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ON A CONJECTURE OF KHOROSHKIN AND TOLSTOY

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Abstract. We prove a *no-go theorem* on the factorization of the lower triangular part in the Gaussian decomposition of the Yangian's universal R–matrix, yielding a negative answer to a conjecture of Khoroshkin and Tolstoy from [*Lett. Math. Phys.* **36** 1996].

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $Y_\hbar(\mathfrak{g})$ the associated Yangian as defined by Drinfeld in [5]. Let $\mathcal{R}(s) \in (Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$ be Drinfeld's universal R–matrix [4]. A constructive proof of the existence of $\mathcal{R}(s)$ was recently obtained by the last two authors and V. Toledano Laredo in [8, Thm. 7.4]. This was achieved by providing a direct construction of the components of the Gaussian decomposition

$$\mathcal{R}(s) = \mathcal{R}^+(s)\mathcal{R}^0(s)\mathcal{R}^-(s).$$

The diagonal part $\mathcal{R}^0(s)$ was first defined by the second author and Toledano Laredo in [7, Thm. 5.9] as a meromorphic function of s acting on the tensor product of any two finite-dimensional representations of $Y_\hbar(\mathfrak{g})$. Its explicit expansion as a formal series in s^{-1} with coefficients in $Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$ was later provided in [8, Thm. 6.7]. The component $\mathcal{R}^-(s)$ is then obtained in [8, Thm. 4.1] as the unique (unipotent, zero–weight) solution to a system of linear equations (see also §2.8 below). Finally, the upper/lower triangular parts are related by $\mathcal{R}^+(s) = \mathcal{R}^-_{21}(-s)^{-1}$.

In this paper, we focus our attention on $\mathcal{R}^-(s)$. When $\mathfrak{g} = \mathfrak{sl}_2$, a closed form formula for $\mathcal{R}^-(s)$ is given in [8, Thm. 5.5] (see also [11, Lemma 5.1]), but no explicit formula is known in higher rank. Following a conjecture of Khoroshkin and Tolstoy [11, Conjecture p. 393], it was expected that $\mathcal{R}^-(s)$ could be expressed as an ordered product over the set of positive roots of $\mathfrak{g}$ of $\mathfrak{sl}_2$-type components (cf. [11, (5.3)], [22, §5] and in particular [21, Thm. 6.2]). The main result of this paper shows that $\mathcal{R}^-(s)$ does not admit any such factorization for $\mathfrak{g} \not\cong \mathfrak{sl}_2$ (Theorem 3.4).

The conjecture of Khoroshkin and Tolstoy mentioned above refers to the lower triangular part of the universal R–matrix of the Yangian double (cf. §4.1). In [11, §2] the latter was conjectured to be isomorphic to the restricted quantum double of $Y_\hbar(\mathfrak{g})$. This conjecture was recently proven by the third author in [24, Thm. 8.4] and led to the identification of the underlying R-matrices (and their components) in [24, Thm. 9.6] (see also [22]). Therefore, relying on these results, the conjecture of Khoroshkin and Tolstoy is directly disproved by our main result.

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Outline of the paper. In Section 2 we give a brief overview of the Yangian $Y_\hbar(\mathfrak{g})$ and its universal R–matrix, with emphasis on the factor $\mathcal{R}^-(s)$. In Section 3, we state and prove our main result (Theorem 3.4). In Section 4, we show that it yields a negative answer to the Khoroshkin–Tolstoy conjecture. Finally, in Section 4.2 we discuss analogies and differences with the factorization problem for quantum affine algebras, referring in particular to the work of Damiani [3, Thm. 2].

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2. Background on Yangians and R–matrices

2.1. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra and $(\cdot,\cdot)$ an invariant symmetric non–degenerate bilinear form on $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i\in\mathbf{I}} \subset \mathfrak{h}^*$ a basis of simple roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $a_{ij} = 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$ the entries of the corresponding Cartan matrix $\mathbf{A}$. Let $\Phi_+ \subset \mathfrak{h}^*$ be the corresponding set of positive roots, and $\mathsf{Q} = \mathbb{Z}\Phi_+ = \bigoplus_{i\in\mathbf{I}} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ the root lattice. The semigroup $\mathbb{Z}_{\geq 0}\Phi_+ \subset \mathsf{Q}$ is denoted by Q_+. We assume that $(\cdot,\cdot)$ is normalised so that the square length of short roots is 2. Set $d_i = (\alpha_i,\alpha_i)/2 \in \{1,2,3\}$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i,j\in\mathbf{I}$. In addition, we set $h_i = \nu^{-1}(\alpha_i)/d_i$ and choose root vectors $x_i^\pm \in \mathfrak{g}_{\pm\alpha_i}$ such that $[x_i^+, x_i^-] = d_i h_i$, where $\nu : \mathfrak{h} \to \mathfrak{h}^*$ is the isomorphism determined by $(\cdot,\cdot)$.

2.2. **The Yangian $Y_\hbar(\mathfrak{g})$ [5].** Let $\hbar \in \mathbb{C}^\times$. The Yangian $Y_\hbar(\mathfrak{g})$ is the unital, associative $\mathbb{C}$–algebra generated by elements $\{x_{i,r}^\pm, \xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geq 0}}$, subject to the following relations:

(Y1) For any $i,j \in \mathbf{I}$, $r,s \in \mathbb{Z}_{\geq 0}$: $[\xi_{i,r}, \xi_{j,s}] = 0$.

(Y2) For $i,j\in\mathbf{I}$ and $s\in\mathbb{Z}_{\geq 0}$: $[\xi_{i,0}, x_{j,s}^\pm] = \pm d_i a_{ij} x_{j,s}^\pm$.

(Y3) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geq 0}$:

$$[\xi_{i,r+1}, x_{j,s}^\pm] - [\xi_{i,r}, x_{j,s+1}^\pm] = \pm\hbar\frac{d_i a_{ij}}{2}\left(\xi_{i,r}x_{j,s}^\pm + x_{j,s}^\pm \xi_{i,r}\right) .$$

(Y4) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geq 0}$:

$$[x_{i,r+1}^\pm, x_{j,s}^\pm] - [x_{i,r}^\pm, x_{j,s+1}^\pm] = \pm\hbar\frac{d_i a_{ij}}{2}\left(x_{i,r}^\pm x_{j,s}^\pm + x_{j,s}^\pm x_{i,r}^\pm\right) .$$

(Y5) For $i,j\in\mathbf{I}$ and $r,s\in\mathbb{Z}_{\geq 0}$: $[x_{i,r}^+, x_{j,s}^-] = \delta_{ij}\xi_{i,r+s}$.

(Y6) Let $i \neq j \in \mathbf{I}$ and set $m = 1 - a_{ij}$. For any $r_1, \cdots, r_m, s \in \mathbb{Z}_{\geq 0}$:
$$\sum_{\pi \in \mathfrak{S}_m} \left[x^{\pm}_{i,r_{\pi(1)}}, \left[x^{\pm}_{i,r_{\pi(2)}}, \left[\cdots, \left[x^{\pm}_{i,r_{\pi(m)}}, x^{\pm}_{j,s}\right] \cdots\right]\right]\right] = 0.$$

We denote by $Y^0_\hbar(\mathfrak{g})$ and $Y^{\pm}_\hbar(\mathfrak{g})$ the unital subalgebras of $Y_\hbar(\mathfrak{g})$ generated by $\{\xi_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geq 0}}$ and $\{x^{\pm}_{i,r}\}_{i\in\mathbf{I}, r\in\mathbb{Z}_{\geq 0}}$, respectively. Let $Y^{\geq}_\hbar(\mathfrak{g})$ (resp. $Y^{\leq}_\hbar(\mathfrak{g})$) denote the subalgebras of $Y_\hbar(\mathfrak{g})$ generated by $Y^0_\hbar(\mathfrak{g})$ and $Y^+_\hbar(\mathfrak{g})$ (resp. $Y^0_\hbar(\mathfrak{g})$ and $Y^-_\hbar(\mathfrak{g})$).

2.3. **Shift automorphism.** The group of translations of the complex plane acts on $Y_\hbar(\mathfrak{g})$ by
$$\tau_a(y_r) = \sum_{s=0}^{r} \begin{pmatrix} r \\ s \end{pmatrix} a^{r-s} y_s$$
where $a \in \mathbb{C}$ and y is one of $\xi_i, x^{\pm}_i$.

2.4. For each $i \in \mathbf{I}$, define $t_{i,1} \in Y^0_\hbar(\mathfrak{g})$ by the formula
$$t_{i,1} := \xi_{i,1} - \frac{\hbar}{2}\xi^2_{i,0} \ . \tag{2.1}$$
The relations (Y2)–(Y3) of $Y_\hbar(\mathfrak{g})$ imply that for any $i, j \in \mathbf{I}$ and $r \in \mathbb{Z}_{\geq 0}$,
$$[t_{i,1}, x^{\pm}_{j,r}] = \pm d_i a_{ij} x^{\pm}_{j,r+1}. \tag{2.2}$$
Hence, the elements $t_{i,1}$ act as shift operators on the generators $x^{\pm}_{j,r}$.

2.5. **Two embeddings $\mathfrak{h} \to Y^0_\hbar(\mathfrak{g})$.** By the Poincaré–Birkhoff–Witt theorem for $Y_\hbar(\mathfrak{g})$ [14] (see also [6, Thm. B.6] and [10, Prop. 2.2]), there is an embedding of $U(\mathfrak{g})$ into $Y_\hbar(\mathfrak{g})$, uniquely determined by
$$x^{\pm}_i \mapsto x^{\pm}_{i,0} \qquad \text{and} \qquad d_i h_i \mapsto \xi_{i,0}$$
for each $i \in \mathbf{I}$. We shall henceforth identify $U(\mathfrak{g}) \subset Y_\hbar(\mathfrak{g})$, with the above embedding implicitly understood. Viewed as a module over $\mathfrak{h} \subset Y_\hbar(\mathfrak{g})$, we then have $Y_\hbar(\mathfrak{g}) = \bigoplus_{\beta\in\mathsf{Q}} Y_\hbar(\mathfrak{g})_\beta$, where
$$Y_\hbar(\mathfrak{g})_\beta = \{y \in Y_\hbar(\mathfrak{g}) : [h, y] = \beta(h)y, \ \forall\ h \in \mathfrak{h}\} \ .$$

A second embedding $\mathrm{T} : \mathfrak{h} \to Y_\hbar(\mathfrak{g})$ is given by setting $\mathrm{T}(d_i h_i) = t_{i,1}$ for all $i \in \mathbf{I}$, where $t_{i,1}$ is defined by (2.1). Using this embedding, (2.2) can be written as
$$[\mathrm{T}(h), x^{\pm}_{i,r}] = \pm\alpha_i(h) x^{\pm}_{i,r+1} \quad \forall\ h \in \mathfrak{h} \ . \tag{2.3}$$

2.6. **Coproduct.** We now recall the definition of the standard coproduct Δ on $Y_\hbar(\mathfrak{g})$. Set
$$\mathsf{r}^- = \hbar \sum_{\beta\in\Phi_+} x^-_{\beta,0} \otimes x^+_{\beta,0}, \tag{2.4}$$
where $x^{\pm}_{\beta,0} \in \mathfrak{g}_{\pm\beta} \subset Y_\hbar(\mathfrak{g})$ are root vectors such that $(x^-_{\beta,0}, x^+_{\beta,0}) = 1$. For any $h \in \mathfrak{h} \subset Y_\hbar(\mathfrak{g})$, define
$$\mathsf{r}^-(h) := \mathrm{ad}(h \otimes 1) \cdot \mathsf{r}^- = -\hbar \sum_{\beta\in\Phi_+} \beta(h) x^-_{\beta,0} \otimes x^+_{\beta,0} \ . \tag{2.5}$$

The coproduct $\Delta : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$ is then uniquely determined by the following formulae, for $i \in \mathbf{I}$ and $h \in \mathfrak{h}$:

$$\Delta(\xi_{i,0}) = \xi_{i,0} \otimes 1 + 1 \otimes \xi_{i,0}, \quad \Delta(x^\pm_{i,0}) = x^\pm_{i,0} \otimes 1 + 1 \otimes x^\pm_{i,0},$$
$$\Delta(\mathrm{T}(h)) = \mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + \mathsf{r}^-(h).$$

We refer the reader to [9, §4.2] for a proof that Δ is an algebra homomorphism. It is immediate that Δ is coassociative (see [9, §4.5]).

2.7. **Drinfeld's universal R–matrix.** Let $\Delta_s := (\tau_s \otimes 1) \circ \Delta$ and $\Delta_s^{\mathrm{op}} := (\tau_s \otimes 1) \circ \Delta^{\mathrm{op}}$. Viewing τ_s as an algebra homomorphism $\tau_s : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})[s]$, Drinfeld [4, Thm. 3] showed that there is a unique $\mathcal{R}(s) \in (Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g}))[\![s^{-1}]\!]$ satisfying the following three conditions:

(1) $\mathcal{R}(s) = 1 \otimes 1 + X(s)$, where $X(s) \in s^{-1} Y_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$.

(2) For every $a \in Y_\hbar(\mathfrak{g})$, we have
$$\Delta_s^{\mathrm{op}}(a) = \mathcal{R}(s)\Delta_s(a)\mathcal{R}(s)^{-1}.$$

(3) The following cabling identities hold:
$$\Delta \otimes 1(\mathcal{R}(s)) = \mathcal{R}_{13}(s)\mathcal{R}_{23}(s)\ ,$$
$$1 \otimes \Delta(\mathcal{R}(s)) = \mathcal{R}_{13}(s)\mathcal{R}_{12}(s)\ .$$

2.8. **Intertwining equation.** As indicated above, a constructive proof of the existence of $\mathcal{R}(s)$ was recently given in [8] by reassembling it from the components in its Gaussian decomposition
$$\mathcal{R}(s) = \mathcal{R}^+(s)\mathcal{R}^0(s)\mathcal{R}^-(s).$$
The lower triangular part $\mathcal{R}^-(s)$ is the main object of study in this paper. Let us recall its defining properties, following [8, Thm. 4.1]. Namely, $\mathcal{R}^-(s)$ is the unique, zero weight element of $\left(Y_\hbar^{\leq}(\mathfrak{g}) \otimes Y_\hbar^{\geq}(\mathfrak{g})\right)[\![s^{-1}]\!]$ satisfying the following two conditions:

(1) Write $\mathcal{R}^-(s) = \sum_{\gamma \in \mathsf{Q}_+} \mathcal{R}^-_\gamma(s)$, where
$$\mathcal{R}^-_\gamma(s) \in \left(Y_\hbar^{\leq}(\mathfrak{g})_{-\gamma} \otimes Y_\hbar^{\geq}(\mathfrak{g})_\gamma\right)[\![s^{-1}]\!].$$
Then $\mathcal{R}^-_0(s) = 1 \otimes 1$.

(2) The following *intertwining equation* holds, for every $h \in \mathfrak{h}$:
$$[\mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + sh \otimes 1, \mathcal{R}^-(s)] = \mathcal{R}^-(s)\mathsf{r}^-(h)\ . \tag{2.6}$$

Here we note that (2.6) is equivalent to the relation (4.1) in [8], which is written in terms of the deformed Drinfeld coproduct on $Y_\hbar(\mathfrak{g})$ and Δ_s from Section 2.7 above. Setting $\mathcal{D}(h;s) = \mathrm{ad}(\mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + sh \otimes 1)$, it may also be written in terms of the elements $\mathcal{R}^-_\gamma(s)$ as
$$\mathcal{D}(h;s) \cdot \mathcal{R}^-_\gamma(s) = -\hbar \sum_{\substack{\alpha \in \Phi_+ \\ \gamma - \alpha \in \mathsf{Q}_+}} \mathcal{R}^-_{\gamma-\alpha}(s) x^-_{\alpha,0} \otimes x^+_{\alpha,0}\ . \tag{2.7}$$

Together with the initial condition $\mathcal{R}_0^-(s) = 1 \otimes 1$, this equation defines $\mathcal{R}_\beta^-(s)$ inductively on the height of β; see [8, Eqn. (4.6)]. The sum $\sum_{\beta\in\mathsf{Q}_+}\mathcal{R}_\beta^-(s)$ is a well–defined element of $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$ which solves (2.6) and, by Part (2) of [8, Thm. 4.1], lies in $\left(Y_\hbar^-(\mathfrak{g})\otimes Y_\hbar^+(\mathfrak{g})\right)[\![s^{-1}]\!]$.

3. Non-existence of $\mathcal{R}^-(s)$ factorizations

The aim of this section is to show that the unique solution $\mathcal{R}^-(s)$ of (2.6) does not admit any factorization over the set of positive roots Φ_+. From now onwards, we assume that $\mathrm{rank}(\mathfrak{g}) > 1$.

3.1. **Notation.** For a given total order $<$ on Φ_+ and a collection of elements $\{A^{(\alpha)}\}_{\alpha\in\Phi_+}$ lying in some associative algebra, we set

$$\prod_{\alpha\in\Phi_+}^{<} A^{(\alpha)} := A^{(\beta_1)}\cdots A^{(\beta_N)}\ ,$$

where $\Phi_+ = \{\beta_1 < \ldots < \beta_N\}$. In addition, for each $\gamma\in\mathsf{Q}_+$ we let $\mathcal{P}(\gamma)$ denote the set of partitions of γ as a sum of positive roots:

$$\mathcal{P}(\gamma) = \left\{\underline{k} = (k_\alpha)_{\alpha\in\Phi_+}\in\mathbb{Z}_{\geq 0}^{\Phi_+} : \gamma = \sum_{\alpha\in\Phi_+} k_\alpha\alpha\right\}.$$

We further record the following symbolic identity for reference later, where $\{X_n^\alpha : \alpha\in\Phi_+, n\in\mathbb{Z}_{\geq 0}\}$ is an arbitrary collection of non–commuting variables:

$$\prod_{\alpha\in\Phi_+}^{<}\left(\sum_{n=0}^{\infty} X_n^{(\alpha)}\right) = \sum_{\gamma\in\mathsf{Q}_+}\left(\sum_{\underline{k}\in\mathcal{P}(\gamma)}\prod_{\alpha\in\Phi_+}^{<} X_{k_\alpha}^{(\alpha)}\right). \tag{3.1}$$

3.2. **Total order.** Fix a total order $\prec$ on Φ_+ satisfying the following condition: there exist two simple roots α_i, α_j such that

- $\alpha_i + \alpha_j \in \Phi_+$ and $\alpha_i + \ell\alpha_j \notin \Phi_+$ for every $\ell\in\mathbb{Z}_{\geq 2}$.
- $\alpha_i \prec \alpha_i + \alpha_j \prec \alpha_j$.

Remark. Recall that a total order $<$ on Φ_+ is said to be *convex* (or *normal*) if for every $\alpha, \beta\in\Phi_+$ such that $\gamma = \alpha+\beta\in\Phi_+$, either $\alpha < \gamma < \beta$, or $\beta < \gamma < \alpha$. Thus, the order $\prec$ considered above could be any convex ordering, if the root system is not B_2 or G_2. In the $\mathsf{B}_2, \mathsf{G}_2$ cases, we can take, for instance, the following convex order, where α_1 is the short simple root:

(B_2) $\alpha_1 \prec 2\alpha_1+\alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_2$.
(G_2) $\alpha_1 \prec 3\alpha_1+\alpha_2 \prec 2\alpha_1+\alpha_2 \prec 3\alpha_1+2\alpha_2 \prec \alpha_1+\alpha_2 \prec \alpha_2$.

3.3. **Block elements.** Assume that we are given an arbitrary collection $\{F_n^{(\alpha)}(s) : \alpha\in\Phi_+, n\in\mathbb{Z}_{\geq 0}\}$, where

$$F_n^{(\alpha)}(s) \in \left(Y_\hbar^{\leq}(\mathfrak{g})_{-n\alpha}\otimes Y_\hbar^{\geq}(\mathfrak{g})_{n\alpha}\right)[\![s^{-1}]\!]$$

and $F_0^{(\alpha)}(s) = 1 \otimes 1$. In accordance with (3.1), define:

$$F_\gamma(s) = \sum_{\underline{k}\in\mathcal{P}(\gamma)} \prod_{\alpha\in\Phi^+}^{\prec} F_{k_\alpha}^{(\alpha)}(s) \ , \qquad F_\gamma(s) \in \left(Y_\hbar^{\leq}(\mathfrak{g})_{-\gamma} \otimes Y_\hbar^{\geq}(\mathfrak{g})_\gamma\right) [\![s^{-1}]\!]. \tag{3.2}$$

3.4. **Main theorem.** To state our main theorem, let us introduce some auxiliary terminology. Let $\mathcal{J}(s)$ be an arbitrary weight zero element of $(Y_\hbar^{\leq}(\mathfrak{g})\otimes Y_\hbar^{\geq}(\mathfrak{g}))[\![s^{-1}]\!]$, and write $\mathcal{J}(s) = \sum_{\gamma\in\mathsf{Q}_+} \mathcal{J}_\gamma(s)$ with

$$\mathcal{J}_\gamma(s) \in (Y_\hbar^{\leq}(\mathfrak{g})_{-\gamma} \otimes Y_\hbar^{\geq}(\mathfrak{g})_\gamma)[\![s^{-1}]\!].$$

Then, for a fixed $\alpha \in \Phi_+$, we say that $\mathcal{J}(s)$ has $\mathbb{Z}\alpha$-*support* if $\mathcal{J}_\gamma(s) = 0$ for $\gamma \notin \mathbb{Z}\alpha$ and, in addition, $\mathcal{J}_0(s) = 1 \otimes 1$.

Theorem. *For any total ordering $\prec$ as in §3.2, and $\{F_n^{(\alpha)}(s)\}_{\alpha\in\Phi_+, n\in\mathbb{Z}_{\geq 0}}$ as in §3.3, the elements $\{F_\gamma(s)\}_{\gamma\in\mathsf{Q}_+}$ do not satisfy the intertwining equation (2.7). Consequently, $\mathcal{R}^-(s)$ does not admit a factorization of the form*

$$\mathcal{R}^-(s) = \prod_{\alpha\in\Phi_+}^{\prec} \mathcal{J}^{(\alpha)}(s),$$

where, for each $\alpha \in \Phi_+$, $\mathcal{J}^{(\alpha)}(s)$ is a weight zero element of $(Y_\hbar^{\leq}(\mathfrak{g}) \otimes Y_\hbar^{\geq}(\mathfrak{g}))[\![s^{-1}]\!]$ with $\mathbb{Z}\alpha$-support.

Proof. The second assertion is an immediate consequence of the first, given the symbolic identity (3.1). The proof of the first statement is by contradiction, which we split into three elementary steps whose details are worked out in §3.5–3.7 below. The struture of our argument is as follows. Assume that (2.7) holds for $\{F_\gamma(s)\}$.

(1) For each simple root α_k, $\{F_n^{(\alpha_k)}(s)\}_{n\in\mathbb{Z}_{\geq 0}}$ can be explicitly computed, as in rank 1 case. We only need the first two terms of $F_1^{(\alpha_k)}(s)$, which are obtained in §3.5.

(2) In §3.6 we show that $F_n^{(\alpha_j)}(s)$ commutes with $x_{ij,0}^- \otimes x_{ij,0}^+$. Here α_i, α_j are the simple roots satisfying conditions imposed on $\prec$ in §3.2 above, and $x_{ij,0}^\pm$ are the root vectors corresponding to $\alpha_i + \alpha_j$.

(3) A simple rank 2 computation is then carried out in §3.7 to show that $x_{ij,0}^- \otimes x_{ij,0}^+$ does not commute with $F_1^{(\alpha_j)}(s)$, thus obtaining the desired contradiction. □

Remark. In the setup of §3.3, we did not assume that $\sum_{\gamma\in\mathsf{Q}_+} F_\gamma(s)$ exists as an element of $Y_\hbar(\mathfrak{g})^{\otimes 2}[\![s^{-1}]\!]$. This, in fact, is a consequence of (2.7), as observed in [8, Eqns. (4.6)–(4.7)]. Namely, if $\{A_\gamma(s)\}_{\gamma\in\mathsf{Q}_+}$ solve (2.7) and $A_0(s) = 1 \otimes 1$, then $A_\gamma(s)$ is divisible by $s^{-\nu(\gamma)}$, where

$$\nu(\gamma) = \min\{k \in \mathbb{Z}_{\geq 0} \mid \gamma = \beta_1 + \cdots + \beta_k, \text{ where } \beta_1, \ldots, \beta_k \in \Phi_+\}.$$

For an interpretation of this fact in terms of dual bases and the Yangian double, we refer the reader to Corollary 9.9 of [24].

3.5. **Simple roots.** Let $\alpha_k \in \Phi_+$ be a simple root. Then the defining equation (3.2) for $F_\gamma(s)$ implies that $F_{n\alpha_k}(s) = F_n^{(\alpha_k)}(s)$, for every $n \in \mathbb{Z}_{\geq 0}$. The intertwining equation (2.7) with $\gamma = n\alpha_k$ then becomes:

$$\mathcal{D}(h;s) \cdot F_n^{(\alpha_k)}(s) = -\hbar\alpha_k(h) F_{n-1}^{(\alpha_k)}(s) x_{k,0}^- \otimes x_{k,0}^+. \tag{3.3}$$

Remark. We would like to point out that this equation is precisely the one defining $\mathcal{R}^-(s)$ for $\mathfrak{sl}_2$ whose explicit formula is given in [8, Thm. 5.5]. For the purposes of our proof, it is enough to know the coefficient of s^{-2} in $F_1^{(\alpha_k)}(s)$. We include this easy computation below, for completeness.

Using $F_0^{(\alpha_k)}(s) = 1 \otimes 1$, the $n = 1$ case of equation (3.3) is the following:

$$[\mathrm{T}(h) \otimes 1 + 1 \otimes \mathrm{T}(h) + sh \otimes 1, F_1^{(\alpha_k)}(s)] = -\hbar\alpha_k(h) x_{k,0}^- \otimes x_{k,0}^+.$$

Comparing coefficients of s^0 and s^{-1}, and using the commutation relation (2.3), we obtain

$$F_1^{(\alpha_k)}(s) = \hbar s^{-1} \left(x_{k,0}^- \otimes x_{k,0}^+ + (-x_{k,1}^- \otimes x_{k,0}^+ + x_{k,0}^- \otimes x_{k,1}^+) s^{-1} + \dots \right). \tag{3.4}$$

In fact, though not needed in the present article, it follows easily from (2.3) and the above relation for $F_1^{(\alpha_k)}(s)$ that

$$F_1^{(\alpha_k)}(s) = \sum_{n \geq 0} x_{k,n}^- \otimes \partial_s^{(n)} x_k^+(s),$$

where $x_k^+(s) = \hbar \sum_{r \geq 0} x_{k,r}^+ s^{-r-1}$ and $\partial_s^{(n)} = \frac{1}{n!}\partial_s^n$, with ∂_s the partial derivative operator with respect to s. This formula may be found in the proof [24, Prop. 7.1], above Remark 7.2 therein.

3.6. **Commutativity relation.** Now, let $\alpha_i, \alpha_j \in \Phi_+$ be the simple roots for which the condition imposed in §3.2 holds. For notational convenience, we will write $\alpha_{ij} = \alpha_i + \alpha_j \in \Phi_+$. Similarly, $x_{ij,0}^\pm$ will denote the root vectors corresponding to the positive root α_{ij}. We will also abbreviate

$$\mathsf{r}_a^-(h) = -\hbar\alpha_a(h) x_{a,0}^- \otimes x_{a,0}^+, \text{ for } a = i, j \text{ or } a = ij.$$

Equations (2.7) and (3.2) for $\gamma = \alpha_i + \ell\alpha_j \in \mathsf{Q}_+$ take the following form:

$$\mathcal{D}(h;s) \cdot F_{\alpha_i + \ell\alpha_j}(s) = F_{\ell\alpha_j}(s)\mathsf{r}_i^-(h) + F_{(\ell-1)\alpha_j}\mathsf{r}_{ij}^-(h) + F_{\alpha_i+(\ell-1)\alpha_j}(s)\mathsf{r}_j^-(h), \tag{3.5}$$

$$F_{\alpha_i+\ell\alpha_j}(s) = F_1^{(\alpha_{ij})}(s) F_{\ell-1}^{(\alpha_j)}(s) + F_1^{(\alpha_i)}(s) F_\ell^{(\alpha_j)}(s). \tag{3.6}$$

Combining these two equations, using the fact that $\mathcal{D}(h;s)$ is a derivation, and (3.3) for $n = 1$ above, we get:

$$\left(\mathcal{D}(h;s) \cdot F_1^{(\alpha_{ij})}(s)\right) F_{\ell-1}^{(\alpha_j)}(s) = F_{\ell-1}^{(\alpha_j)}(s)\mathsf{r}_{ij}^-(h) + \left[F_\ell^{(\alpha_j)}(s), \mathsf{r}_i^-(h)\right].$$

Now choose $h \in \alpha_i^\perp \subset \mathfrak{h}$ so that $\mathsf{r}_i^-(h) = 0$, and take $\ell = 1$ to get $\mathcal{D}(h;s) \cdot F_1^{(\alpha_{ij})}(s) = \mathsf{r}_{ij}^-(h)$. Substitute this back into the equation above to obtain:

$$\left[\mathsf{r}_{ij}^-(h), F_{\ell-1}^{(\alpha_j)}(s)\right] = 0, \ \forall\ h \in \alpha_i^\perp,\ \ell \in \mathbb{Z}_{\geq 1}.$$

Take $h \in \alpha_i^\perp$ such that $\alpha_j(h) \neq 0$ and therefore, $\mathsf{r}_{ij}^-(h)$ is a non–zero scalar multiple of $x_{ij,0}^- \otimes x_{ij,0}^+$. We get:

$$\left[x_{ij,0}^- \otimes x_{ij,0}^+, F_n^{(\alpha_j)}(s)\right] = 0, \ \forall\ n \in \mathbb{Z}_{\geq 0}.$$

Take $n = 1$ and the coefficient of s^{-2} using (3.4) to get:

$$\left[x_{ij,0}^- \otimes x_{ij,0}^+ \ , \ x_{j,0}^- \otimes x_{j,1}^+ - x_{j,1}^- \otimes x_{j,0}^+\right] = 0. \tag{3.7}$$

3.7. **Rank 2 computation.** Let us now restrict our attention to the rank 2 subsystem generated by α_i and α_j. For notational simplicity, we will replace i, j by $1, 2$ and our 2×2 Cartan matrix is of the following form:

$$\mathbf{A} = \begin{bmatrix} 2 & -p \\ -1 & 2 \end{bmatrix}, \quad p = 1, 2 \text{ or } 3.$$

In this case, $d_1 = 1$ and $d_2 = p$. Let us write $\alpha_3 = \alpha_1 + \alpha_2$ and take the following root vectors $x_{3,0}^\pm \in \mathfrak{g}_{\pm\alpha_3}$, so that $(x_{3,0}^+, x_{3,0}^-) = 1$:

$$x_{3,0}^- = [x_{2,0}^-, x_{1,0}^-] \qquad \text{and} \qquad x_{3,0}^+ = \frac{1}{p}[x_{1,0}^+, x_{2,0}^+].$$

Note that, by the Serre relations, $x_{3,0}^\pm$ commutes with $x_{2,0}^\pm$.

Claim. The defining relations of $Y_\hbar(\mathfrak{g})$ imply that

$$[x_{3,0}^- \otimes x_{3,0}^+, x_{2,0}^- \otimes x_{2,1}^+ - x_{2,1}^- \otimes x_{2,0}^+] = -2p\hbar x_{3,0}^- x_{2,0}^- \otimes x_{3,0}^+ x_{2,0}^+ \,, \tag{3.8}$$

which contradicts (3.7).

Proof of the claim. Using (Y4) and (Y6), we obtain the two identities

$$[x_{2,1}^\pm, x_{1,0}^\pm] = [x_{2,0}^\pm, x_{1,1}^\pm] \mp \frac{p\hbar}{2}(x_{2,0}^\pm x_{1,0}^\pm + x_{1,0}^\pm x_{2,0}^\pm),$$
$$[x_{2,1}^\pm, [x_{2,0}^\pm, x_{1,0}^\pm]] = -[x_{2,0}^\pm, [x_{2,1}^\pm, x_{1,0}^\pm]].$$

Combining these two equations, and using the fact that, by the Serre relations, $x_{2,0}^\pm$ commutes with $[x_{2,0}^\pm, x_{1,k}^\pm]$ for all $k \geq 0$, we obtain

$$[x_{2,1}^\pm, [x_{2,0}^\pm, x_{1,0}^\pm]] = \pm\hbar p x_{2,0}^\pm [x_{2,0}^\pm, x_{1,0}^\pm].$$

Now we can carry out the following computations:

$$\begin{aligned}[x_{3,0}^+, x_{2,1}^+] &= \frac{1}{p}[[x_{1,0}^+, x_{2,0}^+], x_{2,1}^+] = \frac{1}{p}[x_{2,1}^+, [x_{2,0}^+, x_{1,0}^+]] = \hbar x_{2,0}^+ [x_{2,0}^+, x_{1,0}^+] \\ &= -p\hbar x_{2,0}^+ x_{3,0}^+,\end{aligned}$$

$$\begin{aligned}[x_{3,0}^-, x_{2,1}^-] &= [[x_{2,0}^-, x_{1,0}^-], x_{2,1}^-] = -[x_{2,1}^-, [x_{2,0}^-, x_{1,0}^-]] = p\hbar x_{2,0}^- [x_{2,0}^-, x_{1,0}^-] \\ &= p\hbar x_{2,0}^+ x_{3,0}^+.\end{aligned}$$

Hence, the left–hand side of (3.8) simplifies to

$$x_{3,0}^- x_{2,0}^- \otimes [x_{3,0}^+, x_{2,1}^+] - [x_{3,0}^-, x_{2,1}^-] \otimes x_{3,0}^+ x_{2,0}^+ = -2p\hbar x_{3,0}^- x_{2,0}^- \otimes x_{3,0}^+ x_{2,0}^+$$

as claimed. □

4. Conclusions

In this last section we discuss the Khoroshkin–Tolstoy conjecture that motivated Theorem 3.4. Moreover, we briefly review the well–known factorization formulae for the universal R–matrices of Drinfeld–Jimbo quantum groups associated to finite or affine Lie algebras. We then observe that our result does not exclude the existence of a factorization formula for the Yangian's R–matrix of the latter kind, which therefore remains an open and challenging problem.

4.1. **The Khoroshkin–Tolstoy conjecture.** Let us now explain how Theorem 3.4 relates to the conjectural formula [11, Eqn. (5.43)]. In this section alone, we assume that $\hbar$ is a formal variable, following the conventions of [24]; see in particular §1.4 therein. We shall return to the case where $\hbar \in \mathbb{C}^\times$ at the end of the section.

Let $\mathrm{D}Y_\hbar(\mathfrak{g})$ denote the Yangian double, as defined [11, Defn. 2.1] and [24, Defn. 4.1]. That is, $\mathrm{D}Y_\hbar(\mathfrak{g})$ is the unital, associative $\mathbb{C}[\![\hbar]\!]$-algebra topologically generated by $\{\xi_{i,r}, x_{i,r}^\pm\}_{i\in\mathbf{I},r\in\mathbb{Z}}$, subject to the same family of relations given in §2.2 above, where the subscripts r, s now range over $\mathbb{Z}$. It was proven in Theorem 8.4 of [24] that $\mathrm{D}Y_\hbar(\mathfrak{g})$ provides a realization of the (restricted) quantum double of the Yangian $Y_\hbar(\mathfrak{g})$, as conjectured in [11, §2]. Let $\mathcal{R}$ be the universal R–matrix of $\mathrm{D}Y_\hbar(\mathfrak{g})$, and let $\mathcal{R}^-$ be the lower triangular factor in its Gaussian decomposition, as first considered in [11, §5]. We refer the reader to [24, §9.3] for the precise definitions of these elements.

Following the conventions of [11, §5], let $\Sigma_- = \{-\gamma + k\delta : \gamma \in \Phi_+, k \geq 0\}$, where δ is the imaginary root of the (affine) root system of $\widehat{\mathfrak{g}}$. Choose an arbitrary total order $<$ on Σ_- satisfying the following two conditions:

- if $\alpha, \beta, \gamma = \alpha + \beta \in \Sigma_-$, then either $\alpha < \gamma < \beta$ or $\beta < \gamma < \alpha$;
- $-\gamma + \ell\delta < -\gamma + k\delta$, for $k < \ell$.

Given such a total order $<$, it was conjectured in [11] (see [11, Eqn. (5.43)]) that $\mathcal{R}^-$ admits a multiplicative factorization

$$\mathcal{R}^- = \prod_{\beta\in\Sigma_-}^{<} \exp\left(-\hbar\Omega_\beta\right), \tag{4.1}$$

where Ω_β is an explicitly defined simple tensor $\Omega_\beta = \omega_\beta^- \otimes \omega_\beta^+$ with $\omega_\beta^- \in Y_\hbar^-(\mathfrak{g}) \subset \mathrm{D}Y_\hbar(\mathfrak{g})$, ω_β^+ an element of the dual Yangian $Y_\hbar^-(\mathfrak{g})^\star \subset \mathrm{D}Y_\hbar(\mathfrak{g})$ (see [24, §6.5]) and, if $\beta = -\gamma + k\delta$, the factors ω_β^- and ω_β^+ have degree $(k, -\gamma)$ and $(-k-1, \gamma)$ with respect to the standard $\mathbb{Z} \times \mathsf{Q}$ grading on $\mathrm{D}Y_\hbar(\mathfrak{g})$, respectively. This conjecture extends Lemma 5.1 of [11], which established (4.1) for $\mathfrak{g} = \mathfrak{sl}_2$. In this case one has $\Omega_{-\alpha_i+k\delta} = x_{i,k}^- \otimes x_{i,-k-1}^+$, where $\mathbf{I} = \{i\}$.

In order to relate this conjectural expression to the results of Section 3 above, we make the following choice for the total order on Σ_-, where we once again assume $\mathfrak{g} \ncong \mathfrak{sl}_2$. If $\mathfrak{g}$ is not of type B_2 or G_2, we fix $<_1$ to be an arbitrary convex order on the set Φ_+. In the B_2 and G_2 cases, we take $<_1$ to be the order given in §3.2. We then extend $<_1$ to Σ_- as follows:

$$-\alpha + \ell\delta <_1 -\gamma + k\delta, \text{ if either } \alpha <_1 \gamma; \text{ or } \alpha = \gamma \text{ and } k < \ell.$$

We remark that this is the same ordering utilized in [21, Thm. 6.2] for $\mathfrak{g} = \mathfrak{sl}_3$. With this choice of ordering, equation (4.1) becomes:

$$\mathcal{R}^- = \prod_{\alpha\in\Phi_+}^{<_1} \left(\prod_{k\geq 0}^{\leftarrow} \exp\left(-\hbar\Omega_{-\alpha+k\delta}\right) \right) = \prod_{\alpha\in\Phi_+}^{<_1} \mathcal{J}^{(\alpha)}, \tag{4.2}$$

where $\mathcal{J}^{(\alpha)}$ is defined to be product over $k \geq 0$ that appears within the parentheses on the right-hand side of the first equality. To pass from this expression back to the setting of Section 3, we recall that the shift homomorphism τ_s from Section 2.7 extends to a ($\mathbb{Z}\times\mathsf{Q}$-graded) algebra homomorphism Φ_s from $\mathrm{D}Y_\hbar(\mathfrak{g})$ into an algebra contained in the space $Y_\hbar(\mathfrak{g})[\![s^{\pm 1}]\!]$; see [23, Thm. 4.3] and [24, Thm. 4.6]. In particular, Φ_s satisfies $\Phi_s(\omega^+_{-\alpha+k\delta}) \in s^{-k-1}Y_\hbar^+(\mathfrak{g})_\alpha[\![s^{-1}]\!]$ for all $\alpha\in\Phi_+$ and $k\geq 0$.

In Theorem 9.6 of [24], it is shown that $(\mathbf{1}\otimes\Phi_{-s})(\mathcal{R}^-)$ coincides with $\mathcal{R}^-(s)$; see also [22] together with [24, §1.3]. Thus, (4.2) implies that $\mathcal{R}^-(s)$ can be written as the ordered product

$$\mathcal{R}^-(s) = \prod_{\alpha\in\Phi_+}^{<_1} \mathcal{J}^{(\alpha)}(s),$$

where, for each $\alpha\in\Phi_+$, $\mathcal{J}^{(\alpha)}(s) := (\mathbf{1}\otimes\Phi_{-s})(\mathcal{J}^{(\alpha)})$. Note that, in the terminology of Section 3.4, $\mathcal{J}^{(\alpha)}(s)$ is necessarily a weight zero element of $(Y_\hbar^-(\mathfrak{g})\otimes Y_\hbar^+(\mathfrak{g}))[\![s^{-1}]\!]$ with $\mathbb{Z}\alpha$-support. Here we emphasize that, by Proposition A.1 of [24], the above factorization must hold regardless of whether $\hbar$ is viewed as a formal variable or an arbitrary non-zero complex number, as in Sections 2 and 3 above. However, this is impossible as shown in Theorem 3.4, proving that (4.1) is false.

4.2. **R–matrices of quantum affine algebras.** For Drinfeld–Jimbo quantum groups, the R–matrix can be expressed as an ordered product of R–matrices of $U_q(\mathfrak{sl}_2)$ associated to positive roots [12, 15, 16]. The root subalgebras are constructed relying on Lusztig's inner braid group action [17, 18], while the factorization of the R–matrix is a consequence of a coproduct formula for the quantum Weyl group operators [12, 18, 19, 20].

For quantum affine algebras, this product ranges over the set of positive affine roots $\widehat{\Phi}_+$, see *e.g.*, [13, 1, 2] and [3, Thm. 2]. Let $\mathbf{R}$ the universal R–matrix of $U_q(\widehat{\mathfrak{g}})$. Then, up to a Cartan correction, which for simplicity is suppressed in the current discussion, one has

$$\mathbf{R} = \prod_{\alpha\in\widehat{\Phi}_+}^{\prec'} \exp_{q_\alpha}\left((q_\alpha - q_\alpha^{-1})E_\alpha\otimes F_\alpha\right). \tag{4.3}$$

For full details about the total order $\prec'$ on $\widehat{\Phi}_+$, we refer the reader to [2, 3]. Here, we only need a few of its salient features. Recall that the set of affine positive roots is given by $\widehat{\Phi}_+ = \widehat{\Phi}_+^{\mathrm{re}} \sqcup \widehat{\Phi}_+^{\mathrm{im}}$, where $\widehat{\Phi}_+^{\mathrm{im}} = \mathbb{Z}_{\geq 1}\delta$ and $\widehat{\Phi}_+^{\mathrm{re}} = \widehat{\Phi}_{+,+}^{\mathrm{re}} \cup \widehat{\Phi}_{+,-}^{\mathrm{re}}$ with

$$\widehat{\Phi}_{+,+}^{\mathrm{re}} = \{\alpha + k\delta : \alpha\in\Phi_+, k\in\mathbb{Z}_{\geq 0}\} \quad \text{and} \quad \widehat{\Phi}_{+,-}^{\mathrm{re}} = \{-\alpha+\ell\delta : \alpha\in\Phi_+, \ell\in\mathbb{Z}_{\geq 1}\}.$$

Then, $\prec'$ has the following properties.

(1) $x \prec' y \prec' z$ for $x\in\widehat{\Phi}_{+,+}^{\mathrm{re}}$, $y\in\widehat{\Phi}_+^{\mathrm{im}}$ and $z\in\widehat{\Phi}_{+,-}^{\mathrm{re}}$.

(2) The total order $\prec'$ restricted to $\widehat{\Phi}^{\mathrm{re}}_{+,-}$ is convex.

(3) For any $z_1, z_2 \in \widehat{\Phi}^{\mathrm{re}}_{+,-}$, the interval $\{z : z_1 \prec' z \prec' z_2\}$ is finite.

Note that the property (3) does not hold for the (general) total orders considered in [11], the ones featuring in Theorem 3.4 and ultimately in the equation (4.2).

By property (1) and equation (4.3), the universal R–matrix $\mathbf{R}$ factors into three components, which we denote by $\mathbf{R}^+$, $\mathbf{R}^0$ and $\mathbf{R}^-$ respectively. By the results of [7], we expect that an analogue of Theorem 3.4 holds for $\mathbf{R}^-$, *i.e.*, $\mathbf{R}^-$ cannot be expressed as an ordered product of the form (4.2).

Conversely, our results do not exclude the possibility of a factorization of $\boldsymbol{\mathcal{R}}^-$ or $\mathcal{R}^-(s)$ similar to that of $\mathbf{R}^-$, *i.e.*, a factorization of the form (4.3) for an order satisfying the properties above. To the best of our knowledge, no such result in the case of Yangians or Yangian doubles is known, and remains an interesting and challenging open problem.

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