Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 366 (2023) 192–248

Journal of Differential Equations

www.elsevier.com/locate/jde

A BSDEs approach to pathwise uniqueness for stochastic evolution equations

Davide Addona [a,*], Federica Masiero [b], Enrico Priola [c]

[a] Dipartimento di Scienza Matematiche, Fisiche e Infomatiche, Università di Parma, Parma, Italy
[b] Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Milano, Italy
[c] Dipartimento di Matematica, Università di Pavia, Pavia, Italy

Received 1 June 2022; revised 3 March 2023; accepted 9 April 2023

Abstract

We prove strong well-posedness for a class of stochastic evolution equations in Hilbert spaces H when the drift term is Hölder continuous. This class includes examples of semilinear stochastic Euler-Bernoulli beam equations which describe elastic systems with structural damping, and semilinear stochastic 3D heat equations. In the deterministic case, there are examples of non-uniqueness in our framework. Strong (or pathwise) uniqueness is restored by means of a suitable additive Wiener noise. The proof of uniqueness relies on the study of related systems of infinite dimensional forward-backward SDEs (FBSDEs). This is a different approach with respect to the well-known method based on the Itô formula and the associated Kolmogorov equation (the so-called Zvonkin transformation or Itô-Tanaka trick). We deal with approximating FBSDEs in which the linear part generates a group of bounded linear operators in H; such approximations depend on the type of SPDEs we are considering. We also prove Lipschitz dependence of solutions from their initial conditions.

MSC: primary 60H15; secondary 35R60

Keywords: Nonlinear stochastic PDEs; Hölder continuous drift; Backward stochastic differential equations; Semilinear stochastic heat equations; Semilinear stochastic Euler-Bernoulli beam equations; Strong uniqueness

---

[*] Corresponding author.
E-mail addresses: davide.addona@unipr.it (D. Addona), federica.masiero@unimib.it (F. Masiero), enrico.priola@unipv.it (E. Priola).

https://doi.org/10.1016/j.jde.2023.04.014
0022-0396/

1. Introduction

1.1. Problem and main result

In this paper we consider the problem of strong well-posedness for a class of stochastic partial differential equations (SPDEs) when the drift term is only Hölder continuous.

In a real and separable Hilbert space H we consider a stochastic evolution equation of the form

$$\begin{cases} dX_\tau = AX_\tau d\tau + G\widetilde{C}(\tau, X_\tau)d\tau + GdW_\tau, & \tau \in [0,T], \\ X_0 = x \in H, \end{cases} \tag{1.1}$$

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$, the operator $G : U \to H$ is a bounded linear operator defined on another real and separable Hilbert space U and W is a cylindrical Wiener process on U (cf. Section 2). Moreover, $\widetilde{C} : [0,T] \times H \to U$ is bounded and continuous and $\widetilde{C}(t,\cdot)$ is β-Hölder continuous, uniformly in $t \in [0,T]$, $\beta \in (0,1)$.

In our framework there are examples of non uniqueness of solutions in the deterministic case (i.e., when $W = 0$ in (1.1)) (see [8] and [26]). Hence our main result on strong uniqueness, see Theorem 4.2, is due to the presence of the Wiener noise (regularization by noise). Clearly, such theorem holds also in the Lipschitz case $\beta = 1$ (in this case the result does not depend on W). Moreover, the boundedness of $\tilde{C}$ can be relaxed by a localization procedure (see Remark 4.3). Our pathwise result also implies the existence of a strong mild solution (cf. Remark 4.4).

Examples of singular semilinear SPDEs of the form (1.1) we can treat are stochastic damped equations which describe elastic systems with structural damping and semilinear stochastic 3D heat equations (see (1.2), (1.3) and Section 6 which is about applications).

In the literature the problem of regularization by noise for stochastic evolution equations (1.1) of parabolic type has been widely studied, see [8], [9], [10], [11], [19], [32] and the references therein. Moreover, we mention [4] for SDEs in Banach space, and [26] and [27], where the semilinear stochastic wave equation is studied. Usually in such papers pathwise uniqueness is obtained by using the Itô formula after solving a Kolmogorov equation. In finite dimension this is the so-called Zvonkin transformation or the Itô-Tanaka trick (see the seminal paper [30], the recent monograph [15] and the references therein).

Note that establishing the Itô formula in infinite dimensions is a delicate issue when the noise is cylindrical (cf. [8], [9], [10] and the references therein). We replace the previous approach with a method based on forward-backward SDEs (FBSDEs). This does not require proving the Itô formula, it works for hyperbolic and parabolic SPDEs and extends the method introduced in [26] for singular semilinear stochastic wave equations.

The techniques of [26] and [27] work only when the operator A appearing in equation (1.1) is the generator of a group of bounded linear operators. Here we introduce approximating FBSDEs in which the linear parts A_n in the backward equation generate a group of bounded linear operators in H (cf. equation (1.6) below); such approximations depend on the type of SPDEs we are considering. Let us explain two examples of singular SPDEs of the form (1.1) we can consider.

The first class of equations we can treat is stochastic semilinear damped Euler-Bernoulli beam equations of the form

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(t) = -\Lambda y(t) - \rho\Lambda^{\alpha}\dfrac{\partial y}{\partial t}(t) + \Lambda^{-\gamma}\widetilde{C}\left(t, y(t), \dfrac{\partial y}{\partial t}(t)\right) + \Lambda^{-\gamma}\dot{W}_t, \quad t \in (0,T], \\ y(0) = y_0, \\ \dfrac{\partial y}{\partial t}(0) = y_1, \end{cases} \tag{1.2}$$

with $\rho, \gamma > 0$ and $\alpha \in [0,1)$. Here, $\Lambda : D(\Lambda) \subset U \to U$ is a positive self-adjoint operator on a separable Hilbert space U, there exists $\Lambda^{-\gamma}$ and it is a trace class operator from U into U. Such equations describe elastic systems with structural damping. For physical motivations we refer to the seminal paper [5] (see also the references therein) for the deterministic equation, and to [2,3,7] for the stochastic counterpart.

The second class of equations we can treat is 3D semilinear stochastic heat equations like

$$\begin{cases} dX_t^x = \Delta X_t^x dt + (-\Delta)^{-\gamma/2}\widetilde{C}(X_t^x)dt + (-\Delta)^{-\gamma/2}dW_t, \quad t \in [0,T], \\ X_0^x = x \in H, \end{cases} \tag{1.3}$$

where we are dealing with the Laplace operator in $H = U = L^2([0,\pi]^d)$ with periodic boundary conditions and $G = (-\Delta)^{-\gamma/2}$ with $\gamma \geq 0$. Such equations are also considered in [8, Example 6.1] with β-Hölder continuous drifts $\widetilde{C}$ (even with $(-\Delta)^{-\gamma/2}\widetilde{C}(X_t^x)$ replaced by the more general term $\widetilde{C}(X_t^x)$). However as the authors explain in [8] for β-Hölder continuous drift terms $\widetilde{C}$ they can only prove uniqueness in dimensions 1 and 2. On the other hand, in Section 6.3 we treat also the dimension $d = 3$.

As in [26] our method allows to establish Lipschitz dependence of solutions from their initial conditions (cf. Theorem 4.2):

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t^{x_1} - X_t^{x_2}|_H^2] \leq c_T |x_1 - x_2|_H^2, \tag{1.4}$$

where X^{x_1} and X^{x_2} denote the weak mild solutions to (1.1) starting at x_1 and $x_2 \in H$, respectively. Estimates like (1.4) have not been proved before in papers on regularization by noise for stochastic evolution equations of parabolic type (cf. [8], [9], [10], [11], [4]).

Note that when $\widetilde{C}$ is only continuous and bounded even for parabolic SPDEs (with A which verifies Hypothesis 5.16) pathwise uniqueness for (1.1) for any initial $x \in H$ is still an open problem (cf. [9], [11] and the references therein).

1.2. Strategy of the proof

First notice the particular structure of equation (1.1), which in the BSDE literature is referred to as *structure condition*: the drift belongs to the image of the diffusion operator G. Under the basic Hypothesis 2.1 the existence of a (unique in law) weak mild solution given by

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}G\widetilde{C}(s, X_s)ds + \int_0^t e^{(t-s)A}G\,dW_s, \quad \mathbb{P}\text{-a.s.}, \tag{1.5}$$

$t \in [0,T]$, directly follows by an infinite dimensional version of the Girsanov theorem, see e.g. [28, Proposition 7.1] and [13, Section 10.3].

In order to prove pathwise uniqueness of solutions to equation (1.1), we complement equation (1.1) with a family of BSDE, which gives the following family of systems of FBSDEs

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + G\widetilde{C}(\tau, X_\tau^{t,x})d\tau + GdW_\tau, & \tau \in [t, \mathcal{T}], \\ X_t^{t,x} = x, & \tau \in [0,t], \\ -dY_\tau^{t,x,n} = -A_nY_\tau^{t,x,n}d\tau + G\widetilde{C}(\tau, X_\tau^{t,x})d\tau - Z_\tau^{t,x,n}dW_\tau, & \tau \in [t, \mathcal{T}], \\ Y_{\mathcal{T}}^{t,x,n} = 0, \end{cases} \tag{1.6}$$

with $\mathcal{T} \in (0, T]$ and $n \in \mathbb{N}$. Here, $(A_n)_{n\in\mathbb{N}}$ is a sequence of linear operators on H and each A_n generates a group of bounded operators $(e^{tA_n})_{t\in\mathbb{R}}$ which pointwise approximates $(e^{tA})_{t\geq 0}$. We stress that the idea of associating to (1.1) a BSDE has been exploited in [26] and [27], but in that case the assumption that A is the generator of a strongly continuous group $(e^{tA})_{t\in\mathbb{R}}$ is fundamental. In the present paper a crucial point is to consider a suitable sequence of approximating BSDEs, where if A is not the generator of a group of operators, each A_n is.

By an infinite dimensional version of Girsanov Theorem, the process

$$\widehat{W}_\tau := W_\tau + \int_0^\tau \widetilde{C}(s, X_s^{t,x})ds,$$

is a cylindrical Wiener process on U up to time T. Under this transformation, system (1.6) reads as

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + Gd\widehat{W}_\tau, & \tau \in [t, \mathcal{T}], \\ X_t^{t,x} = x, & \tau \in [0,t], \\ -dY_\tau^{t,x,n} = -A_nY_\tau^{t,x,n}d\tau + G\widetilde{C}(\tau, X_\tau^{t,x})d\tau + Z_\tau^{t,x,n}\widetilde{C}(\tau, X_\tau^{t,x})d\tau - Z_\tau^{t,x,n}d\widehat{W}_\tau, & \tau \in [t, \mathcal{T}], \\ Y_{\mathcal{T}}^{t,x,n} = 0, \end{cases} \tag{1.7}$$

and this system has a unique solution $(X^{t,x}, Y^{t,x,n}, Z^{t,x,n})$ where $(Y^{t,x,n}, Z^{t,x,n})$ is a pair of predictable processes belonging to $L^2(\Omega; C([0, \mathcal{T}]; H)) \times L^2(\Omega \times [0, \mathcal{T}]; L_2(U; H))$. Further, $\mathbb{P}$-a.s.

$$Y_\tau^{t,x,n} = e^{-(\mathcal{T}-\tau)A_n}u_n^{\mathcal{T}}(\tau, \Xi_\tau^{t,x}), \quad Z_\tau^{t,x,n} = e^{-(\mathcal{T}-\tau)}\nabla^G u_n^{\mathcal{T}}(\tau, \Xi_\tau^{t,x}), \text{ a.e. } \tau \in [t, \mathcal{T}], \tag{1.8}$$

where ∇^G denotes the Gâteaux derivative along the direction of G and $u_n^{\mathcal{T}}$ is the unique solution to

$$v(t,x) = \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_0}G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G v(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{1.9}$$

Here, $(R_t)_{t\geq 0}$ is the Ornstein-Uhlenbeck semigroup defined by $R_t[\Phi](x) := \mathbb{E}[\Phi(\Xi_t^{0,x})]$, for any $\Phi \in B_b(H; H)$, any $t \geq 0$ and any $x \in H$ (see Section 5.1), and $\Xi_\tau^{0,x}$ is the Ornstein-Uhlenbeck process defined by means of (1.1) when $\widetilde{C} = 0$. In mild formulation, the process $Y^{t,x,n}$ satisfies

$$\begin{aligned}Y_\tau^{t,x,n} =& \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} G\widetilde{C}(s, X_s^{t,x})ds + \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} Z_s^{t,x,n}\widetilde{C}(s, X_s^{t,x})ds \\ &- \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} Z_s^{t,x,n} d\widehat{W}_s \\ =& \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} G\widetilde{C}(s, X_s^{t,x})ds - \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} Z_s^{t,x,n} dW_s, \quad \mathbb{P}\text{-a.s.,} \qquad (1.10)\end{aligned}$$

for every $\tau \in [0, \mathcal{T}]$. By setting $t = \tau = 0$, by applying the operator $e^{\mathcal{T}A_n}$ to the first and the last side of (1.10) and by taking (1.8) into account, we get

$$\int_\tau^{\mathcal{T}} e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s, X_s^{t,x})ds = u_n^{\mathcal{T}}(0, x) + \int_0^{\mathcal{T}} e^{(\mathcal{T}-s)A_n} \nabla^G u_n^{\mathcal{T}}(s, X_s^{0,x})dWs, \quad \mathbb{P}\text{-a.s.,}$$

for every $\mathcal{T} \in [0, T]$. By replacing this formula in (1.5) it follows that

$$\begin{aligned}X_t =& e^{tA}x + \int_0^t \left(e^{(t-s)A} - e^{(t-s)A_n}\right) G\widetilde{C}(s, X_s)ds + \int_0^t e^{(t-s)A} GdW_s \\ & u_n^t(0, x) + \int_0^t e^{(t-s)A_n} \nabla^G u_n^t(s, X_s^{0,x})dWs, \quad \mathbb{P}\text{-a.s.,} \qquad (1.11)\end{aligned}$$

for every $t \in [0, T]$. From there, to obtain (1.4) we need that the first integral in the right-hand side of (1.11) converges to 0 as n goes to $+\infty$, and that the function u_n^t is smooth enough in order to get Lipschitz estimates of the addends which involve u_n^t in (1.11). Both these things are a consequence of Hypothesis 2.4, and so estimate (1.4) follows.

1.3. Plan of the paper

The paper is organized as follows. In Section 2 we state the main assumptions on the coefficients of equation (1.1), under which there exists a (unique in law) weak mild solution (X_t) to (1.1) (see Hypothesis 2.1). Further, we provide sufficient conditions which ensure existence and uniqueness of a smooth solution $u_n^{\mathcal{T}}$ to the integral equation (1.9) for any $n \in \mathbb{N}$ and any $\mathcal{T} \in (0, T]$ (see Hypothesis 2.4). We stress that estimates on the Hilbert-Schmidt norm of $\nabla\nabla^G u_n^{\mathcal{T}}$, together with the family of systems of FBSDEs (1.6), are one of the main tool which we need to prove our result. Finally, we prove a generalized Gronwall Lemma which will be applied in the proof of Theorem 4.2.

In Section 3 we consider the family of systems of FBSDEs (1.10) and we show that, under our assumptions, for any $n \in \mathbb{N}$, any $\mathcal{T} \in (0, T]$, any $x \in H$ and any $t \in [0, \mathcal{T})$ there exists a unique mild solution $(X^{t,x}, Y^{t,x,n}, Z^{t,x,n})$ to (1.7) which satisfies (1.8).

In Section 4 we prove the main result of the paper. At first we show that representation (1.11) holds true for the weak mild solution (X_t). Finally, by means of this representation and of Hypothesis 2.4, we prove Theorem 4.2, which states that there exists a positive constant c_T such that for every $x_1, x_2 \in H$ estimate (1.4) is satisfied.

Section 5 is devoted to provide sufficient conditions on A, G and on $\widetilde{C}$ which ensure that Hypothesis 2.4 are verified. We split this section into three parts. In the former we show the existence of a unique smooth solution $u_n^{\mathcal{T}}$ to equation (1.9) for any $n \in \mathbb{N}$ and any $\mathcal{T} \in (0, T]$, while in the second and in the latter we prove the crucial estimate on the Hilbert-Schmidt norm of $\nabla\nabla^G u_n^{\mathcal{T}}$ when $(A_n)_{n\in\mathbb{N}}$ are the Yosida approximants of A and when $(A_n)_{n\in\mathbb{N}}$ are finite dimensional approximations of A, respectively.

Finally, Section 6 concerns with two concrete models to which our abstract results apply:

(i) a stochastic damped Euler-Bernoulli beam equation

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(t) = -\Lambda y(t) - \rho\Lambda^\alpha \dfrac{\partial y}{\partial t}(t) + \Lambda^{-\gamma} c\left(t, y(t), \dfrac{\partial y}{\partial t}(t)\right) + \Lambda^{-\gamma}\dot{W}(t), \quad t \in (0, T], \\ y(0) = y_0, \\ \dfrac{\partial y}{\partial t}(0) = y_1, \end{cases}$$

where $\Lambda : D(\Lambda) \subset U \to U$ is a positive self-adjoint operator such that $\Lambda^{-\gamma}$ is trace class;

(ii) a semilinear stochastic heat equation

$$\begin{cases} dX_t = \Delta X_t dt + (-\Delta)^{-\gamma/2}\widetilde{C}(X_t)dt + (-\Delta)^{-\gamma/2} dW_t, \quad t \in [0, T], \\ X_0 = x \in H, \end{cases}$$

where Δ is the Laplacian operator on $L^2([0, \pi]^d)$ with periodic boundary conditions.

We note that

1. in the case of beam equation, the result is completely new;
2. in the case of the heat equation, the result extends the one from [8] to the dimension $d = 3$.

Appendix contains minimal energy estimates for the beam equation which are necessary for our approach. To the best of our knowledge, these estimates are new and of independent interest. They are inspired by the spectral methods used in [1,22,29].

1.4. Notations

Throughout the paper we denote by H, K, U real and separable Hilbert spaces.

$L(H; K)$ denotes the Banach space of all bounded and linear operators from H into K endowed with the operator norm; we set $L(H) = L(H; H)$. Moreover $L_2(H; K) \subset L(H; K)$ is the Hilbert space of all Hilbert-Schmidt operators, i.e., the space of operators $T \in L(H; K)$ such that

$$\sum_{k\geq 1} |Te_k|_K^2 < +\infty,$$

where (e_k) is an orthonormal basis of H. We introduce the norm $\|\cdot\|_{L_2(H;K)}$ as

$$\|T\|^2_{L_2(H,K)} = \sum_{k\geq 1} |Te_k|^2_K, \quad T \in L_2(H;K).$$

We also set $L_2(H) = L_2(H;H)$.

If $F : H \to K$ is Gâteaux differentiable at $x \in H$, we denote by $\nabla F(x) \in L(H;K)$ its Gâteaux derivative at x and by $\nabla_k F(x)$ its directional derivative in the direction of $k \in H$, i.e.,

$$\nabla_k F(x) := \lim_{t\to 0} \frac{F(x+tk) - F(x)}{t}, \quad x, k \in H.$$

Let $G : U \to H$ be a linear bounded operator. We are interested into differentiating along G-directions, i.e., directions $k \in H$ such that $k = Ga$ for some $a \in U$. We introduce the notation $\nabla^G F(x) := \nabla F(x) G \in L(U;H)$ and

$$\nabla^G_a F(x) G := \nabla_{Ga} F(x) = \lim_{t\to 0} \frac{F(x+tGa) - F(x)}{t}, \quad x \in H, \quad a \in U. \tag{1.12}$$

2. The abstract equation

We fix $T > 0$, and we will consider the following semilinear stochastic differential equation in the real separable Hilbert space H:

$$\begin{cases} dX^{t,x}_\tau = AX^{t,x}_\tau d\tau + G\widetilde{C}(\tau, X^{t,x}_\tau) d\tau + G dW_\tau, & \tau \in [t,T],\ 0 \leq t \leq \tau, \\ X^{t,x}_t = x \in H, \end{cases} \tag{2.1}$$

where $W = (W_t)$ is a cylindrical Wiener process on another real separable Hilbert space U. Recall that W is formally given by "$W_t = \sum_{n\geq 1} W^{(n)}_t e_n$" where $(W^{(n)})_{n\geq 1}$ are independent real Wiener processes and (e_n) is a basis of U; W defines a Wiener process on any Hilbert space $U_1 \supset U$ with Hilbert-Schmidt embedding (cf. Section 4.1.2 in [13] and Section 2 in [16]). Moreover A, $\widetilde{C}$ and G satisfy the following assumptions.

Hypothesis 2.1.

(i) $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(e^{tA})_{t\geq 0}$.

(ii) $G : U \to H$ is a bounded linear operator.

(iii) There exists $\alpha \in (0, 1/2)$ such that

$$\int_0^T t^{-2\alpha} \|e^{tA} G\|^2_{L_2(U;H)}\, dt < +\infty. \tag{2.2}$$

(iv) The function $\widetilde{C} : [0,T] \times H \to U$ is bounded and continuous and, moreover, there exists a positive constant K and $\beta \in (0,1)$ such that

$$|\widetilde{C}(t,x)-\widetilde{C}(t,y)|_U \le K|x-y|_H^{\beta}, \quad x,y\in H,\ t\in[0,T]. \tag{2.3}$$

Note that condition (2.2) implies that for any $t>0$ the linear bounded operator

$$Q_t := \int_0^t e^{sA}GG^*e^{sA^*}ds : H\to H \text{ is a trace class operator.} \tag{2.4}$$

Condition (2.2) is implicitly assumed also in [8]. It ensures that solutions have continuous paths with values in H (cf. Section 5.3 in [13]).

Clearly, $C := G\widetilde{C} : [0,T]\times H\to H$ and $\|C\|_\infty := \sup_{t\in[0,T],x\in H}|C(t,\cdot)|_H < +\infty$.

We underline that in Hypothesis 2.1 it is included the case $U=H$ and $G=I$, if the stochastic convolution Q_t, $t>0$, is a trace class operator and (2.2) holds.

Recall that a (weak) mild solution to (2.1) is given by $(\Omega,\mathcal{F},(\mathcal{F}_\tau),\mathbb{P},W,X)$, where $(\Omega,\mathcal{F},(\mathcal{F}_\tau),\mathbb{P})$ is a stochastic basis on which it is defined a cylindrical U-valued $\mathcal{F}_\tau$-Wiener process W and a continuous $\mathcal{F}_t$-adapted H-valued process $X=(X_\tau)=(X_\tau)_{\tau\in[t,T]}$ such that

$$X_\tau = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-s)A}G\widetilde{C}(s,X_s)ds + \int_t^\tau e^{(\tau-s)A}GdW_s, \quad \tau\in[t,T], \tag{2.5}$$

$\mathbb{P}$-a.s.

We say that equation (2.1) has a strong mild solution if, on every stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t),P)$ on which there is defined a cylindrical $\mathcal{F}_t$-Wiener process W on U, there exists a weak mild solution.

Under Hypothesis 2.1 the existence of a (weak) mild solution to problem (2.1) which is unique in law is a direct consequence of the Girsanov theorem. Recall that

Proposition 2.2. *Under Hypothesis 2.1, for any $x\in H$, $t\in[0.T[$, there exists a unique in law (weak) mild solution to equation (2.1).*

Proof. The result directly follows from an infinite dimensional version of the Girsanov theorem, see e.g. [28, Proposition 7.1] and [13, Section 10.3]. □

For any $x\in H$ we consider the Ornstein-Uhlenbeck process $\Xi=(\Xi_t^{0,x})$, i.e. the unique solution to (2.1) with $\widetilde{C}=0$. The vector-valued Ornstein-Uhlenbeck transition semigroup $(R_t)_{t\ge0}$ associated to Ξ is defined as $R_t[\Phi](x) := \mathbb{E}\big[\Phi(\Xi_t^{0,x})\big]$ for any $t\ge0$, any $\Phi\in B_b(H,H)$ and any $x\in H$ (cf. [8]).

We take any generator A_0 of a C_0-semigroup of linear operators $(e^{tA_0})_{t\ge0}$ in $L(H)$, we fix $\mathcal{T}\in(0,T]$ and consider the integral equation

$$v(t,x) = \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_0}G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G v(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds, \tag{2.6}$$

for any $(t,x)\in[0,\mathcal{T}]\times H$.

We define the space $\mathcal{G}^{0,1}([0,\mathcal{T}]\times H;H)$, see [16] Section 2.2, as the subspace of $C_b([0,\mathcal{T}]\times H;H)$ consisting of all functions f which are Gâteaux differentiable with respect to x and such that the map $\nabla f:[0,\mathcal{T}]\times H\to L(H)$ is strongly continuous and globally bounded.

Definition 2.3. A solution to (2.6) is a mapping $u=u^{\mathcal{T}}\in\mathcal{G}^{0,1}([0,\mathcal{T}]\times H;H)$ which solves (2.6).

To prove our main result we require the existence of a family of operators $(A_n)_{n\in\mathbb{N}}$, generators of C_0-*groups* of linear operators $((e^{tA_n})_{t\in\mathbb{R}})_{n\in\mathbb{N}}$, and for each $n\in\mathbb{N}$ we consider the integral equation (2.6) with A_n in the place of A_0, that is

$$v(t,x)=\int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s,\cdot)\right](x)ds+\int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G v(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{2.7}$$

Note that (2.7) is a mild formulation of a Kolmogorov PDE related to (2.1); on this aspect see Remark 2.10 which also compares such parabolic PDE with the one considered in [8].

We will denote by $u_n^{\mathcal{T}}$ the solution of this equation (2.7). On the family of operators $(A_n)_{n\in\mathbb{N}}$ and on the solution of the above integral equation we make the following assumptions.

Hypothesis 2.4.

(A) For any $n\in\mathbb{N}$ the operator A_n generates a strongly continuous group of linear and bounded operators $(e^{tA_n})_{t\in\mathbb{R}}\subset L(H)$ such that for any $T>0$ we have

$$\sup_{t\in[0,T]}\sup_{n\geq 1}\|e^{tA_n}\|_{L(H)}=K_T<\infty,\qquad \lim_{n\to\infty}e^{tA_n}x=e^{tA}x,\ \ x\in H,\ \ t\geq 0. \tag{2.8}$$

(B) For each $n\in\mathbb{N}$ there exists a unique solution $u_n^{\mathcal{T}}$ to (2.7) which verifies:
(i) there exists $C_T>0$, independent of n and $\mathcal{T}$, such that

$$\sup_{x\in H}|u_n^{\mathcal{T}}(0,x+y)-u_n^{\mathcal{T}}(0,x)|_H\leq C_T\,|y|_H,\qquad y\in H. \tag{2.9}$$

(ii) For any $n\in\mathbb{N}$ and any $(t,x)\in[0,\mathcal{T}]\times H$, the map $k\mapsto\nabla_k^G u_n^{\mathcal{T}}(t,x)\in L_2(U;H)$ and $\nabla_\cdot^G u_n^{\mathcal{T}}\in B_b([0,\mathcal{T}]\times H;L_2(U;H))$. Further, we assume that there exists an integrable function $h:(0,\mathcal{T})\to\mathbb{R}_+$, independent of $n\in\mathbb{N}$, such that for any $n\in\mathbb{N}$ we have

$$\sup_{x\in H}\|\nabla_\cdot^G u_n^{\mathcal{T}}(t,x+y)-\nabla_\cdot^G u_n^{\mathcal{T}}(t,x)\|_{L_2(U;H)}^2\leq h(\mathcal{T}-t)\,|y|_H^2,\qquad t\in(0,\mathcal{T}),\ \ y\in H. \tag{2.10}$$

Further, there exists a positive constant $C=C(T)$ which only depends on T such that $\|h\|_{L^1(0,\mathcal{T})}\leq C$ for any $\mathcal{T}\in(0,T]$.

Remark 2.5. In order to ensure the validity of (i) and (ii) in (B), a sufficient condition for solutions to (2.7) is the following one: $u_n^{\mathcal{T}}(t,\cdot)$ is Fréchet differentiable on H, for any $t\in[0,\mathcal{T}]$, with uniformly bounded Fréchet derivative $\nabla u_n^{\mathcal{T}}(t,\cdot)$, i.e.,

$$\sup_{n\geq 1}\sup_{t\in[0,\mathcal{T}]}\sup_{x\in H}\|\nabla u_n^{\mathcal{T}}(t,x)\|_{L(U,H)}<\infty.$$

Moreover, for any $k\in U$, $t\in[0,\mathcal{T}]$, the map $x\mapsto\nabla_k^G u_n^{\mathcal{T}}(t,x)$ is Fréchet differentiable on H, $\nabla_\cdot^G u_n^{\mathcal{T}}\in B_b([0,\mathcal{T}]\times H;L_2(U;H))$ and

$$\sup_{x\in H}\|\nabla_y\nabla_\cdot^G u_n^{\mathcal{T}}(t,x)\|^2_{L_2(U;H)}\leq h(\mathcal{T}-t)\,|y|_H^2,\quad t\in(0,\mathcal{T}),\ y\in H, \tag{2.11}$$

for some integrable function $h:(0,\mathcal{T})\to\mathbb{R}_+$ with $\|h\|_{L^1(0,\mathcal{T})}\leq C$ for any $\mathcal{T}\in(0,T]$, for some positive constant $C=C(T)$.

Note that for any $x,y\in H$ and $t\in(0,T)$ we have (using an orthonormal basis (f_k) in U)

$$\begin{aligned}\|\nabla_y\nabla_\cdot^G u_n^{\mathcal{T}}(t,x)\|^2_{L_2(U;H)}&=\sum_{k\geq 1}|\nabla_y\nabla_{f_k}^G u_n^{\mathcal{T}}(t,x)|_H^2=\sum_{k\geq 1}|\nabla_y\nabla_{Gf_k}u_n^{\mathcal{T}}(t,x)|_H^2\\&=\sum_{j\geq 1}\sum_{k\geq 1}\langle\nabla_y\nabla_{Gf_k}u_n^{\mathcal{T}}(t,x),e_j\rangle_H^2=\sum_{j\geq 1}\sum_{k\geq 1}[\nabla_y\nabla_{Gf_k}u_{n,j}^{\mathcal{T}}(t,x)]^2\\&=\sum_{j\geq 1}|\nabla_y\nabla^G u_{n,j}^{\mathcal{T}}(t,x)|_U^2,\end{aligned} \tag{2.12}$$

where $u_{n,j}^{\mathcal{T}}:=\langle u_n^{\mathcal{T}},e_j\rangle$ (we are identifying $L(U;\mathbb{R})$ with U). We will prove that the unique mild solution $u_n^{\mathcal{T}}$ of (2.7) satisfies (2.11), and to prove this estimate we will make use of (2.12).

Remark 2.6. In the case of the wave equation as in [26] and [27] we can take $A_n=A$ and so $e^{tA_n}=e^{tA}$ because A is the generator of a group of operators, while in the case of the damped equation A_n will be the Yosida approximations, i.e.,

$$A_n:=nAR(n,A),\quad n\in\mathbb{N},$$

where $R(\lambda,A)$ is the resolvent operator of A, for any λ in the resolvent set of A. Finally, for the stochastic parabolic PDEs considered in [8] we will consider A_n as the finite dimensional approximations of A.
We notice that the stochastic damped equation and the stochastic parabolic PDEs considered in [8] are reformulated as a stochastic evolution equation in H like (2.1), with A generator of a strongly continuous semigroup of linear operators, while the Yosida approximants of A and the finite dimensional approximations of A generate a group of linear operators.

We stress that if the drift C is Lipschitz continuous with respect to x, uniformly in t, existence of a strong mild solution follows in a standard way, see e.g. [13, Theorem 7.6].

Proposition 2.7. *Assume that $C:[0,T]\times U\to H$ is continuous, bounded and $h\mapsto C(\tau,h)$ is Lipschitz continuous, uniformly with respect to τ. Then for any $x\in H$ there exist a unique mild solution to equation* (2.1).

To conclude this section, we provide the following generalization of the Gronwall lemma which will be used in the sequel (see [14, Lemma 3.1]).

Lemma 2.8. *Let* $f, u, v : [0, T] \to \mathbb{R}$ *be bounded measurable functions. Moreover,* $f : [0, T] \to \mathbb{R}_+$*. Let* $C \geq 0$ *and* $g, h : (0, T) \to \mathbb{R}_+$ *be integrable functions.*

(i) If for any $t \in [0, T]$

$$v(t) \leq f(t) + \int_t^T g(s-t)v(s)ds, \quad t \in [0, T],$$

then, for any $t \in [0, T]$,

$$v(t) \leq f(t) + e^{\|g\|_{L^1(0,T)}} \int_t^T f(s)g(s-t)ds.$$

(ii) If for any $t \in [0, T]$

$$u(t) \leq C + \int_0^t h(t-s)u(s)ds, \quad t \in [0, T],$$

then

$$u(t) \leq C\left(1 + \|h\|_{L^1(0,T)} e^{\|h\|_{L^1(0,T)}}\right), \quad t \in [0, T].$$

Proof. Assertion (i) directly follows from [14, Lemma 3.1]. Assertion (ii) can be deduced by (i) as follows. We have $u(t) \leq C + \int_0^t h(r)u(t-r)dr$; define $w(t) = u(T-t)$. From the assumptions the function $s \mapsto h(t-s)w(s) \in L^1(t, T)$ for any $t \in [0, T]$. We get

$$\begin{aligned} w(t) &\leq C + \int_0^{T-t} h(r)u(T-t-r)dr = C + \int_0^{T-t} h(r)w(t+r)dr \\ &= C + \int_t^T h(s-t)w(s)dr, \end{aligned}$$

and (ii) follows by (i). □

Remark 2.9. We recall that under the assumptions of Lemma 2.8, if

$$u(t) \leq C + \int_0^t h(s)u(s)ds, \quad t \in [0, T],$$

then

$$u(t) \le C\left(1 + \|h\|_{L^1(0,T)} e^{\|h\|_{L^1(0,1)}}\right), \quad t \in [0,T].$$

In the following remark we compare the Kolmogorov equations (2.7) used in the present paper and the ones considered in [8].

Remark 2.10. Let $\mathcal{T} \in (0,T]$ and let $n \in \mathbb{N}$. Formal computations give that the H-valued solution $u_n^{\mathcal{T}}$ of (2.7) formally solves the equation

$$\begin{cases} \dfrac{\partial u_n^{\mathcal{T}}(t,x)}{\partial t} + \mathcal{L}_t[u_n^{\mathcal{T}}(t,\cdot)](x) = -e^{(\mathcal{T}-t)A_n} G\widetilde{C}(t,x), & x \in H,\ t \in [0,\mathcal{T}], \\ u_n^{\mathcal{T}}(\mathcal{T},x) = 0, & x \in H, \end{cases} \tag{2.13}$$

where $\mathcal{L}_t f(x) := \frac{1}{2}\mathrm{Tr}[GG^*\nabla^2 f(x)] + \langle Ax, \nabla f(x)\rangle + \langle G\widetilde{C}(t,x), \nabla f(x)\rangle$, for any $t \in [0,T]$ and any $x \in H$, and f is a regular function. Kolmogorov equations similar to (2.13) are considered in [8] for the study of semilinear parabolic SPDEs (however, note that in equation (6) of [8] the term $-e^{(\mathcal{T}-t)A_n}G\widetilde{C}$ is replaced by $G\widetilde{C}$).

On the other hand, the function $v_n := e^{-(\mathcal{T}-t)A_n}u_n^{\mathcal{T}}$ formally solves

$$\begin{cases} \dfrac{\partial v_n^{\mathcal{T}}(t,x)}{\partial t} + \mathcal{L}_t[v_n^{\mathcal{T}}(t,\cdot)](x) = A_n v_n^{\mathcal{T}}(t,x) - G\widetilde{C}(t,x), & x \in H,\ t \in [0,\mathcal{T}], \\ v_n^{\mathcal{T}}(\mathcal{T},x) = 0, & x \in H, \end{cases}$$

which is similar to the equation (formally) solved by the function v considered in [26, Remark 6.2].

3. Ornstein-Uhlenbeck processes and approximated FBSDEs

In this section we consider a family of FBSDEs on the time interval $[t,\mathcal{T}]$, with $0 \le t < \mathcal{T} \le T$. Namely in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for any $n \in \mathbb{N}$ and for $\tau \in [t,\mathcal{T}]$ we consider the following system of FBSDEs with forward and backward equations both taking values in H, given by

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x}d\tau + GdW_\tau, & \tau \in [t,\mathcal{T}], \\ \Xi_\tau^{t,x} = x, & \tau \in [0,t], \\ -dY_\tau^{t,x,n} = -A_nY_\tau^{t,x,n}d\tau + G\widetilde{C}(\tau, \Xi_\tau^{t,x})d\tau + Z_\tau^{t,x,n}\widetilde{C}(\tau,\Xi_\tau^{t,x})d\tau - Z_\tau^{t,x,n}dW_\tau, & \tau \in [0,\mathcal{T}], \\ Y_{\mathcal{T}}^{t,x,n} = 0. \end{cases} \tag{3.1}$$

The solution to the forward equation is the so called Ornstein-Uhenbeck process $\Xi = (\Xi_\tau^{t,x})$ and it is nothing else than equation (2.1) with drift C equal to 0. Moreover G, $\widetilde{C}$ and W are the same as in (2.1). Finally $(A_n)_{n\ge1}$ are given in Hypothesis 2.4, part (A).

Remark 3.1. We recall that under the general assumptions of the present paper we cannot consider the BSDE with A instead of A_n because $-A$ is not the generator of a semigroup of operators (see [17]). This case has been considered in [26] since A is the generator of a group of operators, and here we generalize the method introduced in [26] to the case of A generator of a semigroup of linear operators.

Let $n \in \mathbb{N}$. Concerning the backward equation in the FBSDE (3.1), its precise meaning is given by its mild formulation: $\mathbb{P}$-a.s. the pair of processes $(Y^{t,x,n}, Z^{t,x,n})$ satisfies

$$Y_\tau^{t,x,n} = \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} G\widetilde{C}(s, \Xi_s^{t,x})\, ds + \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} Z_s^{t,x,n}\, \widetilde{C}(s, \Xi_s^{t,x})\, ds - \int_\tau^{\mathcal{T}} e^{-(s-\tau)A_n} Z_s^{t,x,n}\, dW_s, \tag{3.2}$$

for any $\tau \in [t, \mathcal{T}]$ (cf. [16], [17], [21] and the references therein). Notice that in order to give sense to the BSDE in (3.1) we need that $-A_n$ is the generator of a C_0-semigroup of bounded linear operators, and this is true if we assume that Hypothesis 2.4, part (A) is satisfied.

Concerning equation (3.2) recall that we endow $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration $(\mathcal{F}_t^W)$ of W (i.e., $\mathcal{F}_t^W$ is the smallest σ-algebra generated by $W_s^{(n)}$, $n \geq 1$ and $0 \leq s \leq t$) augmented in the usual way with the family of $\mathbb{P}$-null sets of $\mathcal{F}$. All the concepts of measurability, e.g. predictability, are referred to this filtration.

The solution of (3.2) will be a pair of processes $(Y^{t,x,n}, Z^{t,x,n}) \in L^2_{\mathcal{P}}(\Omega; C([0, \mathcal{T}]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0, \mathcal{T}]; L_2(U; H))$ (see Proposition 3.2), where $L^2_{\mathcal{P}}(\Omega; C([0, \mathcal{T}], H))$ is the Banach space of all predictable H-valued processes Y with continuous paths and such that

$$\mathbb{E}[\sup_{\tau\in[0,\mathcal{T}]} |Y_\tau|^2] = \|Y\|^2_{L^2_{\mathcal{P}}(\Omega, C([0,\mathcal{T}]; H))} < \infty,$$

and $L^2_{\mathcal{P}}(\Omega \times [0, \mathcal{T}]; L_2(U, H))$ is the usual L^2-space of predictable processes Z with values in $L_2(U, H)$. We sum up in the following proposition existence results for equation (3.2).

Proposition 3.2. *Assume Hypotheses 2.1 and 2.4 hold true. Then, for any $n \in \mathbb{N}$ the BSDE (3.2) admits a unique solution $(Y^{t,x,n}, Z^{t,x,n}) \in L^2_{\mathcal{P}}(\Omega; C([0, \mathcal{T}]; H)) \times L^2_{\mathcal{P}}(\Omega \times [0, \mathcal{T}]; L_2(U; H))$, satisfying*

$$\mathbb{E}\Big[\sup_{\tau\in[0,\mathcal{T}]} |Y_\tau^{t,x,n}|^2\Big] + \mathbb{E}\int_0^{\mathcal{T}} \|Z_\tau^{t,x,n}\|^2_{L_2(U,H)} d\tau \leq C_{\mathcal{T},n}\|C\|_\infty, \tag{3.3}$$

where $C_{\mathcal{T},n}$ is a positive constant which depends also on $\mathcal{T}$ and n, and the map: $(t, x) \mapsto Y_t^{t,x}$, $[0, \mathcal{T}] \times H \to H$, is deterministic.

Proof. Existence and uniqueness of a solution directly come from Lemma 2.1 and Proposition 2.1 in [21], that we can apply since C is bounded. Estimate (3.3) follows from [18], Remark 4.5, estimate (4.19). Since the process $\Xi^{t,x}$ is $\mathcal{F}^W_{t,T}$-measurable (where $\mathcal{F}^W_{t,T}$ is the σ-algebra generated by $W_r - W_t$, $r \in [t, T]$, augmented with the $\mathbb{P}$-null sets), it turns out that $Y^{t,x}_t$ is measurable both with respect to $\mathcal{F}^W_{t,T}$ and $\mathcal{F}_t$; it follows that $Y^{t,x}_t$ is deterministic. $\square$

Next we prove an identification property that in the present paper we will apply to $u_n^{\mathcal{T}}$, solution to equation (2.7), which is the analogous of the identification formulae proved e.g. in [16] for real valued functions (see also [25] for the case of functions defined on Banach spaces).

Lemma 3.3. *Let $v : [0, \mathcal{T}] \times H \longrightarrow H$ be a continuous function such that for every $t \in [0, \mathcal{T}]$, $v(t, \cdot)$ is Gâteaux differentiable and the map $(t, x) \mapsto \nabla v(t, x)$ is Borel measurable. Let us fix $(t, x) \in [0, \mathcal{T}] \times H$ and let $\Xi^{t,x}$ be the Ornstein-Uhlenbeck process defined in Section 2. If ψ is a square integrable predictable process and $\bar{Z} \in L^2_{\mathcal{P}}(\Omega \times [0, \mathcal{T}]; L_2(U, H))$ and if $v\left(\tau, \Xi^{t,x}_\tau\right)$ admits the representation*

$$v\left(\tau, \Xi^{t,x}_\tau\right) = v\left(\mathcal{T}, \Xi^{t,x}_{\mathcal{T}}\right) + \int_\tau^{\mathcal{T}} \psi_s ds - \int_\tau^{\mathcal{T}} \bar{Z}_s dW_s, \quad \tau \in [0, \mathcal{T}], \tag{3.4}$$

then $\mathbb{P}$-a.s, $\nabla^G v\left(\tau, \Xi^{t,x}_\tau\right) = \bar{Z}_\tau$, for a.e. $\tau \in [0, \mathcal{T}]$.

Proof. The result can be seen as an extension of [16, Proposition 5.6] to the case of an H-valued BSDE, and for this extension we use techniques similar to the ones in [26]. Let $\xi \in U$ and consider the real Wiener process $(W^\xi_\tau)_{\tau \geq 0}$, where

$$W^\xi_\tau := \langle \xi, W_\tau \rangle_U .$$

Let $h \in H$, we set

$$v^h(\tau, x) := \langle v(\tau, x), h \rangle_H, \quad 0 \leq t \leq \tau \leq \mathcal{T}, \, x \in H,$$

and we study the joint quadratic variation between the real process $v^h\left(\cdot, \Xi^{t,x}\right)$ and W^ξ. Since

$$\begin{aligned} v^h\left(\tau, \Xi^{t,x}_\tau\right) =& v^h\left(\mathcal{T}, \Xi^{t,x}_{\mathcal{T}}\right) + \int_\tau^{\mathcal{T}} \langle \psi_s, h \rangle_H \, ds - \int_\tau^{\mathcal{T}} \langle \bar{Z}_s, h \rangle_H \, dW_s \\ =& v^h\left(0, \Xi^{t,x}_0\right) - \int_0^\tau \langle \psi_s, h \rangle_H \, ds + \int_0^\tau \langle \bar{Z}_s, h \rangle_H \, dW_s, \quad \tau \in [0, \mathcal{T}], \end{aligned}$$

we find

$$\langle v^h\left(\cdot, \Xi^{t,x}\right), W^\xi \rangle_\tau = \int_0^\tau \langle \bar{Z}_s \xi, h \rangle_H ds, \;\; \tau \in [0, \mathcal{T}], \; \mathbb{P}\text{-a.s.} \tag{3.5}$$

Now we compute the joint quadratic variation in a different way, arguing as in [16, Lemmata 6.3 & 6.4] and we obtain that the real process $v^h\left(\cdot, \Xi^{t,x}\right)$ admits joint quadratic variation with W^ξ given by

$$\langle v^h(\cdot, \Xi^{t,x}), W^\xi\rangle_\tau = \int_0^\tau \nabla^G v^h(s, \Xi_s^{t,x})\xi\, ds = \int_0^\tau \langle \nabla^G v(s, \Xi_s^{t,x})\xi, h\rangle\, ds, \quad \tau \in [0, \mathcal{T}].$$

Comparing this formula with (3.5) we get that for a.e. $s \in [0, T]$ we have, $\mathbb{P}$-a.s.,

$$\langle \bar{Z}_s\xi, h\rangle = \langle \nabla^G v(s, \Xi_s^{t,x})\xi, h\rangle.$$

Since H is separable, for any $\xi \in U$ we get $\mathbb{P}$-a.s. $\bar{Z}_s\xi = \nabla^G v(s, \Xi_s^{t,x})\xi$, for a.e. $s \in [0, \mathcal{T}]$. The assertion now follows. □

Next we are going to apply the previous Lemma 3.3 to $u_n^{\mathcal{T}}$ solution to equation (2.7). We assume that Hypothesis 2.4, part (B), holds true and recall that the operator A_n generates a group of linear bounded operators $(e^{\sigma A_n})_{\sigma\in\mathbb{R}}$. We want to show that the pair of processes

$$(e^{-(\mathcal{T}-\tau)A_n}u_n^{\mathcal{T}}(\tau, \Xi_\tau^{t,x}), \nabla^G e^{-(\mathcal{T}-\tau)A_n}u_n^{\mathcal{T}}(\tau, \Xi_\tau^{t,x})), \ \tau \in [t, \mathcal{T}], \tag{3.6}$$

satisfy the BSDE (3.2). This is the content of the following proposition.

Proposition 3.4. *Assume Hypotheses 2.1 and 2.4 hold true. Let $u_n^{\mathcal{T}}$ be the unique solution to (2.7). Then the pair of processes defined in (3.6) is the unique solution of the BSDE (3.2). As a consequence, if the pair of processes $(Y^{t,x,n}, Z^{t,x,n})$ is solution to the BSDE (3.2), then we have $Y_t^{t,x,n} = e^{-(\mathcal{T}-t)A_n}u_n^{\mathcal{T}}(t,x)$, $Z_t^{t,x,n} = e^{-(\mathcal{T}-t)A_n}\nabla^G u_n^{\mathcal{T}}(t,x)$, so that*

$$\nabla^G u_n^{\mathcal{T}}(\tau, \Xi_\tau^{t,x}) = e^{(\mathcal{T}-\tau)A_n}Z_\tau^{t,x,n}, \quad \mathbb{P}\text{-a.s.}, \textit{for a.e. } \tau \in [t, \mathcal{T}]. \tag{3.7}$$

Moreover, if a pair of processes $(Y^{t,x,n}, Z^{t,x,n})$ is solution to (3.2), then by setting

$$\widetilde{v}^n(t,x) := e^{(\mathcal{T}-t)A_n}Y_t^{t,x,n}, \ t \in [0, \mathcal{T}],$$

we get that

$$\nabla^G\widetilde{v}^n(t,x) := e^{(\mathcal{T}-t)A_n}Z_t^{t,x,n}, \ t \in [0, \mathcal{T}],$$

and $\widetilde{v}^n(t,x)$, $t \in [0, \mathcal{T}]$, $x \in H$, is solution to equation (2.7), i.e., $\widetilde{v}_n \equiv u_n^{\mathcal{T}}$.

Proof. The arguments are similar to those used in the proof of the uniqueness part of [16, Theorem 6.2], and are adequated to this different context. For reader's convenience, we simply write u_n instead of $u_n^{\mathcal{T}}$.
Starting from equation (2.7), we notice that for any $\tau \in [t, \mathcal{T}]$,

$$u_n(\tau,x) = \mathbb{E}\int_\tau^{\mathcal{T}} \left[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{\tau,x})\right]ds + \mathbb{E}\int_\tau^{\mathcal{T}} \left[\nabla^G u_n(s, \Xi_s^{\tau,x})\widetilde{C}(s, \Xi_s^{\tau,x})\right]ds.$$

Hence, denoting the conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_\tau]$ as $\mathbb{E}^{\mathcal{F}_\tau}[\cdot]$, the random variable $u_n(\tau, \Xi_\tau^{t,x})$ satisfies $\mathbb{P}$-a.s.

$$\begin{aligned}u_n(\tau, \Xi_\tau^{t,x}) :=& \mathbb{E}^{\mathcal{F}_\tau}\int_\tau^{\mathcal{T}}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{\tau,\Xi_\tau^{t,x}})\Big]ds\\ &+\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^{\mathcal{T}}\Big[\nabla^G u_n(\tau, \Xi_s^{\tau,\Xi_\tau^{t,x}})\widetilde{C}(s, \Xi_s^{\tau,\Xi_\tau^{t,x}})\Big]ds\\ &=\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^{\mathcal{T}}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{t,x})\Big]ds+\mathbb{E}^{\mathcal{F}_\tau}\int_\tau^{\mathcal{T}}\Big[\nabla^G u_n(s, \Xi_s^{t,x})\widetilde{C}(s, \Xi_s^{t,x})\Big]ds,\end{aligned} \tag{3.8}$$

since $\Xi_s^{t,x}$ is $\mathcal{F}_s$-measurable $\forall\, \tau \le s \le \mathcal{T}$, and $\Xi_s^{\tau,\Xi_\tau^{t,x}} = \Xi_s^{t,x}$. Setting

$$\xi := \int_t^{\mathcal{T}}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{t,x})\Big]ds+\int_t^{\mathcal{T}}\Big[\nabla^G u_n(s, \Xi_s^{t,x})\widetilde{C}(s, \Xi_s^{t,x})\Big]ds,$$

we can rewrite (3.8) as

$$u_n(\tau, \Xi_\tau^{t,x}) = \mathbb{E}^{\mathcal{F}_\tau}\xi - \int_t^{\tau}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{t,x})\Big]ds - \int_t^{\tau}\Big[\nabla^G u_n(s, \Xi_s^{t,x})\widetilde{C}(s, \Xi_s^{t,x})\Big]ds,$$

for $\tau \in [t, \mathcal{T}]$. By the representation theorem for martingales, see e.g. [13], there exists $(\bar{Z}_s^{t,x})_{s\in[0.\mathcal{T}]} \in L^2_{\mathcal{P}}(\Omega \times [0, \mathcal{T}]; L_2(U, H))$ such that for any $\tau \in [0, \mathcal{T}]$

$$\begin{aligned}u_n(\tau, \Xi_\tau^{t,x}) =& u_n(t, x) + \int_t^{\tau\wedge t}\bar{Z}_s^{t,x}\,dW_s\\ &-\int_t^{\tau\wedge t}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s, \Xi_s^{t,x})\Big]ds - \int_t^{\tau\wedge t}\Big[\nabla^G u_n(s, \Xi_s^{t,x})\widetilde{C}(s, \Xi_s^{t,x})\Big]ds, \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

We conclude that the process $u_n(\tau, \Xi_\tau^{t,x}),\ t \le \tau \le \mathcal{T}$ is a continuous semimartingale with canonical decomposition. By Lemma 3.3, we have that $\bar{Z}_s^{t,x} = \nabla^G u_n(s, \Xi_s^{t,x})$ a.e. $s \in [t, \mathcal{T}]$. Then the previous semimartingale reads as

$$u_n(\tau, \Xi_\tau^{t,x}) = -\int_\tau^{\mathcal{T}}\nabla^G u_n(s, \Xi_s^{t,x})\,dW_s$$

$$+\int_\tau^{\mathcal{T}}\Big[e^{(\mathcal{T}-s)A_n}G\widetilde{C}(s,\Xi_s^{t,x})\Big]ds+\int_\tau^{\mathcal{T}}\Big[\nabla^G u_n(s,\Xi_s^{t,x})\widetilde{C}(s,\Xi_s^{t,x})\Big]ds.$$

If we apply $e^{-(\mathcal{T}-\tau)A_n}$ to both the sides of this equation we get

$$e^{-(\mathcal{T}-\tau)A_n}u_n(\tau,\Xi_\tau^{t,x})=-\int_\tau^{\mathcal{T}}e^{-(s-\tau)A_n}\Big[e^{-(\mathcal{T}-s)A_n}\nabla^G u_n(s,\Xi_s^{t,x})\,dW_s\Big]$$

$$+\int_\tau^{\mathcal{T}}\Big[e^{-(s-\tau)A_n}G\widetilde{C}(s,\Xi_s^{t,x})\Big]ds+\int_t^{\tau}e^{-(s-\tau)A_n}\Big[e^{-(\mathcal{T}-s)A_n}\nabla^G u_n(s,\Xi_s^{t,x})\widetilde{C}(s,\Xi_s^{t,x})\Big]ds,$$

and by comparing this expression with (3.2), it is immediate to see that the pair of processes

$$(e^{-(\mathcal{T}-\tau)A_n}u_n(\tau,\Xi_\tau^{t,x}),e^{-(\mathcal{T}-\tau)A_n}\nabla^G u_n(\tau,\Xi_\tau^{t,x})),\quad \tau\in[t,\mathcal{T}],$$

solves equation (3.2).

The “Moreover” part follows from the fact that, by Proposition 3.2, equation (3.2) admits a unique solution. □

4. Pathwise uniqueness for the nonlinear SDE

Let’s go back to the nonlinear SPDE (2.1), which we rewrite with initial time $t=0$:

$$\begin{cases} dX_\tau^{0,x}=AX_\tau^{0,x}d\tau+G\widetilde{C}(\tau,X_\tau^{0,x})d\tau+GdW_\tau & \tau\in[0,T],\\ X_0^{0,x}=x\in H.\end{cases}$$

The existence of a weak solution to (2.1) has been already discussed in Proposition 2.2. Let us set $X_\tau^x:=X_\tau^{0,x}$ for any $\tau\in[0,T]$ and $x\in H$. Then, $\mathbb{P}$-a.s. we have

$$X_\tau^x=e^{\tau A}x+\int_0^\tau\Big(e^{(\tau-s)A}-e^{(\tau-s)A_n}\Big)G\widetilde{C}(s,X_s^x)ds+\int_0^\tau e^{(\tau-s)A_n}G\widetilde{C}(s,X_s^x)ds$$

$$+\int_0^\tau e^{(\tau-s)A}GdW_s,\qquad \forall\tau\in[0,T]. \tag{4.1}$$

Let $\tau\in[0,T]$. In the next result, for any $n\in\mathbb{N}$ we consider u_n^τ, the regular solution of (2.7) with $\mathcal{T}=\tau$, whose properties are listed in Hypothesis 2.4, part (B).

Proposition 4.1. *Let Hypotheses 2.1 and 2.4 hold true. Then, for any $n\in\mathbb{N}$ and any $\tau\in[0,T]$ we have*

$$\begin{aligned} X_\tau^x =& e^{\tau A}x + \int_0^\tau \left(e^{(\tau-s)A} - e^{(\tau-s)A_n}\right) G\widetilde{C}(s, X_s^x)ds \\ &+ u_n^\tau(0,x) + \int_0^\tau \nabla^G u_n^\tau(s, X_s^x)dW_s + \int_0^\tau e^{(\tau-s)A}GdW_s, \quad \mathbb{P}\text{-a.s..} \end{aligned} \tag{4.2}$$

Proof. Let $0 \le t \le \tau \le T$ and $x \in H$. We consider a (weak) mild solution $X^{t,x}$ to

$$dX_\sigma^{t,x} = AX_\sigma^{t,x}d\sigma + G\widetilde{C}(\sigma, X_\sigma^{t,x})d\sigma + GdW_\sigma, \quad \sigma \in [t,\tau], \quad X_\sigma^{t,x} = x, \quad \sigma \in [0,t],$$

which is given by

$$\begin{aligned} X_\sigma^{t,x} =& e^{(\sigma-t)A}x + \int_t^\sigma e^{(\sigma-s)A}G\widetilde{C}(s, X_s^{t,x})ds + \int_t^\sigma e^{(\sigma-s)A}GdW_s, \quad \mathbb{P}\text{-a.s.}, \quad \sigma \in [t,\tau], \\ X_\sigma^{t,x} =& x, \quad \sigma \in [0,t]. \end{aligned}$$

Such solution is defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, on which it is defined a cylindrical $\mathcal{F}_t$-Wiener process W on U.

Let us set

$$\widehat{W}_\sigma = W_\sigma + \int_0^\sigma \widetilde{C}(s, X_s^{t,x})ds.$$

By the Girsanov theorem (see, for instance, [13, Section 10.3] or the Appendix in [9]) there exists a probability measure $\widehat{\mathbb{P}} = \widehat{\mathbb{P}}_\tau$ on $(\Omega, \mathcal{F}_\tau)$, such that in $(\Omega, \mathcal{F}_\tau, \widehat{\mathbb{P}})$ the process $(\widehat{W}_\sigma)_\sigma$ is a cylindrical Wiener process up to time τ (it is not difficult to prove that $\widehat{\mathbb{P}}$ and $\mathbb{P}$ are equivalent). In $(\Omega, \mathcal{F}_\tau, \widehat{\mathbb{P}})$ the process $X^{t,x}$ solves

$$dX_\sigma^{t,x} = AX_\sigma^{t,x}d\sigma + Gd\widehat{W}_\sigma, \quad X_t^{t,x} = x, \quad \sigma \in [t,\tau].$$

Since for OU stochastic equations pathwise uniqueness holds, we have, in particular, that $(X^{t,x})$ is a predictable process with respect to the completed natural filtration $(\mathcal{F}_s^{\widehat{W}})_{0\le s\le\tau}$ generated by $\widehat{W}$. Let us consider the FBSDE system

$$\begin{cases} dX_\sigma^{t,x} = AX_\sigma^{t,x}d\sigma + Gd\widehat{W}_\sigma, & \sigma \in [t,\tau], \\ X_\sigma^{t,x} = x, & \sigma \in [0,t], \\ -dY_\sigma = -A_nY_\sigma d\sigma + G\widetilde{C}(\sigma, X_\sigma^{t,x})\, d\sigma + Z_\sigma\, \widetilde{C}(\sigma, X_\sigma^{t,x})d\sigma - Z_\sigma\, d\widehat{W}_\sigma, & \sigma \in [0,\tau], \\ Y_\tau = 0. \end{cases} \tag{4.3}$$

From Proposition 3.4, system (4.3) admits a unique solution $(X_\sigma^{t,x}, \widehat{Y}_\sigma^{t,x,n}, \widehat{Z}_\sigma^{t,x,n})$ in $(\Omega, \mathcal{F}_\tau, (\mathcal{F}_s^{\widehat{W}})_{0\le s\le\tau}, \widehat{\mathbb{P}})$. Recall that $\mathcal{F}_s^{\widehat{W}} \subset \mathcal{F}_s$, $s \in [0,\tau]$. Moreover, see (3.6), we have $\widehat{\mathbb{P}}$-a.s in Ω (or equivalently $\mathbb{P}$-a.s. in Ω)

$$\widehat{Y}^{t,x,n}_\sigma = e^{-(\tau-\sigma)A_n} u^\tau_n(\sigma, X^{t,x}_\sigma), \ \ \sigma \in [0,\tau], \ \text{ and } \ \widehat{Z}^{t,x,n}_\sigma = e^{-(\tau-\sigma)A_n} \nabla^G u^\tau_n(\sigma, X^{t,x}_\sigma),$$

for any $\sigma \in [0,\tau]$, a.e. Hence, $\mathbb{P}$-a.s. for any $s \in [0,\tau]$, a.e., we have

$$e^{(\tau-s)A_n} \widehat{Y}^{t,x,n}_s = u^\tau_n(s, X^{t,x}_s), \quad e^{(\tau-s)A_n} \widehat{Z}^{t,x,n}_s = \nabla^G u^\tau_n(s, X^{t,x}_s). \tag{4.4}$$

We recall that

$$\begin{aligned}\widehat{Y}^{t,x,n}_\sigma =& \int_\sigma^\tau e^{-(s-\sigma)A_n} G\widetilde{C}(s, X^{t,x}_s)\, ds + \int_\sigma^\tau e^{-(s-\sigma)A_n} \widehat{Z}^{t,x,n}_s \widetilde{C}(s, X^{t,x}_s) ds \\ & - \int_\sigma^\tau e^{-(s-\sigma)A_n} \widehat{Z}^{t,x,n}_s \, d\widehat{W}_s \\ =& \int_\sigma^\tau e^{-(s-\sigma)A_n} G\widetilde{C}(s, X^{t,x}_s)\, ds \\ & - \int_\sigma^\tau e^{-(s-\sigma)A_n} \widehat{Z}^{t,x,n}_s \, dW_s, \quad \mathbb{P}\text{-a.s.}, \ \sigma \in [0,\tau]. \end{aligned} \tag{4.5}$$

Let us consider now the initial time $t = 0$, and let us write (4.5) with $\sigma = 0$ (we write X^x instead of $X^{0,x}$). We get

$$\int_0^\tau e^{-sA_n} G\widetilde{C}(s, X^x_s) ds = e^{-\tau A_n} u^\tau_n(0, x) + \int_0^\tau e^{-sA_n} \widehat{Z}^{0,x,n}_s dW_s, \quad \mathbb{P}\text{-a.s.},$$

for any $\tau \in [0,T]$, and by applying $e^{\tau A_n}$ to both sides, from (4.4) we deduce that, $\mathbb{P}$-a.s.,

$$\begin{aligned}\int_0^\tau e^{(\tau-s)A_n} G\widetilde{C}(s, X^x_s) ds &= u^\tau_n(0, x) + \int_0^\tau e^{(\tau-s)A_n} \widehat{Z}^{0,x,n}_s dW_s \\ &= u^\tau_n(0, x) + \int_0^\tau \nabla^G u^\tau_n(s, X^x_s) dW_s, \quad \forall \tau \in [0,T]. \end{aligned}$$

By replacing in (4.1) we infer that

$$\begin{aligned} X^x_\tau =& e^{\tau A} x + \int_0^\tau \left(e^{(\tau-s)A} - e^{(\tau-s)A_n} \right) G\widetilde{C}(s, X^x_s) ds \\ & + u^\tau_n(0, x) + \int_0^\tau \nabla^G u^\tau_n(s, X^x_s) dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s, \quad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{4.6}$$

for any $\tau \in [0, T]$, and the proof is finished. □

The next result is our main uniqueness theorem.

Theorem 4.2. *Let Hypotheses 2.1 and 2.4 hold true. Then, there exists a positive constant* $c = c(T) > 0$ *such that for any* $x_1, x_2 \in H$ *we have*

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t^{x_1} - X_t^{x_2}|_H^2] \le c|x_1 - x_2|_H^2, \tag{4.7}$$

where X^{x_1} *and* X^{x_2} *denote (weak) mild solutions to (2.1) starting at* x_1 *and* x_2*, respectively, and defined on the same stochastic basis. In particular, for equation (2.1) pathwise uniqueness holds.*

Proof. Let $x_1, x_2 \in H$ and let us denote by X^1 and X^2 the mild solutions to (2.1) starting at x_1 and x_2, respectively. We fix $t \in]0, T]$. From (4.2) with $\tau = t$, for any $n \in \mathbb{N}$ we have

$$\begin{aligned}\left(X_t^1 - X_t^2\right) =& e^{tA}(x_1 - x_2) + \left(u_n^t(0, x_1) - u_n^t(0, x_2)\right) + \delta_n^1(t) + \delta_n^2(t)\\ &+ \int_0^t \left(\nabla^G u_n^t(s, X_s^1) - \nabla^G u_n^t(s, X_s^2)\right) dW_s, \quad \mathbb{P}\text{-a.s.},\end{aligned} \tag{4.8}$$

where

$$\delta_n^i(t) = \int_0^t \left(e^{(t-s)A} - e^{(t-s)A_n}\right) G\widetilde{C}(s, X_s^i)ds, \quad \mathbb{P}\text{-a.s.}, \ i = 1, 2, \ n \in \mathbb{N}.$$

Notice that in (4.8) we consider the function u_n^t such that $u_n^t(t, \cdot) \equiv 0$.

The crucial point is that estimates on u_n^t are uniform in t (cf. Hypothesis 2.4), part (B). We also note that $x \mapsto e^{tA}x$ is Lipschitz continuous with respect to x, uniformly with respect to t, and taking into account Hypothesis 2.4, part (B), point (i), we know that $x \mapsto u_n^t(0, x)$ is Lipschitz continuous. For what concerns the stochastic integral, note that using Hypothesis 2.4, part (B), point (ii) with $\mathcal{T} = t$, by the Itô isometry we find

$$\mathbb{E}\left[\left|\int_0^t \left(\nabla^G u_n^t(s, X_s^1) - \nabla^G u_n^t(s, X_s^2)\right) dW_s\right|_H^2\right] \le \int_0^t h(t-s)\,\mathbb{E}\left[|X_s^1 - X_s^2|_H^2\right] ds. \tag{4.9}$$

Using (4.9) in (4.8), for any $n \in \mathbb{N}$ we get

$$\begin{aligned}&\mathbb{E}\left[\left|X_t^1 - X_t^2\right|_H^2\right]\\ &\le C_T\left(|x_1 - x_2|_H^2 + \int_0^t h(t-s)\mathbb{E}\left[|X_r^1 - X_r^2|_H^2\right] ds + \mathbb{E}\left[\sup_{t\le T} |\delta_n^1(t)|_H^2\right] + \mathbb{E}\left[\sup_{t\le T} |\delta_n^2(t)|_H^2\right]\right),\end{aligned}$$

where C_T is a positive constant independent of n and t. Note that

$$t \mapsto \mathbb{E}\left[|X_t^1 - X_t^2|_H^2\right],$$

is a bounded function on $[0, T]$. Thus applying the generalized Gronwall lemma we infer

$$\mathbb{E}\left[|X_t^1 - X_t^2|_H^2\right] \le K_T(\mathbb{E}\left[\sup_{t\le T}|\delta_n^1(t)|_H^2\right] + \mathbb{E}\left[\sup_{t\le T}|\delta_n^2(t)|_H^2\right] + |x_1 - x_2|_H^2),$$

where K_T is a positive constant independent of n and t. We need to prove that

$$\mathbb{E}\left[\sup_{t\le T}|\delta_n^1(t)|_H^2\right] + \mathbb{E}\left[\sup_{t\le T}|\delta_n^2(t)|_H^2\right] \to 0, \quad n \to \infty. \tag{4.10}$$

Using the dominated convergence theorem, we get the assertion if we show that, $\mathbb{P}$-a.s.,

$$\lim_{n\to\infty}\sup_{t\le T}|\delta_n^1(t)|_H^2 = 0, \quad \lim_{n\to\infty}\sup_{t\le T}|\delta_n^2(t)|_H^2 = 0. \tag{4.11}$$

Let us consider the first limit in (4.11) (the proof of the second limit is similar).

Let us fix ω, $\mathbb{P}$-a.s.; for any $n \in \mathbb{N}$, we have

$$\delta_n^1(t) = \int_0^t \left(e^{(t-s)A} - e^{(t-s)A_n}\right) g(s)ds = \int_0^t \left(e^{rA} - e^{rA_n}\right) g(t-r)dr, \ \ n \in \mathbb{N},$$

with $g(s) = G\widetilde{C}(s, X_s^1(\omega))$, $s \in [0, T]$, which is continuous from $[0, T]$ with values in H. It is enough to prove that $\sup_{t\le T}|\delta_n^1(t)|_H \to 0$ as $n \to \infty$. We note that, for any compact set $K \subset H$, $r \in [0, T]$, there exists $y = y_{K,r}$ such that

$$\sup_{y\in K}\left|\left(e^{rA} - e^{rA_n}\right)y\right| = \left|\left(e^{rA} - e^{rA_n}\right)y_{K,r}\right|,$$

and so

$$\lim_{n\to\infty}\sup_{y\in K}\left|\left(e^{rA} - e^{rA_n}\right)y\right|_H = 0. \tag{4.12}$$

Let us introduce the compact sets $K_t = \{g(s)\}_{s\in[0,t]} \subset H$, $t \in [0, T]$. From (4.12) it follows that

$$|\delta_n^1(t)|_H \le \int_0^t \sup_{y\in K_t}\left|\left(e^{rA} - e^{rA_n}\right)y\right|_H dr \le \int_0^T \sup_{y\in K_T}\left|\left(e^{rA} - e^{rA_n}\right)y\right|_H dr \to 0, \quad n \to \infty.$$

This shows assertion (4.10) and completes the proof. □

Remark 4.3. We point out that, using a localization argument as in [10] the boundeness of $\widetilde{C}$ can be dispensed. In particular, one can prove strong well-posedness of (1.1), for any $x \in H$, under Hypotheses 2.1 and 2.4 but replacing the condition on $\widetilde{C}$ with the weaker assumption: $\widetilde{C} : [0, T] \times H \to U$ is continuous on $[0, T] \times H$, there exists K_T such that

$$|\widetilde{C}(t, x)|_U \leq K_T(1 + |x|_H), \quad t \in [0, T], \ x \in H, \tag{4.13}$$

and moreover, for any ball $B = B(z, r)$, $z \in H$, $r > 0$, the function $\widetilde{C} : [0, T] \times B \to U$ is β-Hölder continuous, uniformly in $t \in [0, T]$ (the index $\beta \in (0, 1)$ should be the same for any ball B but the Hölder norm may depend on the ball we consider).

Remark 4.4. By Theorem 4.2, using a generalization of the Yamada-Watanabe theorem (see [28] and [23]), one deduces that equation (1.1) has a unique strong mild solution, *for any* $x \in H$.

5. Analytic results on the associated Kolmogorov equation (2.7)

In this section, assuming Hypothesis 2.1 we give *sufficient* conditions on A, G and $\widetilde{C}$ such that Hypothesis 2.4 is satisfied. This will imply the pathwise uniqueness result for equation (2.1) according to Theorem 4.2.

We split this section into two parts: in the former we provide preliminaries results on the equation (2.6) which involves a generator A_0 of a strongly continuous semigroup e^{tA_0} on H; in the second part we will consider Hypothesis 2.4.

5.1. *Preliminary results on equation* (2.6)

Let us assume the following condition on the operator Q_t, $t > 0$, introduced in (2.4).

Hypothesis 5.1 *(Controllability).*

(i) For any $t > 0$ we have $\mathrm{Im}(e^{tA}) \subseteq \mathrm{Im}(Q_t^{1/2})$.

(ii) For any $t > 0$ we set $\Gamma(t) := Q_t^{-1/2} e^{tA} : H \to H$. From (i) it follows that $\Gamma(t)$ is a bounded linear operator for any $t > 0$. We assume that there exist a p ositive constant $C = C_T$ and measurable functions $\Lambda_1, \ \Lambda_2 : (0, T] \to \mathbb{R}_+$ such that $\inf_{t \in (0,T]} \Lambda_1(t), \inf_{t \in (0,T]} \Lambda_2(t) > 0$; further, for any $\varepsilon \in (0, T)$, $\Lambda_1, \ \Lambda_2$ are bounded in the interval $[\varepsilon, T]$, and

$$|\Gamma(t)z|_H \leq C_T \Lambda_1(t)|z|_H, \ \ |\Gamma(t)Gk|_H \leq C_T \Lambda_2(t)|k|_U, \ \ z \in H, \ k \in U, \ t \in (0, T]. \tag{5.1}$$

Note that if $U = H$ and $G = I$ we consider $\Lambda_1 = \Lambda_2$.

Remark 5.2. Since $G \in L(U; H)$ it follows that the best choice of Λ_1 and of Λ_2 in (5.1) and a suitable choice of C_T give $\Lambda_2(t) \leq \Lambda_1(t)$ for any $t \in (0, T]$.

It is well known that for any $t > 0$ condition $\mathrm{Im}(e^{tA}) \subseteq \mathrm{Im}(Q_t^{1/2})$ is related to the null-controllability of the abstract controlled equation

$$\begin{cases} \dot{Y}(t) = AY(t) + Gu(t), & t \in [0, T], \\ Y(0) = y \in H. \end{cases}$$

In the sequel we will also need the following assumption.

Hypothesis 5.3. The function $\Lambda_1^{1-\beta}\Lambda_2 \in L^1(0,T)$, where $\beta \in (0,1)$ is the constant in Hypothesis 2.1(iv) (cf. (5.1)).

By the dominated convergence theorem the previous assumption implies that

$$C_{\gamma,T} := \int_0^T e^{-\gamma t}\,(\Lambda_1(t))^{1-\beta}\,\Lambda_2(t)\,dt \to 0, \qquad \gamma \to +\infty. \tag{5.2}$$

Remark 5.4. The function $\Lambda_1(t) = t^{-\sigma_1}$, $\Lambda_2(t) = t^{-\sigma_2}$, with $\sigma_1 > 0$ and $\sigma_2 \in (0,1)$ satisfy Hypotheses 5.1 and 5.3 if $\beta > \max\{0, (\sigma_1)^{-1}(\sigma_1+\sigma_2-1)\}$. Indeed since $\sigma_2 \in (0,1)$ it follows that $\sigma_1+\sigma_2-1 < \sigma_1$, which implies that $\frac{\sigma_1+\sigma_2-1}{\sigma_1} < 1$. Further, with the condition $\beta > \max\{0, (\sigma_1)^{-1}(\sigma_1+\sigma_2-1)\}$ the product $\Lambda_1^{1-\beta}(t)\Lambda_2(t) = t^{-((1-\beta)\sigma_1+\sigma_2)}$ is integrable as required in (5.2), since $(1-\beta)\sigma_1+\sigma_2 < 1$.

We recall that (R_t) is the Ornstein-Uhlenbeck semigroup defined by $R_t[\Phi](x) := \mathbb{E}[\Phi(\Xi_t^{0,x})]$, for any $\Phi \in B_b(H;H)$, any $t \ge 0$ and any $x \in H$. Under Hypothesis 5.1, from [13, Theorem 9.26] we infer that for any $\Phi \in B_b(H;H)$ the map $x \mapsto R_t[\Phi](x) \in C_b^\infty(H;H)$ and

$$\nabla_k R_t[\Phi](x) = \int_H \langle \Gamma(t)k, Q_t^{-1/2}y\rangle_H \Phi(e^{tA}x+y)\mathcal{N}(0,Q_t)(dy), \tag{5.3}$$

for any $x,k \in H$ and any $t>0$, where $\mathcal{N}(0,Q_t)$ is the Gaussian measure on H with mean 0 and covariance operator Q_t (see for instance [12, Chapter 1]). Estimates (5.1) allow us to repeat verbatim the statements and the proofs of [26, Lemmata 4.1-4.3], and we collect these results in a unique Lemma.

Lemma 5.5. *Let Hypotheses 2.1 and 5.1 hold true, and let R_t be the Ornstein-Uhlenbeck operator defined in Section 2, acting on vector-valued functions.*

(i) For any $\Phi \in B_b(H;H)$ and any $t>0$, the function $x \mapsto R_t[\Phi](x)$ is Gâteaux differentiable and its Gâteaux derivative $\nabla R_t[\Phi](x) \in L(H;H)$ is given by (5.3). Moreover, for any $T>0$ there exists a positive constant $C = C_T$ such that

$$\sup_{x\in H} |\nabla_z R_t[\Phi](x)|_H \le C\|\Phi\|_\infty \Lambda_1(t)|z|_H, \qquad z \in H \tag{5.4}$$

$$\sup_{x\in H} |\nabla_k^G R_t[\Phi](x)|_H \le C\|\Phi\|_\infty \Lambda_2(t)|k|_U, \qquad k \in U. \tag{5.5}$$

If $\Phi \in C_b(H;H)$ then $R_t[\Phi]$ is Fréchet differentiable on H and we have $\nabla R_t[\Phi] \in C_b(H;L(H;H))$ and $\nabla^G R_t[\Phi] \in C_b(H;L(U;H))$.

(ii) For any $\Phi \in C_b(H;H)$, any $t>0$ and any $k \in U$ the function $x \mapsto \nabla_k^G R_t[\Phi](x)$ is Fréchet differentiable on H and

$$\nabla_y \nabla_k^G R_t[\Phi](x) = \int_H \Big(\langle \Gamma(t)y, Q^{-1/2}z\rangle_H \langle \Gamma(t)Gk, Q^{-1/2}z\rangle_H - \langle \Gamma(t)y, \Gamma(t)Gk\rangle_H \Big) \Phi(e^{tA}x+z)\mathcal{N}(0,Q_t)(dz), \tag{5.6}$$

$$\sup_{x\in H} |\nabla_y \nabla_k^G R_t[\Phi](x)|_H \leq C\|\Phi\|_\infty \Lambda_1(t)\Lambda_2(t)|y|_H|k|_U, \quad t\in(0,T], \tag{5.7}$$

and

$$\lim_{x\to 0}\sup_{y\in H} \|\nabla.\nabla_k^G R_t[\Phi](x+y) - \nabla.\nabla_k^G R_t[\Phi](y)\|_{L(H;H)}$$
$$= \lim_{x\to 0}\sup_{y\in H}\sup_{|z|_H=1} |\nabla_z \nabla_k^G R_t[\Phi](x+y) - \nabla_z\nabla_k^G R_t[\Phi](y)|_H = 0, \quad k\in U. \tag{5.8}$$

The last result we need follows from interpolation theory. Let $\beta\in(0,1)$. From [12, Theorem 2.3.3 & example 2.3.4] it follows that

$$(C_b(H), C_b^1(H))_{\beta,\infty} = C_b^\beta(H), \quad \beta\in(0,1), \tag{5.9}$$

with equivalence of the norms (here, $(X,Y)_{\beta,\infty}$ denotes the real interpolation space between the Banach spaces X and Y, for more details see [24]). Further, we denote by $(\mathcal{R}_t)$ the transition semigroup of the Ornstein Uhlenbeck process $\Xi^{0,x}$ acting on Borel measurable real valued functions $\phi : H\to\mathbb{R}$ as

$$\mathcal{R}_t[\phi](x) = \mathbb{E}[\phi(\Xi_t^{0,x})].$$

There is a link between the H-valued Ornstein-Uhlenbeck transition semigroup $(R_t)_{t\geq 0}$ and the scalar Ornstein-Uhlenbeck transition semigroup $(\mathcal{R}_t)_{t\geq 0}$: for any $\Phi\in B_b(H;H)$ and $h\in H$ we set $\Phi_h(x) := \langle\Phi(x), h\rangle_H$ for any $x\in H$. From [8, Section 3] it follows that

$$\langle \nabla_y R_t[\Phi](x), h\rangle_H = \nabla_y \mathcal{R}_t[\Phi_h](x), \quad t>0,\ x,y,h\in H. \tag{5.10}$$

With computations similar to the ones in the proof of [26, Lemma 4.4] we can prove the following result. For reader's convenience we provide a detailed proof in Appendix B.

Lemma 5.6. *Let Hypotheses 2.1 and 5.1 be satisfied. Then, for any $T>0$ there exists a positive constant $C=C_T$ such that for any $\beta\in(0,1)$ any $\Phi\in C_b^\beta(H;H)$ we have*

$$\sup_{x\in H} |\nabla_y R_t[\Phi](x)|_H \leq C\|\Phi\|_{C^\beta(H;H)}\Lambda_1^{1-\beta}(t)|y|_H, \quad y\in H, \tag{5.11}$$

$$\sup_{x\in H} |\nabla_y \nabla_k^G R_t[\Phi](x)|_H \leq C\|\Phi\|_{C^\beta(H;H)}\Lambda_1(t)^{1-\beta}\Lambda_2(t)|y|_H|k|_U, \quad y\in H,\ k\in U, \tag{5.12}$$

for any $t\in(0,T]$.

Remark 5.7. We recall that estimates (5.4), (5.5), (5.7), (5.11) and (5.12) hold true with R_t replaced by $\mathcal{R}_t$ and Φ being a real-valued function.

We go back to the integral equation (2.6) and for any $\mathcal{T} \in [0, T]$ we introduce the spaces $E_0^{\mathcal{T}}$ and $E_{0,\gamma}^{\mathcal{T}}$ as follows.

Definition 5.8. $E_0^{\mathcal{T}}$ is the space of functions $u \in C_b([0, \mathcal{T}] \times H; H)$ such that $u(t, \cdot)$ is Fréchet differentiable on H for any $t \in [0, \mathcal{T}]$ and the map $\nabla u : [0, \mathcal{T}] \times H \to L(H, H)$ is strongly continuous and globally bounded. Moreover, for any $k \in U$ and $t \in [0, \mathcal{T}]$ the map $x \mapsto \nabla_k^G u(t, x)$ is Fréchet differentiable on H.

For any $\gamma \geq 0$, we set

$$E_{0,\gamma}^{\mathcal{T}} := \{u \in E_0^{\mathcal{T}} : \|u\|_{\gamma,\mathcal{T}} < +\infty\}, \tag{5.13}$$

where

$$\begin{aligned} \|u\|_{\gamma,\mathcal{T}} := & \sup_{(t,x)\in[0,\mathcal{T}]\times H} e^{\gamma t}|u(t,x)|_H + \sup_{(t,x)\in[0,\mathcal{T}]\times H} e^{\gamma t}\|\nabla u(t,x)\|_{L(H;H)} \\ & + \sup_{(t,x)\in[0,\mathcal{T}]\times H} \sup_{|k|_U=1} e^{\gamma t}\|\nabla.\nabla_k^G u(t,x)\|_{L(H;H)}. \end{aligned} \tag{5.14}$$

It is easy to prove that $E_{0,\gamma}^{\mathcal{T}}$ is a Banach space for any $\gamma \geq 0$. Some further properties of functions $u \in E_0^{\mathcal{T}}$ are collected in the next remark.

Remark 5.9. (1) If $u \in E_0^{\mathcal{T}}$, $t \in [0, \mathcal{T}]$ and $k \in U$ then

$$\|\nabla_k^G u(t,\cdot)\|_{C^\beta(H,H)} \leq 3 \sup_{x\in H} \|\nabla_k^G u(t,x)\|_H + \sup_{x\in H} \|\nabla.\nabla_k^G u(t,x)\|_{L(H;H)} \tag{5.15}$$

To verify the previous inequality we write for $x \neq y$ (we have to consider $|x-y| \leq 1$ and $|x-y| > 1$)

$$|\nabla_k^G u(t,x) - \nabla_k^G u(t,y)|_H |x-y|^{-\beta} \leq 2 \sup_{x\in H} \|\nabla_k^G u(t,x)\|_H + \sup_{x\in H} \|\nabla.\nabla_k^G u(t,x)\|_{L(H;H)}.$$

(ii) If $u \in E_0^{\mathcal{T}}$ then the mapping

$$t \mapsto \|\nabla^G u(t,\cdot)\|_{C^\beta(H,L(U;H))} \tag{5.16}$$

is Borel measurable on $[0, T]$ (with values in $\mathbb{R}_+$). It is not difficult to prove the measurability of $t \mapsto \sup_{x\in H} \|\nabla^G u(t,x)\|_{L(U;H)}$. In order to show that

$$t \mapsto [\nabla^G u(t,\cdot)]_{C^\beta(H,L(U;H))} \text{ is measurable,} \tag{5.17}$$

we consider a countable dense subset D of $\{u \in U : |u|_U = 1\}$. Let S be a countable dense subset of H. We note that by the continuity property of $\nabla_k^G u$

$$[\nabla^G u(t,\cdot)]_{C^\beta(H,L(U;H))} = \sup_{x,y\in H,\, x\neq y} \sup_{|k|_U=1} \frac{|\nabla^G_k u(t,x) - \nabla^G_k u(t,y)|_H}{|x-y|_H^\beta}$$
$$= \sup_{x,y\in S,\, x\neq y} \sup_{k\in D} \frac{|\nabla^G_k u(t,x) - \nabla^G_k u(t,y)|_H}{|x-y|_H^\beta}.$$

Since for fixed $x, y \in S$, $x \neq y$, $k \in D$, the mapping: $t \mapsto \frac{|\nabla^G_k u(t,x) - \nabla^G_k u(t,y)|_H}{|x-y|_H^\beta}$ is continuous on $[0, T]$ we get assertion (5.17).

In the next result we will also use the Hölder continuity of $\widetilde{C}(t,\cdot)$, $t \in [0, T]$.

Theorem 5.10. *Let Hypotheses 2.1, 5.1 and 5.3 hold true and let A_0 be the generator of a strongly continuous semigroup e^{tA_0} on H.*

Then, there exists a unique solution $u^{\mathcal{T}}$ to (2.6) in the sense of Definition 2.3 which belongs to $E_0^{\mathcal{T}}$ and there exists a positive constant $M = M_T$ which only depends on T, $\sup_{t\in[0,T]} \|e^{tA_0}\|_{L(H)}$ and $\sup_{s\in[0,T]} \|\widetilde{C}(s,\cdot)\|_{C_b^\beta(H;U)}$ but not on $\mathcal{T}$, such that $\|u^{\mathcal{T}}\|_{0,\mathcal{T}} \leq M$.

Proof. Let us introduce the operator $\mathscr{G}$ defined on $E^{\mathcal{T}}_{0,\gamma}$ by

$$(\mathscr{G}u)(t,x) := \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_0} G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G u(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds,$$

for any $(t,x) \in [0,\mathcal{T}] \times H$, with $\gamma > 0$ to be chosen. We proceed in some steps.

Step I. We have to verify that $\mathscr{G} : E^{\mathcal{T}}_{0,\gamma} \to E^{\mathcal{T}}_{0,\gamma}$. We only check the more difficult part, i.e. we only verify that if $u \in E^{\mathcal{T}}_{0,\gamma}$, for a fixed $y \in H$,

$$\nabla_y(\mathscr{G}u) : [0,\mathcal{T}] \times H \to H \quad \text{is continuous on } [0,\mathcal{T}] \times H. \tag{5.18}$$

We will only prove that

$$\nabla_y \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G u(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds \quad \text{is continuous on } [0,\mathcal{T}] \times H, \tag{5.19}$$

the other term $\nabla_y \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_0} G\widetilde{C}(s,\cdot)\right](x)ds$ can be treated in a similar way.

First note that $B(s,x) := \nabla^G u(s,x)\widetilde{C}(s,x)$ is a bounded continuous function on $[0,\mathcal{T}] \times H$ with values in H. We also define $B(s,x) = 0$ for $s \geq \mathcal{T}$, $x \in H$. Using the estimate (5.11) and the fact that

$$\int_0^T \Lambda_1^{1-\beta}(s)ds < \infty$$

we consider the function

$$v(t,x) := \int_t^{\mathcal{T}} \nabla_y R_{s-t}\left[\nabla^G u(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds = \int_0^{\mathcal{T}-t} \nabla_y R_r B(r+t,\cdot)(x)dr,$$

$(t,x) \in [0,\mathcal{T}] \times H$. It is enough to prove that v is continuous on $[0,\mathcal{T}] \times H$. Let us prove the continuity at a fixed (t_0, x_0). We write

$$|v(t,x) - v(t_0,x_0)| \le \left| \int_0^{\mathcal{T}-t} \nabla_y R_r B(r+t,\cdot)(x)dr - \int_0^{\mathcal{T}-t_0} \nabla_y R_r B(r+t,\cdot)(x)dr \right|$$

$$+ \left| \int_0^{\mathcal{T}-t_0} \left[\nabla_y R_r B(r+t,\cdot)(x) - \nabla_y R_r B(r+t_0,\cdot)(x_0)\right]dr \right| = J_1(t,x) + J_2(t,x).$$

Now

$$J_1(t,x) \le \left| \int_{\mathcal{T}-t_0}^{\mathcal{T}-t} |\nabla_y R_r B(r+t,\cdot)(x)|dr \right| \le C \sup_{t\in[0,\mathcal{T}]} \|B(t,\cdot)\|_{C^\beta(H;H)} |y|_H \left| \int_{\mathcal{T}-t_0}^{\mathcal{T}-t} \Lambda_1^{1-\beta}(r)dr \right|$$

and so $\lim_{t\to t_0} \sup_{x\in H} J_1(t,x) = 0$. Concerning J_2 we note that, for any $r \in]0, \mathcal{T} - t_0[$ the mapping

$$(t,x) \mapsto (\nabla_y R_r B(r+t,\cdot)(x) - \nabla_y R_r B(r+t_0,\cdot)(x_0))$$

$$= \int_H \langle \Gamma(r)y, Q_r^{-1/2} z\rangle_H [B(r+t, e^{tA}x + z) - B(r+t_0, e^{t_0 A}x_0 + z)]\mathcal{N}(0, Q_r)(dz)$$

verifies $\lim_{(t,x)\to(t_0,x_0)} |\nabla_y R_r B(r+t,\cdot)(x) - \nabla_y R_r B(r+t_0,\cdot)(x_0)| = 0$ by the dominated convergence theorem. Moreover, using the estimate (5.11) and again the Lebesgue theorem we infer

$$\lim_{(t,x)\to(t_0,x_0)} \int_0^{\mathcal{T}-t_0} |\nabla_y R_r B(r+t,\cdot)(x) - \nabla_y R_r B(r+t_0,\cdot)(x_0)|dr = 0$$

and so $\lim_{(t,x)\to(t_0,x_0)} J_2(t,x) = 0$. This shows (5.18).

Step II. We claim that a suitable choice of γ implies that $\mathcal{G}$ is a contraction on $E_{0,\gamma}^{\mathcal{T}}$. For any $u_1, u_2 \in E_{0,\gamma}^{\mathcal{T}}$ we have to estimate the difference $\|\mathcal{G}u_1 - \mathcal{G}u_2\|_{\gamma,\mathcal{T}}$. Let us only estimate the term

$$e^{\gamma t}\|\nabla.\nabla_k^G\mathscr{G}u_1(t,x)-\nabla.\nabla_k^G\mathscr{G}u_2(t,x)\|_{L(H;H)},\quad t\in[0,\mathcal{T}],\ x\in H,\ k\in U,$$

since the other addends can be estimated in a similar way. We have, for $t\in[0,\mathcal{T}]$, $y\in H$, $|y|_H\leq 1$,

$$e^{\gamma t}|\nabla_y\nabla_k^G\mathscr{G}u_1(t,x)-\nabla_y\nabla_k^G\mathscr{G}u_2(t,x)|_H\leq$$
$$\leq\int_t^{\mathcal{T}}e^{-\gamma(s-t)}|\nabla_y\nabla_k^G R_{s-t}\left[e^{\gamma s}\left(\nabla^G u_1-\nabla^G u_2\right)\widetilde{C}(s,\cdot)\right](x)|_H ds. \tag{5.20}$$

Since $u_1,u_2\in E_{0,\gamma}^{\mathcal{T}}$, the map $x\mapsto\nabla^G u_i(s,x)\widetilde{C}(s,x)$ is β-Hölder continuous from H into H, $i=1,2$, uniformly with respect to $s\in[0,\mathcal{T}]$, and

$$\|\nabla^G u_i(s,\cdot)\widetilde{C}(s,\cdot)\|_{C^\beta(H;H)}\leq 2\sup_{|k|_U=1}\|\nabla_k^G u_i(s,\cdot)\|_{C_b^\beta(U;H)}\|\widetilde{C}(s,\cdot)\|_{C_b^\beta(H;U)},$$

for any $s\in[0,\mathcal{T}]$, with $i=1,2$. By applying (5.12) to (5.20) and taking into account (5.19) we get, uniformly in y and in $(t,x)\in[0,\mathcal{T}]\times H$

$$\begin{aligned}
&\int_t^{\mathcal{T}}e^{-\gamma(s-t)}|\nabla_y\nabla_k^G R_{s-t}\left[e^{\gamma s}\left(\nabla^G u_1-\nabla^G u_2\right)\widetilde{C}(s,\cdot)\right](x)|_H ds\\
&\qquad\leq C\int_t^{\mathcal{T}}e^{-\gamma(s-t)}\left(\Lambda_1(t-s)\right)^{1-\beta}\Lambda_2(s-t)\,ds\,\cdot\\
&\qquad\quad\sup_{r\in[0,T]}\|\widetilde{C}(r,\cdot)\|_{C_b^\beta(H;U)}\sup_{r\in[0,\mathcal{T}]}e^{\gamma r}\sup_{k\in U,\,|k|_U=1}\|\nabla_k^G(u_1-u_2)(r,\cdot)\|_{C_b^\beta(H;H)}\\
&\qquad\leq C_{\gamma,\mathcal{T}}\sup_{r\in[0,T]}\|\widetilde{C}(r,\cdot)\|_{C_b^\beta(H;U)}\|u_1-u_2\|_{\gamma,\mathcal{T}}
\end{aligned}\tag{5.21}$$

(see also (5.15)) where, from Hypothesis 5.3, $C_{\gamma,\mathcal{T}}$ is a positive constant which goes to 0 as $\gamma\to+\infty$, uniformly with respect to $\mathcal{T}\in[0,T]$. We get

$$e^{\gamma t}\|\nabla.\nabla_k^G\mathscr{G}_n u_1(t,x)-\nabla.\nabla_k^G\mathscr{G}_n u_2(t,x)\|_{L(H;H)}\leq C_{\gamma,\mathcal{T}}\sup_{r\in[0,T]}\|\widetilde{C}(r,\cdot)\|_{C_b^\beta(H;U)}\|u_1-u_2\|_{\gamma,\mathcal{T}}$$

Similar arguments applied to the other terms of the norm $\|\mathcal{G}u_1-\mathcal{G}u_2\|_{\gamma,\mathcal{T}}$ give

$$\|\mathscr{G}u_1-\mathscr{G}u_2\|_{\gamma,\mathcal{T}}\leq C_{\gamma,\mathcal{T}}\|u_1-u_2\|_{\gamma,\mathcal{T}}.$$

Choosing γ large enough we deduce that $\mathscr{G}$ is a contraction on $E_{0,\gamma}^{\mathcal{T}}$ and therefore it admits a unique fixed point $u^{\mathcal{T}}$.

Step III. Let us prove the last part of the statement. We will estimate the crucial term

$$\|\nabla.\nabla_k^G u^{\mathcal{T}}(t,x)\|_{L(H;H)},\quad t\in[0,\mathcal{T}],\ x\in H,\ k\in U,$$

with $|k|_U = 1$, since the other addends can be estimated in a similar way. We have, using (5.15), for any $t \in [0, \mathcal{T}]$,

$$\|\nabla^G u^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;U)} \le 3 \sup_{x\in H} \|\nabla^G u^{\mathcal{T}}(t,x)\|_U + \sup_{x\in H} \sup_{|w|_U=1} \|\nabla.\nabla^G_w u^{\mathcal{T}}(t,x)\|_H .$$

Hence, starting from

$$u^{\mathcal{T}}(t,x) := \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_0} G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G u^{\mathcal{T}}(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds$$

and arguing as before (using also that $\inf_{t\in(0,T]} \Lambda_2(t) > 0$) we arrive at

$$\|\nabla^G u^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;U)} \le M_1 + M_1 \int_t^{\mathcal{T}} h(s-t)\,\|\nabla^G u^{\mathcal{T}}(s,\cdot)\|_{C^\beta(H,U)} ds, \quad t \in [0,\mathcal{T}],$$

where the function $r \mapsto h(r) = \Lambda_1(r)^{1-\beta}\Lambda_2(r) \in L^1(0,T)$ by Hypothesis 5.3 and M_1 is a positive constant which depend on T, $\sup_{t\in[0,T]} \|e^{tA_0}\|_{L(H)}$ and $\sup_{s\in[0,T]} \|\widetilde{C}(s,\cdot)\|_{C^\beta_b(H;U)}$, but not on $\mathcal{T}$. The generalized Gronwall's Lemma 2.8 gives

$$\|\nabla^G u^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;U)} \le M_1[1 + \exp\left(\|h\|_{L^1(0,T)}\right)\|h\|_{L^1(0,T)}], \quad t \in [0,\mathcal{T}].$$

Using the previous estimate we can bound $|\nabla_y \nabla^G_k u^{\mathcal{T}}(t,x)|_H$ for any $y \in H$, $|y|_H \le 1$, $|k|_U = 1$, arguing as in (5.21). We obtain

$$\sup_{(t,x)\in[0,\mathcal{T}]\times H} \|\nabla.\nabla^G_k u^{\mathcal{T}}(t,x)\|_{L(H;H)} \le \tilde{M}.$$

Arguing in a similar way, we obtain

$$\|u^{\mathcal{T}}\|_{0,\mathcal{T}} \le \tilde{M}_1$$

with $\tilde{M}_1$ depending on T, $\sup_{t\in[0,T]} \|e^{tA_0}\|_{L(H)}$, $\sup_{s\in[0,T]} \|\widetilde{C}(s,\cdot)\|_{C^\beta_b(H;U)}$, Λ_1 and Λ_2 but not on $\mathcal{T}$. □

Corollary 5.11. *Let Hypotheses 2.1, 5.1 and 5.3 hold true. Consider, for any $n \in \mathbb{N}$, the operator A_n which generates a* strongly continuous group *of linear and bounded operators $(e^{tA_n}) \subset L(H)$ such that for any $T > 0$ we have*

$$\sup_{t\in[0,T]} \sup_{n\ge1} \|e^{tA_n}\|_{L(H)} < \infty, \qquad \lim_{n\to\infty} e^{tA_n}x = e^{tA}x, \;\; x \in H, \;\; t \ge 0. \tag{5.22}$$

(cf. (A) in Hypothesis 2.4). Then, there exist unique solutions $u^{\mathcal{T}}_n$ to (2.7) which belong to $E^{\mathcal{T}}_0$ and there exists a positive constant $M = M_T$, independent of n and $\mathcal{T}$, such that $\|u^{\mathcal{T}}_n\|_{0,\mathcal{T}} \le M$.

Remark 5.12. Note that the previous result implies the validity of (2.9) in Hypothesis 2.4 for any choice of generators (A_n) verifying assertion (A) in Hypothesis 2.4.

In the next two sections we provide sufficient conditions for the validity of (2.10) in Hypothesis 2.4. We will consider two different types of approximations $(A_n)_{n\in\mathbb{N}}$ for A: the Yosida approximations which we use to treat semilinear stochastic damped equations and the finite dimensional approximations which we use to deal with semilinear stochastic heat equations.

5.2. *Sufficient conditions to ensure Hypothesis 2.4, using the Yosida approximations for A*

Let us show that the solutions $u_n^{\mathcal{T}}$ of (2.7) satisfies (2.10) of Hypothesis 2.4 when

$$A_n := nAR(n, A) \tag{5.23}$$

for any $n \in \mathbb{N}$. We notice that with this choice of A_n Hypothesis 2.4, part (A), is fulfilled (cf Remark 5.12). We introduce the following additional assumption which will be verified in Section 6.2 for the damped equation.

Hypothesis 5.13. For any $T > 0$ there exists a positive constant $C = C_T$ such that

$$\|\Gamma(t)G\|_{L_2(U;H)} \le \Lambda_2(t)C_T, \quad t \in (0, T], \tag{5.24}$$

where $\Lambda_2(t)$ is the function introduced in Hypothesis 5.1.

The first estimate of the Hilbert-Schmidt norm of $u_n^{\mathcal{T}}$ follows from the following lemma.

Lemma 5.14. *Let Hypotheses 2.1, 5.1 and 5.13 hold true. Then:*

(i) for any $\Phi \in B_b(H; H)$ we have $\nabla_\cdot^G R_t[\Phi](x) \in L_2(U; H)$, $x \in H$, $t > 0$, and for any $T > 0$ there exists a positive constant $C = C_T$ such that

$$\sup_{x\in H} \|\nabla_\cdot^G R_t[\Phi](x)\|_{L_2(U;H)} \le \Lambda_2(t)C\|\Phi\|_\infty, \quad t \in (0, T]. \tag{5.25}$$

If $\Phi \in C_b(H; H)$ then $\nabla^G R_t[\Phi] \in C_b(H; L_2(U; H))$.

(ii) The map $U \ni k \mapsto \nabla_y\nabla_k^G R_t[\Phi](x) \in L_2(U; H)$ and for any $T > 0$ there exists a positive constant $C = C_T$ such that (cf. Hypothesis 5.1))

$$\sup_{x\in H} \|\nabla_y\nabla_\cdot^G R_t[\Phi](x)\|_{L_2(U;H)} \le \Lambda_1(t)\Lambda_2(t)C\|\Phi\|_\infty|y|_H, \quad t \in (0, T]. \tag{5.26}$$

(iii) For any $T > 0$ there exists a positive constant $C = C_T > 0$ such that for any $\beta \in (0, 1)$ we have

$$\sup_{x\in H} \|\nabla_y\nabla_\cdot^G R_t[\Phi](x)\|_{L_2(U;H)} \le (\Lambda_1(t))^{1-\beta}\, \Lambda_2(t)C\|\Phi\|_{C^\beta(H;H)}|y|_H, \quad t \in (0, T], \tag{5.27}$$

for any $\Phi \in C_b^\beta(H; H)$.

Proof. Let $T > 0$ and let $\{e_k : k \in \mathbb{N}\}$ be an orthonormal basis of U. From (5.3) and (5.24) we get

$$\|\nabla^G_{\cdot} R_t[\Phi](x)\|^2_{L_2(U;H)} = \sum_{k\in\mathbb{N}} |\nabla R_t[\Phi](x)Ge_k|^2_H \leq C\|\Phi\|_\infty \sum_{k\in\mathbb{N}} |\Gamma(t)Ge_k|^2_H \leq \Lambda_2(t)C\|\Phi\|_\infty,$$

for any $\Phi \in C_b(H;H)$, any $x \in H$ and any $t \in (0,T]$, and (i) follows.
To prove (ii) it is enough to consider (5.6) and (5.24), and to argue as in the proof of (i).
It remains to prove (iii). Analogous computations as for (5.12) in the proof of Lemma 5.6 (see Appendix B) give

$$|\nabla_y(\nabla^G_k R_t[\Phi])(x)|_H \leq (\Lambda_1(t))^{1-\beta} C\|\Phi\|_{C^\beta(H;H)}|y|_H|\Gamma(t)Gk|_H, \quad t \in (0,T], \quad \phi \in C^\beta_b(H),$$

for any $T > 0$, anu $\beta \in (0,1)$ and any $\Phi \in C^\beta_b(H;H)$, where $C = C_T$ is a positive constant which only depends on T. To conclude, let us consider an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of $\mathbb{N}$. It follows that, see also the calculations (2.12)

$$\begin{aligned}
\|\nabla_y(\nabla^G_{\cdot} R_t[\Phi])(x)\|^2_{L_2(U;H)} &= \sum_{k\in\mathbb{N}} |\nabla_y(\nabla^G_{f_k} R_t[\Phi])(x)|^2_H \\
&\leq \Lambda_1(t)^{2-2\beta} C^2 \|\phi\|^2_{C^\beta(H;H)} |y|^2_H \sum_{k\in\mathbb{N}} |\Gamma(t)Gf_k|^2_H \\
= \Lambda_1(t)^{2-2\beta} C^2 \|\phi\|^2_{C^\beta(H;H)} |y|^2_H \|\Gamma(t)G\|^2_{L_2(U;H)} &\leq \Lambda_1(t)^{2-2\beta}\Lambda_2(t)^2 C^2 \|\phi\|^2_{C^\beta(H;H)}|y|^2_H,
\end{aligned}$$

which gives the thesis. □

In the next Theorem we investigate further properties of $u^{\mathcal{T}}_n$, the solutions to (2.7) with $A_n = nAR(n,A)$; see Corollary 5.11. Note that (5.28) gives (2.11) with $h = c$.

Theorem 5.15. *Let Hypotheses 2.1, 5.1, 5.3 and 5.13 hold true, and let $u^{\mathcal{T}}_n$ be the solutions to (2.7) with $A_n = nAR(n,A)$.*
Then, $\nabla^G u^{\mathcal{T}}_n \in C_b([0,\mathcal{T}] \times H; L_2(U;H))$. Further, for any $t \in [0,\mathcal{T}]$ and any $x, y \in H$, the map $U \ni k \mapsto \nabla_y \nabla^G_k u^{\mathcal{T}}_n(t,x)$ belongs to $L_2(U;H)$ and there exists a positive constant $c = c(T)$ which depends on T but neither on $\mathcal{T}$ nor on n such that

$$\sup_{(t,x)\in[0,\mathcal{T}]\times H} \|\nabla_y \nabla^G_{\cdot} u^{\mathcal{T}}_n(t,x)\|_{L_2(U;H)} \leq c|y|_H, \quad y \in H. \tag{5.28}$$

Proof. The fact that for each $t \in [0,T]$, $\nabla^G u^{\mathcal{T}}_n(t,\cdot) \in C_b(H, L_2(U;H))$ follows by (i) in Lemma 5.14 taking into account that $B(s,x) := \nabla^G u(s,x)\widetilde{C}(s,x)$ is a bounded continuous function on $[0,\mathcal{T}] \times H$ with values in H.

Arguing as in the first step of the proof of Theorem 5.10 one can show that

$$\nabla^G u^{\mathcal{T}}_n : [0,\mathcal{T}] \times H \to L_2(U;H) \ \text{ is continuous and bounded.}$$

From Theorem 5.10 and estimate (5.27) we infer (see also (5.15))

$$\|\nabla_y \nabla^G_{\cdot} u_n^{\mathcal{T}}(t,x)\|_{L_2(U;H)} \leq C_1 |y|_H + C_2 \|u_n^{\mathcal{T}}\|_{0,\mathcal{T}} |y|_H \int_t^{\mathcal{T}} (\Lambda_1(s))^{1-\beta}\, \Lambda_2(s) ds, \quad y \in H, \tag{5.29}$$

where C_1 and C_2 are positive constants which depend on T, K_T in (2.8) and $\sup_{s\in[0,T]} \|\widetilde{C}(s,\cdot)\|_{C_b^\beta(H;U)}$, but neither on n and $\mathcal{T}$. Corollary 5.11 and (5.29) give the thesis. □

5.3. Sufficient conditions to ensure Hypothesis 2.4, using the finite dimensional approximations for A

Here we assume that

$$U = H$$

and Hypotheses 2.1, 5.1 and 5.3, where in particular it is assumed that $\widetilde{C}(t,\cdot) \in C_b^\beta(H;H)$ for some $0 < \beta < 1$ uniformly in $t \in [0,T]$ (see (2.3)). Moreover we require the following condition:

Hypothesis 5.16.

1. A is self-adjoint, with compact resolvent, $\{e_n : n \in \mathbb{N}\}$ is a complete orthonormal system in H which satisfies $Ae_n = -\alpha_n e_n$, with non-decreasing positive $(\alpha_n)_{n\geq 1}$.
2. We require $G \in L_2(H)$ or, setting $(\widetilde{C})_n := \langle \widetilde{C}, e_n\rangle$,

$$\sum_{n=1}^{\infty} \frac{\sup_{t\in(0,T)} \|(\widetilde{C}(t,\cdot))_n\|^2_{C^\beta(H;\mathbb{R})}}{\alpha_n} < \infty; \tag{5.30}$$

Remark 5.17. We point out that Hypotheses 2.1, 5.1, 5.3 and 5.16 extend assumptions $1-6$ in [8] in the following way.

(i) Assumption (ii) in Hypothesis 5.3 is weaker than assumption 6 in [8] (such assumption 6 corresponds to the case when $\Lambda_1 = \Lambda_2$). Recall that, in general we have $\Lambda_2 \leq \Lambda_1$ (see Remark 5.2). The main consequence of this fact is that our results apply to semilinear stochastic heat equation in dimension $d = 3$ (see Section 6.3), while examples in [8] only cover the cases $d = 1$ and $d = 2$.

(ii) Following [8] one should require a condition like $\int_0^T (\Lambda_1(t))^\beta\, \Lambda_2(t) dt < +\infty$. However we will not impose such condition.

Remark 5.18. In this section we consider the case when G is not necessarily a trace class operator and (5.30) holds true, since if $G \in L_2(H)$ then (5.24) is satisfied with U replaced by H. Indeed, for any $\{h_n : n \in \mathbb{N}\}$ orthonormal basis of H we have

$$\sum_{n\in\mathbb{N}} |\Gamma(t) G h_n|_H^2 \leq C_T \Lambda_2(t) \sum_{n\in\mathbb{N}} |G h_n|_H^2 \leq C_T \Lambda_2(t) \|G\|^2_{L_2(H)},$$

and the estimate (5.28) follows at once. This implies that condition $G \in L_2(H)$ allows to get strong uniqueness by using Yosida approximations and the computations developed in Section 5.2, but for semilinear stochastic heat equation this does not lead to the sharp result.

Let $n \in \mathbb{N}$. We consider $E_n := \operatorname{span}\{e_1, ..., e_n\}$ the finite dimensional linear span generated by $e_1, ..., e_n$ (see Hypothesis 5.16) and we let Π_n be the projection of H onto E_n:

$$\Pi_n : H \to E_n, \quad x \mapsto \sum_{k=1}^{n} \langle x, e_k \rangle_H e_k. \tag{5.31}$$

As approximants of A we will consider in this section the finite dimensional truncations of A, given by

$$A_n = A\Pi_n, \quad n \in \mathbb{N}. \tag{5.32}$$

Let us notice that this family of operators satisfies Hypothesis 2.4, part (A). For any $\mathcal{T} \in (0, T]$ we consider the integral equation (2.7) which we rewrite here for the reader's convenience:

$$u(t,x) := \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G u(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{5.33}$$

We denote by $u_n^{\mathcal{T}}$ the solution of this equation (see Theorem 5.10). Following Remark 2.10 $u_n^{\mathcal{T}}$ solves

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}_t[u(t,\cdot)](x) = -e^{(\mathcal{T}-t)A_n} G\widetilde{C}(t,x), & x \in H,\ t \in [0,\mathcal{T}], \\ u(\mathcal{T},x) = 0, & x \in H, \end{cases} \tag{5.34}$$

where

$$\mathcal{L}_t f(x) := \frac{1}{2}\operatorname{Tr}[GG^*\nabla^2 f(x)] + \langle Ax, \nabla f(x)\rangle + \langle \widetilde{C}(t,x), \nabla^G f(x)\rangle, \quad t \in [0,T],\ x \in H.$$

Indeed in mild formulation equation (5.34) can be rewritten as

$$u_n^{\mathcal{T}}(t,x) = \int_t^{\mathcal{T}} R_{s-t}\left[e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s,\cdot)\right](x)ds + \int_t^{\mathcal{T}} R_{s-t}\left[\nabla^G u_n^{\mathcal{T}}(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{5.35}$$

For every fixed n we let $u_{n,k}^{\mathcal{T}} := \langle u_n^{\mathcal{T}}, e_k\rangle : [0,\mathcal{T}] \times H \to \mathbb{R}$ its k-component, with $k \in \mathbb{N}$. The following lemma states that for any $n \in \mathbb{N}$ we have $u_n^{\mathcal{T}}(t,x) \in E_n$ for any $(t,x) \in [0,\mathcal{T}] \times H$.

Lemma 5.19. *Let $u_{n,k}^{\mathcal{T}}$ be as above. Then:*

(i) For any $k = 1, \ldots, n$ we have

$$u_{n,k}^{\mathcal{T}}(t,x) = \int_t^{\mathcal{T}} \mathcal{R}_{s-t}\left[e^{-(\mathcal{T}-s)\alpha_k}(G\widetilde{C}(s,\cdot))_k\right](x)ds + \int_t^{\mathcal{T}} \mathcal{R}_{s-t}\left[\nabla^G u_{n,k}^{\mathcal{T}}(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{5.36}$$

(ii) For any $k \geq n+1$ *we have* $u^{\mathcal{T}}_{n,k} = 0$.

Proof. Since

$$R_{s-t}\left[e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s,\cdot)\right](x) = \Pi_n \mathbb{E}[e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s, \Xi^{0,x}_s)],$$

formula (5.36) follows. Further, for any $j \geq n+1$, we have

$$u^{\mathcal{T}}_{n,j}(t,x) = \int_t^{\mathcal{T}} \mathcal{R}_{s-t}\left[\nabla^G u^{\mathcal{T}}_{n,j}(s,\cdot)\widetilde{C}(s,\cdot)\right](x)ds. \tag{5.37}$$

Indeed

$$\langle R_{s-t}\left[e^{(\mathcal{T}-s)A_n} G\widetilde{C}(s,\cdot)\right](x), e_k\rangle = 0, \quad k \geq n+1.$$

From (5.5) and Remark 5.7 we infer that, for $k \in H$, $|k|_H = 1$,

$$\|\nabla^G_k u^{\mathcal{T}}_{n,j}(t,\cdot)\|_{C_b(H)} \leq C \int_t^{\mathcal{T}} \Lambda_2(s-t)\|\nabla^G_k u^{\mathcal{T}}_{n,j}(s,\cdot)\|_{C_b(H)}ds \cdot \sup_{s\in[0,T]} \|\widetilde{C}(s,\cdot)\|_{C_b(H;H)}.$$

The generalized Gronwall Lemma 2.8 gives $\nabla^G u^{\mathcal{T}}_{n,j}(t,x) = 0$ for any $t \in [0,\mathcal{T}]$ and any $x \in H$ and from (5.37) we get (ii). □

We notice that, up to revert time, equations (5.33) and (5.36) coincide with the mild integral equations (16) and (15) in the paper [8], respectively, with G and G_k which are given here by $G = e^{sA_n}G\widetilde{C}(\mathcal{T}-s,\cdot)$ and $G_k(s,\cdot) = e^{-s\alpha_k}(G\widetilde{C})_k(\mathcal{T}-s,\cdot)$. We stress that from Corollary 5.11 we already know that $u^{\mathcal{T}}_{n,k} \in E^{\mathcal{T}}_0$ and there exists a positive constant $M = M_T$, independent of $k = 1,\dots,n$, $\mathcal{T}$ and n such that

$$\|u^{\mathcal{T}}_{n,k}\|_{0,\mathcal{T}} \leq M. \tag{5.38}$$

The next result gives a new estimate which is not present in the regularity results of Section 4 in [8]. Indeed in such section estimates on the second derivatives of solutions are given using the operator norm; instead here we consider the stronger Hilbert-Schmidt norm.

Theorem 5.20. *Let Hypotheses 2.1, 5.1, 5.3 and 5.16 be satisfied, and let* $\mathcal{T} \in (0,T]$. *Then,* $\nabla^G u^{\mathcal{T}}_n \in B_b([0,\mathcal{T}] \times H; L_2(H))$ *and there exists* $h : (0,\mathcal{T}) \to \mathbb{R}_+ \in L^1(0,\mathcal{T})$, *independent of* n, *such that for any* $n \in \mathbb{N}$ *and any* $t \in (0,\mathcal{T})$ *we have*

$$\sup_{x\in H} \|\nabla_y \nabla^G_{\cdot} u^{\mathcal{T}}_n(t,x)\|^2_{L_2(H;H)} \leq h(\mathcal{T}-t)\,|y|^2_H, \quad y \in H. \tag{5.39}$$

Further, there exists a positive constant $C = C_T$, *depending on* $\|\Lambda_1^{1-\beta}\Lambda_2\|_{L^1(0,T)}$, *such that for any* $\mathcal{T} \in (0,T]$ *we have*

$$\|h\|_{L^1(0,\mathcal{T})} \leq 2C \sum_{k=1}^{\infty} \frac{\|(\widetilde{C}(s,\cdot))_k\|^2_{C^\beta(H;H)}}{\alpha_k} < +\infty. \tag{5.40}$$

Remark 5.21. Note that (5.39) implies (cf. (2.10))

$$\sup_{x\in H} \|\nabla^G_\cdot u_n^{\mathcal{T}}(t,x+y) - \nabla^G_\cdot u_n^{\mathcal{T}}(t,x)\|^2_{L_2(H)} \leq h(\mathcal{T}-t)\,|y|^2_H, \quad t \in (0,\mathcal{T}),\ y \in H. \tag{5.41}$$

Proof. Let $\mathcal{T} \in (0,T]$ and $n \geq 1$. Let us prove that $\nabla^G u_n^{\mathcal{T}}$ belongs to $B_b([0,\mathcal{T}]\times H; L_2(H))$. We have

$$\nabla^G u_n^{\mathcal{T}}(t,x) = \sum_{k=1}^{n} \nabla^G u_{n,k}^{\mathcal{T}}(t,x) e_k$$

(see Lemma 5.19). Arguing as in (2.12) we have

$$\begin{aligned}\sum_{k\geq 1} |\nabla^G_{e_k} u_n^{\mathcal{T}}(t,x)|^2_H &= \sum_{k\geq 1} |\nabla_{Ge_k} u_n^{\mathcal{T}}(t,x)|^2_H = \sum_{j=1}^{n}\sum_{k\geq 1} \langle \nabla_{Ge_k} u_n^{\mathcal{T}}(t,x), e_j\rangle^2_H \\ &= \sum_{j=1}^{n}\sum_{k\geq 1} [\nabla_{Ge_k} u_{n,j}^{\mathcal{T}}(t,x)]^2 = \sum_{j=1}^{n} |\nabla^G u_{n,j}^{\mathcal{T}}(t,x)|^2_H.\end{aligned}$$

This shows that, for any (t,x), the map: $k \mapsto \nabla^G_k u_n^{\mathcal{T}}(t,x)$ is a Hilbert-Schmidt operator from H into H.

For any $N \geq 1$, $(t,x) \in [0,\mathcal{T}]\times H$, we introduce the approximating mappings:

$$k \mapsto \nabla^G_{\Pi_N k} u_n^{\mathcal{T}}(t,x) = F_N(t,x,k),$$

where Π_n has been defined in (5.31). By the previous calculations and by Theorem 5.10 we deduce that $F_N \in C_b([0,\mathcal{T}]\times H; L_2(H))$ for any $N \geq 1$. Since

$$\lim_{N\to\infty} \|F_N(t,x,\cdot) - \nabla^G u_n^{\mathcal{T}}(t,x)\|_{L_2(H)} = 0,$$

for any $(t,x) \in [0,\mathcal{T}]\times H$, we get the desired measurability property.

In the sequel C is a positive constant which may vary from line to line and which does not depend on n, k and $\mathcal{T}$. In order to prove (5.39) we write as in (2.12)

$$\|\nabla_y \nabla^G_\cdot u_n^{\mathcal{T}}(t,x)\|^2_{L_2(H)} = \sum_{j\geq 1} |\nabla_y \nabla^G_\cdot u_{n,j}^{\mathcal{T}}(t,x)|^2_H.$$

By Lemma 5.19 assertion (5.39) follows if we prove

$$\sum_{k=1}^{n} \sup_{x\in H} \|\nabla_\cdot \nabla^G_\cdot u_{n,k}^{\mathcal{T}}(s,x)\|^2_{L(H;H)} \leq h^{\mathcal{T}}(s), \quad s \in [0,\mathcal{T}], \tag{5.42}$$

for any $n \in \mathbb{N}$. Let us prove estimate (5.42). Arguing as in (5.15) we find, for any $t \in [0, \mathcal{T}]$,

$$\|\nabla_\cdot^G u_{n,k}^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;H)} \le 3 \sup_{x \in H} \|\nabla_\cdot^G u_{n,k}^{\mathcal{T}}(t,x)\|_H + \sup_{x\in H} \sup_{|w|_H=1} \|\nabla_\cdot \nabla_w^G u_{n,k}^{\mathcal{T}}(t,x)\|_H. \tag{5.43}$$

Let us apply ∇^G to (5.36). We will take into account the regularizing properties of $\mathcal{R}_t$ (see (5.5), (5.12) and Remark 5.7), and (5.43).

For any $k \in \{1, \dots, n\}$, define

$$U_{n,k}(t) = \|\nabla_\cdot^G u_{n,k}^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;H)}, \quad t \in [0, \mathcal{T}].$$

Using that $\inf_{t\in(0,T]} \Lambda_2(t) > 0$ and taking into account (5.36) we get

$$U_{n,k}(t) \le C \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|_{C^\beta(H;H)} \int_t^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k} \left(\Lambda_1(r-t)\right)^{1-\beta} \Lambda_2(r-t)dr$$
$$+ C \int_t^{\mathcal{T}} \left(\Lambda_1(r-t)\right)^{1-\beta} \Lambda_2(r-t) U_{n,k}(r)dr.$$

Let

$$g(t) = \left(\Lambda_1(t)\right)^{1-\beta} \Lambda_2(t),$$

$t \in [0, \mathcal{T}]$. By applying the generalized Gronwall Lemma 2.8 we infer

$$\|\nabla_\cdot^G u_{n,k}^{\mathcal{T}}(t,\cdot)\|_{C^\beta(H;H)} \le f(t) + e^{\|g\|_{L^1(0,T)}} \int_t^{\mathcal{T}} f(s) g(s-t) ds, \tag{5.44}$$

for any $t \in [0, \mathcal{T}]$ and $k = 1, \dots, n$, where $f(t) = C \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|_{C^\beta(H;H)} \cdot I_{\mathcal{T}}^k(t)$ and

$$I_{\mathcal{T}}^k(t) := \int_t^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k} g(r-t)dr, \quad t \in [0, \mathcal{T}]. \tag{5.45}$$

Let us estimate $\nabla_\cdot \nabla_\cdot^G u_{n,k}^{\mathcal{T}}$; by the integral equation verified by $u_{n,k}^{\mathcal{T}}$ we get

$$\|\nabla_\cdot \nabla_\cdot^G u_{n,k}^{\mathcal{T}}(t,\cdot)\|_{C_b(H;L(H;H))} \le C \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|_{C^\beta(H;H)} \int_t^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k} g(r-t)dr$$
$$+ C \int_t^{\mathcal{T}} g(r-t) \|\nabla_\cdot^G u_{n,k}^{\mathcal{T}}(r,\cdot)\|_{C^\beta(H;H)} dr$$

$$\leq C \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|_{C^\beta(H;H)}(I^k_{\mathcal{T}}(t)+J^k_{\mathcal{T}}(t)),$$

where in the last passage we have used the definition (5.45). Moreover setting $K_T = e^{\|g\|_{L^1(0,T)}}$ and

$$J^k_{\mathcal{T}}(t) := \int_t^{\mathcal{T}} g(r-t)\Big(I^k_{\mathcal{T}}(r) + K_T \int_r^{\mathcal{T}} I^k_{\mathcal{T}}(\xi) g(\xi - r) d\xi\Big) dr,$$

by (5.44) we have performed the second part of the last inequality. Therefore,

$$\sum_{k=1}^{n} \|\nabla_\cdot \nabla^G_\cdot u^{\mathcal{T}}_{n,k}(t,\cdot)\|^2_{C_b(H;L(H;H))} \leq C \sum_{k=1}^{n} \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|^2_{C^\beta(H;H)} \left((I^k_{\mathcal{T}}(t))^2 + (J^k_{\mathcal{T}}(t))^2\right).$$

Let us prove that for any $k \in \mathbb{N}$ the functions $t \mapsto I^k_{\mathcal{T}}(t)$ and $t \mapsto J^k_{\mathcal{T}}(t)$ only depend on $\mathcal{T} - t$. Indeed, setting $\mathcal{T} - r = s$ in formula (5.45) where $I^k_{\mathcal{T}}(t)$ is defined we get

$$I^k_{\mathcal{T}}(t) := \int_0^{\mathcal{T}-t} e^{-s\alpha_k} g(\mathcal{T} - s - t) ds, \quad t \in [0, \mathcal{T}],$$

and we see that $I^k_{\mathcal{T}}(t)$ depends only on $\mathcal{T} - t$. Analogously setting $\mathcal{T} - r = s$, $\mathcal{T} - \xi = \eta$ in the definition of $J^k_{\mathcal{T}}(t)$ we get

$$J^k_{\mathcal{T}}(t) := \int_0^{\mathcal{T}-t} g(\mathcal{T} - s - t)\Big(I^k_{\mathcal{T}}(\mathcal{T} - s) + K_T \int_s^{\mathcal{T}-t} I^k_{\mathcal{T}}(\mathcal{T} - \eta) g(s-\eta) d\eta\Big) ds,$$

and we see that $J^k_{\mathcal{T}}(t)$ depends only on $\mathcal{T} - t$.

Let us set

$$h(t) := C \sum_{k=1}^{+\infty} \sup_{s\in[0,T]} \|(\widetilde{C}(s,\cdot))_k\|^2_{C^\beta(H;H)} \left((I^k(t))^2 + (J^k(t))^2\right), \quad t \in (0, \mathcal{T}), \tag{5.46}$$

then (5.39) is satisfied. It remains to prove (5.40). To this purpose note that, for each fixed $k \geq 1$, I^k and J^k are bounded function on $[0, \mathcal{T}]$ (uniformly in k). Indeed

$$0 \leq I^k(\mathcal{T} - t) \leq \int_t^{\mathcal{T}} g(r-t) dr \leq \|g\|_{L^1(0,T)} = \|\Lambda_1^{1-\beta}\Lambda_2\|_{L^1(0,T)},$$

and similarly, with a constant $M_0 = M_0(\|g\|_{L^1(0,T)}) > 0$:

$$0 \leq J^k(\mathcal{T} - t) \leq M_0, \ \ t \in [0, \mathcal{T}].$$

We need more precise estimates of the L^1-norms of $(I^k(\mathcal{T}-t))^2$ and $(J^k(\mathcal{T}-t))^2$ to get the estimate (5.42) on $\|\nabla_{\cdot}\nabla^G_{\cdot}u^{\mathcal{T}}_{n,k}(t,\cdot)\|_{C_b(H;L(H;H))}$.
Using that I^k and J^k are uniformly bounded, we concentrate on giving bounds for

$$\|I^k\|_{L^1(0,\mathcal{T})} \text{ and } \|J^k\|_{L^1(0,\mathcal{T})};$$

by the boundednes of I^k and J^k this will imply estimates for $\|(I^k)^2\|_{L^1(0,\mathcal{T})}$ and $\|(J^k)^2\|_{L^1(0,\mathcal{T})}$.
We have by the Fubini theorem

$$\begin{aligned}\int_0^{\mathcal{T}} I^k(\mathcal{T}-t)dt &= \int_0^{\mathcal{T}} dt\int_t^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k}g(r-t)dr = \int_0^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k}\left(\int_0^r g(r-t)dt\right)dr\\ &\le \|g\|_{L^1(0,\mathcal{T})}\int_0^{\mathcal{T}} e^{-(\mathcal{T}-r)\alpha_k}dr \le \frac{\|\Lambda_1^{1-\beta}\Lambda_2\|_{L^1(0,T)}}{\alpha_k},\ \ k\ge 1,\end{aligned}$$

which is the required dependence on α_k. On the other hand,

$$\begin{aligned}\int_0^{\mathcal{T}} J^k(\mathcal{T}-t)dt &= \int_0^{\mathcal{T}} dt\int_t^{\mathcal{T}} g(r-t)\Big(I^k(\mathcal{T}-r)+K_T\int_r^{\mathcal{T}} I^k(\mathcal{T}-\xi)g(\xi-r)d\xi\Big)dr\\ &= \int_0^{\mathcal{T}} dt\int_t^{\mathcal{T}} g(r-t)I^k(\mathcal{T}-r)dr + K_T\int_0^{\mathcal{T}} dt\int_t^{\mathcal{T}} g(r-t)\Big(\int_r^{\mathcal{T}} I^k(\mathcal{T}-\xi)g(\xi-r)d\xi\Big)dr\\ &= \int_0^{\mathcal{T}} I^k(\mathcal{T}-r)dr\int_0^r g(r-t)dt + K_T\int_0^{\mathcal{T}}\Big(\int_r^{\mathcal{T}} I^k(\mathcal{T}-\xi)g(\xi-r)d\xi\Big)dr\int_0^r g(r-t)dt\\ &\le \|g\|_{L^1(0,T)}\Big(\int_0^{\mathcal{T}} I^k(\mathcal{T}-r)dr + K_T\int_0^{\mathcal{T}} dr\int_r^{\mathcal{T}} I^k(\mathcal{T}-\xi)g(\xi-r)d\xi\Big).\end{aligned}$$

Hence, denoting by C_1 a constant depending on $\|g\|_{L^1(0,T)}$ we find

$$\int_0^{\mathcal{T}} J^k(\mathcal{T}-t)dt \le \|g\|_{L^1(0,T)}\Big(\int_0^{\mathcal{T}} I^k(\mathcal{T}-r)dr + K_T\|g\|_{L^1(0,T)}\int_0^{\mathcal{T}} I^k(\mathcal{T}-\xi)d\xi\Big) \le \frac{C_1}{\alpha_k}.$$

Recalling the definition of h in (5.46), collecting the previous estimates it follows that

$$\|h\|_{L^1(0,\mathcal{T})} \le 2C\sum_{k=1}^{\infty}\frac{\|(\widetilde{C}(s,\cdot))_k\|^2_{C^\beta(H;H)}}{\alpha_k} < +\infty.$$

The proof is complete. □

6. Applications

In this section we consider two concrete models to which our results apply: stochastic semilinear damped Euler-Bernoulli beam equation and stochastic semilinear heat equations.

6.1. An example of stochastic damped Euler-Bernoulli beam equation

First we consider a nonlinear stochastic damped Euler-Bernoulli beam equation with the nonlocal term $\left(-\frac{\partial^2}{\partial\xi^2}\right)^{2\alpha}$ and hinged boundary conditions:

$$\begin{cases} \frac{\partial^2}{\partial t^2}y(t,\xi) = -\frac{\partial^4}{\partial\xi^4}y(t,\xi) - \rho\left(-\frac{\partial^2}{\partial\xi^2}\right)^{2\alpha}\frac{\partial}{\partial t}y(t,\xi) \\ \qquad\qquad +\left(-\frac{\partial^2}{\partial\xi^2}\right)^{-2\gamma}c(t,\xi,y(t,\xi)) + \left(-\frac{\partial^2}{\partial\xi^2}\right)^{-2\gamma}\dot{W}(t,\xi), & (t,\xi)\in(0,T]\times(0,1), \\ y(t,0) = y(t,1) = \frac{\partial^2}{\partial x^2}y(t,0) = \frac{\partial^2}{\partial x^2}y(t,1) = 0, & t\in(0,T], \\ y(0,\xi) = y_0(\xi), \ \ \frac{\partial}{\partial t}y(0,\xi) = y_1(\xi) & \xi\in[0,1], \end{cases} \tag{6.1}$$

with $\rho > 0$ and $\alpha \in [0, 1/2]$. Using the terminology of [5] this equation is in the class stochastic Euler-Bernoulli beam equations which describe elastic systems with structural damping.

Here, y_0 describes the initial position and y_1 the initial velocity of the particle, and $\dot{W}(\tau,\xi)$ is a space-time white noise on $[0,T]\times[0,1]$ which describes external random forces.

In Section 6.2 we show that equation (6.1) can be reformulated in an abstract way as a stochastic evolution equation in a suitable space H of the form (1.1).

If the term $c(\cdot,\cdot,\cdot)$ satisfies the next conditions, then we are able to apply our results to equation (6.1) which give the pathwise uniqueness for mild solutions.

The function $c : [0,T]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ *is measurable and, for* $s\in[0,T]$*, a.e.* $\xi\in[0,1]$*, the map* $c(s,\xi,\cdot):\mathbb{R}\to\mathbb{R}$ *is continuous. There exists* c_1 *bounded and measurable on* $[0,1]$*,* $\beta\in(0,1)$*, such that, for* $s\in[0,T]$ *and a.e.* $\xi\in[0,1]$*,*

$$|c(s,\xi,x) - c(s,\xi,y)| \le c_1(\xi)|x-y|^{\beta}, \tag{6.2}$$

$x, y\in\mathbb{R}$*. Moreover* $|c(s,\xi,x)| \le d_2(\xi)$*, for* $s\in[0,T]$*,* $x\in\mathbb{R}$ *and a.e.* $\xi\in[0,1]$*, with* $d_2\in L^2([0,1])$*.*

To deal with equation (6.1) first we have to show the well-posedness when $c = 0$, by proving that the stochastic convolution is well defined in H: this is an easy consequence of the fact that $\Lambda^{-\gamma}$ is a trace class operator on U.

Once that the well-posedness of the linear stochastic damped Euler-Bernoulli beam equation is proved, we investigate the regularizing effects of the associated transition semigroup (R_t) (cf. Section 5). These effects can be proved by means of optimal blow-up rates for the minimal energy associated to null controllability of related linear deterministic control systems. To this purpose we use a spectral approach to the damped elastic operators introduced in [5] and recovered in [22] and [29]. For the stochastic damped Euler-Bernoulli beam equation we will get

$$\Lambda_1(t) = \Lambda_2(t) = t^{-1/2-2\gamma}$$

(cf. Hypotheses 5.1 and 5.13) for every $\gamma \in (1/8, 1/4)$. So we are able to we prove that if

$$\rho^2 \neq 4(\pi^2 n^2)^{1-2\alpha}$$

and β given in (6.2) belongs to $\left(\overline{\beta}, 1\right)$ where $\overline{\beta} = 8\gamma/(1+4\gamma)$, then *pathwise uniqueness holds true for equation* (6.1).

6.2. Semilinear stochastic damped Euler-Bernoulli beam equations in general form

6.2.1. Setting and assumptions

We consider the following nonlinear stochastic damped beam equation which is a general form of (6.1):

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2}(t) = -\Lambda y(t) - \rho\Lambda^{\alpha}\dfrac{\partial y}{\partial t}(t) + \Lambda^{-\gamma}\tilde{C}\left(t, y(t), \dfrac{\partial y}{\partial t}(t)\right) + \Lambda^{-\gamma}\dot{W}_t, \quad t \in (0, T], \\ y(0) = y_0, \\ \dfrac{\partial y}{\partial t}(0) = y_1, \end{cases} \tag{6.3}$$

with $\rho > 0$ and $\alpha \in [0, 1/2]$ and $\gamma \in (1/8, 1/4)$. Here, $\Lambda : D(\Lambda) \subset U \to U$ is a *positive self-adjoint operator* on a separable Hilbert space U such that

$$\Lambda^{-2\gamma} \textit{ which is a trace class operator from } U \textit{ into } U \tag{6.4}$$

and $W = \{W(\tau) : \tau \geq 0\}$ is a cylindrical Wiener process on U.

We aim at formulating this equation as an abstract stochastic evolution equation in the product space $H := V \times U =: D(\Lambda^{1/2}) \times U$.

About $\tilde{C} : [0, T] \times H \to U$ in (6.3) we will assume that it is continuous and bounded and there exists a positive constant K and $\beta \in (0, 1)$ such that

$$|\widetilde{C}(t, h) - \widetilde{C}(t, h')|_U \leq K|h - h'|_H^{\beta}, \quad h, h' \in H,\ t \in [0, T]. \tag{6.5}$$

(cf. (i) in Hypothesis 5.3).

We will assume the following hypothesis.

Hypothesis 6.1. For every $\alpha \in [0, 1/2]$ the index β belongs to $\left(\dfrac{8\gamma}{1+4\gamma}, 1\right)$.

We notice that, since $\gamma < 1/4$, the quantity $8\gamma/(1+4\gamma) < 1$.

Let (μ_n) be the family of eigenvalues of Λ. We are assuming that

$$\sum_{n\geq 1} \mu_n^{-2\gamma} < \infty. \tag{6.6}$$

We also require

Hypothesis 6.2. For any $n \in \mathbb{N}$ $\rho^2 \neq 4\mu_n^{1-2\alpha}$.

Remark 6.3. Concerning the basic example (6.1) in Section 6.1, we have

$$\begin{aligned} &\mathcal{D}(\Lambda) = \{y \in H^2([0,1]) \cap H_0^1([0,1]) : y'' \in H^2([0,1]) \cap H_0^1([0,1])\}, \\ &\Lambda y = y^{(iv)} = (-y'')'' \in L^2([0,1]), \quad \text{for every } y \in \mathcal{D}(\Lambda). \end{aligned}$$

Moreover $y_0 \in V = H^2([0,1]) \cap H_0^1([0,1])$, $y_1 \in U = L^2([0,1])$. Note that $\Lambda^{-2\gamma}$ has *finite trace* for $\gamma > 1/8$ since the eigenvalues of Λ are $\lambda_n = \pi^4 n^4$, $n \geq 1$.

By considering $\widetilde{G} : U \longrightarrow H$, $\widetilde{G}u = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$ for any $u \in U$, $G = \widetilde{G}\Lambda^{-\gamma}$, and for any $h = (h_1, h_2) \in H$ we define

$$C(\tau, h) = G\tilde{C}(\tau, h)(\xi) := \begin{pmatrix} 0 \\ c(\tau, \xi, h_1(\xi)) \end{pmatrix}, \quad \xi \in [0,1], \ \tau \in [0,T]. \tag{6.7}$$

It is easy to see that $\tilde{C}(\tau, h) = c(\tau, \cdot, h_1(\cdot))$ with values in U is β-Hölder continuous in h uniformly in τ (cf. (6.5)). We consider also the operators $\mathcal{A}_{\alpha,\rho} : D(\mathcal{A}_{\alpha,\rho}) \subset H \to H$

$$\mathcal{A}_{\alpha,\rho} := \begin{pmatrix} 0 & I \\ -\Lambda & -\rho\Lambda^{\alpha} \end{pmatrix}. \tag{6.8}$$

Writing $X_\tau(\xi) := \begin{pmatrix} y(\tau,\xi) \\ \frac{\partial}{\partial\tau} y(\tau,\xi) \end{pmatrix}$, it follows that equation (6.3) can be reformulated as a stochastic evolution equation in H:

$$\begin{cases} dX_\tau = \mathcal{A}_{\alpha,\rho} X_\tau d\tau + G\widetilde{C}(\tau, X_\tau) d\tau + G dW_\tau, & \tau \in [0,T], \\ X_0 = \overline{x}_0 := \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} \in H, \end{cases} \tag{6.9}$$

which has the form of (2.1).

In the next Sections 6.2.2 and 6.2.3 we provide preliminary results. Then in Section 6.2.4 we prove that stochastic linear equation (6.9) or (6.3) with $\widetilde{C} \equiv 0$ is well-posed in H. To this purpose we need Hypothesis 6.2 and condition (6.4). Finally, in section 6.2.5 we will formulate our main result on well-posedness for (6.3).

6.2.2. *The operator $\widetilde{\mathcal{A}}_{\alpha,\rho}$*

The techniques of [5] and of [22] can be adapted to our situation. In order to study properties of the operator $\mathcal{A}_{\alpha,\rho}$, we introduce the operator

$$\widetilde{\mathcal{A}}_{\alpha,\rho} := \begin{pmatrix} 0 & \Lambda^{1/2} \\ -\Lambda^{1/2} & -\rho\Lambda^{\alpha} \end{pmatrix} \tag{6.10}$$

on the space $\widetilde{H} := U \times U$, with $D(\widetilde{\mathcal{A}}_{\alpha,\rho}) := D(\Lambda^{1/2}) \times D(\Lambda^{(1/2)\vee\alpha})$. We recall that $D(\Lambda^{1/2}) = V$. When ρ satisfies suitable assumptions (see Hypothesis 6.2) we have two generation results:

from [6, Appendix A] the operator $\mathcal{A}_{\alpha,\rho}$ generates a strongly continuous semigroup $(e^{t\mathcal{A}_{\alpha,\rho}})_{t\geq 0}$ on H which is also analytic for $\alpha \in [\frac{1}{2}, 1)$, and from [22, Section 3] the operator $\widetilde{\mathcal{A}}_{\alpha,\rho}$ generates a strongly continuous semigroup $(e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}})_{t\geq 0}$ on $\widetilde{H}$.

Let us introduce the operator $M : H \to \widetilde{H}$ defined as

$$M := \begin{pmatrix} \Lambda^{1/2} & 0 \\ 0 & I \end{pmatrix}, \quad M\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \Lambda^{1/2}x_1 \\ x_2 \end{pmatrix}. \tag{6.11}$$

We notice that $\widetilde{H} = MH$. Further, since

$$\langle Mx, y\rangle_{\widetilde{H}} = \langle \Lambda^{1/2}x_1, y_1\rangle_U + \langle x_2, y_2\rangle_U = \langle x_1, \Lambda^{-1/2}y_1\rangle_V + \langle x_2, y_2\rangle_U = \langle x, M^* y\rangle_H,$$

it follows that

$$M^* = \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & I \end{pmatrix} = M^{-1} : \widetilde{H} \to H.$$

The operator M is the link between $(e^{t\mathcal{A}_{\alpha,\rho}})_{t\geq 0}$ and $(e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}})_{t\geq 0}$, as the following lemma states.

Lemma 6.4. *Let $\mathcal{A}_{\alpha,\rho}$, $\widetilde{\mathcal{A}}_{\alpha,\rho}$ and M be defined in (6.8), in (6.10) and in (6.11), respectively. Then, for any $t > 0$ and any $y \in H$ we have $Me^{t\mathcal{A}_{\alpha,\rho}}y = e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}My$.*

Proof. By density, to get the thesis it is enough to prove the equality for $y \in D(\Lambda) \times D(\Lambda^{(1/2)\vee\alpha})$. Let $y \in D(\Lambda) \times D(\Lambda^{(1/2)\vee\alpha})$. We set $f(t) := Me^{t\mathcal{A}_{\alpha,\rho}}y$ and $g(t) := e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}My$. For any $t > 0$ we have

$$f'(t) = M\mathcal{A}_{\alpha,\rho}e^{t\mathcal{A}_{\alpha,\rho}}y = \begin{pmatrix} 0 & \Lambda^{1/2} \\ -\Lambda & -\rho\Lambda^{\alpha} \end{pmatrix} M^{-1}f(t) = \widetilde{\mathcal{A}}_{\alpha,\rho}\, f(t),$$

and $f(0) = My$. Let us set $h := f - g$, since $g'(t) = \widetilde{\mathcal{A}}_{\alpha,\rho}\, g(t)$ for any $t > 0$ and $g(0) = My$ it follows that $h \in C^1([0, +\infty), H)$, $h'(t) = \widetilde{\mathcal{A}}_{\alpha,\rho}\, h(t)$ on $(0, T]$ and $h(0) = 0$. This gives $h \equiv 0$ which implies the thesis. □

6.2.3. *Spectral decomposition in $\widetilde{H} = U \times U$*

Only in this subsection and in Appendix A we consider complexified spaces. We do not change the notation to not weigh down them.

Let $\Lambda : D(\Lambda) \subset U \to U$ be as in Section 6.2.1, let (μ_n) be the family of eigenvalues of Λ (without loss of generality we can assume that they are simple, see [22]), and let $\{e_n\}_{n\in\mathbb{N}}$ be a family of corresponding eigenvectors (not-normalized), i.e., $\Lambda e_n = \mu_n e_n$, $n \in \mathbb{N}$; we know that $\mu_n \nearrow +\infty$ as $n \to +\infty$. We notice that $\{e_n\}_{n\in\mathbb{N}}$ forms an orthogonal basis of U. Further, let $\widetilde{G} : U \to \widetilde{H}$ be defined as before, see (6.7).

Now we extend the computations introduced in [5] and recovered in [22] and [29] to the space $\widetilde{H}$; it follows that the operator $\widetilde{\mathcal{A}}_{\alpha,\rho}$ has eigenvalues λ_n^+, λ_n^- in H, $n \in \mathbb{N}$, where

$$\lambda_n^{\pm} := \frac{-\rho\mu_n^{\alpha} \pm \sqrt{\rho^2\mu_n^{2\alpha} - 4\mu_n}}{2}, \quad \lambda_n^+\lambda_n^- = \mu_n, \quad \lambda_n^+ + \lambda_n^- = -\rho\mu_n^{\alpha}, \quad n \in \mathbb{N}.$$

Recall that Hypothesis 6.2 ensures that the eigenvalues $\{\lambda_n^\pm : n \in \mathbb{N}\}$ of $\widetilde{\mathcal{A}}_{\alpha,\rho}$ are simple (see [22, Section 3] and also Section 2.3 in [29]). The normalized corresponding eigenvectors are given by

$$\Phi_n^+ := \begin{pmatrix} \mu_n^{1/2} e_n \\ \lambda_n^+ e_n \end{pmatrix}, \quad \Phi_n^- := \chi_n \begin{pmatrix} \mu_n^{1/2} e_n \\ \lambda_n^- e_n \end{pmatrix}, \tag{6.12}$$

where (possibly replacing e_n by $\gamma_n e_n$) we may assume that

$$(\mu_n + |\lambda_n^+|^2)|e_n|_U^2 = 1, \qquad \chi_n^2(\mu_n + |\lambda_n^-|^2)|e_n|_U^2 = 1, \quad n \in \mathbb{N},$$

with $\chi_n^2 = \frac{\mu_n + |\lambda_n^+|^2}{\mu_n + |\lambda_n^-|^2}$. *Note that* $\{\Phi_n^+ : n \in \mathbb{N}\}$ *and* $\{\Phi_n^- : n \in \mathbb{N}\}$ *each forms an orthonormal family on* $\widetilde{H}$.

On the other hand, $\{\Phi_n^+ : n \in \mathbb{N}\} \cup \{\Phi_n^- : n \in \mathbb{N}\}$ is a complete family on $\widetilde{H}$ under Hypothesis 6.2. Let us set $\widetilde{H}^+ := \overline{\text{span}}\{\Phi_n^+ : n \in \mathbb{N}\}$, $\widetilde{H}^- := \overline{\text{span}}\{\Phi_n^- : n \in \mathbb{N}\}$ and let us consider the decomposition

$$\widetilde{H} = \widetilde{H}^+ \oplus \widetilde{H}^- \quad \text{(non-orthogonal, direct sum)}.$$

We denote by x^+ the projection of x on H^+ and on x^- the projection of x on H^-. From the previous decomposition we have

$$\widetilde{\mathcal{A}}_{\alpha,\rho} x = \sum_{n=1}^{\infty} \left(\lambda_n^+ \langle x^+, \Phi_n^+ \rangle_{\widetilde{H}} \Phi_n^+ + \lambda_n^- \langle x^-, \Phi_n^- \rangle_{\widetilde{H}} \Phi_n^- \right), \quad x \in D(\widetilde{\mathcal{A}}_{\alpha,\rho}), \tag{6.13}$$

$$e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}} x = \sum_{n=1}^{\infty} \left(e^{\lambda_n^+ t} \langle x^+, \Phi_n^+ \rangle_{\widetilde{H}} \Phi_n^+ + e^{\lambda_n^- t} \langle x^-, \Phi_n^- \rangle_{\widetilde{H}} \Phi_n^- \right), \quad t \in [0, +\infty),\ x \in \widetilde{H}. \tag{6.14}$$

Further, for any $a \in U$ we have

$$\widetilde{G} a = \begin{pmatrix} 0 \\ a \end{pmatrix} =: \sum_{n=1}^{\infty} \left(c_n^+ \Phi_n^+ + c_n^- \Phi_n^- \right); \tag{6.15}$$

by considering the orthonormal basis $\{e_n/|e_n|_U\}_{n\in\mathbb{N}}$ of U it follows that

$$c_n^+ + \chi_n c_n^- = 0, \quad (c_n^+ \lambda_n^+ + \chi_n c_n^- \lambda_n^-)|e_n|_U^2 = \langle a, e_n \rangle_U, \quad n \in \mathbb{N},$$

which implies

$$c_n^- = -\frac{c_n^+}{\chi_n}, \quad c_n^+ = \frac{1}{(\lambda_n^+ - \lambda_n^-)|e_n|_U} \langle a, e_n/|e_n|_U \rangle_U, \quad n \in \mathbb{N}.$$

Further, let $\widetilde{\mathcal{A}}^+_{\alpha,\rho} := \widetilde{\mathcal{A}}^{\widetilde{H}^+}_{\alpha,\rho} : D(\widetilde{\mathcal{A}}^+_{\alpha,\rho})(:= D(\widetilde{\mathcal{A}}_{\alpha,\rho}) \cap \widetilde{H}^+) \subset \widetilde{H}^+ \to \widetilde{H}^+$ and $\widetilde{\mathcal{A}}^-_{\alpha,\rho} := \widetilde{\mathcal{A}}^{\widetilde{H}^-}_{\alpha,\rho} : D(\widetilde{\mathcal{A}}^-_{\alpha,\rho})(:= D(\widetilde{\mathcal{A}}_{\alpha,\rho}) \cap \widetilde{H}^-) \subset \widetilde{H}^- \to \widetilde{H}^-$ be the restrictions of $\widetilde{\mathcal{A}}_{\alpha,\rho}$ to $\widetilde{H}^+$ and $\widetilde{H}^-$, respectively. For any $h \in \widetilde{H}$ we denote by h^+ and by h^- its projection on $\widetilde{H}^+$ and $\widetilde{H}^-$, respectively. With respect to this decomposition, the operators

$$\widetilde{\mathcal{A}}_{\alpha,\rho} = \begin{pmatrix} \widetilde{\mathcal{A}}^+_{\alpha,\rho} & 0 \\ 0 & \widetilde{\mathcal{A}}^-_{\alpha,\rho} \end{pmatrix}, \quad e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}} = \begin{pmatrix} e^{t\widetilde{\mathcal{A}}^+_{\alpha,\rho}} & 0 \\ 0 & e^{t\widetilde{\mathcal{A}}^-_{\alpha,\rho}} \end{pmatrix}, \ t \geq 0, \quad \widetilde{G} = \begin{pmatrix} \widetilde{G}^+ \\ \widetilde{G}^- \end{pmatrix},$$

admit the following explicit formulae:

$$\widetilde{\mathcal{A}}^+_{\alpha,\rho} x^+ = \sum_{n=1}^{\infty} \lambda_n^+ \langle x^+, \Phi_n^+ \rangle_{\widetilde{H}} \Phi_n^+, \ x^+ \in D(\widetilde{\mathcal{A}}^+_{\alpha,\rho}),$$

$$\widetilde{\mathcal{A}}^-_{\alpha,\rho} x^- = \sum_{n=1}^{\infty} \lambda_n^- \langle x^-, \Phi_n^- \rangle_{\widetilde{H}} \Phi_n^-, \quad x^- \in D(\widetilde{\mathcal{A}}^-_{\alpha,\rho}),$$

$$e^{t\widetilde{\mathcal{A}^+}_{\alpha,\rho}} x^+ = \sum_{n=1}^{\infty} e^{t\lambda_n^+} \langle x^+, \Phi_n^+ \rangle_{\widetilde{H}} \Phi_n^+, \quad x^+ \in \widetilde{H}^+,$$

$$e^{t\widetilde{\mathcal{A}^-}_{\alpha,\rho}} x^- = \sum_{n=1}^{\infty} e^{t\lambda_n^-} \langle x^-, \Phi_n^- \rangle_{\widetilde{H}} \Phi_n^-, \quad x^- \in \widetilde{H}^-,$$

$$\widetilde{G}^+ a = \sum_{n=1}^{\infty} b_n^+ a_n \Phi_n^+, \quad a \in U,$$

$$\widetilde{G}^- a = \sum_{n=1}^{\infty} b_n^- a_n \Phi_n^-, \quad a \in U,$$

where (cf. (6.15))

$$a_n := \langle a, e_n/|e_n|_U \rangle_U, \quad b_n^+ := \frac{1}{(\lambda_n^+ - \lambda_n^-)|e_n|_U}, \quad b_n^- := -\frac{b_n^+}{\chi_n}, \quad n \in \mathbb{N}. \tag{6.16}$$

From the definition of e_n, $\lambda_n^\pm$, μ_n and $b_n^\pm$ (cf. formulae [29, (2.3.14)-(2.3.18)]) we have

$$|\lambda_n^\pm| \sim \mu_n^{1/2}, \quad |e_n|_U \sim \mu_n^{-1/2}, \quad |\lambda_n^+ - \lambda_n^-| \sim \mu_n^{1/2}, \quad b_n^\pm \sim \text{cost}, \quad \chi_n \sim \text{cost}, \tag{6.17}$$

definitively with respect to $n \in \mathbb{N}$. Finally, since $|\lambda_n^+|, |\lambda_n^-|$ blows up as $n \to +\infty$ and λ_n^+, λ_n^- has negative real part, from (6.13) and (6.14) it follows that $t \mapsto e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}} x$ belongs to $C^1((0,+\infty); \widetilde{H}) \cap C((0,+\infty); D((\widetilde{\mathcal{A}}_{\alpha,\rho})^\eta))$ for any $\eta > 0$ and any $x \in \widetilde{H}$, and for any $T > 0$ and $\eta > 0$ there exists a positive constant $L = L_{T,\eta}$ such that $\|(\widetilde{\mathcal{A}}_{\alpha,\rho})^\eta e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}\|_{L(\widetilde{H})} \leq t^{-\eta} L_T$.

6.2.4. Linear stochastic damped Euler-Bernoulli beam equations

Let us consider the problem

$$\begin{cases} dX_t = \mathcal{A}_{\alpha,\rho} X_t dt + G dW_t, & t \in [0,T], \\ X_0 = \begin{pmatrix} x_0^1 \\ x_0^2 \end{pmatrix} = x \in H = V \times U, \end{cases} \tag{6.18}$$

where $\mathcal{A}_{\alpha,\rho}$ has been defined in (6.8). In the following we will refer to the solution of equation (6.18) as the Ornstein-Uhlenbeck process. We have $\alpha \in [0,1)$ and ρ satisfying Hypothesis 6.2.

We will show that equation (6.18) is well-posed in H, i.e., for any $x \in H$, there exists a unique mild solution having continuous paths with values in H (cf. (2.5)). This is given by

$$X_t = e^{t\mathcal{A}_{\alpha,\rho}} x + \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}} G dW_s, \quad t \in [0,T], \ \mathbb{P} - a.s.. \tag{6.19}$$

To this purpose we need to show that the stochastic convolution

$$W_{\mathcal{A}_{\alpha,\rho}}(t) := \int_0^t e^{(t-s)\mathcal{A}_{\alpha,\rho}} G dW_s, \quad t \in [0,T], \tag{6.20}$$

verifies condition (2.2) (cf. Section 5.3 in [13]). This is an easy consequence of (6.4).

Proposition 6.5. *Assume that Hypothesis 6.2 and condition* (6.4) *are satisfied. Then,*

$$\sup_{t \geq 0} \| e^{t\mathcal{A}_{\alpha\rho}} G \|_{L_2(U,H)} < \infty \tag{6.21}$$

which implies (2.2).

Proof. Formula (6.21) follows from the fact that, from (6.6), $\Lambda^{-2\gamma}$ is a trace-class operator on U and that $G = \widetilde{G}\Lambda^{-\gamma}$. □

6.2.5. Strong uniqueness for nonlinear damped equations

Let us consider the nonlinear equation (6.9) or (6.3) under the hypotheses given in Section 6.2.1. We know that such assumptions implies in particular that Hypothesis 2.1 is satisfied, with $A = \mathcal{A}_{\alpha,\rho}$ and G defined as in Section 6.2.1. Hence, there exists a unique weak solution $X = (X_t)_{t \geq 0}$ to (6.9).

Next we show that the controllability assumption in Hypothesis 5.1(i), the estimates on Λ_1 and Λ_2 in Hypotheses 5.1(ii) and in 5.13 and Hypothesis 5.3 hold true, with $\Gamma(t) = \Gamma_{\alpha,\rho}(t) = (Q_t^{\alpha,\rho})^{-1/2} e^{t\mathcal{A}_{\alpha,\rho}}$ for any $t > 0$.

Proposition 6.6. *Let us take $A = \mathcal{A}_{\alpha,\rho}$ and G as in Section 6.2.1, let $\Gamma(t) = \Gamma_{\alpha,\rho}(t) = (Q_t^{\alpha,\rho})^{-1/2} e^{t\mathcal{A}_{\alpha,\rho}}$ for any $t > 0$ and let β satisfies Hypothesis 6.1. Then, Hypotheses 5.1, 5.3 and 5.13 hold true with*

$$\Lambda_1(t) \sim \Lambda_2(t) \sim t^{-1/2-2\gamma} \tag{6.22}$$

for every $\alpha \in \left[0, \frac{1}{2}\right]$

Proof. Estimates of Λ_1 and Λ_2 follows from Corollary A.2 and Theorem A.3. Further,

$$|\Gamma(t)Ga|_H \leq Ct^{-1/2-2\gamma}|\Lambda^{-\gamma}a|_U, \quad t \in (0, T],$$

for every $\alpha \in [0, 1/2]$. Therefore, for any orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of U we have

$$\|\Gamma(t)G\|^2_{L_2(U;H)} = \sum_{n\in\mathbb{N}} |\Gamma(t)Gh_n|^2_H \leq \frac{C^2}{t^{1+4\gamma}} \sum_{n\in\mathbb{N}} |\Lambda^{-\gamma}h_n|^2_U = \frac{C^2 \mathrm{Tr}(\Lambda^{-2\gamma})}{t^{1+4\gamma}}, \quad t \in (0, T].$$

This implies that (5.24) holds true with $\Lambda_2(t) = t^{-(1/2+2\gamma)}$.

Let us conclude by proving that also Hypothesis 5.3 are satisfied. Indeed, we have

$$\Lambda_1(t)^{1-\beta}\Lambda_2(t) \sim t^{-(1/2+2\gamma)(2-\beta)},$$

and from the choice of β we get $-(1/2 + 2\gamma)(2 - \beta) > -1$. □

The previous results show that we can apply Theorem 4.2 to equation (6.9). We finally get

Theorem 6.7. *Let* $\mathcal{A}_{\alpha,\rho}$ *and* G *be defined as in Section 6.2.1, and let the assumptions of Section 6.2.1 be satisfied. Then, for the nonlinear damped beam equation (6.9) the assertions of Theorem 4.2 hold. In particular, we have pathwise uniqueness for (6.9).*

6.3. Semilinear stochastic heat equations

Let us assume that $A = \Delta$ is the realization of the Laplacian operator in $H = U = L^2([0, \pi]^d)$ with periodic boundary conditions, and let $G = (-\Delta)^{-\gamma/2}$ with $\gamma \geq 0$. We are considering

$$\begin{cases} dX^x_t = \Delta X^x_t dt + C(X^x_t)dt + (-\Delta)^{-\gamma/2}dW_t, & t \in [0, T], \\ X^x_0 = x \in H. \end{cases} \tag{6.23}$$

It is well known that $D(A) = H^2([0, 2\pi]^d)_{\mathrm{per}}$, the classical Sobolev space with periodic boundary conditions. We notice that Hypothesis 5.16-1, is fulfilled. Let W be a cylindrical Wiener process on H, see Section 2. We first note that assumption (2.2) is verified if

$$1 + \gamma > \frac{d}{2} \tag{6.24}$$

(see also the proof of Lemma 9 in [8]). In particular, if $\gamma = 0$, i.e., $G = I$, it is required $d = 1$, as in [8]. Let $R > 0$. We consider, for a fixed $\beta \in (0, 1)$,

$$C(f)(\xi) := g(\xi) \int\limits_{[0,2\pi]^d} h(\xi')\left(|f(\xi')| \wedge R\right)^{\beta} d\xi', \quad \xi \in [0, 2\pi]^d, \tag{6.25}$$

for any $f \in H$ and $g \in H^\gamma([0,2\pi]^d)_{\rm per}$ (see, for instance, Section 6 in [20]) and $h \in L^\infty([0,2\pi]^d)$. The regularity of g implies that $C(f) \in D((-A)^{\gamma/2}) = \mathrm{Im}(G)$ for any $f \in H$. Hence we can set

$$\widetilde{C}(f) = G^{-1}C(f) = (-\Delta)^{\gamma/2}C(f), \quad f \in H,$$

and write $C(f) = (-\Delta)^{-\gamma/2}\widetilde{C}(f)$ in (6.23).

Arguing as in [8, Lemma 8] it follows that there exists a positive constant M such that

$$|\widetilde{C}(f_1) - \widetilde{C}(f_2)|_H \le M|(-\Delta)^{-\gamma/2}g|_H\|h\|_\infty|f_1 - f_2|_H^\beta, \quad f_1, f_2 \in H.$$

On the other hand, using the orthonormal basis (e_n) in Hypothesis 5.16, we have the estimates

$$|\widetilde{C}(f)_n| = |\langle\widetilde{C}(f), e_n\rangle_H|$$
$$= |\langle(-\Delta)^{\gamma/2}g, e_n\rangle_H \int\limits_{[0,2\pi]^d} h(\xi')\sqrt{|f(\xi')| \wedge R}\, d\xi'| \le CR|\langle(-\Delta)^{\gamma/2}g, e_n\rangle_H|\, \|h\|_\infty;$$
$$|\widetilde{C}(f_1)_n - \widetilde{C}(f_2)_n| \le C|\langle(-\Delta)^{\gamma/2}g, e_n\rangle_H|\, \|h\|_\infty\, |f_1 - f_2|_H^\beta, \quad f, f_1, f_2 \in H.$$

Hence, Hypothesis 5.16-2, is satisfied. Indeed we have

$$\sum_{n\in\mathbb{N}} \frac{\|(\widetilde{C}(\cdot))_n\|_\beta^2}{\alpha_n} \le \frac{C_R}{\alpha_1}\|h\|_\infty^2 \sum_{n\in\mathbb{N}} |\langle(-\Delta)^{\gamma/2}g, e_n\rangle_H|^2 \le \frac{C_R'}{\alpha_1}\|h\|_\infty^2\, \|g\|^2_{H^\gamma([0,2\pi]^d)_{\rm per}} < \infty.$$

Let us discuss (ii) in Hypothesis 5.3. With our choice of A, H and G we have

$$\Lambda_1(t) = t^{-1/2-\gamma/2}, \quad \Lambda_2(t) = t^{-1/2}, \quad t \in (0,T].$$

Hence, Hypothesis 5.3 (ii) is satisfied if

$$0 \le \gamma < \frac{\beta}{1-\beta}. \tag{6.26}$$

Since $\beta(1-\beta)^{-1} \to +\infty$ as $\beta \to 1^-$, it follows that the bigger the Hölder exponent β is, the bigger is the bound for γ. In particular (see also the remark below):

(i) if we choose $\beta \in (\frac{1}{3}, 1)$ then (6.26) is satisfied for some $\gamma > \frac{1}{2}$, hence $2(1+\gamma) > 3$ and according to (6.24) we can also consider $d = 3$ for the SPDE (6.23);

(ii) if we take $\beta \in (1/2, 1)$ then (6.26) if fulfilled for some $\gamma > 1$ and we get $2(1+\gamma) > 4$; according to (6.24) we can also consider the case $d = 4$ for the SPDE (6.23).

In the previous two cases we can apply Theorem 4.2 to equation (6.23) with C given in (6.25) and obtain pathwise uniqueness and Lipschitz dependence of initial conditions.

Remark 6.8. Concerning (6.23) the main difference with [8] is that in such paper the authors require the assumption

$$\int_0^T \Lambda_t^{(1+\theta)} dt < +\infty, \quad \theta = \max\{\beta, 1-\beta\},\ \Lambda_t = \Lambda_1(t) = t^{-1/2-\gamma/2}.$$

Hence, the best situation is $\beta = \frac{1}{2}$ and above condition reads as

$$\frac{3}{2}\left(\frac{1}{2} + \frac{\gamma}{2}\right) < 1 \Leftrightarrow \gamma < \frac{1}{3}.$$

According to (6.24) we need $1 + \gamma > \dfrac{d}{2}$. By combining this condition with $\gamma < \frac{1}{3}$ it follows that the case $d = 3$ is not reached in [8].

Data availability

No data was used for the research described in the article.

Acknowledgment

The authors are members of INdAM-GNAMPA.

Appendix A. Energy estimates for the control problem associated to the beam equation

As in Subsection 6.2.3, here we consider complexified spaces. In $H = V \times U$ we consider the following control problem

$$\begin{cases} z_t(t) = \mathcal{A}_{\alpha,\rho} z(t) + \widetilde{G}\Lambda^{-\gamma} u(t), & t \in [0, T], \\ z(0) = (z_0^1, z_0^2) \in H, \end{cases} \tag{A.1}$$

where

$$\mathcal{A}_{\alpha,\rho} := \begin{pmatrix} 0 & I \\ -\Lambda & -\rho\Lambda^{\alpha} \end{pmatrix} : D(\mathcal{A}_{\alpha,\rho}) \subset H \to H, \quad \widetilde{G} := \begin{pmatrix} 0 \\ I \end{pmatrix} : U \to H,$$

and $\Lambda : D(\Lambda) \subset U \to U$, α and ρ satisfy the assumptions in Section 6.2.1.

Let $z(t) = (z_1(t), z_2(t)) \in H$ be the solution to the problem (A.1); arguing as in Section 6.2.2 we infer that $y(t) := (y_1(t), y_2(t)) := (\Lambda^{1/2} z_1(t), z_2(t)) \in \widetilde{H} := U \times U$ is solution to

$$\begin{cases} y_t(t) = \widetilde{\mathcal{A}}_{\alpha,\rho} y(t) + \widetilde{G}\Lambda^{-\gamma} u(t), & t \in [0, T], \\ y(0) = (y_0^1, y_0^2) \in \widetilde{H}, \end{cases} \tag{A.2}$$

where $y_0^1 = \Lambda^{1/2} z_0^1$, $y_0^2 = z_0^2$ and

$$\widetilde{\mathcal{A}}_{\alpha,\rho} := \begin{pmatrix} 0 & \Lambda^{1/2} \\ -\Lambda^{1/2} & -\rho\Lambda^{\alpha} \end{pmatrix} : D(\widetilde{\mathcal{A}}_{\alpha,\rho}) \subset \widetilde{H} \to \widetilde{H}.$$

For every $a \in U$ and $k = Ga = \widetilde{G}\Lambda^{-\gamma}a$, we get $|k|_H = |k|_{\widetilde{H}} = |\Lambda^{-\gamma}a|_U$.

We notice that a control u steers k to 0 at time $t \in (0, T]$ in H in (A.1) if and only if u steers k to 0 at time $t \in (0, T]$ in $\widetilde{H}$ in (A.2). Hence, the energy to steer k to 0 at time t in $\widetilde{H}$, which is given by

$$\widetilde{\mathcal{E}}_C(t,k) := \inf\left\{ \int_0^t |u(s)|_U^2 ds : u \in L^2(0,T;U),\ y \text{ solution to (A.2)},\ k = (y_0^1, y_0^2),\ y(t) = 0 \right\},$$

coincides with the energy to steer k to 0 at time t in H, which is given by

$$\mathcal{E}_C(t,k) := \inf\left\{ \int_0^t |u(s)|_U^2 ds : u \in L^2(0,T;U),\ z \text{ solution to (A.1)},\ k = (z_0^1, z_0^2),\ z(t) = 0 \right\}.$$

We will prove that (A.2) is null controllable and we provide an estimate to $\widetilde{\mathcal{E}}_C(T,k)$.

Theorem A.1. *Let $\mathcal{A}_{\alpha,\rho}$ be defined in (6.8) and let Hypotheses 6.2 and condition (6.4) in Section 6.2.1 hold true. Let $T > 0$. Then, there exists a positive constant $C = C(T)$ such that for any $a \in U$, $k = Ga$, the energy to steer k to 0 at time t can be estimated by*

$$\widetilde{\mathcal{E}}_C(t,k) \le \frac{C|\Lambda^{-\gamma}a|_U^2}{t^{1+4\gamma}} \qquad t \in (0,T].$$

In particular, the equality $\widetilde{\mathcal{E}}_C(t,k) = \mathcal{E}_C(t,k)$ implies that for every $T > 0$ there exists a positive constant $C = C(T)$ such that for any $a \in U$, $k = Ga$, the energy to steer k to 0 at time t can be estimated by

$$\mathcal{E}_C(t,k) \le \frac{C|\Lambda^{-\gamma}a|_U^2}{t^{1+4\gamma}} \qquad t \in (0,T].$$

Proof. Here, we consider the decomposition introduced in Section 6.2.3 and we keep the same notation. We follow the method in [31, Proposition 1.3], which has been extended to infinite dimension in [26]. The idea is the following. If we consider the matrix formulation for $\widetilde{\mathcal{A}}_{\alpha,\rho}$ and G, then the 2×2-matrix $[G|\widetilde{\mathcal{A}}_{\alpha,\rho}G]$ is invertible. We denote by K its inverse matrix and by K_0 and K_1 the rows of K. Then, the control

$$u(t) = K_0(t)\psi(t) + K_1\psi'(t), \qquad t \in [0,T],$$

steers k to 0, where $\psi(t) = -\Phi(t)e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}k$ and $\Phi(t)$ is a suitable smooth function. We construct our control adapting this approach to our situation.

The mild solution to (A.2) is

$$y(t) = e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}} y(0) + \int\limits_0^t e^{(t-s)\widetilde{\mathcal{A}}_{\alpha,\rho}} \widetilde{G}\Lambda^{-\gamma} u(s)ds$$

for every $t \in [0, T]$. Let us fix $T > 0$ and $a \in U$. Further, we set $f_T(t) := t^2(T-t)^2$ and $\phi_T(t) := \|f_T\|^{-1}_{L^1(0,T)} f_T(t)$. It follows that $\|\phi_T\|_{L^1(0,T)} = 1$, that ϕ_T vanishes at 0 and T and that $|\phi_T(t)| \leq \tilde{c}T^{-3}t^2$, $|t\phi_T'(t)| \leq \tilde{c}T^{-3}t^2$ and $|\phi_T'(t)| \leq \tilde{c}T^{-3}t$, for some positive constant $\tilde{c}$ and any $t \in [0, T]$. We claim that the control

$$v(t) := v_0(t) + v_1'(t), \quad t \in (0, T], \qquad v(0) = 0,$$

defined by

$$\langle v_0(t), e_n/|e_n|_U\rangle_U := \mu_n^\gamma \left(\frac{\lambda_n^- e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} - \frac{\lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) \widetilde{a}_n \phi_T(t), \quad n \in \mathbb{N}, \tag{A.3}$$

$$\langle v_1(t), e_n/|e_n|_U\rangle_U := \mu_n^\gamma \left(\frac{-e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) \widetilde{a}_n \phi_T(t), \quad n \in \mathbb{N}, \tag{A.4}$$

for $t \in (0, T]$, steers the initial state Ga at 0 at time T, where $\widetilde{a}_n := \langle \Lambda^{-\gamma}a, e_n/|e_n|_U\rangle_U$ for any $n \in \mathbb{N}$ (we have $v_1' = \frac{dv_1}{dt}$). The series which define $v_0(t)$ and $v_1(t)$ are well-defined since μ_n grows $|\lambda_n^\pm|^2$ (see (6.17)) and for every $t > 0$ we have $e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}$ maps $\widetilde{H}$ onto $D((\widetilde{\mathcal{A}}_{\alpha,\rho})^k)$ for every $k \in \mathbb{N}$.

At first, we notice that $\widetilde{G}\Lambda^{-\gamma} v_1(t) \in D(\widetilde{\mathcal{A}}_{\alpha,\rho})$ for any $t \in [0, T]$. Indeed, from the definitions of G and v_1 we have

$$\begin{aligned} \widetilde{G}\Lambda^{-\gamma} v_1(t) = \phi_T(t) \sum_{n=1}^\infty \Bigg[& \left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) b_n^+ \widetilde{a}_n \Phi_n^+ \\ & + \left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) b_n^- \widetilde{a}_n \Phi_n^- \Bigg]. \end{aligned}$$

For any $N \in \mathbb{N}$ we set

$$(\widetilde{G}\Lambda^{-\gamma} v_1(t))_N = \phi_T(t) \sum_{n=1}^N \left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) \left[b_n^+ a_n \Phi_n^+ + b_n^- a_n \Phi_n^- \right].$$

Clearly, $(\widetilde{G}\Lambda^{-\gamma} v_1(t))_N \to \widetilde{G}\Lambda^{-\gamma} v_1(t)$ as $N \to +\infty$ in $\widetilde{H}$. Further, from the previous decomposition we infer that $(\widetilde{G}\Lambda^{-\gamma} v_1(t))_N \in D(\widetilde{\mathcal{A}}_{\alpha,\rho})$ for any $N \in \mathbb{N}$ and

$$\widetilde{\mathcal{A}}_{\alpha,\rho}(\widetilde{G}\Lambda^{-\gamma} v_1(t))_N = \phi_T(t) \sum_{n=1}^N \left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+} \right) \left[\lambda_n^+ b_n^+ \widetilde{a}_n \Phi_n^+ + \lambda_n^- b_n^- \widetilde{a}_n \Phi_n^- \right].$$

Hence, recalling the definition of Φ_n^+ and of Φ_n^- we get

$$
\begin{aligned}
|\widetilde{\mathcal{A}}_{\alpha,\rho}(\widetilde{G}\Lambda^{-\gamma}v_1(t))_N|_{\widetilde{H}}^2 = \phi_T(t)^2 \sum_{n=1}^{N} \widetilde{a}_n^2 &\left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right)^2 \\
&\times \Big((\lambda_n^+)^2(b_n^+)^2 + (\lambda_n^-)^2((\chi_n b_n^-)^2 \\
&+ 2\chi_n b_n^+ b_n^- \lambda_n^+ \lambda_n^- (\mu_n |e_n|_U^2 + \lambda_n^+\lambda_n^- |e_n|_U^2)\Big).
\end{aligned}
$$

We notice that from (6.17) it follows that for any $t \in (0, T]$

$$
\begin{aligned}
&\left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right)^2 (|\lambda_n^+|^2 + |\lambda_n^-|^2 + |\lambda_n^+||\lambda_n^-|) \sim \text{const}; \\
&(\mu_n |e_n|_U^2 + \lambda_n^+\lambda_n^- |e_n|_U^2) \sim \text{const},
\end{aligned}
$$

and that

$$
\sum_{n=1}^{\infty} \widetilde{a}_n^2 (b_n^+)^2, \quad \sum_{n=1}^{\infty} \widetilde{a}_n^2 (b_n^-)^2 < +\infty.
$$

Since

$$
\begin{aligned}
&|\widetilde{\mathcal{A}}_{\alpha,\rho}(\widetilde{G}\Lambda^{-\gamma}v_1(t))_N - \widetilde{\mathcal{A}}_{\alpha,\rho}(\widetilde{G}\Lambda^{-\gamma}v_1(t))_{N+p}|_{\widetilde{H}}^2 \\
&\leq M\phi_T(t)^2 \sum_{n=N+1}^{N+p} \left(-\frac{e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right)^2 \cdot \\
&(|\lambda_n^+|^2 + |\lambda_n^-|^2 + |\lambda_n^+||\lambda_n^-|) \left(\widetilde{a}_n^2(b_n^+)^2 + \widetilde{a}_n^2(b_n^-)^2\right) \leq M \sum_{n=N+1}^{N+p} \left(\widetilde{a}_n^2(b_n^+)^2 + \widetilde{a}_n^2(b_n^-)^2\right),
\end{aligned}
$$

for some positive constant M, it follows that $\widetilde{\mathcal{A}}_{\alpha,\rho}(Gv_1(t))_N$ converges to

$$
\phi_T(t)\sum_{n=1}^{\infty}\left[\left(-\frac{\lambda_n^+ e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{\lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right) b_n^+ \widetilde{a}_n \Phi_n^+ + \left(-\frac{\lambda_n^- e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{\lambda_n^- e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right) b_n^- \widetilde{a}_n \Phi_n^-\right],
$$

in $\widetilde{H}$ as $N \to +\infty$. Since $\widetilde{\mathcal{A}}_{\alpha,\rho}$ is a closed operator, it follows that $\widetilde{G}\Lambda^{-\gamma}v_1(t) \in D(\widetilde{\mathcal{A}}_{\alpha,\rho})$, for any $t \in [0, T]$, and

$$
\begin{aligned}
\widetilde{\mathcal{A}}_{\alpha,\rho}\widetilde{G}\Lambda^{-\gamma}v_1(t) = \phi_T(t)\sum_{n=1}^{\infty}\Bigg[&\left(-\frac{\lambda_n^+ e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{\lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right) b_n^+ \widetilde{a}_n \Phi_n^+ \\
&+ \left(-\frac{\lambda_n^- e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{\lambda_n^- e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right) b_n^- \widetilde{a}_n \Phi_n^-\Bigg]. \qquad (A.5)
\end{aligned}
$$

Let us consider the integral term in the mild solution. We get

$$\begin{aligned}\int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}\widetilde{G}\Lambda^{-\gamma}v(s)ds &= \int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}\widetilde{G}\Lambda^{-\gamma}v_0(s)ds + \int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}\widetilde{G}\Lambda^{-\gamma}v_1'(s)ds \\ &= \int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}(\widetilde{G}\Lambda^{-\gamma}v_0(s) + \widetilde{\mathcal{A}}_{\alpha,\rho}\widetilde{G}\Lambda^{-\gamma}v_1(s))ds, \qquad \text{(A.6)}\end{aligned}$$

where we have integrated by parts under the second integral and we have used the fact that $\widetilde{G}\Lambda^{-\gamma}v_1(s) \in D(\widetilde{\mathcal{A}}_{\alpha,\rho})$ for any $s \in [0, T]$, and that $\Lambda^{-\gamma}v_1(0) = \Lambda^{-\gamma}v_1(T) = 0$. We notice that for any $s \in [0, T]$ we have

$$\begin{aligned}\widetilde{G}\Lambda^{-\gamma}v_0(s) = \phi_T(s)\sum_{n=1}^{\infty}\Bigg[&\left(\frac{\lambda_n^- e^{\lambda_n^+ s}}{\lambda_n^- - \lambda_n^+} - \frac{\lambda_n^+ e^{\lambda_n^- s}}{\lambda_n^- - \lambda_n^+}\right)b_n^+\widetilde{a}_n\Phi_n^+ \\ &+ \left(\frac{\lambda_n^- e^{\lambda_n^+ s}}{\lambda_n^- - \lambda_n^+} - \frac{\lambda_n^+ e^{\lambda_n^- s}}{\lambda_n^- - \lambda_n^+}\right)b_n^-\widetilde{a}_n\Phi_n^-\Bigg]. \qquad \text{(A.7)}\end{aligned}$$

Hence, (A.5) and (A.7) give

$$\begin{aligned}\widetilde{G}\Lambda^{-\gamma}v_0(s) + \widetilde{\mathcal{A}}_{\alpha,\rho}\widetilde{G}\Lambda^{-\gamma}v_1(s) &= -\phi_T(s)\sum_{n=1}^{\infty}\left(e^{\lambda_n^+ s}b_n^+\widetilde{a}_n\Phi_n^+ + e^{\lambda_n^- s}b_n^-\widetilde{a}_n\Phi_n^-\right) \\ &= -\phi_T(s)e^{s\widetilde{\mathcal{A}}_{\alpha,\rho}}(Ga), \quad s \in [0, T]. \qquad \text{(A.8)}\end{aligned}$$

Replacing (A.8) in (A.6) we get

$$\int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}\widetilde{G}\Lambda^{-\gamma}v(s)ds = -e^{T\widetilde{\mathcal{A}}_{\alpha,\rho}}(Ga).$$

The mild formulation of the solution y to (A.2) implies that

$$y(T) = e^{T\widetilde{\mathcal{A}}_{\alpha,\rho}}(Ga) + \int_0^T e^{(T-s)\widetilde{\mathcal{A}}_{\alpha,\rho}}\widetilde{G}\Lambda^{-\gamma}v(s)ds = e^{T\widetilde{\mathcal{A}}_{\alpha,\rho}}(Ga) - e^{T\widetilde{\mathcal{A}}_{\alpha,\rho}}(Ga) = 0,$$

which gives the claim.

Now we estimate the L^2-norm of the control v. We separately consider the two addends v_0 and v_1.

As far as v_0 is concerned, from (A.3) we get

$$\int_0^T |v_0(t)|_U^2 dt \le \int_0^T (\phi_T(t))^2 \sum_{n=1}^{\infty}\mu_n^{2\gamma}\left|\frac{\lambda_n^- e^{\lambda_n^+ t} - \lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right|^2 |\widetilde{a}_n|^2 dt.$$

From (6.17), for any $n \in \mathbb{N}$ we get

$$\begin{aligned}\mu_n^{2\gamma}\left|\frac{\lambda_n^- e^{\lambda_n^+ t} - \lambda_n^+ e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right|^2 &\leq C\left|\mu_n^{\gamma}\left(e^{\lambda_n^+ t}\right) + \mu_n^{\gamma}\left(e^{\lambda_n^- t}\right)\right|^2 \\ &\leq C\left|(\lambda_n^+)^{2\gamma}\left(e^{\lambda_n^+ t}\right) + (\lambda_n^-)^{2\gamma}\left(e^{\lambda_n^- t}\right)\right|^2 \\ &\leq C|\widetilde{\mathcal{A}}_{\alpha,\rho}^{4\gamma} e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}\Phi_n^+|_{\widetilde{H}}^2 + |\widetilde{\mathcal{A}}_{\alpha,\rho}^{4\gamma} e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}\Phi_n^-|_{\widetilde{H}}^2 \\ &\leq C L_T^2 t^{-4\gamma}, \end{aligned} \tag{A.9}$$

where C is a positive constant which may vary from line to line. Hence,

$$\int_0^T |v_0(t)|_U^2 dt \leq \text{const}\int_0^T |\phi_T(t)|^2 t^{-4\gamma} dt |\Lambda^{-\gamma} a|_U^2 \leq C|\Lambda^{-\gamma} a|_U^2 T^{-1-4\gamma},$$

for some positive constant C.

Let us consider v_1'. From (A.4) we get

$$\begin{aligned}\int_0^T |v_1'(t)|_U^2 dt \leq & 2\int_0^T (\phi_T(t))^2 \sum_{n=1}^{\infty} \mu_n^{2\gamma}\left|\frac{-\lambda_n^+ e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{\lambda_n^- e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right|^2 |\widetilde{a}_n|^2 dt \\ & + 2\int_0^T (\phi_T'(t))^2 \sum_{n=1}^{\infty} \mu_n^{2\gamma}\left|\frac{-e^{\lambda_n^+ t}}{\lambda_n^- - \lambda_n^+} + \frac{e^{\lambda_n^- t}}{\lambda_n^- - \lambda_n^+}\right|^2 |\widetilde{a}_n|^2 dt := I_1 + I_2.\end{aligned}$$

The first integral can be estimated arguing as for v_0. By taking I_2 into account, if we multiply and divide by t^2 under the integral sign we get

$$I_2 = 2\int_0^T (t\phi_T'(t))^2 \sum_{n=1}^{\infty} \mu_n^{2\gamma}\left|\left(\frac{-e^{\lambda_n^+ t} + e^{\lambda_n^- t}}{t(\lambda_n^- - \lambda_n^+)}\right)\widetilde{a}_n\right|^2 dt.$$

Since there exists a positive constant C such that

$$\sup_{t\in(0,T]}\sup_{n\in\mathbb{N}} \mu_n^{2\gamma}\left|\frac{-e^{\lambda_n^+ t} + e^{\lambda_n^- t}}{t(\lambda_n^- - \lambda_n^+)}\right|^2 = \sup_{t\in(0,T]}\sup_{n\in\mathbb{N}} \mu_n^{2\gamma}\left|\frac{e^{\lambda_n^+ t}(e^{(\lambda_n^- - \lambda_n^+)t} - 1)}{t(\lambda_n^- - \lambda_n^+)}\right|^2 \leq C,$$

we infer that $I_2 \leq c|\Lambda^{-\gamma} a|_U^2 T^{-1}$ for some positive constant c. Therefore, we can conclude that

$$\|v\|_{L^2(0,T;U)} \leq \frac{c|\Lambda^{-\gamma} a|_U}{T^{1/2+2\gamma}}. \quad \square$$

Theorem A.1 has an important consequence, due to the fact that $\mathcal{E}_C(t,k) = |Q^{-1/2}e^{t\mathcal{A}_{\alpha,\rho}}k|_H^2$ for any $k \in H$ and any $t > 0$.

Corollary A.2. *Under the assumptions of Theorem A.1, there exists a positive constant C such that for any $a \in U$ and any $t \in (0,T]$ we have*

$$|Q_t^{-1/2}e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}Ga|_H^2 = \mathcal{E}_C(t,Ga) = \mathcal{E}_C(t,\widetilde{G}\Lambda^{-\gamma}a) \le \frac{C|\Lambda^{-\gamma}a|_U^2}{t^{1+4\gamma}}.$$

Now we provide energy estimates in case of general initial datum $k \in H$.

Theorem A.3. *Let $\mathcal{A}_{\alpha,\rho}$ be defined in (6.8) and let Hypotheses 6.2 and condition (6.4) in Section 6.2.1 hold true. There exists a positive constant c such that for every $h \in H$ we have*

$$\mathcal{E}_C(t,h) \le \frac{c|h|_H^2}{t^{1+4\gamma}} \qquad t \in (0,T].$$

In particular, there exists a positive constant c such that for any $h \in H$ and any $t \in (0,T]$

$$|Q_t^{-1/2}e^{t\widetilde{\mathcal{A}}_{\alpha,\rho}}h|_H^2 = \mathcal{E}_C(t,h) \le \frac{c|h|_H^2}{t^{1+4\gamma}}.$$

Proof. Let us apply the method exploited in the proof of Theorem A.1. Let $h \in H$. As above, we provide an estimate of $\widetilde{\mathcal{E}}_C(t,h)$. We set

$$\langle v_0(t), e_n/|e_n|_U\rangle_U := \mu_n^\gamma\left(\lambda_n^- e^{\lambda_n^+ t}h_n^+ + \chi_n\lambda_n^+ e^{\lambda_n^- t}h_n^-\right)|e_n|_U\phi_T(t), \qquad n \in \mathbb{N},$$

$$\langle v_1(t), e_n/|e_n|_U\rangle_U := -\mu_n^\gamma\left(e^{\lambda_n^+ t}h_n^+ + \chi_n e^{\lambda_n^- t}h_n^-\right)|e_n|_U\phi_T(t), \qquad n \in \mathbb{N},$$

for $t \in (0,T]$, and the control

$$v(t) := v_0(t) + v_1'(t), \quad t \in (0,T], \qquad v(0) = 0.$$

Here, $h_n^+ = \langle h^+, \Phi_n^+\rangle_{\widetilde{H}}$ and $h_n^+ = \langle h^-, \Phi_n^-\rangle_{\widetilde{H}}$ for any $n \in \mathbb{N}$. Arguing as in the proof of Theorem A.1 it is possible to prove that v steers the initial state h at 0 at time T. It remains to estimate the L^2-norm of v. Arguing as before we get

$$\|v\|_{L^2(0,T;U)} \le \frac{c|h|_H^2}{T^{1/2+2\gamma}},$$

and we conclude. □

Appendix B. Proof of Lemma 5.6

Proof. In the proof C is a positive constant which may vary from line to line.
Let $t \in (0, T]$, let $y \in H$, and let us consider the linear operators

$$\nabla_y \mathcal{R}_t : C_b^1(H) \to C_b(H), \quad \nabla_y \mathcal{R}_t : C_b(H) \to C_b(H).$$

For any $\phi \in C_b^1(H)$ and any $x, y \in H$ we have

$$\nabla_y \mathcal{R}_t[\phi](x) = \int_H \langle \nabla\phi(e^{tA}x + z), e^{tA}y\rangle_H \mathcal{N}(0, Q_t)(dz),$$

from which it follows that

$$\sup_{x\in H} |\nabla_y \mathcal{R}_t[\phi](x)| \le C_T \|\phi\|_{C_b^1(H)} |y|_H, \quad \phi \in C_b^1(H). \tag{B.1}$$

Further, if $\phi \in C_b(H)$ we have

$$\nabla_y \mathcal{R}_t[\phi](x) = \int_H \langle \Gamma(t)y, Q^{-1/2}z\rangle \phi(e^{tA}x + z)\mathcal{N}(0, Q_t)(dz),$$

which combined with (5.1) implies

$$\sup_{x\in H} |\nabla_y \mathcal{R}_t[\phi](x)| \le C_T \|\phi\|_{C_b(H)} \Lambda_1(t) |y|_H, \quad \phi \in C_b(H). \tag{B.2}$$

Recalling (5.9) and interpolating between (B.1) and (B.2) we infer that

$$\sup_{x\in H} |\nabla_y \mathcal{R}_t[\phi](x)|_H \le C_T \|\phi\|_\beta \Lambda_1^{1-\beta}(t) |y|_H, \quad \phi \in C_b^\beta(H). \tag{B.3}$$

Let us consider $h \in H$ and $\Phi \in C_b^\beta(H; H)$. From (5.10) and (B.3) we have

$$\sup_{x\in H} |\langle \nabla_y R_t[\Phi](x), h\rangle_H| \le C_T \|\Phi_h\|_\beta \Lambda_1^{1-\beta}(t)|y|_H \le C_T \|\Phi\|_\beta \Lambda_1^{1-\beta}(t)|y|_H |h|_H,$$

which gives (5.11). To prove (5.12) we fix $y \in H$ and $k \in U$ and we consider the linear operators

$$\nabla_y \nabla_k^G \mathcal{R}_t : C_b^1(H) \to C_b(H), \quad \nabla_y \nabla_k^G \mathcal{R}_t : C_b(H) \to C_b(H).$$

For any $\phi \in C_b^1(H)$ it follows that

$$\nabla_y \nabla_k^G \mathcal{R}_t[\phi](x) = \int_H \langle \Gamma(t)Gk, Q_t^{-1/2}z\rangle_H \langle \nabla\phi(e^{tA}x + z), e^{tA}y\rangle \mathcal{N}(0, Q_t)(dz),$$

which implies that

$$|\nabla_y(\nabla_k^G \mathcal{R}_t[\phi])(x)|_H \leq C|\Phi\|_{C_b^1(H)}\Lambda_2(t)|y|_H|k|_U, \quad t \in (0,T],\ \phi \in C_b^1(H), \tag{B.4}$$

for any $x, y \in H$ and any $k \in U$. As above, we get

$$|\nabla_y(\nabla_k^G \mathcal{R}_t[\phi])(x)|_H \leq C\|\phi\|_{C_b(H)}\Lambda_1(t)\Lambda_2(t)|y|_H|k|_U, \tag{B.5}$$

for any $k \in U$, $x, y \in H$ and $t \in (0,T]$. Interpolating between (B.4) and (B.5) we infer that

$$|\nabla_y(\nabla_k^G \mathcal{R}_t[\phi])(x)|_H \leq C\|\phi\|_\beta \Lambda_1^{1-\beta}(t)\Lambda_2(t)|y|_H|k|_U, \quad t \in (0,T], \quad \phi \in C_b^\beta(H), \tag{B.6}$$

for any $x, y \in H$ and $k \in U$. Therefore, from (5.10) and (B.6) we infer that

$$\begin{aligned}\sup_{x\in H} |\langle \nabla_y \nabla_k^G R_t[\Phi](x), h\rangle_H| &\leq C_T\|\Phi_h\|_\beta \Lambda_1^{1-\beta}(t)\Lambda_2(t)|y|_H \\ &\leq C_T\|\Phi\|_\beta \Lambda_1^{1-\beta}(t)\Lambda_2(t)|y|_H|h|_H|k|_U. \quad \square\end{aligned}$$

References

[1] G. Avalos, I. Lasiecka, Optimal blowup rates for the minimal energy null control of the strongly damped abstract wave equation, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 5 (2003) 601–616.

[2] Z. Brzeźniak, B. Maslowski, J. Seidler, Stochastic nonlinear beam equations, Probab. Theory Relat. Fields 132 (2005) 119–149.

[3] Z. Brzeźniak, M. Ondreját, J. Seidler, Invariant measures for stochastic nonlinear beam and wave equations, J. Differ. Equ. 260 (2016) 4157–4179.

[4] S. Cerrai, G. Da Prato, F. Flandoli, Pathwise uniqueness for stochastic reaction-diffusion equations in Banach spaces with an Hölder drift component, Stoch. Partial Differ. Equ., Anal. Computat. 1 (2013) 507–551.

[5] G. Chen, D.L. Russell, A mathematical model for linear elastic systems with structural damping, Q. Appl. Math. 39 (1981/1982) 433–454.

[6] S.P. Chen, R. Triggiani, Proof of two conjectures by G. Chen and D. L. Russell on structural damping for elastic systems, in: Approximation and Optimization, Havana 1987, 1988, pp. 234–256.

[7] P.L. Chow, J.L. Menaldi, Stochastic PDE for nonlinear vibration of elastic panels, Differ. Integral Equ. 12 (1999) 419–434.

[8] G. Da Prato, F. Flandoli, Pathwise uniqueness for a class of SDE in Hilbert spaces and applications, J. Funct. Anal. 259 (2010) 243–267.

[9] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, Ann. Probab. 41 (2013) 3306–3344.

[10] G. Da Prato, F. Flandoli, E. Priola, M. Röckner, Strong uniqueness for stochastic evolution equations with unbounded measurable drift term, J. Theor. Probab. 28 (2015) 1571–1600.

[11] G. Da Prato, F. Flandoli, M. Röckner, A.Y. Veretennikov, Strong uniqueness for SDEs in Hilbert spaces with nonregular drift, Ann. Probab. 44 (2016) 1985–2023.

[12] G. Da Prato, J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, London Mathematical Society Note Series, vol. 293, Cambridge University Press, Cambridge, 2002.

[13] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, 2nd ed., Encyclopedia of Mathematics and Its Applications, vol. 152, Cambridge University Press, Cambridge, 2014.

[14] E. Fedrizzi, F. Flandoli, Holder flow and differentiability for SDEs with nonregular drift, Stoch. Anal. Appl. 31 (2013) 708–736.

[15] F. Flandoli, Random perturbation of PDEs andfluid dynamic models, in: Saint Flour Summer School Lectures, in: Lecture Notes in Math., Springer, Berlin, 2015.

[16] M. Fuhrman, G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations, Ann. Probab. 30 (2002) 1397–1465.

[17] G. Guatteri, On a class of forward-backward stochastic differential systems in infinite dimensions, J. Appl. Math. Stoch. Anal. 33pp (2007).

[18] G. Guatteri, G. Tessitore, On the backward stochastic Riccati equation in infinite dimensions, SIAM J. Control Optim. 44 (2005) 159–194.
[19] I. Gyöngy, É. Pardoux, On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations, Probab. Theory Relat. Fields 97 (1993) 211–229.
[20] M. Hairer, An introduction to stochastic PDEs, available at arXiv:0907.4178v1, 2009.
[21] Y. Hu, S. Peng, Adapted solution of a backward semilinear stochastic evolution equation, Stoch. Anal. Appl. 9 (1991) 445–459.
[22] I. Lasiecka, R. Triggiani, Exact null controllability of structurally damped and thermo-elastic parabolic models, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 9 (1998) 43–69.
[23] W. Liu, M. Röckner, Stochastic Partial Differential Equations: an Introduction, Universitext, Springer, Cham, 2015.
[24] A. Lunardi, Interpolation Theory, second edition, Lecture Notes. Scuola Normale Superiore di Pisa (New Series), Edizioni della Normale, Pisa, 2009.
[25] F. Masiero, Stochastic optimal control problems and parabolic equations in Banach spaces, SIAM J. Control Optim. 47 (2008) 251–300.
[26] F. Masiero, E. Priola, Well-posedness of semilinear stochastic wave equations with Hölder continuous coefficients, J. Differ. Equ. 263 (2017) 1773–1812.
[27] F. Masiero, E. Priola, Correction to "Well-posedness of semilinear stochastic wave equations with Hölder continuous coefficients", preprint, arXiv:1607.00029v2, https://arxiv.org/pdf/1607.00029.pdf.
[28] M. Ondreját, Uniqueness for stochastic evolution equations in Banach spaces, Dissertationes Math. (Rozprawy Mat.) 426 (2004) 1–63.
[29] R. Triggiani, Optimal estimates of norms of fast controls in exact null controllability of two non-classical abstract parabolic systems, Adv. Differ. Equ. 8 (2003) 189–229.
[30] A.J. Veretennikov, Strong solutions and explicit formulas for solutions of stochastic integral equations, Mat. Sb. 111 (153) (1980) 434–452.
[31] J. Zabczyk, Mathematical Control Theory, second edition, Birkhäuser, 2008.
[32] A.K. Zvonkin, A transformation of the phase space of a diffusion process that will remove the drift, Mat. Sb. 93 (135) (1974) 129–149.