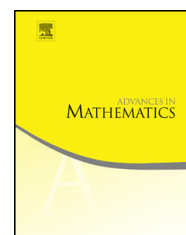




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# Generalized Schur-Weyl dualities for quantum affine symmetric pairs and orientifold KLR algebras <sup>☆</sup>



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## ABSTRACT

Let  $\mathfrak{g}$  be a complex simple Lie algebra and  $U_q L\mathfrak{g}$  the corresponding quantum affine algebra. We construct a functor  ${}^\theta\mathcal{F}$  between finite-dimensional modules over a quantum symmetric pair subalgebra of affine type  $U_q\mathfrak{k} \subset U_q L\mathfrak{g}$  and an orientifold KLR algebra arising from a framed quiver with a contravariant involution, providing a boundary analogue of the Kang-Kashiwara-Kim-Oh generalized Schur-Weyl duality. With respect to their construction, our combinatorial model is further enriched with the poles of a trigonometric K-matrix intertwining the action of  $U_q\mathfrak{k}$  on finite-dimensional  $U_q L\mathfrak{g}$ -modules. By construction,  ${}^\theta\mathcal{F}$  is naturally compatible with the Kang-Kashiwara-Kim-Oh functor in that, while the latter is a functor of monoidal categories,  ${}^\theta\mathcal{F}$  is a functor of module categories. Relying on a suitable isomorphism  $\hat{\alpha}$  Brundan-Kleshchev-Rouquier, we prove that  ${}^\theta\mathcal{F}$  recovers the

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Schur-Weyl dualities due to Fan-Lai-Li-Luo-Wang-Watanabe in quasi-split type AIII.

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## 1. Introduction

1.1. In the present paper, we introduce a *boundary analogue* of Kang-Kashiwara-Kim-Oh generalized Schur-Weyl dualities between quantum affine algebras and Khovanov-Lauda-Rouquier (KLR) algebras (also known as quiver Hecke algebras). More precisely, let  $\mathfrak{g}$  be a complex simple Lie algebra and  $U_q L\mathfrak{g}$  the corresponding quantum affine algebra. Given an affine quantum symmetric pair (QSP) subalgebra  $U_q \mathfrak{k} \subset U_q L\mathfrak{g}$ , we construct a functor

$${}^\theta F : {}^\theta \mathcal{R}_Q(\boldsymbol{\lambda})\text{-mod}_{\text{gr}}^{\text{fd}} \rightarrow U_q \mathfrak{k}\text{-mod}^{\text{fd}}$$

where  ${}^\theta \mathcal{R}_Q(\boldsymbol{\lambda})$  denotes the *orientifold* KLR algebra ( $o$  KLR) associated to a distinguished quiver  $Q$  endowed with a contravariant involution  $\theta$  and a framing  $\boldsymbol{\lambda}$ , which depends upon the choice of a suitable family of finite-dimensional  $U_q L\mathfrak{g}$ -modules. By construction, the functor  ${}^\theta F$  intertwines the standard module category structures on  ${}^\theta \mathcal{R}_Q(\boldsymbol{\lambda})\text{-mod}_{\text{gr}}^{\text{fd}}$  and  $U_q \mathfrak{k}\text{-mod}^{\text{fd}}$ , and it is expected to yield, under a suitable localization, an equivalence of categories with a boundary analogue of the Hernandez-Leclerc category for  $U_q L\mathfrak{g}$ . Finally, we prove that  ${}^\theta F$  recovers the Schur-Weyl dualities constructed by Fan-Lai-Li-Luo-Wang-Watanabe between the (quasi-split type AIII) QSP algebra  $U_q \mathfrak{k}$  and the affine Hecke algebra of type C.

While there is a natural parallelism between the results described above and those in [32], their proof requires several new ideas, which we describe in detail in the rest of this introduction.

1.2. The classical Schur-Weyl duality is a fundamental symmetry, which allows us to identify the category of finite-dimensional representations of the symmetric group

$\mathfrak{S}_\ell$  with the subcategory of  $\mathfrak{sl}_N$ -modules appearing in the decomposition of the  $\ell$ -tensor power of the vector representation  $\mathbb{V} := \mathbb{C}^N$  of  $\mathfrak{sl}_N$  (when  $l \leq N$ ). This symmetry can be realized as a functor  $\mathbb{V}^{\otimes \ell} \otimes_{\mathfrak{S}_\ell} \bullet : \mathfrak{S}_\ell\text{-mod}^{\text{fd}} \rightarrow U\mathfrak{sl}_N\text{-mod}^{\text{fd}}$ , using the fact that  $\mathbb{V}^{\otimes \ell}$  is an  $(U\mathfrak{sl}_N, \mathfrak{S}_\ell)$ -bimodule. The quantum analogue of this construction appears as a duality between quantum groups and Hecke algebras. In the affine setting, it is due to Chari-Pressley [12] and amounts to a functor  $\hat{H}_{\ell, q^2}\text{-mod}^{\text{fd}} \rightarrow U_q L\mathfrak{sl}_N\text{-mod}^{\text{fd}}$  between the affine Hecke algebra  $\hat{H}_{\ell, q^2}$  and the quantum loop algebra  $U_q L\mathfrak{sl}_N$ , arising from their joint action on the  $\ell$ -tensor product of the affinized vector representation  $\mathbb{V} := \mathbb{C}(q)^N[z, z^{-1}]$ .

1.3. More recently, in the series of papers [32,31,33–35], Kang, Kashiwara, Kim and Oh defined a *generalized* version of Chari and Pressley’s Schur-Weyl duality, which goes beyond type A and is expressed in terms of KLR algebras. More precisely, given a complex simple Lie algebra  $\mathfrak{g}$ , the construction depends on a combinatorial datum consisting of a (possibly infinite) set of finite-dimensional  $U_q L\mathfrak{g}$ -representations  $V(i)$ , each decorated with a non-zero scalar  $X(i) \in \mathbb{C}(q)$ . By comparing the order of the poles of the trigonometric R-matrices on  $V(i) \otimes V(j)$  at  $X(j)/X(i)$ , one obtains a quiver  $Q$  for the KLR algebra, which, given a dimension vector  $\beta$ , we denote by  $\mathcal{R}_Q(\beta)$ .

A suitable  $(U_q L\mathfrak{g}, \mathcal{R}_Q(\beta))$  bimodule  $\mathbb{V}^\beta$  is constructed as a direct sum of various tensor products of affinized representations  $V(i)$ . The action of  $\mathcal{R}_Q(\beta)$  on  $\mathbb{V}^\beta$  is given in terms of normalized R-matrices of  $U_q L\mathfrak{g}$ . The bimodule yields a functor  $\mathcal{R}_Q(\beta)\text{-mod}_{\text{gr}}^{\text{fd}} \rightarrow U_q L\mathfrak{g}_Q\text{-mod}^{\text{fd}}$ , and taking the sum over all possible dimension vectors, one obtains a monoidal functor

$$F: \bigoplus_{\beta} \mathcal{R}_Q(\beta)\text{-mod}_{\text{gr}}^{\text{fd}} \rightarrow U_q L\mathfrak{g}\text{-mod}^{\text{fd}} .$$

We emphasize that, *a priori*,  $Q$  is not related to the Dynkin diagram of  $\mathfrak{g}$ . However, it was shown in [31] that there does exist a combinatorial datum such that the two coincide. In that case,  $F$  induces an equivalence with the Hernandez-Leclerc category  $\mathcal{C}_Q \subset \text{Rep}_{\text{fd}}(U_q L\mathfrak{g}_Q)$  [23,52].

1.4. A *quantum affine symmetric pair (QSP) subalgebra* is a distinguished coideal subalgebra  $U_q \mathfrak{k} \subset U_q L\mathfrak{g}$  (also known in the literature as Letzter-Kolb coideal subalgebras or affine  $\iota$  quantum groups) [42]. Building on previous work by Bao-Wang [7] and Balagovic-Kolb [5], the first author and B. Vlaar proved in [3] that QSP subalgebras of arbitrary Kac-Moody type give rise to *universal K-matrices*, *i.e.*, universal solutions of Cherednik’s generalized reflection equation. The latter can be thought of as a boundary analogue of the Yang-Baxter equation, since it arises as a consistency condition in the case of particles moving on a half-line [14,63] and produces representations of type B braid groups (cylindrical braid groups). As in the case of the universal R-matrix, universal K-matrices formally descend to finite-dimensional  $U_q L\mathfrak{g}$ -modules. In the case of irreducible modules, they give rise to *trigonometric K-matrices*, *i.e.*, rational solutions of the reflection equation satisfying a unitarity condition, see [4].

1.5. Our main result is the construction of a boundary analogue of the functor  $F$  for affine QSP subalgebra. The key idea is to enhance the combinatorial model developed by Kang-Kashiwara-Kim in [32] by further taking into account the poles of the trigonometric  $K$ -matrix on the representations  $V(i)$ . Such an enhanced combinatorial datum yields a framed quiver  $Q$  equipped with a contravariant involution. The framing depends on the choice of two (families of) parameters of the QSP subalgebra  $U_q\mathfrak{k}$ , which appear in the expressions of the poles of the  $K$ -matrix. For *generic* parameters, we expect that the framing is trivial.

1.6. In [65], Varagnolo and Vasserot introduced a modified KLR algebra  ${}^\theta\mathcal{R}_Q(\beta; \boldsymbol{\lambda})$ , associated to a framed quiver with an involution. In this paper, we consider a mild generalization, which we call an *orientifold* KLR algebra (*o* KLR) [56,54,53]. By considering a completed tensor product of the  $U_qL\mathfrak{g}$ -modules  $V(i)$ , we construct, in Section 7, a  $(U_q\mathfrak{k}, {}^\theta\mathcal{R}_Q(\beta; \boldsymbol{\lambda}))$ -bimodule  ${}^\theta\nabla^\beta$ , whose *o* KLR algebra action is given in terms of normalized  $R$ - and  $K$ -matrices. This construction yields a functor  ${}^\theta F$ , which intertwines the respective categorical actions, as illustrated below.

$$\begin{array}{ccc} \bigoplus_{\beta} \mathcal{R}_Q(\beta)\text{-mod}_{\text{gr}}^{\text{fd}} & \xrightarrow{F} & U_qL\mathfrak{g}\text{-mod}^{\text{fd}} \\ \downarrow & & \downarrow \\ \bigoplus_{\beta} {}^\theta\mathcal{R}_Q(\beta; \boldsymbol{\lambda})\text{-mod}_{\text{gr}}^{\text{fd}} & \xrightarrow{{}^\theta F} & U_q\mathfrak{k}\text{-mod}^{\text{fd}} \end{array} \quad (1.1)$$

The vertical arrow on the KLR side is given by induction, while that on the quantum affine side is given by restriction.

Moreover, we prove in Section 9 that, in the case of quasi-split QSP subalgebras of type AIII, the functor  ${}^\theta F$  recovers the  $\iota/\jmath$  Schur-Weyl duality between  $U_q\mathfrak{k}$  and the 3-parameter affine Hecke algebra of type C constructed by Fan, Lai, Li, Luo, Wang, and Watanabe in [20]. The proof, which eventually reduces to an explicit and direct computation, relies on a modified Brundan-Kleshchev-Rouquier isomorphism constructed in Section 8 between (certain completions of) *o* KLR algebras and affine Hecke algebras of type C, generalizing similar constructions obtained in [65,54].

1.7. The functor  ${}^\theta F$  does not immediately yield an equivalence, but it is expected to give rise to one under a suitable localization procedure, which we briefly outline below.

In the case of the functor  $F$ , this procedure involves replacing the category of modules over the KLR algebra with a *localized quotient*  $\mathcal{T}$  [32]. Namely, one first mods out the kernel of the functor  $F$  and then localizes the quotient with respect to a family of *commuting objects* whose image through  $F$  is the trivial  $U_qL\mathfrak{g}$ -module. The functor  $F$  factors through  $\mathcal{T}$  and yields an equivalence of monoidal categories  $F : \mathcal{T} \rightarrow \mathcal{C}_{\text{HL}}$ , where  $\mathcal{C}_{\text{HL}}$  is a Hernandez-Leclerc-type subcategory of  $U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  [25,32,23,52].

A similar strategy can be applied towards the construction of a  $\mathcal{T}$ -module category  ${}^\theta\mathcal{T}$ , to which  ${}^\theta\mathbb{F}$  descends, yielding an analogue of the diagram (1.1). The corresponding localization procedure would involve a family of *commuting objects* in a *module* category. This requires the construction of suitable K-matrices for  $o$  KLR algebras, which are expected to correspond, under the functor  ${}^\theta\mathbb{F}$ , to the trigonometric K-matrices for  $U_q\mathfrak{Lg}$ -modules constructed in [4]. The resulting functor is finally expected to give a rise to an equivalence of module categories between  ${}^\theta\mathcal{T}$  and a boundary analogue of the Hernandez-Leclerc category, which will be the subject of future work.

### 1.8. Future directions

Following the seminal work of Bao-Wang [6,7], a general picture has begun to emerge that most fundamental algebraic, geometric, and categorical constructions in quantum groups can be generalized to quantum symmetric pairs. Fundamental results in this direction have been obtained in [17,5]. In the same spirit, the finite-dimensional representation theory of affine QSP subalgebras is expected to be as rich and interesting as that of quantum affine algebras. Several recent advances, such as the new Drinfeld presentation in the split affine QSP case (see, *e.g.*, [47]), and the study of  $\sigma$ -quiver varieties [46], suggest that the algebraic and geometric methods based, respectively, on Drinfeld polynomials and Nakajima quiver varieties are likely to have their QSP analogues.

The results of the present paper show that the recent approach based on Kang-Kashiwara-Kim-Oh Schur-Weyl duality also successfully extends to quantum symmetric pairs. While a direct understanding of their representation theory is, in large part, still out of reach, the boundary Schur-Weyl duality offers an alternative method, exploiting the combinatorial nature of KLR representation theory.

Moreover, the natural grading on  $o$  KLR algebras suggests the existence of a *graded* representation theory of quantum affine symmetric pairs. More specifically, we expect the gradings to manifest themselves through  $q$ -characters and deformations of Grothendieck modules, compatible with analogous deformations of Grothendieck rings in the case of quantum affine algebras [26,51,64]. On the  $o$  KLR side, the Grothendieck module of the category of finite-dimensional representations has been described in terms of certain irreducible highest weight modules  ${}^\theta V(\boldsymbol{\lambda})$  over the Enomoto-Kashiwara algebra [18,19,65]. Following [41,56], we expect that significant new information about the representation theory of quantum affine symmetric pairs may be extracted from the dual canonical basis of  ${}^\theta V(\boldsymbol{\lambda})$ . In rank one, this is expected to be related with the combinatorics of tridiagonal pairs developed by Ito-Terwilliger [27].

### 1.9. Outline

In Sections 2 and 3, we recall the basic definitions of quantum affine algebras and their quantum symmetric pairs (QSP). In Section 4, we briefly discuss the construction of rational K-matrices for finite-dimensional modules over quantum affine algebras. In

Section 5, we review the definition of orientifold KLR algebras ( $o$  KLR) associated to a framed quiver with an involution and we describe the module structure on their category of finite-dimensional graded modules. In Section 6, we introduce the enhanced combinatorial model and discuss few examples. In Section 7, we present the main result of the paper, providing the construction of the Schur-Weyl duality functor  ${}^{\theta}F$  (Theorem 7.2.1) and its compatibility with the Kang-Kashiwara-Kim functor (Theorem 7.3.2). In Section 8, we construct an isomorphism à la Brundan-Kleshchev-Rouquier (BKR) between suitable completions of  $o$  KLR algebras and affine Hecke algebras of type C (Theorem 8.2.1). In Section 9, we prove that, through the BKR isomorphism, the functor  ${}^{\theta}F$  recovers the  $\iota/j$  Schur-Weyl dualities between quasi-split QSP subalgebras of type AIII and affine Hecke algebras of type C constructed in [20] (Theorem 9.5.2).

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## 2. Quantum affine algebras

In this section we recall the definition of quantum affine algebras and basic results on their irreducible finite-dimensional modules.

For any lattice  $\Lambda \subset \mathbb{R}^n$ , we denote by  $\Lambda_+$  its non-negative component. We regard  $q$  as an indeterminate and set  $\mathbb{k} := \overline{\mathbb{C}(q)}$ .

### 2.1. Quantum affine algebras

Let  $\mathfrak{g}$  be a complex finite-dimensional simple Lie algebra with Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ . Let  $I := \{1, 2, \dots, \text{rank}(\mathfrak{g})\}$  be the set of vertices of the corresponding Dynkin diagram,  $A = (a_{ij})_{i,j \in I}$  the Cartan matrix,  $(\cdot, \cdot)$  the normalized invariant bilinear form on  $\mathfrak{g}$ ,  $\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$  a basis of simple roots and  $\Pi^\vee := \{\alpha_i^\vee \mid i \in I\} \subset \mathfrak{h}$  a basis of simple coroots such that  $\alpha_i(\alpha_i^\vee) = a_{ij}$ . Let  $Q := \mathbb{Z}\Pi \subset \mathfrak{h}^*$  and  $Q^\vee := \mathbb{Z}\Pi^\vee \subset \mathfrak{h}$  be the root and coroot lattice, respectively. Let  $\Phi_+ \subset Q_+$  be the set of positive roots. Finally, let  $\Lambda := \{\Lambda_i \mid i \in I\}$  be the set of fundamental weights and  $P := \mathbb{Z}\Lambda \subset \mathfrak{h}^*$  the weight lattice.

Let  $\widehat{\mathfrak{g}}$  be the (untwisted) affine Lie algebra<sup>1</sup> associated to  $\mathfrak{g}$  with affine Cartan subalgebra  $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{g}}$  [29, Ch. 7]. Let  $\widehat{I} := \{0\} \cup I$  be the set of vertices of the affine Dynkin diagram and  $\widehat{A} = (a_{ij})_{i,j \in \widehat{I}}$  the extended Cartan matrix [29, Table Aff. 1]. We denote by  $\widehat{Q}$  and  $\widehat{Q}^\vee$  the affine root and coroot lattices, respectively. Let  $\delta \in \widehat{Q}_+$  and  $c \in \widehat{Q}_+^\vee$  be the unique elements such that

<sup>1</sup> The results of this paper apply to the case of twisted affine Lie algebras. However, in order to simplify the exposition, we consider only the untwisted case.

$$\{\lambda \in \widehat{\mathcal{Q}} \mid \forall i \in \widehat{I}, \lambda(\alpha_i^\vee) = 0\} = \mathbb{Z}\delta \quad \text{and} \quad \{h \in \widehat{\mathcal{Q}}^\vee \mid \forall i \in \widehat{I}, \alpha_i(h) = 0\} = \mathbb{Z}c$$

In particular,  $\delta - \alpha_0 = \sum_{i \in I} a_i \alpha_i \in \Phi_+$  is the highest root,  $c$  is central in  $\widehat{\mathfrak{g}}$  and, under the identification  $\nu : \widehat{\mathfrak{h}} \rightarrow \widehat{\mathfrak{h}}^*$  induced by the bilinear form, one has  $\nu(c) = \delta$ . We fix  $d \in \widehat{\mathfrak{h}}$  such that  $\alpha_i(d) = \delta_{i0}$  for any  $i \in \widehat{I}$ . Note that  $d$  is uniquely defined up to a summand proportional to  $c$  and we obtain a natural identification  $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ . Finally, we set  $\widehat{\mathcal{Q}}_{\text{ext}}^\vee := \widehat{\mathcal{Q}}^\vee \oplus \mathbb{Z}d \subset \widehat{\mathfrak{h}}$ ,  $\widehat{\mathcal{P}} := \{\lambda \in \widehat{\mathfrak{h}}^* \mid \lambda(\widehat{\mathcal{Q}}_{\text{ext}}^\vee) \subseteq \mathbb{Z}\}$ , and  $\widehat{\mathcal{P}}_\delta := \widehat{\mathcal{P}}/(\widehat{\mathcal{P}} \cap \mathbb{Q}\delta)$ .

Fix pairwise coprime non-negative integers  $\{\epsilon_i \mid i \in \widehat{I}\}$  such that the matrix  $(\epsilon_i a_{ij})_{i,j \in \widehat{I}}$  is symmetric and set  $q_i := q^{\epsilon_i}$ . The quantum Kac–Moody algebra associated to  $\widehat{\mathfrak{g}}$  is the algebra  $U_q \widehat{\mathfrak{g}}$  over  $\mathbb{k}$  with generators  $E_i, F_i, i \in \widehat{I}$ , and  $K_h, h \in \widehat{\mathcal{Q}}_{\text{ext}}^\vee$ , subject to the following defining relations. For any  $h, h' \in \widehat{\mathcal{Q}}_{\text{ext}}^\vee$ ,

$$K_h K_{h'} = K_{h+h'} \quad \text{and} \quad K_{\mathbf{0}} = 1,$$

and, for any  $i, j \in \widehat{I}$ ,

$$K_h E_i = q^{\alpha_i(h)} E_i K_h, \quad K_h F_i = q^{-\alpha_i(h)} F_i K_h$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\text{Serre}_{ij}(E_i, E_j) = 0 = \text{Serre}_{ij}(F_i, F_j) \quad (i \neq j)$$

where  $K_i^{\pm 1} := K_{\pm \epsilon_i \alpha_i^\vee}$  and Serre denotes the usual quantum Serre relations (see, e.g., [48]). On  $U_q \widehat{\mathfrak{g}}$  we consider the Hopf algebra structure with coproduct

$$\Delta(K_h) = K_h \otimes K_h, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$$

Finally, the Chevalley involution  $\omega : U_q \widehat{\mathfrak{g}} \rightarrow (U_q \widehat{\mathfrak{g}})^{\text{op}}$  is the isomorphism of Hopf algebras defined by

$$\omega(K_h) = K_{-h}, \quad \omega(E_i) = -F_i, \quad \omega(F_i) = -E_i \tag{2.1}$$

for any  $i \in \widehat{I}$  and  $h \in \widehat{\mathcal{Q}}_{\text{ext}}^\vee$ .

We denote by  $U_q \widehat{\mathfrak{n}}^+$  (resp.  $U_q \widehat{\mathfrak{n}}^-$ ) the subalgebra generated by the elements  $\{E_i\}_{i \in \widehat{I}}$  (resp.  $\{F_i\}_{i \in \widehat{I}}$ ), and we set  $U_q \widehat{\mathfrak{b}}^\pm := U_q \widehat{\mathfrak{n}}^\pm U_q \widehat{\mathfrak{h}}$ , where  $U_q \widehat{\mathfrak{h}}$  is the commutative subalgebra generated by  $K_h, h \in \widehat{\mathcal{Q}}_{\text{ext}}^\vee$ . Similarly, we denote by  $U_q \mathfrak{g}$  the finite-type subalgebra generated by  $E_i, F_i, i \in I$ , and  $K_h, h \in \mathcal{Q}^\vee$ , and by  $U_q \mathfrak{h}, U_q \mathfrak{n}^\pm, U_q \mathfrak{b}^\pm \subset U_q \mathfrak{g}$  their finite-type counterparts. Finally, we denote by  $U_q \widehat{\mathfrak{g}}'$  the subalgebra obtained by replacing the extended coroot lattice  $\widehat{\mathcal{Q}}_{\text{ext}}^\vee$  with the standard coroot lattice  $\widehat{\mathcal{Q}}^\vee$ .

<sup>2</sup> In this paragraph only, we denote the zero element of  $\widehat{\mathcal{Q}}_{\text{ext}}^\vee$  by  $\mathbf{0}$  in order to avoid confusion with the element  $K_0 = K_{\epsilon_0 \alpha_0^\vee}$ .

The quantum loop algebra  $U_qL\mathfrak{g}$  is the quotient of  $U_q\widehat{\mathfrak{g}}'$  by the ideal generated by  $K_c - 1$ , where  $K_c = K_0 \prod_{i \in I} K_i^{a_i}$ . Note that the Hopf algebra structure and the Chevalley involution descend to  $U_qL\mathfrak{g}$ .

## 2.2. Finite-dimensional modules

We assume henceforth that every module is type **1**, *i.e.*, the action of the generators  $K_i$  ( $i \in \widehat{I}$ ) is semisimple with eigenvalues in  $q^{\mathbb{Z}}$ . Recall that the categories of finite-dimensional modules over  $U_q\widehat{\mathfrak{g}}'$  and  $U_qL\mathfrak{g}$  are isomorphic, since the action of the central element  $K_c$  is necessarily trivial. Note also that finite-dimensional modules admit a weight decomposition over  $\widehat{P}_\delta$ .

It is well-known that irreducible modules are classified by rank( $\mathfrak{g}$ )-tuples of monic polynomials in  $\mathbb{C}[u]$  [11, Thm. 12.2.6]. The category  $U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  is monoidal, but it is not semisimple, and it is not braided in the usual sense (see Section 2.3). We denote by  $(U_qL\mathfrak{g})^{\text{fd}}$  the completion of  $U_qL\mathfrak{g}$  with respect to finite-dimensional module, *i.e.*,  $(U_qL\mathfrak{g})^{\text{fd}}$  is the algebra of endomorphisms of the forgetful functor from  $U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  to  $\mathbb{k}$ -vector spaces. The monoidal structure induces on  $(U_qL\mathfrak{g})^{\text{fd}}$  the structure of a cosimplicial algebra (see, *e.g.*, [3, §2.10-2.11] and [2, §8.9]).

## 2.3. The spectral R-matrix

Set  $U_q\widehat{\mathfrak{g}}[z, z^{-1}] := U_q\widehat{\mathfrak{g}} \otimes \mathbb{k}[z, z^{-1}]$  and consider the (homogeneous) grading shift automorphism

$$\Sigma_z : U_q\widehat{\mathfrak{g}}[z, z^{-1}] \rightarrow U_q\widehat{\mathfrak{g}}[z, z^{-1}]$$

given by  $\Sigma_z(K_h) := K_h$ ,  $\Sigma_z(E_i) := z^{\delta_{i0}} E_i$ , and  $\Sigma_z(F_i) := z^{-\delta_{i0}} F_i$ . Note that, by specializing  $z$  in  $\mathbb{k}^\times$ , we obtain a one-parameter family of automorphism of  $U_q\widehat{\mathfrak{g}}$ . Let  $\Delta_z, \Delta_z^{\text{op}} : U_q\widehat{\mathfrak{g}}[z, z^{-1}] \rightarrow (U_q\widehat{\mathfrak{g}} \otimes U_q\widehat{\mathfrak{g}})[z, z^{-1}]$  be the shifted coproduct defined by  $\Delta_z(x) := \text{id} \otimes \Sigma_z(\Delta(x))$  and  $\Delta_z^{\text{op}}(x) := \text{id} \otimes \Sigma_z(\Delta^{\text{op}}(x))$ . The grading shift naturally descends on  $U_qL\mathfrak{g}$ . For any  $V \in U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  with action  $\pi_V : U_qL\mathfrak{g} \rightarrow \text{End}(V)$ , we consider the infinite-dimensional modules  $V(z) := V \otimes \mathbb{k}(z)$  and  $V((z)) := V \otimes \mathbb{k}((z))$  with the natural action given by  $\pi_V(\Sigma_z(x))$ .

By [16],  $U_qL\mathfrak{g}$  has a universal spectral R-matrix, *i.e.*, a formal series  $R(z)$ , whose coefficients belong to suitable completions of  $U_qL\mathfrak{g} \otimes U_qL\mathfrak{g}$ , such that  $\Sigma_a \otimes \Sigma_b(R(z)) = R(\frac{b}{a}z)$  ( $a, b \in \mathbb{k}^\times$ ) and the following identities are satisfied:

$$\begin{aligned} R(z)\Delta_z(x) &= \Delta_z^{\text{op}}(x)R(z), \\ \Delta_z \otimes \text{id}(R(zw)) &= R_{13}(zw)R_{23}(w), \\ \text{id} \otimes \Delta_w(R(z)) &= R_{13}(z)R_{12}(zw). \end{aligned}$$

In particular, the Yang–Baxter equation holds:

$$\mathbf{R}_{12}(z)\mathbf{R}_{13}(zw)\mathbf{R}_{23}(w) = \mathbf{R}_{23}(w)\mathbf{R}_{13}(zw)\mathbf{R}_{12}(z). \tag{2.2}$$

Moreover, for any  $V, W \in U_q\mathbf{Lg}\text{-mod}^{\text{fd}}$ , the operator

$$\mathbf{R}_{VW}(z) := \pi_V \otimes \pi_W(\mathbf{R}(z)) \in \text{End}(V \otimes W)[[z]]$$

is well-defined and yields an intertwiner

$$\mathbf{R}_{VW}^\vee(z) := (1\ 2) \circ \mathbf{R}_{VW}(z) : V \otimes W((z)) \rightarrow W((z)) \otimes V.$$

#### 2.4. The trigonometric R-matrix

In the case of irreducible modules the operator  $\mathbf{R}_{VW}(z)$  is, up to a scalar factor, a *trigonometric R-matrix*.

**Theorem 2.4.1** ([16]). *Let  $V, W \in U_q\mathbf{Lg}\text{-mod}^{\text{fd}}$  be two irreducible modules. There exists a canonical function  $f_{VW}(z) \in \mathbb{k}((z))$  such that*

$$\mathbf{R}_{VW}(z) := f_{VW}(z)^{-1}\mathbf{R}(z) \in \text{End}(V \otimes W)((z))$$

is a rational non-vanishing operator, satisfying the spectral Yang-Baxter equation (2.2) and the unitarity relation

$$\mathbf{R}_{VW}(z)^{-1} = (1\ 2) \circ \mathbf{R}_{WV}(z^{-1}) \circ (1\ 2). \tag{2.3}$$

The proof of the theorem relies on the *generic irreducibility* of the tensor product  $V \otimes W$ , i.e., on the irreducibility of the module  $V \otimes W((z))$  over  $U_q\mathbf{Lg}((z))$  (cf. [38, §4.2] or [10, Thm. 3]). Note also that the function  $f_{VW}(z)$  is uniquely determined by the condition  $\mathbf{R}(z)(v \otimes w) = v \otimes w$ , where  $v \in V$  and  $w \in W$  are highest weight vectors.

#### Remarks 2.4.2.

(1) As before, this yields a rational intertwiner

$$\mathbf{R}_{VW}^\vee(z) := (1\ 2) \circ \mathbf{R}_{VW}(z) : V \otimes W(z) \rightarrow W(z) \otimes V,$$

which is sometimes referred to as the *normalized R-matrix*, see e.g., [32].

(2) If  $V$  is an irreducible *real* module, i.e.,  $V \otimes V$  is irreducible, the normalized R-matrix satisfies  $\mathbf{R}_{VV}^\vee(1) = \text{id}$  (see, e.g., [21]).

(3) By considering the shifted module  $V(w)$ , one obtains an intertwiner

$$\mathbf{R}_{VW}^\vee(w, z) := \mathbf{R}_{V(w)W}^\vee(z) : V(w) \otimes W(z) \rightarrow W(z) \otimes V(w).$$

Since the spectral R–matrix satisfies the identity  $\Sigma_a \otimes \text{id}(\mathbf{R}(z)) = \mathbf{R}(\frac{z}{a})$  ( $a \in \mathbb{k}^\times$ ), it follows that  $\mathbf{R}_{VW}^\vee(w, z)$  depends rationally on  $z/w$  only, and it is denoted  $\mathbf{R}_{VW}^\vee(z/w)$ .  $\nabla$

### 3. Quantum affine symmetric pairs

#### 3.1. Generalized Satake diagrams

Classical and quantum Kac–Moody algebras are defined in terms of a combinatorial datum encoded by the Dynkin diagram and the Cartan matrix. Similarly, classical and quantum symmetric pairs (the latter are also known as *Letzter–Kolb coideal subalgebras* or  *$\iota$  quantum groups*) arise from a refinement of such a datum, see, e.g., [30,45,42].

Let  $\text{Aut}(\widehat{A})$  be the group of *diagram automorphisms* of the affine Cartan matrix, i.e., the group of bijections  $\tau : \widehat{I} \rightarrow \widehat{I}$  such that  $a_{ij} = a_{\tau(i)\tau(j)}$ . Let  $X \subset \widehat{I}$  be a proper subset of indices. Note, in particular, that the corresponding Cartan matrix  $A_X$  is necessarily of finite type. We denote by  $\text{op}_X \in \text{Aut}(A_X)$  the *opposition involution of  $X$* , i.e., the involutive diagram automorphism on  $X$  induced by the action of the longest element  $w_X$  of the parabolic Weyl group  $W_X$  on  $\mathbf{Q}_X$ , see, e.g., [3, §3.11].

Following the recent approach proposed by Regelskis and Vlaar in [59,60], we say that a pair  $(X, \tau)$  is a (*generalized affine*) *Satake diagram* and write  $(X, \tau) \in \text{Sat}(\widehat{A})$  if  $X \subset \widehat{I}$ ,  $\tau$  is an involutive diagram automorphism which preserves  $X$ , and

- (1)  $\tau|_X = \text{op}_X$ ,
- (2) for any  $i \in \widehat{I} \setminus X$  such that  $\tau(i) = i$ , the connected component of  $X \cup \{i\}$  containing  $i$  is not of type  $A_2$ .

A list of Satake diagrams for  $\widehat{A}$  is given in [60, App. A, Tables 5, 6, 7]. Henceforth, we fix an affine Satake diagram  $(X, \tau) \in \text{Sat}(\widehat{A})$ .

#### 3.2. Pseudo–involutions

The diagram automorphism  $\tau \in \text{Aut}(\widehat{A})$  extends canonically to an automorphism of  $\widehat{\mathfrak{g}}'$ , given on the generators by  $\tau(\alpha_i^\vee) := \alpha_{\tau(i)}^\vee$ ,  $\tau(e_i) = e_{\tau(i)}$ , and  $\tau(f_i) = f_{\tau(i)}$ . We then consider the Lie algebra automorphism  $\vartheta : \widehat{\mathfrak{g}}' \rightarrow \widehat{\mathfrak{g}}'$  given by

$$\vartheta := \text{Ad}(\tilde{w}_X) \circ \omega \circ \tau,$$

where  $\omega$  denotes the Chevalley involution and  $\text{Ad}(\tilde{w}_X)$  denotes the automorphism of  $\mathfrak{g}$  given by the triple exponentials.<sup>3</sup> Note that, since  $\alpha_i(\tau(c)) = \alpha_{\tau(i)}(c) = 0$ , one has

<sup>3</sup> More precisely, if  $w_X = s_{i_1} \cdots s_{i_k}$  is a reduced expression of the longest element, then  $\tilde{w}_X = \tilde{s}_{i_1} \cdots \tilde{s}_{i_k}$ , where  $s_i = \exp(e_i) \exp(-f_i) \exp(e_i)$ . It is well–known that the operators  $s_i$  ( $i \in I$ ) induce an action of the

$\tau(c) = c$  and  $\vartheta(c) = -c$ . In [60],  $\vartheta$  is referred to as a *pseudo-involution of  $\widehat{\mathfrak{g}}$  of the second kind* (cf. [30]).

### 3.3. Quantum pseudo-involution

The pseudo-involution  $\vartheta$  has a distinguished lift  $\vartheta_q$  at the quantum level as an algebra automorphism of  $U_q\widehat{\mathfrak{g}}$ . This is obtained by choosing a suitable lift for each of the three factors in  $\vartheta$ . We follow the construction of  $\vartheta_q$  given in [3]. First, one considers the usual Chevalley involution on  $U_q\widehat{\mathfrak{g}}$  given by (2.1). Then, the diagram automorphism  $\tau$  extends canonically to an automorphism of  $U_q\widehat{\mathfrak{g}}$  given on the generators by  $\tau(E_i) = E_{\tau(i)}$ ,  $\tau(F_i) = F_{\tau(i)}$  and  $\tau(K_h) = K_{\tau(h)}$ .

By [48, §5], the quantum Weyl group of  $U_q\widehat{\mathfrak{g}}$  provides a suitable quantization of  $w_X \in W_X$ . Namely, let  $S_X$  be the quantum Weyl group operator corresponding to  $w_X$  acting on any integrable  $U_q\widehat{\mathfrak{g}}$ -module,<sup>4</sup> and set  $\mathcal{S}_\vartheta := \xi_\vartheta \cdot S_X$ . Here  $\xi_\vartheta$  is the operator defined on any weight vector of weight  $\lambda$  as the multiplication by  $q^{(\vartheta(\lambda), \lambda)/2 + (\lambda, \rho_X)}$ , where  $\rho_X$  is the half-sum of the positive roots in  $\Phi_X$ . Therefore,  $\mathcal{S}_\vartheta$  can be thought of as an element of the completion of  $U_q\widehat{\mathfrak{g}}$  with respect to integrable modules. By [3, Lemma 4.3 (iii)],  $\mathcal{T}_\vartheta := \text{Ad } \mathcal{S}_\vartheta$  yields an automorphism of  $U_q\widehat{\mathfrak{g}}$ .

The *quantum pseudo-involution*  $\vartheta_q$  is given by

$$\vartheta_q := \mathcal{T}_\vartheta \circ \omega \circ \tau.$$

Note that, as in the classical case,  $\vartheta_q$  is independent of the order of the three factors. Moreover,  $\vartheta_q$  descends to an automorphism of  $U_qL\mathfrak{g}$ .

### 3.4. QSP subalgebras

It is well-known that there is a *family* of coideal subalgebras of  $U_q\widehat{\mathfrak{g}}$  associated to the pseudo-involution  $\vartheta$ , introduced in [45,42,59] and parametrized by two sets,  $\mathbf{\Gamma} \subset (\mathbb{k}^\times)^{\widehat{I}}$  and  $\mathbf{\Sigma} \subset \mathbb{k}^{\widehat{I}}$  (cf. Remark 3.4.1).

Let  $U_q\widehat{\mathfrak{g}}_X \subset U_q\widehat{\mathfrak{g}}$  be the subalgebra generated by  $E_i, F_i$ , and  $K_i$  ( $i \in X$ ). The *QSP subalgebra* of  $U_q\widehat{\mathfrak{g}}$  corresponding to  $\vartheta$  with parameters  $(\gamma, \sigma) \in \mathbf{\Gamma} \times \mathbf{\Sigma}$  is the subalgebra  $U_q\mathfrak{k} \subseteq U_q\widehat{\mathfrak{g}}$  generated by  $U_q\widehat{\mathfrak{g}}_X$ , the elements  $K_h$  with  $h \in (\widehat{Q}_{\text{ext}}^\vee)^\vartheta$ , and the elements  $B_i$  ( $i \in \widehat{I} \setminus X$ ) given by

$$B_i := F_i + \gamma_i \cdot \vartheta_q(F_i) + \sigma_i \cdot K_i^{-1}.$$

In the following, a morphism of  $U_q\mathfrak{k}$ -modules will be simply referred to as a *QSP intertwiner*.

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braided group on any integrable  $\mathfrak{g}$ -module. In particular, their adjoint action gives rise to an automorphism of  $\mathfrak{g}$ . For a concise summary of the main properties of triple exponentials, we refer the reader to [3, §3.7].

<sup>4</sup> More precisely, given a reduced expression  $s_{i_1} \cdots s_{i_\ell}$  of  $w_X$  in terms of fundamental reflections, set  $S_X := S_{i_1} \cdots S_{i_\ell}$ , where  $S_j = T''_{j,1}$  in the notation from [48, 5.2.1].

**Remarks 3.4.1.**

- (1) Roughly,  $\Gamma \times \Sigma$  is the set of all pairs  $(\gamma, \sigma)$  satisfying  $(\gamma_i, \sigma_i) = (1, 0)$  ( $i \in X$ ) and  $U_q\mathfrak{k} \cap U_q\widehat{\mathfrak{h}} = U_q\widehat{\mathfrak{h}}^\vartheta$ . For the explicit description of the sets  $\Gamma$  and  $\Sigma$  see, e.g., [3, §6.8].
- (2) Following [3, §7.4], we shall regard the tuple  $\gamma$  as a diagonal operator on integrable modules. Namely, we fix henceforth a group homomorphism  $\gamma : \widehat{P} \rightarrow \mathbb{F}^\times$  such that  $\gamma(\alpha_i) := \gamma_i$  ( $i \in \widehat{I}$ ). Then,  $\gamma$  acts on any weight vector of weight  $\lambda$  as the multiplication by  $\gamma(\lambda)$ .
- (3) By [3, §6.2], it follows that  $U_q\mathfrak{k} \subset U_q\widehat{\mathfrak{g}}'$ .  $\nabla$

**3.5. Coideal property and monoidal action**

It is well-known that  $U_q\mathfrak{k}$  is a right coideal subalgebra of  $U_q\widehat{\mathfrak{g}}$ , i.e.,  $\Delta(U_q\mathfrak{k}) \subset U_q\mathfrak{k} \otimes U_q\widehat{\mathfrak{g}}$ , see [42]. The categorical counterpart of this property is described in terms of module categories. Roughly, a module category (resp. a morphism of module categories) is the analogue for a monoidal category (resp. a tensor functor) of what a module (resp. a morphism of modules) is to a ring (resp. a morphism of rings).

We briefly recall these notions following [24]. Let  $\mathcal{A}$  be a monoidal category with tensor product  $\otimes$  and unit object  $\mathbf{1}$ . A (right) *monoidal action* of  $\mathcal{A}$  on a category  $\mathcal{B}$  is a functor  $\triangleleft : \mathcal{B} \times \mathcal{A} \rightarrow \mathcal{B}$  together with an associativity constraint and a unit constraint

$$\Phi : \triangleleft \circ (\text{id} \times \otimes) \rightarrow \triangleleft \circ (\triangleleft \times \text{id}) \quad \text{and} \quad u : \triangleleft \circ (\text{id} \times \mathbf{1}) \rightarrow \text{id},$$

satisfying the analogues of the pentagon axiom and the unit axiom for a monoidal category (see [24, Eqns. (6)–(7)]).

Then, we say that  $\mathcal{B}$  is a *module category* over  $\mathcal{A}$  (or, equivalently, that  $(\mathcal{B}, \mathcal{A})$  is an *action pair*) if it is equipped with a monoidal action of  $\mathcal{A}$ .

**Example 3.5.1.** Set  $\mathcal{A} = U_q\widehat{\mathfrak{g}}\text{-mod}$  and  $\mathcal{B} = U_q\mathfrak{k}\text{-mod}$ . Since  $U_q\mathfrak{k}$  is a coideal subalgebra in  $U_q\widehat{\mathfrak{g}}$ , the standard tensor product induces a functor

$$\triangleleft : U_q\mathfrak{k}\text{-mod} \times U_q\widehat{\mathfrak{g}}\text{-mod} \rightarrow U_q\mathfrak{k}\text{-mod}$$

which is readily verified to be a monoidal action of  $U_q\widehat{\mathfrak{g}}\text{-mod}$  on  $U_q\mathfrak{k}\text{-mod}$ .  $\nabla$

Let now  $(\mathcal{B}, \mathcal{A})$  and  $(\mathcal{B}', \mathcal{A}')$  be two action pairs. A functor  $(\mathcal{B}, \mathcal{A}) \rightarrow (\mathcal{B}', \mathcal{A}')$  is the datum of

- a monoidal functor  $F_A : \mathcal{A} \rightarrow \mathcal{A}'$ ,
- a functor  $F_B : \mathcal{B} \rightarrow \mathcal{B}'$ ,
- a natural isomorphism  $\omega : F_B \circ \triangleleft \rightarrow \triangleleft \circ (F_B \times F_A)$  satisfying the analogue of the consistency condition for the tensor structure of a monoidal functor (see [24, Eqns. (8)–(9)]).

More concisely, we shall say that the functor  $F_B$  intertwines through  $F_A$  the action of  $\mathcal{A}$  on  $\mathcal{B}$  and of  $\mathcal{A}'$  on  $\mathcal{B}'$ .

#### 4. Trigonometric $K$ -matrices

In this section, we briefly review the construction of trigonometric  $K$ -matrices on irreducible finite-dimensional  $U_qL\mathfrak{g}$ -modules. The results of this section are due to the first author and Vlaar [4].

##### 4.1. $\tau$ -invariant shifts

Henceforth, we replace the homogeneous grading shift defined in §2.3 with a fixed  $\tau$ -invariant grading shift. Namely, we fix a morphism  $\Sigma_z^\tau : U_qL\mathfrak{g}[z, z^{-1}] \rightarrow U_qL\mathfrak{g}[z, z^{-1}]$  given by

$$\Sigma_z^\tau(K_h) = K_h, \quad \Sigma_z^\tau(E_i) = z^{\chi_i} E_i, \quad \Sigma_z^\tau(F_i) = z^{-\chi_i} F_i,$$

where  $\chi : \widehat{I} \rightarrow \mathbb{Z}_{\geq 0}$  is a non-zero  $\tau$ -invariant function. In particular,  $\Sigma_z^\tau \circ \tau = \tau \circ \Sigma_z^\tau$ . Examples of  $\tau$ -invariant grading shifts are those determined by the characteristic function of any union of  $\tau$ -orbits (e.g., the principal grading shift). Clearly, the homogeneous grading shift  $\Sigma_z$  is  $\tau$ -invariant if and only if  $\tau(0) = 0$ . In this case, we say that the QSP subalgebra is  $\tau$ -restrictable.

Let  $V \in U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  with action  $\pi_V : U_qL\mathfrak{g} \rightarrow \text{End}(V)$ . As before, we denote by  $\pi_{V,z} := \pi_V \circ \Sigma_z^\tau$  the  $\tau$ -shifted action of  $U_qL\mathfrak{g}$  on the modules  $V(z) := V \otimes \mathbb{k}(z)$  and  $V((z)) := V \otimes \mathbb{k}((z))$ . Note that the  $\tau$ -analogue of Theorem 2.4.1 holds with  $\Delta_z$  replaced by the  $\tau$ -shifted coproduct  $\Delta_z^\tau(x) = \text{id} \otimes \Sigma_z^\tau(\Delta(x))$ . Henceforth, we fix a  $\tau$ -invariant grading shift and drop the superscript  $\tau$ .

##### 4.2. Spectral $K$ -matrices

Fix a QSP subalgebra  $U_q\mathfrak{k} \subset U_q\widehat{\mathfrak{g}}$ . By Remark 3.4.1 (3), every finite-dimensional  $U_qL\mathfrak{g}$ -module  $V$  is naturally acted upon by  $U_q\mathfrak{k}$  through the projection  $U_q\widehat{\mathfrak{g}}' \rightarrow U_qL\mathfrak{g}$ . In the following, we shall consider the restriction of  $V$  to  $U_q\mathfrak{k}$ . With a slight abuse of notation, we will denote by the same symbol an element in  $U_q\mathfrak{k}$  and its image in  $U_qL\mathfrak{g}$ .

Set

$$\mathcal{G} := \{g \in (U_qL\mathfrak{g})^{\text{fd}, \times} \mid \text{Ad}(g)(U_qL\mathfrak{g}) = U_qL\mathfrak{g}, \Sigma_z(g) = g\}.$$

By [4, Thm. 4.2.1],  $U_qL\mathfrak{g}$  has a  $\mathcal{G}$ -family of universal spectral  $K$ -matrices relative to the QSP subalgebra  $U_q\mathfrak{k}$ . Namely, for any  $g \in \mathcal{G}$ , there is a canonical formal series  $K(z) \in (U_qL\mathfrak{g})^{\text{fd}}[[z]]$  such that  $\Sigma_a(K(z)) = K(az)$  ( $a \in \mathbb{k}^\times$ ) and the following properties hold.

(1) For any  $b \in U_q\mathfrak{k}$ ,

$$K(z) \cdot \Sigma_z(b) = \psi(\Sigma_{1/z}(b)) \cdot K(z),$$

where  $\psi = \text{Ad}(g) \circ \vartheta_q^{-1}$ .

(2) Set  $R(z)^\psi := \psi \otimes \text{id}(R(z))$ . Then,

$$\Delta_{w/z}(K_\psi(z)) = R_\psi^{-1} \cdot 1 \otimes K(w) \cdot R(zw)^\psi \cdot K(z) \otimes 1.$$

Moreover,  $K(z)$  is a solution of Cherednik’s reflection equation

$$\begin{aligned} R(w/z)_{21}^{\psi\psi} \cdot 1 \otimes K(w) \cdot R(zw)^\psi \cdot K(z) \otimes 1 &= \\ &= K(z) \otimes 1 \cdot R(zw)_{21}^\psi \cdot 1 \otimes K(w) \cdot R(w/z), \end{aligned} \tag{4.1}$$

where  $R(z)_{21}^\psi := \psi \otimes \text{id}(R(z))_{21}$ .

**Remark 4.2.1.** The result above relies on the construction of universal  $K$ -matrices for quantum Kac–Moody symmetric pairs given in [3, Thm. 8.11-8.12], building on the work of Bao-Wang [7] and Balagović-Kolb [5].

The equation (4.1) was first introduced by Cherednik in [15, Eq. (4.14)].  $\nabla$

### 4.3. Trigonometric $K$ -matrices

A finite-dimensional  $U_qL\mathfrak{g}$ -module  $V$  is

- (1) *QSP irreducible* if it is irreducible as a module over  $U_q\mathfrak{k}$ ;
- (2) *generically QSP irreducible* if  $V((z))$  is irreducible as a module over  $U_q\mathfrak{k}((z))$ .<sup>5</sup>

Such condition is the natural counterpart of the generic irreducibility of the tensor product and yields the following QSP analogue of Theorem 2.4.1, see [4, Thm. 5.2.1].

**Theorem 4.3.1** ([4]).

- (1) *Every finite-dimensional irreducible  $U_qL\mathfrak{g}$ -module is generically QSP irreducible.*
- (2) *Let  $V, W \in U_qL\mathfrak{g}\text{-mod}^{\text{fd}}$  be irreducible modules. There is a formal Laurent series  $g_V(z) \in \mathbb{k}((z))$  and a polynomial non-vanishing operator  $\tilde{K}_V(z) \in \text{End}(V)[z]$  (unique up to a scalar) such that*

$$K_V(z) = g_V(z) \cdot \tilde{K}_V(z),$$

---

<sup>5</sup> Note that  $V((z))$  is equipped with the shifted action (see §4.1).

where  $\mathbf{K}_V(z): V((z)) \rightarrow \psi^*(V)((z^{-1}))$  is the formal QSP intertwiner given by the action of  $\mathbf{K}(z)$  on  $V$ .

- (3) The operators  $\tilde{\mathbf{K}}_V(w)$  and  $\tilde{\mathbf{K}}_W(z)$  satisfy Cherednik’s reflection equation in  $\text{End}(V \otimes W)(z, w)$

$$\begin{aligned} \mathbf{R}_{\psi^*(W)\psi^*(V)}(w/z)_{21} \cdot 1 \otimes \tilde{\mathbf{K}}_W(w) \cdot \mathbf{R}_{\psi^*(V)W}(zw) \cdot \tilde{\mathbf{K}}_V(z) \otimes 1 = \\ = \tilde{\mathbf{K}}_V(z) \otimes 1 \cdot \mathbf{R}_{\psi^*(W)V}(zw)_{21} \cdot 1 \otimes \tilde{\mathbf{K}}_W(w) \cdot \mathbf{R}_{VW}(w/z), \end{aligned} \tag{4.2}$$

where  $\mathbf{R}_{VW}(z)$  is the trigonometric R-matrix (cf. §2.4), and  $\mathbf{R}_{WV}(z)_{21} := (1\ 2) \circ \mathbf{R}_{WV}(z) \circ (1\ 2)$ .

#### 4.4. Unitary K-matrices

Let  $V, W \in U_q\mathbf{Lg}\text{-mod}^{\text{fd}}$  be irreducible modules. By Theorem 2.4.1, the trigonometric R-matrix  $\mathbf{R}_{VW}(z)$  satisfies the unitarity condition  $\mathbf{R}_{VW}(z)^{-1} = (1\ 2) \circ \mathbf{R}_{WV}(z^{-1}) \circ (1\ 2)$ . The analogue result for trigonometric K-matrices requires an additional assumption to hold, see [4, Prop. 5.4.1].

**Theorem 4.4.1** ([4]). *Let  $V \in U_q\mathbf{Lg}\text{-mod}^{\text{fd}}$  be a  $\psi$ -involutive irreducible module, i.e., such that  $(\psi^2)^*(V) \simeq V$ . The trigonometric K-matrices  $\tilde{\mathbf{K}}_V(z)$  and  $\tilde{\mathbf{K}}_{\psi^*(V)}(z)$  from Theorem 4.3.1 give rise through normalization to two non-vanishing QSP intertwiners*

$$\mathbf{K}_V(z) : V(z) \rightarrow \psi^*(V)(z^{-1}) \quad \text{and} \quad \mathbf{K}_{\psi^*(V)}(z) : \psi^*(V)(z) \rightarrow V(z^{-1})$$

satisfying the unitarity condition

$$\mathbf{K}_V(z)^{-1} = \mathbf{K}_{\psi^*(V)}(z^{-1}). \tag{4.3}$$

Moreover, if  $V(\zeta)$  is QSP irreducible for some  $\zeta \in \mathbb{k}^\times$ , then  $\mathbf{K}_V(\zeta)$  is well-defined and invertible.

**Remark 4.4.2.** In contrast with the case of the R-matrix, the normalization yielding the unitary K-matrix  $\mathbf{K}_V(z)$  is non-canonical in general, since it follows from the normalization of the operator  $\tilde{\mathbf{K}}_{\psi^*(V)}(z^{-1}) \circ \tilde{\mathbf{K}}_V(z)$ . In certain cases, however, there is a canonical normalization. We refer the reader to [4, §5.4–5.6].  $\nabla$

#### 4.5. Trigonometric K-matrices for Kirillov–Reshetikhin modules

Kirillov–Reshetikhin modules are minimal affinizations of irreducible  $U_q\mathbf{g}$ -modules whose highest weight is a multiple of a fundamental weight. More precisely, for any  $i \in I$ ,  $k \in \mathbb{Z}_{>0}$ , and  $a \in \mathbb{C}^\times$ , the Kirillov–Reshetikhin module  $W_{k,a}^{(i)}$  is the unique irreducible  $U_q\mathbf{Lg}$ -module whose Drinfeld polynomials are all trivial except for the node

$i$ , where the roots are given by a  $q_i$ -string of length  $k$  starting at  $a$ , see, e.g., [40,11]. The *fundamental representations* of  $U_q\mathbf{Lg}$  are the simplest non-trivial Kirillov–Reshetikhin modules. Specifically, we set  $\mathbf{V}_{\omega_i} := W_{1,1}^{(i)}$ .

Let  $\eta_0 \in \text{Aut}(\widehat{A})$  be the diagram automorphism which fixes the affine node and acts on any other node as the opposition involution  $\text{op}_I$ , see §3.1. From the classification of generalized Satake diagrams of affine type in [60, App. A, Tables 5, 6 and 7] it follows that a QSP subalgebra is  $\tau$ -restrictable (i.e.,  $\tau$  fixes the affine node) if and only if  $\tau$  is either the identity or  $\eta_0$  (except in type  $D_n^{(1)}$  with  $n$  even, where  $\eta_0 = \text{id}$  but there exist nontrivial involutive diagram automorphisms fixing 0). By [4, Thm. 7.8.1], the following holds.

**Theorem 4.5.1** ([4]). *Let  $U_q\mathfrak{k} \subset U_q\mathbf{Lg}$  be a  $\tau$ -restrictable QSP subalgebra. Let  $W$  be a Kirillov–Reshetikhin  $U_q\mathbf{Lg}$ -module.*

(1) *There is a unique QSP intertwiner (up to a scalar multiple)*

$$\mathbf{K}_W(z) : W(z) \rightarrow (\eta_0\tau)^*(W)(z^{-1})$$

*up to a scalar multiple and a unique shift in  $W$ . Moreover,  $\mathbf{K}_W(z)$  is a solution of the diagrammatic reflection equation*

$$\begin{aligned} \mathbf{R}_{WV}(\frac{w}{z})_{21} \cdot \text{id} \otimes \mathbf{K}_W(w) \cdot \mathbf{R}_{(\eta_0\tau)^*(V)W}(zw) \cdot \mathbf{K}_V(z) \otimes \text{id} = \\ = \mathbf{K}_V(z) \otimes \text{id} \cdot \mathbf{R}_{(\eta_0\tau)^*(W)V}(zw)_{21} \cdot \text{id} \otimes \mathbf{K}_W(w) \cdot \mathbf{R}_{VW}(\frac{w}{z}). \end{aligned}$$

(2) *If  $\tau = \eta_0$ , there is a unique QSP intertwiner*

$$\mathbf{K}_W(z) : W(z) \rightarrow W(z^{-1})$$

*up to a scalar multiple and a unique shift in  $W$ . Moreover,  $\mathbf{K}_W(z)$  is a solution of the standard reflection equation*

$$\begin{aligned} \mathbf{R}_{WV}(\frac{w}{z})_{21} \cdot \text{id}_V \otimes \mathbf{K}_W(w) \cdot \mathbf{R}_{VW}(zw) \cdot \mathbf{K}_V(z) \otimes \text{id}_W = \\ = \mathbf{K}_V(z) \otimes \text{id}_W \cdot \mathbf{R}_{WV}(zw)_{21} \cdot \text{id}_V \otimes \mathbf{K}_W(w) \cdot \mathbf{R}_{VW}(\frac{w}{z}) \end{aligned}$$

(3) *If  $\tau = \text{id}$ , there is a unique QSP intertwiner*

$$\mathbf{K}_W(z) : W(z) \rightarrow W^*(z^{-1})$$

*up to a scalar multiple and a unique shift in  $W$ . Moreover, under the identification of  $W$  and  $W^*$  as vector spaces,  $\mathbf{K}_W(z)$  gives rise to a solution of the transposed reflection equation*

$$\begin{aligned} \mathbf{R}_{WV}(\frac{w}{z})_{21}^{\text{tv}^{\text{tw}}} \cdot \text{id}_V \otimes \mathbf{K}_W(w) \cdot (\mathbf{R}_{VW}(zw)^{-1})^{\text{tv}} \cdot \mathbf{K}_V(z) \otimes \text{id}_W = \\ = \mathbf{K}_V(z) \otimes \text{id}_W \cdot (\mathbf{R}_{WV}(zw)_{21}^{-1})^{\text{tw}} \cdot \text{id}_V \otimes \mathbf{K}_W(w) \cdot \mathbf{R}_{VW}(\frac{w}{z}). \end{aligned}$$

**Remark 4.5.2.** The result appears as a direct consequence of Theorem 4.3.1 (3) in the case  $\psi = \omega \circ \tau$ . More precisely, one first observes that, by [10, Eqs. (2.20) and (2.21)], if  $W = W_{k,a}^{(i)}$ , then there is an isomorphism of  $U_q \mathfrak{Lg}$ -modules

$$\omega^*(W) \simeq \eta_0^*(W)(a^{-2}q_i^{-2(k-1)}).$$

This yields (1) and (2). In order to prove (3), one then observes that, by [10, Eqs. (2.20) and (2.21)], there exists an integer  $c \in \mathbb{Z}$  depending only on  $\mathfrak{g}$  such that, for any irreducible finite-dimensional  $U_q \mathfrak{Lg}$ -module  $V$ , one has  $\eta_0^*(V) \simeq V^*(q^c)$ .  $\nabla$

### 5. $\mathfrak{o}$ KLR algebras

In this section, we review the definition and the basic properties of KLR and oriented KLR ( $\mathfrak{o}$  KLR) algebras, in particular their polynomial representation and their convolution product.

#### 5.1. Quiver with an involution

Let  $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \rangle$  denote the symmetric group on  $n$  letters, and  $\mathfrak{W}_n = \langle s_0, s_1, \dots, s_{n-1} \rangle$  the Weyl group of type  $B_n$ , i.e.,  $(\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ .

Let  $\Gamma = (J, \Omega)$  be a quiver with vertices  $J$  and arrows  $\Omega$ . We assume that  $\Gamma$  does not have loops. Given an arrow  $a \in \Omega$ , let  $s(a)$  be its source, and  $t(a)$  its target. If  $i, j \in J$ , let  $\Omega_{ij} \subset \Omega$  be the subset of arrows  $a$  such that  $s(a) = i$  and  $t(a) = j$ . Let  $a_{ij} = |\Omega_{ij}|$  and abbreviate  $\tilde{a}_{ij} = a_{ij} + a_{ji}$ . We assume that  $a_{ij} < \infty$  for all  $i, j \in J$ .

**Definition 5.1.1.** A (contravariant) *involution* of the quiver  $\Gamma$  is a pair of involutions  $\theta: J \rightarrow J$  and  $\theta: \Omega \rightarrow \Omega$  such that:

- (1)  $s(\theta(a)) = \theta(t(a))$  and  $t(\theta(a)) = \theta(s(a))$  for all  $a \in \Omega$ ,
- (2) if  $t(a) = \theta(s(a))$  then  $a = \theta(a)$ .

We denote by  $J^\theta$  the subset of fixed points of  $\theta$ . Let  $\mathbb{N}[J]$  be the commutative semigroup freely generated by  $J$ . We call elements of  $\mathbb{N}[J]$  *dimension vectors*. Given a dimension vector  $\beta = \sum_{i \in J} \beta(i) \cdot i$ , set  $\|\beta\| = \sum_{i \in J} \beta(i)$ .

We call a sequence  $\nu = \nu_1 \cdots \nu_n \in J^n$  a *composition* of  $\beta$  of length  $\ell(\nu) = n$  if  $|\nu| = \sum_{k=1}^n \nu_k = \beta$ . We also set  $\|\nu\| = n$ . Let  $J^\beta$  denote the set of all compositions of  $\beta$ . There is a left action of  $\mathfrak{S}_n$  on  $J^n$  by permutations

$$s_k \cdot \nu_1 \cdots \nu_n = \nu_1 \cdots \nu_{k+1} \nu_k \cdots \nu_n \quad (1 \leq k \leq n-1), \tag{5.1}$$

whose orbits are the sets  $J^\beta$  for all  $\beta$  with  $\|\beta\| = n$ .

Let  $J^\bullet = \bigcup_{\beta \in \mathbb{N}[J]} J^\beta$  be the set of compositions of all dimension vectors. We consider  $J^\bullet$  as a monoid with respect to concatenation:  $\nu\mu = \nu_1 \cdots \nu_{\ell\nu} \mu_1 \cdots \mu_{\ell\mu}$ , with the zero dimension vector composition as the identity.

The involution  $\theta$  induces an involution  $\theta: \mathbb{N}[J] \rightarrow \mathbb{N}[J]$ . We call dimension vectors in  $\mathbb{N}[J]^\theta$  *self-dual*. We will always assume, for any  $\beta \in \mathbb{N}[J]^\theta$ , that if  $i \in J^\theta$  then  $\beta(i)$  is even. Set  $\|\beta\|_\theta = \|\beta\|/2$  and

$$\theta(-): \mathbb{N}[J] \rightarrow \mathbb{N}[J]^\theta, \quad \beta \mapsto {}^\theta\beta = \beta + \theta(\beta). \quad (5.2)$$

We call a sequence  $\nu = \nu_1 \cdots \nu_n \in J^n$  an *isotropic composition* of  $\beta$  if  ${}^\theta|\nu| = \sum_{k=1}^n {}^\theta\nu_k = \beta$ . Let  ${}^\theta J^\beta$  denote the set of all isotropic compositions of  $\beta$ . There is a left action of  $\mathfrak{W}_n$  on  $J^n$  extending (5.1), given by

$$s_0 \cdot \nu_1 \cdots \nu_n = \theta(\nu_1)\nu_2 \cdots \nu_n,$$

whose orbits are the sets  ${}^\theta J^\beta$  for all self-dual  $\beta$  with  $\|\beta\|_\theta = n$ . Let  ${}^\theta J^\bullet = \bigcup_{\beta \in \mathbb{N}[J]^\theta} {}^\theta J^\beta$  be the set of all isotropic compositions of all self-dual dimension vectors.

## 5.2. KLR and o KLR algebras

We recall the definition of orientifold KLR algebras as given by the second author in [56] (see also [65,55,54,53]).

**Definition 5.2.1.** Let  $\Gamma$  be a quiver with a contravariant involution  $\theta$  and a dimension vector  $\lambda \in \mathbb{N}[J]$  such that  $\lambda(i) = 0$  if  $i \in J^\theta$ . Note that  $\lambda$  need not be self-dual. We call  $\lambda$  the *framing dimension vector*, and the datum  $(\Gamma, \theta, \lambda)$  an *enhanced quiver*.

Let  $(\Gamma, \theta, \lambda)$  be a fixed enhanced quiver. Set

$$P_{ij}(u, v) = \delta_{i \neq j} (v - u)^{a_{ij}}, \quad P_i(u) = \delta_{i \neq \theta(i)} (-u)^{\lambda(i)}$$

for  $i, j \in J$ , and define  $(Q, Q')$  as

$$Q_{ij}(u, v) = P_{ij}(u, v)P_{ji}(v, u), \quad Q'_i(u) = P_i(u)P_{\theta(i)}(-u), \quad (i, j \in J).$$

**Definition 5.2.2.** Let  $\beta \in \mathbb{N}[J]^\theta$  with  $\|\beta\|_\theta = n$ , and  $\alpha \in \mathbb{N}[J]$  with  ${}^\theta\alpha = \beta$ .

- (1) The *KLR algebra*  $\mathcal{R}(\alpha)$  associated to  $(\Gamma, \alpha)$  is the  $\mathbb{k}$ -algebra generated by  $e(\nu)$  ( $\nu \in J^\alpha$ ),  $x_l$  ( $1 \leq l \leq n$ ) and  $\tau_k$  ( $1 \leq k \leq n-1$ ), subject to relations (5.3), (5.5), (5.6), (5.8), (5.10) and (5.11).
- (2) The *orientifold KLR algebra*  ${}^\theta\mathcal{R}(\beta; \lambda)$  associated to  $(\Gamma, \theta, \lambda; \beta)$  is the  $\mathbb{k}$ -algebra generated by  $e(\nu)$  ( $\nu \in {}^\theta J^\beta$ ),  $x_l$  ( $1 \leq l \leq n$ ), and  $\tau_k$  ( $0 \leq k \leq n-1$ ), subject to all the relations (5.3)–(5.13).

- *Idempotent relations:*

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad x_l e(\nu) = e(\nu)x_l, \quad \tau_k e(\nu) = e(s_k \cdot \nu)\tau_k, \quad (5.3)$$

$$\tau_0 e(\nu) = e(s_0 \cdot \nu)\tau_0, \quad (5.4)$$

- *Polynomial relations:*

$$x_l x_{l'} = x_{l'} x_l, \quad (5.5)$$

- *Quadratic relations:*

$$\tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_{k+1}, x_k)e(\nu), \quad (5.6)$$

$$\tau_0^2 e(\nu) = Q'_{\nu_1}(-x_1)e(\nu), \quad (5.7)$$

- *Deformed braid relations:*

$$\tau_k \tau_{k'} = \tau_{k'} \tau_k \quad \text{if } k \neq k' \pm 1, \quad (5.8)$$

$$\tau_0 \tau_k = \tau_k \tau_0 \quad \text{if } k \neq 1, \quad (5.9)$$

$$(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k)e(\nu) = \delta_{\nu_k, \nu_{k+2}} \frac{Q_{\nu_k, \nu_{k+1}}(x_{k+1}, x_k) - Q_{\nu_k, \nu_{k+1}}(x_{k+1}, x_{k+2})}{x_k - x_{k+2}} e(\nu), \quad (5.10)$$

$$\begin{aligned} & ((\tau_1 \tau_0)^2 - (\tau_0 \tau_1)^2)e(\nu) = \\ & = \begin{cases} \frac{Q'_{\nu_2}(x_2) - Q'_{\nu_1}(x_1)}{x_1 + x_2} \tau_1 e(\nu) & \text{if } \nu_1 \neq \nu_2, \nu_2 = \theta(\nu_1) \\ \frac{Q_{\nu_1, \nu_2}(x_2, -x_1) - Q_{\nu_1, \nu_2}(-x_2, -x_1)}{x_2} \tau_0 e(\nu) & \text{if } \nu_1 \neq \theta(\nu_1), \nu_2 = \theta(\nu_2), \\ \frac{Q_{\nu_1, \nu_2}(x_2, -x_1) - Q_{\nu_1, \nu_2}(x_2, x_1)}{x_1 x_2} (x_1 \tau_0 + 1)e(\nu) & \text{if } \theta(\nu_1) = \nu_1 \neq \nu_2 = \theta(\nu_2), \\ 0 & \text{else,} \end{cases} \end{aligned}$$

- *Mixed relations:*

$$(\tau_k x_l - x_{s_k(l)} \tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k + 1, \nu_k = \nu_{k+1}, \\ 0 & \text{else,} \end{cases} \quad (5.11)$$

$$(\tau_0 x_1 + x_1 \tau_0)e(\nu) = \begin{cases} 0 & \text{if } \nu_1 \neq \theta(\nu_1), \\ -2e(\nu) & \text{if } \nu_1 = \theta(\nu_1), \end{cases} \quad (5.12)$$

$$\tau_0 x_l = x_l \tau_0 \quad \text{if } l \neq 1, \quad (5.13)$$

where, in all the relations above,  $1 \leq l, l' \leq n$  and  $1 \leq k, k' \leq n - 1$ , except (5.10), where  $1 \leq k \leq n - 2$ .

These algebras are endowed with the following grading:

$$\begin{aligned} \deg e(\nu) &= 0, \\ \deg x_k &= 2, \\ \deg \tau_k e(\nu) &= \begin{cases} -2 & \text{if } \nu_k = \nu_{k+1}, \\ \vec{a}_{\nu_k, \nu_{k+1}} & \text{otherwise,} \end{cases} \\ \deg \tau_0 e(\nu) &= \begin{cases} -2 & \text{if } \theta(\nu_1) = \nu_1, \\ \theta \lambda(\nu_1) & \text{otherwise.} \end{cases} \end{aligned}$$

If  $\lambda = 0$ , we abbreviate

$${}^\theta \mathcal{R}(\beta) = {}^\theta \mathcal{R}(\beta; \lambda).$$

**Remark 5.2.3.** It follows from the PBW theorems (see, e.g., [61, Thm. 3.7], [56, Prop. 2.9]) for the KLR and oKLR algebras that  $\mathcal{R}(\alpha)$  is in fact isomorphic to the subalgebra of  ${}^\theta \mathcal{R}(\beta; \lambda)$  generated by  $e(\nu)$  ( $\nu \in J^\alpha$ ),  $x_l$  ( $1 \leq l \leq n$ ) and  $\tau_k$  ( $1 \leq k \leq n - 1$ ).  $\nabla$

Let  $\mathbb{1}$  and  ${}^\theta \mathbb{1}$  denote the regular representations (in degree zero) of the trivial algebras  $\mathcal{R}(0)$  and  ${}^\theta \mathcal{R}(0; \lambda)$ , respectively. For a fixed  $\lambda \in \mathbb{N}[J]$ , set

$$\mathcal{R}\text{-mod}_{\text{gr}} = \bigoplus_{\alpha \in \mathbb{N}[J]} \mathcal{R}(\alpha)\text{-mod}_{\text{gr}}, \quad {}^\theta \mathcal{R}(\lambda)\text{-mod}_{\text{gr}} = \bigoplus_{\beta \in \mathbb{N}[J]^\theta} {}^\theta \mathcal{R}(\beta; \lambda)\text{-mod}_{\text{gr}},$$

and abbreviate  ${}^\theta \mathcal{R}\text{-mod}_{\text{gr}} = {}^\theta \mathcal{R}(\lambda = 0)\text{-mod}_{\text{gr}}$ . We use analogous notation for direct sums of categories of finite dimensional modules.

### 5.3. Polynomial representation

Set

$$\begin{aligned} \mathbb{P}_\nu &:= \mathbb{k}[x_1, \dots, x_n]e(\nu), & \widehat{\mathbb{P}}_\nu &:= \mathbb{k}[[x_1, \dots, x_n]]e(\nu), & \widehat{\mathbb{K}}_\nu &:= \mathbb{k}((x_1, \dots, x_n))e(\nu), \\ {}^\theta \mathbb{P}_\beta &:= \bigoplus_{\nu \in {}^\theta J^\beta} \mathbb{P}_\nu, & {}^\theta \widehat{\mathbb{P}}_\beta &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widehat{\mathbb{P}}_\nu, & {}^\theta \widehat{\mathbb{K}}_\beta &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widehat{\mathbb{K}}_\nu. \end{aligned}$$

We abbreviate  $x_{-l} = -x_l$  for  $1 \leq l \leq n$ . There is a natural left action of the Weyl group  $\mathfrak{W}_n$  on  $\mathbb{k}((x_1, \dots, x_n))$  given by  $w \cdot x_l = x_{w(l)}$ . This extends to an action on  ${}^\theta \widehat{\mathbb{K}}_\beta$  by

$$w \cdot fe(\nu) = w(f)e(w \cdot \nu),$$

for  $w \in \mathfrak{W}_n$  and  $f \in \mathbb{k}((x_1, \dots, x_n))$ .

**Proposition 5.3.1.** *The algebra  ${}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda})$  has a faithful polynomial representation on  ${}^\theta\mathbb{P}_\beta$ , given by:*

- $e(\nu)$  ( $\nu \in {}^\theta J^\beta$ ) acting as projection onto  $\mathbb{P}_\nu$ ,
- $x_1, \dots, x_n$  acting naturally by multiplication,
- $\tau_1, \dots, \tau_{n-1}$  acting via

$$\tau_k \cdot fe(\nu) = \begin{cases} \frac{s_k(f) - f}{x_k - x_{k+1}} e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ P_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) s_k(f) e(s_k \cdot \nu) & \text{otherwise,} \end{cases}$$

- $\tau_0$  acting via

$$\tau_0 \cdot fe(\nu) = \begin{cases} \frac{s_0(f) - f}{x_1} e(\nu) & \text{if } \theta(\nu_1) = \nu_1, \\ P_{\nu_1}(x_1) s_0(f) e(s_0 \cdot \nu) & \text{otherwise.} \end{cases}$$

**Proof.** See [56, Prop. 2.7].  $\square$

Next, for each  $i, j \in J$ , we choose holomorphic functions  $c_{ij}(u, v) \in \mathbb{k}[[u, v]]$  such that

$$c_{ij}(u, v)c_{ji}(v, u) = 1, \quad c_{ii}(u, v) = 1, \quad c_{ij}(u, v) = c_{\theta(j)\theta(i)}(-v, -u). \tag{5.14}$$

Moreover, for each  $i \in J$ , we also choose holomorphic functions  $c_i \in \mathbb{k}[[u]]$  such that

$$c_i(u)c_{\theta(i)}(-u) = 1, \quad i = \theta(i) \Rightarrow c_i(u) = 1. \tag{5.15}$$

Set

$$\tilde{P}_{ij}(u, v) = P_{ij}(u, v)c_{ij}(u, v), \quad \tilde{P}_i(u) = P_i(u)c_i(u).$$

Note that the corresponding  $(Q, Q')$  remain unchanged.

**Corollary 5.3.2.** *There is an injective  ${}^\theta\mathbb{P}_\beta$ -algebra homomorphism*

$${}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda}) \hookrightarrow {}^\theta\widehat{\mathbb{K}}_\beta \rtimes \mathbb{k}[\mathfrak{W}_n] \tag{5.16}$$

given by

$$\tau_0 e(\nu) = \begin{cases} x_1^{-1}(s_0 - 1)e(\nu) & \text{if } \nu_1 = \theta(\nu_1), \\ \tilde{P}_{\nu_1}(x_1) s_0 e(\nu) & \text{otherwise,} \end{cases}$$

$$\tau_k e(\nu) = \begin{cases} (x_k - x_{k+1})^{-1}(s_k - 1)e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \tilde{P}_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})s_k e(\nu) & \text{otherwise,} \end{cases}$$

for  $1 \leq k \leq n - 1$ .

**Proof.** See [56, Corollary 2.8].  $\square$

Given  $\alpha \in \mathbb{N}[J]$  with  $\theta\alpha = \beta$ , let

$$\mathbb{P}_\alpha := \bigoplus_{\nu \in J^\alpha} \mathbb{P}_\nu, \quad \widehat{\mathbb{P}}_\alpha := \bigoplus_{\nu \in J^\alpha} \widehat{\mathbb{P}}_\nu, \quad \theta\widehat{\mathbb{K}}_\alpha := \bigoplus_{\nu \in J^\alpha} \widehat{\mathbb{K}}_\nu.$$

The embedding (5.16) restricts to a  $\mathbb{P}_\alpha$ -algebra homomorphism

$$\mathcal{R}(\alpha) \hookrightarrow \widehat{\mathbb{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n]. \tag{5.17}$$

#### 5.4. One-dimensional modules

Given  $\mu \in {}^\theta J^\beta$ , let  ${}^\theta L(\mu)$  be the free  $\mathbb{k}$ -module  $\mathbb{k}u_\mu$  with generator  $u_\mu$  of degree zero.

**Lemma 5.4.1. Setting**

$$x_l \cdot u_\mu = 0, \quad \tau_k \cdot u_\mu = 0, \quad e(\nu) \cdot u_\mu = \delta_{\nu, \mu} u_\mu, \tag{5.18}$$

for  $1 \leq l \leq n$ ,  $0 \leq k < n$  and  $\nu \in {}^\theta J^\beta$ , makes  ${}^\theta L(\mu)$  into an  ${}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})$ -module if and only if

- (a)  $\mu_k \neq \mu_{k+1}$  and  $\vec{a}_{\mu_k, \mu_{k+1}} \geq 1$  (for  $1 \leq k < n$ ),
- (b) whenever  $\mu_k = \mu_{k+2}$ , then  $\vec{a}_{\mu_k, \mu_{k+1}} \neq 1$  (for  $1 \leq k < n - 1$ ),
- (c)  $\mu_1 \neq \theta(\mu_1)$  and  ${}^\theta \boldsymbol{\lambda}(\mu_1) \geq 1$ .

**Proof.** The proof is based on a direct inspection of the defining relations of  ${}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})$ . The definition of the action (5.18) implies that the LHS of the relations (5.6)–(5.13) must be zero. The RHS of (5.11) is zero if and only if  $\mu_k \neq \mu_{k+1}$ . Moreover, the RHS of (5.6) is zero if and only if  $\mu_k = \mu_{k+1}$  (which cannot be the case) or the polynomial  $Q_{\mu_k, \mu_{k+1}}(x_{k+1}, x_k)$  has no constant term. The latter is the case if and only if  $\vec{a}_{\mu_k, \mu_{k+1}} \geq 1$ . Similarly, the RHS of (5.10) is zero if and only if condition (b) holds.

Next, the RHS of (5.12) is zero if and only if  $\mu_1 \neq \theta(\mu_1)$ . Moreover, the RHS of (5.7) is zero if and only if  $\mu_1 = \theta(\mu_1)$  (which cannot be the case) or  $Q'_{\mu_1}(-x_1)$  has no constant term. The latter is the case if and only if  ${}^\theta \boldsymbol{\lambda}(\mu_1) \geq 1$ .

Since all the other relations hold without any extra assumptions, we have proven the necessity and sufficiency of conditions (a)–(c).  $\square$

Note that if  $\mu \in J^\alpha$  and conditions (a)–(b) are satisfied then (5.18) (with  $1 \leq k < n$ ) defines the structure of a  $\mathcal{R}(\alpha)$ -module on  $\mathbb{k}u_\mu$ , which we then denote by  $L(\mu)$ .

5.5. Convolution product and monoidal action

We recall the definition of the convolution product of modules over KLR algebras. Let  $\alpha, \alpha' \in \mathbb{N}[J]$  with  $\|\alpha\| = n$  and  $\|\alpha'\| = n'$ . Set

$$e_{\alpha, \alpha'} = \sum_{\substack{\nu \in J^{\alpha + \alpha'} \\ \nu_1 \dots \nu_n \in J^\alpha}} e(\nu) \in \mathcal{R}(\alpha + \alpha').$$

There is a non-unital algebra homomorphism

$$\iota_{\alpha, \alpha'} : \mathcal{R}(\alpha, \alpha') := \mathcal{R}(\alpha) \otimes \mathcal{R}(\alpha') \rightarrow \mathcal{R}(\alpha + \alpha')$$

given by  $e(\nu) \otimes e(\mu) \mapsto e(\nu\mu)$  for  $\nu \in J^\alpha, \mu \in J^{\alpha'}$  and

$$x_l \otimes 1 \mapsto x_l e_{\alpha, \alpha'}, \quad 1 \otimes x_{l'} \mapsto x_{m+l'} e_{\alpha, \alpha'}, \tag{5.19}$$

$$\tau_k \otimes 1 \mapsto \tau_k e_{\alpha, \alpha'}, \quad 1 \otimes \tau_{k'} \mapsto \tau_{m+l} e_{\alpha, \alpha'}, \tag{5.20}$$

where  $1 \leq l \leq n, 1 \leq l' \leq n', 1 \leq k < n,$  and  $1 \leq k' < n'$ . Let  $M$  be a graded  $\mathcal{R}(\alpha)$ -module and  $N$  be a graded  $\mathcal{R}(\alpha')$ -module. Their *convolution product* is defined as

$$M \circ N = \mathcal{R}(\alpha + \alpha') e_{\alpha, \alpha'} \otimes_{\mathcal{R}(\alpha, \alpha')} (M \otimes N).$$

Next, let  $\beta \in \mathbb{N}[J]^\theta$  with  $\|\beta\|_\theta = n$ . Set

$${}^\theta e_{\beta, \alpha'} = \sum_{\substack{\nu \in {}^\theta J^{\beta + \alpha'}, \nu_1 \dots \nu_n \in {}^\theta J^\beta \\ \nu_{n+1} \dots \nu_{n+n'} \in J^{\alpha'}}} e(\nu) \in {}^\theta \mathcal{R}(\beta + \alpha'; \lambda).$$

There is an injective non-unital algebra homomorphism

$${}^\theta \iota_{\beta, \alpha'} : {}^\theta \mathcal{R}(\beta, \alpha'; \lambda) := {}^\theta \mathcal{R}(\beta; \lambda) \otimes \mathcal{R}(\alpha') \rightarrow {}^\theta \mathcal{R}(\beta + \alpha'; \lambda)$$

given by formulae (5.19)-(5.20) (with  $\nu \in {}^\theta J^\beta$  and  $e_{\beta, \alpha'}$  replaced by  ${}^\theta e_{\beta, \alpha'}$ ) and  $\tau_0 \otimes 1 \mapsto \tau_0 {}^\theta e_{\beta, \alpha'}$ . The *convolution action* of  $N \in \mathcal{R}(\alpha')$ -mod<sub>gr</sub> on  $M \in {}^\theta \mathcal{R}(\beta; \lambda)$ -mod<sub>gr</sub> is defined as

$$M \circ N = {}^\theta \mathcal{R}(\beta + \alpha'; \lambda) {}^\theta e(\beta, \alpha') \otimes_{{}^\theta \mathcal{R}(\beta, \alpha'; \lambda)} (M \otimes N).$$

**Proposition 5.5.1.** *The category  $\mathcal{R}$ -mod<sub>gr</sub> is monoidal with product  $\circ$  and unit  $\mathbb{1}$ . Moreover, there is a right monoidal action of  $\mathcal{R}$ -mod<sub>gr</sub> on  ${}^\theta \mathcal{R}(\lambda)$ -mod<sub>gr</sub> via  $\circ$ . The product and the action restrict to the full subcategories  $\mathcal{R}$ -mod<sub>gr</sub><sup>fd</sup> and  ${}^\theta \mathcal{R}(\lambda)$ -mod<sub>gr</sub><sup>fd</sup>, and descend to the corresponding Grothendieck groups.*

**Proof.** In the case of KLR algebras, the proof is well known, see, e.g., [39, §3.1]. In the orientifold case, the argument is analogous, so we only highlight the main points. The crucial fact is that the  ${}^\theta\mathcal{R}(\beta, \alpha'; \boldsymbol{\lambda})$ -module  ${}^\theta\mathcal{R}(\beta + {}^\theta\alpha'; \boldsymbol{\lambda})$  is free of finite rank (see, e.g., [65, Lem. 8.7]). Firstly, it implies that  $M \circledast N$  is finite dimensional whenever  $M$  and  $N$  are. Secondly, it implies that the bifunctor  $- \circledast -$  is biexact, and hence descends to the Grothendieck groups of graded finite dimensional modules. Thirdly, the existence of a suitable associativity constraint (see §3.5) follows from the natural isomorphisms

$$M \circledast (N \circ P) \cong {}^\theta\mathcal{R}(\beta + {}^\theta\alpha^\circ; \boldsymbol{\lambda})e \otimes_{{}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda}) \otimes {}^\theta\mathcal{R}(\alpha') \otimes {}^\theta\mathcal{R}(\alpha'')} (M \otimes N \otimes P) \cong (M \circledast N) \circledast P,$$

where  $\alpha^\circ = \alpha' + \alpha''$  and  $e$  is the sum of all idempotents  $e(\nu)$  such that  $\nu \in {}^\theta J^{\beta + {}^\theta\alpha^\circ}$ ,  $\nu_1 \cdots \nu_n \in {}^\theta J^\beta$ ,  $\nu_{n+1} \cdots \nu_{n+n'} \in J^{\alpha'}$  and  $\nu_{n+n'+1} \cdots \nu_{n+n'+n''} \in J^{\alpha''}$ .  $\square$

### 6. The combinatorial model

Throughout this section, we fix the following data:

- an affine Satake diagram  $(X, \tau)$  with pseudo-involution  $\vartheta$ ;
- a QSP subalgebra  $U_q \mathfrak{k} \subseteq U_q L\mathfrak{g}$ , associated to  $(X, \tau)$ , with parameters  $(\gamma, \sigma)$ ;
- a QSP admissible twisting operator  $\psi \in \text{Aut}(U_q L\mathfrak{g})$  (cf. §4.2).

#### 6.1. $J$ -data

Let  $J$  be a fixed index set. A  $J$ -datum is a choice, for each  $i \in J$ , of a finite-dimensional  $U_q L\mathfrak{g}$ -module  $V(i)$  and a non-zero scalar  $X(i) \in \mathbb{k}^\times$  such that the following properties are satisfied.

(P1) **Admissibility.** For any  $i \in J$ , the module  $V(i)$  is an irreducible *real* module (i.e.,  $V(i) \otimes V(i)$  is irreducible).

(P2) **R-matrices.** For any  $i, j \in J$ , there is a non-vanishing *unitary* trigonometric R-matrix

$$\mathbf{R}_{V(i)V(j)}(w/z) : V(i)_z \otimes V(j)_w \rightarrow V(i)_z \otimes V(j)_w$$

such that  $\mathbf{R}_{V(i)V(i)}^\vee(1) = \text{id}$ , where  $V(i)_z = (V(i))(z)$  (cf. Theorem 2.4.1 and Remarks 2.4.2).

(P3) **Poles.** For any  $i, j \in J$ , the poles of the R-matrix  $\mathbf{R}_{V(i)V(j)}(w/z)$  are in  $q^{1/m} \mathbb{C}[[q^{1/m}]]$  for some  $m > 0$ .

For any  $i, j \in J$ , we denote by  $d_{ij}(z) \in \mathbb{k}[z]$  the *denominator* of the trigonometric R-matrix  $\mathbf{R}_{V(i)V(j)}(z)$ , i.e., the polynomial of smallest degree such that  $d_{ij}(z)\mathbf{R}_{V(i)V(j)}(z)$  is defined over  $\mathbb{k}[z]$ . Moreover, we denote by  $d_{ij} \in \mathbb{Z}_{\geq 0}$  the order of the pole of  $\mathbf{R}_{V(i)V(j)}(z)$  at  $z = X(j)/X(i)$ .

**Remark 6.1.1.** The conditions (P1)–(P3) are easily satisfied.

- Kirillov–Reshetikhin modules are known to satisfy (P1).
- The existence and uniqueness of a unitary trigonometric R-matrix on a tensor product of finite-dimensional irreducible  $U_q\mathfrak{Lg}$ -modules is guaranteed by Theorem 2.4.1. Thus, (P2) is automatically satisfied (see also Remark 2.4.2 (2)).
- The condition (P3) is related to the explicit computation of the denominators  $d_{ij}(z)$ . Unfortunately, this is a difficult, largely open problem for arbitrary irreducible modules. However, it is well-known that (P3) is satisfied by *good* modules, as defined by Kashiwara in [36], *i.e.*, irreducible finite-dimensional modules over  $U_q\mathfrak{Lg}$  with a bar involution, a crystal basis with a simple crystal graph, and a global basis.  $\nabla$

### 6.2. Enhanced $J$ -data

An *enhanced  $J$ -datum* is a  $J$ -datum  $(V(i), X(i))_{i \in J}$  equipped with an involution  $\theta : J \rightarrow J$  such that the following properties are satisfied. Set  $J^\theta = \{i \in J \mid \theta(i) = i\}$ .

- (Q0) **Symmetry.** For any  $i \in J$ , we have  $X(\theta(i)) = X(i)^{-1}$ , and there is an isomorphism of  $U_q\mathfrak{k}$ -modules  $V(\theta(i)) \simeq \psi^*(V(i))$ .
- (Q1) **Admissibility.** For any  $i \in J$ , there is an isomorphism of  $U_q\mathfrak{k}$ -modules  $(\psi^2)^*(V(i)) \simeq V(i)$ .
- (Q2)  **$\theta$ -invariance.** For any  $i \in J^\theta$ , we have  $X(i)^2 = 1$ , and the module  $V(i)$  is *QSP irreducible* (*i.e.*, it is irreducible under restriction to  $U_q\mathfrak{k}$ , cf. §4.3), and there is an isomorphism of  $U_q\mathfrak{k}$ -modules  $\psi^*(V(i)) \simeq V(i)$ .
- (Q3) **K-matrices.** For any  $i \in J$ , there is a non-vanishing *unitary* K-matrix, *i.e.*, a QSP intertwiner

$$\mathbf{K}_{V(i)}(z) : V(i)_z \rightarrow V(\theta(i))_{1/z} \simeq \psi^*(V(i))_{1/z},$$

satisfying Cherednik’s reflection equation (4.2) and the unitarity condition (4.3) (cf. Theorem 4.4.1). Moreover, if  $i \in J^\theta$ , then  $\mathbf{K}_{V(i)}(1) = \text{id}$ .

- (Q4) **Poles.** For any  $i \in J$ , the poles of the K-matrix  $\mathbf{K}_{V(i)}(z)$  are in  $q^{1/m}\mathbb{C}[[q^{1/m}]]$  for some  $m > 0$ .

For any  $i, j \in J$ , we denote by  $d_i(z) \in \mathbb{k}[z]$  the denominator of the trigonometric K-matrix  $\mathbf{K}_{V(i)}(z)$  and by  $d_i \in \mathbb{Z}_{\geq 0}$  the order of the pole of  $\mathbf{K}_{V(i)}(z)$  at  $z = X(i)$ .

**Remark 6.2.1.** As before, the conditions (Q0)–(Q3) are easily satisfied.

- It is useful to observe that the condition (Q0) can be satisfied *by construction*. Namely, given an index set  $J$  with an involution  $\theta$ , choose a representative for any  $\theta$ -orbit and consider the corresponding partition  $J = J_+ \sqcup J^\theta \sqcup J_-$  with  $\theta(J_+) = J_-$ .

Then, choose, for any  $i \in (J_+ \sqcup J^\theta)$ , a finite-dimensional irreducible real  $U_q \mathbf{Lg}$ -module  $V(i)$  and a non-zero scalar  $X(i) \in \mathbb{k}^\times$ . Finally, for any  $\theta(i) \in J_-$ , set  $V(\theta(i)) := \psi^*(V(i))$  and  $X(\theta(i)) := X(i)^{-1}$ .

- The requirement  $(\psi^2)^*(V) \simeq V$  in (Q1) can easily be overcome, since it is always possible to choose involutive twisting operator  $\psi$ , *i.e.*,  $\psi = \omega \circ \tau$ , see [4, Ex. 3.6.3 (2)]. With this choice, Kirillov–Reshetikhin modules satisfy also the condition  $\psi^*(V) \simeq V$  in (Q2) (up to a shift, see [4, Thm. 7.8.1]). Note also that (Q1) follows automatically whenever (Q0) is satisfied.
- By Theorem 4.4.1, the condition  $(\psi^2)^*(V) \simeq V$  guarantees the existence of a unitary K-matrix. Thus, (Q3) is automatically satisfied.

The condition (Q4) is, however, harder to verify, since the properties of the poles of trigonometric K-matrices are at the moment largely unknown.  $\nabla$

### 6.3. Enhanced $J$ -quivers

In [32, §3], Kang, Kashiwara, and Kim defined a quiver attached to a  $J$ -datum, which we refer to as the associated  $J$ -quiver. We extend their construction to a quiver with a framing and a contravariant involution, naturally associated to an enhanced  $J$ -datum.

#### Definition 6.3.1.

- (1) Let  $(V(i), X(i))_{i \in J}$  be a  $J$ -datum. The associated  $J$ -quiver  $\Gamma$  (cf. [32, §3]) is the quiver defined as follows:
  - the set of vertices is  $J$ ;
  - for any  $i, j \in J$ , there are  $d_{ij}$  arrows from  $i$  to  $j$ .
- (2) Let  $(V(i), X(i))_{i \in J}$  be an enhanced  $J$ -datum with involution  $\theta$ . The associated *enhanced*  $J$ -quiver is the  $J$ -quiver associated to the underlying  $J$ -datum, additionally equipped with the framing dimension vector  $\boldsymbol{\lambda} \in \mathbb{N}[J]$  given by  $\boldsymbol{\lambda}(i) = d_{\theta(i)}$ .

The enhanced  $J$ -quiver has several convenient properties, which are easily proved.

**Proposition 6.3.2.** *Let  $\Gamma$  be an enhanced  $J$ -quiver. Then the following holds.*

- (1) *The quiver  $\Gamma$  has neither loops nor cycles.*
- (2) *If  $\boldsymbol{\lambda}(i) \neq 0$  then  $\boldsymbol{\lambda}(\theta(i)) = 0$ . In particular,  $\boldsymbol{\lambda}(i) = 0$  for  $i \in J^\theta$ .*
- (3) *The involution  $\theta$  on  $J$  lifts to a contravariant involution of the quiver  $\Gamma$ .*

**Proof.** Part (1) is proved as in [32]. Namely, since  $V(i)$  is real,  $\mathbf{R}_{V(i)V(i)}(z)$  has no pole at  $z = 1$ , thus  $d_{ii} = 0$ . The condition (P3) then guarantees that no cycle can appear. Part (2) follows similarly from the unitarity condition on the K-matrix (Q3). For part

(3), it is enough to observe that, by definition of the twisting operator  $\psi$ , see [3, §2], one has

$$(\mathbf{R}_{\psi^*(V(i))\psi^*(V(j))}(w/z))_{21} = (F_{V^{(i)}V^{(j)}}^{-1})_{21} \circ \mathbf{R}_{V^{(j)}V^{(i)}}(z/w) \circ F_{V^{(j)}V^{(i)}}$$

where  $F$  is a (constant) Drinfeld twist in  $U_qL\mathfrak{g}$ . Therefore,  $d_{\theta(i)\theta(j)} = d_{ji}$  and the result follows.  $\square$

#### 6.4. Enhanced $J$ -quivers of Dynkin type

It is well-known that every quiver of Dynkin type can be realized as the  $J$ -quiver associated to a  $J$ -datum for a Lie algebra  $\mathfrak{g}$  of the same type (see, e.g., [31,22,23,52]). The analogue result for enhanced  $J$ -quiver is much more restrictive. By Proposition 6.3.2 (3), every enhanced  $J$ -quiver is naturally equipped with a contravariant involution  $\theta$ . However, the latter exists only in the cases of Dynkin quivers of type A, and affine Dynkin quivers of type A and D.

In §6.5–6.7, we provide an explicit realization as enhanced  $J$ -quivers of the following three examples: the linearly oriented quiver of type A; the bipartite quiver of type A; the linearly oriented quiver of affine type D.

We briefly summarize our approach. In analogy with [32,31], the  $J$ -datum  $(V(i), X(i))_{i \in J}$  is given entirely in terms of the fundamental representations  $V_{\omega_i}$  ( $i \in I$ ) of the quantum affine algebra  $U_qL\mathfrak{g}$  of the same type.<sup>6</sup> The definition of the enhanced  $J$ -datum is quite subtle. First, we equip  $J$  with the unique contravariant involution  $\theta$ . Then, we carefully choose the QSP subalgebra in order to satisfy the condition (Q0). We proceed as follows.

- We fix a  $\tau$ -restrictable QSP subalgebra  $U_q\mathfrak{k} \subset U_qL\mathfrak{g}$  with twisting operator  $\psi = \omega \circ \tau$  (cf. §4.5). By Theorem 4.5.1 (1), for any  $i \in J$ , we obtain a trigonometric K-matrix

$$V(i)(z) \rightarrow (\eta_0\tau)^*(V(i))(z^{-1})$$

where  $\eta_0$  is the extension of the opposition involution  $\text{op}_I$ .

- In type A, the contravariant involution  $\theta$  satisfies  $V(\theta(i)) = V(i)$ . Therefore, if  $\tau = \eta_0$ , the symmetry condition (Q0) is automatically satisfied.<sup>7</sup> On the other hand, if  $\tau = \text{id}$ , the same condition fails, since  $\eta_0 \neq \text{id}$  and  $\eta_0^*(V(i)) \not\cong V(i)$ .<sup>8</sup>
- In affine type  $\widehat{D}_N$ , we consider a  $J$ -datum involving only the vector representation  $V_{\omega_1}$  and the two spin representations  $V_{\omega_{N-1}}$  and  $V_{\omega_N}$ . As before, the contravariant involution  $\theta$  satisfies  $V(\theta(i)) = V(i)$ . When  $N$  is even,  $\eta_0 = \text{id}$  and we consider

<sup>6</sup> We shall consider in fact only *small* fundamental representations, i.e., irreducible under restriction to  $U_q\mathfrak{g}$ . In particular, these are isomorphic to their finite type counterparts as  $U_q\mathfrak{g}$ -modules.

<sup>7</sup> Recall that (Q0) requires that, for any  $i \in J$ ,  $V(\theta(i)) \simeq \psi^*(V(i))$ .

<sup>8</sup> This corresponds to the fact that the fundamental representations are not self-dual in type A.

$\tau$ -restrictable QSP subalgebras with  $\tau = \text{id} = \eta_0$ . Thus, (Q0) is automatically satisfied in this case. On the other hand, when  $N$  is odd,  $\eta_0 \neq \text{id}$  and one has  $\eta_0^*(V_{\omega_1}) \simeq V_{\omega_1}$ , but  $\eta_0^*(V_{\omega_{N-1}}) \simeq V_{\omega_N}$ . Thus, in order to satisfy (Q0), in this case we can only consider QSP subalgebras with  $\tau = \eta_0$ .

- In affine type  $\widehat{D}_N$  with  $N$  odd, one can consider a simple modification of the  $J$ -datum such that the two spin representations correspond to each other through  $\theta$ . By the same argument, (Q0) holds for any QSP subalgebra with  $\tau = \text{id}$  (cf. Remark 6.7.2).

Finally, the enhanced  $J$ -datum is obtained by  $\theta$  together with the trigonometric K-matrices given by Theorem 4.5.1 with respect to the QSP subalgebra chosen as above.

**Remark 6.4.1.** In the examples §6.6 and §6.7 below, we do not include a description of the framing.<sup>9</sup> This is due to the fact that, with the exception of the first fundamental representation, an explicit expression for the poles of trigonometric K-matrices is at the moment largely unknown. Even though some progress towards more general Kirillov–Reshetikhin modules has recently been made in [43] from the point of view of crystal combinatorics, no explicit formula for the poles has yet been derived. Nevertheless, the poles are expected to depend rationally on the parameters  $(\gamma, \sigma)$  defining the QSP subalgebra (see, e.g., [57] or also §9.4). In this case, the condition (Q4) would always be satisfied for suitable choices of parameters, yielding a generically trivial framing.  $\nabla$

### 6.5. Example: the linearly oriented quiver of type A

This example is discussed in details in §9.4 in the case of a (not necessarily  $\tau$ -restrictable) quasi-split QSP subalgebra of type AIII. We provide here a brief summary in greater generality.

Let  $\mathfrak{g} = \mathfrak{sl}_{N+1}$  and  $V = V_{\omega_1}$ . Note that, as a  $U_q\mathfrak{g}$ -module,  $V$  is isomorphic to the vector representation. Set  $J := \mathbb{Z}_{\text{odd}}$  and consider the involution  $\theta$  on  $J$  given by  $\theta(n) = -n$ .<sup>10</sup> For any  $i \in J$ , set

$$V(n) := V \quad \text{and} \quad X(n) = q^n.$$

First, we observed that the conditions (P1)–(P3) are clearly satisfied. In particular, (P3) holds, since the R-matrix  $\mathbf{R}_{VV}^\vee(z, w)$  is known to have only one simple pole at  $w/z = q^2$ , see (9.1). The resulting  $J$ -quiver with  $\theta$  is in Fig. 1, where we use the notation  $(\omega_i, p)$  to indicate  $(V_{\omega_i}, p)$ . This is the main example considered in [32].

We fix a QSP subalgebra  $U_q\mathfrak{k} \subset U_qL\mathfrak{g}$  with  $\tau = \eta_0$  and we consider the twisting operator  $\psi = \omega \circ \tau$ . Then, by Theorem 4.5.1, the conditions (Q0), (Q1), (Q2) and (Q3) are automatically satisfied. It remains to verify the condition (Q4). In [58], Regelskis and

<sup>9</sup> For the example §6.5, the framing is described in §9.4 together with an explicit expression of the trigonometric K-matrix for the fundamental representation in type AIII.

<sup>10</sup> In particular,  $J^\theta = \emptyset$ . The case with  $J = \mathbb{Z}_{\text{even}}$  and  $J^\theta = \{0\}$  is analogous, so we omit it.

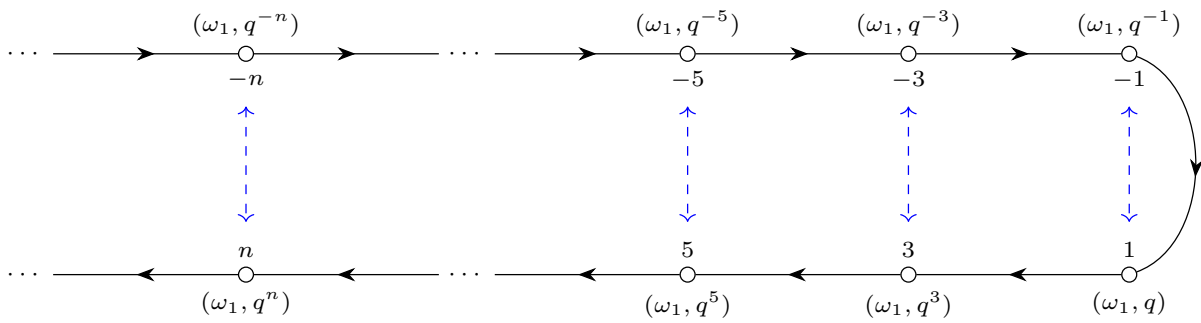


Fig. 1. The case of the linearly oriented  $A_\infty$  quiver.

Vlaar computed explicitly the trigonometric  $K$ -matrices for  $V$ . Therefore, (Q4) holds, provided the QSP parameters are carefully chosen, as outlined in §9.4.

6.6. Example: the bipartite quiver of type A

We shall now discuss the enhancement of an example from [31] involving multiple fundamental representations. Let  $\mathfrak{g} = \mathfrak{sl}_{N+1}$ . Let  $J = \{1, \dots, N\}$  be a set with the involution  $\theta(i) = N + 1 - i$ . We consider  $J$  as the set of vertices of a quiver  $Q$  of type  $A_N$ , with an edge between  $i$  and  $j$  if and only if  $|i - j| = 1$ , and equipped with a bipartite (i.e., alternating) orientation.

Let  $U_q \mathfrak{k} \subset U_q L \mathfrak{g}$  be a QSP subalgebra with  $\tau = \eta_0$  and twisting operator  $\psi = \tau \circ \omega$ . We consider the corresponding trigonometric  $K$ -matrices on the fundamental representations  $V_{\omega_i}$  ( $i \in I$ ) given by Theorem 4.5.1.

**Proposition 6.6.1.** Set  $p = -q$ .

- (1) The quiver  $Q$  admits a contravariant involution if and only if  $N$  is even.
- (2) Let  $N = 2m$  and set, for any  $i \in J$ ,

$$V(i) = \begin{cases} V_{\omega_i} & \text{if } i \text{ is a source in } Q \\ V_{\omega_{\theta(i)}} & \text{if } i \text{ is a sink in } Q \end{cases}$$

$$X(i) = \begin{cases} p^m & \text{if } i \text{ is a source in } Q \\ p^{-m} & \text{if } i \text{ is a sink in } Q \end{cases}$$

If the condition (Q4) is satisfied, the datum  $(V(i), X(i))_{i \in J}$  defines an enhanced  $J$ -datum with respect to the involution  $\theta$ . The resulting  $J$ -quiver  $\Gamma$  is isomorphic to  $Q^{rev}$ , i.e., the quiver obtained from  $Q$  by reversing all the arrows, see Fig. 2.

**Proof.** (1) The existence of an involution when  $N = 2m$  is obvious from Fig. 2. Instead, when  $N = 2m + 1$ , it is enough to observe that the central vertex  $m + 1$  is bound to

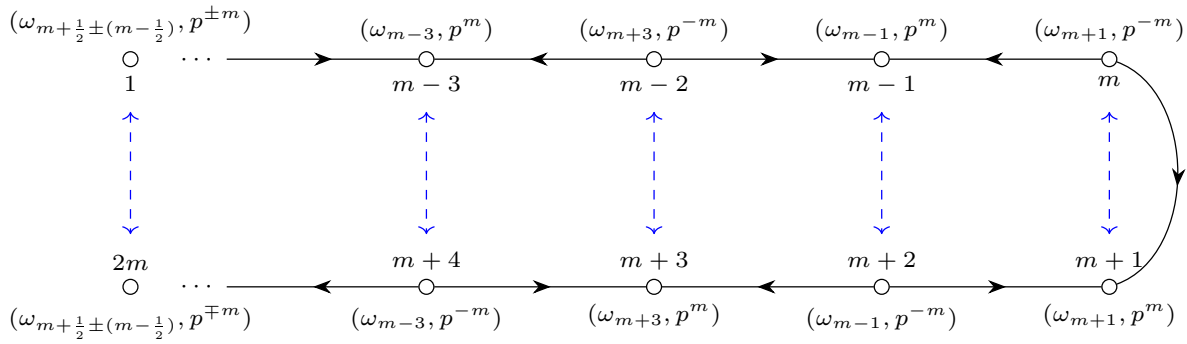


Fig. 2. The case of the bipartite  $A_{2m}$  quiver.

be fixed by the involution. Since  $m + 1$  is supposed to be either a source or a sink, the involution cannot be contravariant.

(2) The proof relies on some results from [31]. Let  $w$  be a Coxeter element adapted to  $Q$ . Without loss of generality, let us assume that  $m$  is odd, and the central arrow sends  $m + 1$  to  $m$ . In that case even vertices are sources and odd vertices are sinks, and

$$w = s_2 s_4 \cdots s_{2m} s_1 s_3 \cdots s_{2m-1}.$$

Choose the height function

$$\xi_i = \begin{cases} m + 1 & \text{if } i \text{ is odd,} \\ m & \text{if } i \text{ is even.} \end{cases}$$

Let  $\Pi_0$  and  $\Delta_+$  denote the sets of simple and positive roots of  $\mathfrak{g}$ , respectively. By [31, Thm. 4.3.1], there exists a  $J$ -datum such that the corresponding  $J$ -quiver  $\Gamma$  is isomorphic to  $Q^{rev}$ . To compute this datum, we need to consider the repetition quiver of  $Q$ , and a certain bijection  $\phi: \widehat{I}_0 \rightarrow \widehat{\Delta}$ , where  $\widehat{I}_0 = \{1, 3, \dots, m\} \times 2\mathbb{Z} \sqcup \{2, 4, \dots, 2m\} \times \mathbb{Z}_{\text{odd}}$  and  $\widehat{\Delta} = \Delta_+ \times \mathbb{Z}$ . The bijection is defined recursively according to the rules in [31, §3.1]. To determine the  $J$ -datum, we need to find the preimage of  $\Pi_0 \times \{0\}$  under  $\phi$ .

For any source vertex  $i$ , we have, by definition,  $\phi(i, m) = (\alpha_i, 0)$ . We also claim that  $w^m(\alpha_i) = \alpha_{\theta(i)}$ , so that, by the recursive rule in the definition of  $\phi$ , we have  $\phi(i, -m) = (\alpha_{\theta(i)}, 0)$ . Then, we have

$$V(i) = \text{pr}_1 \circ \phi^{-1}(\alpha_i, 0), \quad X(i) = \text{pr}_2 \circ \phi^{-1}(\alpha_i, 0),$$

which, according to [31, Thm. 4.3.1], defines the desired  $J$ -datum.

It remains to prove the claim. Write  $i = 2j$  and assume, without loss of generality, that  $i < m$ . An easy calculation shows that  $w^j(\alpha_i) = \alpha_1 + \dots + \alpha_{2i}$ , and that the subsequent  $m - i$  applications of  $w$  shift the string of simple roots by  $2(m - i)$  so that  $w^{m-j}(\alpha_i) = \theta(w^j(\alpha_i))$ . The subsequent  $j$  applications of  $w$  shrink the string to  $\alpha_{\theta(i)}$ .

From Theorem 4.5.1, the conditions (Q0)–(Q3) are automatically satisfied (see also §6.4).  $\square$

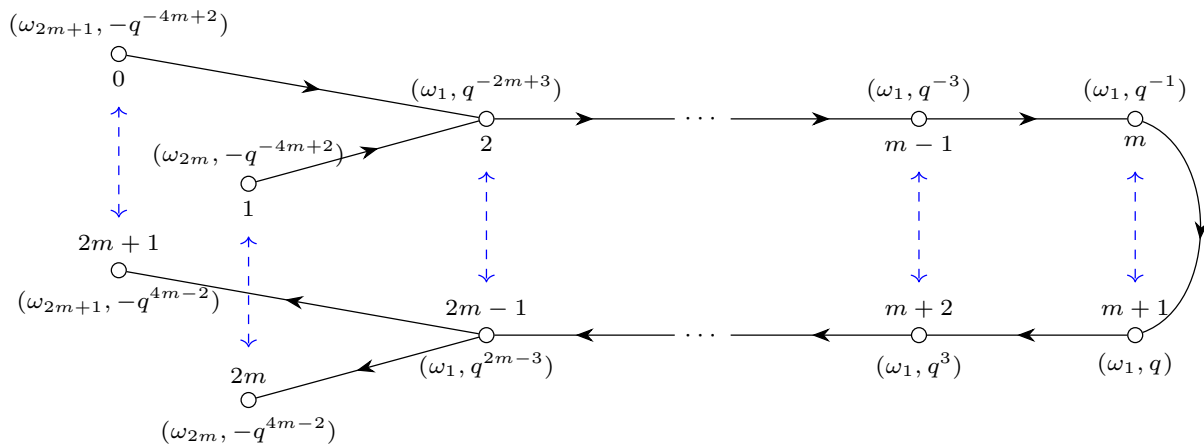


Fig. 3. The case of the linearly oriented affine  $D_{2m+1}$  quiver.

6.7. Example: the quiver of affine type D

We conclude with the example of a quiver of type other than A. Let  $\mathfrak{g} = \mathfrak{so}_{2N}$ . As before, let  $U_q\mathfrak{k} \subset U_qL\mathfrak{g}$  be a QSP subalgebra with  $\tau = \eta_0$ , and set  $\psi = \tau \circ \omega$ . We shall construct an enhanced  $J$ -datum relying on the fundamental representations  $V_{\omega_1}, V_{\omega_{N-1}}, V_{\omega_N}$ , where the vertices of the affine Dynkin diagram of  $\mathfrak{g}$  are numbered as in Fig. 3. Note that all three modules are small, and, as  $U_q\mathfrak{g}$ -modules, are isomorphic to the vector representation and the spin representations, respectively.

Set  $J = \{0, 1, \dots, N\}$  and let  $\theta$  be the involution on  $J$  given by  $\theta(i) = N - i$ . We consider the assignments

$$V(i) = \begin{cases} V_{\omega_{N-i}} & \text{if } i = 0, 1 \\ V_{\omega_i} & \text{if } i = N - 1, N \\ V_{\omega_1} & \text{if } i = 2, \dots, N - 2 \end{cases}$$

and

$$X(i) = \begin{cases} (-1)^N q^{-2(N-2)} & \text{if } i = 0, 1 \\ (-1)^N q^{2(N-2)} & \text{if } i = N - 1, N \\ q^{2i-N} & \text{if } i = 2, \dots, N - 2 \end{cases}$$

**Proposition 6.7.1.** *If the condition (Q4) is satisfied, the datum  $(V(i), X(i))_{i \in J}$  defines an enhanced  $J$ -datum with respect to the involution  $\theta$ . The resulting  $J$ -quiver is represented in Fig. 3 for  $N$  odd, and in Fig. 4 for  $N$  even.*

**Proof.** The proof relies entirely on the explicit computation of the denominators of the  $R$ -matrix between all fundamental representations in affine type D, provided in [31, Thm. A.1.1]. In particular, one has that

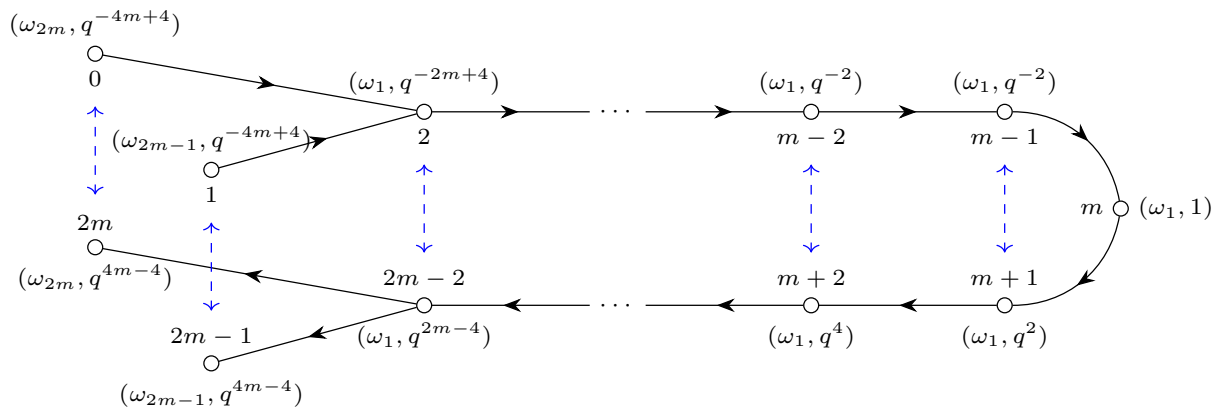


Fig. 4. The case of the linearly oriented affine  $D_{2m}$  quiver.

$$d_{11}(z) = (z - q^2)(z - q^{2N-2}), \quad d_{(N-1)(N-1)}(z) = \prod_{s=1}^{\lfloor \frac{N}{2} \rfloor} (z - (-q)^{4s-2}) = d_{NN}(z),$$

$$d_{1N}(z) = z - (-q)^N = d_{1(N-1)}(z), \quad d_{(N-1)N}(z) = \prod_{s=1}^{\lfloor \frac{N-1}{2} \rfloor} (z - (-q)^{4s}).$$

We immediately observe that, by restriction to the vertices  $J' = \{2, \dots, N - 2\}$ , the datum  $(V(i), X(i))_{i \in J'}$  gives rise to a linearly oriented quiver of type  $A_{N-3}$ , as in the finite analogue of Fig. 1. Then, by the denominators formulae above, the spin representations placed at the four extremal vertices are shown to a single arrow each, towards either incoming to  $V(2)$  or outgoing from  $V(N - 2)$ . The resulting  $J$ -quiver are represented in Figs. 3 and 4 when  $N$  is odd or even, respectively. As before, one finally observes that, provided the condition (Q4) is satisfied, these assignments give rise to an enhanced  $J$ -datum with respect to  $\theta$ .  $\square$

**Remark 6.7.2.** As mentioned in §6.4, the case with  $N$  odd is of further interest, since in this case the spin representations correspond to each other through  $\eta_0$ . Let  $U_q\mathfrak{k} \subset U_qL\mathfrak{g}$  be a  $\tau$ -restrictable QSP subalgebra with  $\tau = \text{id}$  and set

$$\theta'(i) = \begin{cases} N - i & \text{if } i = 2, \dots, N - 2 \\ N - 1 - i & \text{if } i = 0, 1 \\ N + 1 - i & \text{if } i = N - 1, N \end{cases}$$

Then, it follows immediately from Theorem 4.5.1 that, provided the condition (Q4) is satisfied, the previous assignments give rise to an enhanced  $J$ -datum with respect to  $\theta'$ .  $\nabla$

### 7. Boundary Schur–Weyl dualities

In this section, we prove the main result of the paper, consisting in the construction of a *boundary* Schur–Weyl duality functor. We fix the following datum:

- an affine Satake diagram  $(X, \tau)$  with pseudo-involution  $\vartheta$ ;
- a QSP subalgebra  $U_q\mathfrak{k} \subseteq U_qL\mathfrak{g}$  with parameters  $(\gamma, \sigma)$ ;
- a QSP admissible twisting operator  $\psi \in \text{Aut}(U_qL\mathfrak{g})$ ;
- a set  $J$  with an involution  $\theta$ ;
- an enhanced  $J$ -datum  $(V(i), X(i))_{i \in J}$ ;
- the corresponding enhanced  $J$ -quiver  $(\Gamma, \theta, \lambda)$ , see Definition 6.3.1.

Henceforth we will consider  $o$  KLR algebras associated to  $(\Gamma, \theta, \lambda)$  and varying dimension vectors  $\beta \in \mathbb{N}[J]^\theta$ . Moreover, we fix the functions

$$c_{ij}(u, v) \in \mathbb{k}[[u, v]], \quad c_i \in \mathbb{k}[[u]]$$

as in (5.14)–(5.15).

#### 7.1. Polynomial rings

As preparation for the main result, we need to introduce notation for various polynomial rings, their completions, and relate them to the polynomial rings that appeared earlier in §5.3 in the context of  $o$  KLR algebras. Let:

$$\mathcal{O} := \mathbb{k}[X_{\pm 1}, \dots, X_{\pm n}], \quad \mathcal{K} := \mathbb{k}(X_1, \dots, X_n),$$

where  $X_{-l} = X_l^{-1}$ . The group  $\mathfrak{W}_n$  acts on  $\mathcal{O}$  and  $\mathcal{K}$  from the left by  $w \cdot X_l = X_{w(l)}$ .

Given a self-dual dimension vector  $\beta \in \mathbb{N}[J]^\theta$  with  $\|\beta\|_\theta$ , we also set

$$\begin{aligned} \widehat{\mathcal{O}}_\nu &:= \mathbb{k}[[X_1 - X(\nu_1), \dots, X_n - X(\nu_n)]], & \theta \widehat{\mathcal{O}}_\beta &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widehat{\mathcal{O}}_\nu e(\nu), \\ \widehat{\mathcal{K}}_\nu &:= \mathbb{k}((X_1 - X(\nu_1), \dots, X_n - X(\nu_n))), & \theta \widehat{\mathcal{K}}_\beta &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widehat{\mathcal{K}}_\nu e(\nu). \end{aligned}$$

**Lemma 7.1.1.** *There is a  $\mathfrak{W}_n$ -equivariant algebra isomorphism*

$$\theta \widehat{\mathbb{P}}_\beta \xrightarrow{\sim} \theta \widehat{\mathcal{O}}_\beta, \quad e(\nu) \mapsto e(\nu), \quad x_k e(\nu) \mapsto \left( \frac{X(\nu_k)}{X_k} - \frac{X_k}{X(\nu_k)} \right) e(\nu). \tag{7.1}$$

**Proof.** Observe that

$$(X(\nu_k)X_k^{-1} - X(\nu_k)^{-1}X_k) = f(1 - X(\nu_k)^{-1}X_k),$$

where

$$f(z) = z + \frac{z}{1-z} = 2z + \sum_{k \geq 2} z^k \in \mathbb{K}[[z]].$$

Hence (7.1) is well-defined. Since the constant coefficient of  $f$  vanishes and the degree one coefficient is invertible,  $f$  has a composition inverse. Therefore, (7.1) is an isomorphism. The equivariance is clear.  $\square$

We identify

$${}^\theta \widehat{\mathbb{K}}_\beta \rtimes \mathbb{K}[\mathfrak{W}_n] \xrightarrow{\sim} {}^\theta \widehat{\mathcal{K}}_\beta \rtimes \mathbb{K}[\mathfrak{W}_n]$$

using (7.1) and, following (5.16), we regard the oKLR algebra  ${}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})$  as a subalgebra of  ${}^\theta \widehat{\mathcal{K}}_\beta \rtimes \mathbb{K}[\mathfrak{W}_n]$ .

## 7.2. Boundary Schur-Weyl duality functor

We will now construct a  $(U_q \mathfrak{k}, {}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda}))$ -bimodule and the associated Schur-Weyl functor.

For each  $\nu \in {}^\theta J^\beta$ , set

$$V_\nu := V(\nu_1)_{z_{\nu,1}} \otimes \cdots \otimes V(\nu_n)_{z_{\nu,n}} e(\nu).$$

It is a right  $\mathcal{O} \otimes U_q \mathfrak{Lg}$ -module, with  $X_k$  acting as  $z_{\nu,k}$ . Set

$$\begin{aligned} \widehat{V}_\nu &:= \widehat{\mathcal{O}}_\nu \otimes_{\mathcal{O}} V_\nu, & {}^\theta \widehat{V}^{\otimes \beta} &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widehat{V}_\nu, \\ \widetilde{V}_\nu &:= \widehat{\mathcal{K}}_\nu \otimes_{\mathcal{O}} V_\nu, & {}^\theta \widetilde{V}^{\otimes \beta} &:= \bigoplus_{\nu \in {}^\theta J^\beta} \widetilde{V}_\nu. \end{aligned}$$

### Theorem 7.2.1.

- (1) The space  ${}^\theta \widetilde{V}^{\otimes \beta}$  has a natural structure of a  $(U_q \mathfrak{k}, {}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda}))$ -bimodule induced by the action of the trigonometric  $R$ - and  $K$ -matrices.
- (2) The subspace  ${}^\theta \widehat{V}^{\otimes \beta} \subset {}^\theta \widetilde{V}^{\otimes \beta}$  is stable under the right action of the subalgebra  ${}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})$  of  ${}^\theta \widehat{\mathcal{K}}_\beta \rtimes \mathbb{K}[\mathfrak{W}_n]$ . In particular,  ${}^\theta \widehat{V}^{\otimes \beta}$  has the structure of  $(U_q \mathfrak{k}, {}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda}))$ -bimodule.
- (3) There is a right exact functor

$${}^\theta F_\beta: {}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})\text{-mod}_{\text{gr}} \longrightarrow U_q \mathfrak{k}\text{-mod}, \quad M \mapsto {}^\theta \widehat{V}^{\otimes \beta} \otimes_{{}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})} M.$$

Summing over all self-dual dimension vectors, we get

$${}^\theta F := \bigoplus_{\beta \in \mathbb{N}[J]^\theta} {}^\theta F_\beta : {}^\theta \mathcal{R}(\boldsymbol{\lambda})\text{-mod}_{\text{gr}} \longrightarrow U_q \mathfrak{k}\text{-mod}.$$

**Proof.** (1) We first endow  ${}^\theta \widetilde{V}^{\otimes \beta}$  with the structure of a right  ${}^\theta \widehat{\mathcal{K}}_\beta \rtimes \mathbb{k}[\mathfrak{W}_n]$ -module. For each  $\nu \in {}^\theta J^\beta$  and  $k = 1, \dots, n - 1$ , let  $\mathbf{R}_k^\nu : \widetilde{V}_\nu \rightarrow \widetilde{V}_{s_k(\nu)}$  be the intertwiner defined as follows. We first consider the  $\mathbb{k}$ -linear map  $V_\nu \rightarrow V_{s_k(\nu)}$  given by the composition of the operator

$$v_1 \otimes \dots \otimes v_n \mapsto v_1 \otimes \dots \otimes \mathbf{R}_{V(\nu_k), V(\nu_{k+1})}^\nu(z_{\nu, k+1}/z_{\nu, k}) \cdot (v_k \otimes v_{k+1}) \otimes \dots \otimes v_n$$

with the map identifying  $z_{\nu, k}$  (resp.  $z_{\nu, k+1}$ ) with  $z_{s_k(\nu), k+1}$  (resp.  $z_{s_k(\nu), k}$ ) and  $z_{\nu, \ell}$  with  $z_{s_k(\nu), \ell}$  for  $\ell \neq k, k + 1$ . We then extend it by  $\mathcal{O}$ -linearity. Similarly, let  $\mathbf{K}^\nu : \widetilde{V}_\nu \rightarrow \widetilde{V}_{s_0(\nu)}$  be the QSP intertwiner defined as follows. We first consider the  $\mathbb{k}$ -linear map  $V_\nu \rightarrow V_{s_0(\nu)}$  given by the composition of the operator

$$v_1 \otimes \dots \otimes v_n \mapsto \mathbf{K}_{V(\nu_1)}(z_1) \cdot (v_1) \otimes v_2 \otimes \dots \otimes v_n,$$

with the map identifying  $z_{\nu, 1}$  with  $z_{s_0(\nu), 1}^{-1}$  and  $z_{\nu, \ell}$  with  $z_{s_0(\nu), \ell}$  for  $\ell \neq 1$ . We then extend it by  $\mathcal{O}$ -linearity. From the Yang-Baxter equation (2.2) and the unitarity condition (2.3), it follows that

$$\begin{aligned} \mathbf{R}_k^\nu \circ X_l &= X_{s_k(l)} \circ \mathbf{R}_k^\nu, \\ \mathbf{R}_k^{s_k(\nu)} \circ \mathbf{R}_k^\nu &= \text{id}_{V_\nu}, \\ \mathbf{R}_k^{s_{k+1}s_k(\nu)} \circ \mathbf{R}_{k+1}^{s_k(\nu)} \circ \mathbf{R}_k^\nu &= \mathbf{R}_{k+1}^{s_k s_{k+1}(\nu)} \circ \mathbf{R}_k^{s_{k+1}(\nu)} \circ \mathbf{R}_{k+1}^\nu. \end{aligned}$$

Instead, from Cherednik’s generalized reflection equation (4.2) and the unitarity condition (4.3), it follows that

$$\begin{aligned} \mathbf{K}^\nu \circ X_k &= X_{s_0(k)} \circ \mathbf{K}^\nu, \\ \mathbf{K}^{s_0(\nu)} \circ \mathbf{K}^\nu &= \text{id}_{V_\nu}, \\ \mathbf{K}^{s_1 s_0 s_1(\nu)} \circ \mathbf{R}_1^{s_0 s_1(\nu)} \circ \mathbf{K}^{s_1(\nu)} \circ \mathbf{R}_1^\nu &= \mathbf{R}_1^{s_0 s_1 s_0(\nu)} \circ \mathbf{K}^{s_1 s_0(\nu)} \circ \mathbf{R}_1^{s_0(\nu)} \circ \mathbf{K}^\nu. \end{aligned}$$

It follows that letting  ${}^\theta \widehat{\mathcal{K}}_\beta$  act by multiplication and assigning

$$e(\nu)s_0 \mapsto \mathbf{K}^\nu, \quad e(\nu)s_k \mapsto \mathbf{R}_k^\nu$$

makes  ${}^\theta \widetilde{V}^{\otimes \beta}$  into a  $(U_q \mathfrak{k}, {}^\theta \widehat{\mathcal{K}}_\beta \rtimes \mathbb{k}[\mathfrak{W}_n])$ -bimodule.

(2) It is enough to show that  ${}^\theta \widehat{V}^{\otimes \beta}$  is stable under the action of the generators  $\tau_k \in {}^\theta \mathcal{R}(\beta; \boldsymbol{\lambda})$ . First assume that  $1 \leq k \leq n - 1$ . The proof is similar to that of [32, Thm. 3.1.3]. There are two cases to be considered. First suppose that  $\nu_k \neq \nu_{k+1}$ . Then

$$e(\nu)\tilde{P}_{\nu_{k+1},\nu_k}(x_k, x_{k+1}) = e(\nu)d_{\nu_k,\nu_{k+1}}(X_{k+1}/X_k) \frac{(x_k - x_{k+1})^{d_{\nu_k,\nu_{k+1}}} c_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})}{d_{\nu_k,\nu_{k+1}}(X_{k+1}/X_k)}.$$

Let  $f$  denote the fraction on the RHS. One easily calculates that

$$x_k - x_{k+1} = (X_k/X(\nu_{k+1}) + X(\nu_k)/X_{k+1})(X_{k+1}/X_k - X(\nu_{k+1})/X(\nu_k)),$$

where the first factor on the RHS is an invertible element of  $\widehat{\mathcal{O}}_\nu$ . Since  $d_{\nu_k,\nu_{k+1}}$  is the multiplicity of the polynomial  $d_{\nu_k,\nu_{k+1}}(X_{k+1}/X_k)$  at  $X_{k+1}/X_k = X(\nu_{k+1})/X(\nu_k)$ , it follows that  $f \in \widehat{\mathcal{O}}_\nu$ . Hence

$$\widehat{V}_\nu \tau_k \subset \widehat{V}_\nu \tilde{P}_{\nu_{k+1},\nu_k}(x_k, x_{k+1})s_k \subset \widehat{V}_\nu d_{\nu_k,\nu_{k+1}}(X_{k+1}/X_k)s_k,$$

which, by definition, is contained in  $\widehat{V}_{s_k \cdot \nu}$ .

Secondly, suppose that  $\nu_k = \nu_{k+1}$ . By (P2), we have

$$\mathbf{R}_{V(\nu_k),V(\nu_k)}^\vee(X_{k+1}/X_k)|_{X_{k+1}/X_k=1} = \text{id}.$$

Hence  $\mathbf{R}_{V(\nu_k),V(\nu_k)}^\vee(X_{k+1}/X_k) - \text{id}$  has a zero at  $X_{k+1}/X_k = X(\nu_{k+1})/X(\nu_k) = 1$  and so

$$\begin{aligned} \widehat{V}_\nu \tau_k &= \widehat{V}_\nu (x_k - x_{k+1})^{-1}(s_k - 1) \\ &\subset \widehat{V}_\nu (X_{k+1}/X_k - 1)^{-1}(\mathbf{R}_{V(\nu_k),V(\nu_k)}^\vee(X_{k+1}/X_k) - 1) \subset \widehat{V}_\nu. \end{aligned}$$

Next, let  $k = 0$ . Again, there are two cases to be considered. First suppose that  $\theta(\nu_1) \neq \nu_1$ . Then

$$e(\nu)\tilde{P}_{\theta(\nu_1)}(x_1) = e(\nu)d_{\nu_1}(X_1) \frac{x_1^{d_{\nu_1}} c_{\nu_1}(x_1)}{d_{\nu_1}(X_1)}.$$

Let  $f$  denote the fraction on the RHS. Note that

$$x_1 = -(X(\nu_1)X_1)^{-1}(X_1 - X(\nu_1))(X_1 + X(\nu_1)).$$

Since  $d_{\nu_1}$  is the multiplicity of the polynomial  $d_{\nu_1}(X_1)$  at  $X_1 = X(\nu_1)$ , it follows that  $f \in \widehat{\mathcal{O}}_\nu$ . Hence

$$\widehat{V}_\nu \tau_0 \subset \widehat{V}_\nu \tilde{P}_{\theta(\nu_1)}(x_1)s_0 \subset \widehat{V}_\nu d_{\nu_1}(X_1)s_0,$$

which, by definition, is contained in  $\widehat{V}_{s_k \cdot \nu}$ .

Secondly, suppose that  $\nu_1 = \theta(\nu_1)$ . Then (Q0) implies that  $X(\nu_1) = \pm 1$ . By (Q3), we have  $\mathbf{K}_{V(\nu_1)}(X_1)|_{X_1=\pm 1} = \text{id}$ . Hence  $\mathbf{K}_{V(\nu_1)}(X_1) - \text{id}$  has a zero at  $X_1 = X(\nu_1) = \pm 1$  and so

$$\widehat{V}_\nu \tau_0 = \widehat{V}_\nu x_1^{-1} (s_0 - 1) \subset \widehat{V}_\nu (X_1 \mp 1)^{-1} (\mathbf{K}_{V(\nu_1)}(X_1) - 1) \subset \widehat{V}_\nu.$$

Part (3) follows immediately from the bimodule structure.  $\square$

### 7.3. Compatibility with the Kang–Kashiwara–Kim functor

We now consider the relationship between  ${}^\theta\mathbf{F}$  and the Kang–Kashiwara–Kim functor from [32]. Take  $\alpha \in \mathbb{N}[J]$  with  ${}^\theta\alpha = \beta$ . Set

$$\widehat{\mathcal{O}}_\alpha := \bigoplus_{\nu \in J^\alpha} \widehat{\mathcal{O}}_\nu e(\nu), \quad \widehat{\mathcal{K}}_\alpha := \bigoplus_{\nu \in J^\alpha} \widehat{\mathcal{K}}_\nu e(\nu), \quad \widehat{V}^{\otimes \alpha} := \bigoplus_{\nu \in J^\alpha} \widehat{V}_\nu, \quad \widetilde{V}^{\otimes \alpha} := \bigoplus_{\nu \in J^\alpha} \widetilde{V}_\nu$$

Note that there is a  $\mathfrak{S}_n$ -equivariant algebra isomorphism

$$\widehat{\mathbb{P}}_\alpha \xrightarrow{\sim} \widehat{\mathcal{O}}_\alpha, \quad e(\nu) \mapsto e(\nu), \quad x_k e(\nu) \mapsto \left( \frac{X_k}{X(\nu_k)} - 1 \right) e(\nu). \quad (7.2)$$

We identify

$$\widehat{\mathbb{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n] \xrightarrow{\sim} \widehat{\mathcal{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n]$$

using (7.2), and consider  $\mathcal{R}(\alpha)$  as a subalgebra of  $\widehat{\mathcal{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n]$  via (5.17). The right action of  ${}^\theta\widehat{\mathcal{K}}_\beta \rtimes \mathbb{k}[\mathfrak{S}_n]$  on  ${}^\theta\widetilde{V}^{\otimes \beta}$  restricts to a  $\widehat{\mathcal{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n]$ -action on  $\widetilde{V}^{\otimes \alpha}$ . This action commutes with the left action of  $U_q\mathbf{Lg}$  and yields the following result due to Kang, Kashiwara, and Kim [32, Thm. 3.1.3 and 3.2.1].

**Theorem 7.3.1** ([32]).

- (1) *The subspace  $\widehat{V}^{\otimes \alpha} \subset \widetilde{V}^{\otimes \alpha}$  is stable under the right action of the subalgebra  $\mathcal{R}(\alpha)$  of  $\widehat{\mathcal{K}}_\alpha \rtimes \mathbb{k}[\mathfrak{S}_n]$ . In particular,  $\widehat{V}^{\otimes \alpha}$  has the structure of  $(U_q\mathbf{Lg}, \mathcal{R}(\alpha))$ -bimodule and yields a functor*

$$F_\alpha : \mathcal{R}\text{-mod}_{\text{gr}} \longrightarrow U_q\mathbf{Lg}\text{-mod}, \quad M \mapsto \widehat{V}^{\otimes \alpha} \otimes_{\mathcal{R}(\alpha)} M.$$

*Summing over all the dimension vectors, one gets*

$$F := \bigoplus_{\alpha \in \mathbb{N}[J]} F_\alpha : \mathcal{R}\text{-mod}_{\text{gr}} \longrightarrow U_q\mathbf{Lg}\text{-mod}. \quad (7.3)$$

- (2) *The functor (7.3) is a tensor functor, which preserves finite-dimensional modules:*

$$F : \mathcal{R}\text{-mod}_{\text{gr}}^{\text{fd}} \longrightarrow U_q\mathbf{Lg}\text{-mod}^{\text{fd}}.$$

We now prove that the functor  ${}^\theta\mathbf{F}$  is compatible with  $F$ , i.e., it is a functor of module categories over  $\mathcal{R}\text{-mod}_{\text{gr}}^{\text{fd}}$  and  $U_q\mathbf{Lg}\text{-mod}^{\text{fd}}$ , respectively.

**Theorem 7.3.2.** *The functors  $({}^\theta\mathbf{F}, \mathbf{F})$  intertwine the two monoidal actions:*

$${}^\theta\mathcal{R}(\boldsymbol{\lambda})\text{-mod}_{\text{gr}} \curvearrowright \mathcal{R}\text{-mod}_{\text{gr}}, \quad U_q\mathfrak{k}\text{-mod} \curvearrowright U_qL\mathfrak{g}\text{-mod}.$$

**Proof.** We need to show that there are natural isomorphisms

$${}^\theta\mathbf{F}(M \otimes N) \cong {}^\theta\mathbf{F}(M) \otimes \mathbf{F}(N), \quad (7.4)$$

for all  $M \in {}^\theta\mathcal{R}(\boldsymbol{\lambda})\text{-mod}_{\text{gr}}$  and  $N \in \mathcal{R}\text{-mod}_{\text{gr}}$ .

Let  $\beta = \beta_1 + {}^\theta\beta_2$  for some  $\beta_1 \in \mathbb{N}[J]$  and  $\beta_2 \in \mathbb{N}[J]^\theta$  with  $\|\beta_1\|_\theta = m$  and  $\|\beta_2\| = n$ . For each  $\nu \in {}^\theta J^\beta$  with  $\nu' = \nu_1, \dots, \nu_m \in {}^\theta J^{\beta_1}$  and  $\nu'' = \nu_{m+1}, \dots, \nu_{n+m} \in J^{\beta_2}$ , there is an algebra homomorphism  $\widehat{\mathcal{O}}_{\nu'} \otimes \widehat{\mathcal{O}}_{\nu''} \rightarrow \widehat{\mathcal{O}}_\nu$ , restricting to the identity map on  $\widehat{\mathcal{O}}_{\nu'}$  and sending  $1 - X_{m+k}X(\nu_{m+k})^{-1}$  to  $\mathbf{f}(1 - X_{m+k}X(\nu_{m+k})^{-1})$  for  $1 \leq k \leq n$ . For any finite-dimensional  $\widehat{\mathcal{O}}_{\nu'}$ -module  $L_1$  and any finite-dimensional  $\widehat{\mathcal{O}}_{\nu''}$ -module  $L_2$ , the induced morphism

$$L_1 \otimes L_2 \rightarrow \widehat{\mathcal{O}}_\nu \otimes_{\widehat{\mathcal{O}}_{\nu'} \otimes \widehat{\mathcal{O}}_{\nu''}} (L_1 \otimes L_2)$$

is an isomorphism. It follows that for any finite-dimensional  ${}^\theta\widehat{\mathcal{O}}_{\beta_1}$ -module  $L_1$  and any finite-dimensional  $\widehat{\mathcal{O}}_{\beta_2}$ -module  $L_2$ , the induced morphism

$$({}^\theta\widehat{V}^{\otimes\beta_1} \otimes \widehat{V}^{\otimes\beta_2}) \otimes_{\theta\widehat{\mathcal{O}}_{\beta_1} \otimes \widehat{\mathcal{O}}_{\beta_2}} (L_1 \otimes L_2) \rightarrow {}^\theta\widehat{V}^{\otimes\beta} \otimes_{\theta\widehat{\mathcal{O}}_{\beta_1} \otimes \widehat{\mathcal{O}}_{\beta_2}} (L_1 \otimes L_2) \quad (7.5)$$

is also an isomorphism.

The module  ${}^\theta\widehat{V}^{\otimes\beta} \otimes_{\theta\mathcal{R}(\beta; \boldsymbol{\lambda})} (M \otimes N) \cong {}^\theta\widehat{V}^{\otimes\beta} \otimes_{\theta\mathcal{R}(\beta_1, \beta_2; \boldsymbol{\lambda})} (M \otimes N)$  is the quotient of  ${}^\theta\widehat{V}^{\otimes\beta} \otimes_{\theta\widehat{\mathcal{O}}_{\beta_1} \otimes \widehat{\mathcal{O}}_{\beta_2}} (M \otimes N)$  by the submodule generated by  $ur \otimes v - u \otimes rv$ , where  $r \in {}^\theta\mathcal{R}(\beta_1, \beta_2; \boldsymbol{\lambda})$ ,  $u \in M \otimes N$  and  $v \in {}^\theta\widehat{V}^{\otimes\beta}$ . An analogous statement holds for  $({}^\theta\widehat{V}^{\otimes\beta_1} \otimes \widehat{V}^{\otimes\beta_2}) \otimes_{\theta\mathcal{R}(\beta_1, \beta_2; \boldsymbol{\lambda})} (M \otimes N)$ . This, together with (7.5), implies the existence of an isomorphism (7.4). It is routine to check that the conditions from [24, (8)–(9)] are also satisfied.  $\square$

#### 7.4. Basic properties of the Schur–Weyl functor

We now prove several basic properties of  ${}^\theta\mathbf{F}$ .

**Proposition 7.4.1.** *The functor  ${}^\theta\mathbf{F}$  has the following properties.*

- (1)  ${}^\theta\mathbf{F}$  preserves finite-dimensional modules.
- (2) For any  $\mathcal{R}(\alpha)$ -module  $M$ , we have

$${}^\theta\mathbf{F}(\mathbb{1} \otimes M) = \mathbf{F}(M)|_{U_q\mathfrak{k}}.$$

(3) For any  $i \in J$  such that  $i \notin J^\theta$  and  ${}^\theta\lambda(i) \geq 1$ , we have

$${}^\theta F({}^\theta L(i)) \cong \text{coker } K_i(0),$$

where  $K_i(x_1) = x_1^{d_i} c_i(x_1) \mathbf{K}_{V(i)}(x_1)$  and  $x_1 = (X(i)X_1^{-1} - X(i)^{-1}X_1)$ .

**Proof.** Part (1) is obvious and (2) follows immediately from Theorem 7.3.2. Let us prove (3). By Lemma 5.4.1, the  ${}^\theta\mathcal{R}(\beta; \lambda)$ -module  ${}^\theta L(i)$ , with  $\beta = i + \theta(i)$ , is well-defined. Let us abbreviate  $j = \theta(i)$ . By definition,

$${}^\theta F({}^\theta L(i)) = (\widehat{V}_i \oplus \widehat{V}_j) \otimes_{{}^\theta\mathcal{R}(\beta; \lambda)} {}^\theta L(i),$$

which is the quotient of  $(\widehat{V}_i \oplus \widehat{V}_j) \otimes {}^\theta L(i)$  by the subspace  $N$  spanned by elements of the form  $v \otimes r \cdot u_i - v \cdot r \otimes u_i$ , for  $r \in {}^\theta\mathcal{R}(\beta; \lambda)$ . Since  $e(j)$  annihilates  ${}^\theta L(i)$ , it follows that  $\widehat{V}_j \otimes {}^\theta L(i) \subset N$ . Next, since  $x_1 e(i) = (X(i)X_1^{-1} - X(i)^{-1}X_1)e(i)$  annihilates  ${}^\theta L(i)$ , we get that  $(X_1 - X(i))\widehat{V}_i \otimes {}^\theta L(i) \subset N$ . Finally, since  $\tau_0$  annihilates  ${}^\theta L(i)$ , it follows that  $\tau_0(\widehat{V}_j) \otimes {}^\theta L(i) \subset N$ . But  $\tau_0|_{\widehat{V}_j} = K_i(0)$ . Since  ${}^\theta\mathcal{R}(\beta; \lambda)$  is generated by  $e(i), e(j), x_1$  and  $\tau_0$ , we deduce that  $N$  is spanned by the aforementioned subspaces and that  $((\widehat{V}_i \oplus \widehat{V}_j) \otimes {}^\theta L(i))/N \cong \text{coker } K_i(0)$ .  $\square$

### 8. The BKR isomorphism in type C

In this section only, we allow  $q$  to have a finite order, *i.e.*,  $\text{ord}(q) \in \mathbb{Z}_{\geq 3} \cup \{\infty\}$ , and we set

$$\mathbb{k} := \begin{cases} \mathbb{C}[q]/(q^{\text{ord}(q)} - 1) & \text{if } \text{ord}(q) < \infty \\ \overline{\mathbb{C}(q)} & \text{if } \text{ord}(q) = \infty \end{cases}$$

The main result is the construction of an isomorphism between certain completions of the  $o$  KLR algebras and affine Hecke algebras of type C (Theorem 8.2.1) in analogy with a similar result due to Brundan-Kleshchev and Rouquier [9,61].

#### 8.1. Affine Hecke algebras of type C

We recall the definition of the (3-parameter) affine Hecke algebra of type C.

**Definition 8.1.1.** Fix  $p_0, p_1 \in \mathbb{k}^\times$ . The affine Hecke algebra  $\mathcal{H}_{C_n}(p_0, p_1)$  of type  $C_n$  is the  $\mathbb{k}$ -algebra generated by  $T_k$  ( $0 \leq k \leq n-1$ ) and  $X_l^{\pm 1}$  ( $1 \leq l \leq n$ ) subject to the relations:

- quadratic relations:

$$\begin{aligned} (T_k - q)(T_k + q^{-1}) &= 0 & (1 \leq k \leq n-1), \\ (T_0 - p_0)(T_0 + p_1^{-1}) &= 0, \end{aligned}$$

- braid relations:

$$\begin{aligned} T_k T_{k+1} T_k &= T_{k+1} T_k T_{k+1} & (1 \leq k \leq n-1), \\ (T_0 T_1)^2 &= (T_1 T_0)^2, \\ T_k T_{k'} &= T_{k'} T_k & (k \neq k' \pm 1), \end{aligned}$$

- Laurent polynomial relations:

$$\begin{aligned} X_l X_{l'} &= X_{l'} X_l, & (1 \leq l, l' \leq n), \\ X_l X_l^{-1} &= 1 = X_l^{-1} X_l & (1 \leq l \leq n), \end{aligned}$$

- mixed relations:

$$\begin{aligned} T_k X_k T_k &= X_{k+1} & (1 \leq k \leq n-1), \\ T_0 X_1^{-1} T_0 &= p_0 p_1^{-1} X_1 + (p_0 p_1^{-1} - 1) T_0, \\ T_k X_l &= X_l T_k & (l \neq k, k+1). \end{aligned}$$

The affine Hecke algebra of type  $B_n$  is the specialization  $\mathcal{H}_{B_n}(p) = \mathcal{H}_{C_n}(p, p)$ . The finite Hecke algebra  $\mathcal{H}_{B_n}^f(p)$  of type  $B_n$  is the subalgebra of  $\mathcal{H}_{B_n}(p)$  generated by the  $T_k$ .

**Remark 8.1.2.** We use the conventions of [65, §A.1] (the assignment  $q \mapsto p, p_0 \mapsto q_0, p_1 \mapsto q_1$  gives a matching between our parameters and those in [65]). This convention is the same as in [20, §2.3], if one matches the parameters as follows:  $q \mapsto q^{-1}, p_0 \mapsto q_0^{-1}, p_1 \mapsto q_1^{-1}$ . To match our conventions with those of [37, Definition 2.1], one uses the assignment

$$T_k \mapsto \mathbf{q}_2^{-1/2} T_k, \quad T_0 \mapsto \mathbf{q}_1^{-1} T_n, \quad p_1 \mapsto \mathbf{q}_1, \quad p_0 \mapsto -\mathbf{q}_0.$$

∇

The intertwiners  $\Phi_k \in \mathcal{H}_{C_n}(p_0, p_1) \otimes_{\mathcal{O}} \mathcal{K}$  are defined as (see, e.g., [65, §A.3]):

$$\Phi_k = 1 + \frac{X_k - X_{k+1}}{q X_k - q^{-1} X_{k+1}} (T_k - q) \tag{8.1} \quad (1 \leq k \leq n-1),$$

$$\Phi_0 = 1 + p_1 \frac{X_1^2 - 1}{(X_1 + p_0)(X_1 - p_1)} (T_0 - p_0). \tag{8.2}$$

**Proposition 8.1.3.** *The following hold.*

- (1) *There is an isomorphism of  $\mathcal{K}$ -algebras*

$$\mathbb{k}[\mathfrak{W}_n] \times \mathcal{K} \xrightarrow{\sim} \mathcal{H}_{C_n}(p_0, p_1) \otimes_{\mathcal{O}} \mathcal{K}, \quad s_k \mapsto \Phi_k \quad (0 \leq k \leq n-1).$$

- (2) The affine Hecke algebra  $\mathcal{H}_{\mathbb{C}_n}(p_0, p_1)$  has a faithful representation on  $\mathcal{O}$  given by
- $X_{\pm 1}, \dots, X_{\pm n}$  acting naturally by multiplication,
  - $T_1, \dots, T_{n-1}$  acting via

$$(T_k - q) \cdot f = \frac{qX_k - q^{-1}X_{k+1}}{X_k - X_{k+1}}(s_k(f) - f),$$

- $T_0$  acting via

$$(T_0 - p_0) \cdot f = p_1^{-1} \frac{(X_1 + p_0)(X_1 - p_1)}{X_1^2 - 1}(s_0(f) - f).$$

**Proof.** The first statement can be found in, e.g., [65, §A.3]. The second statement is [37, Thm. 2.7], taking into account the difference in conventions explained in Remark 8.1.2.  $\square$

### 8.2. BKR-type isomorphism

We establish a Brundan–Kleshchev–Rouquier-style isomorphism between completions of orientifold KLR algebras and affine Hecke algebras of type  $\mathbb{C}$ , generalizing [65, Thm. A.4] and [54, Thm. 1.1].

Assume  $p_0, p_1 \neq \pm 1$ . We now determine the datum defining  ${}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda})$ . Let  $\xi \in \mathbb{k}^\times$  and set

$$J = \{\xi^{\pm 1}q^{2k} \mid k \in \mathbb{Z}\}, \quad \theta: i \mapsto i^{-1}, \quad a_{ij} = \delta_{j=q^2i}, \quad \boldsymbol{\lambda}(i) = \delta_{i=p_1} + \delta_{i=-p_0}.$$

The resulting quiver  $\Gamma = (J, \Omega)$  can, depending on  $\xi$  and the order of  $q$ , be of the following types:

	ord( $q$ )	$\xi$	$\Gamma$	$J^\theta$
(1)	$\infty$	1	$A_\infty$	{1}
(2)	$\infty$	$q$	$A_\infty$	$\emptyset$
(3)	$\infty$	$\notin \{\pm q^{\mathbb{Z}}\}$	$A_\infty \times A_\infty$	$\emptyset$
(4)	$2m$	1	$A_m^{(1)}$	{ $\pm 1$ }
(5)	$2m$	$q$	$A_m^{(1)}$	$\emptyset$
(6)	$2m$	$\notin \{q^{\mathbb{Z}}\}$	$A_m^{(1)} \times A_m^{(1)}$	$\emptyset$
(7)	$m$ odd	1	$A_m^{(1)}$	{1}
(8)	$m$ odd	$\notin \{\pm q^{\mathbb{Z}}\}$	$A_m^{(1)} \times A_m^{(1)}$	$\emptyset$

Given a self-dual dimension vector  $\beta$  with  $\|\beta\|_\theta = n$ , let  ${}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda})$  be the orientifold KLR algebra associated to the datum  $(\Gamma, \theta, \beta, \boldsymbol{\lambda})$ . Choose  $X(-)$  in the  $J$ -datum to be the identity function.

**Theorem 8.2.1.** *The isomorphism (7.1) extends to an algebra isomorphism*

$${}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda}) \otimes_{\theta\mathbb{P}_\beta} {}^\theta\widehat{\mathbb{P}}_\beta \xrightarrow{\sim} \mathcal{H}_{C_n}(p_0, p_1) \otimes_{\mathcal{O}} {}^\theta\widehat{\mathcal{O}}_\beta \quad (8.3)$$

given by:

$$\begin{aligned} \tau_k e(\nu) &\mapsto \begin{cases} \left( \frac{X_k}{X(\nu_{k+1})} + \frac{X(\nu_k)}{X_{k+1}} \right)^{-1} \left( \frac{X_{k+1}}{X_k} - 1 \right)^{-1} (\Phi_k - 1) e(\nu) & \text{if } \nu_k = \nu_{k+1}, \\ \left( \frac{X_k}{X(\nu_{k+1})} + \frac{X(\nu_k)}{X_{k+1}} \right) \left( \frac{X(\nu_{k+1})}{X(\nu_k)} - \frac{X_{k+1}}{X_k} \right) \Phi_k e(\nu) & \text{if } \nu_{k+1} = q^2 \nu_k, \\ \Phi_k e(\nu) & \text{otherwise,} \end{cases} \\ \tau_0 e(\nu) &\mapsto \begin{cases} X(\nu_1) (X_1^{-1} - X_1)^{-1} (\Phi_0 - 1) e(\nu) & \text{if } \nu_1 = \theta(\nu_1), \\ \left( \frac{X_1}{X(\nu_1)} - \frac{X(\nu_1)}{X_1} \right)^{\boldsymbol{\lambda}(\nu_1)} \Phi_0 e(\nu) & \text{otherwise.} \end{cases} \end{aligned}$$

**Proof.** The completed algebra on the LHS of (8.3) has a faithful representation on  ${}^\theta\widehat{\mathbb{P}}_\beta$  while the completed algebra on the RHS has a faithful representation on  ${}^\theta\widehat{\mathcal{O}}_\beta$ . Therefore, it suffices to check that the actions of the generators agree under the isomorphism (7.1). This follows by a direct calculation using Proposition 5.3.1, Proposition 8.1.3 and (8.1)–(8.2).  $\square$

**Remark 8.2.2.** A weaker statement about a Morita equivalence between orientifold KLR algebras and blocks of affine Hecke algebras of type C, in the case when  $\theta$  has no fixed points, can be found in [65, Thm. A.4]. A proof of the isomorphism of cyclotomic quotients, in type B case, for any  $\theta$ , appeared in [54, Thm. 1.1]. The idea of comparing the polynomial representations goes back to [61] (see also [50,49]).  $\nabla$

## 9. Boundary Schur-Weyl duality in type A

In this section we study a particular instance of the boundary Schur-Weyl duality from Section 7 in the case  $\mathfrak{g} = \mathfrak{sl}_N$ . Specifically, we consider only quasi-split affine QSP subalgebras of type AIII with the unitary K-matrices satisfying the standard reflection equation on the first fundamental representation.

9.1. *The fundamental representation*

Recall that the first fundamental representation  $V_{\omega_1}$  of  $U_q\mathfrak{sl}_N$  extends, through the evaluation morphism, to a representation of the quantum loop algebra  $U_qL\mathfrak{sl}_N$ . More precisely, we denote by  $V$  the  $N$ -dimensional  $U_qL\mathfrak{sl}_N$ -module with basis  $u_1, \dots, u_N$  and action given by

$$\begin{aligned} E_i \cdot u_r &= \delta(r \equiv i + 1 \pmod N) u_i, \\ F_i \cdot u_r &= \delta(r \equiv i \pmod N) u_{i+1}, \\ K_i \cdot u_r &= q^{\delta(r \equiv i \pmod N) - \delta(r \equiv i+1 \pmod N)} u_r, \end{aligned}$$

where for simplicity we are adopting the cyclic notation on the indices of the Chevalley generators and the basis vectors  $u_k$  (e.g.,  $u_N = u_0$ ,  $E_N = E_0$ , and so on). More precisely,  $V = \text{ev}_1^*(V_{\omega_1})$ , where  $\text{ev}_1: U_qL\mathfrak{sl}_N \rightarrow U_q\mathfrak{sl}_N$  is the evaluation morphism at  $z = 1$ . We set  $V_z := V \otimes \mathbb{k}[z, z^{-1}]$  and  $V(z) := V \otimes \mathbb{k}(z)$  endowed with the shifted  $U_qL\mathfrak{sl}_N$ -action (cf. §2.3).

The explicit formulae for the unitary R-matrix on  $V$  is well-known and due to Jimbo [28]. More precisely,

$$\mathbf{R}^V(z, w): V(z) \otimes V(w) \rightarrow V(w) \otimes V(z)$$

is given by  $u_r \otimes u_s \mapsto u_r \otimes u_s$  and

$$u_r \otimes u_s \mapsto \frac{(1 - q^2)w^{\delta(r>s)}z^{\delta(r<s)}}{w - q^2z} \cdot u_r \otimes u_s + \frac{q(w - z)}{w - q^2z} \cdot u_s \otimes u_r \quad (r \neq s) \quad (9.1)$$

In particular,  $\mathbf{R}^V(z, w)$  is a rational function in  $w/z$  with only one simple pole at  $w/z = q^2$ .

9.2. *Normalized parameters and K-matrices*

Let  $(X, \tau)$  be an affine Satake diagram of type  $A$  and  $U_q\mathfrak{k} \subset U_qL\mathfrak{sl}_N$  the corresponding QSP subalgebra with parameters  $(\gamma, \sigma) \in \Gamma \times \Sigma$  (cf. §3.4). We are interested in the K-matrices on  $V(z)$  supported on  $U_q\mathfrak{k}$ . From the point of view of the combinatorial model described in §6.2, it is convenient to consider  $U_qL\mathfrak{sl}_N$ -modules  $V$  that are *fixed* by the chosen twisting operator, i.e.,  $\psi^*(V) = V$ , or conversely to consider only twisting operators that fix a given module. In both cases, the corresponding K-matrix gives rise to a solution of the *standard* reflection equation (see, e.g., §4.5). Since  $V$  is small, there exists a distinguished twisting operator  $\psi$  which fixes it. Note, however, that this requires to impose a normalization condition  $\gamma(\delta) = \beta$  for a uniquely determined  $\beta \in \mathbb{k}^\times$ , cf. [4, §7].

Therefore, for any QSP subalgebra in  $U_q \mathfrak{Lsl}_N$  with normalized parameter  $\gamma$ , we obtain a rational QSP intertwiner

$$\mathbf{K}(z): \mathbf{V}(z) \rightarrow \mathbf{V}(z^{-1})$$

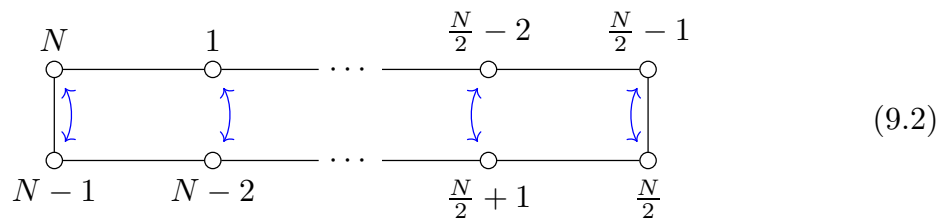
which satisfies the standard reflection equation. By generic QSP irreducibility of the first fundamental representation, we recover in this way the QSP intertwiners explicitly described by Regelskis-Vlaar in [57,58]. By direct inspection, it then follows that the operators  $\mathbf{K}(z)$  can be normalized to be non-vanishing, unitary and such that  $\mathbf{K}(1) = \text{id}$ . In particular, the condition (Q3) holds.

### 9.3. Quasi-split affine QSP subalgebras of type AIII

We provide the explicit formulae of the unitary K-matrix for the first fundamental representation in the case of quasi-split affine QSP subalgebra of type AIII. Our main reference is [57,58] as explained above. We note however that the K-matrices in this case first appeared in [1].

#### 9.3.1. The non-restrictable case

Recall that a quasi-split QSP subalgebra is simply determined by a non-trivial involution on the Dynkin diagram. We first consider the case where the affine node, which is numbered  $N$  according to our cyclic notation, is not fixed by the involution, *i.e.*, we consider the case of  $N$  even and Satake diagram



For any  $1 \leq r \leq N$ , we set  $\bar{r} := N - r$ . The corresponding QSP subalgebra  $U_q \mathfrak{k}$  is generated by  $K_r K_{\bar{r}-1}^{-1}$  ( $0 \leq r \leq N/2 - 1$ ) and

$$B_r := F_r + \gamma_r E_{\bar{r}-1} K_r^{-1} + \sigma_r K_r^{-1} \quad (1 \leq r \leq N)$$

where the parameters  $\gamma, \sigma$  are determined according to Remark 3.4.1(1) by the following assignments  $(\lambda, \mu \in \mathbb{k}^\times)$ <sup>11</sup>:

$$\gamma_N = q, \quad \gamma_{N-1} = \mu^{-2}, \quad \gamma_{N/2+1} = q\mu\lambda^{-1}, \quad \gamma_{N/2-1} = \lambda\mu,$$

<sup>11</sup> Note that, both here and in §9.3.2, with respect to the formulae given in [57, §9.3], we are setting the shifting parameter  $\eta$  equal to 1 and we are choosing suitable dressing parameters  $\omega_i$ .

$$\sigma_N = \frac{\mu - \mu^{-1}}{q - q^{-1}}, \quad \sigma_{N/2} = \frac{\lambda - \lambda^{-1}}{q - q^{-1}}, \quad \sigma_r = 0 \quad \left(0 \leq r < \frac{N}{2}\right)$$

Following [57, §9], the compact form of the K-matrix  $\mathbf{K}(z)$  on  $\mathbf{V}$  supported on  $U_q\mathfrak{k}$  is given by:

$$\mathbf{K}(z) = \text{id} + \frac{z - z^{-1}}{(\lambda\mu - z)} \left( E_{NN} + \frac{M_2}{(\lambda^{-1} + (\mu z)^{-1})} \right) \tag{9.3}$$

where

$$M_2 = \sum_{1 \leq i \leq N/2-1} (\lambda E_{ii} + \lambda^{-1} E_{N-i, N-i} + \mu E_{i, N-i} + \mu^{-1} E_{N-i, i}).$$

It is convenient for us to consider the parameters  $p_0 = \lambda\mu^{-1}$  and  $p_1 = \lambda\mu$ . Then, formula (9.3) reads  $u_{N/2} \mapsto u_{N/2}$ ,  $u_N \mapsto \left(1 + \frac{z-z^{-1}}{p_1-z}\right) \cdot u_N$ , and

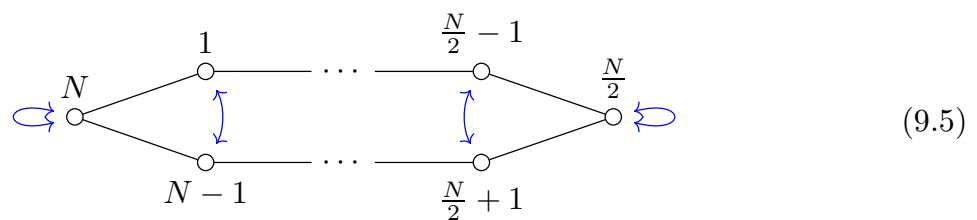
$$u_r \mapsto \frac{(p_0^{-1}p_1 - 1) + (p_1 - p_0^{-1})z^{\delta(r < \bar{r})}z^{-\delta(r > \bar{r})}}{(p_1 - z)(p_0^{-1} + z^{-1})} \cdot u_r + \frac{(p_0^{-1}p_1)^{\delta(r < \bar{r})}(z - z^{-1})}{(p_1 - z)(p_0^{-1} + z^{-1})} \cdot u_{\bar{r}} \tag{9.4}$$

if  $1 \leq r < N/2$ .

**Remark 9.3.1.** Since the QSP subalgebra is non-restrictable, the K-matrix  $\mathbf{K}(z)$  does not fit into the framework described in Section 4. It is expected, however, that  $\mathbf{K}(z)$  is also induced by the action of a universal K-matrix.  $\nabla$

9.3.2. *The restrictable case*

For  $N$  even, we consider the Satake diagram



The corresponding QSP subalgebra  $U_q\mathfrak{k}$  is generated by  $K_r K_{\bar{r}-1}^{-1}$  ( $1 \leq r < N/2$ ) and

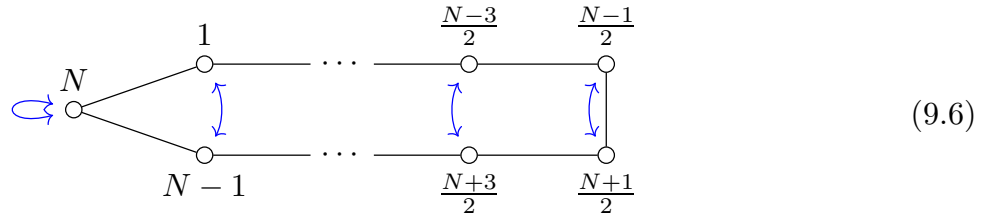
$$B_r = F_r + \gamma_r E_{\bar{r}-1} K_r^{-1} + \sigma_r K_r^{-1} \quad (1 \leq r \leq N),$$

where  $\bar{r} = N + 1 - r$  and the parameters  $\gamma$  and  $\sigma$  are determined by the assignments:

$$\gamma_N = q^{-1}\mu^{-2}, \quad \gamma_{N/2} = q^{-1}\mu^2, \quad \gamma_r = 1 \quad \left(1 \leq r < \frac{N}{2}\right)$$

$$\sigma_N = \frac{1 - \mu^2}{q - q^{-1}}, \quad \sigma_{\frac{N}{2}} = \frac{(\lambda\mu)^{-1} - \lambda\mu^{-1}}{q - q^{-1}}, \quad \sigma_r = 0 \quad \left(0 \leq r < \frac{N}{2}\right)$$

For  $N$  odd, we consider the Satake diagram



In this case, the parameters of  $U_q\mathfrak{k}$  are determined by

$$\gamma_N = q^{-1}\mu^2, \quad \gamma_{N/2} = q^{-1}\lambda\mu^{-1}, \quad \gamma_r = 1 \quad \left(1 \leq r < \frac{N}{2}\right)$$

$$\sigma_N = \frac{1 - \mu^2}{q - q^{-1}}, \quad \sigma_{\frac{N}{2}} = \frac{(\lambda\mu)^{-1} - \lambda\mu^{-1}}{q - q^{-1}}, \quad \sigma_r = 0 \quad \left(0 \leq r < \frac{N}{2}\right)$$

In both cases, the K-matrix is given by

$$\mathbf{K} = \text{id} + \frac{(z - z^{-1})}{(\lambda\mu - z)} \frac{M_2}{(\lambda^{-1} + (\mu z)^{-1})}, \tag{9.7}$$

where

$$M_2 = \sum_{1 \leq i \leq N/2} (\lambda E_{ii} + \lambda^{-1} E_{N+1-i, N+1-i} + \mu E_{i, N+1-i} + \mu^{-1} E_{N+1-i, i}).$$

As before, we set  $p_0 = \lambda\mu^{-1}$  and  $p_1 = \lambda\mu$ . Then, the formula (9.7) reads  $u_r \mapsto u_{\bar{r}}$  if  $r = \bar{r}$  and

$$u_r \mapsto \frac{(p_0^{-1}p_1 - 1) + (p_1 - p_0^{-1})z^{\delta(r < \bar{r})}z^{-\delta(r > \bar{r})}}{(p_1 - z)(p_0^{-1} + z^{-1})}u_r + \frac{(p_0^{-1}p_1)^{\delta(r < \bar{r})}(z - z^{-1})}{(p_1 - z)(p_0^{-1} + z^{-1})}u_{\bar{r}} \tag{9.8}$$

if  $r \neq \bar{r}$ .

Note that, in the case  $\mu = 1$ , we get  $p_0 = p_1 =: p$  and (9.8) simplifies to  $u_r \mapsto u_{\bar{r}}$  if  $r = \bar{r}$  and

$$u_r \mapsto \frac{(1 - p^2)z^{\delta(r < \bar{r})}z^{-\delta(r > \bar{r})}}{z - p^2z^{-1}}u_r + \frac{p(z - z^{-1})}{z - p^2z^{-1}}u_{\bar{r}} \tag{9.9}$$

if  $r \neq \bar{r}$ .

### 9.4. The boundary Schur-Weyl functor

As anticipated in 6.5, we briefly describe here one example of the combinatorial model from §6.2 in this setting.

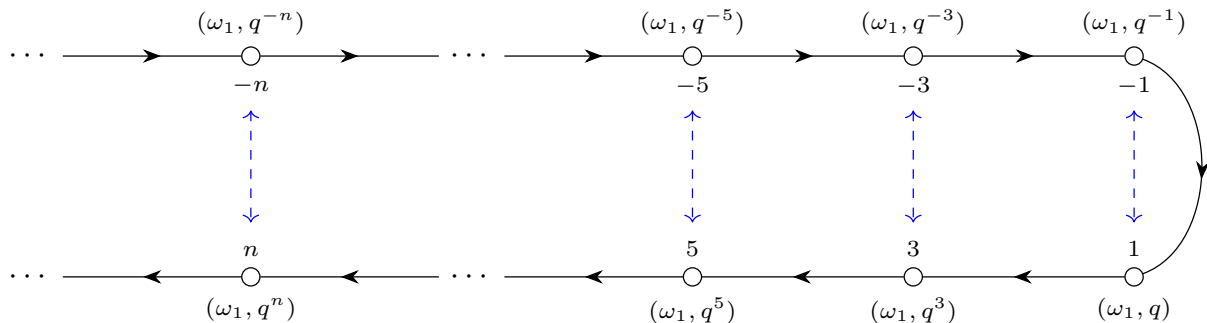
We consider a quasi-split affine QSP subalgebra of type AIII with a choice of parameters as described in §9.3 such that

- (1) the twisting  $\psi$  satisfying  $\psi^*(\mathbf{V}) = \mathbf{V}$  is QSP-admissible (cf. §4.2),
- (2)  $\lambda, \mu \in \mathbb{k}^\times$  are such that  $\lambda\mu, \lambda\mu^{-1} \in q^{1/m}\mathbb{C}[[q^{1/m}]]$  for some  $m > 0$ .

Set  $J := \mathbb{Z}_{\text{odd}}$  and consider the involution  $\theta$  on  $J$  given by  $\theta(n) = -n$ . Note that  $J^\theta = \emptyset$ . For any  $i \in J$ , we set  $V(n) := \mathbf{V}$  and  $X(n) = q^n$ .

As observed in 6.5, the conditions (Q0)–(Q3) are easily verified to hold. Moreover, by (9.3)–(9.8), the K-matrix has only two simple poles at  $z = \lambda\mu^{-1}$  and  $z = \lambda\mu$ . Thus, the condition (Q4) follows from the condition (2) above.

By Definition 6.3.1, we obtain a quiver  $\Gamma$  whose nodes are indexed by odd numbers, there is an edge between any two consecutive numbers, and it is equipped with a natural contravariant involution induced from  $\theta$ , i.e.,  $\Gamma$  is an  $A_\infty$  quiver with a non-trivial involution and no fixed points:



Finally, the framing dimension vector  $\lambda$  on  $\Gamma$  is non-trivial if and only if  $\lambda = q^n \mu^{\pm 1}$  for some  $n \in \mathbb{Z}_{\text{odd}}$ . In this case, the framing dimension vector is given by  $\lambda(m) = \delta_{m, -n}$  for any  $m \in \mathbb{Z}_{\text{odd}}$ .

### 9.5. Hecke algebras and $v/\mathfrak{g}$ Schur–Weyl dualities

It is proved in [32,44] that the functor  $F$  recovers Chari-Pressley’s quantum affine Schur-Weyl duality [12] through the BKR isomorphism between KLR algebras and affine Hecke algebras. We shall prove the analogue result for the functor  ${}^\theta F$ .

More precisely, let  $U_q \mathfrak{k} \subseteq U_q L\mathfrak{sl}_N$  be a quasi-split QSP subalgebra of type AIII with the choice of parameters described above. A Schur-Weyl duality functor between  $U_q \mathfrak{k}$  and the 2-parameters affine Hecke algebra of type C (cf. §8.1) was constructed in [20], relying on an explicit action of the latter on  $\mathbf{V}_z^n := \mathbf{V}(z_1) \otimes \cdots \otimes \mathbf{V}(z_n)$ .

**Lemma 9.5.1.** *The action of  $\mathcal{H}_{C_n}(p_0, p_1)$  on  $V_z^n$ , determined by the assignment*

$$\Phi_k \mapsto \mathbf{R}_{k, k+1}^\vee(z_k, z_{k+1}), \quad \Phi_0 \mapsto \mathbf{K}_1(z_1), \quad X_l \mapsto z_l,$$

*coincides with the action from [20, Prop. 2.5].*

**Proof.** A formula for the action of the generators  $T_i$  is obtained by substituting the expressions (9.1), (9.4) and (9.8) for R- and K-matrices into the formulas (8.1)–(8.2) for the intertwiners. The lemma then follows from a straightforward comparison with the formulae in [20, §2.4].<sup>12</sup>  $\square$

Let

$$F_{\text{AHC}} : \mathcal{H}_{C_n}(p_0, p_1)\text{-mod} \rightarrow U_q\mathfrak{k}\text{-mod}, \quad M \mapsto V_z^n \otimes_{\mathcal{H}_{C_n}(p_0, p_1)} M$$

be the functor defined by the bimodule  $V_z^n$ . For any  $\beta \in N[J]^\theta$ , let  ${}^\theta\mathcal{R}(\beta; \lambda)\text{-mod}_0$  be the full subcategory of  ${}^\theta\mathcal{R}(\beta; \lambda)\text{-mod}$  whose objects are modules with a locally nilpotent action of the elements  $x_i$ . Also let  $\mathcal{H}_{C_n}(p_0, p_1)\text{-mod}_\beta$  be the full subcategory of  $\mathcal{H}_{C_n}(p_0, p_1)\text{-mod}$  such that the action of the  $X_i^{\pm 1}$  is locally nilpotent and their multiset of eigenvalues equals  $\beta$ .

**Theorem 9.5.2.** *Let  $\|\beta\|_\theta = n$ . The following diagram commutes*

$$\begin{array}{ccc}
 {}^\theta\mathcal{R}(\beta; \lambda) \otimes_{\theta\mathbb{P}_\beta} {}^\theta\widehat{\mathbb{P}}_\beta\text{-mod} & \xrightarrow{\sim} & \mathcal{H}_{C_n}(p_0, p_1) \otimes_{\mathcal{O}} {}^\theta\widehat{\mathcal{O}}_\beta\text{-mod} \\
 \uparrow \wr & \swarrow \text{---} & \downarrow \wr \\
 {}^\theta\mathcal{R}(\beta; \lambda)\text{-mod}_0 & & \mathcal{H}_{C_n}(p_0, p_1)\text{-mod}_\beta \\
 \searrow \theta F_\beta & \swarrow F_{\text{AHC}} & \\
 & U_q\mathfrak{k}\text{-mod} & 
 \end{array}$$

**Proof.** The diagram is composed of three triangles. The commutativity of the side ones follows immediately from the definition of the completions  ${}^\theta\widehat{\mathbb{P}}_\beta$  and  ${}^\theta\widehat{\mathcal{O}}_\beta$  (cf. §7.1). The commutativity of the triangle in the middle square follows directly from the BKR-type isomorphism constructed in Theorem 8.2.1 and Lemma 9.5.1.  $\square$

In the case of a restrictable QSP subalgebra, we further recover a finite-type Schur-Weyl duality between the Hecke algebra of type B and a finite-type QSP subalgebra in  $U_q\mathfrak{sl}_N$ . More precisely, let  $U_q\mathfrak{k} \in U_qL\mathfrak{sl}_N$  be a restrictable affine QSP subalgebra from

<sup>12</sup> More precisely, it is also necessary to take into account that the formulae in [20] are given in terms of the left-coideal subalgebra  $\omega(U_q\mathfrak{k})$ .

§9.3.2. Then,  $U_q\mathfrak{k}^{\text{fin}} := U_q\mathfrak{k} \cap U_q\mathfrak{sl}_N$  is the QSP subalgebra corresponding to the finite-type Satake diagram obtained from (9.5) (for  $N$  even) or (9.6) (for  $N$  odd) by removing the affine node.

Consider the operators  $\mathbf{R}^{\text{fin}}: \mathbb{V}^{\otimes 2} \rightarrow \mathbb{V}^{\otimes 2}$  and  $\mathbf{K}^{\text{fin}}: \mathbb{V} \rightarrow \mathbb{V}$  on the fundamental representation given respectively by

$$u_r \otimes u_s \mapsto \begin{cases} u_s \otimes u_r & \text{if } r > s, \\ u_s \otimes u_r + (q - q^{-1})u_r \otimes u_s & \text{if } r < s, \\ qu_r \otimes u_r & \text{if } r = s, \end{cases} \tag{9.10}$$

$$u_r \mapsto \begin{cases} u_{\bar{r}} & \text{if } r < \bar{r}, \\ u_{\bar{r}} + (p - p^{-1})u_r & \text{if } r > \bar{r}, \\ pu_r & \text{if } r = \bar{r}. \end{cases} \tag{9.11}$$

By [8, Thms. 2.6 and 4.4], the assignment

$$T_0 \mapsto \mathbf{K}_1^{\text{fin}}, \quad T_k \mapsto \mathbf{R}_{k,k+1}^{\text{fin}} \quad (1 \leq k \leq n - 1)$$

defines a  $(U_q\mathfrak{k}^{\text{fin}}, \mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p))$ -bimodule structure on  $\mathbb{V}^{\otimes n}$ , where  $\mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)$  is the finite Hecke algebra of type  $\mathbb{B}_n$ . Let

$$F_{\text{HB}}: \mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)\text{-mod} \rightarrow U_q\mathfrak{k}^{\text{fin}}\text{-mod}, \quad M \mapsto \mathbb{V}^{\otimes n} \otimes_{\mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)} M$$

be the functor induced by the bimodule structure on  $\mathbb{V}^{\otimes n}$  (cf. [66]).

**Corollary 9.5.3.** *The diagram*

$$\begin{array}{ccc} {}^\theta\mathcal{R}(\beta; \boldsymbol{\lambda})\text{-mod}_0 & \xrightarrow{\theta F_\beta} & U_q\mathfrak{k}\text{-mod} \\ \downarrow \wr & & \parallel \\ \mathcal{H}_{\mathbb{C}_n}(p_0, p_1)\text{-mod}_\beta & \xrightarrow{F_{\text{AHC}}} & U_q\mathfrak{k}\text{-mod} \\ \text{res} \downarrow p_0=p_1 & & \text{res} \downarrow \mu=1 \\ \mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)\text{-mod} & \xrightarrow{F_{\text{HB}}} & U_q\mathfrak{k}^{\text{fin}}\text{-mod} \end{array}$$

where the upper vertical arrow on the left is as in Theorem 9.5.2 and the bottom vertical arrows are given by restriction under the additional assumption  $p_0 = p_1$  and  $\mu = 1$  (cf. §9.3.2), is commutative.

**Proof.** Let  ${}^\theta\widehat{\mathbb{V}}^{\otimes \beta}$  be the module defined in §7.2 with respect to the combinatorial model described in §9.4. Relying on (9.1), (9.9), Theorem 8.2.1 and (9.10)-(9.11), one shows by direct inspection that

$${}^\theta\widehat{\mathbb{V}}^{\otimes \beta} \cong \mathbb{V}^{\otimes n} \otimes_{\mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)} \mathcal{H}_{\mathbb{B}_n}(p) \otimes_{\mathcal{O}} {}^\theta\widehat{\mathcal{O}}_\beta$$

as  $(U_q^{\mathfrak{k}^{\text{fin}}}, \mathcal{H}_{\mathbb{B}_n}(p) \otimes_{\mathcal{O}} \widehat{\mathcal{O}}_{\beta})$ -bimodules. Hence, for any  $M$  in  ${}^{\theta}\mathcal{R}(\beta; \boldsymbol{\lambda})\text{-mod}_0$ , we have

$${}^{\theta}\mathbf{F}_{\beta}(M) = {}^{\theta}\widehat{\mathbf{V}}^{\otimes \beta} \otimes_{\mathcal{H}_{\mathbb{B}_n}(p) \otimes_{\mathcal{O}} \widehat{\mathcal{O}}_{\beta}} M \cong \mathbf{V}^{\otimes n} \otimes_{\mathcal{H}_{\mathbb{B}_n}^{\text{fin}}(p)} M = \mathbf{F}_{\text{HB}}(M). \quad \square$$

**Remark 9.5.4.** Relying on the explicit formulae for the K-matrix provided in [57, §9.3] and [62], a similar computation shows that the same result holds for arbitrary affine QSP subalgebras of type AIII. It is also expected that one can recover the Schur-Weyl functor with the 2-parameter affine Hecke algebra of type B defined in [13]. This is, however, harder to verify since [13] relies on the FRT presentation of QSP subalgebras.  $\nabla$

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