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Abstract

In the p-wave spin-triplet pairing model of superfluid Helium-3, at each point x of the region Ω occupied by the system, the order parameter field is described by a 3×3 complex matrix A encoding the orientation of the spin and orbital angular momentum of the Cooper pairs of Helium-3 atoms. The transition of liquid Helium-3 to a superfluid state is associated with a spontaneous breaking of the overall symmetry group $\mathcal{G} = SO(3) \times SO(3) \times U(1)$ of the system. In the Ginzburg–Landau regime (i.e., in regions near to the critical phase-transition temperature), the free-energy density f of superfluid Helium-3 is expanded into powers of the components $A_{\mu j}$ of A and $A_{\mu j,k}$ of its gradient ∇A, and can be decomposed in the sum $f(A, \nabla A) = f_B(A) + f_{\text{grad}}(A, \nabla A)$ of the bulk part, f_B, and the gradient part, f_{grad}. The free-energy density f must be invariant under the action of $\mathcal{G}$ defined by $A \mapsto A' = \exp(i\phi)R_1 A R_2^T$, where ϕ is a phase, and R_1, R_2 are elements of $SO(3)$, that is, it must be invariant against gauge transformations and against rotations in spin space and ordinary (orbital) space, separately. We address the question of $\mathcal{G}$-invariance for a general free-energy density in the Ginzburg–Landau energy functional and determine all linearly independent quartic terms of the form $AA^*\nabla A(\nabla A)^*$ in the expansion of the gradient free-energy density. It is known that the superfluid phases of Helium-3 near the critical temperature correspond to the minima of the bulk free energy and that the absolute minimum corresponds to a stable equilibrium phase. In zero magnetic field, there are two distinct superfluid phases, A and B, which exhibit an absolute minimum of the bulk free energy in different regions of the phase diagram. Explicit expressions for the generalized gradient energy densities are provided for both the A and B phases. Finally, a unified approach to A and B phases is proposed, which involves an auxiliary control parameter. In this framework, the extremal properties of A and B phases are recovered and a transition between the two phases is observed in dependence of pressure.

1 Introduction

The ^{3}He (Helium-3) stable isotope of the Helium element has a boiling temperature equal to 3.19 K at 1 atm. Below this temperature one finds liquid ^{3}He. For $T > T_c$, where the critical phase-transition temperature T_c depends on the pressure and lies below the boiling temperature, liquid ^{3}He has all the symmetries allowed in condensed matter. Superfluidity only appears at $0 < T < T_c$ (cf. [30, 31]).

The known superfluid phases of ^{3}He are characterized by a condensation of Cooper pairs of ^{3}He atoms into a state characterized by the quantum numbers $\ell = 1$ and $S = 1$ (this type of pairing is referred to as a p-wave, spin triplet pairing). Here ℓ refers to the internal angular momentum operator $\mathbf{L}$ of the Cooper pairs, which describes the relative orbital motion of the ^{3}He atoms in a pair, while S refers to the total nuclear spin operator $\mathbf{S}$ of a pair.[1]

[1]We recall that for an abstract angular momentum operator $\mathbf{M}$, a state is said to have angular momentum j if it is an eigenstate of the operator $\mathbf{M}^2 = \mathbf{M}_x^2 + \mathbf{M}_y^2 + \mathbf{M}_z^2$ with eigenvalue $j(j+1)\hbar^2$, where $\hbar$ is the reduced Plank constant. The possible values for the quantum number j are $j = 0, 1/2, 1, 3/2, 2, \ldots$. Having angular momentum j means that an arbitrary axis component of the angular momentum $\mathbf{M}$, say $\mathbf{M}_z$, has simple eigenvalues $m\hbar^2$, where $m \in E_j := \{-j, -j+1, \ldots, j\}$. The corresponding $2j+1$ eigenvectors v_m, $m \in E_j$, span a $(2j+1)$-dimensional subspace $\mathcal{V}_{2j+1}$ which defines an irreducible representation ρ_j of the rotation group $SO(3)$. Moreover, all vectors v_m, $m \in E_j$, are eigenvectors of the operator $\mathbf{M}^2$; indeed, $\mathbf{M}^2(v_m) = j(j+1)\hbar^2 v_m$. For the orbital angular momentum $\mathbf{L}$, the possible values of the orbital quantum number ℓ are $\ell = 0, 1, 2, \ldots$. This implies that an arbitrary axis component of the angular momentum $\mathbf{L}$, say $\mathbf{L}_z$, has simple eigenvalues $m\hbar$, $m = -\ell, -\ell+1, \ldots, \ell$. If $\ell = 1$, the z-axis component $\mathbf{L}_z$ has only three eigenvalues, namely $-\hbar$, 0, and $\hbar$. The spin quantum number S refers to the total nuclear spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$ of the two ^{3}He atoms. In this case the nuclear spin of either atom in the pair is $s = 1/2$. If $S = 1$, the z-axis component $\mathbf{S}_z$ has only three eigenvalues, namely $-\hbar$, 0, and $\hbar$. For more details, we refer to [21].

The transition of liquid ^{3}He to a superfluid state is associated with a spontaneous breaking of symmetry, as in any other phase transition of condensed matter into an ordered state (cf. [16]). To a good approximation, above the critical phase-transition temperature T_c, liquid ^{3}He is symmetric under independent spin and space (orbital) rotations, as well as under gauge transformations (phase changes).[2] Thus the symmetry group $\mathcal{G}$ relevant to the superfluid transition of ^{3}He is given by the product

$$\mathcal{G} = SO(3)_{\mathbf{S}} \times SO(3)_{\mathbf{L}} \times U(1)_{\Phi} \tag{1.1}$$

where the subscripts $\mathbf{S}$ and $\mathbf{L}$ denote the infinitesimal generators of the groups of rotations in spin space and in ordinary (orbital) space, and Φ denotes the generator of the gauge group $U(1)$.

Within the Landau theory of phase transitions [10, 16], if T is close enough to T_c, the order parameter describing the presence of superfluid states (or phases) is given by a 2×2 complex matrix of the form [20, 29, 30][3]

$$\boldsymbol{\Delta}(\widehat{\mathbf{k}}) = A_{\mu j} \widehat{\mathbf{k}}_j \sigma_\mu \, \mathsf{i} \, \sigma_2 \quad (\mu, j = 1, 2, 3) \tag{1.2}$$

where μ labels the *spin* indices and j the *orbital* ones, $\widehat{\mathbf{k}} \in \mathbb{S}^2$ is the orbital unit vector, or momentum, σ_μ are the Pauli spin matrices,[4] and $A = (A_{\mu j})$ is a general 3×3 complex matrix. More explicitly, we have

$$\boldsymbol{\Delta}(\widehat{\mathbf{k}}) = \begin{pmatrix} (-A_{1j} + \mathsf{i}\, A_{2j}) \, \widehat{\mathbf{k}}_j & A_{3j} \widehat{\mathbf{k}}_j \\ A_{3j} \widehat{\mathbf{k}}_j & (A_{1j} + \mathsf{i}\, A_{2j}) \, \widehat{\mathbf{k}}_j \end{pmatrix}.$$

The 3×3 complex matrix $A = (A_{\mu j})$ involved in (1.2) completely determines $\boldsymbol{\Delta}(\widehat{\mathbf{k}})$ and can then be considered as the order parameter of the system, describing the presence of superfluid states (or phases) in a certain domain $\Omega \subset \mathbb{R}^3$ of ^{3}He. The matrix A transforms like a vector under a spin rotation, for a given orbital index j, and like a vector under an orbital rotation, for a given spin index μ. Hence A may be expressed as a second order tensor of the form

$$A = \sum_{\mu, j = 1}^{3} A_{\mu j} \mathbf{e}_\mu \otimes \mathbf{e}_j : \Omega \subset \mathbb{R}^3 \to \mathbb{C}^3 \otimes \mathbb{C}^3$$

where $\{\mathbf{e}_\mu = \mathbf{e}_\mu^{(S)}\}$ and $\{\mathbf{e}_j = \mathbf{e}_j^{(L)}\}$ denote the unit vectors of the Cartesian coordinate systems in the spin space and the ordinary (orbital) space (in principle, they may be chosen to be different).[5] Correspondingly, the action on $A = (A_{\mu j})$ of an element $G := (R^{(S)}, R^{(L)}, \phi) \in \mathcal{G}$ is given by

$$(G \star A)_{\mu j} := \left(\mathsf{e}^{\mathsf{i}\phi} R^{(S)} A {R^{(L)}}^T \right)_{\mu j} = \mathsf{e}^{\mathsf{i}\phi} R^{(S)}_{\mu\nu} R^{(L)}_{jk} A_{\nu k} \tag{1.3}$$

where ϕ is a phase, $R^{(S)} = (R^{(S)}_{\mu\nu}) \in SO(3)_{\mathbf{S}}$ and $R^{(L)} = (R^{(L)}_{jk}) \in SO(3)_{\mathbf{L}}$.

In the Ginzburg–Landau regime, that is, in regions near the critical temperature T_c, the free-energy density f in the Ginzburg–Landau integral functional is constructed using truncated expansions involving powers of the components of the tensor order parameter A and its gradient ∇A, subject to appropriate symmetry conditions. The free-energy density can be decomposed as

$$f(A, \nabla A) = f_B(A) + f_{\text{grad}}(A, \nabla A)$$

where $f_B(A)$ is the "bulk part" and $f_{\text{grad}}(A, \nabla A)$ is the "gradient part". The invariance under the physical symmetries $\mathcal{G}$, see (1.1), imposes certain restrictions on the form of the bulk and gradient parts of the

[2] The invariance under a phase transformation $\phi \mapsto \phi'$ is mathematically equivalent to a $U(1)$-symmetry.

[3] Here and in the following we adopt the Einstein summation convention by which repeated indices are implicitly summed. Moreover, i stands for the imaginary unit.

[4] I.e., the Hermitian matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -\mathsf{i} \\ \mathsf{i} & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which are the real generators of the Lie algebra $(-\mathsf{i})\mathfrak{su}(2) \subset \mathrm{End}(\mathbb{C}^2)$.

[5] In this paper, we think of vectors as column vectors. If $\mathbf{n}, \mathbf{m} \in \mathbb{R}^3$, the tensor product $\mathbf{n} \otimes \mathbf{m}$ is the matrix $\mathbf{n}\mathbf{m}^T$, so that if $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)^T$ and $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)^T$, then $(\mathbf{n} \otimes \mathbf{m})_{ij} = \mathbf{n}_i \mathbf{m}_j$.

free energy. More details on the question of the invariance of the free-energy density will be discussed in Section 2.

According to [2, 7, 20, 29], the expression of the bulk part $f_B(A)$ up to fourth order terms is given by[6]

$$\begin{aligned} f_B(A) := -\alpha\,\mathrm{tr}(AA^\dagger) + \beta_1\,|\mathrm{tr}(AA^T)|^2 + \beta_2\,[\mathrm{tr}(AA^\dagger)]^2 + \beta_3\,\mathrm{tr}[(AA^T)(AA^T)^*] \\ + \beta_4\,\mathrm{tr}[(AA^\dagger)^2] + \beta_5\,\mathrm{tr}[(AA^\dagger)(AA^\dagger)^*]\,, \end{aligned} \tag{1.4}$$

where the coefficient $\alpha = \alpha_0(1-T/T_c)$ changes sign at T_c, while α_0 and the coefficients $\beta_1,\ldots,\beta_5$ depend on the pressure, the temperature, and on the details of the interaction of the ^{3}He atoms. The expression for f_B is obtained from $A = (A_{\mu j})$ by contracting all indices in second and fourth order terms in order to guarantee rotational invariance in spin space and orbital space, separately. Moreover, each term in the expansion should contain an equal number of A and A^* to satisfy gauge invariance. The quadric expression (1.4) is the simplest form of f_B that allows for multiple critical points. The critical points fall into two classes: the so-called *inert* and *non-inert* phases. The inert phases are those in which the order parameter, specified by the matrix A, does not vary with the parameters $\beta_1,\ldots,\beta_5$. In other words, in the space of the $\beta_1,\ldots,\beta_5$, there exist domains such that, in each domain, the bulk part of the free energy is minimized by a matrix A remaining fixed throughout the domain. Among the inert phases, the A and B phases exhibit absolute minima of the bulk free energy in different region of the phase diagram. Thus, the phases A and B are those more relevant from the physical point of view. In this sense, the phases A and B are reminiscent of the uniaxial nematic phases $s_\pm(n\otimes n - \frac{1}{3}\mathrm{Id})$ in the Q-tensor theory of liquid crystal (cf. [3, 23]). For the study of more general invariant free energy densities involving higher order terms, we refer to [15]. This study is similar in spirit to that in [18] for the simpler case of Q-tensor theory of liquid crystals. The invariance under $\mathcal{G}$ of the bulk free-energy density f_B is expressed by the condition

$$f_B(G\star A) = f_B(A), \quad \text{for every } G\in\mathcal{G}.$$

On the other hand, according to [1], it is assumed that $f_{\mathrm{grad}}(A,\nabla A)$ is quadratic in ∇A and has the form

$$f_{\mathrm{grad}}(A,\nabla A) := \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_3$$

where γ_i are non-negative physical constants and

$$I_1 := \partial_k A_{\mu j}\partial_k A^*_{\mu j}\,, \quad I_2 := \partial_k A_{\mu j}\partial_j A^*_{\mu k}\,, \quad I_3 := \partial_k A_{\mu k}\partial_j A^*_{\mu j}\,.$$

As it will be explained in Section 2, for every $G = (S,L,\phi)\in\mathcal{G}$, the invariance of $f_{\mathrm{grad}}(A,\nabla A)$ under G is expressed by the condition

$$f_{\mathrm{grad}}(A,D) = f_{\mathrm{grad}}(G\star A, G\star\nabla A)$$

where $(G\star A) = \mathrm{e}^{\mathrm{i}\phi}SAL^T$ and $(G\star\nabla A)_{\mu jk} = \mathrm{e}^{\mathrm{i}\phi}S_{\mu\nu}L_{jb}L_{kc}\partial_c A_{\nu b}$. The above condition can be seen as the analog of the frame-indifference condition in the framework of the Landau–de Gennes theory of liquid crystals [3, 4, 14].

In Section 2, we shall determine all linearly independent quartic terms of the form $AA^*\nabla A(\nabla A)^*$ in the expansion of the gradient free-energy density. Observe that in the expression for the gradient free energy density f_{grad}, the odd powers of the gradients are not compatible with the rotational and gauge invariance of the free energy.

The equilibrium order parameters correspond to the minima of the bulk free energy f_B [29, 30, 31]. Above T_c, the absolute minimum of the bulk energy f_B is $A = 0$, which corresponds to the normal liquid ^{3}He. In the superfluid state, i.e., below T_c, there are many critical points $A\neq 0$ of f_B. Depending on the relations among the parameters $\beta_1,\ldots,\beta_5$, some of them correspond to the local minima of f_B. If some $A\neq 0$ attains the minimum of f_B, then, according to the invariance under the symmetry group $\mathcal{G}$ of the functional f_B, also the state $G\star A$ obtained from A by the action of any element G of $\mathcal{G}$ has the same energy, and hence it is an equilibrium state. All these states correspond to the same superfluid phase but to different degenerate physical states of the phase.

In other words, the values of the order parameter A corresponding to a certain fixed superfluid phase belong to a $\mathcal{G}$-orbit, or more generally, to a stratum of $\mathcal{G}$-orbits in the space of all 3×3 complex matrices

[6] We adopt the notation $A^\dagger := (A^T)^*$, where A^T and A^* denote, respectively, the transpose and the complex conjugate of A, so that we e.g. have $|A|^2 := \mathrm{tr}(AA^\dagger)$.

acted upon by the symmetry group $\mathcal{G}$. Any $\mathcal{G}$-orbit describes the manifold of the internal (degenerate) states of the corresponding phase. Two different $\mathcal{G}$-orbits describe two different phases, which differ one from the other by the symmetry and by the manifolds of their internal states.

Accordingly, below the superfluid transition temperature T_c the symmetry group $\mathcal{G}$ is spontaneously broken (cf. [16, 22]).

For a given equilibrium state (or vacuum state) A of an ordered superfluid phase ($\mathcal{G}$-orbit), the elements of the *isotropy group*

$$\mathcal{H}_A := \{G \in \mathcal{G} \mid G \star A = A\}$$

of $\mathcal{G}$ at A leave the order parameter invariant. The group $\mathcal{H}$ is known as the *residual symmetry* of the equilibrium state A. All the most important physical properties of a given superfluid phase are mostly determined by the residual symmetry $\mathcal{H}$ of its equilibrium state.

Among the different superfluid phases, the inert phases are indeed good candidates for providing the absolute minimum of the energy [6, 7]. They correspond to maximal subgroups $\mathcal{H}$ of $\mathcal{G}$, and hence, besides the normal state of liquid ^{3}He, they are the most symmetric vacuum states (cf. [28, 31]). Near T_c, they are always critical points for the Ginzburg–Landau functional corresponding the bulk energy density (1.4), for any $\beta_1, \ldots, \beta_5$. For more details on the inert phases and the Ginzburg–Landau equations, we refer to [12, 15, 30, 31].

Symmetry classification of superfluid phases. Since there are many subgroups $\mathcal{H}$ of $\mathcal{G}$, it is natural to expect that there are many types of vacuum states, which in turn correspond to different superfluid phases with different groups of residual symmetries.

In principle the symmetry classification of all possible superfluid phases reduces to the enumeration of all subgroups $\mathcal{H} \preceq \mathcal{G}$ not containing $U(1)$ as a subgroup. Significant progress in the enumeration of the different classes of superfluids was obtained by Bruder–Vollhardt (cf. [8] and [30]).

For a continuous subgroup $\mathcal{H} \preceq \mathcal{G}$, every element $h \in \mathcal{H}$ can be written as

$$h = \mathsf{e}^{-(\mathsf{i}/\hbar)\boldsymbol{\epsilon}\cdot\mathbf{T}} \simeq I - \frac{\mathsf{i}}{\hbar}\boldsymbol{\epsilon}\cdot\mathbf{T} \tag{1.5}$$

where $\boldsymbol{\epsilon}$ is an *arbitrary* vector of infinitesimal length and

$$\mathbf{T} = (\mathbf{T}_x, \mathbf{T}_y, \mathbf{T}_z) = (\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$$

are the generators of the infinitesimal transformations of the group $\mathcal{H}$. The infinitesimal generators $\mathbf{T}$ can be written in principle as linear combinations of the generators of $SO(3)_{\mathbf{L}}$, $SO(3)_{\mathbf{S}}$ and $U(1)_{\Phi}$, i.e.,

$$(\mathbf{T}_x, \mathbf{T}_y, \mathbf{T}_z) = (a_x\mathbf{L}_x + b_x\mathbf{S}_x, a_y\mathbf{L}_y + b_y\mathbf{S}_y, a_z\mathbf{L}_z + b_z\mathbf{S}_z + c\Phi) \tag{1.6}$$

where $(a_x, a_y, a_z) = \mathbf{a}$, $(b_x, b_y, b_z) = \mathbf{b}$, c are real parameters, $(\mathbf{L}_x, \mathbf{L}_y, \mathbf{L}_z)$ and $(\mathbf{S}_x, \mathbf{S}_y, \mathbf{S}_z)$ are the components of the orbital and spin angular momentum operators, $\mathbf{L}$ and $\mathbf{S}$, given respectively by[7] $(\mathbf{L}_i)_{jk} = -\mathsf{i}\,\hbar\,\varepsilon_{ijk}$, $(\mathbf{S}_\alpha)_{\beta\gamma} = -\mathsf{i}\,\hbar\,\varepsilon_{\alpha\beta\gamma}$, and $\Phi = -\mathsf{i}\,\hbar\,\partial/\partial\phi$ is the operator for a gauge transformation, with $\Phi A = A$ and $\Phi A^* = -A^*$.

In order to obtain all possible continuous subgroups $\mathcal{H}$ of $\mathcal{G}$ one can list all possible linear combinations of infinitesimal generators. Once a subgroup $\mathcal{H} \preceq \mathcal{G}$ is given, to determine an order parameter A such that the isotropy group at A coincides with $\mathcal{H}$, that is, $\mathcal{H}_A = \mathcal{H}$, one proceeds as follows. If $\mathcal{H} \preceq \mathcal{G}$ is the isotropy group at A, then

$$h \star A = A$$

for every $h \in \mathcal{H}$. By (1.5), the order parameter A is then obtained as a solution of the *symmetry equations*

$$\mathbf{T}_i\, A = 0\,, \quad i = 1, 2, 3.$$

According to [30, Eq. (6.20)], the symmetry equations lead to the following system of 3×9 homogeneous equations in the nine complex components $A_{\mu j}$ of the order parameter A,

$$a_i \sum_k \varepsilon_{ijk} A_{\mu k} + b_i \sum_\nu \varepsilon_{i\mu\nu} A_{\nu j} + \mathsf{i}\,\delta_{i3}\, c\, A_{\mu j} = 0 \qquad (i = 1, 2, 3)\,. \tag{1.7}$$

[7] We denote by $\varepsilon_{\alpha\beta\gamma}$ the Levi-Civita symbols.

Equations (1.7) are used to compute the components $A_{\mu j}$ of A.

In this setting, we briefly describe the two physically more relevant phases of superfluid ^{3}He: the inert phases ^{3}He-A and ^{3}He-B (cf. also Remark 5.1).

Example 1.1 [The B phase] According to [8, 30], the B phase of superfluid ^{3}He, ^{3}He-B, corresponds to the choice of $\mathbf{T} = \mathbf{S} + \mathbf{L}$, i.e., $c = 0$, $a_i = b_\mu = 1$. Solving the symmetry equations $\mathbf{T}_i\, A = 0$, $i = 1, 2, 3$, that is, (1.7), among the unit matrices ($|A|^2 := \mathrm{tr}(AA^\dagger) = 1$) one obtains the order parameter

$$A^0 = (A^0_{\mu j}) := \frac{1}{\sqrt{3}}\,(\delta_{\mu j}) = \frac{1}{\sqrt{3}}\, I_3 = \frac{1}{\sqrt{3}}(\mathbf{e}_1\mathbf{e}_1^T + \mathbf{e}_2\mathbf{e}_2^T + \mathbf{e}_3\mathbf{e}_3^T)$$

which describes the B phase [29, 31]. The manifold of all degenerate equilibrium states of the B phase corresponds to the orbit under $\mathcal{G}$ of the initial state given by A^0, that is,

$$\mathcal{G} \star A^0 = \{G \star A^0 \mid G \in \mathcal{G}\}$$

where if $G = (R^{(S)}, R^{(L)}, \phi)$ one has

$$(G \star A^0)_{\mu j} = \frac{1}{\sqrt{3}}\, \mathrm{e}^{\mathrm{i}\phi}\, R_{\mu j}\,, \qquad (R_{\mu j}) := R^{(S)} {R^{(L)}}^T \in SO(3) \simeq \mathbb{R}P^3\,. \tag{1.8}$$

The isotropy group at A^0 of $\mathcal{G}$ is

$$\mathcal{H}_B = \{G \in \mathcal{G} \mid G \star A^0 = A^0\} = \{(M, M, 1) \in \mathcal{G} \mid M \in SO(3)\} =: SO(3)_{\mathbf{S}+\mathbf{L}} \cong SO(3)$$

which implies that the orbit is the 4-dimensional manifold

$$\mathcal{G} \star A^0 \cong \mathcal{G}/\mathcal{H}_B \cong SO(3) \times U(1)\,.$$

Example 1.2 [The A phase] According to [8, 30], the A phase of superfluid ^{3}He, ^{3}He-A, corresponds to the choice of $\mathbf{T}_z = b_z\mathbf{S}_z + a_z(\mathbf{L}_z - \Phi)$. In this case, solving the equations $\mathbf{T}_i\, A = 0$, $i = 1, 2, 3$, i.e., the equations (1.7), among the unit matrices one obtains[8] the order parameter

$$A^{\pi/2} := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \mathrm{i} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{e}_3(\mathbf{e}_1^T + \mathrm{i}\mathbf{e}_2^T) \tag{1.9}$$

which describes the A phase [29, 31]. The manifold of all degenerate equilibrium states of the A phase corresponds to the orbit under $\mathcal{G}$ of the initial state described by $A^{\pi/2}$, that is,

$$\mathcal{G} \star A^{\pi/2} = \{G \star A^{\pi/2} \mid G \in \mathcal{G}\}.$$

This time, for any $G = (R^{(S)}, R^{(L)}, \phi) \in \mathcal{G}$,

$$(G \star A^{\pi/2})_{\mu j} = \frac{1}{\sqrt{2}}\, \mathrm{e}^{\mathrm{i}\phi}\, R^{(S)}_{\mu 3}\, \big(R^{(L)}_{j1} + \mathrm{i} R^{(L)}_{j2}\big)\,. \tag{1.10}$$

The isotropy group at $A^{\pi/2}$ of $\mathcal{G}$ is computed to be (cf. [31])

$$\mathcal{H}_A = U(1)_{\mathbf{S}_z} \times U(1)_{\mathbf{L}_z - \Phi} \times \mathbb{Z}_2\,.$$

The $\mathbb{Z}_2$ factor reflects the discrete *combined spin-gauge symmetry* of the A phase: the state (1.9) does not change if the spin rotation, which transforms $\mathbf{e}_3$ into $-\mathbf{e}_3$, is combined with the gauge transformation $\phi \mapsto \phi + \pi$, which takes $\mathbf{e}_1^T + \mathrm{i}\mathbf{e}_2^T$ to $-(\mathbf{e}_1^T + \mathrm{i}\mathbf{e}_2^T)$. This additional $\mathbb{Z}_2$ discrete symmetry of the A phase cannot be obtained by the symmetry classification described above (cf. [8, 30]). Then, the $\mathcal{G}$-orbit of $A^{\pi/2}$ is the 5-dimensional manifold

$$\mathcal{G} \star A^{\pi/2} \cong \mathcal{G}/\mathcal{H}_A \cong \left(\frac{SO(3)_{\mathbf{S}}}{U(1)_{\mathbf{S}_z}} \times \frac{SO(3)_{\mathbf{L}} \times U(1)_{\Phi}}{U(1)_{\mathbf{L}_z - \Phi}}\right)/\mathbb{Z}_2 \cong \big(\mathbb{S}^2 \times SO(3)\big)/\mathbb{Z}_2\,.$$

[8] The notation $A^0 = (A^0_{\mu j})$ and $A^{\pi/2} = (A^{\pi/2}_{\mu j})$ will be clear from the ansatz (1.11).

Remark 1.3 The $\mathbf{e}_3$ vector in the A phase state (1.9) may be considered as a "director" in all cases in which the change of phase ϕ is unimportant. In this sense, $\mathbf{e}_3$ resembles the director in the theory of uniaxial nematic liquid crystals [4, 9, 24, 25, 26]. Accordingly, an orientability problem similar to that occurring in the Q-tensor theory of uniaxial nematic systems as treated in [5] could be envisaged for the superfluid A phase.

Description of results and organization of the paper. In Section 2, we address the question of the invariance under the symmetry group $\mathcal{G}$ of a general free-energy density for the Ginzburg–Landau functional and consider a generalized gradient energy density expansion including quartic terms. More precisely, we determine all linearly independent quartic terms I_α (cf. (2.7) and (2.8)) of the form $AA^*\nabla A(\nabla A)^*$ that are quadratic in ∇A. Their linear independence is discussed in the Appendix. In Section 3, we analyze the gradient energy density for the B phase and explicitly write the gradient terms I_α. In Section 4, we discuss the gradient energy density for the A phase and explicitly write the gradient terms I_α. In Section 5, we propose a unified approach to the study of the A and B superfluid phases. Namely, we assume

$$A = \Delta(T,P)\, A^\theta \in \mathbb{C}^3 \otimes \mathbb{C}^3$$

where the amplitude $\Delta(T,P) > 0$ depends on temperature T and pressure P and A^θ is a unit matrix parametrized by an auxiliary control parameter $\theta \in [0,\pi/2]$ relating the unit states describing the A and B phases considered in Examples 1.1 and 1.2,

$$A^\theta := \cos\theta \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin\theta \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \mathrm{i} & 0 \end{pmatrix}, \qquad \theta \in [0,\pi/2]\,. \tag{1.11}$$

We will see that when $\theta \in]0,\pi/2[$, the isotropy group $\mathcal{H}$ at A^θ is trivial (the absence of discrete symmetries is also checked). Thus, for $\theta \in]0,\pi/2[$, the $\mathcal{G}$-orbit of A^θ is diffeomorphic to the overall symmetry group $\mathcal{G}$. We shall then explicitly write the fourth order expansion of the bulk energy density (1.4) in our ansatz, obtaining the formulas in (5.1) and (5.2). In Proposition 5.3, we establish the optimal values of the control parameter θ and the corresponding relations among the coefficients β_i in the expression of the bulk free-energy density (1.4).

Provided that the inhomogeneity region is large enough (cf. [29, Section II-D] and the discussion at the end of Section 5), the gradient terms may be neglected, and the superfluid phases are determined by the minimization of the bulk energy (1.4) alone. In this case, only the *maximal isotropy groups* $\mathcal{H}$ of $\mathcal{G}$ may produce inert phases (cf. [8, 29, 30]). In particular, the A and B phases are the only minimizers in absence of an external magnetic field. This property is recovered in our ansatz, when the coefficients β_i are chosen according to the weak or strong coupling theories, see Proposition 5.4. More precisely, by choosing the coefficients β_i in accordance with Anderson–Brinkman [2], see the formulas in (5.3), a phase transition is observed in dependence of the spin-fluctuation parameter δ. Our results are consistent with the ones reported by Salomaa–Volovik in [29, Section II-D].

2 Bulk and gradient free energy densities

In this section, we discuss the invariance properties of a general free-energy density for the Ginzburg–Landau energy functional, and in particular of the bulk and gradient energy densities. We then determine all linearly independent quartic terms of the form $AA^*\nabla A(\nabla A)^*$ in the expansion of the gradient energy density.

Bulk energy density. According to Mermin–Stare [20], Anderson–Brinkman [2] and Salomaa–Volovik [29], the relevant fourth order expansion of the bulk condensation term in the Ginzburg–Landau free energy is given by the expression (1.4).

In the following, we will occasionally take advantage of the possibility of writing the order parameter $A \in \mathbb{C}^3 \otimes \mathbb{C}^3$ as

$$A = \Delta\,\hat{A}\,, \quad \text{with} \quad \Delta = |A|\,, \quad \hat{A} \in \mathbb{C}^3 \otimes \mathbb{C}^3\,, \quad |\hat{A}| = 1 \tag{2.1}$$

where in general the size (amplitude) $\Delta = \Delta(T,P)$ only depends on temperature and pressure. In this case, the bulk free energy density (1.4) takes the form

$$f_B(A) = -\alpha\,\Delta^2\,\mathrm{tr}(\hat{A}\hat{A}^\dagger) + \Delta^4\Big(\beta_1\,|\mathrm{tr}(\hat{A}\hat{A}^T)|^2 + \beta_2\,[\mathrm{tr}(\hat{A}\hat{A}^\dagger)]^2 + \beta_3\,\mathrm{tr}[(\hat{A}\hat{A}^T)(\hat{A}\hat{A}^T)^*] \\ + \beta_4\,\mathrm{tr}[(\hat{A}\hat{A}^\dagger)^2] + \beta_5\,\mathrm{tr}[(\hat{A}\hat{A}^\dagger)(\hat{A}\hat{A}^\dagger)^*]\Big) \tag{2.2}$$

where α and β_i are real constants depending on temperature and pressure, see Section 5 below.

Gradient energy density. When spatial variations of the order parameter $A = (A_{\mu j})$ are allowed, the free energy density in the Ginzburg–Landau functional will contain extra terms depending on the components of the gradient ∇A of A, with respect to the *ordinary* (*orbital*) *space* coordinates x_1, x_2, x_3. We recall that the gradient ∇A is a third order tensor, whose components are given by $(\nabla A)_{\mu jk} = A_{\mu j,k} = \partial_k A_{\mu j} := \frac{\partial}{\partial x_k} A_{\mu j}$.

According to de Gennes *et al.* (cf. [1]), the gradient part of the free energy density of an anisotropic superfluid in the Ginzburg–Landau functional has leading $\mathcal{G}$-invariant terms quadratic in the components of ∇A of the form

$$f_{\mathrm{grad}}(A, \nabla A) := \gamma_1 I_1 + \gamma_2 I_2 + \gamma_3 I_3 \tag{2.3}$$

where γ_i are non-negative physical constants (see also Section 5) and

$$I_1 := \partial_k A_{\mu j}\partial_k A^*_{\mu j}\,, \quad I_2 := \partial_k A_{\mu j}\partial_j A^*_{\mu k}\,, \quad I_3 := \partial_k A_{\mu k}\partial_j A^*_{\mu j}\,.$$

These energy densities can alternatively be expressed by (cf. [1])

$$I_1 = \sum_{\mu=1}^{3} |\nabla \underline{A}_\mu|^2\,, \quad I_2 = \sum_{\mu=1}^{3} \mathrm{tr}[\nabla \underline{A}_\mu \nabla \underline{A}^*_\mu]\,, \quad I_3 = \sum_{\mu=1}^{3} |\,\mathrm{div}\,\underline{A}_\mu|^2$$

where $\underline{A}_\mu = A_{\mu j}\mathbf{e}_j^T$ are the *row vectors* of the matrix A (see also Remark 3.1). Notice that $I_1 = |\nabla A|^2$ is the Dirichlet energy and $I_2 - I_3$ is a null Lagrangian, since

$$\partial_k(A_{\mu j}\partial_j A^*_{\mu k} - A_{\mu k}\partial_j A^*_{\mu j}) = I_2 - I_3\,. \tag{2.4}$$

Invariance. Let $D := \nabla_x A$ denote the third order tensor corresponding to the gradient of A with respect to the ordinary spatial coordinates x. In components, this amounts to saying that $D_{\mu jk} = A_{\mu j,k} = \partial_k A_{\mu j}$.

Let $G := (R^{(S)}, R^{(L)}, \phi)$ be an element of $\mathcal{G}$ where, for simplicity, we let $S = R^{(S)}$ and $L = R^{(L)}$. According to the action (1.3), for any $x \in \Omega$, we have $(G \star A)(x) = \mathsf{e}^{\mathrm{i}\phi} S A(x) L^T$. Next, let $z = Lx$ be the coordinates obtained by rotating by L the ordinary spatial coordinates x.

We can state the following.

Proposition 2.1 *Let A, $G \star A$, and $D = \nabla_x A$ be as above. Let $G \star D$ be the third order tensor $G \star D := \nabla_z(G \star A)$. Then*

$$(G \star D)_{\mu jk} = \mathsf{e}^{\mathrm{i}\phi} S_{\mu\nu} L_{jb} L_{kc} D_{\nu bc}\,. \tag{2.5}$$

Proof: Since $z = Lx$, we have

$$\frac{\partial x_i}{\partial z_k} = (L^T)_{ik} = L_{ki}\,.$$

With $(G \star A)_{\mu j} = (\mathsf{e}^{\mathrm{i}\phi} S A L^T)_{\mu j} = \mathsf{e}^{\mathrm{i}\phi} S_{\mu\nu} A_{\nu b} L_{jb}$, we have

$$\begin{aligned} \frac{\partial}{\partial z_k}[(G \star A)_{\mu j})](x) &= \frac{\partial x_c}{\partial z_k}\frac{\partial}{\partial x_c}[\mathsf{e}^{\mathrm{i}\phi} S_{\mu\nu} A_{\nu b} L_{jb}](x) \\ &= \mathsf{e}^{\mathrm{i}\phi} L_{kc} S_{\mu\nu}\frac{\partial}{\partial x_c}[A_{\nu b}(x)]\,L_{jb} \\ &= \mathsf{e}^{\mathrm{i}\phi} S_{\mu\nu}\partial_c A_{\nu b}(x)\,L_{kc}\,L_{jb} \end{aligned}$$

which implies (2.5), as claimed. $\square$

Therefore, it turns out that, for every $G = (S, L, \phi) \in \mathcal{G}$, a general free energy density I with the correct invariance properties under gauge transformations and spin and spatial rotations must satisfy the condition

$$I(A, D) = I(G \star A, G \star D)\,, \tag{2.6}$$

where $(G \star A) = \mathsf{e}^{\mathsf{i}\phi} S A L^T$ and $(G \star D)_{\mu jk} := \mathsf{e}^{\mathsf{i}\phi} S_{\mu\nu} L_{jb} L_{kc} D_{\nu bc}$.

For example, as for the density I_2, we have $I_2(A, D) = D_{\mu jk} D^*_{\mu kj}$, whereas

$$\begin{aligned} I_2(G \star A, G \star D) &= (G \star D)_{\mu jk} (G \star D)^*_{\mu kj} = \mathsf{e}^{\mathsf{i}\phi} S_{\mu\nu} L_{jb} L_{kc} D_{\nu bc} \mathsf{e}^{-\mathsf{i}\phi} S_{\mu\alpha} L_{k\ell} L_{jm} D^*_{\alpha\ell m} \\ &= \delta_{\nu\alpha} \delta_{bm} \delta_{c\ell} D_{\nu bc} D^*_{\alpha\ell m} = D_{\nu bc} D^*_{\nu cb} \\ &= I_2(A, D)\,. \end{aligned}$$

Remark 2.2 The above invariance condition (2.6) (as well as its derivation) is reminiscent of the frame-indifference condition in the Landau–de Gennes Q-tensor theory of liquid crystals [3, 4].

Quartic terms. We now wish to consider the quartic terms of the form AA^*DD^* in the expansion of the gradient free energy. The most symmetric of such terms is

$$I_4 = I_4(A, D) := A_{\mu\ell} A^*_{\nu\ell} \partial_k A_{\mu j} \partial_k A^*_{\nu j}\,.$$

More generally, in order to satisfy the invariance property (2.6), any such term must have the structure:

$$A_{\mu a} D_{\mu bc} A^*_{\nu d} D^*_{\nu ef}$$

where the indices a, b, c, d, e, f are pairwise equal and range on $\{1, 2, 3\}$. This yields fifteen different terms. However, since the energy densities must be real valued, some of them have to be suitably combined together.

Accordingly, we obtain seven invariants, five of which are readily recognized in terms of the row vectors $\underline{A}_\mu$ (where by $\bullet$ we denote the scalar product), namely

$$\begin{aligned} I_4 &:= A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu\ell} \partial_k A^*_{\nu j} &&= (\underline{A}_\mu \bullet \underline{A}^*_\nu)\,(\partial_k \underline{A}_\mu \bullet \partial_k \underline{A}^*_\nu) \\ I_5 &:= A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu\ell} \partial_j A^*_{\nu k} &&= (\underline{A}_\mu \bullet \underline{A}^*_\nu)\, \mathrm{tr}(\nabla \underline{A}_\mu \nabla \underline{A}^*_\nu) \\ I_6 &:= A_{\mu\ell} \partial_j A_{\mu j} A^*_{\nu\ell} \partial_k A^*_{\nu k} &&= (\underline{A}_\mu \bullet \underline{A}^*_\nu)\, \mathrm{div}\, \underline{A}_\mu\, \mathrm{div}\, \underline{A}^*_\nu \\ I_7 &:= A_{\mu\ell} \partial_k A_{\mu\ell} A^*_{\nu j} \partial_k A^*_{\nu j} &&= (\underline{A}_\mu \bullet \partial_k \underline{A}_\mu)(\underline{A}^*_\nu \bullet \partial_k \underline{A}^*_\nu) \\ I_8 &:= A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu j} \partial_k A^*_{\nu\ell} &&= (\underline{A}_\mu \bullet \partial_k \underline{A}^*_\nu)(\underline{A}^*_\nu \bullet \partial_k \underline{A}_\mu) \\ I_9 &:= A_{\mu\ell} \partial_\ell A_{\mu j} A^*_{\nu k} \partial_k A^*_{\nu j} \\ I_{10} &:= A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu k} \partial_\ell A^*_{\nu j} \end{aligned} \tag{2.7}$$

and four additional combined invariants, namely

$$\begin{aligned} I_{11} &:= A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu j} \partial_\ell A^*_{\nu k} + A_{\mu\ell} \partial_k A_{\mu j} A^*_{\nu k} \partial_j A^*_{\nu\ell} \\ I_{12} &:= A_{\mu\ell} \partial_j A_{\mu\ell} A^*_{\nu j} \partial_k A^*_{\nu k} + A_{\mu\ell} \partial_j A_{\mu j} A^*_{\nu k} \partial_\ell A^*_{\nu k} \\ I_{13} &:= A_{\mu\ell} \partial_j A_{\mu\ell} A^*_{\nu k} \partial_k A^*_{\nu j} + A_{\mu\ell} \partial_\ell A_{\mu j} A^*_{\nu k} \partial_j A^*_{\nu k} \\ I_{14} &:= A_{\mu\ell} \partial_\ell A_{\mu j} A^*_{\nu j} \partial_k A^*_{\nu k} + A_{\mu\ell} \partial_j A_{\mu j} A^*_{\nu k} \partial_k A^*_{\nu\ell}\,. \end{aligned} \tag{2.8}$$

The above ones are all the quartic terms satisfying (separately) the invariance property (2.6) and such that $I_\alpha = I^*_\alpha$, for $\alpha = 4, \ldots, 14$. Moreover, differently from what happens for the quadratic terms I_2 and I_3, see (2.4), by means of a long computation (which will be reproduced in the Appendix) we will check that no one of the above invariants can be written as a linear combination of the other terms, up to a null Lagrangian.

Remark 2.3 When writing $A = \Delta\hat{A}$, according to the factorization (2.1), the amplitude $\Delta = \Delta(T, P)$ of the matrix A does not depend on the position $x \in \Omega$. We thus have

$$(\nabla A)(x) = \Delta(T, P) \cdot (\nabla \hat{A})(x)\,, \quad x \in \Omega$$

and hence, in general,

$$I_\alpha(A, \nabla A) = \begin{cases} \Delta(T, P)^2\, I_\alpha(\hat{A}, \nabla\hat{A}) & \text{for } i = 1, 2, 3 \\ \Delta(T, P)^4\, I_\alpha(\hat{A}, \nabla\hat{A}) & \text{for } i = 4, \ldots, 14 \end{cases} \quad \text{if} \quad A(x) = \Delta(T, P)\, \hat{A}(x)\,. \tag{2.9}$$

3 The B phase

In this section, we discuss the gradient energy density in the B phase and write explicit expressions for the gradient terms I_α. Taking into account that the general form of the order parameter in the B phase is given by (cf. [29, 31])

$$\Delta_B(T, P)\, A, \quad A_{\mu j} = \frac{1}{\sqrt{3}}\, \mathsf{e}^{\mathsf{i}\phi}\, R_{\mu j}, \quad R = (R_{\mu j}) \in SO(3)\,, \tag{3.1}$$

we will compute the gradient terms I_α for the unit matrix A. The gradient terms I_α for the general form of the order parameter in the B phase are then obtained by the homogeneity formulas (2.9) in Remark 2.3.

Remark 3.1 Let $\mathbf{r} : \mathbb{R}^3 \to \mathbb{S}^2$ a smooth unit vector field. Since $\mathbf{r}_i\mathbf{r}_i = 1$, we have $\mathbf{r}_i\mathbf{r}_{i,k} = 0$ for each k, where "$f_{,k}$" denotes the partial derivative $\partial_k f$. Moreover,

$$\begin{array}{rcl} |\operatorname{curl} \mathbf{r}|^2 & = & (\mathbf{r} \cdot \operatorname{curl} \mathbf{r})^2 + |\mathbf{r} \times \operatorname{curl} \mathbf{r}|^2\,, \\ |\nabla \mathbf{r}|^2 & = & \operatorname{tr}[(\nabla \mathbf{r})^2] + |\operatorname{curl} \mathbf{r}|^2\,, \\ |\mathbf{r} \times \operatorname{curl} \mathbf{r}|^2 & = & \mathbf{r}_l \mathbf{r}_k \mathbf{r}_{i,l} \mathbf{r}_{i,k}\,, \\ \operatorname{tr}[(\nabla \mathbf{r})^2] & = & \mathbf{r}_{k,j} \mathbf{r}_{j,k}\,, \\ (\operatorname{div} \mathbf{r})^2 & = & \mathbf{r}_{k,k} \mathbf{r}_{j,j}\,. \end{array} \tag{3.2}$$

We also recall that the term

$$\operatorname{tr}[(\nabla \mathbf{r})^2] - (\operatorname{div} \mathbf{r})^2 = (\mathbf{r}_{j,k}\mathbf{r}_k - \mathbf{r}_{k,k}\mathbf{r}_j)_{,j} = \operatorname{div}[(\nabla \mathbf{r})\mathbf{r} - (\operatorname{div} \mathbf{r})\mathbf{r})] \tag{3.3}$$

is a null-Lagrangian. With this notation, for a unit vector field $\mathbf{r}$ we thus have

$$|\partial_{\mathbf{r}}\mathbf{r}|^2 = \partial_{\mathbf{r}}\mathbf{r}_j \partial_{\mathbf{r}}\mathbf{r}_j = \mathbf{r}_\ell \mathbf{r}_{j,\ell} \mathbf{r}_k \mathbf{r}_{j,k} = |\mathbf{r} \times \operatorname{curl} \mathbf{r}|^2\,.$$

In a similar way, given another unit vector field $\mathbf{s} : \mathbb{R}^3 \to \mathbb{S}^2$ we have:

$$\partial_{\mathbf{s}}\mathbf{r} \bullet \partial_{\mathbf{r}}\mathbf{s} = \partial_{\mathbf{s}}\mathbf{r}_j \partial_{\mathbf{r}}\mathbf{s}_j = \mathbf{r}_k \mathbf{s}_\ell \mathbf{r}_{j,\ell} \mathbf{s}_{j,k} \qquad \partial_{\mathbf{r}}\mathbf{r} \bullet \partial_{\mathbf{s}}\mathbf{s} = \partial_{\mathbf{r}}\mathbf{r}_j \partial_{\mathbf{s}}\mathbf{s}_j = \mathbf{r}_k \mathbf{s}_\ell \mathbf{r}_{j,k} \mathbf{s}_{j,\ell}\,.$$

Explicit formulas. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, and consider the smooth function

$$\Omega \ni x \mapsto A_{\mu j} = \frac{1}{\sqrt{3}}\, \mathsf{e}^{\mathsf{i}\phi}\, R_{\mu j} \tag{3.4}$$

where $\phi(x) \in \mathbb{R}$ and $R(x) \in SO(3)$ account for the four degrees of freedom in the B phase.

We shall denote by $\mathbf{n}$, $\mathbf{m}$ and $\boldsymbol{\ell} = \mathbf{n} \times \mathbf{m}$ the orthonormal unit vector fields given by the rows of the matrix R, so that (in terms of column vectors)

$$R^T(x) = (\mathbf{n}(x) | \mathbf{m}(x) | \boldsymbol{\ell}(x))\,, \qquad x \in \Omega\,. \tag{3.5}$$

Following the notation in [1], the row vectors $\underline{A}_\mu$ then become

$$\underline{A}_1 = \frac{1}{\sqrt{3}}\, \mathsf{e}^{\mathsf{i}\phi}\, \mathbf{n}^T\,, \quad \underline{A}_2 = \frac{1}{\sqrt{3}}\, \mathsf{e}^{\mathsf{i}\phi}\, \mathbf{m}^T\,, \quad \underline{A}_3 = \frac{1}{\sqrt{3}}\, \mathsf{e}^{\mathsf{i}\phi}\, \boldsymbol{\ell}^T$$

and we correspondingly denote

$$\begin{array}{rcl} |\nabla R|^2 & := & |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2 + |\nabla \boldsymbol{\ell}|^2 \\ \operatorname{tr}[(\nabla R)^2] & := & \operatorname{tr}[(\nabla \mathbf{n})^2] + \operatorname{tr}[(\nabla \mathbf{m})^2] + \operatorname{tr}[(\nabla \boldsymbol{\ell})^2] \\ (\operatorname{div} R)^2 & := & (\operatorname{div} \mathbf{n})^2 + (\operatorname{div} \mathbf{m})^2 + (\operatorname{div} \boldsymbol{\ell})^2\,. \end{array}$$

As for the Dirichlet term $I_1 := \partial_k A_{\mu j} \partial_k A^*_{\mu j}$, using that $R_{\mu j} R_{\mu j} = 3$, we have

$$\begin{aligned}
3\, I_1 = &\ \partial_k(\mathsf{e}^{\mathsf{i}\phi} R_{\mu j})\, \partial_k(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu j}) = \mathsf{e}^{\mathsf{i}\phi}(R_{\mu j,k} + \mathsf{i}\phi_{,k} R_{\mu j})\, \mathsf{e}^{-\mathsf{i}\phi}(R_{\mu j,k} - \mathsf{i}\phi_{,k} R_{\mu j}) \\
= &\ (R_{\mu j,k} R_{\mu j,k} + \phi_{,k}\phi_{,k} R_{\mu j} R_{\mu j}) + \mathsf{i}(\phi_{,k} R_{\mu j} R_{\mu j,k} - \phi_{,k} R_{\mu j} R_{\mu j,k}) \\
= &\ |\nabla R|^2 + 3|\nabla\phi|^2 = |\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2 + |\nabla \boldsymbol{\ell}|^2 + 3|\nabla\phi|^2 .
\end{aligned}$$

As for the term $I_2 := \partial_k A_{\mu j} \partial_j A^*_{\mu k}$, using that $R_{\mu j} R_{\mu k} = \delta_{jk}$, we compute

$$\begin{aligned}
3\, I_2 = &\ \partial_k(\mathsf{e}^{\mathsf{i}\phi} R_{\mu j})\, \partial_j(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu k}) = \mathsf{e}^{\mathsf{i}\phi}(R_{\mu j,k} + \mathsf{i}\phi_{,k} R_{\mu j})\, \mathsf{e}^{-\mathsf{i}\phi}(R_{\mu k,j} - \mathsf{i}\phi_{,j} R_{\mu k}) \\
= &\ (R_{\mu j,k} R_{\mu k,j} + \phi_{,k}\phi_{,j} R_{\mu j} R_{\mu k}) + \mathsf{i}(\phi_{,k} R_{\mu j} R_{\mu k,j} - \phi_{,j} R_{\mu k} R_{\mu j,k}) \\
= &\ \mathbf{n}_{j,k}\mathbf{n}_{k,j} + \mathbf{m}_{j,k}\mathbf{m}_{k,j} + \boldsymbol{\ell}_{j,k}\boldsymbol{\ell}_{k,j} + \phi_{,k}\phi_{,j}\delta_{jk} + 0\,\mathsf{i} \\
= &\ \mathrm{tr}[(\nabla\mathbf{n})^2] + \mathrm{tr}[(\nabla\mathbf{m})^2] + \mathrm{tr}[(\nabla\boldsymbol{\ell})^2] + |\nabla\phi|^2 .
\end{aligned}$$

Finally, for the term $I_3 := \partial_k A_{\mu k} \partial_j A^*_{\mu j}$, we similarly obtain

$$\begin{aligned}
3\, I_3 = &\ \partial_k(\mathsf{e}^{\mathsf{i}\phi} R_{\mu k})\, \partial_j(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu j}) = \mathsf{e}^{\mathsf{i}\phi}(R_{\mu k,k} + \mathsf{i}\phi_{,k} R_{\mu k})\, \mathsf{e}^{-\mathsf{i}\phi}(R_{\mu j,j} - \mathsf{i}\phi_{,j} R_{\mu j}) \\
= &\ (R_{\mu k,k} R_{\mu j,j} + \phi_{,k}\phi_{,j} R_{\mu j} R_{\mu k}) + \mathsf{i}(\phi_{,k} R_{\mu k} R_{\mu j,j} - \phi_{,j} R_{\mu j} R_{\mu k,k}) \\
= &\ \mathbf{n}_{k,k}\mathbf{n}_{j,j} + \mathbf{m}_{k,k}\mathbf{m}_{j,j} + \boldsymbol{\ell}_{k,k}\boldsymbol{\ell}_{j,j} + \phi_{,k}\phi_{,j}\delta_{jk} + 0\,\mathsf{i} \\
= &\ (\mathrm{div}\,\mathbf{n})^2 + (\mathrm{div}\,\mathbf{m})^2 + (\mathrm{div}\,\boldsymbol{\ell})^2 + |\nabla\phi|^2 .
\end{aligned}$$

In conclusion, taking into account the general form (3.1) of the order parameter in the B phase, it follows from the homogeneity formulas (2.9) that the gradient energy density (2.3) in the B phase takes the form

$$3\,\Delta_B(T,P)^{-2}\, f_{\mathrm{grad}} = \gamma_1\, |\nabla R|^2 + \gamma_2\, \mathrm{tr}[(\nabla R)^2] + \gamma_3\, (\mathrm{div}\, R)^2 + (3\gamma_1 + \gamma_2 + \gamma_3)|\nabla\phi|^2 . \tag{3.6}$$

Quartic invariants. In accordance with the comments made at the beginning of the section, we now consider the invariant energy densities I_α, $\alpha = 4, \ldots, 8$, introduced in (2.7). These are the most symmetric quartic real valued invariants we found in the previous section.

In the B phase, concerning the term $I_4 := A_{\mu\ell}\partial_k A_{\mu j} A^*_{\nu\ell} \partial_k A^*_{\nu j}$ we first observe that $3(\underline{A}_\mu \bullet \underline{A}^*_\nu) = \delta_{\mu\nu}$, whence arguing as above we get

$$9\, I_4 = \partial_k(\mathsf{e}^{\mathsf{i}\phi} R_{\mu j})\partial_k(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu j}) = (\mathsf{i}\phi_{,k} R_{\mu j} + R_{\mu j,k})(-\mathsf{i}\phi_{,k} R_{\mu j} + R_{\mu j,k}) = 3|\nabla\phi|^2 + |\nabla R|^2 .$$

As for the term $I_5 := A_{\mu\ell}\partial_k A_{\mu j} A^*_{\nu\ell} \partial_j A^*_{\nu k}$, we similarly write

$$9\, I_5 = \partial_k(\mathsf{e}^{\mathsf{i}\phi} R_{\mu j})\partial_j(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu k}) = (\mathsf{i}\phi_{,k} R_{\mu j} + R_{\mu j,k})(-\mathsf{i}\phi_{,j} R_{\mu k} + R_{\mu k,j}) = |\nabla\phi|^2 + \mathrm{tr}[(\nabla R)^2]$$

whereas for the term $I_6 := A_{\mu\ell}\partial_j A_{\mu j} A^*_{\nu\ell} \partial_k A^*_{\nu k}$ we obtain

$$9\, I_6 = \partial_j(\mathsf{e}^{\mathsf{i}\phi} R_{\mu j})\partial_k(\mathsf{e}^{-\mathsf{i}\phi} R_{\mu k}) = (\mathsf{i}\phi_{,j} R_{\mu j} + R_{\mu j,j})(-\mathsf{i}\phi_{,k} R_{\mu k} + R_{\mu k,k}) = |\nabla\phi|^2 + (\mathrm{div}\, R)^2 .$$

Concerning the term $I_7 := A_{\mu\ell}\partial_k A_{\mu\ell} A^*_{\nu j} \partial_k A^*_{\nu j}$, we have

$$\begin{aligned}
3\,(\underline{A}_\mu \bullet \partial_k \underline{A}_\mu) = &\ \mathsf{e}^{2\mathsf{i}\phi} R_{\mu\ell}(\mathsf{i}\phi_{,k} R_{\mu\ell} + R_{\mu\ell,k}) \\
3\,(\underline{A}^*_\nu \bullet \partial_k \underline{A}^*_\nu) = &\ \mathsf{e}^{-2\mathsf{i}\phi} R_{\nu j}(-\mathsf{i}\phi_{,k} R_{\nu j} + R_{\nu j,k}) .
\end{aligned}$$

Recalling that I_7 is real valued, and using that $R_{\nu j} R_{\nu j} = 3$ and $R_{\nu j} R_{\nu j,k} = 0$, we get

$$9\, I_7 = R_{\mu\ell} R_{\nu j}(\phi_{,k}\phi_{,k} R_{\mu\ell} R_{\nu j} + R_{\mu\ell,k} R_{\nu j,k}) = 9\,|\nabla\phi|^2 .$$

As for the term $I_8 := A_{\mu\ell}\partial_k A_{\mu j} A^*_{\nu j} \partial_k A^*_{\nu\ell}$, we have

$$\begin{aligned}
3\,(\underline{A}_\mu \bullet \partial_k \underline{A}^*_\nu) = &\ R_{\mu\ell}(-\mathsf{i}\phi_{,k} R_{\nu\ell} + R_{\nu\ell,k}) = -\mathsf{i}\phi_{,k}\delta_{\mu\nu} + R_{\mu\ell} R_{\nu\ell,k} \\
3\,(\underline{A}^*_\nu \bullet \partial_k \underline{A}_\nu) = &\ R_{\nu j}(\mathsf{i}\phi_{,k} R_{\mu j} + R_{\mu j,k}) = \mathsf{i}\phi_{,k}\delta_{\mu\nu} + R_{\nu j} R_{\mu j,k} .
\end{aligned}$$

Using that I_8 is real valued and that $R_{\nu j} R_{\mu j,k} = -R_{\mu j} R_{\nu j,k}$ we get

$$\begin{aligned}
9\, I_8 = &\ (-\mathsf{i}\phi_{,k}\delta_{\mu\nu} + R_{\mu\ell} R_{\nu\ell,k})(\mathsf{i}\phi_{,k}\delta_{\mu\nu} + R_{\nu j} R_{\mu j,k}) \\
= &\ 3|\nabla\phi|^2 + R_{\mu\ell} R_{\nu\ell,k} R_{\nu j} R_{\mu j,k} \\
= &\ 3|\nabla\phi|^2 - R_{\mu\ell} R_{\nu\ell,k} R_{\mu j} R_{\nu j,k} \\
= &\ 3|\nabla\phi|^2 - \delta_{\ell j} R_{\nu\ell,k} R_{\nu j,k} = 3|\nabla\phi|^2 - |\nabla R|^2 .
\end{aligned}$$

Remark 3.2 In the B phase, the above computation shows that for $\alpha = 4, \ldots, 8$, the quartic invariants I_α are expressed in terms of the same Cartesian invariants involved in the expressions of the quadratic invariants I_1, I_2, and I_3, namely $|\nabla R|^2$, $\operatorname{tr}[(\nabla R)^2]$, $(\operatorname{div} R)^2$, and $|\nabla \phi|^2$.

On the other hand, for $\alpha = 9, \ldots, 14$, the quartic invariants I_α in general mix together the row vectors of the matrix R. For example, as for $I_9 := A_{\mu\ell}\partial_\ell A_{\mu j} A^*_{\nu k}\partial_k A^*_{\nu j}$, in the B phase we have

$$\begin{aligned} 9\, I_9 = &\quad R_{\mu\ell} R_{\nu k}(\mathrm{i}\phi_{,\ell} R_{\mu j} + R_{\mu j,\ell})(-\mathrm{i}\phi_{,k} R_{\nu j} + R_{\nu j,k}) \\ = &\quad \delta_{j\ell}\delta_{jk}\phi_{,\ell}\phi_{,k} + R_{\mu\ell} R_{\mu j,\ell} R_{\nu k} R_{\nu j,k}\,. \end{aligned}$$

Using the notation (3.5), we observe that e.g. for $\mu = 1$

$$R_{1\ell} R_{1j,\ell} = \mathbf{n}_\ell \mathbf{n}_{j,\ell} = \mathbf{n} \bullet \nabla \mathbf{n}_j =: \partial_{\mathbf{n}} \mathbf{n}_j$$

so that we readily obtain

$$9 I_9 = |\nabla\phi|^2 + |\partial_{\mathbf{n}}\mathbf{n} + \partial_{\mathbf{m}}\mathbf{m} + \partial_{\boldsymbol{\ell}}\boldsymbol{\ell}|^2$$

where the last term can be expressed (up to a null Lagrangian term) using only the twelve independent quadratic first order invariants

$$\begin{array}{ccc} |\mathbf{n}\times \operatorname{curl}\mathbf{n}|^2\,, & |\mathbf{m}\times \operatorname{curl}\mathbf{m}|^2\,, & |\boldsymbol{\ell}\times \operatorname{curl}\boldsymbol{\ell}|^2\,, \\ (\operatorname{div}\mathbf{n})^2\,, & (\operatorname{div}\mathbf{m})^2\,, & (\operatorname{div}\boldsymbol{\ell})^2\,, \\ (\mathbf{n}\cdot \operatorname{curl}\mathbf{n})^2\,, & (\mathbf{m}\cdot \operatorname{curl}\mathbf{m})^2\,, & (\boldsymbol{\ell}\cdot \operatorname{curl}\boldsymbol{\ell})^2\,, \\ (\mathbf{m}\cdot \operatorname{curl}\mathbf{n})^2\,, & (\boldsymbol{\ell}\cdot \operatorname{curl}\mathbf{m})^2\,, & (\mathbf{n}\cdot \operatorname{curl}\boldsymbol{\ell})^2\,, \end{array} \tag{3.7}$$

compare e.g. [24, 25] and [18]. In a similar way, as for $I_{10} := A_{\mu\ell}\partial_k A_{\mu j} A^*_{\nu k}\partial_\ell A^*_{\nu j}$, in the B phase we have

$$\begin{aligned} 9\, I_{10} = &\quad R_{\mu\ell} R_{\nu k}(\mathrm{i}\phi_{,k} R_{\mu j} + R_{\mu j,k})(-\mathrm{i}\phi_{,\ell} R_{\nu j} + R_{\nu j,\ell}) \\ = &\quad \delta_{j\ell}\delta_{jk}\phi_{,k}\phi_{,\ell} + R_{\mu\ell} R_{\nu k} R_{\mu j,k} R_{\nu j,\ell}\,. \end{aligned}$$

When $\mu = \nu = 1$, we e.g. have

$$R_{\mu\ell} R_{\nu k} R_{\mu j,k} R_{\nu j,\ell} = \mathbf{n}_\ell \mathbf{n}_k \mathbf{n}_{j,\ell} \mathbf{n}_{j,k} = |\mathbf{n}\times \operatorname{curl}\mathbf{n}|^2 = |\partial_{\mathbf{n}}\mathbf{n}|^2$$

and when e.g. $\mu = 1$ and $\nu = 2$,

$$R_{\mu\ell} R_{\nu k} R_{\mu j,k} R_{\nu j,\ell} = \mathbf{n}_\ell \mathbf{m}_k \mathbf{n}_{j,k} \mathbf{m}_{j,\ell} = \partial_{\mathbf{n}}\mathbf{m}_j \partial_{\mathbf{m}}\mathbf{n}_j = \partial_{\mathbf{n}}\mathbf{m} \bullet \partial_{\mathbf{m}}\mathbf{n}$$

so that we get

$$\begin{aligned} 9\, I_9 = &\quad |\nabla\phi|^2 + |\partial_{\mathbf{n}}\mathbf{n}|^2 + |\partial_{\mathbf{m}}\mathbf{m}|^2 + |\partial_{\boldsymbol{\ell}}\boldsymbol{\ell}|^2 + \partial_{\mathbf{n}}\mathbf{m}\bullet\partial_{\mathbf{m}}\mathbf{n} + \partial_{\mathbf{m}}\boldsymbol{\ell}\bullet\partial_{\boldsymbol{\ell}}\mathbf{m} + \partial_{\boldsymbol{\ell}}\mathbf{n}\bullet\partial_{\mathbf{n}}\boldsymbol{\ell} \\ = &\quad |\nabla\phi|^2 + \frac{1}{2}\left(|\mathbf{n}\times \operatorname{curl}\mathbf{n}|^2 + |\mathbf{m}\times \operatorname{curl}\mathbf{m}|^2 + |\mathbf{m}\times \operatorname{curl}\mathbf{m}|^2\right) + \frac{1}{2}\,|\partial_{\mathbf{n}}\mathbf{n} + \partial_{\mathbf{m}}\mathbf{m} + \partial_{\boldsymbol{\ell}}\boldsymbol{\ell}|^2\,. \end{aligned}$$

Similar calculations, which we omit, can be made in the B phase for the coupled invariants I_α, for $\alpha = 11, \ldots, 14$.

4 The A phase

In this section, we discuss the gradient energy density in the A phase and write explicit expressions for the gradient terms I_α.

Referring to the notation used in Example 1.2, and denoting this time

$$R^{(S)} = \left(\mathbf{n}^{(S)}|\mathbf{m}^{(S)}|\boldsymbol{\ell}^{(S)}\right), \quad R^{(L)T} = \left(\mathbf{n}^{(L)}|\mathbf{m}^{(L)}|\boldsymbol{\ell}^{(L)}\right),$$

it turns out that the complex matrix $(G \star A^{\pi/2})$ only depends on the third column vector $\mathbf{d} := \boldsymbol{\ell}^{(S)}$ of the spin matrix $R^{(S)}$, and on the first two row vectors $\mathbf{n}^T$ and $\mathbf{m}^T$ of the orbital matrix $R^{(L)}$, i.e., $\mathbf{n} := \mathbf{n}^{(L)}$ and $\mathbf{m} := \mathbf{m}^{(L)}$. With this notation, one can thus write

$$(G \star A^{\pi/2})_{\mu j} = \frac{1}{\sqrt{2}}\, \mathrm{e}^{\mathrm{i}\phi}\, \mathbf{d}_\mu\, (\mathbf{n}_j + \mathrm{i}\, \mathbf{m}_j) \tag{4.1}$$

where $\mathbf{d}$ specifies the direction of the *spontaneous magnetic anisotropy axis*. The third row vector of the orbit matrix $R^{(L)}$ is called the *orbital vector* $\boldsymbol{\ell}^T$. One has $\boldsymbol{\ell} = \boldsymbol{\ell}^{(L)} = \mathbf{n}^{(L)} \times \mathbf{m}^{(L)}$, and the matrix $G \star A^{\pi/2}$ is described by the phase ϕ, the spin vector $\mathbf{d}$, and the orbital vector $\boldsymbol{\ell}$. In fact, notice that a rotation by an angle α about $\boldsymbol{\ell}$ takes the orbital part $\mathbf{n} + \mathrm{i}\,\mathbf{m}$ of the order parameter (4.1) to $\mathsf{e}^{-\mathrm{i}\alpha}(\mathbf{n} + \mathrm{i}\,\mathbf{m})$. Provided that this rotation is combined with a gauge transformation $\phi \mapsto \phi + \alpha$, the order parameter (4.1) does not change (this amounts to the combined gauge-orbital symmetry with the generator $\mathbf{L}_z - \Phi$). The phase ϕ and the vectors $\mathbf{d}$, $\boldsymbol{\ell}$ account then for the five degrees of freedom in the A phase, modulo an additional $\mathbb{Z}_2$-discrete symmetry, which reflects the discrete spin-gauge symmetry of the A phase: the spin rotation $\mathbf{d} \mapsto -\mathbf{d}$, combined with the gauge transformation $\phi \to \phi + \pi$, does not change the state (4.1) (cf. Example 1.2).

A further and more significant notation will be used, by setting

$$\mathbf{u} + \mathrm{i}\,\mathbf{v} := \frac{1}{\sqrt{2}} \mathsf{e}^{\mathrm{i}\phi} (\mathbf{n} + \mathrm{i}\,\mathbf{m}) \in \mathbb{C}^3$$

so that

$$|\mathbf{u} + \mathrm{i}\,\mathbf{v}| = 1\,, \quad (\mathbf{u} + \mathrm{i}\,\mathbf{v})^2 = |\mathbf{u}|^2 - |\mathbf{v}|^2 + 2\,\mathrm{i}\,\mathbf{u} \bullet \mathbf{v} = 0 \tag{4.2}$$

and

$$(G \star A^{\pi/2})_{\mu j} = \mathbf{d}_\mu\,(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j)\,.$$

As in the previous section, without loss of generality, taking into account that the general form of the order parameter in the A phase is given by (cf. [29, 31])

$$\Delta_A(T, P)\,A, \quad A_{\mu j} = \frac{1}{\sqrt{2}}\,\mathsf{e}^{\mathrm{i}\phi}\,\mathbf{d}_\mu\,(\mathbf{n}_j + \mathrm{i}\,\mathbf{m}_j) = \mathbf{d}_\mu\,(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j)\,, \tag{4.3}$$

we will compute the gradient terms I_α for the unit matrix A. The gradient terms I_α for the general form of the order parameter in the A phase are then obtained by the homogeneity formulas (2.9) in Remark 2.3.

Explicit formulas. Consider the function

$$\Omega \ni x \mapsto A_{\mu j} := \frac{1}{\sqrt{2}}\,\mathsf{e}^{\mathrm{i}\phi}\,\mathbf{d}_\mu\,(\mathbf{n}_j + \mathrm{i}\,\mathbf{m}_j)$$

where $\phi(x) \in \mathbb{R}$ and $\mathbf{n}(x)$, $\mathbf{m}(x)$, $\mathbf{d}(x) \in \mathbb{S}^2$, with the orthogonality condition $\mathbf{n} \bullet \mathbf{m} = 0$ (but with $\mathbf{d}$ independent of $\mathbf{n}$ and $\mathbf{m}$), or equivalently,

$$\Omega \ni x \mapsto A_{\mu j} := \mathbf{d}_\mu\,(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j) \tag{4.4}$$

where $\mathbf{d}(x) \in \mathbb{S}^2$ and $\mathbf{u}(x) + \mathrm{i}\,\mathbf{v}(x) \in \mathbb{C}^3$ satisfies (4.2) for each $x \in \Omega$. Following the notation in [1], the row vectors of A become

$$\underline{A}_\mu = \mathbf{d}_\mu\,(\mathbf{u} + \mathrm{i}\,\mathbf{v})^T\,, \qquad \mu = 1, 2, 3\,.$$

As for the Dirichlet term $I_1 := \partial_k A_{\mu j} \partial_k A^*_{\mu j}$, using that $\mathbf{d}_\mu \mathbf{d}_\mu = 1$ and $\mathbf{d}_\mu \mathbf{d}_{\mu,k} = 0$, on account of (4.2), we have

$$\begin{aligned} I_1 = {} & \partial_k\big(\mathbf{d}_\mu(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j)\big)\,\partial_k\big(\mathbf{d}_\mu(\mathbf{v}_j - \mathrm{i}\,\mathbf{v}_j)\big) \\ = {} & \big(\mathbf{d}_{\mu,k}(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j) + \mathbf{d}_\mu(\mathbf{u}_{j,k} + \mathrm{i}\,\mathbf{v}_{j,k})\big)\big)\,\big(\mathbf{d}_{\mu,k}(\mathbf{u}_j - \mathrm{i}\,\mathbf{v}_j) + \mathbf{d}_\mu(\mathbf{u}_{j,k} - \mathrm{i}\,\mathbf{v}_{j,k})\big) \\ = {} & \mathbf{d}_{\mu,k}\mathbf{d}_{\mu,k}(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j)(\mathbf{u}_j - \mathrm{i}\,\mathbf{v}_j) + (\mathbf{u}_{j,k} + \mathrm{i}\,\mathbf{v}_{j,k})(\mathbf{u}_{j,k} - \mathrm{i}\,\mathbf{v}_{j,k}) \\ = {} & |\nabla \mathbf{d}|^2 + |\nabla \mathbf{u}|^2 + |\nabla \mathbf{v}|^2\,. \end{aligned}$$

Notice that in terms of ϕ, $\mathbf{n}$, and $\mathbf{m}$, since

$$\mathbf{u} = \frac{1}{\sqrt{2}}\,(\cos\phi\,\mathbf{n} - \sin\phi\,\mathbf{m})\,, \quad \mathbf{v} = \frac{1}{\sqrt{2}}\,(\sin\phi\,\mathbf{n} + \cos\phi\,\mathbf{m})$$

one may explicitly compute

$$|\nabla \mathbf{u}|^2 + |\nabla \mathbf{v}|^2 = \frac{1}{2}\,(|\nabla \mathbf{n}|^2 + |\nabla \mathbf{m}|^2) + \phi_{,k}(\mathbf{n}_j \mathbf{m}_{j,k} - \mathbf{m}_j \mathbf{n}_{j,k})\,.$$

As for the term $I_2 := \partial_k A_{\mu j}\partial_j A^*_{\mu k}$, we similarly obtain

$$\begin{aligned} I_2 = {} & \partial_k\big(\mathbf{d}_\mu(\mathbf{v}_j + \mathrm{i}\,\mathbf{v}_j)\big)\,\partial_j\big(\mathbf{d}_\mu(\mathbf{u}_k - \mathrm{i}\,\mathbf{v}_k)\big) \\ = {} & \big(\mathbf{d}_{\mu,k}(\mathbf{u}_j + \mathrm{i}\,\mathbf{v}_j) + \mathbf{d}_\mu(\mathbf{u}_{j,k} + \mathrm{i}\,\mathbf{v}_{j,k})\big)\,\big(\mathbf{d}_{\mu,j}(\mathbf{u}_k - \mathrm{i}\,\mathbf{v}_k) + \mathbf{d}_\mu(\mathbf{u}_{k,j} - \mathrm{i}\,\mathbf{v}_{k,j})\big) \\ = {} & \mathbf{d}_{\mu,k}\mathbf{d}_{\mu,j}\big((\mathbf{u}_j\mathbf{u}_k + \mathbf{v}_j\mathbf{v}_k) + \mathrm{i}(\mathbf{u}_k\mathbf{v}_j - \mathbf{u}_j\mathbf{v}_k)\big) \\ & +(\mathbf{u}_{j,k}\mathbf{u}_{k,j} + \mathbf{v}_{j,k}\mathbf{v}_{k,j}) + \mathrm{i}(\mathbf{u}_{k,j}\mathbf{v}_{j,k} - \mathbf{u}_{j,k}\mathbf{v}_{k,j})\,. \end{aligned}$$

On account of (3.2), using that $\mathbf{d}_{\mu,k}\mathbf{r}_k = \partial_{\mathbf{r}}\mathbf{d}_\mu$ we thus get $\Im I_2 = 0$ and

$$I_2 = |\partial_{\mathbf{u}}\mathbf{d}|^2 + |\partial_{\mathbf{v}}\mathbf{d}|^2 + \mathrm{tr}[(\nabla\mathbf{u})^2] + \mathrm{tr}[(\nabla\mathbf{v})^2]$$

where more explicitly we have

$$2I_2 = |\partial_{\mathbf{n}}\mathbf{d}|^2 + |\partial_{\mathbf{m}}\mathbf{d}|^2 + (\partial_{\mathbf{n}}\phi)^2 + (\partial_{\mathbf{m}}\phi)^2 + \mathrm{tr}[(\nabla\mathbf{n})^2] + \mathrm{tr}[(\nabla\mathbf{m})^2] + 2\,\phi_{,k}(\mathbf{n}_j\mathbf{m}_{k,j} - \mathbf{m}_j\mathbf{n}_{k,j})\,.$$

Analogously, concerning the term $I_3 := \partial_k A_{\mu k}\partial_j A^*_{\mu j}$ we get

$$\begin{aligned} I_3 = {} & \partial_k\big(\mathbf{d}_\mu(\mathbf{u}_k + \mathrm{i}\,\mathbf{v}_k)\big)\,\partial_j\big(\mathbf{d}_\mu(\mathbf{u}_j - \mathrm{i}\,\mathbf{v}_j)\big) \\ = {} & \big(\mathbf{d}_{\mu,k}(\mathbf{u}_k + \mathrm{i}\,\mathbf{v}_k) + \mathbf{d}_\mu(\mathbf{u}_{k,k} + \mathrm{i}\,\mathbf{v}_{k,k})\big)\,\big(\mathbf{d}_{\mu,j}(\mathbf{u}_j - \mathrm{i}\,\mathbf{v}_j) + \mathbf{d}_\mu(\mathbf{u}_{j,j} - \mathrm{i}\,\mathbf{v}_{j,j})\big) \\ = {} & \mathbf{d}_{\mu,k}\mathbf{d}_{\mu,j}\big((\mathbf{u}_j\mathbf{u}_k + \mathbf{v}_j\mathbf{v}_k) + \mathrm{i}(\mathbf{v}_k\mathbf{u}_j - \mathbf{u}_k\mathbf{v}_j)\big) \\ & +(\mathbf{u}_{k,k}\mathbf{u}_{j,j} + \mathbf{v}_{k,k}\mathbf{v}_{j,j}) + \mathrm{i}(\mathbf{u}_{j,j}\mathbf{v}_{k,k} - \mathbf{u}_{k,k}\mathbf{v}_{j,j}) \end{aligned}$$

and hence $\Im I_3 = 0$ and

$$I_3 = |\partial_{\mathbf{u}}\mathbf{d}|^2 + |\partial_{\mathbf{v}}\mathbf{d}|^2 + (\mathrm{div}\,\mathbf{u})^2 + (\mathrm{div}\,\mathbf{v})^2$$

where more explicitly we can write

$$2I_3 = |\partial_{\mathbf{n}}\mathbf{d}|^2 + |\partial_{\mathbf{m}}\mathbf{d}|^2 + (\partial_{\mathbf{n}}\phi)^2 + (\partial_{\mathbf{m}}\phi)^2 + (\mathrm{div}\,\mathbf{n})^2 + (\mathrm{div}\,\mathbf{m})^2 + 2\,\phi_{,k}(\mathbf{n}_k\mathbf{m}_{j,j} - \mathbf{m}_k\mathbf{n}_{j,j})\,.$$

Remark 4.1 We recover that $I_2 - I_3$ is a null Lagrangian, as by (3.3) we have

$$\begin{aligned} I_2 - I_3 = {} & \mathrm{tr}[(\nabla\mathbf{u})^2] + \mathrm{tr}[(\nabla\mathbf{v})^2] - (\mathrm{div}\,\mathbf{u})^2 - (\mathrm{div}\,\mathbf{v})^2 \\ = {} & \mathrm{div}[(\nabla\mathbf{u})\mathbf{u} + (\nabla\mathbf{v})\mathbf{v} - (\mathrm{div}\,\mathbf{u})\mathbf{u} - (\mathrm{div}\,\mathbf{v})\mathbf{v}]\,. \end{aligned}$$

In conclusion, taking into account the general form (4.3) of the order parameter in the A phase, it follows from the homogeneity formulas (2.9) that the gradient energy density (2.3) in the A phase takes the form

$$\begin{aligned} \Delta_A(T,P)^{-2}\,f_{\mathrm{grad}} = {} & \gamma_1|\nabla\mathbf{d}|^2 + (\gamma_2+\gamma_3)\big(|\partial_{\mathbf{u}}\mathbf{d}|^2 + |\partial_{\mathbf{v}}\mathbf{d}|^2\big) \\ + {} & \gamma_1\big(|\nabla\mathbf{u}|^2 + |\nabla\mathbf{v}|^2\big) + \gamma_2\big(\mathrm{tr}[(\nabla\mathbf{u})^2] + \mathrm{tr}[(\nabla\mathbf{v})^2]\big) + \gamma_3\big((\mathrm{div}\,\mathbf{u})^2 + (\mathrm{div}\,\mathbf{v})^2\big)\,. \end{aligned} \tag{4.5}$$

Quartic invariants. We now deal with the invariant energy densities I_α in (2.7), for $\alpha = 4,\dots,10$.

Concerning the term $I_4 := A_{\mu\ell}\partial_k A_{\mu j}A^*_{\nu\ell}\partial_k A^*_{\nu j}$, using the notation $\mathbf{w} = \mathbf{u} + \mathrm{i}\,\mathbf{v}$, in the A phase we observe that

$$(\underline{A}_\mu \bullet \underline{A}^*_\nu) = \mathbf{d}_\mu\mathbf{w}_j\mathbf{d}_\nu\mathbf{w}^*_j = \mathbf{d}_\mu\mathbf{d}_\nu|\mathbf{w}|^2 = \mathbf{d}_\mu\mathbf{d}_\nu$$

whence arguing as above, and using that $\mathbf{d}_\mu\mathbf{d}_\mu = 1$ and $\mathbf{d}_\mu\mathbf{d}_{\mu,k} = 0$, we get

$$I_4 = \mathbf{d}_\mu\mathbf{d}_\nu(\mathbf{d}_{\mu,k}\mathbf{w}_j + \mathbf{d}_\mu\mathbf{w}_{j,k})(\mathbf{d}_{\nu,k}\mathbf{w}^*_j + \mathbf{d}_\nu\mathbf{w}^*_{j,k}) = \mathbf{w}_{j,k}\mathbf{w}^*_{j,k} = |\nabla\mathbf{u}|^2 + |\nabla\mathbf{v}|^2\,.$$

As for the term $I_5 := A_{\mu\ell}\partial_k A_{\mu j}A^*_{\nu\ell}\partial_j A^*_{\nu k}$, we similarly write

$$I_5 = \mathbf{d}_\mu\mathbf{d}_\nu(\mathbf{d}_{\mu,k}\mathbf{w}_j + \mathbf{d}_\mu\mathbf{w}_{j,k})(\mathbf{d}_{\nu,j}\mathbf{w}^*_k + \mathbf{d}_\nu\mathbf{w}^*_{k,j}) = \mathbf{w}_{j,k}\mathbf{w}^*_{k,j} = \mathrm{tr}[(\nabla\mathbf{u})^2] + \mathrm{tr}[(\nabla\mathbf{v})^2]$$

whereas for the term $I_6 := A_{\mu\ell}\partial_j A_{\mu j}A^*_{\nu\ell}\partial_k A^*_{\nu k}$ we obtain

$$I_6 = \mathbf{d}_\mu\mathbf{d}_\nu(\mathbf{d}_{\mu,j}\mathbf{w}_j + \mathbf{d}_\mu\mathbf{w}_{j,j})(\mathbf{d}_{\nu,k}\mathbf{w}^*_k + \mathbf{d}_\nu\mathbf{w}^*_{k,k}) = \mathbf{w}_{j,j}\mathbf{w}^*_{k,k} = (\mathrm{div}\,\mathbf{u})^2 + (\mathrm{div}\,\mathbf{v})^2\,.$$

Concerning the term $I_7 := A_{\mu\ell}\partial_k A_{\mu\ell}A^*_{\nu j}\partial_k A^*_{\nu j}$, we instead have $I_7 \equiv 0$, as e.g.

$$(\underline{A}_\mu \bullet \partial_k\underline{A}_\mu) = \mathbf{d}_\mu\mathbf{w}_\ell(\mathbf{d}_{\mu,k}\mathbf{w}_\ell + \mathbf{d}_\mu\mathbf{w}_{\ell,k}) = \mathbf{w}_\ell\mathbf{w}_{\ell,\mathbf{k}} = 0\,.$$

Remark 4.2 In the A phase, it follows from the above computation that for $\alpha = 4, \dots, 7$, the quartic invariants I_α are expressed in terms of the same Cartesian invariants involved in the quadratic invariants I_1, I_2, and I_3.

As for the term $I_8 := A_{\mu\ell}\partial_k A_{\mu j} A^*_{\nu j}\partial_k A^*_{\nu\ell}$, we have

$$I_8 = \mathbf{d}_\mu \mathbf{d}_\nu \mathbf{w}_\ell \mathbf{w}_j^*(\mathbf{d}_{\mu,k}\mathbf{w}_j + \mathbf{d}_\mu \mathbf{w}_{j,k})(\mathbf{d}_{\nu,k}\mathbf{w}_\ell^* + \mathbf{d}_\nu \mathbf{w}^*_{\ell,k}) = \mathbf{w}_\ell \mathbf{w}^*_{\ell,k}\mathbf{w}_j^*\mathbf{w}_{j,k}\,.$$

Using that $\mathbf{u}_\ell \mathbf{v}_{\ell,k} + \mathbf{v}_\ell \mathbf{u}_{\ell,k} = 0$, we compute

$$\mathbf{w}_\ell \mathbf{w}^*_{\ell,k} = -2\mathrm{i}\,\mathbf{u}_\ell \mathbf{v}_{\ell,k} \qquad \mathbf{w}_j^*\mathbf{w}_{j,k} = 2\mathrm{i}\,\mathbf{u}_j \mathbf{v}_{j,k}$$

and hence

$$I_8 = 4\mathbf{u}_\ell \mathbf{v}_{\ell,k}\mathbf{u}_j \mathbf{v}_{j,k} = 4\mathbf{u}^T\nabla\mathbf{v}(\nabla\mathbf{v})^T\mathbf{u}\,.$$

For $\alpha = 9, \dots, 14$, the quartic invariants I_α in general mix together the derivatives of the row vectors $\mathbf{u}$, $\mathbf{v}$ of the orbit matrix $R^{(L)}$. For example, as for $I_9 := A_{\mu\ell}\partial_\ell A_{\mu j}A^*_{\nu k}\partial_k A^*_{\nu j}$ we have

$$\begin{aligned} I_9 = &\ \mathbf{d}_\mu \mathbf{w}_\ell \mathbf{d}_\nu \mathbf{w}_k^*(\mathbf{d}_{\mu,\ell}\mathbf{w}_j + \mathbf{d}_\mu \mathbf{w}_{j,\ell})(\mathbf{d}_{\nu,k}\mathbf{w}_j^* + \mathbf{d}_\nu \mathbf{w}_{j,k}) \\ = &\ \mathbf{w}_\ell \mathbf{w}_k^* \mathbf{w}_{j,\ell}\mathbf{w}_{j,k} \\ = &\ \big((\mathbf{u}_\ell \mathbf{u}_k + \mathbf{v}_\ell \mathbf{v}_k) + \mathrm{i}\,(\mathbf{u}_k \mathbf{v}_\ell - \mathbf{u}_\ell \mathbf{v}_k)\big)\big((\mathbf{u}_{j,\ell}\mathbf{u}_{j,k} + \mathbf{v}_{j,\ell}\mathbf{v}_{j,k}) + \mathrm{i}\,(\mathbf{u}_{j,k}\mathbf{v}_{j,\ell} - \mathbf{u}_{j,\ell}\mathbf{v}_{j,k})\big) \end{aligned}$$

from which we recover the condition $\Im I_9 = 0$ and we find

$$\begin{aligned} I_9 \ = \ &\mathbf{u}_\ell \mathbf{u}_k \mathbf{u}_{j,\ell}\mathbf{u}_{j,k} + \mathbf{v}_\ell \mathbf{v}_k \mathbf{v}_{j,\ell}\mathbf{v}_{j,k} + \mathbf{u}_\ell \mathbf{u}_k \mathbf{v}_{j,\ell}\mathbf{v}_{j,k} + \mathbf{v}_\ell \mathbf{v}_k \mathbf{u}_{j,\ell}\mathbf{u}_{j,k} \\ &+ \mathbf{u}_k \mathbf{v}_\ell \mathbf{u}_{j,\ell}\mathbf{v}_{j,k} - \mathbf{u}_k \mathbf{v}_\ell \mathbf{u}_{j,k}\mathbf{v}_{j,\ell} + \mathbf{u}_\ell \mathbf{v}_k \mathbf{u}_{j,k}\mathbf{v}_{j,\ell} - \mathbf{u}_\ell \mathbf{v}_k \mathbf{u}_{j,\ell}\mathbf{v}_{j,k}\,. \end{aligned}$$

Now, the first term is $|\mathbf{u}\times \operatorname{curl} u|^2 = |\partial_{\mathbf{u}}\mathbf{u}|^2$ and e.g. the fifth term

$$\mathbf{u}_k \mathbf{v}_\ell \mathbf{u}_{j,\ell}\mathbf{v}_{j,k} = \partial_{\mathbf{v}}\mathbf{u}_j \partial_{\mathbf{u}}\mathbf{v}_j = \partial_{\mathbf{v}}\mathbf{u} \bullet \partial_{\mathbf{u}}\mathbf{v}\,.$$

In this way we get

$$\begin{aligned} I_9 \ &= \ |\partial_{\mathbf{u}}\mathbf{u}|^2 + |\partial_{\mathbf{v}}\mathbf{v}|^2 + |\partial_{\mathbf{u}}\mathbf{v}|^2 + |\partial_{\mathbf{v}}\mathbf{u}|^2 + 2\,\partial_{\mathbf{v}}\mathbf{u}\bullet\partial_{\mathbf{u}}\mathbf{v} - 2\,\partial_{\mathbf{u}}\mathbf{u}\bullet\partial_{\mathbf{v}}\mathbf{v} \\ &= \ |\partial_{\mathbf{u}}\mathbf{u} - \partial_{\mathbf{v}}\mathbf{v}|^2 + |\partial_{\mathbf{u}}\mathbf{v} + \partial_{\mathbf{v}}\mathbf{u}|^2\,. \end{aligned}$$

Similarly, for $I_{10} := A_{\mu\ell}\partial_k A_{\mu j}A^*_{\nu k}\partial_\ell A^*_{\nu j}$ we have

$$\begin{aligned} I_{10} = &\ \mathbf{d}_\mu \mathbf{w}_\ell \mathbf{d}_\nu \mathbf{w}_k^*(\mathbf{d}_{\mu,k}\mathbf{w}_j + \mathbf{d}_\mu \mathbf{w}_{j,k})(\mathbf{d}_{\nu,\ell}\mathbf{w}_j^* + \mathbf{d}_\nu \mathbf{w}^*_{j,\ell}) \\ = &\ \mathbf{w}_\ell \mathbf{w}^*_{j,\ell}\mathbf{w}_k^*\mathbf{w}_{j,k} \\ = &\ \big((\partial_{\mathbf{u}}\mathbf{u}_j + \partial_{\mathbf{v}}\mathbf{v}_j) + \mathrm{i}\,(\partial_{\mathbf{v}}\mathbf{u}_j - \partial_{\mathbf{u}}\mathbf{v}_j)\big)\big((\partial_{\mathbf{u}}\mathbf{u}_j + \partial_{\mathbf{v}}\mathbf{v}_j) - \mathrm{i}\,(\partial_{\mathbf{v}}\mathbf{u}_j - \partial_{\mathbf{u}}\mathbf{v}_j)\big) \\ = &\ |\partial_{\mathbf{u}}\mathbf{u}|^2 + |\partial_{\mathbf{v}}\mathbf{v}|^2 + 2\partial_{\mathbf{u}}\mathbf{u}\bullet\partial_{\mathbf{v}}\mathbf{v} + |\partial_{\mathbf{v}}\mathbf{u}|^2 + |\partial_{\mathbf{u}}\mathbf{v}|^2 - 2\partial_{\mathbf{v}}\mathbf{u}\bullet\partial_{\mathbf{u}}\mathbf{v} \\ = &\ |\partial_{\mathbf{u}}\mathbf{u} + \partial_{\mathbf{v}}\mathbf{v}|^2 + |\partial_{\mathbf{v}}\mathbf{u} - \partial_{\mathbf{u}}\mathbf{v}|^2\,. \end{aligned}$$

The terms I_8, I_9, and I_{10} can be expressed (up to a null Lagrangian) as linear combinations of the first order invariants from (3.7), where we take $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$ instead of $\mathbf{n}$, $\mathbf{m}$, and $\boldsymbol{\ell}$, respectively. Finally, again we omit the explicit computation in the A phase of the coupled invariants I_α, for $\alpha = 11, \dots, 14$.

5 A unified approach to the A and B phases

In this section, we introduce a class of matrices that allows a unified treatment of both the A and B phases. In our ansatz, in fact, we are able to compute analytically the bulk energy minimum and the phase transition parameters in accordance with the results for the Ginzburg–Landau functional collected in [29, Section II-D].

We assume

$$A = \Delta(T,P)\,A^\theta \in \mathbb{C}^3\otimes\mathbb{C}^3$$

where the amplitude $\Delta(T,P) > 0$ depends on temperature T and pressure P, and A^θ is the unit matrix parametrized by an auxiliary control parameter θ relating the unit states describing the A and B phases considered in Examples 1.1 and 1.2,

$$A^\theta := \cos\theta\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} + \sin\theta\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & \mathrm{i} & 0\end{pmatrix}, \qquad \theta\in[0,\pi/2]\,.$$

Therefore, in our ansatz we have $|A^\theta|^2 := \mathrm{tr}(A^\theta(A^\theta)^\dagger) = 1$, for every θ, whereas in components

$$A^\theta_{\mu j} = \frac{\cos\theta}{\sqrt{3}}\,\delta_{\mu j} + \frac{\sin\theta}{\sqrt{2}}\,\delta_{\mu 3}(\delta_{j1} + \delta_{j2}\,\mathsf{i})\,.$$

GEOMETRY. For $\theta \in]0,\pi/2[$, we show that the isotropy group $\mathcal{H}$ at A^θ is trivial. In fact, the system (1.7), when $A = A^\theta$ and $\theta \in]0,\pi/2[$, yields 3×9 equations in the real part:

$$\frac{\cos\theta}{\sqrt{3}}\,(a_i - b_i)\,\varepsilon_{ij\mu} + \frac{\sin\theta}{\sqrt{2}}\,\big(\delta_{\mu 3}(a_i\varepsilon_{ij1} - c\,\delta_{i3}\delta_{j2}) + b_i\,\varepsilon_{i\mu 3}\delta_{j1}\big) = 0\,, \quad i,\mu,j = 1,2,3$$

and other 3×9 in the imaginary part:

$$\frac{\cos\theta}{\sqrt{3}}\,c\,\delta_{i3}\delta_{\mu j} + \frac{\sin\theta}{\sqrt{2}}\,\big(\delta_{\mu 3}(a_i\varepsilon_{ij2} + c\,\delta_{i3}\delta_{j1}) + b_i\,\varepsilon_{i\mu 3}\delta_{j2}\big) = 0\,, \quad i,\mu,j = 1,2,3\,.$$

This over-determined system implies that $\mathbf{a} = \mathbf{b} = 0_{\mathbb{R}^3}$ and $c = 0$, whence $\mathcal{H} = \{e\}$ and $\mathcal{G}\star A^\theta \cong \mathcal{G}$.

Remark 5.1 If $\theta = 0$, equations $(a_i - b_i)\,\varepsilon_{ij\mu} = 0$ and $c\,\delta_{i3}\delta_{\mu j} = 0$ yield $c = 0$ and $\mathbf{a} = \mathbf{b}$, so that we can take $\mathbf{T} = \mathbf{S} + \mathbf{L}$, and hence the B phase, see Example 1.1. Similarly, if $\theta = \pi/2$, equations $\delta_{\mu 3}(a_i\varepsilon_{ij1} - c\,\delta_{i3}\delta_{j2}) + b_i\,\varepsilon_{i\mu 3}\delta_{j1} = 0$ and $\delta_{\mu 3}(a_i\varepsilon_{ij2} + c\,\delta_{i3}\delta_{j1}) + b_i\,\varepsilon_{i\mu 3}\delta_{j2} = 0$ imply $a_3 + c = 0$ and $a_1 = a_2 = b_1 = b_2 = 0$, yielding $\mathbf{T}_z = b_z\mathbf{S}_z + a_z(\mathbf{L}_z - \Phi)$, and hence the A phase, see Example 1.2.

According to (1.3), the action of an element $G := (R^{(S)}, R^{(L)}, \phi) \in \mathcal{G}$ on A^θ is given by

$$(G\star A^\theta)_{\mu j} = \mathsf{e}^{\mathsf{i}\phi}\Big(\frac{\cos\theta}{\sqrt{3}} R^{(S)}_{\mu\alpha}R^{(L)}_{j\alpha} + \frac{\sin\theta}{\sqrt{2}} R^{(S)}_{\mu 3}(R^{(L)}_{j1} + R^{(L)}_{j2}\,\mathsf{i})\Big)\,.$$

Now, acting by the element $G_R := (R, R, 0)$, with $R = (R^{(S)})^T$, we get

$$(G_R \star (G\star A^\theta)) = \mathsf{e}^{\mathsf{i}\phi}A^\theta M\,, \qquad M := (R^{(L)})^T R^{(S)} \in SO(3)$$

and, denoting by $\mathbf{n}^T, \mathbf{m}^T, \boldsymbol{\ell}^T$ the row vectors of M, $M^T = \big(\mathbf{n}|\mathbf{m}|\boldsymbol{\ell}\big)$, we obtain

$$\mathsf{e}^{\mathsf{i}\phi}A^\theta M = \mathsf{e}^{\mathsf{i}\phi}\Big\{\frac{\cos\theta}{\sqrt{3}}\,\big(\mathbf{n}|\mathbf{m}|\boldsymbol{\ell}\big)^T + \frac{\sin\theta}{\sqrt{2}}\big(0_{\mathbb{R}^3}|0_{\mathbb{R}^3}|\mathbf{n} + \mathsf{i}\,\mathbf{m}\big)^T\Big\}.$$

We thus conclude that the absence of discrete symmetries can be verified by observing that, for $\theta \in]0,\pi/2[$, the only matrix $M \in SO(3)$ satisfying $\mathsf{e}^{\mathsf{i}\phi}A^\theta M = A^\theta$ is the identity matrix, with $\phi = 0$.

Remark 5.2 Concerning the gradient energy densities I_α, we recall that the amplitude $\Delta(T,P)$ does not depend on the position, so that the formulas (2.9) hold. In the above coordinates, we e.g. readily check that

$$I_1(\mathsf{e}^{\mathsf{i}\phi}A^\theta M, \nabla(\mathsf{e}^{\mathsf{i}\phi}A^\theta M)) = |\nabla\phi|^2 + \frac{\cos^2\theta}{3}\,(|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2 + |\nabla\boldsymbol{\ell}|^2) + \frac{\sin^2\theta}{2}\,(|\nabla\mathbf{n}|^2 + |\nabla\mathbf{m}|^2)\,.$$

BULK ENERGY. We now write explicitly the fourth order expansion of the bulk energy density (1.4) in our ansatz. For this purpose, we first recall that by the factorization formula (2.1), we can rewrite the bulk energy as in (2.2). Moreover, compare [13, 20, 30], we make use of the explicit formulas:

$$\mathrm{tr}(AA^\dagger) = A^*_{\mu j}A_{\mu j}\,, \qquad |\mathrm{tr}(AA^T)|^2 = A^*_{\mu j}A^*_{\mu j}A_{\nu k}A_{\nu k}\,,$$

$$[\mathrm{tr}(AA^\dagger)]^2 = A^*_{\mu j}A_{\mu j}A^*_{\nu k}A_{\nu k}\,, \qquad \mathrm{tr}[(AA^T)(AA^T)^*] = A^*_{\mu j}A^*_{\nu j}A_{\mu k}A_{\nu k}\,,$$

$$\mathrm{tr}[(AA^\dagger)^2] = A^*_{\mu j}A_{\nu j}A^*_{\nu k}A_{\mu k}\,, \qquad \mathrm{tr}[(AA^\dagger)(AA^\dagger)^*] = A^*_{\mu j}A_{\nu j}A_{\nu k}A^*_{\mu k}\,.$$

Using that $(\delta_{\mu 1} + \delta_{\mu 2}\mathsf{i})(\delta_{\mu 1} + \delta_{\mu 2}\mathsf{i}) = 0$ and $(\delta_{\mu 1} + \delta_{\mu 2}\mathsf{i})(\delta_{\mu 1} - \delta_{\mu 2}\mathsf{i}) = 2$, we compute

$$A^{\theta *}_{\mu j}A^\theta_{\mu j} = 1\,, \quad A^\theta_{\nu k}A^\theta_{\nu k} = \cos^2\theta\,, \quad A^{\theta *}_{\mu j}A^{\theta *}_{\mu j}A^\theta_{\nu k}A^\theta_{\nu k} = \cos^4\theta\,, \quad A^{\theta *}_{\mu j}A^\theta_{\mu j}A^{\theta *}_{\nu k}A^\theta_{\nu k} = 1$$

and also

$$A^\theta_{\mu k}A^\theta_{\nu k} = \frac{\cos^2\theta}{3}\,\delta_{\mu\nu} + \frac{\cos\theta\sin\theta}{\sqrt{6}}\,\big(\delta_{\nu 3}(\delta_{\mu 1} + \delta_{\mu 2}\mathsf{i}) + \delta_{\mu 3}(\delta_{\nu 1} + \delta_{\nu 2}\mathsf{i})\big)$$

$$A^{\theta*}_{\mu j}A^{\theta*}_{\nu j} = \frac{\cos^2\theta}{3}\,\delta_{\mu\nu} + \frac{\cos\theta\sin\theta}{\sqrt{6}}\,\big(\delta_{\nu 3}(\delta_{\mu 1}-\delta_{\mu 2}\mathrm{i}) + \delta_{\mu 3}(\delta_{\nu 1}-\delta_{\nu 2}\mathrm{i})\big)\,,$$

from which we obtain

$$A^{\theta*}_{\mu j}A^{\theta*}_{\nu j}A^{\theta}_{\mu k}A^{\theta}_{\nu k} = \frac{\cos^4\theta}{3} + \frac{2}{3}\,\cos^2\theta\sin^2\theta = \frac{1-\sin^4\theta}{3}\,.$$

In a similar way, we have

$$A^{\theta*}_{\mu j}A^{\theta}_{\nu j} = \frac{\cos^2\theta}{3}\,\delta_{\mu\nu} + \sin^2\theta\,\delta_{\mu 3}\delta_{\nu 3} + \frac{\cos\theta\sin\theta}{\sqrt{6}}\,\big(\delta_{\nu 3}(\delta_{\mu 1}+\delta_{\mu 2}\mathrm{i}) + \delta_{\mu 3}(\delta_{\nu 1}-\delta_{\nu 2}\mathrm{i})\big)$$

and $A^{\theta*}_{\nu k}A^{\theta}_{\mu k} = \big(A^{\theta*}_{\mu j}A^{\theta}_{\nu j}\big)^*$, so that

$$A^{\theta*}_{\mu j}A^{\theta}_{\nu j}A^{\theta*}_{\nu k}A^{\theta}_{\mu k} = \frac{\cos^4\theta}{3} + \sin^4\theta + \frac{4}{3}\,\cos^2\theta\sin^2\theta = \frac{2-\cos^4\theta+\sin^4\theta}{3}\,.$$

Finally, since $A^{\theta}_{\nu k}A^{\theta*}_{\mu k} = A^{\theta*}_{\mu j}A^{\theta}_{\nu j}$, we get

$$A^{\theta*}_{\mu j}A^{\theta}_{\nu j}A^{\theta}_{\nu k}A^{\theta*}_{\mu k} = \frac{\cos^4\theta}{3} + \sin^4\theta + \frac{2}{3}\,\cos^2\theta\sin^2\theta = \frac{1+2\sin^4\theta}{3}\,.$$

Therefore, introducing the notation

$$F_\Delta(\theta) := f_B(\Delta A^\theta)\,,\quad \Delta = \Delta(T,P)$$

and replacing the above terms in formulas (2.2), for $\theta\in[0,\pi/2]$ and $\Delta>0$, we correspondingly obtain

$$F_\Delta(\theta) = -\Delta^2\,\alpha + \Delta^4\Big(\beta_2 + \beta_1\,\cos^4\theta + \beta_3\,\frac{1-\sin^4\theta}{3} + \beta_4\,\frac{2-\cos^4\theta+\sin^4\theta}{3} + \beta_5\,\frac{1+2\sin^4\theta}{3}\Big)\,. \tag{5.1}$$

In particular, using the shorthand notation $\beta_{ij} := \beta_i+\beta_j$ and $\beta_{ijk} := \beta_i+\beta_j+\beta_k$, compare [29], we have

$$F_\Delta(0) = -\Delta^2\,\alpha + \Delta^4\Big(\beta_{12} + \frac{1}{3}\,\beta_{345}\Big)\,,\qquad F_\Delta(\pi/2) = -\Delta^2\alpha + \Delta^4\beta_{245}\,. \tag{5.2}$$

As for the minimum of the functional $\theta\mapsto F_\Delta(\theta)$, in general we can state the following.

Proposition 5.3 *Let $\Delta>0$ be a given constant and let $M(\Delta) := \min\{F_\Delta(\theta)\mid\theta\in[0,\pi/2]\}$, where $F_\Delta(\theta)$ is as in* (5.1)*. If $\beta_3-2\beta_5<\beta_4<3\beta_1$, then the minimum $M(\Delta)$ is attained for some $\theta\in]0,\pi/2[$. Otherwise, the minimum is always attained at the boundary of the interval $[0,\pi/2]$. More precisely, if $\beta_3-2\beta_5\geq\beta_4$ or $\beta_4\geq 3\beta_1$, then:*

(1) $M(\Delta) = F_\Delta(0) = f_B(\Delta\,A^0)$ *in the case* $3\beta_1+\beta_3\leq 2(\beta_4+\beta_5)$

(2) $M(\Delta) = F_\Delta(\pi/2) = f_B(\Delta\,A^{\pi/2})$ *in the case* $3\beta_1+\beta_3\geq 2(\beta_4+\beta_5)$

where, we recall,

$$A^0 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\,,\qquad A^{\pi/2} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0\\0&0&0\\1&\mathrm{i}&0\end{pmatrix}\,.$$

Proof: For any fixed $\Delta>0$, we compute:

$$\frac{d}{d\theta}F_\Delta(\theta) = \frac{4}{3}\,\Delta^4\,\sin\theta\cos\theta\,\big((3\beta_1+2\beta_5-\beta_3)\,\sin^2\theta - (3\beta_1-\beta_4)\big)\,.$$

Therefore, setting $D := (3\beta_1+2\beta_5-\beta_3)$ and $N := (3\beta_1-\beta_4)$, it turns out that F_Δ is increasing when $D\sin^2\theta > N$. Now, if $N>0$, $D>0$, and $0<N/D<1$, the minimum $M(\Delta)$ of the energy $F_\Delta(\theta)$ is attained at the interior of the interval $[0,\pi/2]$. In all the other cases, it is readily checked that the minimum is always attained at the boundary of the interval $[0,\pi/2]$. Finally, by (5.2) we deduce that $F_\Delta(0)<F_\Delta(\pi/2) \iff 3\,\beta_1+\beta_3<2(\beta_4+\beta_5)$, as required. □

Physics. Provided that the gradient part of the energy may be neglected, see the discussion below, the superfluid phases are determined by the minimization of the bulk energy (1.4) alone [8, 29, 30]. In particular, it is known that in absence of an external magnetic field the A and B phases are the only minimizers. This property is recovered in the following.

Proposition 5.4 *In our ansatz, the minimum in Proposition* 5.3 *is always attained by the A or B phases, provided that the coefficients in* (2.2) *agree with the physical constants, according to the weak or strong coupling theories.*

Proof: Following e.g. [29], the coefficient α of the first addendum (the second order term) in the bulk energy density (1.4), depends on the absolute temperature T through the formula $\alpha = N(0) \cdot (1 - T/T_c)/3$, where T_c is the critical phase-transition temperature, whence it is positive when $T < T_c$.[9]

In the so called weak coupling theory, one has $\beta_2 = \beta_3 = \beta_4 = -\beta_5 = -2\beta_1$, where $\beta_1 = -\beta_0$ for some positive constant β_0.[10] Therefore, by Proposition 5.3 the minimum of the functional $\theta \mapsto F_\Delta(\theta)$ is always attained at $\theta = 0$, where by (5.2) we get $F_\Delta(0) = f_B(\Delta\, A^0) = -\alpha\, \Delta^2 + 5\, \Delta^4 \beta_0/3$.

In the more consistent strong coupling theory, following Anderson–Brinkman [2] (cf. also [29]) one has

$$\begin{gathered}\beta_1 = -\Big(1 + \frac{\delta}{10}\Big)\, \beta_0\,, \qquad \beta_2 = \Big(2 + \frac{\delta}{5}\Big)\, \beta_0\,, \qquad \beta_3 = \Big(2 - \frac{\delta}{20}\Big)\, \beta_0\,, \\ \beta_4 = \Big(2 - \frac{11}{20}\,\delta\Big)\, \beta_0\,, \qquad \beta_5 = -\Big(2 + \frac{7}{10}\,\delta\Big)\, \beta_0\end{gathered} \tag{5.3}$$

where $\delta > 0$ is the spin-fluctuation (paramagnon) parameter, which increases as pressure increases (notice that for $\delta = 0$, the coefficients β_i correspond to the ones in the weak coupling theory).

Again, by Proposition 5.3 we infer that the minimum of the functional $\theta \mapsto F_\Delta(\theta)$ is attained at the boundary of $[0, \pi/2]$. Moreover, since by (5.2),

$$F_\Delta(0) = f_B(\Delta\, A^0) = -\Delta^2\alpha + \Delta^4\beta_0\, \Big(\frac{5-\delta}{3}\Big)\,, \qquad F_\Delta(\pi/2) = f_B(\Delta\, A^{\theta/2}) = -\Delta^2\alpha + \Delta^4\beta_0\, \Big(2 - \frac{21}{20}\,\delta\Big)\,,$$

in the strong coupling theory we deduce that the minimum of the bulk energy is attained at the B phase, i.e., $M(\Delta) = F_\Delta(0)$, if $0 < \delta < 20/43$. If $\delta > 20/43$, instead, $M(\Delta) = F_\Delta(\pi/2)$, and the mimimum is attained at the A phase. □

Remark 5.5 Since $20/43 \simeq 0.46$, our previous result is in line with [29, p. 541]. Moreover, the spin-fluctuation parameter δ being increasing with the pressure, in our ansatz the A phase happens at higher pressure, according e.g. to the phase diagram at page 536 in [29].

Energy minima. In the regimes of temperature and pressure in which the minimum of the bulk energy is attained by the B phase, i.e., when $\theta = 0$, or by the A phase, i.e., when $\theta = \pi/2$, by computing the derivative w.r.t. the amplitude $\Delta = \Delta(T, P) > 0$ of the functions $\Delta \mapsto F_\Delta(0)$ and $\Delta \mapsto F_\Delta(\pi/2)$ from (5.2), we respectively obtain:

$$\frac{d}{d\Delta}\, F_\Delta(0) = -2\Delta\, \alpha + 4\Delta^3\Big(\beta_{12} + \frac{1}{3}\, \beta_{345}\Big)\,, \qquad \frac{d}{d\Delta}\, F_\Delta(\pi/2) = -2\Delta\, \alpha + 4\Delta^3\beta_{245}$$

and hence the corresponding bulk energy minima are respectively attained at the amplitudes

$$\Delta_B^2(T, P) := \frac{3\alpha}{2(\beta_{345} + 3\beta_{12})}\,, \qquad \Delta_A^2(T, P) := \frac{\alpha}{2\beta_{245}}\,.$$

These amplitudes may be compared with the ones for $\Delta_B(T, P)$ and $\Delta_A(T, P)$ obtained in [29, p. 541].

The corresponding energy minima are then given by:

$$F_{\Delta_B(T,P)}(0) = -\frac{3}{4}\, \frac{\alpha^2}{\beta_{345} + 3\beta_{12}}\,, \qquad F_{\Delta_A(T,P)}(\pi/2) = -\frac{1}{4}\, \frac{\alpha^2}{\beta_{245}}\,.$$

In the weak coupling theory, where $\beta_2 = \beta_3 = \beta_4 = -\beta_5 = -2\beta_1$, and $\beta_1 = -\beta_0$, one has $\beta_{345} = 2\beta_0$, $\beta_{12} = \beta_0$, and $\beta_{245} = 2\beta_0$. In the strong coupling theory, instead, by the formulas (5.3) one obtains:

$$F_{\Delta_B(T,P)}(0) = -\frac{3}{4}\, \frac{\alpha^2}{\beta_0(5 - \delta)}\,, \qquad F_{\Delta_A(T,P)}(\pi/2) = -\frac{5\alpha^2}{\beta_0(40 - 21\delta)}\,. \tag{5.4}$$

[9] The constant $N(0)$ is the density of the ^3He quasiparticle states for one spin projection at the Fermi level.

[10] $\beta_0 = \dfrac{7N(0)\, \zeta(3)}{240\, (\pi T)^2}$, where the constant $\zeta(3)$ is independent of temperature and pressure.

Phase transition. We now consider the energy

$$F(A) := \int_\Omega \big(f_{\rm grad}(A,\nabla A) + f_B(A)\big)\,dx$$

on smooth maps $\Omega \ni x \mapsto A(x)$, where $\Omega \subset \mathbb{R}^3$ is a smooth domain, and $f_{\rm grad}$ and f_B are given by (2.3) and (1.4). In the weak coupling theory, one has $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_0$, where $\gamma_0 := N(0)\,{\xi_0}^2/3$. The number ξ_0 is of the order of several hundred angstroms, and defines the coherence length through the formula

$$\xi_{GL}(T) := \sqrt{\gamma_0/\alpha} = \frac{\xi_0}{(1-T/T_c)^{1/2}}$$

(cf. [29]), which defines the spatial extent of an inhomogeneity, at which the gradient energy becomes comparable with the bulk energy. The same feature holds in the strong coupling assumption. As a consequence, if the size of the inhomogeneity region Ω is larger than $\xi_{GL}(T)$, the gradient term may be neglected. In this case, the minimum essentially depends on the bulk energy.

In our ansatz, we have seen that the minimum is attained in the B phase, at small pressure, and in the A phase, at higher pressure. In particular, by increasing the spin-fluctuation parameter $\delta > 0$, a phase transition is observed.

More precisely, by the formulas (5.4), when $\delta = 0$, i.e., in the framework of the weak coupling theory, one has a bulk energy minimum equal to $-3\alpha^2/(20\beta_0)$ and given by the B phase.

In the strong coupling theory, instead, for $0 < \delta < 20/43$, the bulk energy minimum is attained by the B phase, for $\delta = 20/43$ one has

$$F_{\Delta_B(T,P)}(0) = F_{\Delta_A(T,P)}(\pi/2) = -\frac{43\alpha^2}{260\beta_0}$$

and for $20/43 < \delta \ll 40/21$, the bulk energy minimum is attained by the A phase. Finally, by (5.4) it turns out that the energy minimum is an increasing function of the spin-fluctuation parameter $\delta \in [0, 40/21[$.

Remark 5.6 The above considerations on the smallness of the gradient term have an analogy in the Q-tensor theory of liquid crystals. This has been physically unravelled in [11] and also analytically treated in [19], where it is shown that the gradient term has a significant effect even if it has a small coefficient in front.

Appendix A Independence of quartic invariants

Denote by $f_{,jk} = \partial_k(\partial_j f)$ the second partial derivative of f in the (j,k)-th coordinate directions. We first observe that by computing e.g. the k-th partial derivative of the quantity $A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu j,k}$ (and summating on $k = 1,2,3$) four terms appear: I_4, I_8, and two terms of the type $AAA^*_{,k}A^*_{,k}$ and $AAA^*A^*_{,kk}$ that fail to be real valued. In a similar way, starting from $A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu k,j}$ one obtains the term I_5, the first addendum of I_{11}, and two complex-valued terms of the type $AAA^*_{,k}A^*_{,j}$ and $AAA^*A^*_{,jk}$. Since moreover $I_5 = I_5^*$, it turns out that I_5 is also obtained when starting from the conjugate quantity $(A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu k,j})^*$, a second term being the second addendum of I_{11}, and the other two terms being of the type $A_{,k}A_{,j}A^*A^*$ and $AA_{,jk}A^*A^*$. We thus infer that in order to try and express a quartic invariant I_α as a linear combination of the other invariants plus a divergence term, in general we cannot separate the derivatives of the above couples of conjugated quantities, but we have to add them together.

With this preliminary consideration and referring to formulas (2.7) and (2.8), we can write

$$\begin{aligned}(A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu j,k} + (A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu j,k})^*)_{,k} &= 2I_4 + 2I_8 \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu\ell,k}A^*_{\nu j,k} + A^*_{\nu\ell}A^*_{\nu j,kk}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu\ell,k}A_{\nu j,k} + A_{\nu\ell}A_{\nu j,kk})\end{aligned} \tag{A.1}$$

where the left-hand side is a null-Lagrangian term, and also

$$\begin{aligned}(A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu k,j} + (A_{\mu\ell}A_{\mu j}A^*_{\nu\ell}A^*_{\nu k,j})^*)_{,k} &= 2I_5 + I_{11} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu\ell,k}A^*_{\nu k,j} + A^*_{\nu\ell}A^*_{\nu k,jk}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu\ell,k}A_{\nu k,j} + A_{\nu\ell}A_{\nu k,jk})\,.\end{aligned} \tag{A.2}$$

Looking at I_6, we similarly obtain:

$$\begin{aligned}(A_{\mu\ell}A_{\mu j,j}A^*_{\nu\ell}A^*_{\nu k} + (A_{\mu\ell}A_{\mu j,j}A^*_{\nu\ell}A^*_{\nu k})^*)_{,k} &= 2I_6 + I_{14} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu\ell,j}A^*_{\nu k,k} + A^*_{\nu\ell}A^*_{\nu k,\ell k}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu\ell,j}A_{\nu k,k} + A_{\nu\ell}A_{\nu k,\ell k})\,.\end{aligned} \tag{A.3}$$

Looking at I_9, we get:

$$\begin{aligned}(A_{\mu\ell}A_{\mu j,\ell}A^*_{\nu k}A^*_{\nu j} + (A_{\mu\ell}A_{\mu j,\ell}A^*_{\nu k}A^*_{\nu j})^*)_{,k} &= 2I_9 + I_{14} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu k,\ell}A^*_{\nu j,k} + A^*_{\nu k}A^*_{\nu j,\ell k}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu k,\ell}A_{\nu j,k} + A_{\nu k}A_{\nu j,\ell k})\,.\end{aligned} \tag{A.4}$$

Looking instead at I_{10}, we have:

$$\begin{aligned}(A_{\mu\ell}A_{\mu j}A^*_{\nu k}A^*_{\nu j,\ell} + (A_{\mu\ell}A_{\mu j}A^*_{\nu k}A^*_{\nu j,\ell})^*)_{,k} &= 2I_{10} + I_{11} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu k,k}A^*_{\nu\ell,j} + A^*_{\nu k}A^*_{\nu\ell,jk}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu k,k}A_{\nu\ell,j} + A_{\nu k}A_{\nu\ell,jk})\,.\end{aligned} \tag{A.5}$$

Now, looking at the two complex terms in I_{12}, we also obtain:

$$\begin{aligned}(A_{\mu\ell}A_{\mu k}A^*_{\nu j}A^*_{\nu j,\ell} + (A_{\mu\ell}A_{\mu k}A^*_{\nu j}A^*_{\nu j,\ell})^*)_{,k} &= I_{12} + I_{13} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu k,j}A^*_{\nu k,\ell} + A^*_{\nu k}A^*_{\nu k,\ell j}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu k,j}A_{\nu k,\ell} + A_{\nu k}A_{\nu k,\ell j})\,.\end{aligned} \tag{A.6}$$

In a similar way, looking at the two complex terms in I_{14}, we get:

$$\begin{aligned}(A_{\mu\ell}A_{\mu j,j}A^*_{\nu k}A^*_{\nu j} + (A_{\mu\ell}A_{\mu j,j}A^*_{\nu k}A^*_{\nu j})^*)_{,k} &= I_{13} + I_{14} \\ +A_{\mu\ell}A_{\mu j}(A^*_{\nu k,\ell}A^*_{\nu j,k} + A^*_{\nu k}A^*_{\nu j,\ell k}) &+ A^*_{\mu\ell}A^*_{\mu j}(A_{\nu k,\ell}A_{\nu j,k} + A_{\nu k}A_{\nu j,\ell k})\,.\end{aligned} \tag{A.7}$$

With a bit of care, one can check that there are no other independent formulas yielding to remainder terms whose pre-factor is equal to $A_{\mu\ell}A_{\mu j}$ or to its conjugate. In fact, concerning I_7 we get:

$$\begin{aligned}(A_{\mu\ell}A_{\mu\ell}A^*_{\nu j}A^*_{\nu j,k} + (A_{\mu\ell}A_{\mu\ell}A^*_{\nu j}A^*_{\nu j,k})^*)_{,k} &= 4I_7 \\ +A_{\mu\ell}A_{\mu\ell}(A^*_{\nu j,k}A^*_{\nu j,k} + A^*_{\nu j}A^*_{\nu j,kk}) &+ A^*_{\mu\ell}A^*_{\mu\ell}(A_{\nu j,k}A_{\nu j,k} + A_{\nu j}A_{\nu j,kk})\end{aligned} \tag{A.8}$$

whereas, looking separately at the two terms of I_{12} we can write:

$$\begin{aligned}(A_{\mu\ell}A_{\mu\ell}A^*_{\nu k}A^*_{\nu j,j} + (A_{\mu\ell}A_{\mu\ell}A^*_{\nu k}A^*_{\nu j,j})^*)_{,k} &= 2I_{12} \\ +A_{\mu\ell}A_{\mu\ell}(A^*_{\nu k,k}A^*_{\nu j,j} + A^*_{\nu k}A^*_{\nu j,jk}) &+ A^*_{\mu\ell}A^*_{\mu\ell}(A_{\nu k,k}A_{\nu j,j} + A_{\nu k}A_{\nu j,jk})\end{aligned} \tag{A.9}$$

and in a similar way we finally write:

$$\begin{aligned}(A_{\mu\ell}A_{\mu\ell}A^*_{\nu j}A^*_{\nu k,j} + (A_{\mu\ell}A_{\mu\ell}A^*_{\nu j}A^*_{\nu k,j})^*)_{,k} &= 2I_{13} \\ +A_{\mu\ell}A_{\mu\ell}(A^*_{\nu j,k}A^*_{\nu k,j} + A^*_{\nu j}A^*_{\nu k,jk}) &+ A^*_{\mu\ell}A^*_{\mu\ell}(A_{\nu j,k}A_{\nu k,j} + A_{\nu j}A_{\nu k,jk})\,.\end{aligned} \tag{A.10}$$

Also, it turns out that there are no other independent formulas yielding to remainder terms whose pre-factor is equal to $A_{\mu\ell}A_{\mu\ell}$ or to its conjugate.

Now, concerning the seven equations (A.1)–(A.7), it can be checked that there is no way to combine them linearly in such a way that one can cancel the remainder terms at the second line the above formulas. The same fact can be checked when starting from the three equations (A.8)–(A.10).

This way, one concludes that no one of the quartic invariants I_α, for $\alpha = 4, \ldots, 14$, can be written as a linear combination of the other ones plus a null-Lagrangian term, as required.

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Declarations.

Conflict of Interest. The authors declare that there is no conflict of interest.

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