2c^2 \geq 0,$$

631 and equality holds if and only if  $(\mathbf{v}_1 - \bar{\mathbf{x}})$  and  $(\mathbf{v}_2 - \bar{\mathbf{x}})$  are linearly dependent. Thus,  
 632 the optimal value of (4.8) is always strictly positive unless the two vectors  $(\mathbf{v}_1 - \bar{\mathbf{x}})$   
 633 and  $(\mathbf{v}_2 - \bar{\mathbf{x}})$  lie along the same direction. More precisely, the optimal value is null  
 634 only if the two vectors have the same direction but opposite sign. Indeed, let

$$\mathbf{v}_1 - \bar{\mathbf{x}} = \gamma(\mathbf{v}_2 - \bar{\mathbf{x}}).$$

635 Then, we have  $b = \gamma^2 a$  and  $c = \gamma a$ . If  $\gamma$  is positive, then either  $b \leq c$  (if  $\gamma \leq 1$ ), or  $a \leq c$   
 636 (if  $\gamma \geq 1$ ) occurs, so that the optimal value is equal to  $a$  or  $b$  and is, thus, positive. If  
 637  $(\mathbf{v}_1 - \bar{\mathbf{x}})$  is not a negative multiple of  $(\mathbf{v}_2 - \bar{\mathbf{x}})$ , we are able to locally adjust  $\bar{\mathbf{x}}$  along  
 638 direction

$$\mathbf{d} = \beta^*(\mathbf{v}_1 - \bar{\mathbf{x}}) + (1 - \beta^*)(\mathbf{v}_2 - \bar{\mathbf{x}}).$$

639 **4.3.  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})$  is an infinite connected set.** In this case we need to solve the  
 640 following optimization problem:

$$(4.10) \quad \max_{\mathbf{d} \in \mathbb{R}^n} \min_{\mathbf{v} \in P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})} -\mathbf{d}^\top \mathbf{A} \mathbf{d} + 2\mathbf{d}^\top \mathbf{A}(\mathbf{v} - \bar{\mathbf{x}}).$$

641 An improving direction exists if and only if the optimal value of this problem is  
 642 strictly positive (note that the optimal value is always nonnegative since the inner  
 643 minimization problem has optimal value 0 for  $\mathbf{d} = \mathbf{0}$ ). We first remark that the  
 644 problem is convex. Indeed, for each fixed  $\mathbf{v}$ , we have a concave function with respect  
 645 to  $\mathbf{d}$ , and the minimum of an infinite set of concave functions is itself a concave  
 646 function (to be maximized, so that the problem is convex). The inner minimization  
 647 problem can be solved in closed form. After removing the terms which do not depend  
 648 on  $\mathbf{v}$ , the inner problem to be solved is

$$\min_{\mathbf{v} \in P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})} 2\mathbf{d}^\top \mathbf{A} \mathbf{v}.$$

649 According to Case 2.3.2 in section 3,  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}})$  can be written as in (3.3) and the  
 650 minimization problem can be reduced to the computation of the minimum of a linear  
 651 function over the unit sphere:

$$\min_{\boldsymbol{\xi} \in \mathbb{R}^q : \|\boldsymbol{\xi}\|^2=1} \bar{\mathbf{c}}(\mathbf{d})^\top \boldsymbol{\xi},$$

652 where  $\bar{\mathbf{c}}(\mathbf{d})$  is some linear function of  $\mathbf{d}$  and  $q \geq 2$  is the multiplicity of the minimum  
 653 eigenvalue of the matrix  $\mathbf{V}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{V}$ , corresponding to the Hessian of the objective  
 654 function of problem (3.11). The optimal solution of this problem is

$$\boldsymbol{\xi}^* = -\frac{\bar{\mathbf{c}}(\mathbf{d})}{\|\bar{\mathbf{c}}(\mathbf{d})\|},$$

655 while the optimal value is  $-\|\bar{\mathbf{c}}(\mathbf{d})\|$ .

656 **4.4. An algorithm for the refinement of the bound.** Let  $\bar{\mathbf{x}}$  and  $\lambda_{\mathbb{R}^n}$  be  
 657 defined as in section 3. We propose Algorithm 4.1 for a bound based on successive  
 658 local adjustments of the linear cut. In line 2, Algorithm 2.1 is run with input  $X = \Omega_{\bar{\mathbf{x}}}$   
 659 and  $\lambda_{\mathbb{R}^n}$ . Note that with a slight abuse here we are assuming that the algorithm  
 660 returns  $\lambda_{\Omega_{\bar{\mathbf{x}}}}$  and the related points  $\mathbf{z}_1$  and  $\mathbf{z}_2$ , while in practice close approximations  
 661 of these quantities are returned, namely  $\lambda^{\max}$ ,  $\mathbf{z}_1(\lambda^{\max})$ , and  $\mathbf{z}_2(\lambda^{\min})$ . In line 3,  $\mathbf{z}$  is  
 662 initialized with the input point  $\bar{\mathbf{x}}$  itself and the direction  $\mathbf{d}^*$ , following the discussion  
 663 in section 4.1, is set equal to the difference between  $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ , the point outside  $H$   
 664 returned by Algorithm 2.1, and  $\bar{\mathbf{x}}$ . The outer while loop of the algorithm (lines 4–  
 665 20) is repeated until the bound is improved by at least a tolerance value  $tol$ . Inside  
 666 this loop, in line 5 the initial step size  $\eta = 1$  is set and a new incumbent  $\mathbf{y} \in \partial H$   
 667 is computed. The inner while loop (lines 7–15) computes the step size: until the  
 668 optimal value of problem (3.9) with  $\bar{\mathbf{x}} = \mathbf{y}$  and  $\lambda = \lambda_{\Omega_{\mathbf{z}}}$ , denoted by  $opt$ , is lower  
 669 than the current lower bound  $Lb$ , we need to decrease the step size and recompute a  
 670 new incumbent  $\mathbf{y}$  (lines 10–11). If the step size falls below a given tolerance value,  
 671 we exit the inner loop and also the outer one. Otherwise, we have identified a new  
 672 valid incumbent and we set to 1 the exit flag  $stop$  for the inner loop (line 13), so  
 673 that, later on, a new linear inequality (3.7) with  $\bar{\mathbf{x}} = \mathbf{y}$  will be computed. Then, at  
 674 line 17 we run Algorithm 2.1 with input  $X = \Omega_{\mathbf{y}}$  and  $\lambda_{\Omega_{\mathbf{z}}}$ . Finally, in line 18, we  
 675 update point  $\mathbf{z}$  and the direction  $\mathbf{d}^*$ . We remark that at each iteration  $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}})$  is  
 676 *one* optimal solution of the current subproblem (3.9) with  $\lambda = \lambda_{\Omega_{\mathbf{z}}}$  lying outside  $H$   
 677 and at which the linear cut of the subproblem is active, i.e.,  $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}}) \in P_1^*(\lambda_{\Omega_{\mathbf{z}}})$ . As  
 678 seen in section 4.1, if  $|P_1^*(\lambda_{\Omega_{\mathbf{z}}})| = 1$ , i.e.,  $\mathbf{z}_2(\lambda_{\Omega_{\mathbf{z}}})$  is the unique optimal solution of the  
 679 current subproblem (3.9) with  $\lambda = \lambda_{\Omega_{\mathbf{z}}}$  lying outside  $H$ , then, in view of Proposition  
 680 2.11, the local adjustment employed in Algorithm 4.1 is guaranteed to improve the  
 681 bound. However, as seen in sections 4.2 and 4.3, if  $P_1^*(\lambda_{\Omega_{\mathbf{z}}})$  contains more than one  
 682 point, than the proposed local adjustment is not guaranteed to improve the bound.  
 683 Sections 4.2 and 4.3 suggest how to define perturbing directions which still allow  
 684 one to improve the bound, in case they exist. However, as we will see through the  
 685 computational experiments, Algorithm 4.1 turns out to be time-consuming, and it  
 686 is more convenient to improve the bound by adding a further linear cut, as we do  
 687 in section 5, rather than further locally adjusting the current linear cut. In order to  
 688 clarify this point, we can make a comparison with Integer Linear Programming (ILP).  
 689 In ILP problems, once a linear relaxation is solved, a valid cut removes *one* optimal  
 690 solution of the relaxation. If the optimal solution is unique, then after the addition  
 691 of the valid cut, the bound improves. But if the linear relaxation has got multiple  
 692 solutions, then the valid cut is not guaranteed to remove all of them and, thus, the  
 bound may not improve. It is possible to try to strengthen the valid cut in such a way

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**Algorithm 4.1** Bound improvement through a local adjustment of the linear cut.

---

**Input:**  $\bar{\mathbf{x}}, \lambda_{\mathbb{R}^n}$

```

1: Set  $Lb_{old} = -\infty$ 
2: Let  $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \mathbf{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n})$ 
3: Set  $\mathbf{z} = \bar{\mathbf{x}}$  and  $\mathbf{d}^* = \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) - \bar{\mathbf{x}}$ 
4: while  $Lb - Lb_{old} > tol$  do
5:   Set  $Lb_{old} = Lb$ ,  $\eta = 1$  and  $\mathbf{y} = \Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{z} + \mathbf{d}^*) \in \partial H$ 
6:   Set  $stop = 0$ 
7:   while  $stop = 0$  and  $\eta > \varepsilon$  do
8:     Solve problem (3.9) with  $\bar{\mathbf{x}} = \mathbf{y}$  and  $\lambda = \lambda_{\Omega_{\mathbf{z}}}$ , and let  $opt$  be its optimal
     value
9:     if  $opt < Lb$  then
10:       Set  $\eta = \eta/2$ 
11:       Set  $\mathbf{y} = \Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{z} + \eta \mathbf{d}^*) \in \partial H$ 
12:     else
13:       Set  $stop = 1$ 
14:     end if
15:   end while
16:   if  $stop = 1$  then
17:     Let  $[Lb, \lambda_{\Omega_{\mathbf{y}}}, \mathbf{z}_1(\lambda_{\Omega_{\mathbf{y}}}), \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}})] = \mathbf{DualLagrangian}(\Omega_{\mathbf{y}}, \lambda_{\Omega_{\mathbf{z}}})$ 
18:     Set  $\mathbf{z} = \mathbf{y}$ ,  $\mathbf{d}^* = \mathbf{z}_2(\lambda_{\Omega_{\mathbf{y}}}) - \mathbf{z}$ 
19:   end if
20: end while
21: return  $Lb$ 

```

---

693 that all optimal solutions of the linear relaxations are removed. But, more commonly,  
694 new linear cuts are added.

695 Now we apply Algorithm 4.1 to our example.

696 *Example 4.3.* We have that  $\mathbf{z}$  is initialized with  $(-0.7901, 0.3565)$  and  $Lb$  with  
697  $-4.0971$ . During the execution of Algorithm 4.1,  $\mathbf{z}$  and  $Lb$  are updated as indicated  
698 in Table 1.

699 Interestingly, the best bound obtained in the example is exactly the one obtained  
700 for the same problem by the approach proposed in [12], based on the addition of  
701 SOC-RLT constraints. Figure 3 displays the situation at the last iteration of Algorithm  
702 4.1. Problem (3.9) has three optimal solutions, one in  $int(H)$  and two outside  $H$ . The

TABLE 1  
Iterations of Algorithm 4.1 over the example.

Iteration	$\mathbf{z}$	$Lb$
1	$(-0.7204, 0.6658)$	$-4.0850$
2	$(-0.7742, 0.4493)$	$-4.0638$
3	$(-0.7481, 0.5665)$	$-4.0477$
4	$(-0.7607, 0.5136)$	$-4.0416$
5	$(-0.7556, 0.5361)$	$-4.0378$
6	$(-0.7571, 0.5296)$	$-4.0364$
7	$(-0.7568, 0.5309)$	$-4.0362$

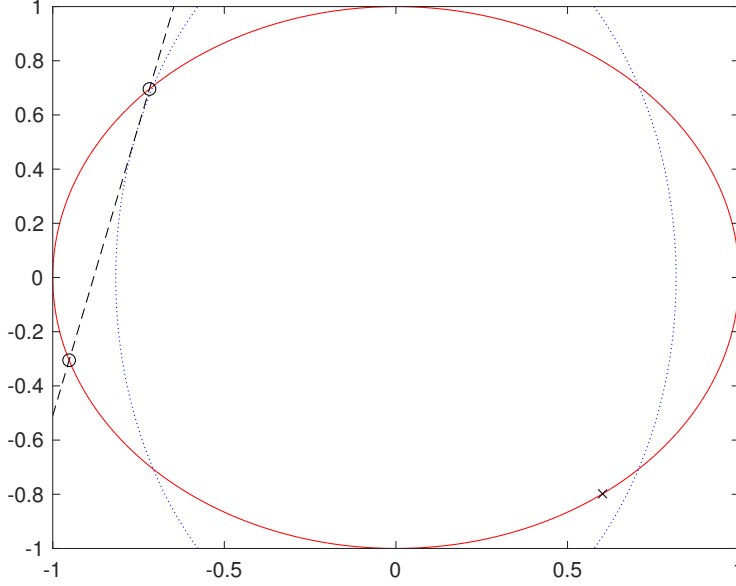


FIG. 3. Final linear cut after running Algorithm 4.1. Problem (3.9) has three optimal solutions, one in  $\text{int}(H)$  and two outside  $H$ . The latter solutions are opposite to each other with respect to the final vector  $\mathbf{z}$ .

703 two optimal solutions outside  $H$  are opposite to each other with respect to the final  
 704 vector  $\mathbf{z}$ , so that, as discussed in section 4.2, no further local adjustment is possible  
 705 to improve the bound in this case.

### 706 5. Bound improvement through the addition of a further linear cut.

707 Another possible way to improve the bound is by adding a further linear cut to (3.9).  
 708 Let  $\bar{\mathbf{x}}$  and  $\lambda_{\mathbb{R}^n}$  be defined as in section 3. In line 2 of Algorithm 4.1, we compute  
 709  $[Lb, \lambda_{\Omega_{\bar{\mathbf{x}}}}, \mathbf{z}_1(\lambda_{\Omega_{\bar{\mathbf{x}}}}), \mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})] = \text{DualLagrangian}(\Omega_{\bar{\mathbf{x}}}, \lambda_{\mathbb{R}^n})$ , and, later on, we try to  
 710 locally adjust  $\bar{\mathbf{x}}$ . Rather than doing that, we can add a further linear cut, cutting  
 711  $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}}) \notin H$  away. In particular, we add the one obtained through the projection over  
 712  $\partial H$  of  $\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})$ . Let  $\tilde{\mathbf{x}} = \Pi_{\mathbf{A}, \mathbf{a}}(\mathbf{z}_2(\lambda_{\Omega_{\bar{\mathbf{x}}}})) \in \partial H$  be such projection. Then, we define the  
 713 following problem:

$$(5.1) \quad \begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top \mathbf{x} \leq 1 \\ & (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \\ & (2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \tilde{\mathbf{x}}) \leq 0, \end{aligned}$$

714 which is equivalent to problem (3.2) where

$$X = \Omega_{\bar{\mathbf{x}}} \cap \Omega_{\tilde{\mathbf{x}}} = \{\mathbf{x} : (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0, (2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \tilde{\mathbf{x}}) \leq 0\} \supset H.$$

715 A convex reformulation as the one proposed in [12, 23] for problem (3.9) is not available  
 716 in this case (unless the two linear inequalities do not intersect in the interior of the  
 717 unit ball). But in this case the alternative procedure discussed in section 3 turns out  
 718 to be useful. As before, for each value  $\lambda$  in the while loop of Algorithm 2.1 we can first  
 719 check whether a local and nonglobal optimal solution of problem (3.2) with  $X = \mathbb{R}^n$   
 720 exists, by exploiting the necessary and sufficient condition stated in [24]. If it exists,

721 and belongs to  $\Omega_{\bar{x}} \cap \Omega_{\tilde{x}}$ , we denote it by  $\mathbf{z}_1(\lambda)$ . Next, we need to compute the optimal  
 722 value of (5.1) when at least one of the two linear constraints is active, i.e., we need to  
 723 solve the following problem:

$$(5.2) \quad \begin{aligned} \min_{\mathbf{x}} \quad & \mathbf{x}^\top (\mathbf{Q} + \lambda \mathbf{A}) \mathbf{x} + (\mathbf{q} + \lambda \mathbf{a})^\top \mathbf{x} - \lambda a_0 \\ & \mathbf{x}^\top \mathbf{x} \leq 1 \\ & (2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \\ & (2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \tilde{\mathbf{x}}) \leq 0 \\ & [(2\mathbf{A}\bar{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \bar{\mathbf{x}})] [(2\mathbf{A}\tilde{\mathbf{x}} + \mathbf{a})^\top (\mathbf{x} - \tilde{\mathbf{x}})] = 0. \end{aligned}$$

724 A convex reformulation of this problem has been proposed in [26]. Alternatively, one  
 725 can solve two distinct problems, each imposing that one of the two linear inequalities  
 726 is active. Each of these problems can be converted into a trust region problem with  
 727 an additional linear inequality, which can be solved in polynomial time through the  
 728 already mentioned convex reformulation proposed in [12, 23]. Thus, we compute the  
 729 set  $P_1^*(\lambda) \subseteq \partial\Omega_{\bar{x}} \cap \Omega_{\tilde{x}}$  of optimal solutions of (5.1) for which the first linear cut is  
 730 active, and then the set  $P_2^*(\lambda) \subseteq \Omega_{\bar{x}} \cap \partial\Omega_{\tilde{x}}$  of optimal solutions of (5.1) for which  
 731 the second linear cut is active. Finally, the optimal values of these problems are  
 732 compared with the value of the local and nonglobal minimizer (if it exists) in order  
 733 to identify the set  $P_X(\lambda)$  of optimal solutions of (5.1). At this point we are able to  
 734 compute  $h_X^{\min}(\lambda), h_X^{\max}(\lambda)$  and update  $\lambda^{\min}$  and  $\lambda^{\max}$  accordingly. If for some  $\lambda$  we  
 735 have that  $\mathbf{z}_1(\lambda) \in P_X(\lambda)$  and  $P_X(\lambda) \cap [P_1^*(\lambda) \cup P_2^*(\lambda)] \neq \emptyset$ , i.e., problem (5.1) has  
 736 an optimal solution in  $\text{int}(H)$  and (at least) one optimal solution outside  $H$ , then  
 737  $0 \in [h_X^{\min}(\lambda), h_X^{\max}(\lambda)]$  and Algorithm 2.1 stops. We illustrate all this on Example 3.3.

739 *Example 5.1.* We add a second linear cut obtained through the projection over  
 740  $\partial H$  of the optimal solution of problem (3.9) with  $\lambda_{\Omega_{\bar{x}}} = 0.726$  outside  $H$ . This leads  
 741 to a further improvement with  $\lambda_{\Omega_{\bar{x}} \cap \Omega_{\tilde{x}}} \approx 0.39$  and  $p_{\Omega_{\bar{x}} \cap \Omega_{\tilde{x}}}(\lambda_{\Omega_{\bar{x}} \cap \Omega_{\tilde{x}}}) \approx -4.005$ , which  
 742 almost closes the gap. In Figure 4 we show the two linear cuts and the two new optimal

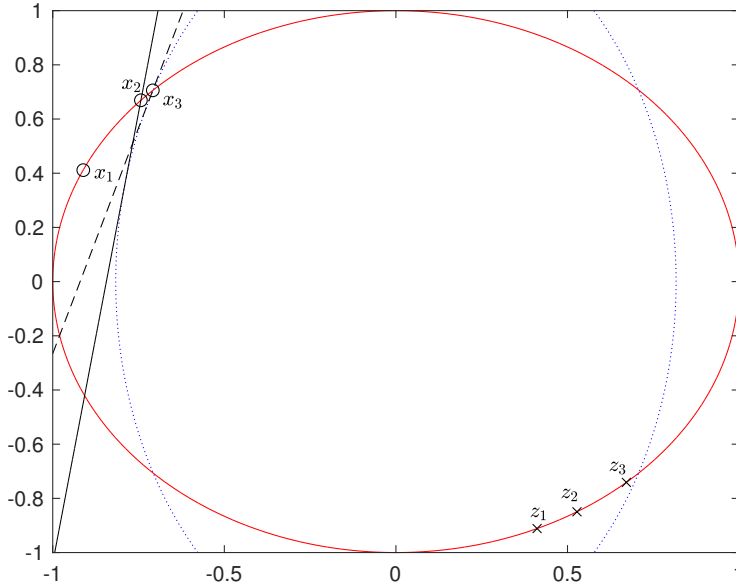


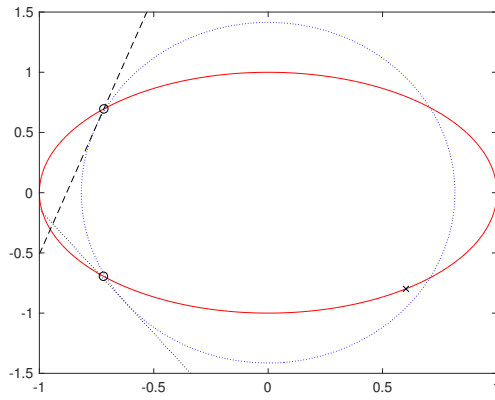
FIG. 4. Two linear cuts and the two optimal solutions outside  $H$  ( $x_3$ ) and in  $\text{int}(H)$  ( $z_3$ ), denoted by  $\circ$  and  $\times$ , respectively.

743 solutions, one outside  $H$  and one belonging to  $\text{int}(H)$  ( $x_3$  and  $z_3$ , respectively). Again,  
 744 we also report the previous pairs of optimal solutions in order to show the progress.

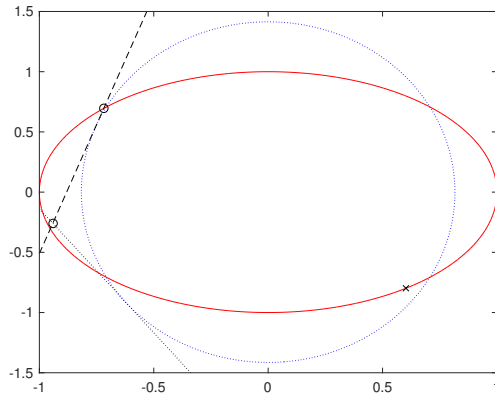
745 Now, assume that the returned bound is not exact. Also in this case  $\bar{\mathbf{x}}$  and  $\tilde{\mathbf{x}}$  can be  
 746 locally adjusted. One can combine the techniques presented in section 4 and in the  
 747 current section, by using a technique similar to the one described in the former section  
 748 to improve the pair of points  $\bar{\mathbf{x}}$  and  $\tilde{\mathbf{x}}$ . In particular, at  $\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}}}$  we have one optimal  
 749 solution of problem 5.1 belonging to  $\text{int}(H)$ , namely the local and nonglobal optimal  
 750 solution of problem (3.2) with  $X = \mathbb{R}^n$ , and at least another one outside  $H$ . We denote  
 751 the latter by  $\mathbf{v}$  and we observe that at least one of the two linear cuts is active at this  
 752 point, i.e., either  $\mathbf{v} \in \partial\Omega_{\bar{\mathbf{x}}}$  or  $\mathbf{v} \in \partial\Omega_{\tilde{\mathbf{x}}}$  (or both). Then, if only the first cut is active at  
 753  $\mathbf{v}$ , we update  $\bar{\mathbf{x}}$  as follows:  $\bar{\mathbf{x}}' = \Pi_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}} + \eta(\mathbf{v} - \bar{\mathbf{x}}))$  for a sufficiently small  $\eta$  value, while  
 754  $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}}$ . If only the second cut is active, we update  $\tilde{\mathbf{x}}$  as follows:  $\tilde{\mathbf{x}}' = \Pi_{\mathbf{A},\mathbf{a}}(\tilde{\mathbf{x}} + \eta(\mathbf{v} - \tilde{\mathbf{x}}))$   
 755 for a sufficiently small  $\eta$  value, while  $\bar{\mathbf{x}}' = \bar{\mathbf{x}}$ . Finally, if both are active we select one  
 756 of the two cuts and perturb it. After the perturbation, we run again Algorithm 2.1  
 757 with input  $X = \Omega_{\bar{\mathbf{x}}'} \cap \Omega_{\tilde{\mathbf{x}}'}$  and  $\lambda_{\Omega_{\bar{\mathbf{x}}'}\cap\Omega_{\tilde{\mathbf{x}}'}}$ , and we repeat this procedure until there is  
 758 a significant reduction of the bound. Note, however, that it might happen that no  
 759 improvement is possible. In case  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 1$  and  $P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} = \emptyset$  (similar  
 760 for  $|P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 1$  and  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} = \emptyset$ ), then the proposed perturbation  $\bar{\mathbf{x}}' =$   
 761  $\Pi_{\mathbf{A},\mathbf{a}}(\bar{\mathbf{x}} + \eta(\mathbf{v} - \bar{\mathbf{x}}))$  for  $\eta$  sufficiently small allows improvement of the bound. Indeed,  
 762 in such cases the local adjustment is able to cut the unique solution outside  $H$  away.  
 763 In order to illustrate other different cases we employ Figures 5a–5c. As usual, in these  
 764 figures the point in  $\text{int}(H)$  is denoted by  $\times$ , while the others (outside  $H$ ) are denoted  
 765 by  $\circ$ . If  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = |P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 1$  and  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} \cap P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} = \emptyset$  (see  
 766 Figure 5a), or  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 2$ ,  $|P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 1$ , and  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} \cap P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} \neq \emptyset$   
 767 (see Figure 5b), then it is not possible to remove all the solutions outside  $H$  by  
 768 perturbing a single linear cut. Indeed, in both cases the perturbation of a single  
 769 linear cut is able to remove just one of the two optimal solutions outside  $H$ . But it is  
 770 possible to remove both by perturbing both linear cuts. Instead, Figure 5c illustrates  
 771 a case where  $|P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = |P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})}| = 2$  and  $P_1^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} \cap P_2^*(\lambda_{\Omega_{\bar{\mathbf{x}}}\cap\Omega_{\tilde{\mathbf{x}}})} \neq \emptyset$ .  
 772 In this case even the perturbation of both linear cuts is unable to remove all three  
 773 solutions outside  $H$ . The only way to remove all three solutions outside  $H$  is through  
 774 the addition of a further linear cut, but, of course, this leads to a more complex  
 775 problem with one trust region constraint and three linear inequalities.

776 *Example 5.2.* In our example, this refinement is finally able to close the gap  
 777 and return the exact optimal value  $-4$ . In Figure 6 we report the result of the first  
 778 perturbation of the linear cuts. Since only the second linear cut is active at  $x_3$ , in  
 779 this case the second linear cut is slightly perturbed and becomes equivalent to the  
 780 tangent to  $H$  at the optimal solution  $(-\sqrt{2}/2, \sqrt{2}/2)$  of the original problem (1.1). It  
 781 is interesting to note that the new optimal solution outside  $H$ , indicated by  $x_4$ , lies  
 782 in a different region with respect to the previous ones and is further from  $\partial H$  with  
 783 respect to  $x_2$  and  $x_3$  (the reduction of  $\lambda$  reduces the penalization of points outside  
 784  $H$ ). Such a solution is cut by the new linear inequality, obtained by a (not so small)  
 785 perturbation of the first linear cut, displayed in Figure 7, together with the two new  
 786 optimal solutions ( $x_5$  and  $z_5$ ), now corresponding to the two optimal solutions of  
 787 problem (1.1).

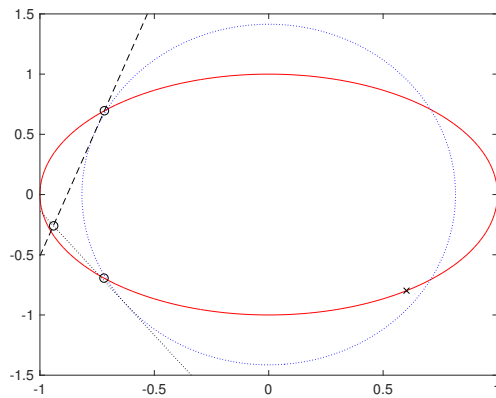
788 **6. Computational experiments.** In this section we report the computational  
 789 results for the proposed bounds over the set of hard instances selected from the random  
 790 ones generated in [12] and inspired by [18]. More precisely, in [12] 1000 random



(a) Three optimal solutions, none with both linear cuts active.



(b) Three optimal solutions, one with both linear cuts active.



(c) Four optimal solutions.

FIG. 5.

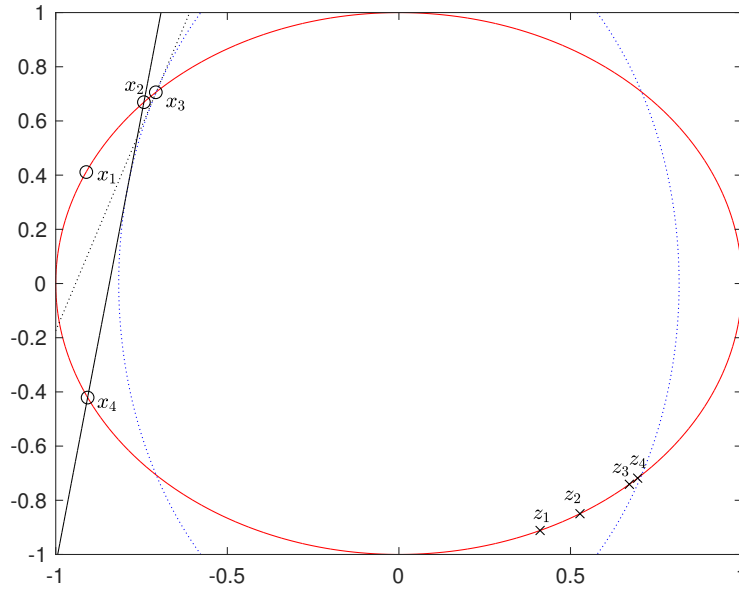


FIG. 6. Perturbation of the second linear cut and the two new optimal solutions outside  $H$  ( $x_4$ ) and in  $\text{int}(H)$  ( $z_4$ ), denoted by  $\circ$  and  $\times$ , respectively.

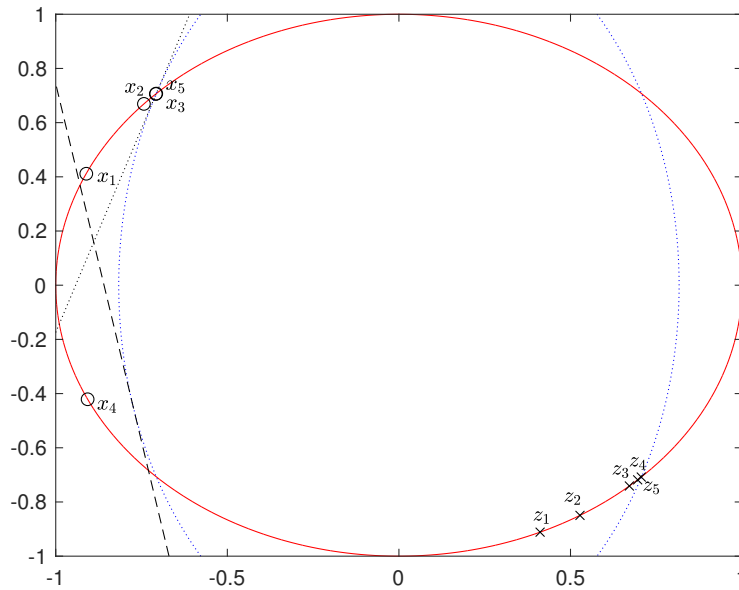


FIG. 7. Perturbation of the first linear cut and the two optimal solutions outside  $H$  ( $x_5$ ) and in  $\text{int}(H)$  ( $z_5$ ), denoted by  $\circ$  and  $\times$ , respectively.

791 instances were generated for each size  $n = 5, 10, 20$ . Some of these instances have been  
 792 declared hard ones, namely those for which the bound obtained by adding **SOC-RLT**  
 793 constraints was not exact. In particular, these are 38 instances with  $n = 5$ , 70 instances  
 794 with  $n = 10$ , and 104 instances with  $n = 20$ . Such instances have been made available  
 795 in **GAMS**, **AMPL**, and **COCOUNT** formats in [19]. We tested our bounds on such instances.

796 All tests have been performed on an Intel Core i7 running at 1.8 GHz with 16 GB of  
797 RAM. All bounds have been coded in MATLAB.

798 We computed the following bounds:

- 800 • **LbDual**, the dual Lagrangian bound computed through Algorithm 2.1 with  
801 input  $X = \mathbb{R}^n$ ;
- 802 • **LbOneCut**, the bound obtained by adding a single linear cut and computed  
803 through Algorithm 2.1 with input  $X = \Omega_{\bar{x}}$ ;
- 804 • **LbOneAdj**, the bound obtained by local adjustments of the added linear cut  
805 as indicated in Algorithm 4.1;
- 806 • **LbTwoCut**, the bound obtained by adding two linear cuts;
- 807 • **LbTwoAdj**, the bound obtained by adjusting the two linear cuts.

808 According to what was done in [3, 12, 25], an instance is considered to be ‘solved’  
809 when the relative gap between the lower bound, say  $LB$ , and the upper bound, say  
810  $UB$ , is not larger than  $10^{-4}$ , i.e.,

$$\frac{UB - LB}{|UB|} \leq 10^{-4}.$$

811 We set  $UB$  equal to the lowest value obtained by running, after the addition of the  
812 first linear cut, two local searches for the original problem (1.1), one from the optimal  
813 solution  $\mathbf{z}_1(\lambda_{\Omega_{\bar{x}}}) \in \text{int}(H)$  of (3.9) returned at the end of Algorithm 2.1, and the other  
814 from an optimal solution of the same problem outside  $H$ . In Tables 2–4 we report the  
815 average and maximum relative gaps for each bound, and the average and maximum  
816 computing times for  $n = 5, 10, 20$ , respectively. In the last line of the tables we also  
817 report the same values for the **SOC-RLT** bound presented in [12], computed by Mosek.<sup>1</sup>  
818 Note that the average gap is taken only over the instances which were *not* solved (in  
819 the sense specified above) by a given bound. Moreover, the average computing time  
820 for bound **LbTwoAdj** is computed only over the instances (87 overall, as we will see)  
821 which are *not* solved by bound **LbTwoCut**.

822 We remark that the bound **LbTwoCut** is computed by adding the first cut as in  
823 bound **LbOneCut**, i.e., the supporting hyperplane at  $\bar{x} \in \partial H$ , and then adding a further  
824 linear cut through the projection of an optimal solution outside  $H$  obtained when com-  
825 puting bound **LbOneCut**, i.e., point  $\mathbf{z}_2(\lambda_{\min})$  returned by procedure **DualLagrangian**  
826 with input  $X = \Omega_{\bar{x}}$ . We could as well choose the adjusted cut computed by bound  
827 **LbOneAdj** as the first cut for bound **LbTwoCut**, but we observed that with this choice  
828 no improvement over **LbOneAdj** is obtained. This is related to what already observed

TABLE 2

*Average and maximum relative gaps and computing times (in seconds) for the instances with  $n = 5$ .*

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
<b>LbDual</b>	0.90 %	2.97 %	0.013	0.015
<b>LbOneCut</b>	0.31 %	1.27 %	0.035	0.040
<b>LbOneAdj</b>	0.130 %	0.548 %	0.266	0.388
<b>LbTwoCut</b>	0.07 %	0.21 %	0.089	0.108
<b>LbTwoAdj</b>	0 %	0 %	0.146	0.281
<b>SOC-RLT</b>	0.131 %	0.548 %	1.435	2.080

<sup>1</sup>The authors are very grateful to Professor Samuel Burer for providing the MATLAB code for the computation of the **SOC-RLT** bound.

TABLE 3

Average and maximum relative gaps and computing times (in seconds) for the instances with  $n = 10$ .

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
<b>LbDual</b>	0.41 %	1.57 %	0.014	0.022
<b>LbOneCut</b>	0.14 %	0.81 %	0.039	0.057
<b>LbOneAdj</b>	0.073 %	0.478 %	0.339	0.574
<b>LbTwoCut</b>	0.05 %	0.24 %	0.101	0.173
<b>LbTwoAdj</b>	0 %	0 %	0.197	0.670
<b>SOC-RLT</b>	0.073 %	0.474%	1.228	2.898

TABLE 4

Average and maximum relative gaps and computing times (in seconds) for the instances with  $n = 20$ .

Bound	Average relative gap (%)	Max relative gap (%)	Average time	Max time
<b>LbDual</b>	0.20 %	0.59 %	0.019	0.027
<b>LbOneCut</b>	0.08 %	0.29 %	0.057	0.079
<b>LbOneAdj</b>	0.054 %	0.166 %	0.539	0.926
<b>LbTwoCut</b>	0.03 %	0.09 %	0.148	0.199
<b>LbTwoAdj</b>	0.05 %	0.05 %	0.350	1.574
<b>SOC-RLT</b>	0.053 %	0.166%	2.266	3.983

829 in Figure 3: bound **LbOneAdj** cannot be improved any more when there are (at least)  
 830 two optimal solutions outside  $H$  (besides the one in  $int(H)$ ). Thus, the second cut  
 831 is able to remove one of such optimal solutions but not the other, so that the bound  
 832 cannot be improved. Similarly, for bound **LbTwoAdj** the two initial cuts are the ones  
 833 computed for bound **LbTwoCut**.

834 For what concerns the computing times, we observe that these are lower than  
 835 those reported in [25] for the bound obtained by adding lifted-RLT cuts (around  
 836 92s for an instance with  $n = 20$ ). They are also lower than those reported in [3]  
 837 for the bound obtained by adding KSOC cuts (up to 2s for  $n = 20$  instances). For  
 838 the sake of correctness, we point out that the computing times reported in those  
 839 papers have been obtained with different processors. However, such processors have  
 840 comparable performance with respect to the one employed for the computational  
 841 experiments in this paper. In general, the proposed bounds are very cheap. Only  
 842 for two instances with  $n = 20$ , **LbTwoAdj** required times above 1s (around 1.5s in  
 843 both cases). Usually the computing times are (largely) below 1s. Both the dual  
 844 Lagrangian bound and the bound obtained by a single linear cut are pretty cheap but  
 845 with poorer performance in terms of relative gap. The bound obtained by Algorithm  
 846 4.1 with a local adjustment of the linear cut is better than the two previous ones in  
 847 terms of gap but is also more expensive (although still cheap). The bound **LbTwoCut**  
 848 offers a good combination between quality and cheap computing time. But a more  
 849 careful choice of the two linear cuts, through a local adjustment, improves the quality  
 850 without compromising the computing times. This is confirmed by the results reported  
 851 for **LbTwoAdj**. Although this bound is more expensive than the others, the additional  
 852 search for adjusted linear cuts further increases the quality of the bound. In Table 5  
 853 we report the number of solved instances for **LbTwoCut** and **LbTwoAdj**. According to  
 854 what was reported in [3], the total number of unsolved instances out of the 212 hard

TABLE 5  
*Number of solved instances for the bounds LbTwoCut and LbTwoAdj.*

Bound	$n = 5$ (out of 38)	$n = 10$ (out of 70)	$n = 20$ (out of 104)
LbTwoCut	14	41	70
LbTwoAdj	38	70	103

TABLE 6  
*Minimum, average, and maximum PercDiff values, where PercDiff is defined in (6.1).*

$n$	Minimum	Average	Maximum
5	-0.0020%	0.0009 %	0.0058 %
10	-0.0046 %	-0.0001 %	0.0024 %
20	-0.0022 %	-0.0003 %	0.0004 %

855 instances is equal to the following: 133 for the bound proposed in [25] (18 with  $n = 5$ ,  
856 49 with  $n = 10$ , and 66 with  $n = 20$ ); 85 for the bound proposed in [3] (18 with  $n = 5$ ,  
857 22 with  $n = 10$ , and 45 with  $n = 20$ ); 56 by considering the best bound between the  
858 one in [25] and the one in [3] (10 with  $n = 5$ , 15 with  $n = 10$ , and 31 with  $n = 20$ ).  
859 For bound LbTwoCut the total number of unsolved instances reduces to 87 (24, 29,  
860 and 34 for  $n = 5$ ,  $n = 10$ , and  $n = 20$ , respectively). Finally, for bound LbTwoAdj  
861 we have the remarkable outcome that there is just one unsolved instance (namely,  
862 instance '20'628). For the sake of correctness, we should warn that the value  $UB$  in [3,  
863 25] is not computed by running two local searches as done in this paper. It is instead  
864 computed from the final solution of the relaxed problem, so that it could be slightly  
865 worse and justify the larger number of unsolved instances. All the same, the quality  
866 of the proposed bounds appears to be quite good.

867 We still need to compare our bounds with the SOC-RLT bound (last line in Ta-  
868 bles 2–4). In terms of computing times we notice that both the average and the  
869 maximum computing times of the SOC-RLT bound are larger than those of all the pro-  
870 posed bounds. But we believe that the most interesting observation is that, in terms  
871 of average and maximum gap, the SOC-RLT bound is almost identical to the LbOneAdj  
872 bound. In order to better investigate the relation between the two bounds, in Table 6  
873 we report the minimum, average, and maximum percentages difference between the  
874 two bounds, i.e., the quantity

$$(6.1) \quad \text{PercDiff} = 100 * \frac{\text{LbOneAdj} - \text{SOC} - \text{RLT}}{|\text{LbOneAdj}|} \%$$

875 We notice that the difference is sometimes positive and sometimes negative, suggesting  
876 that none of the two bounds dominate the other. But the differences are also so small  
877 (below the tolerance value under which an instance is declared to be ‘‘solved’’ by a  
878 given bound) that they could also be numerical differences due to the tolerance values  
879 employed in the solvers. We believe that an interesting question for future research  
880 is to establish whether these two bounds are, in fact, equivalent, which would lead to  
881 a new interpretation of the SOC-RLT bound proposed in [12].

882 **6.1. Investigating the hardest instance.** As a final experiment, we investi-  
883 gate the behavior of bound LbTwoAdj over the hardest instance with  $n = 20$ , the one

884 for which the relative error is above  $10^{-4}$ . For this instance, at the last iteration  
 885 we recorded the following objective function values, corresponding to values of local  
 886 minimizers of problem (5.1), which certainly include the global minimizer(s) of such  
 887 a problem:

- 889 • the value at the optimal solution of problem (5.1) belonging to  $\text{int}(H)$ ;
- 890 • the value at a globally optimal solution of the trust region problem obtained  
 891 by fixing in problem (5.1) the first linear cut to an equality, in case such  
 892 solution fulfills the second linear cut, or, alternatively, the value at the local  
 893 and nonglobal solution of the same problem, in case such solution exists and  
 894 fulfills the second linear cut (if the global minimizer does not fulfill the second  
 895 linear cut and the local and nonglobal minimizer does not exist or does not  
 896 fulfill the second linear cut, then the value is left undefined);
- 897 • the same value as above but after fixing the second linear cut to an equality  
 898 in problem (5.1);
- 899 • the value at a globally optimal solution of the trust region problem obtained  
 900 by fixing both cuts to equalities in problem (5.1).

901 Note that two of the four values must be equal. In particular, one of the two  
 902 equal values is always the first one, attained in  $\text{int}(H)$ . But for the hardest instance  
 903 we observed that all four values are very close to each other and all of them are lower  
 904 than the  $UB$  value. Thus, it appears that for this instance a situation like the one  
 905 displayed in Figure 5c occurs. In this case even the perturbation of both linear cuts  
 906 is unable to remove all of the three solutions outside  $H$ .

907 **7. Conclusions.** In this paper we discussed the CDT problem. First, we derived  
 908 some theoretical results for a class of problems which includes the CDT problem as  
 909 a special case. Then, from the theory developed for such class, we have rederived a  
 910 necessary and sufficient condition for the exactness of the Shor relaxation and of the  
 911 equivalent dual Lagrangian bound for the CDT problem. The condition is based on  
 912 the existence of multiple solutions for a Lagrangian relaxation. Based on such con-  
 913 dition, we proposed to strengthen the dual Lagrangian bound by adding one or two  
 914 linear cuts. These cuts are based on supporting hyperplanes of one of the two qua-  
 915 dratic constraints, and they are, thus, redundant for the original CDT problem (1.1).  
 916 However, the cuts are not redundant for the Lagrangian relaxation and their addition  
 917 allows one to improve the bound. We ran different computational experiments over  
 918 the 212 hard test instances selected from the three thousand ones randomly gener-  
 919 ated in [12], reporting gaps and computing times. We have shown that the bounds  
 920 are computationally cheap and are quite effective. In particular, one of them, based  
 921 on the addition of two linear cuts, is able to solve all but one of the hard instances.  
 922 We have also investigated more in detail such hardest instance for which the bound  
 923 is not exact (though quite close to the optimal value). An interesting topic for fu-  
 924 ture research could be that of establishing the relations between the bounds proposed  
 925 in this work and those presented in the recent literature (in particular, as already  
 926 mentioned, it would be interesting to establish whether bound `LbOneAdj` is equivalent  
 927 to the `SOC-RLT` bound introduced in [12]). Moreover, it would also be interesting  
 928 to develop procedures which are able to generate CDT instances for which the bound  
 929 `LbTwoAdj` is unable to return the optimal value. Finally, it would be interesting to  
 930 see if the results presented in this work could be extended to QP problems with more  
 931 than two constraints. Some preliminary studies, which will appear elsewhere, show  
 932 that for such problems it is sometimes possible to improve the dual Lagrangian bound  
 933 with the addition of a linear cut, but it may be hard to identify it and it is not even  
 934 guaranteed to exist.

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