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# Advances in Water Resources

## Relaxation-induced flow in a smooth fracture for Ellis rheology

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<b>Abstract:</b>	<p>Hydraulic fracturing is a process aimed at improving the productivity of oil, gas or geothermal reservoirs. During hydrofracturing, backflow follows injection and represents the second phase of the process, when part of the fracturing fluid returns from fractures to well, and from well to surface. A conceptual model is presented to grasp the essential features of the phenomenon, conceiving the draining subsurface domain as a planar and rigid fracture. Backflow against an outlet pressure in the injection well is induced by the relaxation of the fracture wall, exerting a force on the fluid proportional to <math>h^\lambda</math>, with <math>h</math> the time-variable aperture and <math>\lambda</math> a non-negative exponent; an overload on the fracture may contribute to slowing or accelerating the closure process. The fluid rheology is described by the three-parameter Ellis constitutive equation, well representing the shear-thinning rheology typical of hydrofracturing fluids and coupling Newtonian and power-law behaviour. The interplay between these tendencies is modulated by a dimensionless number <math>N</math> encapsulating most problem parameters; the range of variation of <math>N</math> is discussed and found to vary around unity. The time-variable aperture and discharge rate, the space-time variable pressure field, and the time to drain a specified fraction of the fracture volume are derived as functions of geometry (length and initial aperture), wall elastic parameters, fluid properties, outlet pressure <math>p_e</math> and overload <math>f_0</math>. The late-time behaviour of the system is practically independent from rheology as the Newtonian nature of the fluid prevails at low shear stress. In particular, aperture and discharge scale asymptotically with time as <math>t^{-1/(\lambda+2)}</math> and <math>t^{-1/(\lambda+3)}</math> for <math>p_{e-f_0}=0</math>; else, the aperture tends to a constant, residual value proportional to <math>(p_{e-f_0})^\lambda</math>. A case study with equally spaced fractures adopting realistic geometric, mechanical and rheological parameters is examined: two fluids normally used in fracking technology show completely different behaviours, with backflow dynamics and drainage times initially not dissimilar, later varying by orders of magnitude.</p>
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<b>Response to Reviewers:</b>	Dear Editor: We would like to thank you, the anonymous Reviewer and the Associate Editor for the positive assessment of our work and for the suggestions, which have helped us to further improve the presentation. Our detailed response to the Reviewer is given in a separate file. Her/his comments are reproduced in italic. The corresponding changes in the manuscript are typeset in blue. Sincerely yours, Valentina Ciriello Alessandro Lenci Sandro Longo Vittorio Di Federico

# 1 Relaxation-induced flow in a smooth fracture for Ellis 2 rheology

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## 7 **Abstract**

8 Hydraulic fracturing is a process aimed at improving the productivity of oil,  
9 gas or geothermal reservoirs. During hydrofracturing, backflow follows injec-  
10 tion and represents the second phase of the process, when part of the fractur-  
11 ing fluid returns from fractures to well, and from well to surface. A concep-  
12 tual model is presented to grasp the essential features of the phenomenon,  
13 conceiving the draining subsurface domain as a planar and rigid fracture.  
14 Backflow against an outlet pressure in the injection well is induced by the  
15 relaxation of the fracture wall, exerting a force on the fluid proportional to  
16  $h^\lambda$ , with  $h$  the time-variable aperture and  $\lambda$  a non-negative exponent; an  
17 overload on the fracture may contribute to slowing or accelerating the clo-  
18 sure process. The fluid rheology is described by the three-parameter Ellis  
19 constitutive equation, well representing the shear-thinning rheology typical  
20 of hydrofracturing fluids and coupling Newtonian and power-law behaviour.  
21 The interplay between these tendencies is modulated by a dimensionless num-  
22 ber  $N$  encapsulating most problem parameters; the range of variation of  $N$   
23 is discussed and found to vary around unity. The time-variable aperture  
24 and discharge rate, the space-time variable pressure field, and the time to  
25 drain a specified fraction of the fracture volume are derived as functions of  
26 geometry (length and initial aperture), wall elastic parameters, fluid prop-  
27 erties, outlet pressure  $p_e$  and overload  $f_0$ . The late-time behaviour of the  
28 system is practically independent from rheology as the Newtonian nature of  
29 the fluid prevails at low shear stress. In particular, aperture and discharge  
30 scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  $p_e - f_0 = 0$ ; else,  
31 the aperture tends to a constant, residual value proportional to  $(p_e - f_0)^\lambda$ .  
32 A case study with equally spaced fractures adopting realistic geometric, me-

33 chanical and rheological parameters is examined: two fluids normally used  
34 in fracking technology show completely different behaviours, with backflow  
35 dynamics and drainage times initially not dissimilar, later varying by orders  
36 of magnitude.

37 *Keywords:*

38 Hydraulic fracturing, Non-Newtonian, Ellis rheology, elastic wall, backflow

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## 39 1. Introduction

40 Hydraulic fracturing is a process aimed at improving the productivity of  
41 oil, gas or geothermal reservoirs. Analysis of the different phases of hydraulic  
42 fracturing is of particular modeling and experimental interest [e.g. 1, 2].

43 An understanding of fractured media flow induced by the relaxation of  
44 elastic fracture walls is crucial in modeling fracturing fluid backflow, a com-  
45 plicated phenomenon involving hydrodynamic, mechanical and chemical pro-  
46 cesses. Backflow is typically the final phase of the hydraulic fracturing pro-  
47 cess: in the first one, fracturing fluid is injected at high pressure in a rock  
48 mass, forming new fractures and enlarging existing ones; in the second phase,  
49 proppant is introduced in the subsurface environment to prop fractures open;  
50 then when the injection ceases, the pressure drops, existing and new fractures  
51 tend to close, and a portion of the injected fracturing fluid, often mixed  
52 with proppant [3], flows back towards the injection well and interact with  
53 the relaxing walls of the fractures. As the retention of fracturing fluid in  
54 the fracture network impairs the fracture conductivity reducing the wellbore  
55 productivity [4], and favours migration in the subsurface environment along  
56 different pathways [5], it is of utmost interest to optimize the amount of fluid  
57 recovered, irrespective of the reservoir product, be it oil [6], gas [7] or heat  
58 [8].

59 The scientific literature offers two main approaches to modeling backflow:  
60 (i) detailed numerical simulations involving single fractures [9], fracture net-  
61 works [10] or dual or triple porosity models [11], or (ii) conceptual models  
62 capturing the main features of the interaction between fracture flow and  
63 wall relaxation [12], including the effects of branching networks described at  
64 different degrees of complexity [13, 14]. A recent addition to the modeling  
65 effort is the influence of fluid rheology, following the notion that the backflow  
66 fluid is non-Newtonian in the widest sense [15], as not only the relationship  
67 between shear stress and shear rate is nonlinear, but also exhibits normal

68 stress and temperature dependency, as well as viscoelasticity, thixotropy,  
69 and nonzero yield stress [16]. At the same time, non-Newtonian fluids allow  
70 achieving several engineering objectives, such as (i) minimize the pressure-  
71 drop in the entire process; (ii) carry suspended proppant; (iii) minimize the  
72 leak-off within the formation; (iv) adapt their characteristics to different en-  
73 vironments in terms of temperature and chemical composition; and (v) flow  
74 back easily towards the wellbore. Given their versatility and economic value,  
75 these fluids are typically treated for reuse once recovered, removing contam-  
76 inants they may have transported to the surface [17]. The recovery ratios of  
77 backflow fluid vary between 2% and 48% according to Ipatova and Chuprakov  
78 [18], with considerable economic value.

79 Modeling non Newtonian backflow is in its early stage, in variance with  
80 the injection and fracture formation stage, for which several conceptualiza-  
81 tions and models are available: see Detournay [19] for a review and the recent  
82 work by Wrobel [20] comparing different rheological models for fracturing  
83 fluids. To the best of our knowledge, only Chiapponi et al. [21] considered  
84 non-Newtonian fluids in the context of backflow modeling: these authors  
85 examined flow of a power-law fluid towards a wellbore in a single fracture  
86 of annular geometry, supporting their theoretical findings with laboratory  
87 experiments. The present paper develops the analysis of non-Newtonian  
88 backflow for a smooth fracture, common in field applications [22], and adds  
89 realism by employing a three-parameter Ellis model, that well represents the  
90 rheology of hydrofracturing [23] and drilling fluids [24]. The Ellis model tends  
91 to Newtonian for low shear rates, to power-law for high shear rates and al-  
92 lows avoiding the unphysical effect of infinite apparent viscosity at zero shear  
93 rate that is typical of the power-law model [25]. We note in passing that our  
94 results are of a general nature for Newtonian pressurized flow in ducts of  
95 variable width and may be of interest for, and be applied also to, deformable  
96 microfluidic [26] and biological [27] systems.

97 The plan of the paper is as follows. Section 2 formulates the problem  
98 of relaxation-induced backflow of an Ellis fluid in a fracture with nonlin-  
99 ear wall reaction and subject to overload. Numerical results obtained are  
100 presented and discussed in Section 3 as a function of dimensionless groups  
101 characterizing the system: the indicial exponent  $\alpha$  quantifying the degree  
102 of shear-thinning behaviour of the Ellis fluid, the non-negative exponent  $\lambda$   
103 modulating the fracture wall reaction, and a further group  $N$  encapsulating  
104 most problem parameters. Section 4 illustrates an hypothetical case study  
105 adopting realistic geometric and mechanical parameters and two real hy-

106 drofracturing fluids described by the Ellis model. Section 5 reports the main  
107 conclusions and perspectives for future work. In Appendix A the special case  
108 of a Newtonian fluid is examined, obtaining results that generalize those of  
109 Dana et al. [13] to a nonlinear wall reaction, while Appendix B presents  
110 an alternative expression for the dimensionless number  $N$ , shown to be a  
111 combination of well-known dimensionless groups in fluid mechanics.

## 112 2. Material and methods

### 113 2.1. Problem statement

114 A rock fracture produced by hydrofracturing, though of irregular geom-  
115 etry, is often conceptualized for modeling purposes as a 3-D space of length  
116  $L$ , width  $W$ , and aperture  $h$  between two parallel walls [28]; the Cartesian  
117 coordinate system  $x, y, z$  is illustrated in Figure 1 and the fracture is subject  
118 to a pressure gradient  $\nabla p' \equiv (\partial p'/\partial x, 0, 0)$  in the  $x$  direction. In horizontal  
119 fractures, the additional gravity-induced pressure gradient is perpendicular  
120 the flow plane and has no effect on the flow field. If the  $(x, y)$  plane is not  
121 horizontal, the  $z$  direction perpendicular to the walls is not vertical and grav-  
122 ity effects can be included in a reduced pressure term  $p$ , thus leading to a  
123 mathematical treatment with no gravity term to consider. For instance, for  
124 the Figure 3 below representing multiple vertical fractures backflowing to an  
125 horizontal well, the reduced pressure  $p$  is equal to  $p = p' + \rho gy$ .

126 The walls are taken to be rigid, so that the aperture  $h(t)$  is solely a  
127 function of time, and the deformation is concentrated for mathematical con-  
128 venience in the upper wall, that behaves as a nonlinear elastic foundation  
129 exerting a reaction on the fluid. At  $t = 0$  the relaxation of the wall in-  
130 duces a backflow in the negative  $x$  direction, and the fracture begins to drain  
131 subject to a constant outlet pressure  $p_e$  at  $x = 0$  and to a no-flow bound-  
132 ary condition at the upstream end  $x = L$ . Three further hypotheses are  
133 adopted: i) the flow is quasi-steady, allowing to neglect the time derivative  
134 of the velocity in the momentum equation; ii) the fracture aspect ratio is  
135 small,  $h_0/L \ll 1$ , warranting the lubrication approximation, and iii) the flow  
136 is essentially one-dimensional along  $x$ ,  $L \gg W$ . The latter conceptualiza-  
137 tion is usually adopted in hydrogeology also when the two dimensions are  
138 comparable, as it is often the case for rock fractures [11].

139 The flowback fluid is taken to be incompressible of density  $\rho$ , non-Newtonian  
140 shear-thinning [15] and described by the Ellis three-parameter model [29].  
141 Under the above assumptions, the fluid undergoes simple shear flow in the

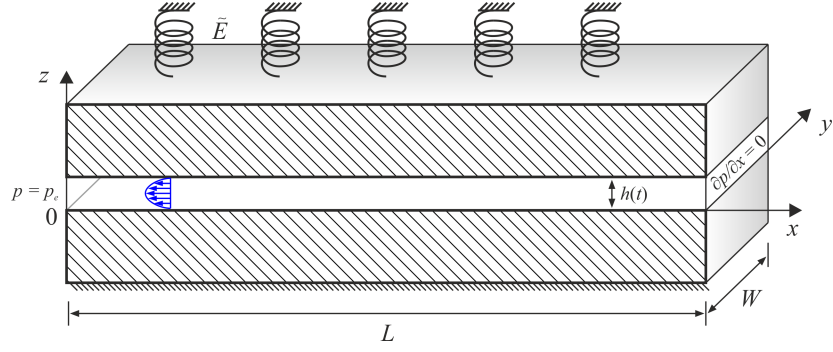


Figure 1: Layout of a plane fracture of variable uniform aperture  $h(t)$ .

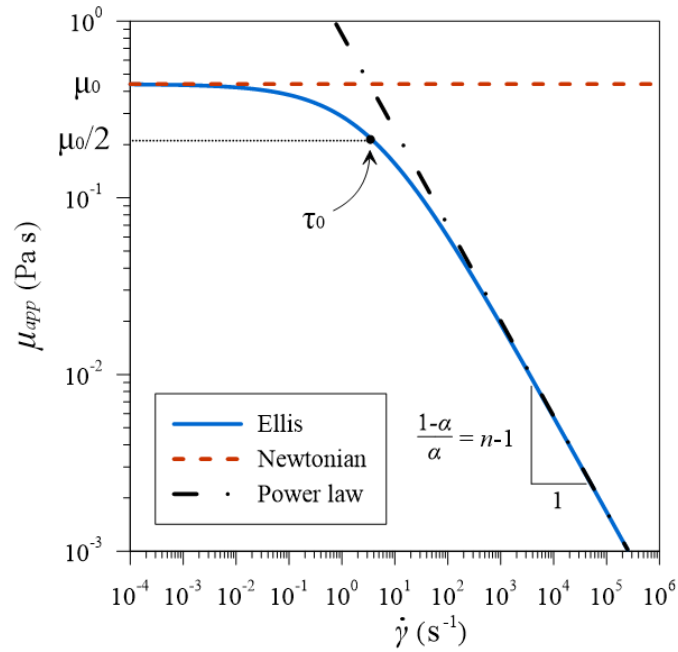


Figure 2: Apparent viscosity for three rheological models: Ellis (blue solid line) of parameters  $\mu_0$ ,  $\tau_0$ ,  $\alpha$ ; Newtonian (red dashed line) of viscosity  $\mu_0$ ; power-law (black dot-dashed line) of consistency index  $m$  and rheological index  $n$ . The comparison with the latter is drawn assuming:  $\alpha = 1/n$  and  $\tau_0 = (m/\mu_0^n)^{n/(1-n)}$ .

142  $x$  direction, and the Ellis rheology is described by the following relationship  
 143 between shear stress  $\tau_{zx}$  (hereinafter  $\tau$ ) and shear rate  $\dot{\gamma}_{zx}$  (hereinafter simply  
 144  $\dot{\gamma}$ )

$$\tau = \frac{\mu_0}{1 + (\tau/\tau_0)^{\alpha-1}} \dot{\gamma}; \quad \dot{\gamma} = \frac{\partial u}{\partial z}, \quad (1)$$

145 where  $u$  is the velocity in the  $x$  direction. The rheological law (1) features a  
 146 viscosity parameter  $\mu_0$ , a constant  $\tau_0$  defined as the shear stress corresponding  
 147 to apparent viscosity  $\mu_0/2$ , and an indicial parameter  $\alpha$ , typically larger than  
 148 one as the fluid is shear-thinning. For  $\alpha = 1$ , a pseudo-Newtonian behaviour  
 149 with dynamic viscosity  $\mu_0/2$  is recovered, see Figure 2 showing the apparent  
 150 viscosity  $\mu_{app} = \tau/\dot{\gamma}$  for the Ellis model compared to Newtonian and power-  
 151 law models. Newtonian behaviour in the form of a plateau for low shear rates  
 152 is also observed for  $\alpha \rightarrow 0$ . For high shear rates the behaviour is power-law,  
 153 and its two parameters can be determined from the Ellis model parameters,  
 154 see Appendix A in Balhoff and Thompson [28]; in particular, the rheological  
 155 index is  $n = 1/\alpha$  [30]. Note that when curve fitting is performed on real data,  
 156  $n$  and  $1/\alpha$  may significantly differ [23], as two different models are fitted  
 157 to the same data set. It is also seen that the Ellis model allows avoiding  
 158 the unphysical effect of infinite apparent viscosity at zero shear rate that is  
 159 typical of power-law fluids [25]. In the following, we will consider  $\alpha > 1$ ,  
 160 dealing with the case  $\alpha = 1$  in the Appendix, and the parameters  $\mu_0$  and  $\tau_0$   
 161 to be finite and positive. Couette-Poiseuille slit flow of an Ellis fluid under  
 162 a constant pressure gradient was studied extensively by Steller [31], listing  
 163 all combinations of parameters leading to Newtonian or pseudo-Newtonian  
 164 behaviour. In particular, the **negative** velocity  $u(z)$  under a **positive** reduced  
 165 pressure gradient  $\partial p/\partial x$  in the  $x$  direction is

$$u(z, t) = -\frac{1}{8\mu_0} \left[ h^2 - (2z - h)^2 \right] \frac{\partial p}{\partial x} + \frac{1}{(\alpha + 1)2^{\alpha+1}\mu_0\tau_0^{\alpha-1}} \left[ h^{\alpha+1} - |2z - h|^{\alpha+1} \right] \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}. \quad (2)$$

166 The corresponding average velocity  $\bar{u}$  and flow per unit width  $q_x$  in the  $x$   
 167 direction are

$$\bar{u} = -\frac{h^2}{12\mu_0} \frac{\partial p}{\partial x} - \frac{h^{\alpha+1}}{2^{\alpha+1}(\alpha + 2)\mu_0\tau_0^{\alpha-1}} \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}; \quad q_x = \bar{u}h. \quad (3)$$

168 For the Newtonian case ( $\alpha = 1$ ) the latter equation reduces to the clas-  
 169 sical “cubic law” [32] written for a fluid with viscosity  $\mu_0/2$ . The continuity  
 170 equation reads [13]

$$\frac{dh}{dt} + h(t) \frac{\partial \bar{u}}{\partial x} = 0, \quad (4)$$

171 and substituting eq. (3) in eq. (4) gives

$$\frac{dh}{dt} = \frac{h^3}{12\mu_0} \frac{\partial^2 p}{\partial x^2} + \frac{\alpha h^{\alpha+2}}{2^{\alpha+1}(\alpha+2)\mu_0\tau_0^{\alpha-1}} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1} \frac{\partial^2 p}{\partial x^2}. \quad (5)$$

172 The problem formulation is completed by the force balance, expressed per  
 173 unit width of fracture, among the fluid pressure and the elastic reaction of the  
 174 upper wall, taken to be proportional to aperture  $h$ ; an overload at the upper  
 175 wall  $f_0$  (a force per unit width) is included in the balance for generality [21];  
 176 the overload represents an additional force exerted by the walls and usually  
 177 opposing the fracture opening due, e.g., to a residual stress state generated  
 178 by the load history of the rocks. It is assumed constant and independent  
 179 from the fracture aperture. The balance reads

$$\int_0^L p(x, t) dx = \tilde{E}Lh(t) + f_0, \quad (6)$$

180 where the constant of proportionality  $\tilde{E}$  has dimensions  $[ML^{-2}T^{-2}]$ ; for a  
 181 linear elastic foundation, called a Winkler soil in geotechnical applications,  
 182  $\tilde{E}$  is equal, for a thin elastic layer of thickness  $l$ , to the ratio between the  
 183 Young modulus of the layer’s material  $E$   $[ML^{-1}T^{-2}]$  and  $l$ ,  $\tilde{E} = E/l$ . In  
 184 the context of hydraulic fracturing,  $l$  may be identified with the fracture  
 185 spacing [13, 21], a design parameter that depends, among others, on the  
 186 type of rock; in hydraulically fractured shales, values of  $l/L$  equal to 0.057,  
 187 0.28, and 0.029 are reported, respectively, by Ghanbari and Dehghanpour [7],  
 188 Wang et al. [11], and Wang et al. [33]. In the case of vertical/sub-vertical  
 189 fractures perpendicular to a horizontal/sub-horizontal well or borehole, the  
 190 geometry of the idealized system is described by Figure 3, showing the two  
 191 wings of equally spaced planar fractures of half-length  $L$ , width  $W$ , aperture  
 192  $h$  and spacing  $l$ . Albeit the flow very close to the well is radial, the influence  
 193 of the boundary condition at the well decreases rapidly with distance, and  
 194 flow in most of the fracture half-length  $L$  is uniform, consistently with the  
 195 assumption  $L \gg W$ . Hence, as an approximation the boundary condition of  
 196 assigned pressure  $p_e$  at the well is extended to a segment of height  $W$ . In the

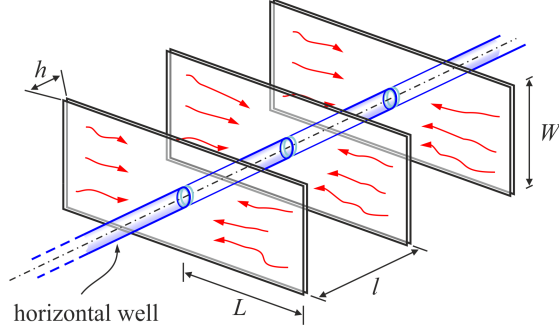


Figure 3: Typical scheme for bi-wing planar fractures around a horizontal borehole;  $L$ ,  $W$  and  $h$  are the fracture length, width and aperture,  $l$  is the fracture spacing.

197 case of planar vertical fractures parallel to, and propagating from, a vertical  
 198 well, the geometry of the flow is plane without using this approximation.

199 A further issue deserving investigation is the linearity of the relationship  
 200 between the wall reaction and the fracture aperture. In fact, a nonlinear  
 201 elastic behaviour can be the result of the pervasive damage of rocks by micro-  
 202 cracks and voids, which determines nonlinearity even for infinitesimal strain,  
 203 also with an incremental jump in the elastic modulus from tension to com-  
 204 pression [34, 35]. In this case the Young modulus of the material is a function  
 205 of the strain rate,  $E = E_0(h/l)$ , and assuming that the latter dependence is  
 206 expressed with a power-law function one has

$$E = E_0 \left( \frac{h}{l} \right)^{\lambda-1}, \quad (7)$$

207 where  $\lambda$  is a non-negative exponent modulating the nature of the reaction: for  
 208  $\lambda = 1$  a constant Young modulus is recovered, while  $0 < \lambda < 1$  is associated  
 209 to a softening behaviour, and  $\lambda > 1$  to a stiffening one. The assumption  
 210 results in

$$\tilde{E} = \frac{E_0}{l} \left( \frac{h}{l} \right)^{\lambda-1} \equiv \hat{E} h^{\lambda-1}, \quad (8)$$

211 and eq. (6) is modified as

$$\int_0^L p(x, t) dx = \hat{E} L h^\lambda(t) + f_0, \quad (9)$$

212 with  $\hat{E} = E_0 l^{-\lambda}$  of dimensions  $[ML^{-1-\lambda}T^{-2}]$ .

213 Equations (5) and (9) are subject to the following initial and boundary  
214 conditions

$$h(0) = h_0, \quad \frac{\partial p(x, t)}{\partial x}(L, t) = 0, \quad p(0, t) = p_e, \quad (10)$$

215  $h_0$  being the initial fracture aperture, and  $p_e$  the exit pressure at the well.

216 The solution to the above problem yields two relevant quantities expressed  
217 per unit width, the flowrate exiting the fracture at the well,  $q(t)$ , and the  
218 residual volume of the fracture at a given time,  $v(t)$ ; these are easily derivable  
219 as

$$q(t) = L \frac{dh(t)}{dt}, \quad v(t) = Lh(t). \quad (11)$$

## 220 2.2. Dimensionless form

221 Dimensionless quantities are defined as

$$\begin{aligned} X = x/L, \quad H = h/h_0, \quad T = t/t_c, \quad P = (p - p_e)/p_c, \quad P_e = p_e/p_c, \\ Q = qt_c/(h_0L) = q/(u_0h_0), \quad V = v/(h_0L), \end{aligned} \quad (12)$$

222 where the scales for pressure and time are

$$p_c = \hat{E}h_0^\lambda, \quad t_c = \frac{(2 + \alpha)}{\alpha} \left( \frac{2L}{h_0} \right)^{1+\alpha} \frac{1}{h_0^{\alpha\lambda}} \frac{\mu_0\tau_0^{\alpha-1}}{\hat{E}^\alpha}, \quad (13)$$

223 and  $u_0 = L/t_c$  is a velocity scale. This leads to the dimensionless counterpart  
224 of eq. (5)

$$\frac{dH}{dT} = NH^3 \frac{\partial^2 P}{\partial X^2} + H^{\alpha+2} \left( \frac{\partial P}{\partial X} \right)^{\alpha-1} \frac{\partial^2 P}{\partial X^2}, \quad (14)$$

225 where the pure number

$$N = \frac{2 + \alpha}{3\alpha} \left( \frac{2\tau_0 L}{\hat{E}h_0^{\lambda+1}} \right)^{\alpha-1} = \frac{2 + \alpha}{3\alpha} \left[ \frac{2\tau_0}{p_c(h_0/L)} \right]^{\alpha-1} \quad (15)$$

226 modulates the relative importance of the Newtonian behaviour of the Ellis  
227 fluid at low shear rate, expressed by the first term on the r.h.s. of eq. (14),  
228 with respect to the second term, the power-law behaviour at high shear rate.  
229 For a Newtonian fluid ( $\alpha = 1$ )  $N$  reduces to unity; for a shear-thinning fluid  
230 ( $\alpha > 1$ ),  $N$  is zero for  $\tau_0 = 0$  and/or a rigid wall ( $\hat{E} = E_0/l^\lambda \rightarrow \infty$ ), but  
231 the latter case renders the scales (13) meaningless. In eq. (15) defining  $N$ ,

232 the quantity within brackets represents the ratio between the characteristic  
 233 shear stress  $\tau_0$  of the Ellis fluid and the pressure scale  $p_c = \hat{E}h_0^\lambda$  associated  
 234 with the elastic reaction of the fracture wall; the ratio is in turn corrected by  
 235 the initial aspect ratio of the fracture  $h_0/L$ . This formulation of  $N$  includes  
 236 only parameters defined at the single fracture scale. Note that if the scheme  
 237 of multiple fractures with spacing  $l$  depicted in Figure 3 is considered, eq.  
 238 (15) may be rewritten as

$$N = \frac{2 + \alpha}{3\alpha} \left[ \frac{2 \left( \frac{\tau_0}{E_0} \right) \left( \frac{l}{L} \right) \left( \frac{l}{L} \right)^{\lambda-1}}{\left( \frac{h_0}{L} \right)^2 \left( \frac{h_0}{L} \right)^{\lambda-1}} \right]^{\alpha-1}, \quad (16)$$

239 where  $\tau_0/E_0$  is the ratio between the representative shear stress of the fluid  
 240 and the Young modulus of the host rock, and  $l/L$  is the dimensionless fracture  
 241 spacing. The terms to the power  $(\lambda - 1)$  represent the contribution due  
 242 to non-linear elastic behaviour of the walls, and disappear for  $\lambda = 1$ . An  
 243 alternative formulation of  $N$  as a function of Cauchy, Reynolds, and Ellis  
 244 dimensionless groups is reported in Appendix B. To grasp the order of  
 245 magnitude of  $N$ , we recall that  $l/L$  may be taken to vary between 0.03 and  
 246 0.3 (with  $l/L \approx 0.1$  being appropriate for an order of magnitude analysis),  
 247 while the initial fracture aspect ratio  $h_0/L$ , a number much smaller than 1,  
 248 may be considered of order  $10^{-3} - 10^{-5}$  [7, 11, 33]. The latter reference also  
 249 reports  $E_0 = 2.5 \cdot 10^{10}$  Pa for the rock elastic modulus in fractured shales;  
 250 quite close values,  $E_0 = 3 \cdot 10^{10}$  Pa and  $E_0 = 2.76 \cdot 10^{10}$  Pa are reported in  
 251 [19] and [36], hence reference values  $E_0 = 2.5 - 3.0 \cdot 10^{10}$  Pa are considered.

252 Actual values of rheological parameters for Ellis fluids are quite scarce  
 253 in the literature. A reference specific to fracking is [23], where the Ellis  
 254 parameters are reported for two fracturing fluids, HPG (Hydroxypropylguar)  
 255 and VES (viscoelastic surfactant). For the first,  $\mu_0 = 0.44$  Pa  $\cdot$  s,  $\tau_0 = 2.01$   
 256 Pa, and  $\alpha = 1.22$ ; for the second,  $\mu_0 = 49$  Pa  $\cdot$  s,  $\tau_0 = 8.836$  Pa, and  $\alpha = 12$ .  
 257 Adopting as reference geometrical parameters  $l/L = 0.1$  and  $h_0/L = 10^{-4}$ ,  
 258 and a young modulus of  $E_0 = 2.75 \cdot 10^{10}$  Pa for the host rock, one obtains  
 259  $N = 0.209$  for HPG and  $N \simeq 0$  for VES, indicating that for the latter fluid  
 260 the Newtonian component of rheological behaviour is negligible. A further  
 261 consideration is that VES is very strongly shear-thinning ( $\alpha \gg 1$ ), therefore  
 262 the value of  $N$  is extremely sensitive to variations in parameters: adopting  
 263 for example  $l/L = 0.125$ ,  $h_0/L = 10^{-5}$ , and  $E_0 = 2.5 \cdot 10^{10}$  Pa, again realistic

264 values, one obtains  $N = 0.100$  for VES and  $N = 0.618$  for HPG. This second  
 265 set of parameters is adopted for later reference in Section 4 describing a case  
 266 study and is shown there in dimensional form (see Table 1). Trying further  
 267 combinations of realistic values for fluid and rock properties, it is seen that  
 268  $N$  may take values smaller or larger than unity, the former case being more  
 269 frequent. This indicates a certain prevalence of the power-law component of  
 270 rheology over the Newtonian one, although the asymptotic system behaviour  
 271 is dominated by the latter, as will be shown in the next section. We bear in  
 272 mind that a large variety of combinations is possible for the two parameters  
 273  $N$  and  $\alpha$  depending on geometry and properties of fluid and rock, but with  
 274 the constraint from the definition (15) that for  $\alpha = 1$  it must be  $N = 1$ .

275 The dynamic boundary condition (9) and the boundary conditions (10)  
 276 transform as

$$\int_0^1 P(X, T) dX = H^\lambda - P_e + F_0, \quad (17)$$

277

$$H(0) = 1, \quad \frac{\partial P}{\partial X}(1, T) = 0, \quad P(0, T) = 0. \quad (18)$$

### 278 2.3. Solution

279 A solution to eq. (14) is sought by integrating in two steps the pressure  
 280 of the fluid and the fracture aperture. Posing

$$U(X, T) = \frac{\partial P}{\partial X}, \quad \dot{H} = \frac{dH}{dT}, \quad (19)$$

281 eq. (14) can be written as

$$B(1 + AU^{\alpha-1}) \frac{\partial U}{\partial X} = \dot{H} \quad (20)$$

282 where

$$A = A(T) = \frac{(H)^{\alpha-1}}{N}, \quad B = B(T) = NH^3, \quad (21)$$

283 while the second boundary condition in eq. (18) becomes

$$U(1, T) = 0. \quad (22)$$

284 Separating variables in eq. (20), and integrating with the boundary condition  
 285 (22) leads to

$$\frac{BU(AU^{\alpha-1} + \alpha)}{\alpha} = -\dot{H}(1 - X). \quad (23)$$

286 Eq. (23) can be rewritten as

$$U^\alpha + CU + D(1 - X) = 0 \quad (24)$$

287 where

$$C = C(T) = \frac{\alpha N}{H^{\alpha-1}}, \quad D = D(T) = \frac{\alpha \dot{H}}{H^{2+\alpha}}. \quad (25)$$

288 Eq. (25) is algebraic in  $U$  and admits an analytical solution for  $\alpha = 1, 2, 3$   
 289 and for  $\alpha = 1/2, 1/3$  in the form of a combination of functions of  $H$  and  $\dot{H}$ .  
 290 This solution can be integrated once in space, with the boundary condition  
 291  $P(0, T) = 0$ , obtaining the pressure field. The pressure field is finally inte-  
 292 grated in  $X \in [0, 1]$  and the integral in eq. (17) is computed as a function  
 293 of  $H$  and  $\dot{H}$ . Then eq. (17) is transformed in a nonlinear ODE which is  
 294 numerically integrated with the initial condition  $H(0) = 1$ .

295 These solutions are analytical in [the  \$x\$  coordinate](#) and numerical in [the](#)  
 296 [time domain](#) and seem quite cumbersome, while their accuracy is comparable  
 297 to that of a fully numerical solution in space and time; the latter also has  
 298 the advantage of a free selection of the indicial parameter  $\alpha$ . Among the  
 299 many possible numerical schemes, we adopt a finite difference in time and  
 300 an implicit resolver in space, with a step size reduction to track solution  
 301 accurately.

302 The code is written in Mathematica, introducing a parametric solver for  
 303 the function  $U(X, T)$  as a function of  $N, \alpha, H_{i+1}, H_i, \Delta t$ , where  $H_{i+1}$  and  
 304  $H_i$  are the values at time  $(i + 1)\Delta t$  and  $i\Delta t$ , respectively; the only free  
 305 parameter is  $H_{i+1}$ , all the other parameters are given.

306 Each time iteration includes the following steps:

- 307 • The function  $U(X)_{i+1}$  is estimated by solving eq. (20) in parametric  
 308 form, with  $\dot{H} \approx (H_{i+1} - H_i)/\Delta t$ , with the term  $H$  taken to be the  
 309 average between  $H_{i+1}$  and  $H_i$  and with the b.c.  $U(1)_{i+1} = 0$ , where  
 310  $H_{i+1}$  is the free parameter;  $H_0 = 1$  is assumed at the first step.
- 311 • The space values of  $U$ , known in parametric form, are used to solve  
 312 the differential problem  $\partial P(X)_{i+1}/\partial X = U(X)_{i+1}$ , with  $P(0)_{i+1} = 0$ ,  
 313 obtaining the pressure  $P(X)_{i+1}$ .
- 314 • The pressure field is numerically integrated (in parametric form) in the  
 315 domain  $[0, 1]$ .

- 316 • The parametric integral is inserted in eq. (17), and the equality is  
317 forced with a Newton method for finding the value of the parameter  
318  $H_{i+1}$ .
- 319 • The procedure is repeated for the next time step, shifting the values  
320  $H_{i+1}$ .

321 Once the pressure  $P(X, T)$  and aperture  $H(T)$  fields are known, the di-  
322 mensionless flowrate and fracture volume are given by

$$Q(T) = \frac{dH(T)}{dT} = \dot{H}, \quad V(T) = H(T). \quad (26)$$

323 Hence at late-time the fracture volume and flowrate behave like the aperture  
324 and its time derivative, respectively; for zero borehole pressure and overload  
325 the corresponding time scalings are  $T^{-1/(\lambda+2)}$  and  $T^{-1/(\lambda+3)}$ .

### 326 3. Results and discussion

327 Figure 4 shows the results of the numerical computation for the fracture  
328 aperture and different  $\alpha$  values, with the analytical solution  $H = (1+9T)^{-1/3}$   
329 valid for the Newtonian case and a linearly elastic fracture [13], corresponding  
330 to  $\alpha = 1$ ,  $N = 1$ , and  $\lambda = 1$ . Note that the values  $\alpha = 1$ ,  $N = 1$  imply  
331 Newtonian behaviour but with a viscosity equal to  $\mu_0/2$ , thus halving the  
332 time scale  $t_c$  in eq. (13); this requires doubling the dimensionless time  $T$   
333 in eq. (12) to compare results of equations having a different time scale.  
334 The time integration was performed with a time step  $\Delta t = 0.01$ . Since  
335 the results of the numerical integration using this fully explicit scheme fit  
336 exceedingly well the analytical solution, it was not necessary to adopt higher  
337 order schemes, even considering that the solution has no singularity and  
338 behaves rather smoothly.

339 The asymptotic behaviour of the solution  $H(T)$  is dictated by the inter-  
340 play between the two terms on the r.h.s. of eq. (14): the second term scales  
341 with the gradient pressure (decaying in time) and with a power of  $H$  always  
342 larger than 3, since  $\alpha > 1$ , whereas the first term scales with the third power  
343 of  $H$  and has  $N$  as a coefficient. Since  $H \leq 1$  and the gradient pressure  
344 quickly decays to values less than unity, the dominant term is the first one,  
345 which entails the asymptotic behaviour  $H \sim T^{-1/(2+\lambda)}$ , see Figure 5 where  
346 different values of  $\alpha$ , for  $N = 1$  and  $P_e = 0$ , produce almost parallel curves  
347 for large  $T$ . Figure 5 also shows how variations in  $\lambda$  significantly affect the

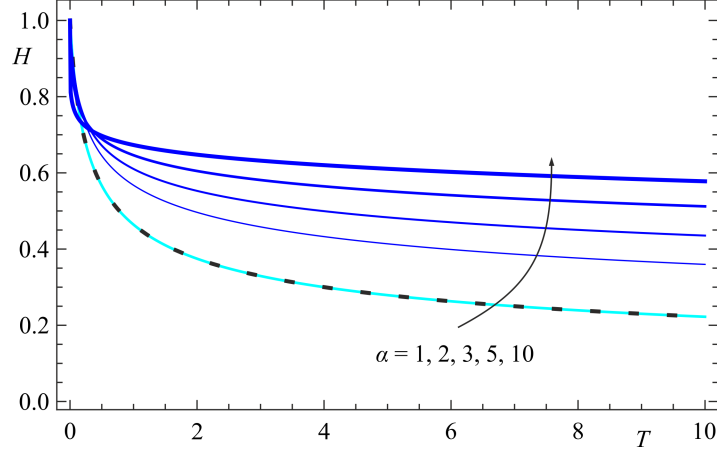


Figure 4: Time variation of the fracture aperture  $H$  for  $N = 0$ ,  $\lambda = 1$ ,  $P_e - F_0 = 0$  and different  $\alpha$  values. The black dotted curve refers to the analytical solution for a Newtonian fluid,  $H = (1 + 9T)^{-1/3}$ . Due to the different time scales adopted for a Newtonian fluid and for the present model, comparison is feasible if the dimensionless time  $T$  in the solution for the Newtonian fluid is doubled.

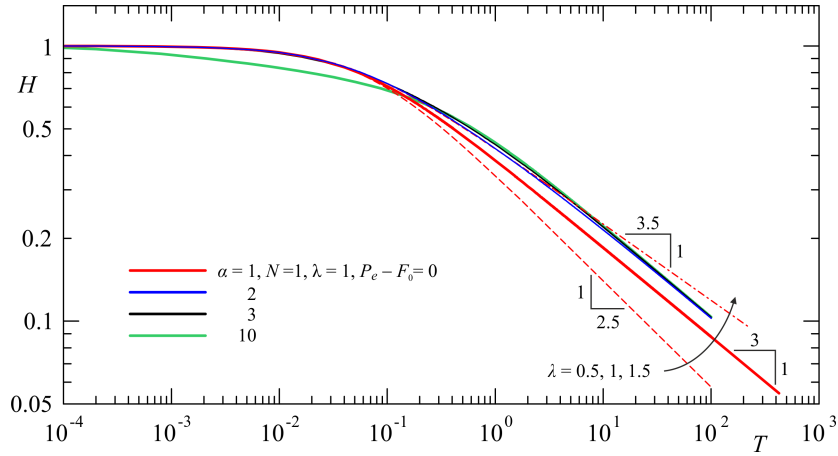


Figure 5: Time variation of the fracture aperture  $H$  for  $N = 1$ ,  $\lambda = 1$  and different  $\alpha$  values. For one case ( $\alpha = 1$ ) the effects of a softening/stiffening wall is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively. The asymptotic behaviour is  $H \sim T^{-1/(2+\lambda)}$ , independent on  $\alpha$ .

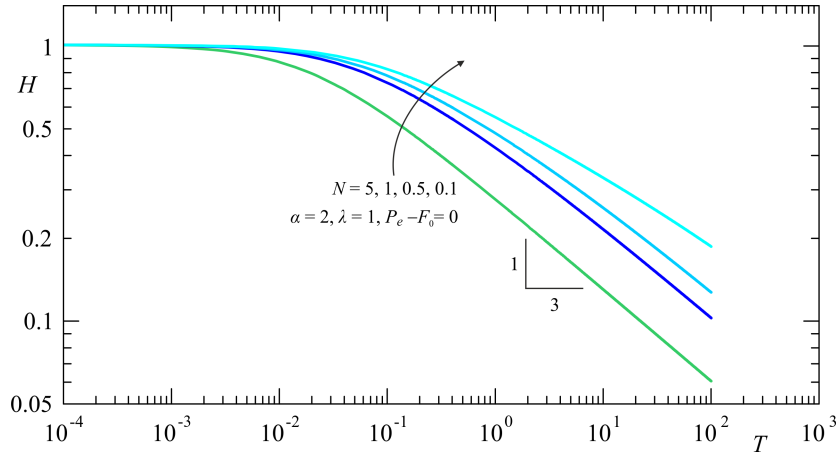


Figure 6: Time variation of the fracture aperture  $H$  for  $\alpha = 2$ ,  $\lambda = 1$  and different  $N$ .

348 late-time behaviour for fixed  $\alpha$ : a stiffening ( $\lambda > 1$ )/softening ( $\lambda < 1$ ) elastic  
 349 reaction of the walls delays/facilitates the drainage. It is also seen that the  
 350 parameter  $\alpha$  mainly controls the early stage, the parameter  $\lambda$  the late stage  
 351 of the backflow process. Figure 6 shows results for a fixed  $\alpha = 2$ ,  $\lambda = 1$ ,  
 352 and different  $N$  values; the asymptote is reached much faster for larger  $N$ .  
 353 In sum, the early time behaviour for zero external pressure at the well is  
 354 in general dominated by the second term in eq. (14) unless the coefficient  
 355  $N \gg 1$ ; in the latter case both terms substantially contribute to the time  
 356 evolution of  $H$ .

357 In presence of a non-zero external pressure ( $P_e > 0$ ) or a negative overload  
 358  $F_0$  (an additional force per unit of wall surface acting in the same direction  
 359 of the internal pressure), the asymptotic residual aperture is equal to  $(P_e -$   
 360  $F_0)^{1/\lambda}$ , see Figure 7 where both effects are included. The curves coalesce  
 361 to the asymptote faster for larger  $N$  values, implying a dominance of the  
 362 Newtonian behaviour, while for small  $N$  the power-law behaviour prevails  
 363 and the asymptote is reached for larger dimensionless times. Upon plotting  
 364 results for  $\alpha = 3$  (not shown) the main curves for  $\lambda = 1$  and the secondary  
 365 curves for  $\lambda \neq 1$  are very similar to those for  $\alpha = 2$ .

366 Figure 8 shows the pressure distribution for two different combinations  
 367 of the parameters and a shear-thinning fluid with  $\alpha = 2$ . Results for other  
 368 combinations are similar (and thus not shown), with a pressure decay in  
 369 space/time quicker or slower depending on the parameter values; at all times

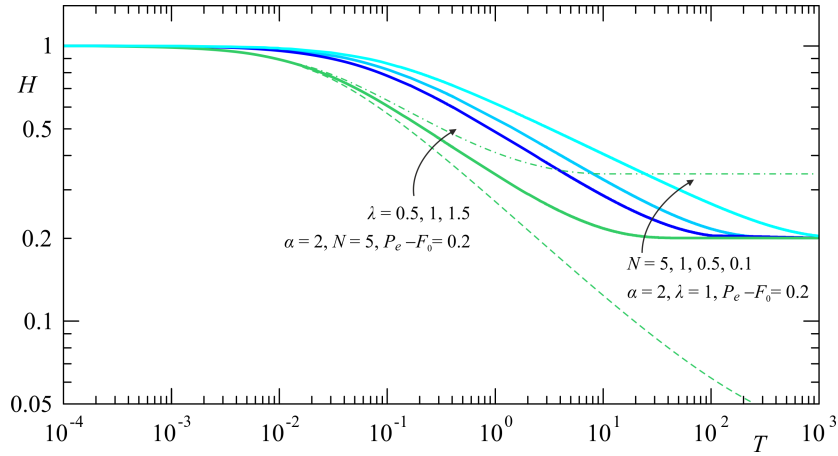


Figure 7: Time variation of the fracture aperture  $H$  for  $\alpha = 2$  and different  $N$  values, with given difference between external pressure and overload  $P_e - F_0 = 0.2$ . For one case ( $N = 5$ ) the effects of a stiffening/softening elastic reaction of the walls is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively.

370 the residual pressure within the fracture increases with smaller  $N$  values,  
 371 implying a behaviour closer to Newtonian, and with smaller  $\lambda$  values, i.e. a  
 372 softening wall; however when the fluid is closer to Newtonian the effect of a  
 373  $\lambda$  variation is irrelevant.

374 An important quantity characterizing the performance of the backflow  
 375 process is the time required to recover the fluid injected in the fracture net-  
 376 work and not lost in the form of leakoff. Here the network is conceptualized  
 377 as a single fracture and fluid losses are not explicitly represented (they are  
 378 assumed to take place in the upstream network), however the time  $T_Y$  needed  
 379 to recover  $Y\%$  of the fracture volume provides an indication of how rapid the  
 380 recovery is. Contour maps in the  $(\alpha, N)$  space of the dimensionless time  $T_{90}$   
 381 needed to recover 90% of the fluid are depicted in Figure 9 for a linear wall  
 382 reaction ( $\lambda = 1$ ). As the degree of shear-thinning behaviour rises with  $\alpha$   
 383 for constant  $N$ , there is a sharp increase in dimensionless  $T_Y$  for  $N < 0.5$ ,  
 384 while  $T_Y$  is almost independent on  $\alpha$  for  $N > 2$ . Conversely,  $T_Y$  for constant  
 385  $\alpha$  decreases with larger  $N$  values, i.e. as the fluid behaviour is closer to  
 386 Newtonian; this effect is more evident for larger  $\alpha$ . Highest values of  $T_Y$  are  
 387 attained for large  $\alpha$  and low  $N$ , lowest values for small  $\alpha$  and large  $N$ , the  
 388 two combinations farthest and closest to Newtonian behaviour. The effect

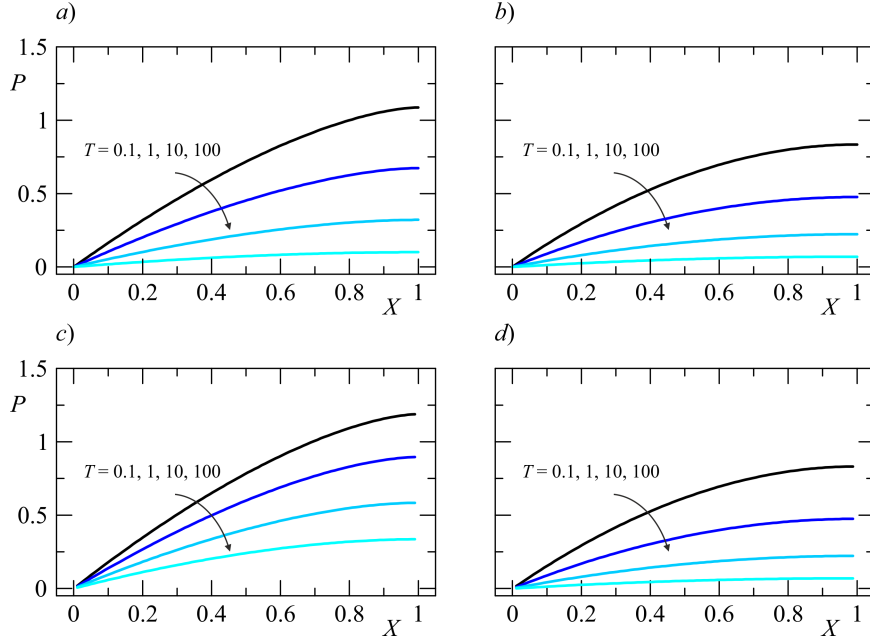


Figure 8: Pressure along the fracture at different times for  $P_e - F_0 = 0.2$  and a shear-thinning fluid with  $\alpha = 2$ . Results for a)  $N = 0.1$  and  $\lambda = 1$ ; b)  $N = 5$  and  $\lambda = 1$ ; c)  $N = 0.1$  and  $\lambda = 0.5$ ; d)  $N = 5$ ,  $\lambda = 0.5$ .

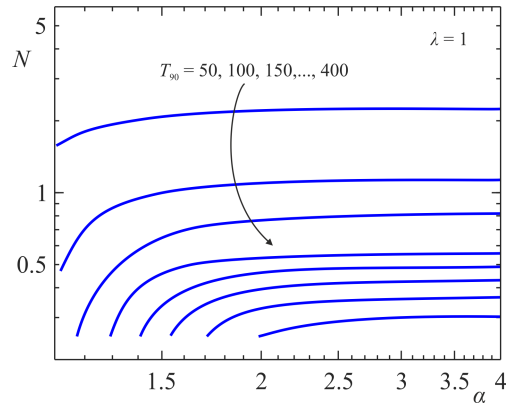


Figure 9: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1$  and  $P_e - F_0 = 0$ .

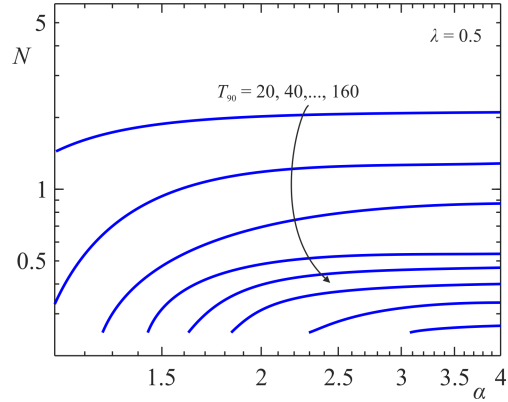


Figure 10: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 0.5$  and  $P_e - F_0 = 0$ .

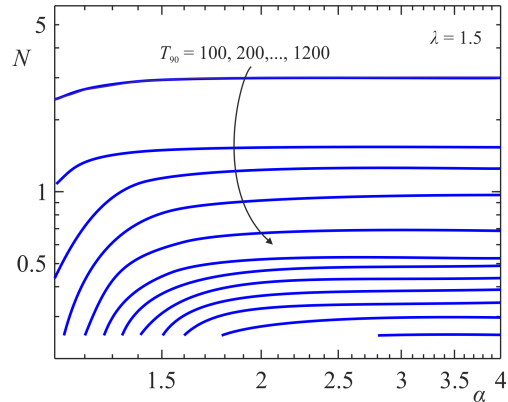


Figure 11: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1.5$  and  $P_e - F_0 = 0$ .

Fluid	$\mu_0$ (Pa s)	$\tau_0$ (Pa)	$\alpha$	$L$ (m)	$l$ (m)	$h_0$ (mm)	$E$ (Pa)	$\lambda$	$N$
HPG	0.44	2.01	1.22	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.618
VES	49.00	8.836	12.00	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.100

Table 1: Reference parameters for case study:  $\mu_0$ ,  $\tau_0$  and  $\alpha$  are the reference viscosity, shear stress and indicial exponent of the Ellis fluid,  $L$  is the fracture length,  $l$  is the fracture spacing,  $h_0$  is the fracture initial height,  $E$  is the rock modulus of elasticity,  $\lambda$  is the exponent of the rock wall reaction,  $N$  is the dimensionless number governing the interplay between Newtonian and power-law behaviour in an Ellis fluid.

389 of a sublinear wall reaction ( $\lambda = 0.5$ ) is depicted in Figure 10, that of a  
390 supralinear wall reaction in Figure 11. The dimensionless time to recover the  
391 bulk of the stored fluid is decidedly faster or slower with a softening or stiff-  
392 ening wall, demonstrating once again the decisive influence of the parameter  
393  $\lambda$  modulating the wall reaction at late time.

394 A word of caution is needed when drawing comparisons between non-  
395 Newtonian fluids with different rheology as the models are semi-empirical  
396 and the time scale used for the dimensionless formulation depends upon the  
397 rheological parameters of the Ellis model and is particularly sensitive to the  
398 value of the indicial exponent  $\alpha$ . Hence model outputs are best compared in  
399 dimensional coordinates when quantitative results are needed.

#### 400 4. A case study

401 A case study is illustrated by comparing the performance of two real  
402 hydrofracturing fluids [23], HPG (Hydroxypropylguar) and VES (viscoelastic  
403 surfactant) in a realistic setting. The rheological parameters according with  
404 the Ellis model are reported for both fluids in Table 1, together with realistic  
405 geometric and mechanical parameters within plausible ranges deduced from  
406 the literature, see the earlier discussion in Section 2.2. It is seen that HPG  
407 is relatively close to Newtonian in behaviour, while VES is extremely shear-  
408 thinning, with an equivalent rheological index  $n$  less than 0.1 when expressed  
409 according to the power-law model.

410 Figure 12 shows the relaxation of the fracture aperture for the two fluids:

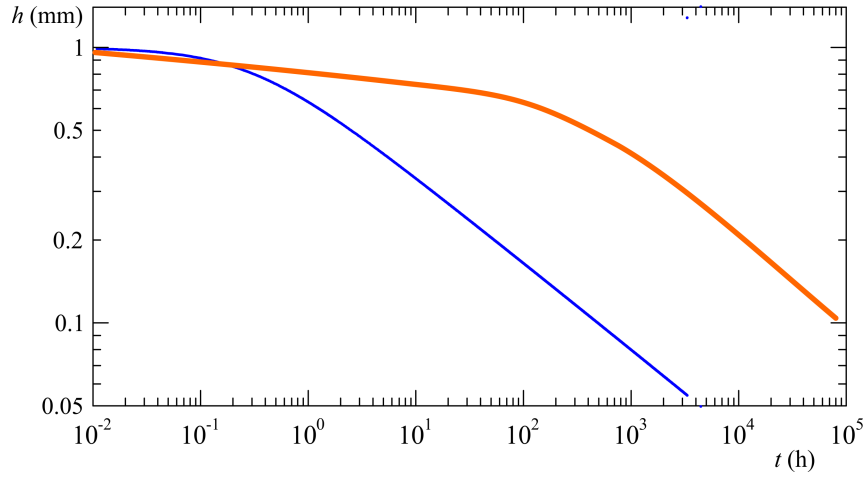


Figure 12: Time variation of the fracture aperture  $h$  for the HPG (thin line) and VES (thick line) fluids.

411 the aperture for the HPG is only initially slightly larger than for the VES,  
 412 but then closes more rapidly, reaching one tenth of the initial value at a time  
 413 around 500 hours. The closure is much more gradual for the VES, requiring  
 414 about a year to reach the same stage. The difference between corresponding  
 415 pressure profiles, illustrated in Figure 13, shows a decidedly sharper pressure  
 416 decrease for HPG than for VES in the initial stage.

417 Figure 14 shows the time to recover the volume stored in the fracture  
 418 for the two fluids. Following the same trend manifested for the evolution of

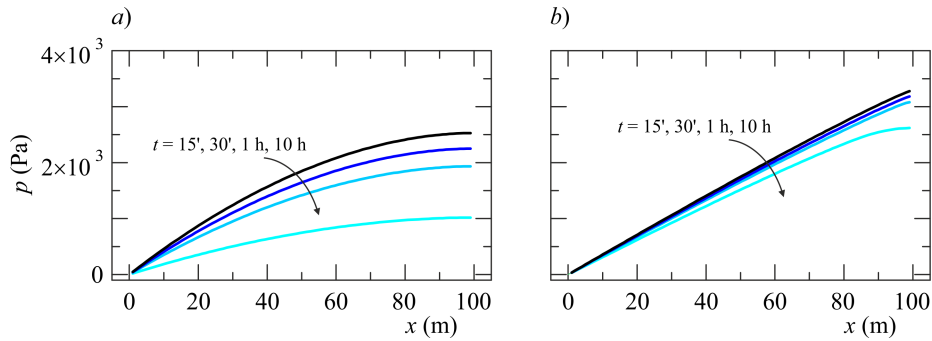


Figure 13: Pressure distribution at different time *a)* for HPG fluid, and *b)* for VES.

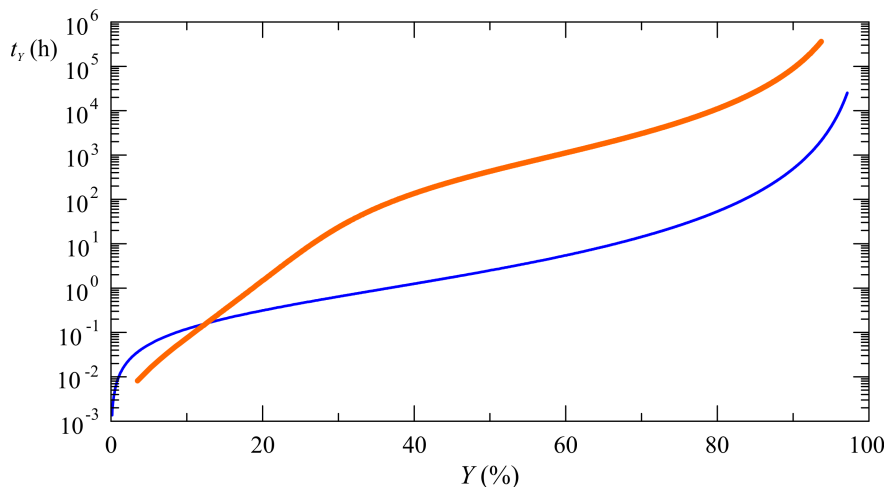


Figure 14: Time to recover the fracture volume  $Y\%$  for the HPG (thin line) and VES (thick line) fluids.

419 fracture opening, VES demonstrates a higher drainage capacity than HPG in  
 420 the very early phase, for  $Y < 15\%$ ; subsequently it is much less efficient, and  
 421 requires an extra time at least three orders of magnitude larger to drain the  
 422 same percentage of fluid than HPG. Overall the large difference in rheology,  
 423 mainly encapsulated in the  $\alpha$  value, translates into corresponding wide dif-  
 424 ferences in terms of aperture, pressure, and drainage time. This is so because  
 425 the value of the dimensionless group  $N$  is very low for VES, thus allowing the  
 426 fluid to manifest its essentially power-law nature. We tried a number of other  
 427 combinations of parameters and found that for very shear thinning fluids like  
 428 VES the results are very sensitive to relatively small changes in parameters:  
 429 slightly increasing the modulus of elasticity  $E$  to  $3 \cdot 10^{10}$  and increasing the  
 430 spacing to 20 m, leaving the other parameters in Table 1 unchanged, leads  
 431 to  $N(\text{HPG}) = 0.659$  and  $N(\text{VES}) = 2.360$ . While the change in the  $N$   
 432 value associated to HPG is modest (6.6%) and implies the system behaviour  
 433 is essentially unchanged with respect to the reference case, the increase in  
 434  $N$  for the VES is dramatic (2260%) and entails a fluid behaviour closer to  
 435 Newtonian despite the exceedingly high value of  $\alpha$ . Upon plotting the aper-  
 436 ture variation over time for this case (not shown) the two fluids exhibit a  
 437 similar behaviour, with only modest differences (less than 10%) in the frac-  
 438 ture aperture at early times and an almost identical behaviour later on. The  
 439 pressure profiles do not show any significant differences.

440 **5. Conclusions**

441 A conceptual model for backflow of non-Newtonian fluid from a closing  
 442 rock fracture was presented in this paper. Under the assumption of Ellis rhe-  
 443 ology and elastic, but non-deformable wall, the problem in plane geometry is  
 444 tractable in semi-analytical form to yield the time-variable fracture aperture  
 445  $h(t)$ , pressure field  $p(x, t)$  and discharge rate  $q(t)$ , as well as the drainage  
 446 time  $t_Y$  for a specified recovery rate  $Y$ , outlet pressure  $p_e$  and overload  $f_0$ .

447 Our results lead to the following specific conclusions:

- 448 • The Ellis model adopted herein to describe shear-thinning rheology  
 449 couples Newtonian and power-law behaviour. When an Ellis fluid back-  
 450 flows from a relaxing fracture the interplay between the two natures  
 451 is modulated by a dimensionless group  $N$  encapsulating the main prob-  
 452 lem parameters.  $N$  can be expressed in terms of i) the indicial exponent  
 453  $\alpha$  of the Ellis rheology, ii) the parameter  $\lambda$  governing the wall relaxation  
 454 process, iii) the ratio between the characteristic shear stress of the Ellis  
 455 fluid  $\tau_0$  and the rock modulus of elasticity  $E$ , iv) two geometric ratios,  
 456 the fracture initial aspect ratio  $h_0/L$  and dimensionless spacing  $l/L$ .  
 457 An alternative format of  $N$  is a modified ratio between the Cauchy  
 458 number and the product of Reynolds and Ellis numbers.
- 459 • The factors  $N$  and  $\alpha$  mostly influence the early and intermediate time  
 460 evolution of the system: when  $N < 1$  the power-law behaviour prevails;  
 461 for  $N = 1$  the pure Newtonian case is recovered ( $\alpha = 1$  entails  $N = 1$ ),  
 462 while for  $N \gg 1$  the behaviour is mixed.
- 463 • For late-time the system behaviour tends to Newtonian, is independent  
 464 of  $N$  and is governed by the wall relaxation parameter  $\lambda$ : aperture and  
 465 discharge scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  
 466  $p_e - f_0 = 0$ ; else, the aperture tends asymptotically to a constant value  
 467 proportional to  $(p_e - f_0)^{1/\lambda}$ .
- 468 • Very shear-thinning fluids (larger  $\alpha$ ) and reactive walls (larger  $\lambda$ ) are  
 469 associated with a more gradual closure of the aperture.
- 470 • The residual pressure within the fracture increases with smaller  $N$  val-  
 471 ues and with a softening wall ( $\lambda < 1$ ); when the fluid is close to New-  
 472 tonian the effect of a  $\lambda$  variation is almost irrelevant.

- 473 • The dimensionless drainage time  $T_Y$  attains the largest values for large  
474  $\alpha$  and low  $N$ , the lowest values for small  $\alpha$  and large  $N$ , the two com-  
475 binations farthest and closest to Newtonian behaviour. A non-linear  
476 reaction of the walls result in a faster/slower recovery for  $\lambda < 1$  (soft-  
477 ening) and  $\lambda > 1$  (stiffening). For recovery values close to 100%,  $T_Y$  is  
478 very sensitive to variations of model parameters.
- 479 • Results are discussed in dimensional form for a case study to reinforce  
480 the notion that dimensionless results need to be compared with caution  
481 as scales include fluid rheological parameters. Realistic geometric and  
482 mechanical parameters are adopted for a system of equally spaced frac-  
483 tures, and results are compared for two fluids, HPG and VES, normally  
484 used in fracking technology. The time evolution of the aperture and  
485 the dependence of the drainage time upon the recovery ratio are similar  
486 at early times, then differ by orders of magnitude at intermediate and  
487 late times.

488 The developments presented, together with earlier results [13, 21], provide  
489 an overview of the backflow phenomenon in the two basic geometric configu-  
490 rations for a single fracture, plane and radial, and for three rheological models  
491 of increasing complexity: Newtonian, power-law, and Ellis. Further improve-  
492 ments of the model remain open in several directions, e.g.: i) a more complex  
493 geometry, considering nonplanar fractures with non-negligible curvature; ii)  
494 the combination of non-Newtonian rheology with multiple fracture systems,  
495 adopting the asymptotic viewpoint of Dana et al. [14]; iii) the incorporation  
496 of particle transport to simulate the settling of solid proppant.

## 497 **Acknowledgments**

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500 conflicts of interest to declare. There are no data sharing issues since all of  
501 the numerical information is provided in the figures produced by solving the  
502 equations in the paper.

## 503 **Appendix A. The Newtonian case ( $n = 1$ )**

504 For  $\alpha = 1$  and  $N = 1$  eq. (25) reduces to

$$C = 1, \quad D = \frac{\dot{H}}{H^3}, \quad (\text{A.1})$$

505 and integrating eq. (24) using these expressions yields

$$P(X, T) = \frac{\dot{H}}{4H^3} [(X - 1)^2 - 1]. \quad (\text{A.2})$$

506 Substituting in eq. (17) and integrating  $P(X, T)$  over  $X$  gives

$$-\frac{\dot{H}}{3H^3} = H^\lambda - P_e + F_0, \quad (\text{A.3})$$

507 generalizing eq. (2.14) of Dana et al. [13], where  $\lambda = 1$  and  $F_0 = 0$ , to non-  
 508 linear wall reaction and non-zero overload. Now define an effective pressure  
 509  $\tilde{P}_e = P_e - F_0$  at the fracture outflow: this symbol will be used for brevity in  
 510 the sequel. Consider first the case  $\tilde{P}_e = 0$ . Integration of eq. (A.3) over time  
 511  $T$  yields, with the first b.c. in eq. (18),

$$H(T) = [1 + 3(2 + \lambda)T]^{-\frac{1}{2+\lambda}}, \quad (\text{A.4})$$

512 that for  $\lambda = 1$  gives back eq. (2.15) of [13].

Consider now the case  $\tilde{P}_e > 0$ . Integration with the help of Mathematica and using transformation formulae for the analytic continuation of hypergeometric functions [37] yields for generic  $\lambda$  the following implicit equation

$$T = \frac{1}{3(\lambda + 2)} \left[ \frac{1}{H^{\lambda+2}} {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \frac{\tilde{P}_e}{H^\lambda} \right) + \right. \\ \left. - {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \tilde{P}_e \right) \right], \quad (\text{A.5})$$

where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is the hypergeometric function of parameters  $\alpha$ ,  $\beta$ ,  $\gamma$ , and argument  $z$ . Specific results for  $\lambda = 1/2$ ,  $\lambda = 1$ ,  $\lambda = 2$ , i.e. a sublinear, linear or supralinear wall reaction, can be obtained as

$$T = \frac{1}{18\tilde{P}_e^5} \left[ 12 \ln \left( \frac{H^{1/2}(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{12\tilde{P}_e}{H^{1/2}} - \frac{6\tilde{P}_e^2}{H} - \frac{4\tilde{P}_e^3}{H^{3/2}} - \frac{3\tilde{P}_e^4}{H^2} + \right. \\ \left. + 12\tilde{P}_e + 6\tilde{P}_e^2 + 4\tilde{P}_e^3 + 3\tilde{P}_e^4 \right], \quad (\text{A.6})$$

513

$$T = \frac{1}{6\tilde{P}_e^3} \left[ 2 \ln \left( \frac{H(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{2\tilde{P}_e}{H} - \frac{\tilde{P}_e^2}{H^2} + 2\tilde{P}_e + \tilde{P}_e^2 \right], \quad (\text{A.7})$$

514

$$T = \frac{1}{6\tilde{P}_e^2} \left[ \ln \left( \frac{H^2(1 - \tilde{P}_e)}{H^2 - \tilde{P}_e} \right) - \frac{\tilde{P}_e}{H^2} + \tilde{P}_e \right], \quad (\text{A.8})$$

515 either by direct integration of eq. (A.5) or using transformations involving  
 516 the hypergeometric functions [37]. Eq. (A.7) valid for  $\lambda = 1$  is identical  
 517 to Eq. (2.18) of Dana et al. [13]. Other results in terms of trascendental  
 518 and algebraic functions can be obtained for other special values of  $\lambda \in \mathbb{N}$  or  
 519  $1/\lambda \in \mathbb{N}$  but are too cumbersome to report and/or of little technical interest.

520 Expressions (A.5)-(A.8), when evaluated for for given  $\tilde{P}_e$ , allow deriving  
 521  $H(T)$  and the drainage time  $T_Y$  needed to drain  $Y\%$  of the fracture volume.  
 522 As the latter quantity is given in dimensionless form by  $H$  according to (26),  
 523 to derive  $T_Y$  it is sufficient to evaluate (A.5) and its special cases (15)-(A.8)  
 524 for  $H = (100 - Y)/100$ .

525 Finally, it is worthwhile to derive the asymptotic behaviour of the general  
 526 equation (A.5) for the limit case  $\lambda \rightarrow 0$ . According to eq. (9),  $\lambda = 0$  implies  
 527 a wall reaction constant over time rather than dependent from the fracture  
 528 aperture. Integrating (A.3) for  $H^\lambda = 1$  gives

$$H = \frac{1}{[1 + 6(1 - \tilde{P}_e)T]^{1/2}}, \quad (\text{A.9})$$

529 a result that can be simplified for large time to  $H = 1/[6(1 - \tilde{P}_e)T]^{1/2}$  and  
 530 further for  $\tilde{P}_e = 0$  to  $H = 1/(6T)^{1/2}$ . Equation (A.9) can be also obtained  
 531 directly from eq. (A.5) for  $\lambda \rightarrow 0$  on the basis of eq. (9.121.1) in [37]. The  
 532 late-time scaling for a Newtonian fluid and a wall with constant reaction  
 533 ( $\lambda = 0$ ) is therefore  $H \propto T^{-1/2}$ , a result coinciding with the scaling  $H \propto$   
 534  $T^{-1/(2+\lambda)}$  implied by Figure 5 for a Newtonian fluid with  $N = 1$ ,  $\alpha = 1$ .

## 535 Appendix B. The dimensionless group $N$

536 The pure number  $N$  may be expressed as a function of well-known dimen-  
 537 sionless groups in fluid mechanics [see, e.g., 38]. Multiplying and dividing  
 538 eq. (16) by  $\rho\mu_0 h_0 u_0^3$ , where  $u_0$  is the reference velocity defined in (12), yields

$$N = K \left( \frac{\text{Ca}}{\text{Re} \cdot \text{El}} \right)^{\alpha-1}; \quad \text{Ca} = \frac{\rho u_0^2}{E}; \quad \text{Re} = \frac{2\rho u_0 h_0}{\mu_0}; \quad \text{El} = \frac{\mu_0 u_0}{\tau_0 h_0}, \quad (\text{B.1})$$

539

$$K = K(\alpha, \lambda, l/L, h_0/L) = \frac{2 + \alpha}{3\alpha} \left[ \frac{4 \left( \frac{l}{L} \right)^\lambda}{\left( \frac{h_0}{L} \right)^{\lambda+1}} \right]^{\alpha-1} \quad (\text{B.2})$$

540 where Ca, Re, and El are the Cauchy, Reynolds, and Ellis numbers, and  $K$   
 541 a geometric factor correcting the ratio  $\text{Ca}/(\text{Re} \cdot \text{El})$ . In turn, Ca is the ratio  
 542 between inertial forces and elastic forces transmitted by solid walls, Re is the  
 543 ratio between inertial and viscous forces, while El is the ratio between the  
 544 viscous stress associated with the low shear rate Newtonian behaviour and  
 545 the shear stress  $\tau_0$  associated with high shear rate non-Newtonian (power-  
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Dear Editor:

We would like to thank you, the anonymous Reviewer and the Associate Editor for the positive assessment of our work and for the suggestions, which have helped us to further improve the presentation. Our detailed response to the Reviewer is given below. Her/his comments are reproduced in *italic*. The corresponding changes in the manuscript are typeset in blue.

Sincerely yours,

Valentina Ciriello  
Alessandro Lenci  
Sandro Longo  
Vittorio Di Federico

Bologna, April 4th 2021

## Reply to Reviewer #1

*The manuscript presents an analytical solution for back-flow of a fracking fluid towards a well. The analysis is elegant and relevant to several subsurface applications. It is also innovative and well-described. For these reasons I recommend to publish this manuscript after the following issues are resolved.*

We thank the Referee for this succinct summary of our contribution and for the positive assessment of our work. All the minor editorial changes suggested are incorporated in the revision.

*The authors argue that "gravity effects are usually neglected when compared to pressure gradients" (page 4). What is a typical range of the Froude number encountered in this application?*

We note that the gravity term can be included in a pressure term corrected by adding the hydrostatic pressure; this leads to a mathematical treatment with no gravity term to consider. We do so in the revised manuscript. For instance, for the Figure 3 representing multiple vertical fractures backflowing to an horizontal well, the reduced pressure  $p$  is equal to  $p = p' + \rho gy$ , with  $p'$  being pressure. The objection is then mute as no approximation is needed. *The authors assume that "the deformation is concentrated for convenience in the upper wall" (page 4). Convenience is not an adequate justification for the model construction. Perhaps the lower wall can be thought of as a line of symmetry (the middle of a fracture) instead?*

In a real fracture the deformation would be concentrated at both the upper and the lower wall. Under the one-dimensional approximation in the  $x$  direction adopted in the model, this makes no real difference as fluid velocities in perpendicular directions are neglected, and considering the squeezing of both walls would only complicate the notation. An alternative formulation is suggested by the reviewer, considering the lower wall as a line of symmetry and the fracture having aperture  $2h$ . As the word "convenience" by itself may appear misleading, for brevity and to maintain consistency of the model formulation with other references, we have changed the wording to "'mathematical convenience'".

*Since the authors are concerned with back-flow towards a well where flow is divergent, treating it as a "planar flow" (do you mean uniform?) requires a better justification than "convenience" (see above).*

The referee is right in asking for a clarification, and the word "simplicity" in the original manuscript was misleading. We revised the whole paragraph explaining the approximation adopted for vertical fractures perpendicular to the horizontal borehole axis, and introducing the case of vertical fractures parallel to the borehole axis, where no approximation is needed. We also note that there was a typo in the "Problem statement" that read  $L \ll W$  while the correct assumption to be listed is  $L \gg W$ . The typo has been corrected and may have contributed to the reviewer's remark.

*What does "the overload at the upper wall" mean in this context?*

We added an additional sentence to better explain its meaning.

*Abstract. Replace "independent on rheology" with "independent from rheology".*

Done.

*The phrase "parallelepiped space" (page 4) is suspect.*

The word is correct English, we substituted it with "3-D space" for ease of reading.

*Replace "following Figure 3" with "Figure 3 below".*

Done.

*The phrase "shear flow of coordinates" (page 4) doesn't make sense.*

We thank the reviewer, we corrected the sentence.

*Replace "Figure eq. (2)" (page 6) with "Figure 2"*

Done.

*Replace "Figure (3)" (page 7) with "Figure 3"*

Done.

*Remove the indent in the lines following Eqs. 13 and 14.*

Done.

*What does "pure number" (page 9) mean?*

An adimensional quantity in physics, a synonymous is "dimensionless", used elsewhere in the manuscript. We prefer to leave the sentence as is.

*What does the phrase "solutions are analytical in space and numerical in time" mean?*

We reformulated the sentence for ease of reading.

## Highlights

Our model handles realistic shear-thinning behaviour with upper-bounded apparent viscosity

A dimensionless number reveals the relative importance of governing parameters during backflow

Time to recover the fluid is compared for two hydrofracturing fluids showing significant differences

# 1 Relaxation-induced flow in a smooth fracture for Ellis 2 rheology

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## 7 Abstract

8 Hydraulic fracturing is a process aimed at improving the productivity of oil,  
9 gas or geothermal reservoirs. During hydrofracturing, backflow follows injec-  
10 tion and represents the second phase of the process, when part of the fractur-  
11 ing fluid returns from fractures to well, and from well to surface. A concep-  
12 tual model is presented to grasp the essential features of the phenomenon,  
13 conceiving the draining subsurface domain as a planar and rigid fracture.  
14 Backflow against an outlet pressure in the injection well is induced by the  
15 relaxation of the fracture wall, exerting a force on the fluid proportional to  
16  $h^\lambda$ , with  $h$  the time-variable aperture and  $\lambda$  a non-negative exponent; an  
17 overload on the fracture may contribute to slowing or accelerating the clo-  
18 sure process. The fluid rheology is described by the three-parameter Ellis  
19 constitutive equation, well representing the shear-thinning rheology typical  
20 of hydrofracturing fluids and coupling Newtonian and power-law behaviour.  
21 The interplay between these tendencies is modulated by a dimensionless num-  
22 ber  $N$  encapsulating most problem parameters; the range of variation of  $N$   
23 is discussed and found to vary around unity. The time-variable aperture  
24 and discharge rate, the space-time variable pressure field, and the time to  
25 drain a specified fraction of the fracture volume are derived as functions of  
26 geometry (length and initial aperture), wall elastic parameters, fluid prop-  
27 erties, outlet pressure  $p_e$  and overload  $f_0$ . The late-time behaviour of the  
28 system is practically independent from rheology as the Newtonian nature of  
29 the fluid prevails at low shear stress. In particular, aperture and discharge  
30 scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  $p_e - f_0 = 0$ ; else,  
31 the aperture tends to a constant, residual value proportional to  $(p_e - f_0)^\lambda$ .  
32 A case study with equally spaced fractures adopting realistic geometric, me-

chanical and rheological parameters is examined: two fluids normally used in fracking technology show completely different behaviours, with backflow dynamics and drainage times initially not dissimilar, later varying by orders of magnitude.

*Key words:*

Hydraulic fracturing, Non-Newtonian, Ellis rheology, elastic wall, backflow

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## 1. Introduction

Hydraulic fracturing is a process aimed at improving the productivity of oil, gas or geothermal reservoirs. Analysis of the different phases of hydraulic fracturing is of particular modeling and experimental interest [e.g. ? ? ].

An understanding of fractured media flow induced by the relaxation of elastic fracture walls is crucial in modeling fracturing fluid backflow, a complicated phenomenon involving hydrodynamic, mechanical and chemical processes. Backflow is typically the final phase of the hydraulic fracturing process: in the first one, fracturing fluid is injected at high pressure in a rock mass, forming new fractures and enlarging existing ones; in the second phase, proppant is introduced in the subsurface environment to prop fractures open; then when the injection ceases, the pressure drops, existing and new fractures tend to close, and a portion of the injected fracturing fluid, often mixed with proppant [? ], flows back towards the injection well and interact with the relaxing walls of the fractures. As the retention of fracturing fluid in the fracture network impairs the fracture conductivity reducing the wellbore productivity [? ], and favours migration in the subsurface environment along different pathways [? ], it is of utmost interest to optimize the amount of fluid recovered, irrespective of the reservoir product, be it oil [? ], gas [? ] or heat [? ].

The scientific literature offers two main approaches to modeling backflow: (i) detailed numerical simulations involving single fractures [? ], fracture networks [? ] or dual or triple porosity models [? ], or (ii) conceptual models capturing the main features of the interaction between fracture flow and wall relaxation [? ], including the effects of branching networks described at different degrees of complexity [? ? ]. A recent addition to the modeling effort is the influence of fluid rheology, following the notion that the backflow fluid is non-Newtonian in the widest sense [? ], as not only the relationship between shear stress and shear rate is nonlinear, but also exhibits normal

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10 stress and temperature dependency, as well as viscoelasticity, thixotropy,  
11 and nonzero yield stress [? ]. At the same time, non-Newtonian fluids allow  
12 achieving several engineering objectives, such as (i) minimize the pressure-  
13 drop in the entire process; (ii) carry suspended proppant; (iii) minimize the  
14 leak-off within the formation; (iv) adapt their characteristics to different  
15 environments in terms of temperature and chemical composition; and (v)  
16 flow back easily towards the wellbore. Given their versatility and economic  
17 value, these fluids are typically treated for reuse once recovered, removing  
18 contaminants they may have transported to the surface [? ]. The recovery  
19 ratios of backflow fluid vary between 2% and 48% according to Ipatova and  
20 Chuprakov [? ], with considerable economic value.

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23 Modeling non Newtonian backflow is in its early stage, in variance with  
24 the injection and fracture formation stage, for which several conceptualiza-  
25 tions and models are available: see Detournay [? ] for a review and the recent  
26 work by Wrobel [? ] comparing different rheological models for fracturing  
27 fluids. To the best of our knowledge, only Chiapponi et al. [? ] considered  
28 non-Newtonian fluids in the context of backflow modeling: these authors  
29 examined flow of a power-law fluid towards a wellbore in a single fracture  
30 of annular geometry, supporting their theoretical findings with laboratory  
31 experiments. The present paper develops the analysis of non-Newtonian  
32 backflow for a smooth fracture, common in field applications [? ], and adds  
33 realism by employing a three-parameter Ellis model, that well represents the  
34 rheology of hydrofracturing [? ] and drilling fluids [? ]. The Ellis model  
35 tends to Newtonian for low shear rates, to power-law for high shear rates  
36 and allows avoiding the unphysical effect of infinite apparent viscosity at  
37 zero shear rate that is typical of the power-law model [? ]. We note in pass-  
38 ing that our results are of a general nature for Newtonian pressurized flow  
39 in ducts of variable width and may be of interest for, and be applied also to,  
40 deformable microfluidic [? ] and biological [? ] systems.

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46 The plan of the paper is as follows. Section 2 formulates the problem  
47 of relaxation-induced backflow of an Ellis fluid in a fracture with nonlin-  
48 ear wall reaction and subject to overload. Numerical results obtained are  
49 presented and discussed in Section 3 as a function of dimensionless groups  
50 characterizing the system: the indicial exponent  $\alpha$  quantifying the degree  
51 of shear-thinning behaviour of the Ellis fluid, the non-negative exponent  $\lambda$   
52 modulating the fracture wall reaction, and a further group  $N$  encapsulating  
53 most problem parameters. Section 4 illustrates an hypothetical case study  
54 adopting realistic geometric and mechanical parameters and two real hy-  
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106 drofracturing fluids described by the Ellis model. Section 5 reports the main  
107 conclusions and perspectives for future work. In Appendix A the special case  
108 of a Newtonian fluid is examined, obtaining results that generalize those of  
109 Dana et al. [?] to a nonlinear wall reaction, while Appendix B presents  
110 an alternative expression for the dimensionless number  $N$ , shown to be a  
111 combination of well-known dimensionless groups in fluid mechanics.

## 112 2. Material and methods

### 113 2.1. Problem statement

114 A rock fracture produced by hydrofracturing, though of irregular geom-  
115 etry, is often conceptualized for modeling purposes as a 3-D space of length  
116  $L$ , width  $W$ , and aperture  $h$  between two parallel walls [? ]; the Cartesian  
117 coordinate system  $x, y, z$  is illustrated in Figure ?? and the fracture is subject  
118 to a pressure gradient  $\nabla p' \equiv (\partial p'/\partial x, 0, 0)$  in the  $x$  direction. In horizontal  
119 fractures, the additional gravity-induced pressure gradient is perpendicular  
120 the flow plane and has no effect on the flow field. If the  $(x, y)$  plane is not  
121 horizontal, the  $z$  direction perpendicular to the walls is not vertical and grav-  
122 ity effects can be included in a reduced pressure term  $p$ , thus leading to a  
123 mathematical treatment with no gravity term to consider. For instance, for  
124 the Figure ?? below representing multiple vertical fractures backflowing to  
125 an horizontal well, the reduced pressure  $p$  is equal to  $p = p' + \rho gy$ .

126 The walls are taken to be rigid, so that the aperture  $h(t)$  is solely a  
127 function of time, and the deformation is concentrated for mathematical con-  
128 venience in the upper wall, that behaves as a nonlinear elastic foundation  
129 exerting a reaction on the fluid. At  $t = 0$  the relaxation of the wall in-  
130 duces a backflow in the negative  $x$  direction, and the fracture begins to drain  
131 subject to a constant outlet pressure  $p_e$  at  $x = 0$  and to a no-flow bound-  
132 ary condition at the upstream end  $x = L$ . Three further hypotheses are  
133 adopted: i) the flow is quasi-steady, allowing to neglect the time derivative  
134 of the velocity in the momentum equation; ii) the fracture aspect ratio is  
135 small,  $h_0/L \ll 1$ , warranting the lubrication approximation, and iii) the flow  
136 is essentially one-dimensional along  $x$ ,  $L \gg W$ . The latter conceptualiza-  
137 tion is usually adopted in hydrogeology also when the two dimensions are  
138 comparable, as it is often the case for rock fractures [? ].

139 The flowback fluid is taken to be incompressible of density  $\rho$ , non-Newtonian  
140 shear-thinning [?] and described by the Ellis three-parameter model [? ].  
141 Under the above assumptions, the fluid undergoes simple shear flow in the

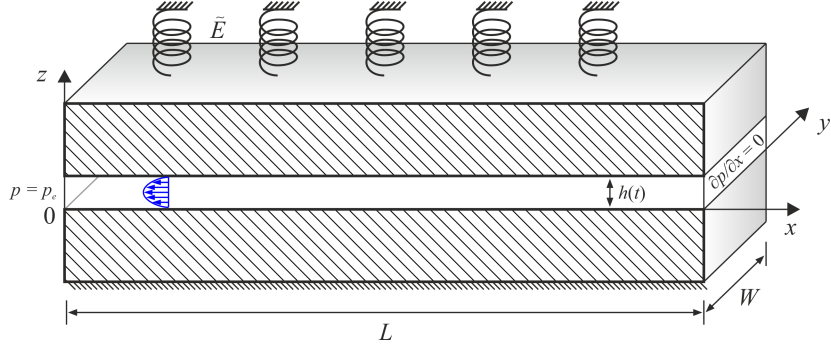


Figure 1: Layout of a plane fracture of variable uniform aperture  $h(t)$ .

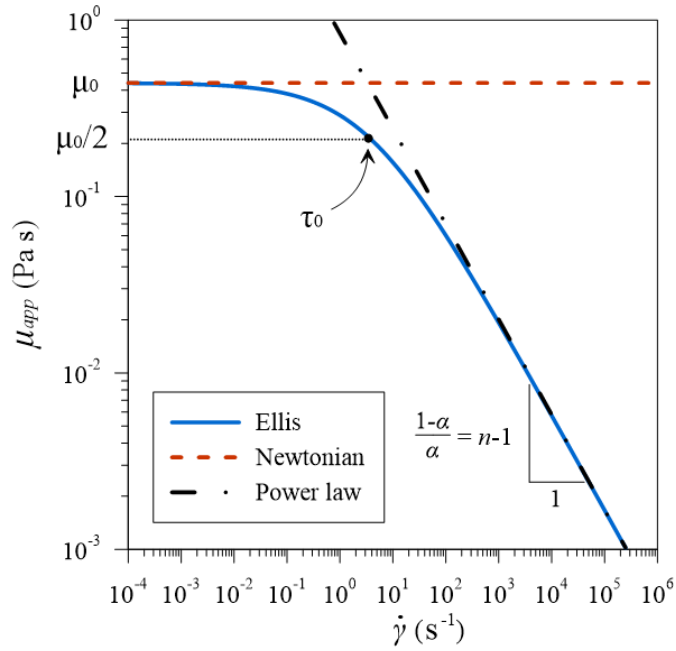


Figure 2: Apparent viscosity for three rheological models: Ellis (blue solid line) of parameters  $\mu_0$ ,  $\tau_0$ ,  $\alpha$ ; Newtonian (red dashed line) of viscosity  $\mu_0$ ; power-law (black dot-dashed line) of consistency index  $m$  and rheological index  $n$ . The comparison with the latter is drawn assuming:  $\alpha = 1/n$  and  $\tau_0 = (m/\mu_0^n)^{n/(1-n)}$ .

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142  $x$  direction, and the Ellis rheology is described by the following relationship  
143 between shear stress  $\tau_{zx}$  (hereinafter  $\tau$ ) and shear rate  $\dot{\gamma}_{zx}$  (hereinafter simply  
144  $\dot{\gamma}$ )

$$\tau = \frac{\mu_0}{1 + (\tau/\tau_0)^{\alpha-1}} \dot{\gamma}; \quad \dot{\gamma} = \frac{\partial u}{\partial z}, \quad (1)$$

145 where  $u$  is the velocity in the  $x$  direction. The rheological law (??) features a  
146 viscosity parameter  $\mu_0$ , a constant  $\tau_0$  defined as the shear stress corresponding  
147 to apparent viscosity  $\mu_0/2$ , and an indicial parameter  $\alpha$ , typically larger than  
148 one as the fluid is shear-thinning. For  $\alpha = 1$ , a pseudo-Newtonian behaviour  
149 with dynamic viscosity  $\mu_0/2$  is recovered, see Figure ?? showing the apparent  
150 viscosity  $\mu_{app} = \tau/\dot{\gamma}$  for the Ellis model compared to Newtonian and power-  
151 law models. Newtonian behaviour in the form of a plateau for low shear rates  
152 is also observed for  $\gamma \rightarrow 0$ . For high shear rates the behaviour is power-law,  
153 and its two parameters can be determined from the Ellis model parameters,  
154 see Appendix A in Balhoff and Thompson [? ]; in particular, the rheological  
155 index is  $n = 1/\alpha$  [? ]. Note that when curve fitting is performed on real data,  
156  $n$  and  $1/\alpha$  may significantly differ [? ], as two different models are fitted  
157 to the same data set. It is also seen that the Ellis model allows avoiding  
158 the unphysical effect of infinite apparent viscosity at zero shear rate that is  
159 typical of power-law fluids [? ]. In the following, we will consider  $\alpha > 1$ ,  
160 dealing with the case  $\alpha = 1$  in the Appendix, and the parameters  $\mu_0$  and  $\tau_0$   
161 to be finite and positive. Couette-Poiseuille slit flow of an Ellis fluid under  
162 a constant pressure gradient was studied extensively by Steller [? ], listing  
163 all combinations of parameters leading to Newtonian or pseudo-Newtonian  
164 behaviour. In particular, the negative velocity  $u(z)$  under a positive reduced  
165 pressure gradient  $\partial p/\partial$  in the  $x$  direction is

$$u(z, t) = -\frac{1}{8\mu_0} \left[ h^2 - (2z - h)^2 \right] \frac{\partial p}{\partial x} + \frac{1}{(\alpha + 1)2^{\alpha+1}\mu_0\tau_0^{\alpha-1}} \left[ h^{\alpha+1} - |2z - h|^{\alpha+1} \right] \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}. \quad (2)$$

166 The corresponding average velocity  $\bar{u}$  and flow per unit width  $q_x$  in the  $x$   
167 direction are

$$\bar{u} = -\frac{h^2}{12\mu_0} \frac{\partial p}{\partial x} - \frac{h^{\alpha+1}}{2^{\alpha+1}(\alpha + 2)\mu_0\tau_0^{\alpha-1}} \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}; \quad q_x = \bar{u}h. \quad (3)$$

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10 For the Newtonian case ( $\alpha = 1$ ) the latter equation reduces to the classical  
11 “cubic law” [?] written for a fluid with viscosity  $\mu_0/2$ . The continuity  
12 equation reads [? ]

$$\frac{dh}{dt} + h(t) \frac{\partial \bar{u}}{\partial x} = 0, \quad (4)$$

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15 and substituting eq. (??) in eq. (??) gives

$$\frac{dh}{dt} = \frac{h^3}{12\mu_0} \frac{\partial^2 p}{\partial x^2} + \frac{\alpha h^{\alpha+2}}{2^{\alpha+1}(\alpha+2)\mu_0\tau_0^{\alpha-1}} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1} \frac{\partial^2 p}{\partial x^2}. \quad (5)$$

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18 The problem formulation is completed by the force balance, expressed per  
19 unit width of fracture, among the fluid pressure and the elastic reaction of the  
20 upper wall, taken to be proportional to aperture  $h$ ; an overload at the upper  
21 wall  $f_0$  (a force per unit width) is included in the balance for generality [? ];  
22 the overload represents an additional force exerted by the walls and usually  
23 opposing the fracture opening due, e.g., to a residual stress state generated  
24 by the load history of the rocks. It is assumed constant and independent  
25 from the fracture aperture. The balance reads

$$\int_0^L p(x, t) dx = \tilde{E}Lh(t) + f_0, \quad (6)$$

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28 where the constant of proportionality  $\tilde{E}$  has dimensions  $[ML^{-2}T^{-2}]$ ; for a  
29 linear elastic foundation, called a Winkler soil in geotechnical applications,  
30  $\tilde{E}$  is equal, for a thin elastic layer of thickness  $l$ , to the ratio between the  
31 Young modulus of the layer’s material  $E$   $[ML^{-1}T^{-2}]$  and  $l$ ,  $\tilde{E} = E/l$ . In  
32 the context of hydraulic fracturing,  $l$  may be identified with the fracture  
33 spacing [? ? ], a design parameter that depends, among others, on the type  
34 of rock; in hydraulically fractured shales, values of  $l/L$  equal to 0.057, 0.28,  
35 and 0.029 are reported, respectively, by Ghanbari and Dehghanpour [? ],  
36 Wang et al. [? ], and Wang et al. [? ]. In the case of vertical/sub-vertical  
37 fractures perpendicular to a horizontal/sub-horizontal well or borehole, the  
38 geometry of the idealized system is described by Figure ??, showing the two  
39 wings of equally spaced planar fractures of half-length  $L$ , width  $W$ , aperture  
40  $h$  and spacing  $l$ . Albeit the flow very close to the well is radial, the influence  
41 of the boundary condition at the well decreases rapidly with distance, and  
42 flow in most of the fracture half-length  $L$  is uniform, consistently with the  
43 assumption  $L \gg W$ . Hence, as an approximation the boundary condition of  
44 assigned pressure  $p_e$  at the well is extended to a segment of height  $W$ . In the

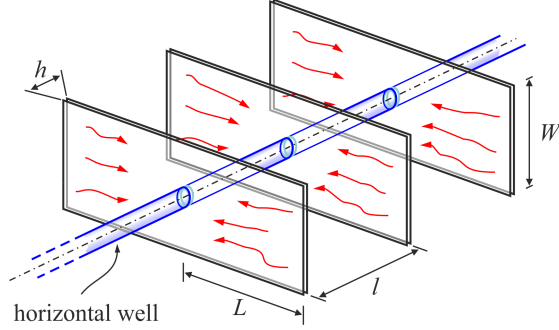


Figure 3: Typical scheme for bi-wing planar fractures around a horizontal borehole;  $L$ ,  $W$  and  $h$  are the fracture length, width and aperture,  $l$  is the fracture spacing.

197 case of planar vertical fractures parallel to, and propagating from, a vertical  
 198 well, the geometry of the flow is plane without using this approximation.

199 A further issue deserving investigation is the linearity of the relationship  
 200 between the wall reaction and the fracture aperture. In fact, a nonlinear  
 201 elastic behaviour can be the result of the pervasive damage of rocks by micro-  
 202 cracks and voids, which determines nonlinearity even for infinitesimal strain,  
 203 also with an incremental jump in the elastic modulus from tension to com-  
 204 pression [? ?]. In this case the Young modulus of the material is a function  
 205 of the strain rate,  $E = E_0(h/l)$ , and assuming that the latter dependence is  
 206 expressed with a power-law function one has

$$E = E_0 \left( \frac{h}{l} \right)^{\lambda-1}, \quad (7)$$

207 where  $\lambda$  is a non-negative exponent modulating the nature of the reaction: for  
 208  $\lambda = 1$  a constant Young modulus is recovered, while  $0 < \lambda < 1$  is associated  
 209 to a softening behaviour, and  $\lambda > 1$  to a stiffening one. The assumption  
 210 results in

$$\tilde{E} = \frac{E_0}{l} \left( \frac{h}{l} \right)^{\lambda-1} \equiv \hat{E} h^{\lambda-1}, \quad (8)$$

211 and eq. (??) is modified as

$$\int_0^L p(x, t) dx = \hat{E} L h^\lambda(t) + f_0, \quad (9)$$

212 with  $\hat{E} = E_0 l^{-\lambda}$  of dimensions  $[ML^{-1-\lambda}T^{-2}]$ .

Equations (??) and (??) are subject to the following initial and boundary conditions

$$h(0) = h_0, \quad \frac{\partial p(x, t)}{\partial x}(L, t) = 0, \quad p(0, t) = p_e, \quad (10)$$

$h_0$  being the initial fracture aperture, and  $p_e$  the exit pressure at the well.

The solution to the above problem yields two relevant quantities expressed per unit width, the flowrate exiting the fracture at the well,  $q(t)$ , and the residual volume of the fracture at a given time,  $v(t)$ ; these are easily derivable as

$$q(t) = L \frac{dh(t)}{dt}, \quad v(t) = Lh(t). \quad (11)$$

## 2.2. Dimensionless form

Dimensionless quantities are defined as

$$\begin{aligned} X = x/L, \quad H = h/h_0, \quad T = t/t_c, \quad P = (p - p_e)/p_c, \quad P_e = p_e/p_c, \\ Q = qt_c/(h_0L) = q/(u_0h_0), \quad V = v/(h_0L), \end{aligned} \quad (12)$$

where the scales for pressure and time are

$$p_c = \hat{E}h_0^\lambda, \quad t_c = \frac{(2 + \alpha)}{\alpha} \left( \frac{2L}{h_0} \right)^{1+\alpha} \frac{1}{h_0^{\alpha\lambda}} \frac{\mu_0\tau_0^{\alpha-1}}{\hat{E}^\alpha}, \quad (13)$$

and  $u_0 = L/t_c$  is a velocity scale. This leads to the dimensionless counterpart of eq. (??)

$$\frac{dH}{dT} = NH^3 \frac{\partial^2 P}{\partial X^2} + H^{\alpha+2} \left( \frac{\partial P}{\partial X} \right)^{\alpha-1} \frac{\partial^2 P}{\partial X^2}, \quad (14)$$

where the pure number

$$N = \frac{2 + \alpha}{3\alpha} \left( \frac{2\tau_0L}{\hat{E}h_0^{\lambda+1}} \right)^{\alpha-1} = \frac{2 + \alpha}{3\alpha} \left[ \frac{2\tau_0}{p_c(h_0/L)} \right]^{\alpha-1} \quad (15)$$

modulates the relative importance of the Newtonian behaviour of the Ellis fluid at low shear rate, expressed by the first term on the r.h.s. of eq. (??), with respect to the second term, the power-law behaviour at high shear rate. For a Newtonian fluid ( $\alpha = 1$ )  $N$  reduces to unity; for a shear-thinning fluid ( $\alpha > 1$ ),  $N$  is zero for  $\tau_0 = 0$  and/or a rigid wall ( $\hat{E} = E_0/l^\lambda \rightarrow \infty$ ), but the latter case renders the scales (??) meaningless. In eq. (??) defining  $N$ , the quantity within brackets represents the ratio between the characteristic

233 shear stress  $\tau_0$  of the Ellis fluid and the pressure scale  $p_c = \hat{E}h_0^\lambda$  associated  
 234 with the elastic reaction of the fracture wall; the ratio is in turn corrected by  
 235 the initial aspect ratio of the fracture  $h_0/L$ . This formulation of  $N$  includes  
 236 only parameters defined at the single fracture scale. Note that if the scheme  
 237 of multiple fractures with spacing  $l$  depicted in Figure ?? is considered, eq.  
 238 (??) may be rewritten as

$$N = \frac{2 + \alpha}{3\alpha} \left[ \frac{2 \left( \frac{\tau_0}{E_0} \right) \left( \frac{l}{L} \right) \left( \frac{l}{L} \right)^{\lambda-1}}{\left( \frac{h_0}{L} \right)^2 \left( \frac{h_0}{L} \right)^{\lambda-1}} \right]^{\alpha-1}, \quad (16)$$

239 where  $\tau_0/E_0$  is the ratio between the representative shear stress of the fluid  
 240 and the Young modulus of the host rock, and  $l/L$  is the dimensionless fracture  
 241 spacing. The terms to the power  $(\lambda - 1)$  represent the contribution due  
 242 to non-linear elastic behaviour of the walls, and disappear for  $\lambda = 1$ . An  
 243 alternative formulation of  $N$  as a function of Cauchy, Reynolds, and Ellis  
 244 dimensionless groups is reported in ?. To grasp the order of magnitude  
 245 of  $N$ , we recall that  $l/L$  may be taken to vary between 0.03 and 0.3 (with  
 246  $l/L \approx 0.1$  being appropriate for an order of magnitude analysis), while the  
 247 initial fracture aspect ratio  $h_0/L$ , a number much smaller than 1, may be  
 248 considered of order  $10^{-3} - 10^{-5}$  [? ? ?]. The latter reference also reports  
 249  $E_0 = 2.5 \cdot 10^{10}$  Pa for the rock elastic modulus in fractured shales; quite close  
 250 values,  $E_0 = 3 \cdot 10^{10}$  Pa and  $E_0 = 2.76 \cdot 10^{10}$  Pa are reported in [?] and [?  
 251 ], hence reference values  $E_0 = 2.5 - 3.0 \cdot 10^{10}$  Pa are considered.

252 Actual values of rheological parameters for Ellis fluids are quite scarce  
 253 in the literature. A reference specific to fracking is [?], where the Ellis  
 254 parameters are reported for two fracturing fluids, HPG (Hydroxypropylguar)  
 255 and VES (viscoelastic surfactant). For the first,  $\mu_0 = 0.44$  Pa  $\cdot$  s,  $\tau_0 = 2.01$   
 256 Pa, and  $\alpha = 1.22$ ; for the second,  $\mu_0 = 49$  Pa  $\cdot$  s,  $\tau_0 = 8.836$  Pa, and  $\alpha = 12$ .  
 257 Adopting as reference geometrical parameters  $l/L = 0.1$  and  $h_0/L = 10^{-4}$ ,  
 258 and a young modulus of  $E_0 = 2.75 \cdot 10^{10}$  Pa for the host rock, one obtains  
 259  $N = 0.209$  for HPG and  $N \simeq 0$  for VES, indicating that for the latter fluid  
 260 the Newtonian component of rheological behaviour is negligible. A further  
 261 consideration is that VES is very strongly shear-thinning ( $\alpha \gg 1$ ), therefore  
 262 the value of  $N$  is extremely sensitive to variations in parameters: adopting  
 263 for example  $l/L = 0.125$ ,  $h_0/L = 10^{-5}$ , and  $E_0 = 2.5 \cdot 10^{10}$  Pa, again realistic  
 264 values, one obtains  $N = 0.100$  for VES and  $N = 0.618$  for HPG. This second

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 265 set of parameters is adopted for later reference in Section ?? describing a case  
 266 study and is shown there in dimensional form (see Table ??). Trying further  
 267 combinations of realistic values for fluid and rock properties, it is seen that  
 268  $N$  may take values smaller or larger than unity, the former case being more  
 269 frequent. This indicates a certain prevalence of the power-law component of  
 270 rheology over the Newtonian one, although the asymptotic system behaviour  
 271 is dominated by the latter, as will be shown in the next section. We bear in  
 272 mind that a large variety of combinations is possible for the two parameters  
 273  $N$  and  $\alpha$  depending on geometry and properties of fluid and rock, but with  
 274 the constraint from the definition (??) that for  $\alpha = 1$  it must be  $N = 1$ .

275 The dynamic boundary condition (??) and the boundary conditions (??)  
 276 transform as

$$\int_0^1 P(X, T) dX = H^\lambda - P_e + F_0, \quad (17)$$

$$H(0) = 1, \quad \frac{\partial P}{\partial X}(1, T) = 0, \quad P(0, T) = 0. \quad (18)$$

### 278 2.3. Solution

279 A solution to eq. (??) is sought by integrating in two steps the pressure  
 280 of the fluid and the fracture aperture. Posing

$$U(X, T) = \frac{\partial P}{\partial X}, \quad \dot{H} = \frac{dH}{dT}, \quad (19)$$

281 eq. (??) can be written as

$$B(1 + AU^{\alpha-1}) \frac{\partial U}{\partial X} = \dot{H} \quad (20)$$

282 where

$$A = A(T) = \frac{(H)^{\alpha-1}}{N}, \quad B = B(T) = NH^3, \quad (21)$$

283 while the second boundary condition in eq. (??) becomes

$$U(1, T) = 0. \quad (22)$$

284 Separating variables in eq. (??), and integrating with the boundary condition  
 285 (??) leads to

$$\frac{BU(AU^{\alpha-1} + \alpha)}{\alpha} = -\dot{H}(1 - X). \quad (23)$$

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286 Eq. (??) can be rewritten as

$$U^\alpha + CU + D(1 - X) = 0 \quad (24)$$

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14 287 where

$$C = C(T) = \frac{\alpha N}{H^{\alpha-1}}, \quad D = D(T) = \frac{\alpha \dot{H}}{H^{2+\alpha}}. \quad (25)$$

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18 Eq. (??) is algebraic in  $U$  and admits an analytical solution for  $\alpha = 1, 2, 3$   
19 and for  $\alpha = 1/2, 1/3$  in the form of a combination of functions of  $H$  and  $\dot{H}$ .  
20 This solution can be integrated once in space, with the boundary condition  
21  $P(0, T) = 0$ , obtaining the pressure field. The pressure field is finally inte-  
22 grated in  $X \in [0, 1]$  and the integral in eq. (??) is computed as a function  
23 of  $H$  and  $\dot{H}$ . Then eq. (??) is transformed in a nonlinear ODE which is  
24 numerically integrated with the initial condition  $H(0) = 1$ .  
25  
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27 These solutions are analytical in the  $x$  coordinate and numerical in the  
28 time domain and seem quite cumbersome, while their accuracy is comparable  
29 to that of a fully numerical solution in space and time; the latter also has  
30 the advantage of a free selection of the indicial parameter  $\alpha$ . Among the  
31 many possible numerical schemes, we adopt a finite difference in time and  
32 an implicit resolver in space, with a step size reduction to track solution  
33 accurately.  
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36 The code is written in Mathematica, introducing a parametric solver for  
37 the function  $U(X, T)$  as a function of  $N, \alpha, H_{i+1}, H_i, \Delta t$ , where  $H_{i+1}$  and  
38  $H_i$  are the values at time  $(i + 1)\Delta t$  and  $i\Delta t$ , respectively; the only free  
39 parameter is  $H_{i+1}$ , all the other parameters are given.  
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41 306 Each time iteration includes the following steps:

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- The function  $U(X)_{i+1}$  is estimated by solving eq. (??) in parametric form, with  $\dot{H} \approx (H_{i+1} - H_i)/\Delta t$ , with the term  $H$  taken to be the average between  $H_{i+1}$  and  $H_i$  and with the b.c.  $U(1)_{i+1} = 0$ , where  $H_{i+1}$  is the free parameter;  $H_0 = 1$  is assumed at the first step.
  - The space values of  $U$ , known in parametric form, are used to solve the differential problem  $\partial P(X)_{i+1}/\partial X = U(X)_{i+1}$ , with  $P(0)_{i+1} = 0$ , obtaining the pressure  $P(X)_{i+1}$ .
  - The pressure field is numerically integrated (in parametric form) in the domain  $[0, 1]$ .

- 316 • The parametric integral is inserted in eq. (??), and the equality is  
317 forced with a Newton method for finding the value of the parameter  
318  $H_{i+1}$ .
- 319 • The procedure is repeated for the next time step, shifting the values  
320  $H_{i+1}$ .

321 Once the pressure  $P(X, T)$  and aperture  $H(T)$  fields are known, the di-  
322 mensionless flowrate and fracture volume are given by

$$323 \quad Q(T) = \frac{dH(T)}{dT} = \dot{H}, \quad V(T) = H(T). \quad (26)$$

324 Hence at late-time the fracture volume and flowrate behave like the aperture  
325 and its time derivative, respectively; for zero borehole pressure and overload  
the corresponding time scalings are  $T^{-1/(\lambda+2)}$  and  $T^{-1/(\lambda+3)}$ .

### 326 3. Results and discussion

327 Figure ?? shows the results of the numerical computation for the fracture  
328 aperture and different  $\alpha$  values, with the analytical solution  $H = (1+9T)^{-1/3}$   
329 valid for the Newtonian case and a linearly elastic fracture [? ], corresponding  
330 to  $\alpha = 1$ ,  $N = 1$ , and  $\lambda = 1$ . Note that the values  $\alpha = 1$ ,  $N = 1$  imply  
331 Newtonian behaviour but with a viscosity equal to  $\mu_0/2$ , thus halving the  
332 time scale  $t_c$  in eq. (??); this requires doubling the dimensionless time  $T$   
333 in eq. (??) to compare results of equations having a different time scale.  
334 The time integration was performed with a time step  $\Delta t = 0.01$ . Since  
335 the results of the numerical integration using this fully explicit scheme fit  
336 exceedingly well the analytical solution, it was not necessary to adopt higher  
337 order schemes, even considering that the solution has no singularity and  
338 behaves rather smoothly.

339 The asymptotic behaviour of the solution  $H(T)$  is dictated by the inter-  
340 play between the two terms on the r.h.s. of eq. (??): the second term scales  
341 with the gradient pressure (decaying in time) and with a power of  $H$  always  
342 larger than 3, since  $\alpha > 1$ , whereas the first term scales with the third power  
343 of  $H$  and has  $N$  as a coefficient. Since  $H \leq 1$  and the gradient pressure  
344 quickly decays to values less than unity, the dominant term is the first one,  
345 which entails the asymptotic behaviour  $H \sim T^{-1/(2+\lambda)}$ , see Figure ?? where  
346 different values of  $\alpha$ , for  $N = 1$  and  $P_e = 0$ , produce almost parallel curves  
347 for large  $T$ . Figure ?? also shows how variations in  $\lambda$  significantly affect the

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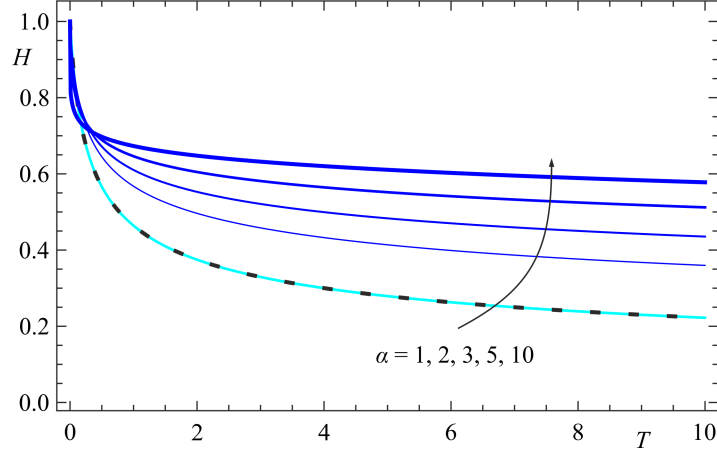


Figure 4: Time variation of the fracture aperture  $H$  for  $N = 0$ ,  $\lambda = 1$ ,  $P_e - F_0 = 0$  and different  $\alpha$  values. The black dotted curve refers to the analytical solution for a Newtonian fluid,  $H = (1+9T)^{-1/3}$ . Due to the different time scales adopted for a Newtonian fluid and for the present model, comparison is feasible if the dimensionless time  $T$  in the solution for the Newtonian fluid is doubled.

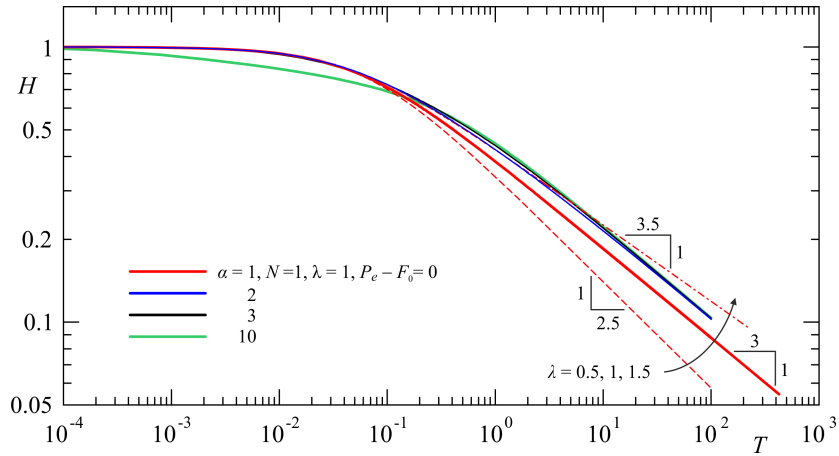


Figure 5: Time variation of the fracture aperture  $H$  for  $N = 1$ ,  $\lambda = 1$  and different  $\alpha$  values. For one case ( $\alpha = 1$ ) the effects of a softening/stiffening wall is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively. The asymptotic behaviour is  $H \sim T^{-1/(2+\lambda)}$ , independent on  $\alpha$ .

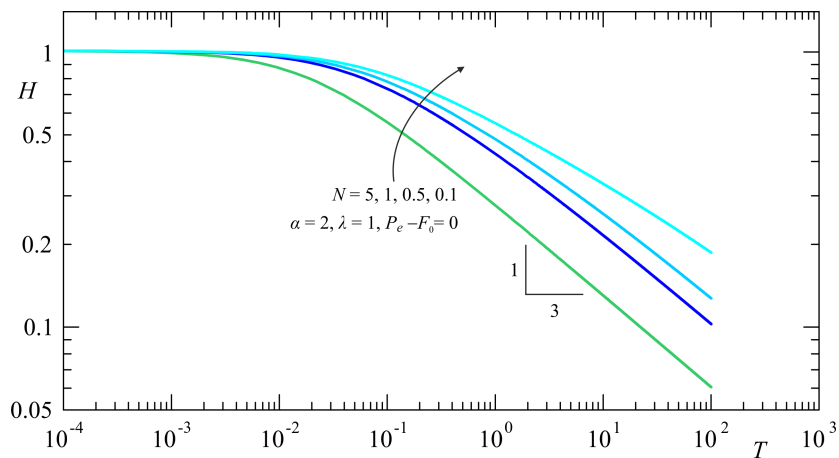


Figure 6: Time variation of the fracture aperture  $H$  for  $\alpha = 2$ ,  $\lambda = 1$  and different  $N$ .

late-time behaviour for fixed  $\alpha$ : a stiffening ( $\lambda > 1$ )/softening ( $\lambda < 1$ ) elastic reaction of the walls delays/facilitates the drainage. It is also seen that the parameter  $\alpha$  mainly controls the early stage, the parameter  $\lambda$  the late stage of the backflow process. Figure ?? shows results for a fixed  $\alpha = 2$ ,  $\lambda = 1$ , and different  $N$  values; the asymptote is reached much faster for larger  $N$ . In sum, the early time behaviour for zero external pressure at the well is in general dominated by the second term in eq. (??) unless the coefficient  $N \gg 1$ ; in the latter case both terms substantially contribute to the time evolution of  $H$ .

In presence of a non-zero external pressure ( $P_e > 0$ ) or a negative overload  $F_0$  (an additional force per unit of wall surface acting in the same direction of the internal pressure), the asymptotic residual aperture is equal to  $(P_e - F_0)^{1/\lambda}$ , see Figure ?? where both effects are included. The curves coalesce to the asymptote faster for larger  $N$  values, implying a dominance of the Newtonian behaviour, while for small  $N$  the power-law behaviour prevails and the asymptote is reached for larger dimensionless times. Upon plotting results for  $\alpha = 3$  (not shown) the main curves for  $\lambda = 1$  and the secondary curves for  $\lambda \neq 1$  are very similar to those for  $\alpha = 2$ .

Figure ?? shows the pressure distribution for two different combinations of the parameters and a shear-thinning fluid with  $\alpha = 2$ . Results for other combinations are similar (and thus not shown), with a pressure decay in space/time quicker or slower depending on the parameter values; at all times the residual pressure within the fracture increases with smaller  $N$  values,

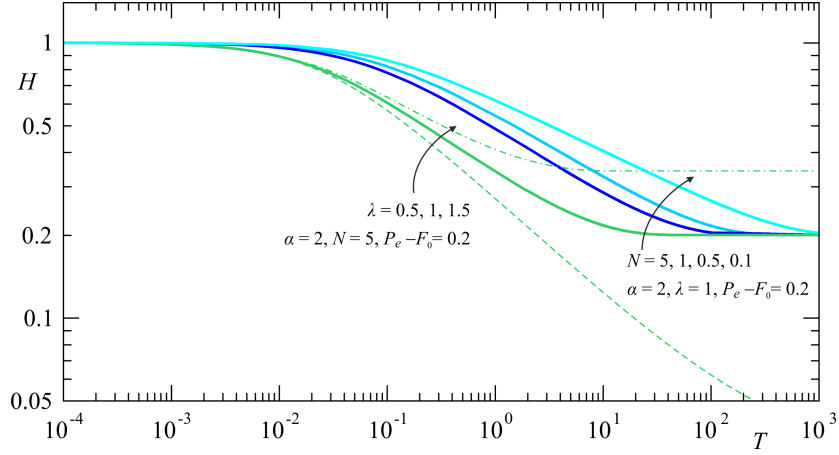


Figure 7: Time variation of the fracture aperture  $H$  for  $\alpha = 2$  and different  $N$  values, with given difference between external pressure and overload  $P_e - F_0 = 0.2$ . For one case ( $N = 5$ ) the effects of a stiffening/softening elastic reaction of the walls is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively.

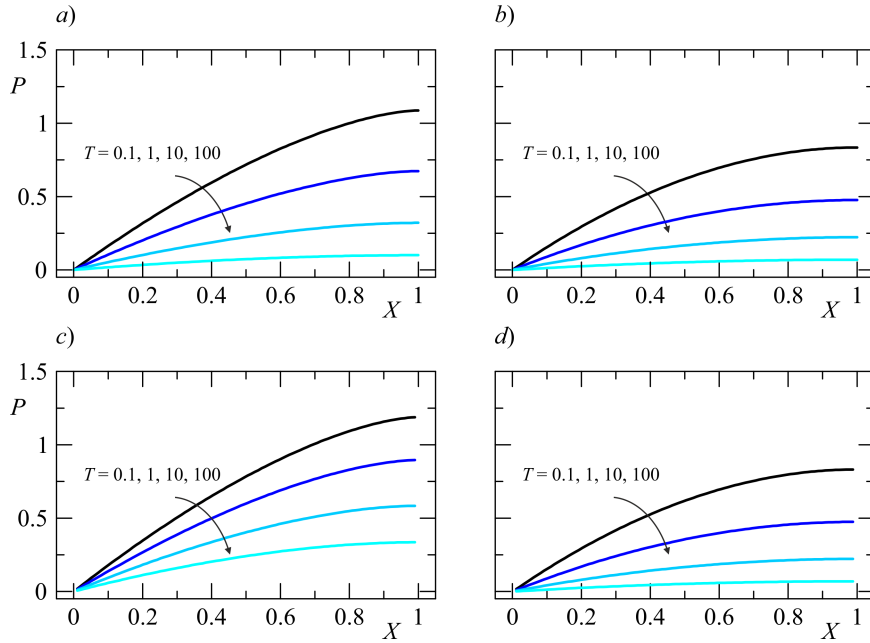


Figure 8: Pressure along the fracture at different times for  $P_e - F_0 = 0.2$  and a shear-thinning fluid with  $\alpha = 2$ . Results for a)  $N = 0.1$  and  $\lambda = 1$ ; b)  $N = 5$  and  $\lambda = 1$ ; c)  $N = 0.1$  and  $\lambda = 0.5$ ; d)  $N = 5$ ,  $\lambda = 0.5$ .

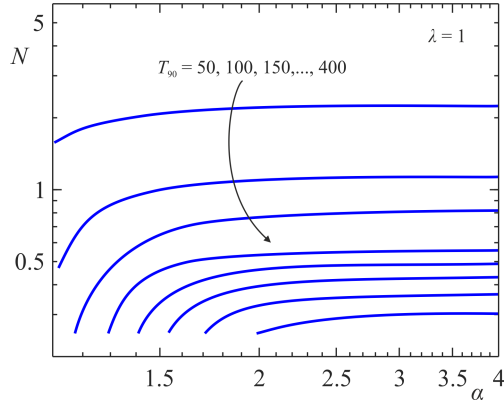


Figure 9: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1$  and  $P_e - F_0 = 0$ .

implying a behaviour closer to Newtonian, and with smaller  $\lambda$  values, i.e. a softening wall; however when the fluid is closer to Newtonian the effect of a  $\lambda$  variation is irrelevant.

An important quantity characterizing the performance of the backflow process is the time required to recover the fluid injected in the fracture network and not lost in the form of leakoff. Here the network is conceptualized as a single fracture and fluid losses are not explicitly represented (they are assumed to take place in the upstream network), however the time  $T_Y$  needed to recover  $Y\%$  of the fracture volume provides an indication of how rapid the recovery is. Contour maps in the  $(\alpha, N)$  space of the dimensionless time  $T_{90}$  needed to recover 90% of the fluid are depicted in Figure ?? for a linear wall reaction ( $\lambda = 1$ ). As the degree of shear-thinning behaviour rises with  $\alpha$  for constant  $N$ , there is a sharp increase in dimensionless  $T_Y$  for  $N < 0.5$ , while  $T_Y$  is almost independent on  $\alpha$  for  $N > 2$ . Conversely,  $T_Y$  for constant  $\alpha$  decreases with larger  $N$  values, i.e. as the fluid behaviour is closer to Newtonian; this effect is more evident for larger  $\alpha$ . Highest values of  $T_Y$  are attained for large  $\alpha$  and low  $N$ , lowest values for small  $\alpha$  and large  $N$ , the two combinations farthest and closest to Newtonian behaviour. The effect of a sublinear wall reaction ( $\lambda = 0.5$ ) is depicted in Figure ??, that of a supra-linear wall reaction in Figure ?. The dimensionless time to recover the bulk of the stored fluid is decidedly faster or slower with a softening or stiffening wall, demonstrating once again the decisive influence of the parameter  $\lambda$  modulating the wall reaction at late time.

A word of caution is needed when drawing comparisons between non-

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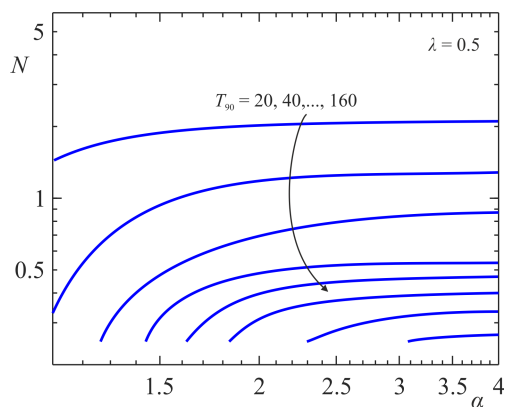


Figure 10: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 0.5$  and  $P_e - F_0 = 0$ .

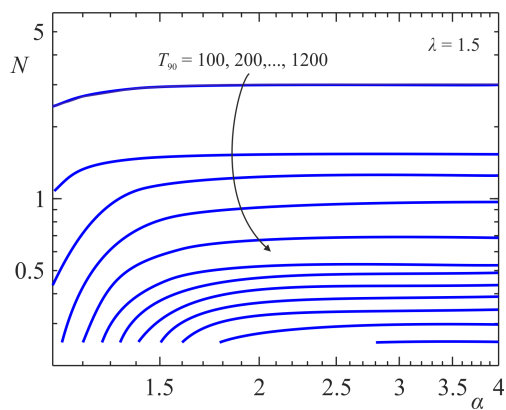


Figure 11: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1.5$  and  $P_e - F_0 = 0$ .

Fluid	$\mu_0$ (Pa s)	$\tau_0$ (Pa)	$\alpha$	$L$ (m)	$l$ (m)	$h_0$ (mm)	$E$ (Pa)	$\lambda$	$N$
HPG	0.44	2.01	1.22	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.618
VES	49.00	8.836	12.00	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.100

Table 1: Reference parameters for case study:  $\mu_0$ ,  $\tau_0$  and  $\alpha$  are the reference viscosity, shear stress and indicial exponent of the Ellis fluid,  $L$  is the fracture length,  $l$  is the fracture spacing,  $h_0$  is the fracture initial height,  $E$  is the rock modulus of elasticity,  $\lambda$  is the exponent of the rock wall reaction,  $N$  is the dimensionless number governing the interplay between Newtonian and power-law behaviour in an Ellis fluid.

Newtonian fluids with different rheology as the models are semi-empirical and the time scale used for the dimensionless formulation depends upon the rheological parameters of the Ellis model and is particularly sensitive to the value of the indicial exponent  $\alpha$ . Hence model outputs are best compared in dimensional coordinates when quantitative results are needed.

#### 4. A case study

A case study is illustrated by comparing the performance of two real hydrofracturing fluids [? ], HPG (Hydroxypropylguar) and VES (viscoelastic surfactant) in a realistic setting. The rheological parameters according with the Ellis model are reported for both fluids in Table ??, together with realistic geometric and mechanical parameters within plausible ranges deduced from the literature, see the earlier discussion in Section ?. It is seen that HPG is relatively close to Newtonian in behaviour, while VES is extremely shear-thinning, with an equivalent rheological index  $n$  less than 0.1 when expressed according to the power-law model.

Figure ?? shows the relaxation of the fracture aperture for the two fluids: the aperture for the HPG is only initially slightly larger than for the VES, but then closes more rapidly, reaching one tenth of the initial value at a time around 500 hours. The closure is much more gradual for the VES, requiring about a year to reach the same stage. The difference between corresponding pressure profiles, illustrated in Figure ??, shows a decidedly sharper pressure decrease for HPG than for VES in the initial stage.

Figure ?? shows the time to recover the volume stored in the fracture for the two fluids. Following the same trend manifested for the evolution of

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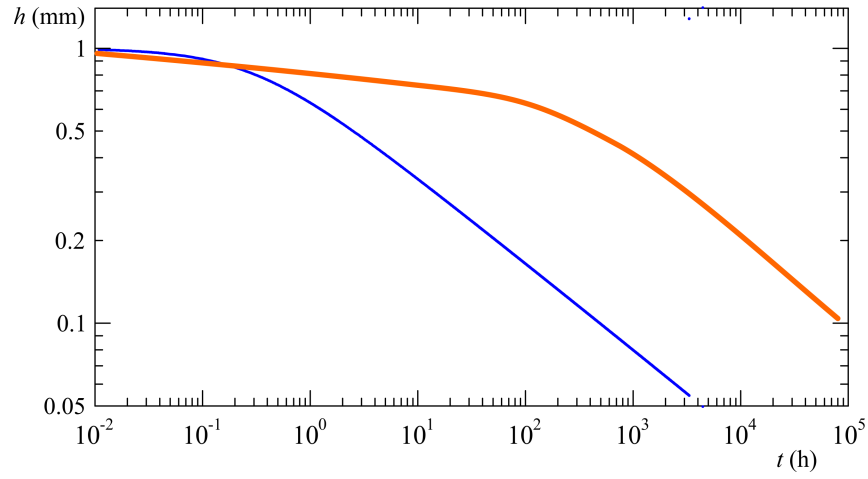


Figure 12: Time variation of the fracture aperture  $h$  for the HPG (thin line) and VES (thick line) fluids.

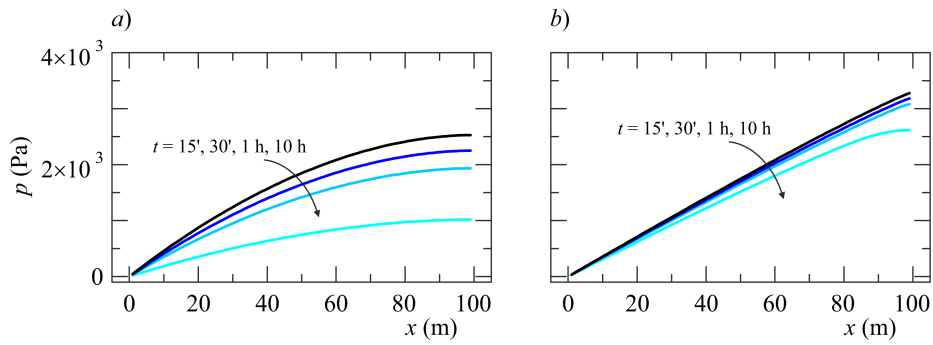


Figure 13: Pressure distribution at different time *a)* for HPG fluid, and *b)* for VES.

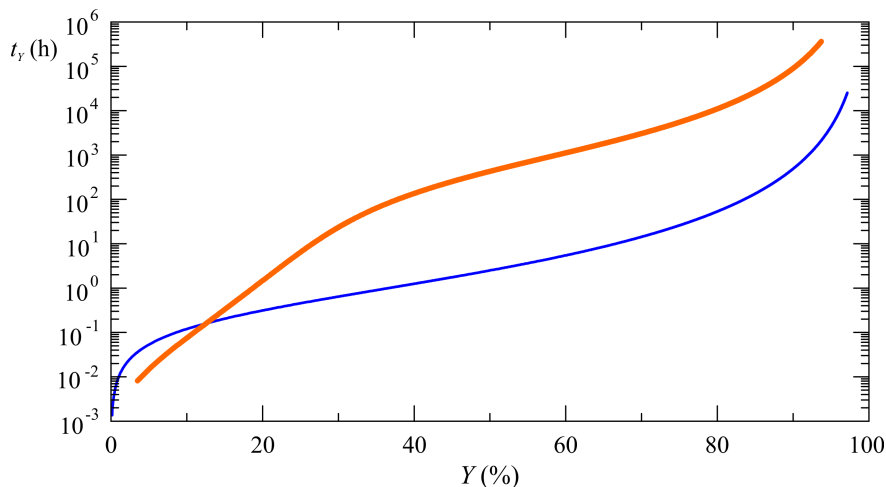


Figure 14: Time to recover the fracture volume  $Y\%$  for the HPG (thin line) and VES (thick line) fluids.

419 fracture opening, VES demonstrates a higher drainage capacity than HPG in  
 420 the very early phase, for  $Y < 15\%$ ; subsequently it is much less efficient, and  
 421 requires an extra time at least three orders of magnitude larger to drain the  
 422 same percentage of fluid than HPG. Overall the large difference in rheology,  
 423 mainly encapsulated in the  $\alpha$  value, translates into corresponding wide differ-  
 424 ences in terms of aperture, pressure, and drainage time. This is so because  
 425 the value of the dimensionless group  $N$  is very low for VES, thus allowing the  
 426 fluid to manifest its essentially power-law nature. We tried a number of other  
 427 combinations of parameters and found that for very shear thinning fluids like  
 428 VES the results are very sensitive to relatively small changes in parameters:  
 429 slightly increasing the modulus of elasticity  $E$  to  $3 \cdot 10^{10}$  and increasing the  
 430 spacing to 20 m, leaving the other parameters in Table ?? unchanged, leads  
 431 to  $N(HPG) = 0.659$  and  $N(VES) = 2.360$ . While the change in the  $N$   
 432 value associated to HPG is modest (6.6%) and implies the system behaviour  
 433 is essentially unchanged with respect to the reference case, the increase in  
 434  $N$  for the VES is dramatic (2260%) and entails a fluid behaviour closer to  
 435 Newtonian despite the exceedingly high value of  $\alpha$ . Upon plotting the aper-  
 436 ture variation over time for this case (not shown) the two fluids exhibit a  
 437 similar behaviour, with only modest differences (less than 10%) in the frac-  
 438 ture aperture at early times and an almost identical behaviour later on. The  
 439 pressure profiles do not show any significant differences.

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440 **5. Conclusions**

441 A conceptual model for backflow of non-Newtonian fluid from a closing  
442 rock fracture was presented in this paper. Under the assumption of Ellis rhe-  
443 ology and elastic, but non-deformable wall, the problem in plane geometry is  
444 tractable in semi-analytical form to yield the time-variable fracture aperture  
445  $h(t)$ , pressure field  $p(x, t)$  and discharge rate  $q(t)$ , as well as the drainage  
446 time  $t_Y$  for a specified recovery rate  $Y$ , outlet pressure  $p_e$  and overload  $f_0$ .

447 Our results lead to the following specific conclusions:

- 448 • The Ellis model adopted herein to describe shear-thinning rheology  
449 couples Newtonian and power-law behaviour. When an Ellis fluid back-  
450 flows from a relaxing fracture the interplay between the two natures  
451 is modulated by a dimensionless group  $N$  encapsulating the main prob-  
452 lem parameters.  $N$  can be expressed in terms of i) the indicial exponent  
453  $\alpha$  of the Ellis rheology, ii) the parameter  $\lambda$  governing the wall relaxation  
454 process, iii) the ratio between the characteristic shear stress of the Ellis  
455 fluid  $\tau_0$  and the rock modulus of elasticity  $E$ , iv) two geometric ratios,  
456 the fracture initial aspect ratio  $h_0/L$  and dimensionless spacing  $l/L$ .  
457 An alternative format of  $N$  is a modified ratio between the Cauchy  
458 number and the product of Reynolds and Ellis numbers.
- 459 • The factors  $N$  and  $\alpha$  mostly influence the early and intermediate time  
460 evolution of the system: when  $N < 1$  the power-law behaviour prevails;  
461 for  $N = 1$  the pure Newtonian case is recovered ( $\alpha = 1$  entails  $N = 1$ ),  
462 while for  $N \gg 1$  the behaviour is mixed.
- 463 • For late-time the system behaviour tends to Newtonian, is independent  
464 of  $N$  and is governed by the wall relaxation parameter  $\lambda$ : aperture and  
465 discharge scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  
466  $p_e - f_0 = 0$ ; else, the aperture tends asymptotically to a constant value  
467 proportional to  $(p_e - f_0)^{1/\lambda}$ .
- 468 • Very shear-thinning fluids (larger  $\alpha$ ) and reactive walls (larger  $\lambda$ ) are  
469 associated with a more gradual closure of the aperture.
- 470 • The residual pressure within the fracture increases with smaller  $N$  val-  
471 ues and with a softening wall ( $\lambda < 1$ ); when the fluid is close to New-  
472 tonian the effect of a  $\lambda$  variation is almost irrelevant.

- 473 • The dimensionless drainage time  $T_Y$  attains the largest values for large  
474  $\alpha$  and low  $N$ , the lowest values for small  $\alpha$  and large  $N$ , the two com-  
475 binations farthest and closest to Newtonian behaviour. A non-linear  
476 reaction of the walls result in a faster/slower recovery for  $\lambda < 1$  (soft-  
477 ening) and  $\lambda > 1$  (stiffening). For recovery values close to 100%,  $T_Y$  is  
478 very sensitive to variations of model parameters.
- 479 • Results are discussed in dimensional form for a case study to reinforce  
480 the notion that dimensionless results need to be compared with caution  
481 as scales include fluid rheological parameters. Realistic geometric and  
482 mechanical parameters are adopted for a system of equally spaced frac-  
483 tures, and results are compared for two fluids, HPG and VES, normally  
484 used in fracking technology. The time evolution of the aperture and  
485 the dependence of the drainage time upon the recovery ratio are similar  
486 at early times, then differ by orders of magnitude at intermediate and  
487 late times.

488 The developments presented, together with earlier results [? ? ], provide  
489 an overview of the backflow phenomenon in the two basic geometric configu-  
490 rations for a single fracture, plane and radial, and for three rheological models  
491 of increasing complexity: Newtonian, power-law, and Ellis. Further improve-  
492 ments of the model remain open in several directions, e.g.: i) a more complex  
493 geometry, considering nonplanar fractures with non-negligible curvature; ii)  
494 the combination of non-Newtonian rheology with multiple fracture systems,  
495 adopting the asymptotic viewpoint of Dana et al. [? ]; iii) the incorporation  
496 of particle transport to simulate the settling of solid proppant.

## 497 **Acknowledgments**

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500 conflicts of interest to declare. There are no data sharing issues since all of  
501 the numerical information is provided in the figures produced by solving the  
502 equations in the paper.

## 503 **Appendix A. The Newtonian case ( $n = 1$ )**

504 For  $\alpha = 1$  and  $N = 1$  eq. (??) reduces to

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$$C = 1, \quad D = \frac{\dot{H}}{H^3}, \quad (\text{A.1})$$

505 and integrating eq. (??) using these expressions yields

$$P(X, T) = \frac{\dot{H}}{4H^3} [(X - 1)^2 - 1]. \quad (\text{A.2})$$

506 Substituting in eq. (??) and integrating  $P(X, T)$  over  $X$  gives

$$-\frac{\dot{H}}{3H^3} = H^\lambda - P_e + F_0, \quad (\text{A.3})$$

507 generalizing eq. (2.14) of Dana et al. [? ], where  $\lambda = 1$  and  $F_0 = 0$ ,  
508 to nonlinear wall reaction and non-zero overload. Now define an effective  
509 pressure  $\tilde{P}_e = P_e - F_0$  at the fracture outflow: this symbol will be used for  
510 brevity in the sequel. Consider first the case  $\tilde{P}_e = 0$ . Integration of eq. (??)  
511 over time  $T$  yields, with the first b.c. in eq. (??),

$$H(T) = [1 + 3(2 + \lambda)T]^{-\frac{1}{2+\lambda}}, \quad (\text{A.4})$$

512 that for  $\lambda = 1$  gives back eq. (2.15) of [? ].

Consider now the case  $\tilde{P}_e > 0$ . Integration with the help of Mathematica  
and using transformation formulae for the analytic continuation of hypergeometric  
functions [? ] yields for generic  $\lambda$  the following implicit equation

$$T = \frac{1}{3(\lambda + 2)} \left[ \frac{1}{H^{\lambda+2}} {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \frac{\tilde{P}_e}{H^\lambda} \right) + \right. \\ \left. - {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \tilde{P}_e \right) \right], \quad (\text{A.5})$$

where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is the hypergeometric function of parameters  $\alpha, \beta, \gamma$ ,  
and argument  $z$ . Specific results for  $\lambda = 1/2, \lambda = 1, \lambda = 2$ , i.e. a sublinear,  
linear or supralinear wall reaction, can be obtained as

$$T = \frac{1}{18\tilde{P}_e^5} \left[ 12 \ln \left( \frac{H^{1/2}(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{12\tilde{P}_e}{H^{1/2}} - \frac{6\tilde{P}_e^2}{H} - \frac{4\tilde{P}_e^3}{H^{3/2}} - \frac{3\tilde{P}_e^4}{H^2} + \right. \\ \left. + 12\tilde{P}_e + 6\tilde{P}_e^2 + 4\tilde{P}_e^3 + 3\tilde{P}_e^4 \right], \quad (\text{A.6})$$

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$$T = \frac{1}{6\tilde{P}_e^3} \left[ 2 \ln \left( \frac{H(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{2\tilde{P}_e}{H} - \frac{\tilde{P}_e^2}{H^2} + 2\tilde{P}_e + \tilde{P}_e^2 \right], \quad (\text{A.7})$$

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$$T = \frac{1}{6\tilde{P}_e^2} \left[ \ln \left( \frac{H^2(1 - \tilde{P}_e)}{H^2 - \tilde{P}_e} \right) - \frac{\tilde{P}_e}{H^2} + \tilde{P}_e \right], \quad (\text{A.8})$$

515 either by direct integration of eq. (??) or using transformations involving  
516 the hypergeometric functions [? ]. Eq. (??) valid for  $\lambda = 1$  is identical  
517 to Eq. (2.18) of Dana et al. [? ]. Other results in terms of trascendental  
518 and algebraic functions can be obtained for other special values of  $\lambda \in \mathbb{N}$  or  
519  $1/\lambda \in \mathbb{N}$  but are too cumbersome to report and/or of little technical interest.

520 Expressions (??)-(??), when evaluated for for given  $\tilde{P}_e$ , allow deriving  
521  $H(T)$  and the drainage time  $T_Y$  needed to drain  $Y\%$  of the fracture volume.  
522 As the latter quantity is given in dimensionless form by  $H$  according to (??),  
523 to derive  $T_Y$  it is sufficient to evaluate (??) and its special cases (??)-(??)  
524 for  $H = (100 - Y)/100$ .

525 Finally, it is worthwhile to derive the asymptotic behaviour of the general  
526 equation (??) for the limit case  $\lambda \rightarrow 0$ . According to eq. (??),  $\lambda = 0$  implies  
527 a wall reaction constant over time rather than dependent from the fracture  
528 aperture. Integrating (??) for  $H^\lambda = 1$  gives

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$$H = \frac{1}{[1 + 6(1 - \tilde{P}_e)T]^{1/2}}, \quad (\text{A.9})$$

530 a result that can be simplified for large time to  $H = 1/[6(1 - \tilde{P}_e)T]^{1/2}$  and  
531 further for  $\tilde{P}_e = 0$  to  $H = 1/(6T)^{1/2}$ . Equation (??) can be also obtained  
532 directly from eq. (??) for  $\lambda \rightarrow 0$  on the basis of eq. (9.121.1) in [? ]. The late-  
533 time scaling for a Newtonian fluid and a wall with constant reaction ( $\lambda = 0$ )  
534 is therefore  $H \propto T^{-1/2}$ , a result coinciding with the scaling  $H \propto T^{-1/(2+\lambda)}$   
implied by Figure ?? for a Newtonian fluid with  $N = 1$ ,  $\alpha = 1$ .

535 **Appendix B. The dimensionless group  $N$**

536 The pure number  $N$  may be expressed as a function of well-known dimen-  
537 sionless groups in fluid mechanics [see, e.g., ? ]. Multiplying and dividing  
538 eq. (??) by  $\rho\mu_0 h_0 u_0^3$ , where  $u_0$  is the reference velocity defined in (??), yields

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$$N = K \left( \frac{\text{Ca}}{\text{Re} \cdot \text{El}} \right)^{\alpha-1}; \quad \text{Ca} = \frac{\rho u_0^2}{E}; \quad \text{Re} = \frac{2\rho u_0 h_0}{\mu_0}; \quad \text{El} = \frac{\mu_0 u_0}{\tau_0 h_0}, \quad (\text{B.1})$$

539

$$K = K(\alpha, \lambda, l/L, h_0/L) = \frac{2 + \alpha}{3\alpha} \left[ \frac{4 \left( \frac{l}{L} \right)^\lambda}{\left( \frac{h_0}{L} \right)^{\lambda+1}} \right]^{\alpha-1} \quad (\text{B.2})$$

540 where Ca, Re, and El are the Cauchy, Reynolds, and Ellis numbers, and  $K$   
541 a geometric factor correcting the ratio  $\text{Ca}/(\text{Re} \cdot \text{El})$ . In turn, Ca is the ratio  
542 between inertial forces and elastic forces transmitted by solid walls, Re is the  
543 ratio between inertial and viscous forces, while El is the ratio between the  
544 viscous stress associated with the low shear rate Newtonian behaviour and  
545 the shear stress  $\tau_0$  associated with high shear rate non-Newtonian (power-  
546 law) behaviour.

1 Relaxation-induced flow in a smooth fracture for Ellis  
 2 rheology

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7 **Abstract**

8 Hydraulic fracturing is a process aimed at improving the productivity of oil,  
 9 gas or geothermal reservoirs. During hydrofracturing, backflow follows injec-  
 10 tion and represents the second phase of the process, when part of the fractur-  
 11 ing fluid returns from fractures to well, and from well to surface. A concep-  
 12 tual model is presented to grasp the essential features of the phenomenon,  
 13 conceiving the draining subsurface domain as a planar and rigid fracture.  
 14 Backflow against an outlet pressure in the injection well is induced by the  
 15 relaxation of the fracture wall, exerting a force on the fluid proportional to  
 16  $h^\lambda$ , with  $h$  the time-variable aperture and  $\lambda$  a non-negative exponent; an  
 17 overload on the fracture may contribute to slowing or accelerating the clo-  
 18 sure process. The fluid rheology is described by the three-parameter Ellis  
 19 constitutive equation, well representing the shear-thinning rheology typical  
 20 of hydrofracturing fluids and coupling Newtonian and power-law behaviour.  
 21 The interplay between these tendencies is modulated by a dimensionless num-  
 22 ber  $N$  encapsulating most problem parameters; the range of variation of  $N$   
 23 is discussed and found to vary around unity. The time-variable aperture  
 24 and discharge rate, the space-time variable pressure field, and the time to  
 25 drain a specified fraction of the fracture volume are derived as functions of  
 26 geometry (length and initial aperture), wall elastic parameters, fluid prop-  
 27 erties, outlet pressure  $p_e$  and overload  $f_0$ . The late-time behaviour of the  
 28 system is practically independent from rheology as the Newtonian nature of  
 29 the fluid prevails at low shear stress. In particular, aperture and discharge  
 30 scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  $p_e - f_0 = 0$ ; else,  
 31 the aperture tends to a constant, residual value proportional to  $(p_e - f_0)^\lambda$ .  
 32 A case study with equally spaced fractures adopting realistic geometric, me-

33 chanical and rheological parameters is examined: two fluids normally used  
34 in fracking technology show completely different behaviours, with backflow  
35 dynamics and drainage times initially not dissimilar, later varying by orders  
36 of magnitude.

37 *Key words:*

38 Hydraulic fracturing, Non-Newtonian, Ellis rheology, elastic wall, backflow

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## 39 1. Introduction

40 Hydraulic fracturing is a process aimed at improving the productivity of  
41 oil, gas or geothermal reservoirs. Analysis of the different phases of hydraulic  
42 fracturing is of particular modeling and experimental interest [e.g. 1, 2].

43 An understanding of fractured media flow induced by the relaxation of  
44 elastic fracture walls is crucial in modeling fracturing fluid backflow, a com-  
45 plicated phenomenon involving hydrodynamic, mechanical and chemical pro-  
46 cesses. Backflow is typically the final phase of the hydraulic fracturing pro-  
47 cess: in the first one, fracturing fluid is injected at high pressure in a rock  
48 mass, forming new fractures and enlarging existing ones; in the second phase,  
49 proppant is introduced in the subsurface environment to prop fractures open;  
50 then when the injection ceases, the pressure drops, existing and new fractures  
51 tend to close, and a portion of the injected fracturing fluid, often mixed  
52 with proppant [3], flows back towards the injection well and interact with  
53 the relaxing walls of the fractures. As the retention of fracturing fluid in  
54 the fracture network impairs the fracture conductivity reducing the wellbore  
55 productivity [4], and favours migration in the subsurface environment along  
56 different pathways [5], it is of utmost interest to optimize the amount of fluid  
57 recovered, irrespective of the reservoir product, be it oil [6], gas [7] or heat  
58 [8].

59 The scientific literature offers two main approaches to modeling backflow:  
60 (i) detailed numerical simulations involving single fractures [9], fracture net-  
61 works [10] or dual or triple porosity models [11], or (ii) conceptual models  
62 capturing the main features of the interaction between fracture flow and  
63 wall relaxation [12], including the effects of branching networks described at  
64 different degrees of complexity [13, 14]. A recent addition to the modeling  
65 effort is the influence of fluid rheology, following the notion that the backflow  
66 fluid is non-Newtonian in the widest sense [15], as not only the relationship  
67 between shear stress and shear rate is nonlinear, but also exhibits normal

68 stress and temperature dependency, as well as viscoelasticity, thixotropy,  
69 and nonzero yield stress [16]. At the same time, non-Newtonian fluids allow  
70 achieving several engineering objectives, such as (i) minimize the pressure-  
71 drop in the entire process; (ii) carry suspended proppant; (iii) minimize the  
72 leak-off within the formation; (iv) adapt their characteristics to different en-  
73 vironments in terms of temperature and chemical composition; and (v) flow  
74 back easily towards the wellbore. Given their versatility and economic value,  
75 these fluids are typically treated for reuse once recovered, removing contam-  
76 inants they may have transported to the surface [17]. The recovery ratios of  
77 backflow fluid vary between 2% and 48% according to Ipatova and Chuprakov  
78 [18], with considerable economic value.

79 Modeling non Newtonian backflow is in its early stage, in variance with  
80 the injection and fracture formation stage, for which several conceptualiza-  
81 tions and models are available: see Detournay [19] for a review and the recent  
82 work by Wrobel [20] comparing different rheological models for fracturing  
83 fluids. To the best of our knowledge, only Chiapponi et al. [21] considered  
84 non-Newtonian fluids in the context of backflow modeling: these authors  
85 examined flow of a power-law fluid towards a wellbore in a single fracture  
86 of annular geometry, supporting their theoretical findings with laboratory  
87 experiments. The present paper develops the analysis of non-Newtonian  
88 backflow for a smooth fracture, common in field applications [22], and adds  
89 realism by employing a three-parameter Ellis model, that well represents the  
90 rheology of hydrofracturing [23] and drilling fluids [24]. The Ellis model tends  
91 to Newtonian for low shear rates, to power-law for high shear rates and al-  
92 lows avoiding the unphysical effect of infinite apparent viscosity at zero shear  
93 rate that is typical of the power-law model [25]. We note in passing that our  
94 results are of a general nature for Newtonian pressurized flow in ducts of  
95 variable width and may be of interest for, and be applied also to, deformable  
96 microfluidic [26] and biological [27] systems.

97 The plan of the paper is as follows. Section 2 formulates the problem  
98 of relaxation-induced backflow of an Ellis fluid in a fracture with nonlin-  
99 ear wall reaction and subject to overload. Numerical results obtained are  
100 presented and discussed in Section 3 as a function of dimensionless groups  
101 characterizing the system: the indicial exponent  $\alpha$  quantifying the degree  
102 of shear-thinning behaviour of the Ellis fluid, the non-negative exponent  $\lambda$   
103 modulating the fracture wall reaction, and a further group  $N$  encapsulating  
104 most problem parameters. Section 4 illustrates an hypothetical case study  
105 adopting realistic geometric and mechanical parameters and two real hy-

106 drofracturing fluids described by the Ellis model. Section 5 reports the main  
107 conclusions and perspectives for future work. In Appendix A the special case  
108 of a Newtonian fluid is examined, obtaining results that generalize those of  
109 Dana et al. [13] to a nonlinear wall reaction, while Appendix B presents  
110 an alternative expression for the dimensionless number  $N$ , shown to be a  
111 combination of well-known dimensionless groups in fluid mechanics.

## 112 2. Material and methods

### 113 2.1. Problem statement

114 A rock fracture produced by hydrofracturing, though of irregular geom-  
115 etry, is often conceptualized for modeling purposes as a 3-D space of length  
116  $L$ , width  $W$ , and aperture  $h$  between two parallel walls [28]; the Cartesian  
117 coordinate system  $x, y, z$  is illustrated in Figure 1 and the fracture is subject  
118 to a pressure gradient  $\nabla p' \equiv (\partial p'/\partial x, 0, 0)$  in the  $x$  direction. In horizontal  
119 fractures, the additional gravity-induced pressure gradient is perpendicular  
120 the flow plane and has no effect on the flow field. If the  $(x, y)$  plane is not  
121 horizontal, the  $z$  direction perpendicular to the walls is not vertical and grav-  
122 ity effects can be included in a reduced pressure term  $p$ , thus leading to a  
123 mathematical treatment with no gravity term to consider. For instance, for  
124 the Figure 3 below representing multiple vertical fractures backflowing to an  
125 horizontal well, the reduced pressure  $p$  is equal to  $p = p' + \rho gy$ .

126 The walls are taken to be rigid, so that the aperture  $h(t)$  is solely a  
127 function of time, and the deformation is concentrated for mathematical con-  
128 venience in the upper wall, that behaves as a nonlinear elastic foundation  
129 exerting a reaction on the fluid. At  $t = 0$  the relaxation of the wall in-  
130 duces a backflow in the negative  $x$  direction, and the fracture begins to drain  
131 subject to a constant outlet pressure  $p_e$  at  $x = 0$  and to a no-flow bound-  
132 ary condition at the upstream end  $x = L$ . Three further hypotheses are  
133 adopted: i) the flow is quasi-steady, allowing to neglect the time derivative  
134 of the velocity in the momentum equation; ii) the fracture aspect ratio is  
135 small,  $h_0/L \ll 1$ , warranting the lubrication approximation, and iii) the flow  
136 is essentially one-dimensional along  $x$ ,  $L \gg W$ . The latter conceptualiza-  
137 tion is usually adopted in hydrogeology also when the two dimensions are  
138 comparable, as it is often the case for rock fractures [11].

139 The flowback fluid is taken to be incompressible of density  $\rho$ , non-Newtonian  
140 shear-thinning [15] and described by the Ellis three-parameter model [29].  
141 Under the above assumptions, the fluid undergoes simple shear flow in the

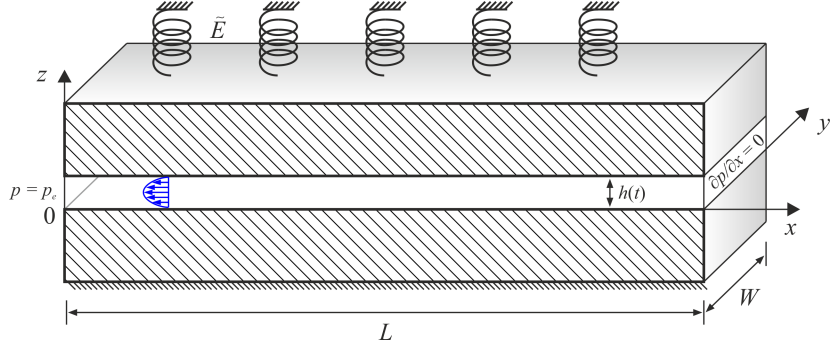


Figure 1: Layout of a plane fracture of variable uniform aperture  $h(t)$ .

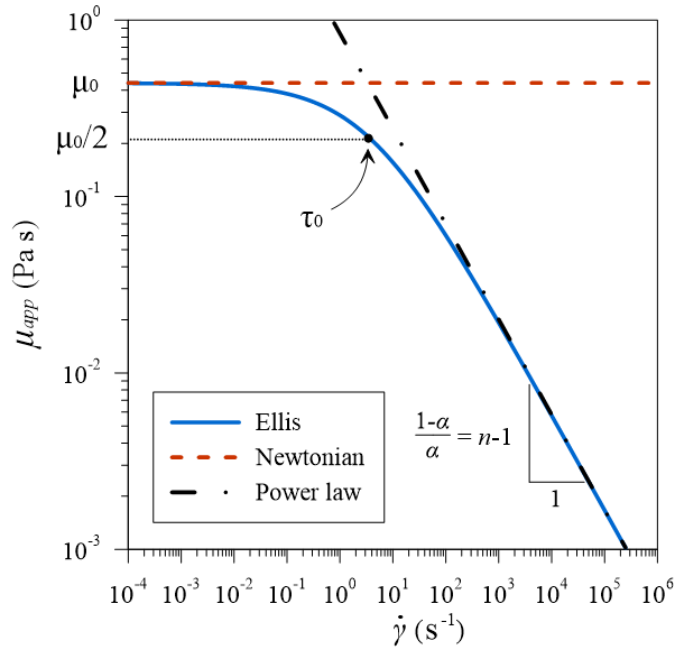


Figure 2: Apparent viscosity for three rheological models: Ellis (blue solid line) of parameters  $\mu_0$ ,  $\tau_0$ ,  $\alpha$ ; Newtonian (red dashed line) of viscosity  $\mu_0$ ; power-law (black dot-dashed line) of consistency index  $m$  and rheological index  $n$ . The comparison with the latter is drawn assuming:  $\alpha = 1/n$  and  $\tau_0 = (m/\mu_0^n)^{n/(1-n)}$ .

142  $x$  direction, and the Ellis rheology is described by the following relationship  
 143 between shear stress  $\tau_{zx}$  (hereinafter  $\tau$ ) and shear rate  $\dot{\gamma}_{zx}$  (hereinafter simply  
 144  $\dot{\gamma}$ )

$$\tau = \frac{\mu_0}{1 + (\tau/\tau_0)^{\alpha-1}} \dot{\gamma}; \quad \dot{\gamma} = \frac{\partial u}{\partial z}, \quad (1)$$

145 where  $u$  is the velocity in the  $x$  direction. The rheological law (1) features a  
 146 viscosity parameter  $\mu_0$ , a constant  $\tau_0$  defined as the shear stress corresponding  
 147 to apparent viscosity  $\mu_0/2$ , and an indicial parameter  $\alpha$ , typically larger than  
 148 one as the fluid is shear-thinning. For  $\alpha = 1$ , a pseudo-Newtonian behaviour  
 149 with dynamic viscosity  $\mu_0/2$  is recovered, see Figure 2 showing the apparent  
 150 viscosity  $\mu_{app} = \tau/\dot{\gamma}$  for the Ellis model compared to Newtonian and power-  
 151 law models. Newtonian behaviour in the form of a plateau for low shear rates  
 152 is also observed for  $\alpha \rightarrow 0$ . For high shear rates the behaviour is power-law,  
 153 and its two parameters can be determined from the Ellis model parameters,  
 154 see Appendix A in Balhoff and Thompson [28]; in particular, the rheological  
 155 index is  $n = 1/\alpha$  [30]. Note that when curve fitting is performed on real data,  
 156  $n$  and  $1/\alpha$  may significantly differ [23], as two different models are fitted  
 157 to the same data set. It is also seen that the Ellis model allows avoiding  
 158 the unphysical effect of infinite apparent viscosity at zero shear rate that is  
 159 typical of power-law fluids [25]. In the following, we will consider  $\alpha > 1$ ,  
 160 dealing with the case  $\alpha = 1$  in the Appendix, and the parameters  $\mu_0$  and  $\tau_0$   
 161 to be finite and positive. Couette-Poiseuille slit flow of an Ellis fluid under  
 162 a constant pressure gradient was studied extensively by Steller [31], listing  
 163 all combinations of parameters leading to Newtonian or pseudo-Newtonian  
 164 behaviour. In particular, the negative velocity  $u(z)$  under a positive reduced  
 165 pressure gradient  $\partial p/\partial$  in the  $x$  direction is

$$u(z, t) = -\frac{1}{8\mu_0} \left[ h^2 - (2z - h)^2 \right] \frac{\partial p}{\partial x} + \frac{1}{(\alpha + 1)2^{\alpha+1}\mu_0\tau_0^{\alpha-1}} \left[ h^{\alpha+1} - |2z - h|^{\alpha+1} \right] \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}. \quad (2)$$

166 The corresponding average velocity  $\bar{u}$  and flow per unit width  $q_x$  in the  $x$   
 167 direction are

$$\bar{u} = -\frac{h^2}{12\mu_0} \frac{\partial p}{\partial x} - \frac{h^{\alpha+1}}{2^{\alpha+1}(\alpha + 2)\mu_0\tau_0^{\alpha-1}} \frac{\partial p}{\partial x} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1}; \quad q_x = \bar{u}h. \quad (3)$$

168 For the Newtonian case ( $\alpha = 1$ ) the latter equation reduces to the clas-  
 169 sical “cubic law” [32] written for a fluid with viscosity  $\mu_0/2$ . The continuity  
 170 equation reads [13]

$$\frac{dh}{dt} + h(t) \frac{\partial \bar{u}}{\partial x} = 0, \quad (4)$$

171 and substituting eq. (3) in eq. (4) gives

$$\frac{dh}{dt} = \frac{h^3}{12\mu_0} \frac{\partial^2 p}{\partial x^2} + \frac{\alpha h^{\alpha+2}}{2^{\alpha+1}(\alpha+2)\mu_0\tau_0^{\alpha-1}} \left| \frac{\partial p}{\partial x} \right|^{\alpha-1} \frac{\partial^2 p}{\partial x^2}. \quad (5)$$

172 The problem formulation is completed by the force balance, expressed per  
 173 unit width of fracture, among the fluid pressure and the elastic reaction of the  
 174 upper wall, taken to be proportional to aperture  $h$ ; an overload at the upper  
 175 wall  $f_0$  (a force per unit width) is included in the balance for generality [21];  
 176 the overload represents an additional force exerted by the walls and usually  
 177 opposing the fracture opening due, e.g., to a residual stress state generated  
 178 by the load history of the rocks. It is assumed constant and independent  
 179 from the fracture aperture. The balance reads

$$\int_0^L p(x, t) dx = \tilde{E}Lh(t) + f_0, \quad (6)$$

180 where the constant of proportionality  $\tilde{E}$  has dimensions  $[ML^{-2}T^{-2}]$ ; for a  
 181 linear elastic foundation, called a Winkler soil in geotechnical applications,  
 182  $\tilde{E}$  is equal, for a thin elastic layer of thickness  $l$ , to the ratio between the  
 183 Young modulus of the layer’s material  $E$   $[ML^{-1}T^{-2}]$  and  $l$ ,  $\tilde{E} = E/l$ . In  
 184 the context of hydraulic fracturing,  $l$  may be identified with the fracture  
 185 spacing [13, 21], a design parameter that depends, among others, on the  
 186 type of rock; in hydraulically fractured shales, values of  $l/L$  equal to 0.057,  
 187 0.28, and 0.029 are reported, respectively, by Ghanbari and Dehghanpour [7],  
 188 Wang et al. [11], and Wang et al. [33]. In the case of vertical/sub-vertical  
 189 fractures perpendicular to a horizontal/sub-horizontal well or borehole, the  
 190 geometry of the idealized system is described by Figure 3, showing the two  
 191 wings of equally spaced planar fractures of half-length  $L$ , width  $W$ , aperture  
 192  $h$  and spacing  $l$ . Albeit the flow very close to the well is radial, the influence  
 193 of the boundary condition at the well decreases rapidly with distance, and  
 194 flow in most of the fracture half-length  $L$  is uniform, consistently with the  
 195 assumption  $L \gg W$ . Hence, as an approximation the boundary condition of  
 196 assigned pressure  $p_e$  at the well is extended to a segment of height  $W$ . In the

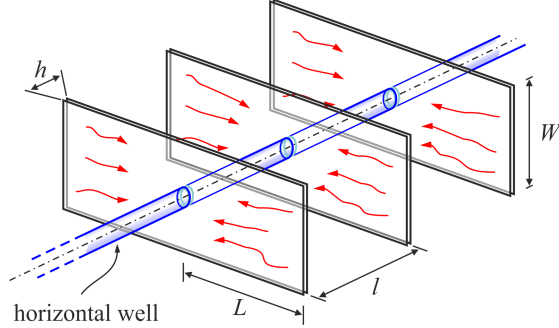


Figure 3: Typical scheme for bi-wing planar fractures around a horizontal borehole;  $L$ ,  $W$  and  $h$  are the fracture length, width and aperture,  $l$  is the fracture spacing.

197 case of planar vertical fractures parallel to, and propagating from, a vertical  
 198 well, the geometry of the flow is plane without using this approximation.

199 A further issue deserving investigation is the linearity of the relationship  
 200 between the wall reaction and the fracture aperture. In fact, a nonlinear  
 201 elastic behaviour can be the result of the pervasive damage of rocks by micro-  
 202 cracks and voids, which determines nonlinearity even for infinitesimal strain,  
 203 also with an incremental jump in the elastic modulus from tension to com-  
 204 pression [34, 35]. In this case the Young modulus of the material is a function  
 205 of the strain rate,  $E = E_0(h/l)$ , and assuming that the latter dependence is  
 206 expressed with a power-law function one has

$$E = E_0 \left( \frac{h}{l} \right)^{\lambda-1}, \quad (7)$$

207 where  $\lambda$  is a non-negative exponent modulating the nature of the reaction: for  
 208  $\lambda = 1$  a constant Young modulus is recovered, while  $0 < \lambda < 1$  is associated  
 209 to a softening behaviour, and  $\lambda > 1$  to a stiffening one. The assumption  
 210 results in

$$\tilde{E} = \frac{E_0}{l} \left( \frac{h}{l} \right)^{\lambda-1} \equiv \hat{E} h^{\lambda-1}, \quad (8)$$

211 and eq. (6) is modified as

$$\int_0^L p(x, t) dx = \hat{E} L h^\lambda(t) + f_0, \quad (9)$$

212 with  $\hat{E} = E_0 l^{-\lambda}$  of dimensions  $[ML^{-1-\lambda}T^{-2}]$ .

213 Equations (5) and (9) are subject to the following initial and boundary  
 214 conditions

$$h(0) = h_0, \quad \frac{\partial p(x, t)}{\partial x}(L, t) = 0, \quad p(0, t) = p_e, \quad (10)$$

215  $h_0$  being the initial fracture aperture, and  $p_e$  the exit pressure at the well.

216 The solution to the above problem yields two relevant quantities expressed  
 217 per unit width, the flowrate exiting the fracture at the well,  $q(t)$ , and the  
 218 residual volume of the fracture at a given time,  $v(t)$ ; these are easily derivable  
 219 as

$$q(t) = L \frac{dh(t)}{dt}, \quad v(t) = Lh(t). \quad (11)$$

## 220 2.2. Dimensionless form

221 Dimensionless quantities are defined as

$$\begin{aligned} X = x/L, \quad H = h/h_0, \quad T = t/t_c, \quad P = (p - p_e)/p_c, \quad P_e = p_e/p_c, \\ Q = qt_c/(h_0L) = q/(u_0h_0), \quad V = v/(h_0L), \end{aligned} \quad (12)$$

222 where the scales for pressure and time are

$$p_c = \hat{E}h_0^\lambda, \quad t_c = \frac{(2 + \alpha)}{\alpha} \left( \frac{2L}{h_0} \right)^{1+\alpha} \frac{1}{h_0^{\alpha\lambda}} \frac{\mu_0\tau_0^{\alpha-1}}{\hat{E}^\alpha}, \quad (13)$$

223 and  $u_0 = L/t_c$  is a velocity scale. This leads to the dimensionless counterpart  
 224 of eq. (5)

$$\frac{dH}{dT} = NH^3 \frac{\partial^2 P}{\partial X^2} + H^{\alpha+2} \left( \frac{\partial P}{\partial X} \right)^{\alpha-1} \frac{\partial^2 P}{\partial X^2}, \quad (14)$$

225 where the pure number

$$N = \frac{2 + \alpha}{3\alpha} \left( \frac{2\tau_0 L}{\hat{E}h_0^{\lambda+1}} \right)^{\alpha-1} = \frac{2 + \alpha}{3\alpha} \left[ \frac{2\tau_0}{p_c(h_0/L)} \right]^{\alpha-1} \quad (15)$$

226 modulates the relative importance of the Newtonian behaviour of the Ellis  
 227 fluid at low shear rate, expressed by the first term on the r.h.s. of eq. (14),  
 228 with respect to the second term, the power-law behaviour at high shear rate.  
 229 For a Newtonian fluid ( $\alpha = 1$ )  $N$  reduces to unity; for a shear-thinning fluid  
 230 ( $\alpha > 1$ ),  $N$  is zero for  $\tau_0 = 0$  and/or a rigid wall ( $\hat{E} = E_0/l^\lambda \rightarrow \infty$ ), but  
 231 the latter case renders the scales (13) meaningless. In eq. (15) defining  $N$ ,  
 232 the quantity within brackets represents the ratio between the characteristic

233 shear stress  $\tau_0$  of the Ellis fluid and the pressure scale  $p_c = \hat{E}h_0^\lambda$  associated  
 234 with the elastic reaction of the fracture wall; the ratio is in turn corrected by  
 235 the initial aspect ratio of the fracture  $h_0/L$ . This formulation of  $N$  includes  
 236 only parameters defined at the single fracture scale. Note that if the scheme  
 237 of multiple fractures with spacing  $l$  depicted in Figure 3 is considered, eq.  
 238 (15) may be rewritten as

$$N = \frac{2 + \alpha}{3\alpha} \left[ \frac{2 \left( \frac{\tau_0}{E_0} \right) \left( \frac{l}{L} \right) \left( \frac{l}{L} \right)^{\lambda-1}}{\left( \frac{h_0}{L} \right)^2 \left( \frac{h_0}{L} \right)^{\lambda-1}} \right]^{\alpha-1}, \quad (16)$$

239 where  $\tau_0/E_0$  is the ratio between the representative shear stress of the fluid  
 240 and the Young modulus of the host rock, and  $l/L$  is the dimensionless fracture  
 241 spacing. The terms to the power  $(\lambda-1)$  represent the contribution due to non-  
 242 linear elastic behaviour of the walls, and disappear for  $\lambda = 1$ . An alternative  
 243 formulation of  $N$  as a function of Cauchy, Reynolds, and Ellis dimensionless  
 244 groups is reported in Appendix B. To grasp the order of magnitude of  $N$ ,  
 245 we recall that  $l/L$  may be taken to vary between 0.03 and 0.3 (with  $l/L \approx$   
 246 0.1 being appropriate for an order of magnitude analysis), while the initial  
 247 fracture aspect ratio  $h_0/L$ , a number much smaller than 1, may be considered  
 248 of order  $10^{-3} - 10^{-5}$  [7, 11, 33]. The latter reference also reports  $E_0 = 2.5 \cdot 10^{10}$   
 249 Pa for the rock elastic modulus in fractured shales; quite close values,  $E_0 =$   
 250  $3 \cdot 10^{10}$  Pa and  $E_0 = 2.76 \cdot 10^{10}$  Pa are reported in [19] and [36], hence reference  
 251 values  $E_0 = 2.5 - 3.0 \cdot 10^{10}$  Pa are considered.

252 Actual values of rheological parameters for Ellis fluids are quite scarce  
 253 in the literature. A reference specific to fracking is [23], where the Ellis  
 254 parameters are reported for two fracturing fluids, HPG (Hydroxypropylguar)  
 255 and VES (viscoelastic surfactant). For the first,  $\mu_0 = 0.44$  Pa  $\cdot$  s,  $\tau_0 = 2.01$   
 256 Pa, and  $\alpha = 1.22$ ; for the second,  $\mu_0 = 49$  Pa  $\cdot$  s,  $\tau_0 = 8.836$  Pa, and  $\alpha = 12$ .  
 257 Adopting as reference geometrical parameters  $l/L = 0.1$  and  $h_0/L = 10^{-4}$ ,  
 258 and a young modulus of  $E_0 = 2.75 \cdot 10^{10}$  Pa for the host rock, one obtains  
 259  $N = 0.209$  for HPG and  $N \simeq 0$  for VES, indicating that for the latter fluid  
 260 the Newtonian component of rheological behaviour is negligible. A further  
 261 consideration is that VES is very strongly shear-thinning ( $\alpha \gg 1$ ), therefore  
 262 the value of  $N$  is extremely sensitive to variations in parameters: adopting  
 263 for example  $l/L = 0.125$ ,  $h_0/L = 10^{-5}$ , and  $E_0 = 2.5 \cdot 10^{10}$  Pa, again realistic  
 264 values, one obtains  $N = 0.100$  for VES and  $N = 0.618$  for HPG. This second

265 set of parameters is adopted for later reference in Section 4 describing a case  
 266 study and is shown there in dimensional form (see Table 1). Trying further  
 267 combinations of realistic values for fluid and rock properties, it is seen that  
 268  $N$  may take values smaller or larger than unity, the former case being more  
 269 frequent. This indicates a certain prevalence of the power-law component of  
 270 rheology over the Newtonian one, although the asymptotic system behaviour  
 271 is dominated by the latter, as will be shown in the next section. We bear in  
 272 mind that a large variety of combinations is possible for the two parameters  
 273  $N$  and  $\alpha$  depending on geometry and properties of fluid and rock, but with  
 274 the constraint from the definition (15) that for  $\alpha = 1$  it must be  $N = 1$ .

275 The dynamic boundary condition (9) and the boundary conditions (10)  
 276 transform as

$$\int_0^1 P(X, T) dX = H^\lambda - P_e + F_0, \quad (17)$$

$$277 \quad H(0) = 1, \quad \frac{\partial P}{\partial X}(1, T) = 0, \quad P(0, T) = 0. \quad (18)$$

### 278 2.3. Solution

279 A solution to eq. (14) is sought by integrating in two steps the pressure  
 280 of the fluid and the fracture aperture. Posing

$$U(X, T) = \frac{\partial P}{\partial X}, \quad \dot{H} = \frac{dH}{dT}, \quad (19)$$

281 eq. (14) can be written as

$$B(1 + AU^{\alpha-1}) \frac{\partial U}{\partial X} = \dot{H} \quad (20)$$

282 where

$$A = A(T) = \frac{(H)^{\alpha-1}}{N}, \quad B = B(T) = NH^3, \quad (21)$$

283 while the second boundary condition in eq. (18) becomes

$$U(1, T) = 0. \quad (22)$$

284 Separating variables in eq. (20), and integrating with the boundary condition  
 285 (22) leads to

$$\frac{BU(AU^{\alpha-1} + \alpha)}{\alpha} = -\dot{H}(1 - X). \quad (23)$$

286 Eq. (23) can be rewritten as

$$U^\alpha + CU + D(1 - X) = 0 \quad (24)$$

287 where

$$C = C(T) = \frac{\alpha N}{H^{\alpha-1}}, \quad D = D(T) = \frac{\alpha \dot{H}}{H^{2+\alpha}}. \quad (25)$$

288 Eq. (25) is algebraic in  $U$  and admits an analytical solution for  $\alpha = 1, 2, 3$   
 289 and for  $\alpha = 1/2, 1/3$  in the form of a combination of functions of  $H$  and  $\dot{H}$ .  
 290 This solution can be integrated once in space, with the boundary condition  
 291  $P(0, T) = 0$ , obtaining the pressure field. The pressure field is finally inte-  
 292 grated in  $X \in [0, 1]$  and the integral in eq. (17) is computed as a function  
 293 of  $H$  and  $\dot{H}$ . Then eq. (17) is transformed in a nonlinear ODE which is  
 294 numerically integrated with the initial condition  $H(0) = 1$ .

295 These solutions are analytical in the  $x$  coordinate and numerical in the  
 296 time domain and seem quite cumbersome, while their accuracy is comparable  
 297 to that of a fully numerical solution in space and time; the latter also has  
 298 the advantage of a free selection of the indicial parameter  $\alpha$ . Among the  
 299 many possible numerical schemes, we adopt a finite difference in time and  
 300 an implicit resolver in space, with a step size reduction to track solution  
 301 accurately.

302 The code is written in Mathematica, introducing a parametric solver for  
 303 the function  $U(X, T)$  as a function of  $N, \alpha, H_{i+1}, H_i, \Delta t$ , where  $H_{i+1}$  and  
 304  $H_i$  are the values at time  $(i + 1)\Delta t$  and  $i\Delta t$ , respectively; the only free  
 305 parameter is  $H_{i+1}$ , all the other parameters are given.

306 Each time iteration includes the following steps:

- 307 • The function  $U(X)_{i+1}$  is estimated by solving eq. (20) in parametric  
 308 form, with  $\dot{H} \approx (H_{i+1} - H_i)/\Delta t$ , with the term  $H$  taken to be the  
 309 average between  $H_{i+1}$  and  $H_i$  and with the b.c.  $U(1)_{i+1} = 0$ , where  
 310  $H_{i+1}$  is the free parameter;  $H_0 = 1$  is assumed at the first step.
- 311 • The space values of  $U$ , known in parametric form, are used to solve  
 312 the differential problem  $\partial P(X)_{i+1}/\partial X = U(X)_{i+1}$ , with  $P(0)_{i+1} = 0$ ,  
 313 obtaining the pressure  $P(X)_{i+1}$ .
- 314 • The pressure field is numerically integrated (in parametric form) in the  
 315 domain  $[0, 1]$ .

- 316 • The parametric integral is inserted in eq. (17), and the equality is  
317 forced with a Newton method for finding the value of the parameter  
318  $H_{i+1}$ .
- 319 • The procedure is repeated for the next time step, shifting the values  
320  $H_{i+1}$ .

321 Once the pressure  $P(X, T)$  and aperture  $H(T)$  fields are known, the di-  
322 mensionless flowrate and fracture volume are given by

$$Q(T) = \frac{dH(T)}{dT} = \dot{H}, \quad V(T) = H(T). \quad (26)$$

323 Hence at late-time the fracture volume and flowrate behave like the aperture  
324 and its time derivative, respectively; for zero borehole pressure and overload  
325 the corresponding time scalings are  $T^{-1/(\lambda+2)}$  and  $T^{-1/(\lambda+3)}$ .

### 326 3. Results and discussion

327 Figure 4 shows the results of the numerical computation for the fracture  
328 aperture and different  $\alpha$  values, with the analytical solution  $H = (1+9T)^{-1/3}$   
329 valid for the Newtonian case and a linearly elastic fracture [13], corresponding  
330 to  $\alpha = 1$ ,  $N = 1$ , and  $\lambda = 1$ . Note that the values  $\alpha = 1$ ,  $N = 1$  imply  
331 Newtonian behaviour but with a viscosity equal to  $\mu_0/2$ , thus halving the  
332 time scale  $t_c$  in eq. (13); this requires doubling the dimensionless time  $T$   
333 in eq. (12) to compare results of equations having a different time scale.  
334 The time integration was performed with a time step  $\Delta t = 0.01$ . Since  
335 the results of the numerical integration using this fully explicit scheme fit  
336 exceedingly well the analytical solution, it was not necessary to adopt higher  
337 order schemes, even considering that the solution has no singularity and  
338 behaves rather smoothly.

339 The asymptotic behaviour of the solution  $H(T)$  is dictated by the inter-  
340 play between the two terms on the r.h.s. of eq. (14): the second term scales  
341 with the gradient pressure (decaying in time) and with a power of  $H$  always  
342 larger than 3, since  $\alpha > 1$ , whereas the first term scales with the third power  
343 of  $H$  and has  $N$  as a coefficient. Since  $H \leq 1$  and the gradient pressure  
344 quickly decays to values less than unity, the dominant term is the first one,  
345 which entails the asymptotic behaviour  $H \sim T^{-1/(2+\lambda)}$ , see Figure 5 where  
346 different values of  $\alpha$ , for  $N = 1$  and  $P_e = 0$ , produce almost parallel curves  
347 for large  $T$ . Figure 5 also shows how variations in  $\lambda$  significantly affect the

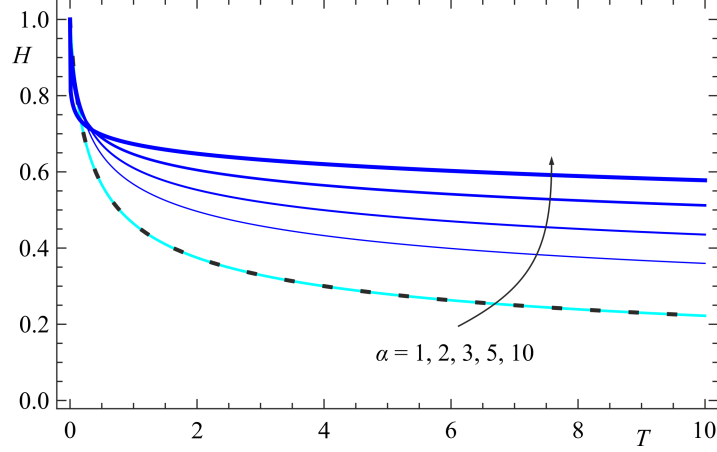


Figure 4: Time variation of the fracture aperture  $H$  for  $N = 0$ ,  $\lambda = 1$ ,  $P_e - F_0 = 0$  and different  $\alpha$  values. The black dotted curve refers to the analytical solution for a Newtonian fluid,  $H = (1+9T)^{-1/3}$ . Due to the different time scales adopted for a Newtonian fluid and for the present model, comparison is feasible if the dimensionless time  $T$  in the solution for the Newtonian fluid is doubled.

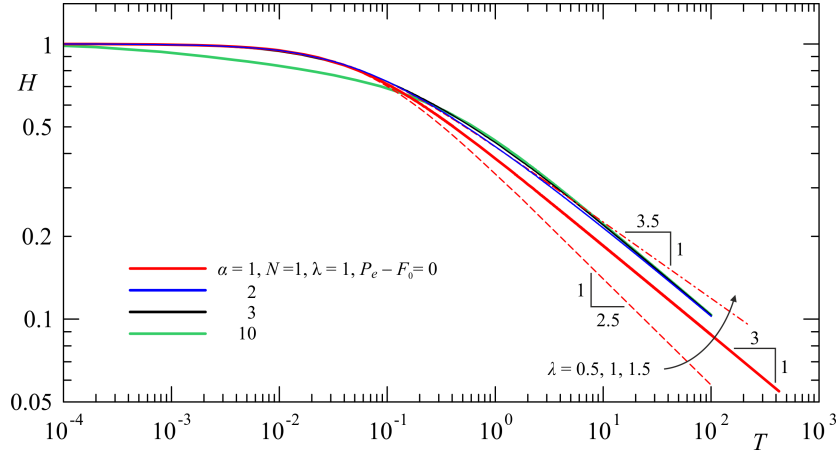


Figure 5: Time variation of the fracture aperture  $H$  for  $N = 1$ ,  $\lambda = 1$  and different  $\alpha$  values. For one case ( $\alpha = 1$ ) the effects of a softening/stiffening wall is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively. The asymptotic behaviour is  $H \sim T^{-1/(2+\lambda)}$ , independent on  $\alpha$ .

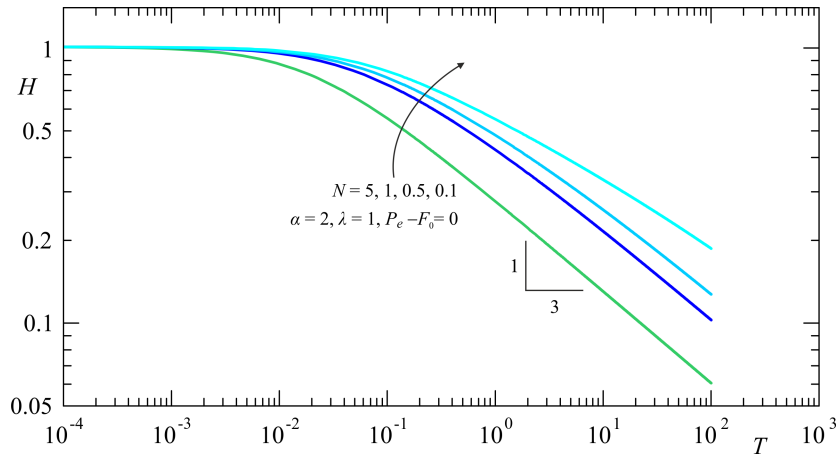


Figure 6: Time variation of the fracture aperture  $H$  for  $\alpha = 2$ ,  $\lambda = 1$  and different  $N$ .

348 late-time behaviour for fixed  $\alpha$ : a stiffening ( $\lambda > 1$ )/softening ( $\lambda < 1$ ) elastic  
 349 reaction of the walls delays/facilitates the drainage. It is also seen that the  
 350 parameter  $\alpha$  mainly controls the early stage, the parameter  $\lambda$  the late stage  
 351 of the backflow process. Figure 6 shows results for a fixed  $\alpha = 2$ ,  $\lambda = 1$ ,  
 352 and different  $N$  values; the asymptote is reached much faster for larger  $N$ .  
 353 In sum, the early time behaviour for zero external pressure at the well is  
 354 in general dominated by the second term in eq. (14) unless the coefficient  
 355  $N \gg 1$ ; in the latter case both terms substantially contribute to the time  
 356 evolution of  $H$ .

357 In presence of a non-zero external pressure ( $P_e > 0$ ) or a negative overload  
 358  $F_0$  (an additional force per unit of wall surface acting in the same direction  
 359 of the internal pressure), the asymptotic residual aperture is equal to  $(P_e -$   
 360  $F_0)^{1/\lambda}$ , see Figure 7 where both effects are included. The curves coalesce  
 361 to the asymptote faster for larger  $N$  values, implying a dominance of the  
 362 Newtonian behaviour, while for small  $N$  the power-law behaviour prevails  
 363 and the asymptote is reached for larger dimensionless times. Upon plotting  
 364 results for  $\alpha = 3$  (not shown) the main curves for  $\lambda = 1$  and the secondary  
 365 curves for  $\lambda \neq 1$  are very similar to those for  $\alpha = 2$ .

366 Figure 8 shows the pressure distribution for two different combinations  
 367 of the parameters and a shear-thinning fluid with  $\alpha = 2$ . Results for other  
 368 combinations are similar (and thus not shown), with a pressure decay in  
 369 space/time quicker or slower depending on the parameter values; at all times  
 370 the residual pressure within the fracture increases with smaller  $N$  values,

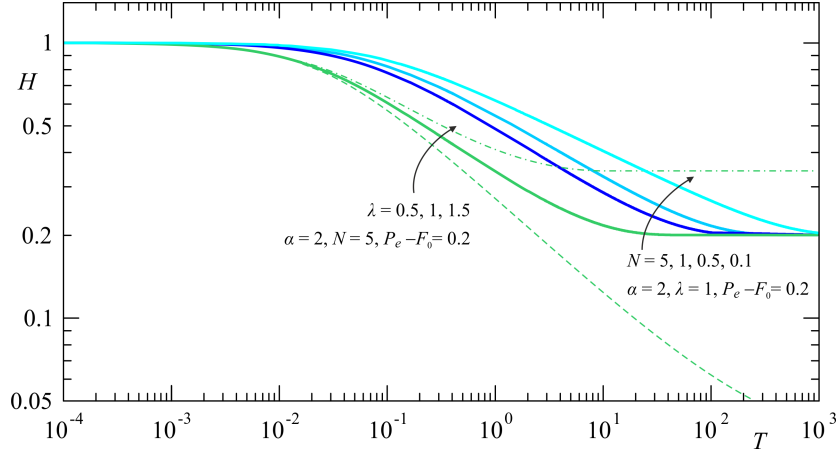


Figure 7: Time variation of the fracture aperture  $H$  for  $\alpha = 2$  and different  $N$  values, with given difference between external pressure and overload  $P_e - F_0 = 0.2$ . For one case ( $N = 5$ ) the effects of a stiffening/softening elastic reaction of the walls is explored, see the dashed and dash-dotted thin curves for  $\lambda = 0.5 - 1.5$ , respectively.

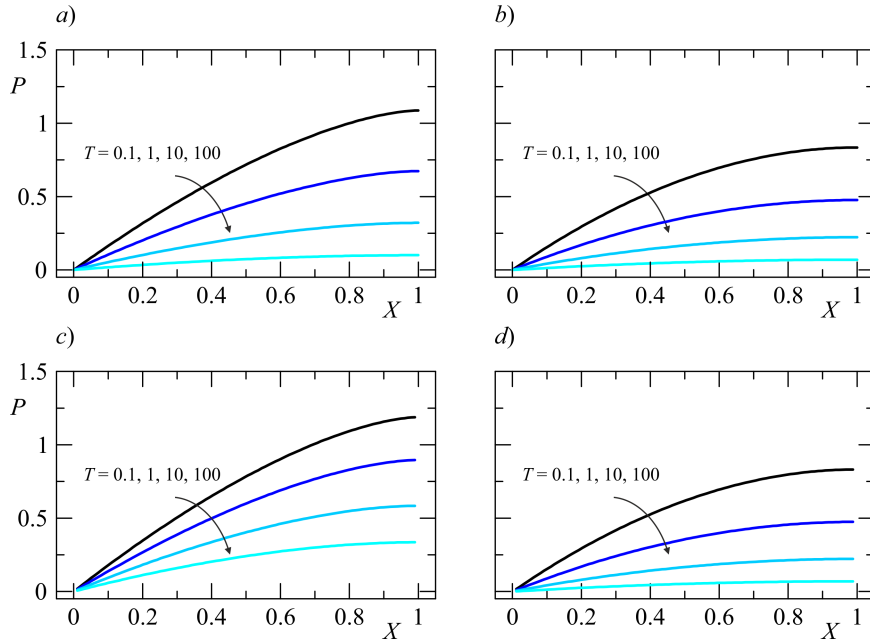


Figure 8: Pressure along the fracture at different times for  $P_e - F_0 = 0.2$  and a shear-thinning fluid with  $\alpha = 2$ . Results for a)  $N = 0.1$  and  $\lambda = 1$ ; b)  $N = 5$  and  $\lambda = 1$ ; c)  $N = 0.1$  and  $\lambda = 0.5$ ; d)  $N = 5$ ,  $\lambda = 0.5$ .

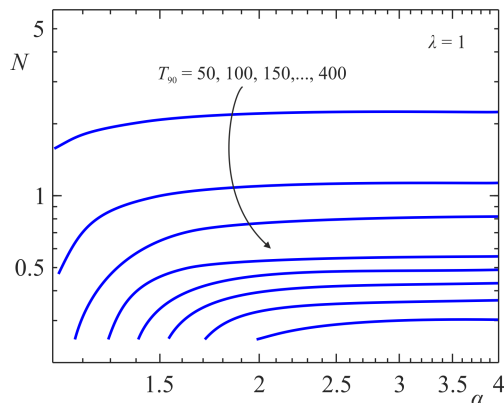


Figure 9: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1$  and  $P_e - F_0 = 0$ .

371 implying a behaviour closer to Newtonian, and with smaller  $\lambda$  values, i.e. a  
 372 softening wall; however when the fluid is closer to Newtonian the effect of a  
 373  $\lambda$  variation is irrelevant.

374 An important quantity characterizing the performance of the backflow  
 375 process is the time required to recover the fluid injected in the fracture net-  
 376 work and not lost in the form of leakoff. Here the network is conceptualized  
 377 as a single fracture and fluid losses are not explicitly represented (they are  
 378 assumed to take place in the upstream network), however the time  $T_Y$  needed  
 379 to recover  $Y\%$  of the fracture volume provides an indication of how rapid the  
 380 recovery is. Contour maps in the  $(\alpha, N)$  space of the dimensionless time  $T_{90}$   
 381 needed to recover 90% of the fluid are depicted in Figure 9 for a linear wall  
 382 reaction ( $\lambda = 1$ ). As the degree of shear-thinning behaviour rises with  $\alpha$   
 383 for constant  $N$ , there is a sharp increase in dimensionless  $T_Y$  for  $N < 0.5$ ,  
 384 while  $T_Y$  is almost independent on  $\alpha$  for  $N > 2$ . Conversely,  $T_Y$  for constant  
 385  $\alpha$  decreases with larger  $N$  values, i.e. as the fluid behaviour is closer to  
 386 Newtonian; this effect is more evident for larger  $\alpha$ . Highest values of  $T_Y$   
 387 are attained for large  $\alpha$  and low  $N$ , lowest values for small  $\alpha$  and large  $N$ , the  
 388 two combinations farthest and closest to Newtonian behaviour. The effect  
 389 of a sublinear wall reaction ( $\lambda = 0.5$ ) is depicted in Figure 10, that of a  
 390 supralinear wall reaction in Figure 11. The dimensionless time to recover the  
 391 bulk of the stored fluid is decidedly faster or slower with a softening or stiff-  
 392 ening wall, demonstrating once again the decisive influence of the parameter  
 393  $\lambda$  modulating the wall reaction at late time.

394 A word of caution is needed when drawing comparisons between non-

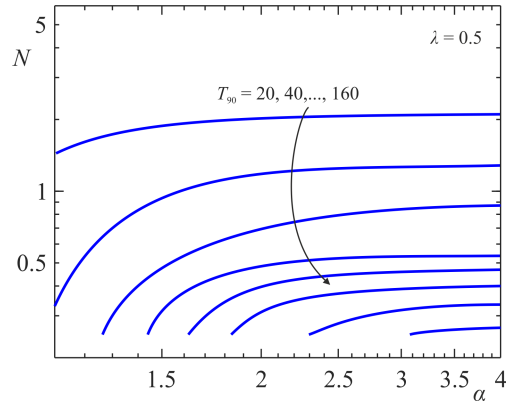


Figure 10: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 0.5$  and  $P_e - F_0 = 0$ .

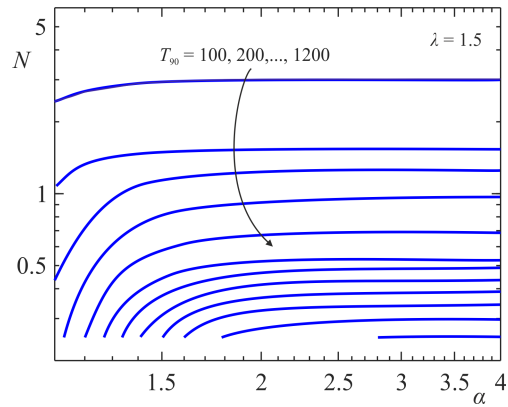


Figure 11: Time to recover 90% of the fluid as a function of  $\alpha$  and  $N$ , with  $\lambda = 1.5$  and  $P_e - F_0 = 0$ .

Fluid	$\mu_0$ (Pa s)	$\tau_0$ (Pa)	$\alpha$	$L$ (m)	$l$ (m)	$h_0$ (mm)	$E$ (Pa)	$\lambda$	$N$
HPG	0.44	2.01	1.22	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.618
VES	49.00	8.836	12.00	100	12.5	1.00	$2.5 \cdot 10^{10}$	1.00	0.100

Table 1: Reference parameters for case study:  $\mu_0$ ,  $\tau_0$  and  $\alpha$  are the reference viscosity, shear stress and indicial exponent of the Ellis fluid,  $L$  is the fracture length,  $l$  is the fracture spacing,  $h_0$  is the fracture initial height,  $E$  is the rock modulus of elasticity,  $\lambda$  is the exponent of the rock wall reaction,  $N$  is the dimensionless number governing the interplay between Newtonian and power-law behaviour in an Ellis fluid.

395 Newtonian fluids with different rheology as the models are semi-empirical  
396 and the time scale used for the dimensionless formulation depends upon the  
397 rheological parameters of the Ellis model and is particularly sensitive to the  
398 value of the indicial exponent  $\alpha$ . Hence model outputs are best compared in  
399 dimensional coordinates when quantitative results are needed.

#### 400 4. A case study

401 A case study is illustrated by comparing the performance of two real  
402 hydrofracturing fluids [23], HPG (Hydroxypropylguar) and VES (viscoelastic  
403 surfactant) in a realistic setting. The rheological parameters according with  
404 the Ellis model are reported for both fluids in Table 1, together with realistic  
405 geometric and mechanical parameters within plausible ranges deduced from  
406 the literature, see the earlier discussion in Section 2.2. It is seen that HPG  
407 is relatively close to Newtonian in behaviour, while VES is extremely shear-  
408 thinning, with an equivalent rheological index  $n$  less than 0.1 when expressed  
409 according to the power-law model.

410 Figure 12 shows the relaxation of the fracture aperture for the two fluids:  
411 the aperture for the HPG is only initially slightly larger than for the VES,  
412 but then closes more rapidly, reaching one tenth of the initial value at a time  
413 around 500 hours. The closure is much more gradual for the VES, requiring  
414 about a year to reach the same stage. The difference between corresponding  
415 pressure profiles, illustrated in Figure 13, shows a decidedly sharper pressure  
416 decrease for HPG than for VES in the initial stage.

417 Figure 14 shows the time to recover the volume stored in the fracture  
418 for the two fluids. Following the same trend manifested for the evolution of

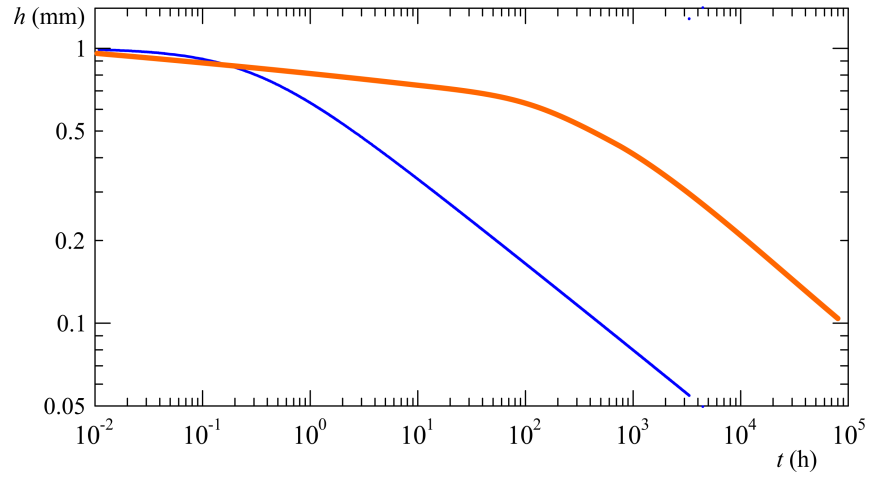


Figure 12: Time variation of the fracture aperture  $h$  for the HPG (thin line) and VES (thick line) fluids.

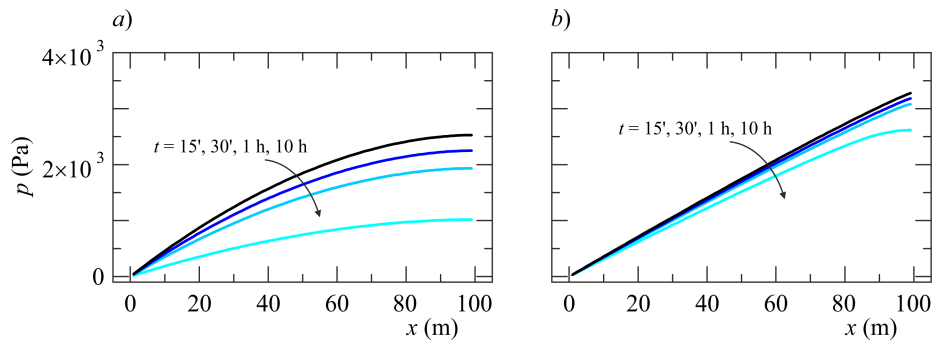


Figure 13: Pressure distribution at different time *a)* for HPG fluid, and *b)* for VES.

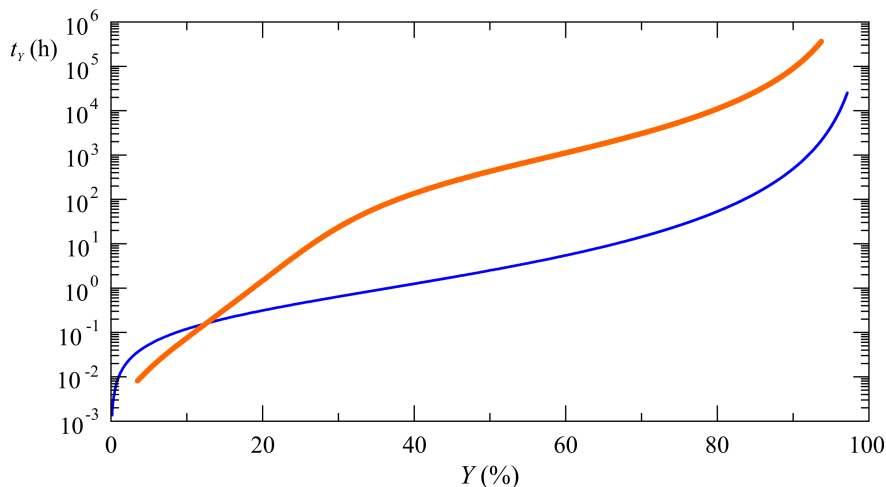


Figure 14: Time to recover the fracture volume  $Y\%$  for the HPG (thin line) and VES (thick line) fluids.

419 fracture opening, VES demonstrates a higher drainage capacity than HPG in  
 420 the very early phase, for  $Y < 15\%$ ; subsequently it is much less efficient, and  
 421 requires an extra time at least three orders of magnitude larger to drain the  
 422 same percentage of fluid than HPG. Overall the large difference in rheology,  
 423 mainly encapsulated in the  $\alpha$  value, translates into corresponding wide dif-  
 424 ferences in terms of aperture, pressure, and drainage time. This is so because  
 425 the value of the dimensionless group  $N$  is very low for VES, thus allowing the  
 426 fluid to manifest its essentially power-law nature. We tried a number of other  
 427 combinations of parameters and found that for very shear thinning fluids like  
 428 VES the results are very sensitive to relatively small changes in parameters:  
 429 slightly increasing the modulus of elasticity  $E$  to  $3 \cdot 10^{10}$  and increasing the  
 430 spacing to 20 m, leaving the other parameters in Table 1 unchanged, leads  
 431 to  $N(HPG) = 0.659$  and  $N(VES) = 2.360$ . While the change in the  $N$   
 432 value associated to HPG is modest (6.6%) and implies the system behaviour  
 433 is essentially unchanged with respect to the reference case, the increase in  
 434  $N$  for the VES is dramatic (2260%) and entails a fluid behaviour closer to  
 435 Newtonian despite the exceedingly high value of  $\alpha$ . Upon plotting the aper-  
 436 ture variation over time for this case (not shown) the two fluids exhibit a  
 437 similar behaviour, with only modest differences (less than 10%) in the frac-  
 438 ture aperture at early times and an almost identical behaviour later on. The  
 439 pressure profiles do not show any significant differences.

440 **5. Conclusions**

441 A conceptual model for backflow of non-Newtonian fluid from a closing  
 442 rock fracture was presented in this paper. Under the assumption of Ellis rhe-  
 443 ology and elastic, but non-deformable wall, the problem in plane geometry is  
 444 tractable in semi-analytical form to yield the time-variable fracture aperture  
 445  $h(t)$ , pressure field  $p(x, t)$  and discharge rate  $q(t)$ , as well as the drainage  
 446 time  $t_Y$  for a specified recovery rate  $Y$ , outlet pressure  $p_e$  and overload  $f_0$ .

447 Our results lead to the following specific conclusions:

- 448 • The Ellis model adopted herein to describe shear-thinning rheology  
 449 couples Newtonian and power-law behaviour. When an Ellis fluid back-  
 450 flows from a relaxing fracture the interplay between the two natures  
 451 is modulated by a dimensionless group  $N$  encapsulating the main prob-  
 452 lem parameters.  $N$  can be expressed in terms of i) the indicial exponent  
 453  $\alpha$  of the Ellis rheology, ii) the parameter  $\lambda$  governing the wall relaxation  
 454 process, iii) the ratio between the characteristic shear stress of the Ellis  
 455 fluid  $\tau_0$  and the rock modulus of elasticity  $E$ , iv) two geometric ratios,  
 456 the fracture initial aspect ratio  $h_0/L$  and dimensionless spacing  $l/L$ .  
 457 An alternative format of  $N$  is a modified ratio between the Cauchy  
 458 number and the product of Reynolds and Ellis numbers.
- 459 • The factors  $N$  and  $\alpha$  mostly influence the early and intermediate time  
 460 evolution of the system: when  $N < 1$  the power-law behaviour prevails;  
 461 for  $N = 1$  the pure Newtonian case is recovered ( $\alpha = 1$  entails  $N = 1$ ),  
 462 while for  $N \gg 1$  the behaviour is mixed.
- 463 • For late-time the system behaviour tends to Newtonian, is independent  
 464 of  $N$  and is governed by the wall relaxation parameter  $\lambda$ : aperture and  
 465 discharge scale asymptotically with time as  $t^{-1/(\lambda+2)}$  and  $t^{-1/(\lambda+3)}$  for  
 466  $p_e - f_0 = 0$ ; else, the aperture tends asymptotically to a constant value  
 467 proportional to  $(p_e - f_0)^{1/\lambda}$ .
- 468 • Very shear-thinning fluids (larger  $\alpha$ ) and reactive walls (larger  $\lambda$ ) are  
 469 associated with a more gradual closure of the aperture.
- 470 • The residual pressure within the fracture increases with smaller  $N$  val-  
 471 ues and with a softening wall ( $\lambda < 1$ ); when the fluid is close to New-  
 472 tonian the effect of a  $\lambda$  variation is almost irrelevant.

- 473 • The dimensionless drainage time  $T_Y$  attains the largest values for large  
474  $\alpha$  and low  $N$ , the lowest values for small  $\alpha$  and large  $N$ , the two com-  
475 binations farthest and closest to Newtonian behaviour. A non-linear  
476 reaction of the walls result in a faster/slower recovery for  $\lambda < 1$  (soft-  
477 ening) and  $\lambda > 1$  (stiffening). For recovery values close to 100%,  $T_Y$  is  
478 very sensitive to variations of model parameters.
- 479 • Results are discussed in dimensional form for a case study to reinforce  
480 the notion that dimensionless results need to be compared with caution  
481 as scales include fluid rheological parameters. Realistic geometric and  
482 mechanical parameters are adopted for a system of equally spaced frac-  
483 tures, and results are compared for two fluids, HPG and VES, normally  
484 used in fracking technology. The time evolution of the aperture and  
485 the dependence of the drainage time upon the recovery ratio are similar  
486 at early times, then differ by orders of magnitude at intermediate and  
487 late times.

488 The developments presented, together with earlier results [13, 21], provide  
489 an overview of the backflow phenomenon in the two basic geometric configu-  
490 rations for a single fracture, plane and radial, and for three rheological models  
491 of increasing complexity: Newtonian, power-law, and Ellis. Further improve-  
492 ments of the model remain open in several directions, e.g.: i) a more complex  
493 geometry, considering nonplanar fractures with non-negligible curvature; ii)  
494 the combination of non-Newtonian rheology with multiple fracture systems,  
495 adopting the asymptotic viewpoint of Dana et al. [14]; iii) the incorporation  
496 of particle transport to simulate the settling of solid proppant.

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500 conflicts of interest to declare. There are no data sharing issues since all of  
501 the numerical information is provided in the figures produced by solving the  
502 equations in the paper.

## 503 **Appendix A. The Newtonian case ( $n = 1$ )**

504 For  $\alpha = 1$  and  $N = 1$  eq. (25) reduces to

$$C = 1, \quad D = \frac{\dot{H}}{H^3}, \quad (\text{A.1})$$

505 and integrating eq. (24) using these expressions yields

$$P(X, T) = \frac{\dot{H}}{4H^3} [(X - 1)^2 - 1]. \quad (\text{A.2})$$

506 Substituting in eq. (17) and integrating  $P(X, T)$  over  $X$  gives

$$-\frac{\dot{H}}{3H^3} = H^\lambda - P_e + F_0, \quad (\text{A.3})$$

507 generalizing eq. (2.14) of Dana et al. [13], where  $\lambda = 1$  and  $F_0 = 0$ , to non-  
 508 linear wall reaction and non-zero overload. Now define an effective pressure  
 509  $\tilde{P}_e = P_e - F_0$  at the fracture outflow: this symbol will be used for brevity in  
 510 the sequel. Consider first the case  $\tilde{P}_e = 0$ . Integration of eq. (A.3) over time  
 511  $T$  yields, with the first b.c. in eq. (18),

$$H(T) = [1 + 3(2 + \lambda)T]^{-\frac{1}{2+\lambda}}, \quad (\text{A.4})$$

512 that for  $\lambda = 1$  gives back eq. (2.15) of [13].

Consider now the case  $\tilde{P}_e > 0$ . Integration with the help of Mathematica and using transformation formulae for the analytic continuation of hypergeometric functions [37] yields for generic  $\lambda$  the following implicit equation

$$T = \frac{1}{3(\lambda + 2)} \left[ \frac{1}{H^{\lambda+2}} {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \frac{\tilde{P}_e}{H^\lambda} \right) + \right. \\ \left. - {}_2F_1 \left( 1, \frac{\lambda + 2}{\lambda}; \frac{2(\lambda + 1)}{\lambda}; \tilde{P}_e \right) \right], \quad (\text{A.5})$$

where  ${}_2F_1(\alpha, \beta; \gamma; z)$  is the hypergeometric function of parameters  $\alpha$ ,  $\beta$ ,  $\gamma$ , and argument  $z$ . Specific results for  $\lambda = 1/2$ ,  $\lambda = 1$ ,  $\lambda = 2$ , i.e. a sublinear, linear or supralinear wall reaction, can be obtained as

$$T = \frac{1}{18\tilde{P}_e^5} \left[ 12 \ln \left( \frac{H^{1/2}(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{12\tilde{P}_e}{H^{1/2}} - \frac{6\tilde{P}_e^2}{H} - \frac{4\tilde{P}_e^3}{H^{3/2}} - \frac{3\tilde{P}_e^4}{H^2} + \right. \\ \left. + 12\tilde{P}_e + 6\tilde{P}_e^2 + 4\tilde{P}_e^3 + 3\tilde{P}_e^4 \right], \quad (\text{A.6})$$

513

$$T = \frac{1}{6\tilde{P}_e^3} \left[ 2 \ln \left( \frac{H(1 - \tilde{P}_e)}{H - \tilde{P}_e} \right) - \frac{2\tilde{P}_e}{H} - \frac{\tilde{P}_e^2}{H^2} + 2\tilde{P}_e + \tilde{P}_e^2 \right], \quad (\text{A.7})$$

514

$$T = \frac{1}{6\tilde{P}_e^2} \left[ \ln \left( \frac{H^2(1 - \tilde{P}_e)}{H^2 - \tilde{P}_e} \right) - \frac{\tilde{P}_e}{H^2} + \tilde{P}_e \right], \quad (\text{A.8})$$

515 either by direct integration of eq. (A.5) or using transformations involving  
 516 the hypergeometric functions [37]. Eq. (A.7) valid for  $\lambda = 1$  is identical  
 517 to Eq. (2.18) of Dana et al. [13]. Other results in terms of trascendental  
 518 and algebraic functions can be obtained for other special values of  $\lambda \in \mathbb{N}$  or  
 519  $1/\lambda \in \mathbb{N}$  but are too cumbersome to report and/or of little technical interest.

520 Expressions (A.5)-(A.8), when evaluated for for given  $\tilde{P}_e$ , allow deriving  
 521  $H(T)$  and the drainage time  $T_Y$  needed to drain  $Y\%$  of the fracture volume.  
 522 As the latter quantity is given in dimensionless form by  $H$  according to (26),  
 523 to derive  $T_Y$  it is sufficient to evaluate (A.5) and its special cases (15)-(A.8)  
 524 for  $H = (100 - Y)/100$ .

525 Finally, it is worthwhile to derive the asymptotic behaviour of the general  
 526 equation (A.5) for the limit case  $\lambda \rightarrow 0$ . According to eq. (9),  $\lambda = 0$  implies  
 527 a wall reaction constant over time rather than dependent from the fracture  
 528 aperture. Integrating (A.3) for  $H^\lambda = 1$  gives

$$H = \frac{1}{[1 + 6(1 - \tilde{P}_e)T]^{1/2}}, \quad (\text{A.9})$$

529 a result that can be simplified for large time to  $H = 1/[6(1 - \tilde{P}_e)T]^{1/2}$  and  
 530 further for  $\tilde{P}_e = 0$  to  $H = 1/(6T)^{1/2}$ . Equation (A.9) can be also obtained  
 531 directly from eq. (A.5) for  $\lambda \rightarrow 0$  on the basis of eq. (9.121.1) in [37]. The  
 532 late-time scaling for a Newtonian fluid and a wall with constant reaction  
 533 ( $\lambda = 0$ ) is therefore  $H \propto T^{-1/2}$ , a result coinciding with the scaling  $H \propto$   
 534  $T^{-1/(2+\lambda)}$  implied by Figure 5 for a Newtonian fluid with  $N = 1$ ,  $\alpha = 1$ .

## 535 Appendix B. The dimensionless group $N$

536 The pure number  $N$  may be expressed as a function of well-known dimen-  
 537 sionless groups in fluid mechanics [see, e.g., 38]. Multiplying and dividing  
 538 eq. (16) by  $\rho\mu_0 h_0 u_0^3$ , where  $u_0$  is the reference velocity defined in (12), yields

$$N = K \left( \frac{\text{Ca}}{\text{Re} \cdot \text{El}} \right)^{\alpha-1}; \quad \text{Ca} = \frac{\rho u_0^2}{E}; \quad \text{Re} = \frac{2\rho u_0 h_0}{\mu_0}; \quad \text{El} = \frac{\mu_0 u_0}{\tau_0 h_0}, \quad (\text{B.1})$$

539

$$K = K(\alpha, \lambda, l/L, h_0/L) = \frac{2 + \alpha}{3\alpha} \left[ \frac{4 \left( \frac{l}{L} \right)^\lambda}{\left( \frac{h_0}{L} \right)^{\lambda+1}} \right]^{\alpha-1} \quad (\text{B.2})$$

540 where Ca, Re, and El are the Cauchy, Reynolds, and Ellis numbers, and  $K$   
 541 a geometric factor correcting the ratio  $\text{Ca}/(\text{Re} \cdot \text{El})$ . In turn, Ca is the ratio  
 542 between inertial forces and elastic forces transmitted by solid walls, Re is the  
 543 ratio between inertial and viscous forces, while El is the ratio between the  
 544 viscous stress associated with the low shear rate Newtonian behaviour and  
 545 the shear stress  $\tau_0$  associated with high shear rate non-Newtonian (power-  
 546 law) behaviour.

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**Declaration of interests**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

## Author statement

**Valentina Ciriello:** Methodology, Writing - Original Draft. **Alessandro Lenci:** Methodology, Visualization. **Sandro Longo:** Methodology, Software, Writing - Review & Editing, Visualization. **Vittorio Di Federico:** Conceptualization, Writing - Review & Editing, Project administration, Funding acquisition.