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On Kodaira dimension of almost complex 4-dimensional solvmanifolds without complex structures / Cattaneo, A.; Nannicini, A.; Tomassini, A.. - In: INTERNATIONAL JOURNAL OF MATHEMATICS. - ISSN 0129-167X. - (2021), p. 2150075.2150075. [10.1142/S0129167X21500750]

Availability:

This version is available at: 11381/2896339 since: 2022-01-20T18:27:00Z

Publisher:

World Scientific

Published

DOI:10.1142/S0129167X21500750

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02 May 2026

ON KODAIRA DIMENSION OF ALMOST COMPLEX 4-DIMENSIONAL SOLVMANIFOLDS WITHOUT COMPLEX STRUCTURES

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ABSTRACT. The aim of this paper is to continue the study of Kodaira dimension for almost complex manifolds, focusing on the case of compact 4-dimensional solvmanifolds without any integrable almost complex structure. According to the classification theory we consider: $\mathfrak{r}_{3,-1}$, \mathfrak{nil}^4 and $\mathfrak{r}_{4,\lambda,-(1+\lambda)}$ with $-1 < \lambda < -\frac{1}{2}$. For the first solvmanifold we introduce special families of almost complex structures, compute the corresponding Kodaira dimension and show that it is no longer a deformation invariant. Moreover we prove Ricci flatness of the canonical connection for the almost Kähler structure. Regarding the other two manifolds we compute the Kodaira dimension for certain almost complex structures. Finally we construct a natural hypercomplex structure providing a twistorial description.

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2010 *Mathematics Subject Classification.* 53C55, 53C25.

Key words and phrases. Kodaira dimension, almost complex manifolds, almost Kähler manifolds, canonical connection.

This work was partially supported by the Project PRIN 2017 “Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics” and by GNSAGA of INdAM. When the major part of this work was done Andrea Cattaneo was supported by the ‘Grant de Bartolomeis’, a fellowship of Università di Firenze in memory of Prof. Paolo de Bartolomeis.

1. INTRODUCTION

Let M be a smooth $2n$ -dimensional manifold endowed with an almost complex structure J , that is a smooth $(1,1)$ -tensor field such that $J^2 = -\text{id}_M$. According to the celebrated Newlander–Nirenberg Theorem, the almost complex structure J is induced by a structure of complex manifold on M , that is J is *integrable*, if and only if the torsion tensor of J , namely the *Nijenhuis tensor* of J , vanishes. The investigation of the existence of new almost complex invariants of (M, J) is a natural problem in almost complex geometry. Recently, in this direction, H. Chen and W. Zhang in [CZ18] and [CZ20], starting with a $2n$ -dimensional compact almost complex manifold (M, J) , introduce the notion of *plurigenera* and give the definition of *Kodaira dimension* of (M, J) , denoted as $P_m(M, J)$, $\text{kod}(M, J)$ respectively. They prove that, given any 4-dimensional compact almost complex manifold, the plurigenera and the Kodaira dimension are birational invariants in the almost complex category. Furthermore, they show that such invariants are not stable under small deformations of the almost complex structure. For this purpose, they construct in [CZ18, §6.1] a 1-parameter family $\{J_a\}_{a \in \mathbb{R}^*}$ of almost complex structures on the Kodaira–Thurston manifold X such that

$$\text{kod}(X, J_a) = \begin{cases} 0 & a \in \pi\mathbb{Q} \setminus \{0\}, \\ -\infty & a \notin \pi\mathbb{Q}. \end{cases}$$

In [CNT20] the authors compute the Kodaira dimension of special families of 6-dimensional almost complex manifolds and of their deformations. More in particular, starting with the smooth manifold underlying the Nakamura complex 3-fold, they compute the Kodaira dimension of an almost Kähler deformation, showing that the possible values are 0 or $-\infty$; furthermore, it is proved that the Ricci curvature of the canonical connection ∇^c vanishes.

The aim of this paper is to continue the study of the Kodaira dimension in the almost complex category. More specifically, we focus on compact 4-dimensional solvmanifolds M , that is, compact quotients of simply connected solvable Lie groups by discrete cocompact subgroups, i.e., by lattices, without any complex structure. According to the classification theory (see [ABP20]) these are the solvable Lie algebras whose corresponding simply connected Lie groups admit compact quotients by a lattice:

- (1) $\mathfrak{r}_{3,-1}$;
- (2) \mathfrak{nil}^4 ;
- (3) $\mathfrak{r}_{4,\lambda,-(1+\lambda)}$ with $-1 < \lambda \leq -\frac{1}{2}$.

Notice that, with the only exception of $\mathfrak{r}_{4,-\frac{1}{2},-\frac{1}{2}}$, none of the 4-dimensional compact quotients as above carries any integrable almost complex structure. Therefore, the notion by Chen and Zhang of Kodaira dimension appears as one useful tool for the study of the almost complex geometry of such manifolds.

The paper is organized as follows. After we recall the definition of Kodaira dimension for almost complex manifolds in Section 2, in Section 3 we consider the families of solvmanifolds $M(k)$, \mathcal{N} and $\mathcal{M}(\lambda)$ associated respectively to the solvable Lie algebras $\mathfrak{r}_{3,-1}$, \mathfrak{nil}^4 and $\mathfrak{r}_{4,\lambda,-(1+\lambda)}$ with $-1 < \lambda < -\frac{1}{2}$ and we recall their constructions. Sections 4, 5 and 6 are devoted to the computations of the Kodaira dimension of special families of almost complex structures on $M(k)$, \mathcal{N} and $\mathcal{M}(\lambda)$: see Propositions 4.1 and 4.2 for the structures on $M(k)$, Propositions 5.1, 5.3 and 5.5 for the structures on \mathcal{N} and Proposition 6.1 for the one on $\mathcal{M}(\lambda)$. The main tool for our computations is an ad hoc method involving Fourier series: for each example we view functions on the solvmanifolds as functions defined on the universal cover which are invariant with respect to a suitable lattice. Then we find

a sublattice with respect to which such functions are periodic, and then we express them in Fourier series. Finally, we compute the Fourier coefficients by solving some system of partial differential equations and then obtain the Kodaira dimension.

Furthermore, the Ricci curvature of the canonical Chern connection is computed in the case of almost Kähler structures. In particular, for the almost Kähler family (J_s, g_s) on $M(k)$ (see (4.1)), the Ricci curvature vanishes (see Proposition 4.4). It is worth to note that there exists a non Chern–Ricci flat almost Kähler metric, with vanishing scalar curvature on (\mathcal{N}, J, g_J) , such that $\text{kod}(\mathcal{N}, J) = -\infty$ (see Proposition 5.6). Notice that the Kodaira dimension of $M(k)$ is no longer a deformation invariant.

In Section 7, we construct a natural hypercomplex structure providing a twistorial description. Finally, in Section 8 we describe some natural Norden structures.

Acknowledgement — The authors are grateful to Weiyi Zhang for a interesting discussions and suggestions during the workshop ‘Geometria in Bicocca 2019’ held in Milan, which led to the preparation of this paper. They also want to thank Scott O. Wilson for having attracted their attention on an inaccuracy in § 3.3. The authors want also to express their gratitude to the useful suggestions of the Referee, which led to a substantial improvement of some of the results presented in the paper.

2. PRELIMINARIES AND NOTATION

We start by fixing some notation and recalling the definition of Kodaira dimension in the almost complex category.

Let M be a compact $2n$ -dimensional smooth manifold endowed with an almost complex structure J . Denote by N_J the *Nijenhuis tensor* of J , that is, the smooth $(1, 1)$ -tensor field defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

for every pair of vector fields X, Y on M . The Newlander–Nirenberg Theorem states that J is integrable if and only if N_J vanishes.

Following [CZ18], we briefly recall the definition of *Kodaira dimension* of (M, J) .

Denote by $\Lambda_J^{p,q}M$ the bundle of smooth (p, q) -forms on (M, J) and by $A_J^{p,q}(M) = \Gamma(M, \Lambda_J^{p,q}M)$ the space of smooth (p, q) -forms on (M, J) . Then the exterior differential d satisfies

$$d(A_J^{p,q}(M)) \subset A_J^{p+2,q-1}(M) + A_J^{p+1,q}(M) + A_J^{p,q+1}(M) + A_J^{p-1,q+2}(M),$$

and, consequently, d decomposes as

$$d = \mu_J + \partial_J + \bar{\partial}_J + \bar{\mu}_J,$$

where $\mu_J = \pi^{p+2,q-1} \circ d$ and $\bar{\mu}_J = \pi^{p,q+1} \circ d$. Let $\mathcal{K}_X = \Lambda_J^{n,0}M$ be the *canonical bundle* of the almost complex manifold $X = (M, J)$. Then \mathcal{K}_X is a complex line bundle over X and the $\bar{\partial}_J$ -operator on (M, J) gives rise to a pseudoholomorphic structure on \mathcal{K}_X , i.e., a differential operator (still denoted by $\bar{\partial}_J$)

$$\bar{\partial}_J : \Gamma(M, \mathcal{K}_X) \rightarrow \Gamma(M, \Lambda_J^{0,1}M \otimes \mathcal{K}_X)$$

satisfying the Leibniz rule

$$\bar{\partial}_J(f\sigma) = \bar{\partial}_J f \otimes \sigma + f \bar{\partial}_J \sigma,$$

for every smooth function f and section σ .

As a consequence of Hodge Theory (see [CZ18, Theorem 1.1]), the space $H^0(M, \mathcal{K}_X^{\otimes m})$ of pseudoholomorphic pluricanonical sections is a finite dimensional complex vector space for every $m \geq 1$.

Definition ([CZ18, Definition 1.2]) — The m^{th} -*plurigenus* of (M, J) is

$$P_m(M, J) := \dim_{\mathbb{C}} H^0(M, \mathcal{K}_X^{\otimes m}).$$

The *Kodaira dimension* of (M, J) is defined as

$$\text{kod}(M, J) := \begin{cases} -\infty & \text{if } P_m(M, J) = 0 \text{ for every } m \geq 1, \\ \limsup_{m \rightarrow +\infty} \frac{\log P_m(M, J)}{\log m} & \text{otherwise.} \end{cases}$$

3. 4-SOLVMANIFOLDS WITHOUT COMPLEX STRUCTURES

3.1. The solvmanifolds $M(k)$. Based on the classification of the pairs consisting of a connected, simply connected, (real) three dimensional solvable Lie group and a discrete cocompact lattice made in [AGH61], Fernández and Gray produced in [FG90] an example of compact four dimensional manifold admitting no integrable almost complex structures. In this section we review the construction of this manifold, with the purpose of introducing the notation.

Fix $k \in \mathbb{R} \setminus \{0\}$ such that $e^k + e^{-k} \in \mathbb{Z}$, and consider the group

$$G(k) = \left\{ \left(\begin{array}{cccc} e^{kz} & 0 & 0 & x \\ 0 & e^{-kz} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{array} \right) \middle| x, y, z \in \mathbb{R} \right\}.$$

Then we can see $G(k) \simeq \mathbb{R}^2 \rtimes_{\phi_k} \mathbb{R}$, where the semidirect product is taken with respect to the action of \mathbb{R} on \mathbb{R}^2 given by

$$\begin{aligned} \phi_k : \mathbb{R} &\longrightarrow \text{GL}(2, \mathbb{R}) \\ z &\longmapsto \begin{pmatrix} e^{kz} & 0 \\ 0 & e^{-kz} \end{pmatrix}. \end{aligned}$$

It is then easy to show that $G(k)$ is a solvable, non nilpotent, Lie group.

Fix now two linearly independent vectors $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ and an integer $n \in \mathbb{Z}$, and let $D = D(u, v, n)$ be the subgroup of $G(k)$ generated by

$$\begin{aligned} E_u &= \begin{pmatrix} 1 & 0 & 0 & u_1 \\ 0 & 1 & 0 & u_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & E_v &= \begin{pmatrix} 1 & 0 & 0 & v_1 \\ 0 & 1 & 0 & v_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ E_n &= \begin{pmatrix} e^{nk} & 0 & 0 & 0 \\ 0 & e^{-nk} & 0 & 0 \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Then D is a discrete cocompact lattice in $G(k)$ (see [AGH61, Theorem 5(4)]).

On the product $G(k) \times \mathbb{R}$ we have the action of $D \times \mathbb{Z}$, where D acts on $G(k)$ by multiplication *on the left*, while \mathbb{Z} acts on \mathbb{R} by translations. Let $M(k)$ be the

quotient of $G(k) \times \mathbb{R}$ by this diagonal action, then $M(k)$ is a compact manifold, admitting the following 1-forms:

- e^1 induced by the invariant form $e^{-kz} dx$,
- e^2 induced by the invariant form $e^{kz} dy$,
- e^3 induced by the invariant form dz ,
- e^4 induced by the invariant form dt .

Dually, the corresponding fields are

- e_1 induced by the invariant field $e^{kz} \frac{\partial}{\partial x}$,
 - e_2 induced by the invariant field $e^{-kz} \frac{\partial}{\partial y}$,
 - e_3 induced by the invariant field $\frac{\partial}{\partial z}$,
 - e_4 induced by the invariant field $\frac{\partial}{\partial t}$.
- (3.1)

The structure equations for $M(k)$ and the only non trivial commutators are

$$de^1 = ke^1 \wedge e^3, \quad de^2 = -ke^2 \wedge e^3, \quad de^3 = 0, \quad de^4 = 0$$

and

$$[e_1, e_3] = -ke_1, \quad [e_2, e_3] = ke_2$$

respectively.

We will consider on $M(k)$ the following additional structures:

- the 2-form

$$\omega = e^1 \wedge e^2 + e^3 \wedge e^4,$$

which is easily seen to be a symplectic form on $M(k)$;

- the Riemannian metric

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4.$$

Remark 3.1 — The manifold $M(k)$ is (real) parallelizable, since the vectors $e_1(P)$, $e_2(P)$, $e_3(P)$ and $e_4(P)$ span the tangent space $T_P M(k)$ for every $P \in M(k)$. As a consequence, we can define an almost complex structure on $M(k)$ by simply specifying its action on $H^0(M(k), TM(k))$.

Remark 3.2 — Observe that a smooth function $f : M(k) \rightarrow \mathbb{R}$ can be identified with a smooth function (still denoted by f) on the universal cover \mathbb{R}^4 such that

$$f(x, y, z, t) = f(e^{kn\gamma} x + \alpha u_1 + \beta v_1, e^{-kn\gamma} y + \alpha u_2 + \beta v_2, z + n\gamma, t + \varepsilon)$$

for every $\alpha, \beta, \gamma, \varepsilon \in \mathbb{Z}$. In particular, for $\gamma = 0$ we see that f is \mathbb{Z}^3 -periodic, and so it admits a Fourier series expansion of the form

$$(3.2) \quad f(x, y, z, t) = \sum_{I=(a,b,c) \in \mathbb{Z}^3} f_I(z) e^{2\pi\sqrt{-1}\frac{1}{3}((av_2 - bu_2)x + (-av_1 + bu_1)y + c\delta t)},$$

where $\delta = u_1 v_2 - u_2 v_1 \neq 0$.

3.2. The nilmanifold \mathcal{N} . Let \mathfrak{G} be the group $(\mathbb{R}^4, *)$, where the product of $(a, b, c, d), (x, y, z, t) \in \mathbb{R}^4$ is defined as

$$(a, b, c, d) * (x, y, z, t) = \left(x + a, y + b, z + ay + c, t + \frac{1}{2}a^2y + az + d \right).$$

It is then straightforward to verify that the set of 1-forms

$$\begin{aligned} e^1 &= dx, \\ e^2 &= dy, \\ e^3 &= dz - xdy, \\ e^4 &= dt + \frac{1}{2}x^2dy - xdz \end{aligned}$$

on \mathfrak{G} is a basis for the space $\Gamma(\mathfrak{G}, T\mathfrak{G})$, and that these forms are left invariant with respect to translations by elements in the subgroup consisting of 4-tuples with *even* integral entries.

Call \mathcal{N} the quotient of \mathfrak{G} by such an action, then the previous forms descend to \mathcal{N} and their valuations at any given point span the cotangent space at that point. The dual tangent frame to \mathcal{N} is given by the fields

$$(3.3) \quad \begin{aligned} e_1 &= \frac{\partial}{\partial x}, \\ e_2 &= \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + \frac{1}{2}x^2\frac{\partial}{\partial t}, \\ e_3 &= \frac{\partial}{\partial z} + x\frac{\partial}{\partial t}, \\ e_4 &= \frac{\partial}{\partial t}. \end{aligned}$$

The structure equations for \mathcal{N} and the only non trivial commutators are

$$(3.4) \quad de^1 = 0, \quad de^2 = 0, \quad de^3 = -e^1 \wedge e^2, \quad de^4 = -e^1 \wedge e^3,$$

and

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4,$$

respectively.

Remark 3.3 — The 2-form

$$\xi = e^1 \wedge e^4 + e^2 \wedge e^3$$

is a symplectic form on \mathcal{N} . Indeed, $\xi \wedge \xi = 2e^1 \wedge e^2 \wedge e^3 \wedge e^4$ and by the structure equations $d\xi = 0$.

Remark 3.4 — As \mathcal{N} is a quotient of \mathbb{R}^4 by the action of a suitable lattice, we can see a smooth function $f : \mathcal{N} \rightarrow \mathbb{R}$ as an invariant function on \mathbb{R}^4 , where invariance is meant in terms of the lattice:

$$(3.5) \quad f(x, y, z, t) = f(x + 2\alpha, y + 2\beta, z + 2\alpha y + 2\gamma, t + 2\alpha^2 y + 2\alpha z + 2\delta) \quad \forall \alpha, \beta, \gamma, \delta \in \mathbb{Z}.$$

In particular f is periodic of period 2 in y, z, t , hence it admits a Fourier expansion of the form

$$(3.6) \quad f(x, y, z, t) = \sum_{I=(a,b,c) \in \mathbb{Z}^3} f_I(x) e^{\sqrt{-1}\pi(a y + b z + c t)}.$$

3.3. The solvmanifolds $\mathcal{M}(\lambda)$. Let $\lambda \in (-1, -\frac{1}{2})$ and define $\mathcal{G}(\lambda)$ as the group

$$\mathcal{G}(\lambda) = \left\{ \left(\begin{array}{ccccc} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^t & 0 & 0 & y \\ 0 & 0 & e^{-(1+\lambda)t} & 0 & z \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{array} \right) \middle| x, y, z, t \in \mathbb{R} \right\}.$$

Then $\mathcal{G}(\lambda)$ is a simply connected solvable Lie group isomorphic to

$$\mathbb{R}^3 \rtimes_{\varphi} \mathbb{R}, \text{ where } \varphi(t) = \begin{pmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-(1+\lambda)t} \end{pmatrix}$$

and we can characterize the values of the parameter λ for which it admits a lattice. In fact, by [LLSY07, Proposition 2.1] we have that $\mathcal{G}(\lambda)$ admits a lattice if and only if there exist integers m, n with $m \neq n$ such that the equation $x^3 - mx^2 + nx - 1 = 0$ has 3 real, positive and distinct solutions $\alpha_1 > \alpha_2 > \alpha_3$. Assume this is the case, and define

$$(\lambda, t_0) = \begin{cases} \left(\frac{\log \alpha_3}{\log \alpha_1}, \log \alpha_1 \right), & \text{if } \alpha_1 > 1 > \alpha_2 > \alpha_3, \\ \left(\frac{\log \alpha_1}{\log \alpha_3}, \log \alpha_3 \right), & \text{if } \alpha_1 > \alpha_2 > 1 > \alpha_3; \end{cases}$$

then there exists a matrix $A \in \text{GL}(3, \mathbb{Z})$ which is similar to the matrix $\varphi(t_0)$. Let $P \in \text{GL}(3, \mathbb{R})$ be such that $\varphi(t_0) = P \cdot A \cdot P^{-1}$. Let \mathbb{Z} act on \mathbb{Z}^3 via the matrix A and consider the semidirect product $\mathbb{Z}^3 \rtimes \mathbb{Z}$, it is possible to embed this group in $\mathcal{G}(\lambda)$ as a lattice in the following way: we send the canonical generators of \mathbb{Z}^3 to the vectors in \mathbb{R}^3 corresponding to the columns of the matrix P and the generator of \mathbb{Z} corresponding to A to t_0 .

We let $\mathcal{M}(\lambda)$ be the quotient of $\mathcal{G}(\lambda)$ by the *left* action of translations by elements of a lattice in $\mathcal{G}(\lambda)$ (being understood that λ is chosen in such a way that a lattice exists).

The 1-forms

$$\begin{aligned} e^1 &= dt, \\ e^2 &= e^{-t} dy, \\ e^3 &= e^{-\lambda t} dx, \\ e^4 &= e^{(1+\lambda)t} dz, \end{aligned}$$

generate the cotangent space at each point of $\mathcal{M}(\lambda)$, and it is easy to see that they satisfy the structure equations

$$(3.7) \quad de^1 = 0, \quad de^2 = -e^1 \wedge e^2, \quad de^3 = -\lambda e^1 \wedge e^3, \quad de^4 = (1 + \lambda) e^1 \wedge e^4.$$

The dual picture on the tangent bundle is as follows: the dual tangent frame is

$$(3.8) \quad \begin{aligned} e_1 &= \frac{\partial}{\partial t}, \\ e_2 &= e^t \frac{\partial}{\partial y}, \\ e_3 &= e^{\lambda t} \frac{\partial}{\partial x}, \\ e_4 &= e^{-(1+\lambda)t} \frac{\partial}{\partial z}, \end{aligned}$$

and the non trivial commutators are

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = \lambda e_3, \quad [e_1, e_4] = -(1 + \lambda) e_4.$$

Hence $\mathcal{M}(\lambda)$ is a 2-step (non nilpotent) solvmanifold. It is known (cf. [Has05, Section 4]) that $\mathcal{M}(\lambda)$ admits no integrable almost complex structures.

Remark 3.5 — As $\mathcal{M}(\lambda)$ is a quotient of \mathbb{R}^4 by the action of a suitable lattice Γ , we can see a smooth function $f : \mathcal{M}(\lambda) \rightarrow \mathbb{R}$ as an invariant function on \mathbb{R}^4 , where invariance is meant in terms of the lattice:

$$(3.9) \quad f(x, y, z, t) = f(e^{\lambda \delta} x + \alpha, e^{\delta} y + \beta, e^{-(1+\lambda)\delta} z + \gamma, t + \delta) \quad \forall (\alpha, \beta, \gamma, \delta) \in \Gamma.$$

In particular, f is periodic in (x, y, z) with periods given by the columns of the matrix P , hence f admits a Fourier expansion of the form

$$(3.10) \quad f(x, y, z, t) = \sum_{I=(a,b,c) \in \mathbb{Z}^3} f_I(t) e^{2\pi\sqrt{-1}(\lambda_I x + \mu_I y + \nu_I z)},$$

with $(\lambda_I, \mu_I, \nu_I) = I \cdot P^{-1} = (a, b, c) \cdot P^{-1}$.

4. SPECIAL FAMILIES OF ALMOST COMPLEX STRUCTURES ON $M(k)$

Consider the following families of endomorphisms of the tangent bundle of $M(k)$:

$$(4.1) \quad J_s = \begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 \\ \gamma_1 & -\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & \gamma_2 & -\alpha_2 \end{pmatrix},$$

$$\text{with } \begin{cases} s = (r_1, s_1, r_2, s_2) \in \mathbb{R}^4, \\ \alpha_i = -\frac{2s_i}{r_i^2 + s_i^2 - 1}, \\ \beta_i = \frac{r_i^2 + 2r_i + s_i^2 + 1}{r_i^2 + s_i^2 - 1}, \\ \gamma_i = -\frac{r_i^2 - 2r_i + s_i^2 + 1}{r_i^2 + s_i^2 - 1}, \end{cases}$$

and

$$J_r = \alpha \begin{pmatrix} 0 & -(1-r)^2 & 2r^2 & -2r(1-r) \\ (1-r)^2 & 0 & -2r(1-r) & -2r^2 \\ -2r^2 & 2r(1-r) & 0 & -(1-r)^2 \\ 2r(1-r) & 2r^2 & (1-r)^2 & 0 \end{pmatrix},$$

$$\text{with } \begin{cases} r \in \mathbb{R}, \\ \alpha = \frac{1}{(1-r)^2 + 2r^2} = \frac{1}{3r^2 - 2r + 1}. \end{cases}$$

It is then easy to see that $\alpha_i^2 + \beta_i \gamma_i = -1$ for $i = 1, 2$ and that both J_s and J_r define smooth families of almost complex structures on $M(k)$, as $J_s^2 = J_r^2 = -\text{id}_{TM(k)}$. We recall here that since $M(k)$ can not admit an integrable almost complex structure, none of the J_s 's nor of the J_r 's is integrable.

The almost complex structures J_s behave well with respect to the symplectic form $\omega = e^1 \wedge e^2 + e^3 \wedge e^4$, at least for small values of the parameter $s = (r_1, s_1, r_2, s_2)$: we have in fact that $\omega(J_s \cdot, J_s \cdot) = \omega(\cdot, \cdot)$, and that the symmetric bilinear form $g_s(\cdot, \cdot) = \omega(\cdot, J_s \cdot)$ is positive definite (hence it is a Riemannian metric on $M(k)$). The matrix representation for the metric g_s in the basis $\{e_1, e_2, e_3, e_4\}$ is

$$g_s = \begin{pmatrix} \gamma_1 & -\alpha_1 & 0 & 0 \\ -\alpha_1 & -\beta_1 & 0 & 0 \\ 0 & 0 & \gamma_2 & -\alpha_2 \\ 0 & 0 & -\alpha_2 & -\beta_2 \end{pmatrix}.$$

Since the fundamental form of all the almost Hermitian manifolds $(M(k), g_s, J_s)$ is ω , we have a family of almost Kähler manifolds.

On the other hand, the almost complex structures J_r are all compatible with the fixed metric g , as $g(J_r \cdot, J_r \cdot) = g(\cdot, \cdot)$, hence we have an almost Hermitian manifold $(M(k), g, J_r)$ for all $r \in \mathbb{R}$. By a direct computation, it is easy to see that the fundamental form $\omega_r(\cdot, \cdot) = g(J_r \cdot, \cdot)$ is d -closed if and only if $r = 0$. So, apart for $r = 0$, none of the manifolds in this family is almost Kähler.

We remark that both families $\{J_s\}$ and $\{J_r\}$ are deformations of the almost complex structure J_0 defined by $s = 0$ in the first family and by $r = 0$ in the second family.

4.1. Kodaira dimension. We focus first on the family $(M(k), J_s)$.

We can then find a g_s -orthonormal frame $\{\varepsilon_1(s), \varepsilon_2(s), \varepsilon_3(s), \varepsilon_4(s)\}$ for the tangent bundle (to simplify the notation, from now on we will drop the explicit dependence on s):

$$\begin{aligned} \varepsilon_1 &= \frac{1}{\sqrt{\gamma_1}} e_1, & \varepsilon_3 &= \frac{1}{\sqrt{\gamma_2}} e_3, \\ \varepsilon_2 &= \frac{1}{\sqrt{\gamma_1}} (\alpha_1 e_1 + \gamma_1 e_2), & \varepsilon_4 &= \frac{1}{\sqrt{\gamma_2}} (\alpha_2 e_3 + \gamma_2 e_4), \end{aligned}$$

whose corresponding dual frame is

$$\begin{aligned} \varepsilon^1 &= \sqrt{\gamma_1} \left(e^1 - \frac{\alpha_1}{\gamma_1} e^2 \right), & \varepsilon^3 &= \sqrt{\gamma_2} \left(e^3 - \frac{\alpha_2}{\gamma_2} e^4 \right), \\ \varepsilon^2 &= \frac{1}{\sqrt{\gamma_1}} e^2, & \varepsilon^4 &= \frac{1}{\sqrt{\gamma_2}} e^4. \end{aligned}$$

The commutators of the fields and, dually, the differentials of these forms are respectively given by

$$\begin{aligned} [\varepsilon_1, \varepsilon_2] &= 0, & [\varepsilon_1, \varepsilon_3] &= -\frac{k}{\sqrt{\gamma_2}} \varepsilon_1, & [\varepsilon_1, \varepsilon_4] &= -\frac{k\alpha_2}{\sqrt{\gamma_2}} \varepsilon_1, \\ [\varepsilon_2, \varepsilon_3] &= -2\frac{k\alpha_1}{\sqrt{\gamma_2}} \varepsilon_1 + \frac{k}{\sqrt{\gamma_2}} \varepsilon_2, & [\varepsilon_2, \varepsilon_4] &= -2\frac{k\alpha_1\alpha_2}{\sqrt{\gamma_2}} \varepsilon_1 + \frac{k\alpha_2}{\sqrt{\gamma_2}} \varepsilon_2, & [\varepsilon_3, \varepsilon_4] &= 0, \end{aligned}$$

and by

$$\begin{aligned} d\varepsilon^1 &= \frac{k}{\sqrt{\gamma_2}} \varepsilon^1 \wedge \varepsilon^3 + \frac{k\alpha_2}{\sqrt{\gamma_2}} \varepsilon^1 \wedge \varepsilon^4 + 2\frac{k\alpha_1}{\sqrt{\gamma_2}} \varepsilon^2 \wedge \varepsilon^3 + 2\frac{k\alpha_1\alpha_2}{\sqrt{\gamma_2}} \varepsilon^2 \wedge \varepsilon^4, & d\varepsilon^3 &= 0, \\ d\varepsilon^2 &= -\frac{k}{\sqrt{\gamma_2}} \varepsilon^2 \wedge \varepsilon^3 - \frac{k\alpha_2}{\sqrt{\gamma_2}} \varepsilon^2 \wedge \varepsilon^4, & d\varepsilon^4 &= 0. \end{aligned}$$

Now, we switch from the real to the complex formalism. Let h_s be the Hermitian metric induced by g_s , we consider the h_s -unitary frame of fields of type $(1, 0)$

$$\mathcal{X}_1 = \frac{\sqrt{2}}{2} (\varepsilon_1 - \sqrt{-1}\varepsilon_2), \quad \mathcal{X}_2 = \frac{\sqrt{2}}{2} (\varepsilon_3 - \sqrt{-1}\varepsilon_4),$$

and its dual frame of $(1, 0)$ -forms

$$\varphi^1 = \frac{\sqrt{2}}{2} (\varepsilon^1 + \sqrt{-1}\varepsilon^2), \quad \varphi^2 = \frac{\sqrt{2}}{2} (\varepsilon^3 + \sqrt{-1}\varepsilon^4).$$

It is in fact easy to see that the almost complex structure J_s takes the standard form on the basis $\{\varepsilon_1(s), \varepsilon_2(s), \varepsilon_3(s), \varepsilon_4(s)\}$. It follows that

$$\begin{aligned} d\varphi^1 &= \frac{\sqrt{2}k}{2\sqrt{\gamma_2}} (-\sqrt{-1}\alpha_1(1 - \sqrt{-1}\alpha_2)\varphi^1 \wedge \varphi^2 + \\ &\quad -\sqrt{-1}\alpha_1(1 + \sqrt{-1}\alpha_2)\varphi^1 \wedge \bar{\varphi}^2 + \\ &\quad -(1 + \sqrt{-1}\alpha_1)(1 - \sqrt{-1}\alpha_2)\varphi^2 \wedge \bar{\varphi}^1 + \\ &\quad +(1 + \sqrt{-1}\alpha_1)(1 + \sqrt{-1}\alpha_2)\bar{\varphi}^1 \wedge \bar{\varphi}^2), \\ d\varphi^2 &= 0, \end{aligned}$$

from which we deduce that

$$\begin{aligned}\bar{\partial}\varphi^1 &= -\frac{\sqrt{2k}}{2\sqrt{\gamma_2}}(\sqrt{-1}\alpha_1(1+\sqrt{-1}\alpha_2)\varphi^1\wedge\bar{\varphi}^2+(1+\sqrt{-1}\alpha_1)(1-\sqrt{-1}\alpha_2)\varphi^2\wedge\bar{\varphi}^1), \\ \bar{\partial}\varphi^2 &= 0.\end{aligned}$$

We observe that $\varphi^1\wedge\varphi^2$ is a smooth section of the canonical bundle of $(M(k), J_s)$ and

$$\bar{\partial}(\varphi^1\wedge\varphi^2) = \frac{\sqrt{2k}}{2\sqrt{\gamma_2}}\sqrt{-1}\alpha_1(1+\sqrt{-1}\alpha_2)\varphi^1\wedge\varphi^2\wedge\bar{\varphi}^2.$$

Proposition 4.1 — *In a suitable neighbourhood of the origin the Kodaira dimension of $(M(k), J_s)$ is*

$$\text{kod}(M(k), J_s) = \begin{cases} -\infty & \text{if } \alpha_1 \neq 0, \\ 0 & \text{if } \alpha_1 = 0. \end{cases}$$

Proof. Let $f\varphi^1\wedge\varphi^2$ be a smooth section of the canonical bundle. This section is pseudoholomorphic if and only if

$$0 = \bar{\partial}(f\varphi^1\wedge\varphi^2) = \bar{\mathcal{X}}_1(f)\varphi^1\wedge\varphi^2\wedge\bar{\varphi}^1 + \bar{\mathcal{X}}_2(f)\varphi^1\wedge\varphi^2\wedge\bar{\varphi}^2 + f\bar{\partial}(\varphi^1\wedge\varphi^2),$$

hence if and only if

$$(4.2) \quad \begin{cases} \bar{\mathcal{X}}_1(f) = 0 \\ \bar{\mathcal{X}}_2(f) + \frac{\sqrt{2k}}{2\sqrt{\gamma_2}}\sqrt{-1}f\alpha_1(1+\sqrt{-1}\alpha_2) = 0. \end{cases}$$

Write $f = u + \sqrt{-1}v$, where we see u and v as real functions of the *real* variables x, y, z and t , which are defined on \mathbb{R}^4 and are periodic with respect to the action of the lattice $D \times \mathbb{Z}$.

The first equation in (4.2) then becomes

$$0 = \bar{\mathcal{X}}_1(f) = \frac{\sqrt{2}}{2}(\varepsilon_1 + \sqrt{-1}\varepsilon_2)(u + \sqrt{-1}v) = \frac{\sqrt{2}}{2}(\varepsilon_1(u) - \varepsilon_2(v) + \sqrt{-1}(\varepsilon_1(v) + \varepsilon_2(u))),$$

which leads us to the system

$$\begin{cases} e^{kz}\frac{\partial u}{\partial x} - \alpha_1 e^{kz}\frac{\partial v}{\partial x} - \gamma_1 e^{-kz}\frac{\partial v}{\partial y} = 0 \\ e^{kz}\frac{\partial v}{\partial x} + \alpha_1 e^{kz}\frac{\partial u}{\partial x} + \gamma_1 e^{-kz}\frac{\partial u}{\partial y} = 0. \end{cases}$$

We can then express $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in terms of $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$:

$$\begin{cases} \frac{\partial u}{\partial x} = \alpha_1 \frac{\partial v}{\partial x} + \gamma_1 e^{-2kz} \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = e^{2kz} \beta_1 \frac{\partial v}{\partial x} - \alpha_1 \frac{\partial v}{\partial y}, \end{cases}$$

and so once we take the derivative with respect to y of the first relation and with respect to x of the second one, we see that the relation

$$\alpha_1 \frac{\partial^2 v}{\partial x \partial y} + \gamma_1 e^{-2kz} \frac{\partial^2 v}{\partial y^2} = e^{2kz} \beta_1 \frac{\partial^2 v}{\partial x^2} - \alpha_1 \frac{\partial^2 v}{\partial x \partial y}$$

must hold. Hence v is a solution of the elliptic differential equation

$$\left(-e^{2kz} \beta_1 \frac{\partial^2}{\partial x^2} + 2\alpha_1 \frac{\partial^2}{\partial x \partial y} + e^{-2kz} \gamma_1 \frac{\partial^2}{\partial y^2} \right) v = 0,$$

and so it must be constant with respect to x and y (because of the periodicity). An analogous argument shows that also u must be constant with respect to x and y .

Consider the Fourier series expansion (3.2) of u and v . Since we know that they do not depend on x and y , this expansion can be simplified further to

$$(4.3) \quad u(x, y, z, t) = \sum_{I=(a,b) \in \mathbb{Z}^2} u_I e^{2\pi\sqrt{-1}(\frac{a}{n}z+bt)}, \quad u_I \in \mathbb{C}$$

and similarly for v .

The second equation in (4.2) is equivalent to the system

$$\begin{cases} \frac{\partial u}{\partial z} - \alpha_2 \frac{\partial v}{\partial z} - \gamma_2 \frac{\partial v}{\partial t} = k\alpha_1(\alpha_2 u + v) \\ \frac{\partial v}{\partial z} + \alpha_2 \frac{\partial u}{\partial z} + \gamma_2 \frac{\partial u}{\partial t} = k\alpha_1(\alpha_2 v - u), \end{cases}$$

in view of (4.3) we obtain the following system for (u_I, v_I)

$$\begin{cases} 2\pi\sqrt{-1}\frac{a}{n}u_I - 2\pi\sqrt{-1}\frac{a}{n}\alpha_2 v_I - 2\pi\sqrt{-1}b\gamma_2 v_I - k\alpha_1\alpha_2 u_I - k\alpha_1 v_I = 0 \\ 2\pi\sqrt{-1}\frac{a}{n}v_I + 2\pi\sqrt{-1}\frac{a}{n}\alpha_2 u_I + 2\pi\sqrt{-1}b\gamma_2 u_I - k\alpha_1\alpha_2 v_I + k\alpha_1 u_I = 0. \end{cases}$$

This is a homogeneous linear system, whose representing matrix has determinant

$$(4.4) \quad -4\pi^2 \frac{a^2}{n^2} + k^2 \alpha_1^2 (\alpha_2^2 + 1) - 4\pi^2 \left(\frac{a}{n} \alpha_2 + b\gamma_2 \right)^2 + 4\pi\sqrt{-1}bk\alpha_1\gamma_2.$$

We want to determine when this determinant vanishes.

- (1) If $\alpha_1 = 0$, the determinant (4.4) vanishes if and only if $a = b = 0$.
In fact for $\alpha_1 = 0$ the imaginary part of (4.4) is zero and its real part is $-4\pi^2 \left(\frac{a^2}{n^2} + \left(\frac{a}{n} \alpha_2 + b\gamma_2 \right)^2 \right)$. So this determinant vanishes if and only if $a = b\gamma_2 = 0$, which happens if and only if $a = b = 0$ in a suitable neighbourhood of the origin.
- (2) If $\alpha_1 \neq 0$, the determinant (4.4) is always non-zero.
In fact, if $\alpha_1 \neq 0$ then in a neighbourhood of the origin the imaginary part of (4.4) vanishes if and only if $b = 0$. In this case the real part of 4.4 is

$$(1 + \alpha_2^2) \left(k\alpha_1 + 2\pi\frac{a}{n} \right) \left(k\alpha_1 - 2\pi\frac{a}{n} \right),$$

which is zero if and only if $a = \pm \frac{k\alpha_1 n}{2\pi}$. In a suitable neighbourhood of the origin we have $|\frac{k\alpha_1 n}{2\pi}| < 1$, so on such a neighbourhood the only possibility is $a = 0$. But then this implies that $\alpha_1 = 0$, which is not the case.

We can summarize this result saying that f is a solution of (4.2) if and only if

$$\begin{cases} f = \text{const} & \text{if } \alpha_1 = 0, \\ f = 0 & \text{if } \alpha_1 \neq 0. \end{cases}$$

To conclude the computation of the Kodaira dimension of $(M(k), J_s)$, we observe that it is not difficult to see that

$$\bar{\partial}((\varphi^1 \wedge \varphi^2)^{\otimes m}) = \frac{\sqrt{2}k}{2\sqrt{\gamma_2}} \sqrt{-1}\alpha_1 (1 + \sqrt{-1}\alpha_2) m \bar{\varphi}^2 \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}$$

as a section of $T^{*0,1}M(k) \otimes (\wedge^{2,0}T^*M(k))^{\otimes m}$. Hence $\bar{\partial}(f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}) = 0$ leads us to a situation which is completely analogous to the one described in (4.2). As a consequence, we can claim that

$$P_m(M(k), J_s) = \begin{cases} 0 & \text{if } \alpha_1 \neq 0, \\ 1 & \text{if } \alpha_1 = 0, \end{cases}$$

and so

$$\kappa(M(k), J_s) = \begin{cases} -\infty & \text{if } \alpha_1 \neq 0, \\ 0 & \text{if } \alpha_1 = 0. \end{cases}$$

□

Consider now the family $(M(k), J_r)$. A g_0 -orthonormal basis in which the almost complex structures J_r are in canonical form is then given by

$$\begin{aligned} \varepsilon_1 &= \sqrt{\alpha}((1-r)e_1 + re_2 - re_3), & \varepsilon_3 &= \sqrt{\alpha}(re_1 + (1-r)e_3 + re_4) \\ \varepsilon_2 &= \sqrt{\alpha}(-re_1 + (1-r)e_2 + re_4), & \varepsilon_4 &= \sqrt{\alpha}(-re_2 - re_3 + (1-r)e_4), \end{aligned}$$

whose corresponding dual frame is

$$\begin{aligned} \varepsilon^1 &= \sqrt{\alpha}((1-r)e^1 + re^2 - re^3), & \varepsilon^3 &= \sqrt{\alpha}(re^1 + (1-r)e^3 + re^4) \\ \varepsilon^2 &= \sqrt{\alpha}(-re^1 + (1-r)e^2 + re^4), & \varepsilon^4 &= \sqrt{\alpha}(-re^2 - re^3 + (1-r)e^4). \end{aligned}$$

It follows that

$$\mathcal{X}_1 = \frac{\sqrt{2}}{2} (\varepsilon_1 - \sqrt{-1}\varepsilon_2), \quad \mathcal{X}_2 = \frac{\sqrt{2}}{2} (\varepsilon_3 - \sqrt{-1}\varepsilon_4)$$

is an h -unitary frame of fields of type $(1, 0)$, where h is the Hermitian form associated to g and J_r , and that

$$\varphi^1 = \frac{\sqrt{2}}{2} (\varepsilon^1 + \sqrt{-1}\varepsilon^2), \quad \varphi^2 = \frac{\sqrt{2}}{2} (\varepsilon^3 + \sqrt{-1}\varepsilon^4)$$

is the corresponding coframe of forms of type $(1, 0)$. A computation yields

$$\begin{aligned} d\varphi^1 &= \frac{\sqrt{2}}{2}\alpha\sqrt{\alpha}k(1-r-\sqrt{-1}r)(r^2\varphi^1 \wedge \varphi^2 + \\ &\quad + r(1-r-\sqrt{-1}r)\varphi^1 \wedge \bar{\varphi}^1 + \\ &\quad - ((1-r)^2 + 2r^2)\varphi^2 \wedge \bar{\varphi}^1 + \\ &\quad + r(1-r-\sqrt{-1}r)\varphi^2 \wedge \bar{\varphi}^2 + \\ &\quad + (1-r-\sqrt{-1}r)^2\bar{\varphi}^1 \wedge \bar{\varphi}^2), \\ d\varphi^2 &= \frac{\sqrt{2}}{2}\alpha\sqrt{\alpha}kr((1-r+\sqrt{-1}r)^2\varphi^1 \wedge \varphi^2 + \\ &\quad - r(1-r+\sqrt{-1}r)\varphi^1 \wedge \bar{\varphi}^1 + \\ &\quad + \frac{1}{\alpha}\varphi^1 \wedge \bar{\varphi}^2 + \\ &\quad - r(1-r+\sqrt{-1}r)\varphi^2 \wedge \bar{\varphi}^2 + \\ &\quad + r^2\bar{\varphi}^1 \wedge \bar{\varphi}^2). \end{aligned}$$

We can then see that

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = \underbrace{-\frac{\sqrt{2}}{2}\alpha\sqrt{\alpha}kr((1-r-\sqrt{-1}r)^2\bar{\varphi}^1 - r(1-r+\sqrt{-1}r)\bar{\varphi}^2)}_{\tau} \wedge \varphi^1 \wedge \varphi^2.$$

As a consequence

$$\bar{\partial}((\varphi^1 \wedge \varphi^2)^{\otimes m}) = m\tau \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}, \quad \text{for all } m \geq 1,$$

and so we see that the smooth pluricanonical section $(\varphi^1 \wedge \varphi^2)^{\otimes m}$ is pseudoholomorphic if and only if $r = 0$.

Proposition 4.2 — *We have the following:*

$$P_m(M(k), J_r) = \begin{cases} 0 & \text{if } r = 1, \\ 1 & \text{if } r = 0, \end{cases}$$

and so

$$\text{kod}(M(k), J_r) = \begin{cases} -\infty & \text{if } r = 1, \\ 0 & \text{if } r = 0. \end{cases}$$

Proof. This proposition is a particular case of Proposition 7.7 (see also Remark 7.8) for $r = 1$ and a particular case of Proposition 4.1 for $r = 0$. \square

4.2. Curvature of the canonical connection. The manifold $(M(k), J_s, g_s)$ is an almost Kähler manifold for every (small) value of the parameter s . As a consequence, we can use the same strategy used in [CNT20] to compute the Chern–Ricci tensor and the scalar curvature of the canonical connection of these almost Kähler metrics.

Direct computations give:

Lemma 4.3 — *The Nijenhuis tensor of J_s is the following:*

$$\begin{aligned} N_{J_s}(\varepsilon_1, \varepsilon_2) &= 0, \\ N_{J_s}(\varepsilon_1, \varepsilon_3) &= 2\frac{k}{\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)\varepsilon_1 + 2\frac{k}{\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)\varepsilon_2, \\ N_{J_s}(\varepsilon_1, \varepsilon_4) &= 2\frac{k}{\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)\varepsilon_1 - 2\frac{k}{\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)\varepsilon_2, \\ N_{J_s}(\varepsilon_2, \varepsilon_3) &= 2\frac{k}{\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)\varepsilon_1 - 2\frac{k}{\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)\varepsilon_2, \\ N_{J_s}(\varepsilon_2, \varepsilon_4) &= -2\frac{k}{\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)\varepsilon_1 - 2\frac{k}{\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)\varepsilon_2, \\ N_{J_s}(\varepsilon_3, \varepsilon_4) &= 0. \end{aligned}$$

By [CNT20, Corollary 3.8] we can then deduce the real torsion forms of the canonical connection:

$$\begin{aligned} \Theta_{\mathbb{R}}^1 &= \frac{k}{2\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)(\varepsilon^1 \wedge \varepsilon^3 - \varepsilon^2 \wedge \varepsilon^4) + \frac{1}{2\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)(\varepsilon^1 \wedge \varepsilon^4 + \varepsilon^2 \wedge \varepsilon^3), \\ \Theta_{\mathbb{R}}^2 &= \frac{k}{2\sqrt{\gamma_2}}(\alpha_1 + \alpha_2)(\varepsilon^1 \wedge \varepsilon^3 - \varepsilon^2 \wedge \varepsilon^4) - \frac{1}{2\sqrt{\gamma_2}}(1 - \alpha_1\alpha_2)(\varepsilon^1 \wedge \varepsilon^4 + \varepsilon^2 \wedge \varepsilon^3), \\ \Theta_{\mathbb{R}}^3 &= 0, \\ \Theta_{\mathbb{R}}^4 &= 0. \end{aligned}$$

The complex torsion forms are easy to compute from these ones:

$$\begin{aligned} \Theta^1 &= \frac{\sqrt{2}}{2} (\Theta_{\mathbb{R}}^1 + \sqrt{-1}\Theta_{\mathbb{R}}^2) = \frac{\sqrt{2}k}{2\sqrt{\gamma_2}}(1 + \sqrt{-1}\alpha_1)(1 + \sqrt{-1}\alpha_2)\bar{\varphi}^1 \wedge \bar{\varphi}^2, \\ \Theta^2 &= \frac{\sqrt{2}}{2} (\Theta_{\mathbb{R}}^3 + \sqrt{-1}\Theta_{\mathbb{R}}^4) = 0. \end{aligned}$$

To deduce the complex connection forms, we have to solve the system given by the first structure equations:

$$\begin{cases} d\varphi^1 + \vartheta_1^1 \wedge \varphi^1 + \vartheta_2^1 \wedge \varphi^2 = \Theta^1 \\ d\varphi^2 + \vartheta_1^2 \wedge \varphi^1 + \vartheta_2^2 \wedge \varphi^2 = \Theta^2 \\ \vartheta_1^1 + \overline{\vartheta_1^1} = \vartheta_1^2 + \overline{\vartheta_1^2} = \vartheta_2^2 + \overline{\vartheta_2^2} = 0, \end{cases}$$

whose solution is

$$\begin{aligned}\vartheta_1^1 &= -\frac{\sqrt{2}k}{2\sqrt{\gamma_2}}\sqrt{-1}\alpha_1(1-\sqrt{-1}\alpha_2)\varphi^2 - \frac{\sqrt{2}k}{2\sqrt{\gamma_2}}\sqrt{-1}\alpha_1(1+\sqrt{-1}\alpha_2)\bar{\varphi}^2, \\ \vartheta_2^1 &= -\frac{\sqrt{2}k}{2\sqrt{\gamma_2}}(1+\sqrt{-1}\alpha_1)(1+\sqrt{-1}\alpha_2)\bar{\varphi}^1, \\ \vartheta_1^2 &= \frac{\sqrt{2}k}{2\sqrt{\gamma_2}}(1-\sqrt{-1}\alpha_1)(1-\sqrt{-1}\alpha_2)\varphi^1, \\ \vartheta_2^2 &= 0.\end{aligned}$$

From the knowledge of the connection forms, we can compute the curvature forms ψ_j^i by means of the second structure equations, and we see that

$$\begin{aligned}\psi_1^1 &= \frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\alpha_2^2)\varphi^1 \wedge \bar{\varphi}^1, \\ \psi_1^2 &= -\frac{k^2}{\gamma_2}\sqrt{-1}\alpha_1(1-\sqrt{-1}\alpha_1)(1-\sqrt{-1}\alpha_2)^2\varphi^1 \wedge \varphi^2 + \\ &\quad -\frac{k^2}{\gamma_2}\sqrt{-1}\alpha_1(1-\sqrt{-1}\alpha_1)(1+\alpha_2^2)\varphi^1 \wedge \bar{\varphi}^2 + \\ &\quad -\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1-\sqrt{-1}\alpha_2)^2\varphi^2 \wedge \bar{\varphi}^1 + \\ &\quad +\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\alpha_2^2)\bar{\varphi}^1 \wedge \bar{\varphi}^2, \\ \psi_2^1 &= -\bar{\psi}_1^2, \\ \psi_2^2 &= -\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\alpha_2^2)\varphi^1 \wedge \bar{\varphi}^1.\end{aligned}$$

We can state the following.

Proposition 4.4 — *All the manifolds $(M(k), J_s, g_s)$ in our family are Chern–Ricci flat, hence they also have vanishing scalar curvature.*

Proof. To compute the component $R_{k\bar{l}}$ of the Chern–Ricci tensor we recall that $R_{jk\bar{l}}$ is the coefficient of $\varphi^k \wedge \bar{\varphi}^l$ in ψ_j^i and that $R_{k\bar{l}} = \sum_i R_{ik\bar{l}}$. But the only non-vanishing coefficients among the $R_{jk\bar{l}}$ are

$$\begin{aligned}R_{11\bar{1}}^1 &= \frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\alpha_2^2), & R_{21\bar{2}}^1 &= -\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\sqrt{-1}\alpha_2)^2, \\ R_{22\bar{1}}^1 &= \frac{k^2}{\gamma_2}\sqrt{-1}\alpha_1(1+\sqrt{-1}\alpha_1)(1+\alpha_2^2), & R_{11\bar{2}}^2 &= -\frac{k^2}{\gamma_2}\sqrt{-1}\alpha_1(1-\sqrt{-1}\alpha_1)(1+\alpha_2^2), \\ R_{12\bar{1}}^2 &= -\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1-\sqrt{-1}\alpha_2)^2, & R_{21\bar{1}}^2 &= -\frac{k^2}{2\gamma_2}(1+\alpha_1^2)(1+\alpha_2^2),\end{aligned}$$

hence it is now easy to see that $R_{k\bar{l}} = 0$ for all $k, l = 1, 2$. \square

5. SPECIAL ALMOST COMPLEX STRUCTURES ON \mathcal{N}

5.1. Kodaira dimension. Let J be the almost complex structure on \mathcal{N} which acts on the tangent fields (3.3) as

$$(5.1) \quad Je_1 = e_4, \quad Je_2 = e_3, \quad Je_3 = -e_2, \quad Je_4 = -e_1.$$

We can then define the fields of type $(1, 0)$

$$(5.2) \quad \mathcal{X}_1 = \frac{1}{2}(e_1 - \sqrt{-1}e_4), \quad \mathcal{X}_2 = \frac{1}{2}(e_2 - \sqrt{-1}e_3),$$

and the corresponding dual $(1, 0)$ -forms

$$\varphi^1 = e^1 + \sqrt{-1}e^4, \quad \varphi^2 = e^2 + \sqrt{-1}e^3.$$

It is then a straightforward computation with the structure equations (3.4) to see that

$$\begin{aligned}d\varphi^1 &= -\frac{1}{4}(\varphi^1 \wedge \varphi^2 - \varphi^1 \wedge \bar{\varphi}^2 - \varphi^2 \wedge \bar{\varphi}^1 - \bar{\varphi}^1 \wedge \bar{\varphi}^2), \\ d\varphi^2 &= -\frac{\sqrt{-1}}{4}(\varphi^1 \wedge \varphi^2 + \varphi^1 \wedge \bar{\varphi}^2 - \varphi^2 \wedge \bar{\varphi}^1 + \bar{\varphi}^1 \wedge \bar{\varphi}^2),\end{aligned}$$

hence that J is not integrable, and

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = -\frac{1}{4}(\sqrt{-1}\bar{\varphi}^1 + \bar{\varphi}^2) \wedge \varphi^1 \wedge \varphi^2.$$

Proposition 5.1 — *We have the following:*

$$\text{kod}(\mathcal{N}, J) = -\infty.$$

Proof. By a direct computation, a smooth pluricanonical section $f(\varphi^1 \wedge \varphi^2)^{\otimes m}$ is pseudoholomorphic if and only if $\bar{\partial}f - \frac{1}{4}mf(\sqrt{-1}\bar{\varphi}^1 + \bar{\varphi}^2) = 0$, which is equivalent to the system

$$\begin{cases} \mathcal{X}_1(f) - \frac{1}{4}\sqrt{-1}mf = 0 \\ \mathcal{X}_2(f) - \frac{1}{4}mf = 0. \end{cases}$$

Writing $f = u + \sqrt{-1}v$ and using (5.2) and (3.3) we find that the previous system is equivalent to

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial t} + \frac{1}{2}mv = 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial t} - \frac{1}{2}mu = 0 \\ \frac{\partial u}{\partial y} + x\frac{\partial u}{\partial z} + \frac{1}{2}x^2\frac{\partial u}{\partial t} - \frac{\partial v}{\partial z} - x\frac{\partial v}{\partial t} - \frac{1}{2}mu = 0 \\ \frac{\partial v}{\partial y} + x\frac{\partial v}{\partial z} + \frac{1}{2}x^2\frac{\partial v}{\partial t} + \frac{\partial u}{\partial z} + x\frac{\partial u}{\partial t} - \frac{1}{2}mv = 0. \end{cases}$$

Using the Fourier expansion (3.6), the last two equations in the last system become

$$\begin{cases} \sqrt{-1}\pi au_I + \sqrt{-1}\pi bxu_I + \sqrt{-1}\pi c\frac{1}{2}x^2u_I - \sqrt{-1}\pi bv_I - \sqrt{-1}\pi cv_I - \frac{1}{2}mu_I = 0 \\ \sqrt{-1}\pi av_I + \sqrt{-1}\pi bxv_I + \sqrt{-1}\pi c\frac{1}{2}x^2v_I + \sqrt{-1}\pi bu_I + \sqrt{-1}\pi cxu_I - \frac{1}{2}mv_I = 0, \end{cases}$$

which is a homogeneous linear system for the pair (u_I, v_I) . The determinant of the matrix representing this last system is

$$-\pi^2 \left(a + bx + \frac{1}{2}x^2c + \frac{\sqrt{-1}}{2\pi}m + \sqrt{-1}(b + cx) \right) \left(a + bx + \frac{1}{2}x^2c + \frac{\sqrt{-1}}{2\pi}m - \sqrt{-1}(b + cx) \right).$$

This determinant vanishes if and only if

$$\begin{cases} a + bx + \frac{1}{2}x^2c = 0 \\ \frac{m}{2\pi} + b + cx = 0 \end{cases} \quad \text{or} \quad \begin{cases} a + bx + \frac{1}{2}x^2c = 0 \\ \frac{m}{2\pi} - b - cx = 0, \end{cases}$$

and we want to determine explicitly when this happens.

- (1) If $c = 0$, both systems are impossible since π is irrational.
- (2) If $c \neq 0$, we can solve the second equation finding that $x = \frac{m}{2\pi c} - \frac{b}{c}$ (we focus on the second system, but the situation for the first one is completely analogous). We can then substitute this expression in the first equation, and after we clean the denominators we find the equation

$$4(2ca - b^2)\pi + m^2 = 0.$$

We have then two further subcases:

- (a) if $b^2 = 2ca$, this equation is never satisfied, since $m \neq 0$;
- (b) if $b^2 \neq 2ca$, this equation is never satisfied, since π is irrational.

In any case, the determinant is non-zero for every value of $I = (a, b, c)$ and so u_I and v_I must always be identically zero. Obviously $u_I = v_I = 0$ is a solution of the system. As a consequence $u = v = 0$, which implies that the Kodaira dimension of (\mathcal{N}, J) is $-\infty$. \square

Remark 5.2 — Observe that with respect to the symplectic form ξ we have that

$$\xi(J\cdot, J\cdot) = \xi(\cdot, \cdot),$$

and that $\xi = \frac{1}{2}\sqrt{-1}(\varphi^1 \wedge \bar{\varphi}^1 + \varphi^2 \wedge \bar{\varphi}^2)$ is of pure type $(1, 1)$. Moreover, ξ is the $(1, 1)$ -form naturally associated with the metric

$$g' = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4$$

on \mathcal{N} , and so (\mathcal{N}, J, g') is an almost Kähler manifold.

We consider now a second almost complex structure on \mathcal{N} : the one defined by

$$J'e_1 = e_2, \quad J'e_2 = -e_1, \quad J'e_3 = e_4, \quad J'e_4 = -e_3.$$

In this case we have

$$\varphi^1 = e^1 + \sqrt{-1}e^2, \quad \varphi^2 = e^3 + \sqrt{-1}e^4,$$

and so

$$\begin{aligned} d\varphi^1 &= 0, \\ d\varphi^2 &= -\frac{1}{4}\sqrt{-1}(\varphi^1 \wedge \varphi^2 + 2\varphi^1 \wedge \bar{\varphi}^1 + \varphi^1 \wedge \bar{\varphi}^2 - \varphi^2 \wedge \bar{\varphi}^1 + \bar{\varphi}^1 \wedge \bar{\varphi}^2). \end{aligned}$$

As a consequence

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = -\frac{1}{4}\sqrt{-1}\bar{\varphi}^1 \wedge \varphi^1 \wedge \varphi^2,$$

so for any smooth function $f : \mathcal{N} \rightarrow \mathbb{C}$ we have

$$\begin{aligned} (5.3) \quad \bar{\partial}(f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}) &= \bar{\partial}f \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m} - \frac{1}{4}\sqrt{-1}mf\bar{\varphi}^1 \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m} = \\ &= (\bar{\partial}f - \frac{1}{4}\sqrt{-1}mf\bar{\varphi}^1) \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}. \end{aligned}$$

Proposition 5.3 — *The Kodaira dimension of (\mathcal{N}, J') is*

$$\text{kod}(\mathcal{N}, J') = -\infty.$$

Proof. The dual frame of vector fields of type $(1, 0)$ on (\mathcal{N}, J') is

$$\mathcal{X}_1 = \frac{1}{2}(e_1 - \sqrt{-1}e_2), \quad \mathcal{X}_2 = \frac{1}{2}(e_3 - \sqrt{-1}e_4),$$

so after we write $f = u + \sqrt{-1}v$ we see from (5.3) that the smooth pluricanonical section $f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}$ is pseudoholomorphic if and only if

$$\begin{cases} e_1(u) - e_2(v) + \frac{1}{2}mv = 0 \\ e_1(v) + e_2(u) - \frac{1}{2}mu = 0 \\ e_3(u) - e_4(v) = 0 \\ e_3(v) + e_4(u) = 0. \end{cases}$$

By (3.3) this system becomes

$$(5.4) \quad \begin{cases} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - x\frac{\partial v}{\partial z} - \frac{1}{2}x^2\frac{\partial v}{\partial t} + \frac{1}{2}mv = 0 \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} + x\frac{\partial u}{\partial z} + \frac{1}{2}x^2\frac{\partial u}{\partial t} - \frac{1}{2}mu = 0 \\ \frac{\partial u}{\partial z} + x\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} = 0 \\ \frac{\partial v}{\partial z} + x\frac{\partial v}{\partial t} + \frac{\partial u}{\partial t} = 0. \end{cases}$$

Recall from (3.6) that u and v can be expressed as Fourier series: in particular, the third and fourth equation of (5.4) give us the system for the Fourier coefficients u_I and v_I

$$\begin{cases} (b+cx)u_I - cv_I = 0 \\ (b+cx)v_I + cu_I = 0 \end{cases} \quad \text{i.e.} \quad \begin{pmatrix} b+cx & -c \\ c & b+cx \end{pmatrix} \begin{pmatrix} u_I \\ v_I \end{pmatrix} = 0.$$

As the determinant of the above matrix is $(b+cx)^2 + c^2$, which is the sum of two *real* squares, we argue that it vanishes if and only if $b = c = 0$. As a consequence

$$b \neq 0 \text{ or } c \neq 0 \implies u_I = v_I = 0$$

and so u and v do not depend on z and t . It is then easy to check that they admit a ‘new’ Fourier series expansion as

$$u(x, y) = \sum_{(\lambda, \mu) \in \mathbb{Z}^2} u_{\lambda\mu} e^{\sqrt{-1}\pi(\lambda x + \mu y)}, \quad u_{\lambda\mu} \in \mathbb{C}$$

and similarly for v . Moreover, the first and second equations of (5.4) simplify to

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} - \frac{1}{2}mv \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} + \frac{1}{2}mu \end{cases}$$

and so

$$\frac{\partial^2 v}{\partial y^2} - \frac{1}{2}m \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y \partial x} = \frac{\partial u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}m \left(\frac{\partial v}{\partial y} - \frac{1}{2}mv \right).$$

So, both u and v are solution of the second order operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - m \frac{\partial}{\partial y} + \frac{1}{4}m^2,$$

which implies that the Fourier coefficients satisfy

$$\left(-\pi^2 \lambda^2 - \pi^2 \mu^2 + \frac{1}{4}m^2 - \sqrt{-1}\pi m \mu \right) u_{\lambda\mu} = 0.$$

Observe that if $\mu \neq 0$ then the coefficient in the above equation has non-vanishing imaginary part, which forces $u_{\lambda\mu} = 0$. Assume finally that $\mu = 0$: the previous relation simplifies to $(-\pi^2 \lambda^2 + \frac{1}{4}m^2) u_{\lambda 0} = 0$, which is still satisfied only if $u_{\lambda 0} = 0$ by the irrationality of π . This means that $u = 0$ and analogously for v .

To sum up, we have shown that if $f = u + \sqrt{-1}v$ is such that $f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}$ is a pseudoholomorphic pluricanonical section then $f = 0$, which means that

$$P_m(\mathcal{N}, J') = 0 \quad \forall m \geq 1$$

and so

$$\text{kod}(\mathcal{N}, J') = -\infty.$$

□

Remark 5.4 — With respect to this second almost complex structure we have

$$\xi = -\frac{1}{2}\sqrt{-1}(\varphi^1 \wedge \varphi^2 - \bar{\varphi}^1 \wedge \bar{\varphi}^2)$$

and so $\xi(J' \cdot, J' \cdot) = -\xi(\cdot, \cdot)$.

Finally, consider the following family of almost complex structures on \mathcal{N} :

$$J_a e_1 = \frac{1}{a} e_3, \quad J_a e_2 = e_4, \quad J_a e_3 = -a e_1, \quad J_a e_4 = -e_2.$$

where $a \in \mathbb{R} \setminus \{0\}$.

In this case we have the $(1, 0)$ -fields

$$\mathcal{X}_{a1} = \frac{1}{2} \left(e_1 - \frac{\sqrt{-1}}{a} e_3 \right), \quad \mathcal{X}_{a2} = \frac{1}{2} (e_2 - \sqrt{-1} e_4)$$

with dual $(1, 0)$ -forms

$$\varphi_a^1 = e^1 + \sqrt{-1} a e^3, \quad \varphi_a^2 = e^2 + \sqrt{-1} e^4.$$

In this case, the canonical section $\varphi_a^1 \wedge \varphi_a^2$ is not pseudoholomorphic as

$$\bar{\partial}(\varphi_a^1 \wedge \varphi_a^2) = \frac{a}{4} \sqrt{-1} \varphi_a^1 \wedge \varphi_a^2 \wedge \bar{\varphi}_a^2.$$

We compute the Kodaira dimension of (\mathcal{N}, J_a) in the next proposition.

Proposition 5.5 — *We have*

$$\text{kod}(\mathcal{N}, J_a) = \begin{cases} -\infty & \text{if } a \notin 2\pi\mathbb{Q}, \\ 0 & \text{if } a \in 2\pi\mathbb{Q} \setminus \{0\}. \end{cases}$$

Proof. It is easy to see that a pluricanonical form $f(\varphi_a^1 \wedge \varphi_a^2)^{\otimes k}$ is pseudoholomorphic if and only if

$$\begin{cases} \bar{\mathcal{X}}_{a1}(f) = 0 \\ \bar{\mathcal{X}}_{a2}(f) + \frac{1}{4} \sqrt{-1} a k f = 0. \end{cases}$$

Writing $f = u + \sqrt{-1}v$ and using (3.3), the previous systems is equivalent to

$$\begin{cases} \frac{\partial u}{\partial x} - \frac{1}{a} \left(\frac{\partial v}{\partial z} + x \frac{\partial v}{\partial t} \right) = 0 \\ \frac{\partial v}{\partial x} + \frac{1}{a} \left(\frac{\partial u}{\partial z} + x \frac{\partial u}{\partial t} \right) = 0 \\ 2 \frac{\partial u}{\partial y} + 2x \frac{\partial u}{\partial z} + x^2 \frac{\partial u}{\partial t} - 2 \frac{\partial v}{\partial t} - kav = 0 \\ 2 \frac{\partial v}{\partial y} + 2x \frac{\partial v}{\partial z} + x^2 \frac{\partial v}{\partial t} + 2 \frac{\partial u}{\partial t} + kau = 0. \end{cases}$$

By (3.6) the last differential system becomes then a differential system for the Fourier coefficients $u_N(x)$ and $v_N(x)$: denoting by $'$ the derivative with respect to x we have

$$\begin{cases} u'_N - \frac{1}{a} \sqrt{-1} \pi m v_N - \frac{1}{a} \sqrt{-1} \pi p x v_N = 0 \\ v'_N + \frac{1}{a} \sqrt{-1} \pi m u_N + \frac{1}{a} \sqrt{-1} \pi p x u_N = 0 \\ 2\pi \sqrt{-1} n u_N + 2\pi \sqrt{-1} m x u_N + \pi \sqrt{-1} p x^2 u_N - 2\pi \sqrt{-1} p v_N - kav_N = 0 \\ 2\pi \sqrt{-1} n v_N + 2\pi \sqrt{-1} m x v_N + \pi \sqrt{-1} p x^2 v_N + 2\pi \sqrt{-1} p u_N + kau_N = 0. \end{cases}$$

The last two equations involve only u_N and v_N

$$\begin{pmatrix} \sqrt{-1} \pi (2n + 2mx + px^2) & -(ka + 2\pi \sqrt{-1} p) \\ ka + 2\pi \sqrt{-1} p & \sqrt{-1} \pi (2n + 2mx + px^2) \end{pmatrix} \begin{pmatrix} u_N \\ v_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and so we can deduce what follows.

- (1) If $p \neq 0$, then $u_N = v_N = 0$. In fact, the determinant of the previous matrix is not zero, as its imaginary part is $4\pi p \neq 0$.
- (2) If $p = 0$ and $m \neq 0$, then $u_N = v_N = 0$. In this case we see that the last system becomes

$$\begin{cases} v_N = \frac{2\pi}{ka} \sqrt{-1} (n + mx) u_N \\ (k^2 a^2 - 4\pi^2 (n + mx)^2) u_N = 0. \end{cases}$$

From the second equation we deduce that u_N must be zero, except possibly for $x = -\frac{2\pi n + ka}{2\pi m}$ where the first function vanishes. As u_N is continuous, we deduce that $u_N = v_N = 0$.

- (3) If $m = p = 0$, then the determinant of the previous matrix equals $k^2 a^2 - 4\pi^2 n^2$. This is non zero for $a \notin 2\pi\mathbb{Q} \setminus \{0\}$, so in this case we have that $u_N = v_N = 0$.

This shows that if $a \notin 2\pi\mathbb{Q}$ then $u_N = v_N = 0$ for every $N \in \mathbb{Z}^3$ and so $\text{kod}(\mathcal{N}, J_a) = 0$.

Assume now that $a \in 2\pi\mathbb{Q} \setminus \{0\}$, hence that $a = 2\pi\frac{n'}{k'}$ for some $n', k' \in \mathbb{Z}$ with $k' > 0$ and $\text{gcd}(n', k') = 1$. What we did in the previous section allow us to conclude the following:

- (1) if k is not a multiple of k' , then $u_N = v_N = 0$ for every $N \in \mathbb{Z}^3$ and so $P_k(\mathcal{N}, J_a) = 0$;
- (2) if k is a multiple of k' , say $k = qk'$ for a suitable positive integer q , then $u_N = v_N = 0$ for every $N \in \mathbb{Z}^3$, possibly with the exception of $N = (n, m, p) = (\pm qn', 0, 0)$.

Hence we focus on this last case. For $k = qk'$ and $N = (n, m, p) = (qn', 0, 0)$ we have the system

$$\begin{cases} u'_N = 0 \\ v'_N = 0 \\ \sqrt{-1}u_N - v_N = 0 \\ \sqrt{-1}v_N + u_N = 0 \end{cases} \longrightarrow \begin{cases} u_N(x) \equiv u_N \in \mathbb{C} \\ v_N(x) \equiv v_N \in \mathbb{C} \\ v_N = \sqrt{-1}u_N. \end{cases}$$

The analogous system for $N = (-qn', 0, 0)$ has $v_N = -\sqrt{-1}u_N$ as solution, where $u_N(x) \equiv u_N$ and $v_N(x) \equiv v_N$ are again constant.

Since u and v are real valued functions, we have that $u_{-N} = \bar{u}_N$ and $v_{-N} = \bar{v}_N$, so

$$\begin{aligned} u(x, y, z, t) &= u_{(-qn', 0, 0)} e^{-\sqrt{-1}\pi qn' y} + u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} \\ &= \bar{u}_{(qn', 0, 0)} e^{-\sqrt{-1}\pi qn' y} + u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} \\ &= 2 \operatorname{Re} \left(u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} \right), \\ v(x, y, z, t) &= -\sqrt{-1}u_{(-qn', 0, 0)} e^{-\sqrt{-1}\pi qn' y} + \sqrt{-1}u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} \\ &= \sqrt{-1} \left(u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} - \bar{u}_{(qn', 0, 0)} e^{-\sqrt{-1}\pi qn' y} \right) \\ &= -2 \operatorname{Im} \left(u_{(qn', 0, 0)} e^{\sqrt{-1}\pi qn' y} \right). \end{aligned}$$

As a consequence we can deduce that

$$f(x, y, z, t) = 2\bar{u}_{(qn', 0, 0)} e^{-\sqrt{-1}\pi qn' y},$$

and it is now easy to verify that such a function satisfies the periodicity prescribed by (3.5) and so descends from \mathbb{R}^4 to \mathcal{N} .

To sum up, we have shown that for the almost complex manifold $X_a = (\mathcal{N}, J_a)$, if $a = 2\pi\frac{n'}{k'}$ for coprime integers n', k' with $k' > 0$ then for every positive $q \in \mathbb{Z}$

$$H^0(X_a, \omega_{X_a}^{\otimes qk'}) = \mathbb{C} \cdot e^{-\sqrt{-1}\pi qn' y}.$$

Hence $P_{qk'}(\mathcal{N}, J_a) = 1$ and so

$$\text{kod}(\mathcal{N}, J_a) = 0.$$

□

5.2. Curvature of the canonical connection. Let J be the almost complex structure on \mathcal{N} defined by (5.1). Then the action of J on the the dual coframe $\{e^1, \dots, e^4\}$ of $\{e_1, \dots, e_4\}$ is given by

$$Je^1 = -e^4, \quad Je^2 = -e^3, \quad Je^3 = e^2, \quad Je^4 = e^1.$$

Then, setting

$$\omega = e^1 \wedge e^4 + e^2 \wedge e^3, \quad g_J(\cdot, \cdot) = \omega(\cdot, J\cdot),$$

the pair (J, g_J) gives rise to an almost Hermitian metric on \mathcal{N} . A straightforward computation yields to the following

Proposition 5.6 — *On the nilmanifold \mathcal{N} consider the almost Hermitian metric (J, g_J) . Then (J, g_J) is a non Chern–Ricci flat almost Kähler metric with vanishing scalar curvature on \mathcal{N} .*

Proof. The coframe given by $\Phi^1 = \frac{\sqrt{2}}{2}\varphi^1$ and $\Phi^2 = \frac{\sqrt{2}}{2}\varphi^2$ is unitary with respect to the Hermitian metric induced by (J, g_J) .

One can then compute the Nijenhuis tensor of J and use it to compute the real torsion forms of the canonical connection (see [CNT20, Corollary 3.8]). Hence the complex torsion forms are

$$\Theta^1 = \frac{\sqrt{2}}{4}\bar{\Phi}^1 \wedge \bar{\Phi}^2, \quad \text{and} \quad \Theta^2 = -\frac{\sqrt{2}}{4}\sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2.$$

Solving the primary structure equations to find the connection forms ϑ_j^i we see that these are given by

$$\begin{aligned} \vartheta_1^1 &= -\frac{\sqrt{2}}{4}\bar{\Phi}^2 + \frac{\sqrt{2}}{4}\bar{\Phi}^2, & \vartheta_2^1 &= -\frac{\sqrt{2}}{4}\sqrt{-1}\bar{\Phi}^2 + \frac{\sqrt{2}}{4}\bar{\Phi}^1, \\ \vartheta_1^2 &= -\frac{\sqrt{2}}{4}\bar{\Phi}^1 - \frac{\sqrt{2}}{4}\sqrt{-1}\bar{\Phi}^2, & \vartheta_2^2 &= \frac{\sqrt{2}}{4}\sqrt{-1}\bar{\Phi}^1 + \frac{\sqrt{2}}{4}\bar{\Phi}^1. \end{aligned}$$

The secondary structure equations allow us to compute the curvature forms ψ_j^i :

$$\begin{aligned} \psi_1^1 &= \frac{1}{8}(\sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2 + \bar{\Phi}^1 \wedge \bar{\Phi}^1 + 2\sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2 - 2\sqrt{-1}\bar{\Phi}^2 \wedge \bar{\Phi}^1 - \bar{\Phi}^2 \wedge \bar{\Phi}^2 + \sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2), \\ \psi_2^1 &= -\frac{1}{8}(\bar{\Phi}^1 \wedge \bar{\Phi}^2 + \sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^1 + 2\bar{\Phi}^1 \wedge \bar{\Phi}^2 - \sqrt{-1}\bar{\Phi}^2 \wedge \bar{\Phi}^2 + 3\bar{\Phi}^1 \wedge \bar{\Phi}^2), \\ \psi_1^2 &= \frac{1}{8}(3\bar{\Phi}^1 \wedge \bar{\Phi}^2 + \sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^1 - 2\bar{\Phi}^2 \wedge \bar{\Phi}^1 - \sqrt{-1}\bar{\Phi}^2 \wedge \bar{\Phi}^2 + \bar{\Phi}^1 \wedge \bar{\Phi}^2), \\ \psi_2^2 &= \frac{1}{8}(\sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2 - \bar{\Phi}^1 \wedge \bar{\Phi}^1 + \bar{\Phi}^2 \wedge \bar{\Phi}^2 + \sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2). \end{aligned}$$

and finally the Chern–Ricci tensor of the canonical connection, which is:

$$\frac{1}{4}\sqrt{-1}\bar{\Phi}^1 \wedge \bar{\Phi}^2 - \frac{1}{4}\sqrt{-1}\bar{\Phi}^2 \wedge \bar{\Phi}^1.$$

□

6. A SPECIAL ALMOST COMPLEX STRUCTURE ON $\mathcal{M}(\lambda)$

Let J be the almost complex structure on $\mathcal{M}(\lambda)$ defined on the tangent fields (3.8) by

$$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3,$$

and let $X = (\mathcal{M}(\lambda), J)$ be the corresponding almost complex manifold. From the complex point of view, we introduce the $(0, 1)$ -forms

$$\varphi^1 = e^1 + \sqrt{-1}e^2, \quad \varphi^2 = e^3 + \sqrt{-1}e^4,$$

and thanks to the structure equations (3.7) we see that

$$\begin{aligned} d\varphi^1 &= \frac{1}{2}\varphi^1 \wedge \bar{\varphi}^1, \\ d\varphi^2 &= \frac{1}{4}\varphi^1 \wedge \varphi^2 - \frac{1}{4}(1 + 2\lambda)\varphi^1 \wedge \bar{\varphi}^2 - \frac{1}{4}\varphi^2 \wedge \bar{\varphi}^1 - \frac{1}{4}(1 + 2\lambda)\bar{\varphi}^1 \wedge \bar{\varphi}^2. \end{aligned}$$

Hence

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = -\frac{1}{4}\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1,$$

and so for a smooth complex valued function $f : \mathcal{M}(\lambda) \rightarrow \mathbb{C}$ we have that

$$(6.1) \quad \bar{\partial}(f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}) = \left(\bar{\partial}f - \frac{1}{4}mf\bar{\varphi}^1 \right) \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}.$$

Proposition 6.1 — *The Kodaira dimension of $(\mathcal{M}(\lambda), J)$ is*

$$\text{kod}(\mathcal{M}(\lambda), J) = -\infty.$$

Proof. It follows from (6.1) that a smooth pluricanonical section of the form $f \cdot (\varphi^1 \wedge \varphi^2)^{\otimes m}$, with $m \geq 1$, is pseudoholomorphic if and only if $\bar{\partial}f - \frac{1}{4}mf\bar{\varphi}^1 = 0$. As the vector fields of type $(1, 0)$ corresponding to φ^1, φ^2 are

$$\mathcal{X}_1 = \frac{1}{2}(e_1 - \sqrt{-1}e_2), \quad \mathcal{X}_2 = \frac{1}{2}(e_3 - \sqrt{-1}e_4),$$

the previous equation is equivalent to the system

$$\begin{cases} \bar{\mathcal{X}}_1(f) - \frac{1}{4}mf = 0 \\ \bar{\mathcal{X}}_2(f) = 0. \end{cases}$$

Using the definition (3.8) for the fields e_i and writing $f = u + \sqrt{-1}v$, once we take the real and the imaginary parts of the equations in the previous system we find that it becomes equivalent to

$$(6.2) \quad \begin{cases} \frac{\partial u}{\partial t} - e^t \frac{\partial v}{\partial y} - \frac{1}{2}mu = 0 \\ \frac{\partial v}{\partial t} + e^t \frac{\partial u}{\partial y} - \frac{1}{2}mv = 0 \\ e^{\lambda t} \frac{\partial u}{\partial x} - e^{-(1+\lambda)t} \frac{\partial v}{\partial z} = 0 \\ e^{\lambda t} \frac{\partial v}{\partial x} + e^{-(1+\lambda)t} \frac{\partial u}{\partial z} = 0. \end{cases}$$

From the third and fourth equations of this system we see that

$$e^{(1+2\lambda)t} \frac{\partial^2 v}{\partial x^2} + e^{-(1+2\lambda)t} \frac{\partial^2 v}{\partial z^2} = 0$$

and similarly for u . As a consequence, both u and v do not depend neither on x nor on z .

Thanks to this information, the Fourier series expansion (3.10) of u and v simplify to

$$u = \sum_{k \in \mathbb{Z}} u_k(t) e^{2\pi\sqrt{-1}ky}, \quad v = \sum_{k \in \mathbb{Z}} v_k(t) e^{2\pi\sqrt{-1}ky}.$$

In fact, if $u_I(t) \neq 0$ (resp. $v_I(t) \neq 0$) then we must have $\lambda_I = \nu_I = 0$ because otherwise u (resp. v) would depend on x or z . Hence the sum is taken over $I \in \mathbb{Z}^3$ such that $(\lambda_I, \mu_I, \nu_I) = (0, k, 0)$, i.e., over all the integral multiples of the second row of the matrix P associated to the lattice. The first two equations of (6.2) then give us the differential system for the Fourier coefficients u_k, v_k

$$\begin{pmatrix} \frac{\partial u_k}{\partial t} \\ \frac{\partial v_k}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}m & 2\pi\sqrt{-1}e^tk \\ -2\pi\sqrt{-1}e^tk & \frac{1}{2}m \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix}.$$

If we make the substitution

$$\begin{pmatrix} \xi_k \\ \zeta_k \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{-1} \\ -1 & \sqrt{-1} \end{pmatrix} \begin{pmatrix} u_k \\ v_k \end{pmatrix},$$

then the system decouples and we obtain

$$\begin{cases} \frac{\partial \xi_k}{\partial t} = (\frac{1}{2}m + 2\pi k e^t) \xi_k \\ \frac{\partial \zeta_k}{\partial t} = (\frac{1}{2}m - 2\pi k e^t) \zeta_k, \end{cases}$$

which can easily be solved. Explicitly, the solution (u_k, v_k) is

$$\begin{aligned} u_k(t) &= \frac{1}{2}e^{\frac{1}{2}mt} \left(c_k e^{2\pi k e^t} - d_k e^{-2\pi k e^t} \right), \\ v_k(t) &= -\frac{\sqrt{-1}}{2}e^{\frac{1}{2}mt} \left(c_k e^{2\pi k e^t} + d_k e^{-2\pi k e^t} \right), \end{aligned}$$

with $c_k, d_k \in \mathbb{C}$. We can then see that

$$f = f(y, t) = \sum_{k \in \mathbb{Z}} f_k(t) e^{2\pi \sqrt{-1}ky}, \quad f_k(t) = u_k(t) + \sqrt{-1}v_k(t) = e^{\frac{1}{2}mt} c_k e^{2\pi k e^t}.$$

Recall from (3.9) that in order for f to define a function on $\mathcal{M}(\lambda)$ we must have that $f(e^\delta y, t + \delta) = f(y, t)$ for every $\delta \in \mathbb{Z} \cdot t_0$. For any fixed $\delta \in \mathbb{Z} \cdot t_0$ we can write

$$f(e^\delta y, t + \delta) = \sum_{h \in \mathbb{Z}} g_h^\delta(t) e^{2\pi \sqrt{-1}hy},$$

where the Fourier coefficients $g_h^\delta(t)$ can be explicitly computed:

$$g_h^\delta(t) = \int_0^1 f(e^\delta y, t + \delta) e^{-2\pi \sqrt{-1}hy} dy = e^{\frac{1}{2}m(t+\delta)} \sum_{k \in \mathbb{Z}} c_k e^{2\pi k e^{t+\delta}} I(k e^\delta - h),$$

where we put

$$I(\eta) = \int_0^1 e^{2\pi \sqrt{-1}\eta y} dy = \begin{cases} 1 & \text{for } \eta = 0, \\ \frac{1}{2\pi \sqrt{-1}\eta} (e^{2\pi \sqrt{-1}\eta} - 1) & \text{for } \eta \neq 0. \end{cases}$$

Because of the invariance with respect to the lattice we have $f_h(t) = g_h^\delta(t)$ for every $\delta \in \mathbb{Z} \cdot t_0$, so in particular $f_h = \lim_{\delta \rightarrow -\infty} g_h^\delta$. Since

$$\begin{aligned} e^{\frac{1}{2}m(t+\delta)} &\longrightarrow 0 \\ e^{2\pi k e^{t+\delta}} &\longrightarrow 1 \\ I(k e^\delta - h) &\longrightarrow I(-h) = \begin{cases} 1 & \text{for } h = 0, \\ 0 & \text{for } h \neq 0 \end{cases} \end{aligned}$$

we deduce that

$$f_h(t) = \lim_{\delta \rightarrow -\infty} g_h^\delta(t) = 0.$$

It follows that then $f \equiv 0$, the plurigenera of $(\mathcal{M}(\lambda), J)$ are

$$P_m(\mathcal{M}(\lambda)) = 0 \quad \forall m \geq 1,$$

hence

$$\text{kod}(\mathcal{M}(\lambda), J) = -\infty.$$

□

7. A TWISTORIAL APPROACH

In the following we will denote by M one of the three four dimensional solv-manifolds without complex structures described before, unless otherwise specified. Moreover we will denote by $\{e_1, e_2, e_3, e_4\}$ and $\{e^1, e^2, e^3, e^4\}$ the global frame for TM and T^*M respectively.

7.1. **Almost hypercomplex structure on $M(k)$, \mathcal{N} , $\mathcal{M}(\lambda)$.** Consider the following endomorphisms of the tangent bundle of M :

$$J_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $J_0 J_1 = -J_1 J_0$, thus we define:

$$J_2 = J_0 J_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and we get that J_0, J_1, J_2 define an almost hypercomplex structure on M .

Let ω_0 and g_0 be defined by:

$$\begin{aligned} \omega_0 &= e^1 \wedge e^2 + e^3 \wedge e^4 \\ g_0 &= e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4. \end{aligned}$$

We have

$$g_0(\cdot, \cdot) = \omega_0(\cdot, J_0 \cdot).$$

Then we consider

$$\omega_1(\cdot, \cdot) = g_0(J_1 \cdot, \cdot) = e^1 \wedge e^3 - e^2 \wedge e^4$$

and

$$\omega_2(\cdot, \cdot) = g_0(J_2 \cdot, \cdot) = e^1 \wedge e^4 + e^2 \wedge e^3.$$

A direct computation gives the following results.

Lemma 7.1 — $g_0(J_i \cdot, J_i \cdot) = g_0(\cdot, \cdot)$ for $i = 0, 1, 2$.

Lemma 7.2 — $\omega_0, \omega_1, \omega_2$ are self-dual.

Lemma 7.3 — If $M = M(k)$ then $d\omega_0 = 0, d\omega_1 = ke^2 \wedge e^3 \wedge e^4, d\omega_2 = -ke^1 \wedge e^3 \wedge e^4$.
If $M = \mathcal{N}$ then $d\omega_0 = -e^1 \wedge e^2 \wedge e^4, d\omega_1 = e^1 \wedge e^2 \wedge e^3, d\omega_2 = 0$.
If $M = \mathcal{M}(\lambda)$ then $d\omega_0 = e^1 \wedge e^3 \wedge e^4, d\omega_1 = -\lambda e^1 \wedge e^2 \wedge e^4, d\omega_2 = -(1+\lambda)e^1 \wedge e^2 \wedge e^3$.

7.2. **The twistor bundle of $M(k)$, \mathcal{N} , $\mathcal{M}(\lambda)$.** Let us consider the Riemannian manifold (M, g_0) with the orientation defined by e_1, e_2, e_3, e_4 . Let $P_{g_0} = P_{g_0}(M, SO(4))$ be the $SO(4)$ -principal bundle of oriented g_0 -orthonormal frames on M . $SO(4)$ acts on the right on P_{g_0} and on the left on $SO(4)/U(2)$. The *twistor space* of (M, g_0) is the associated bundle to P_{g_0} defined as the quotient $Z_{g_0} = Z(M, g_0)$ of $P_{g_0} \times SO(4)/U(2)$ with respect to previous action. Z_{g_0} is a trivial bundle over M with fibre $SO(4)/U(2)$. Let $x \in M$, the fibre $(Z_{g_0})_x$ parametrises the complex structures on $T_x M$ compatible with the metric g_0 and the fixed orientation. A global section is an almost complex structure on M compatible with the metric and the orientation. For more details on the construction and main properties of the twistor bundle, we address the interested reader to [dBN98].

By using the twistorial description we have immediately that any almost complex structure J on M , compatible with the metric g_0 and the fixed orientation, is given by:

$$J = \alpha J_0 + \beta J_1 + \gamma J_2$$

where α, β, γ are smooth functions on M such that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

Proposition 7.4 — *Let J be an almost complex structure on $M=M(k)$ compatible with the metric g_0 and the given orientation and let ω be the Kähler form of J , $\omega(\cdot, \cdot) := g_0(J\cdot, \cdot)$, then $d\omega = 0$ if and only if $\omega = c\omega_0$ for some real constant c .*

Proof. Let $J = \alpha J_0 + \beta J_1 + \gamma J_2$, then $\omega = \alpha\omega_0 + \beta\omega_1 + \gamma\omega_2$. In particular $d\omega = 0$ implies $d(\star\omega) = 0$, where \star is the Hodge operator defined by g_0 . Hence ω is harmonic and then $\omega = c_1 e^1 \wedge e^2 + c_2 e^3 \wedge e^4$, for some real constants c_1, c_2 . On the other hand $\omega = \alpha e^1 \wedge e^2 + \alpha e^3 \wedge e^4$, then $\alpha = c_1 = c_2 = c$ is constant and $\omega = c\omega_0$. \square

Analogously

Proposition 7.5 — *Let J be an almost complex structure on $M=\mathcal{N}$ compatible with the metric g_0 and the given orientation and let ω be the Kähler form of J , $\omega(\cdot, \cdot) := g_0(J\cdot, \cdot)$, then $d\omega = 0$ if and only if $\omega = c\omega_2$ for some real constant c .*

Proof. Let $J = \alpha J_0 + \beta J_1 + \gamma J_2$, then $\omega = \alpha\omega_0 + \beta\omega_1 + \gamma\omega_2$. In particular $d\omega = 0$ implies $d(\star\omega) = 0$, where \star is the Hodge operator defined by g_0 . Hence ω is harmonic and then $\omega = c_1 e^1 \wedge e^4 + c_2 e^2 \wedge e^3$, for some real constants c_1, c_2 . On the other hand $\omega = -\gamma e^1 \wedge e^4 - \gamma e^2 \wedge e^3$, then $\gamma = -c_1 = -c_2 = c$ is constant and $\omega = c\omega_2$. \square

Remark 7.6 — *For $M=\mathcal{M}(\lambda)$ it is impossible to have J such that $d\omega = 0$, so there is not an analogous proposition in this case.*

7.3. Kodaira dimension of a special family. Define

$$J_{(a,b,c)} = aJ_0 + bJ_1 + cJ_2, \quad a, b, c \in \mathbb{R} \text{ such that } a^2 + b^2 + c^2 = 1.$$

Then $J_{(a,b,c)}$ is a (constant) section of the twistor bundle Z_{g_0} and so defines a g_0 -compatible almost complex structure. Among the members of this family we can find the almost complex structures J_0 (for $(a, b, c) = (1, 0, 0)$), J_1 (for $(a, b, c) = (0, 1, 0)$) and J_2 (for $(a, b, c) = (0, 0, 1)$), and the only almost Kähler ones are $J_0, -J_0$ for $M=M(k)$ and $J_2, -J_2$ for $M=\mathcal{N}$. As we know that $\text{kod}(M(k), \pm J_0) = 0$, $\text{kod}(\mathcal{N}, \pm J_0) = -\infty$ and $\text{kod}(\mathcal{M}(\lambda), \pm J_0) = -\infty$ we can (and will) assume that $a \neq \pm 1$.

We observe that a J -adapted basis of vector fields is given by

$$\begin{aligned} \varepsilon_1 &= be_1 + ce_2 - ae_4, & \varepsilon_3 &= ce_1 - be_2 + ae_3, \\ \varepsilon_2 &= e_3, & \varepsilon_4 &= e_4, \end{aligned}$$

hence dually we have

$$\begin{aligned} \varepsilon^1 &= \frac{b}{1-a^2}e^1 + \frac{c}{1-a^2}e^2, & \varepsilon^3 &= \frac{c}{1-a^2}e^1 - \frac{b}{1-a^2}e^2 \\ \varepsilon^2 &= -\frac{ac}{1-a^2}e^1 + \frac{ab}{1-a^2}e^2 + e^3, & \varepsilon^4 &= \frac{ab}{1-a^2}e^1 + \frac{ac}{1-a^2}e^2 + e^4. \end{aligned}$$

We can then consider the vector fields and 1-forms of type $(1, 0)$ given respectively by

$$\mathcal{X}_1 = \frac{1}{2}(\varepsilon_1 - \sqrt{-1}\varepsilon_2), \quad \mathcal{X}_2 = \frac{1}{2}(\varepsilon_3 - \sqrt{-1}\varepsilon_4)$$

and

$$\varphi^1 = \varepsilon^1 + \sqrt{-1}\varepsilon^2, \quad \varphi^2 = \varepsilon^3 + \sqrt{-1}\varepsilon^4.$$

In the following subsections we will compute the Kodaira dimension of $(M(k), J_{(a,b,0)})$, $(\mathcal{N}, J_{(a,b,0)})$ and $(\mathcal{M}(\lambda), J_{(a,b,0)})$. The computations for the general case (with $J_{(a,b,c)}$) are more complicated but, in principle, they can be treated with the same methods we have presented so far. We plan to come back on this subject in a future work.

7.3.1. *Kodaira dimension of $(M(k), J_{(a,b,c)})$.* First of all we compute the Kodaira dimension of $(M(k), J_{(a,b,c)})$.

We have that

$$\begin{aligned} \bar{\partial}\varphi^1 &= \frac{k}{2(1-a^2)}((2acb + \sqrt{-1}(b^2 - c^2))\varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1}bc(1 - a^2)\varphi^1 \wedge \bar{\varphi}^2 + \\ &\quad - (a(b^2 - c^2) - \sqrt{-1}bc(1 + a^2))\varphi^2 \wedge \bar{\varphi}^1) \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}\varphi^2 &= \frac{k}{4(1-a^2)}(2(-a(b^2 - c^2) + 2\sqrt{-1}bc)\varphi^1 \wedge \bar{\varphi}^1 + \sqrt{-1}(b^2 - c^2)(a^2 - 1)\varphi^1 \wedge \bar{\varphi}^2 + \\ &\quad - (4abc + \sqrt{-1}(b^2 - c^2)(a^2 + 1))\varphi^2 \wedge \bar{\varphi}^1), \end{aligned}$$

from which

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = -\frac{k}{4}\sqrt{-1}((b^2 - c^2)\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1 + 2bc\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^2).$$

As a section of $\Omega_X^{0,1} \otimes \Omega_X^{2,0}$ this corresponds to $\bar{\partial}(\varphi^1 \wedge \varphi^2) = \omega \otimes (\varphi^1 \wedge \varphi^2)$, where

$$\omega = -\frac{k}{4}\sqrt{-1}((b^2 - c^2)\bar{\varphi}^1 + 2bc\bar{\varphi}^2)$$

and so

$$\bar{\partial}((\varphi^1 \wedge \varphi^2)^{\otimes m}) = m\omega \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}.$$

Proposition 7.7 — *We have the following:*

$$\text{kod}(M(k), J_{(a,b,0)}) = \begin{cases} 0 & \text{if } (a, b, c) = (\pm 1, 0, 0), \\ -\infty & \text{otherwise.} \end{cases}$$

Proof. Let $f = u + \sqrt{-1}v$ be a smooth complex-valued function. The condition that $f(\varphi^1 \wedge \varphi^2)^{\otimes m}$ is a pseudo-holomorphic pluricanonical section translates in $\bar{\partial}f + mf\omega = 0$, which in turns is equivalent to

$$(7.1) \quad \begin{cases} \bar{\mathcal{X}}_1(f) - \sqrt{-1}\frac{mk}{4}(b^2 - c^2)f = 0 \\ \bar{\mathcal{X}}_2(f) - \sqrt{-1}\frac{mk}{2}bcf = 0, \end{cases}$$

hence to

$$\begin{cases} be_1(u) + ce_2(u) - ae_4(u) - e_3(v) + \frac{1}{2}mk(b^2 - c^2)v = 0 \\ be_1(v) + ce_2(v) - ae_4(v) + e_3(u) - \frac{1}{2}mk(b^2 - c^2)u = 0 \\ ce_1(u) - be_2(u) + ae_3(u) - e_4(v) + mkbcv = 0 \\ ce_1(v) - be_2(v) + ae_3(v) + e_4(u) - mkbcu = 0. \end{cases}$$

For $c = 0$ and using the Fourier series expansion (3.2) we get the following system for the coefficients u_I and v_I (with $I = (A, B, C) \in \mathbb{Z}^3$)

$$(7.2) \quad \begin{pmatrix} 2\pi\sqrt{-1}(be^{kz}\frac{1}{\delta}(Av_2 - Bu_2) - aC) & \frac{1}{2}mkb^2 & 0 & -1 \\ -\frac{1}{2}mkb^2 & 2\pi\sqrt{-1}(be^{kz}\frac{1}{\delta}(Av_2 - Bu_2) - aC) & 1 & 0 \\ -2\pi\sqrt{-1}be^{-kz}\frac{1}{\delta}(-Av_1 + Bu_1) & -2\pi\sqrt{-1}C & a & 0 \\ 2\pi\sqrt{-1}C & -2\pi\sqrt{-1}be^{-kz}\frac{1}{\delta}(-Av_1 + Bu_1) & 0 & a \end{pmatrix} \begin{pmatrix} u_I \\ v_I \\ u'_I \\ v'_I \end{pmatrix} = 0$$

where ' denotes the derivative with respect to z . The real part of the determinant of this matrix is

$$-4\pi^2 \left(\frac{1}{\delta}e^{kz}ab(Av_2 - Bu_2) + b^2C \right)^2 - 4\pi^2 \frac{1}{\delta^2}e^{-2kz}b^2(Av_1 - Bu_1)^2 + \frac{1}{4}m^2k^2a^2b^4$$

while its imaginary part is

$$2\pi e^{-kz} \frac{1}{\delta} mkab^3 (Av_1 - Bu_1).$$

We analyze several different cases separately.

- (1) If $b = 0$ then $(a, b, c) = (\pm 1, 0, 0)$ and so $J_{(a,b,c)} = \pm J_0$. We computed that $\text{kod}(M(k), J_{(\pm 1, 0, 0)}) = 0$ in Proposition 4.1 (it is the fibre over the origin). From now on we will then assume that $b \neq 0$.
- (2) If $a \neq 0$ and $I = (A, B, C) \in \mathbb{Z}^3$ is such that $Av_1 - Bu_1 \neq 0$, then the imaginary part of the determinant is non zero. In particular, the determinant itself is non zero and so $u_I = v_I = 0$ is the only solution.
- (3) If $a = 0$, then $b = \pm 1$ and the matrix of the system (7.2) has determinant

$$-4\pi^2 C^2 - 4\pi^2 \frac{1}{\delta^2} e^{-2kz} (Av_1 - Bu_1)^2.$$

It follows that if $I = (A, B, C) \in \mathbb{Z}^3$ is such that either $C \neq 0$ or $Av_1 - Bu_1 \neq 0$, then the solution of system (7.2) is $u_I = v_I = 0$. We are then left with the case where $C = Av_1 - Bu_1 = 0$.

Assume first that $Av_1 - Bu_1 = 0$ is possible only for $A = B = 0$. In this case we can conclude from our discussion that $u = u_{(0,0,0)}$, $v = v_{(0,0,0)}$ are constant. So the same is true for f , which must then be zero because of system (7.1). So $\text{kod}(M(k), J_{(0,\pm 1,0)}) = -\infty$ in this case.

Assume now that $C = Av_1 - Bu_1 = 0$ for some $(A, B) \neq (0, 0)$. Then there exists a pair $(A', B') \neq (0, 0)$ such that $A'v_1 - B'u_1 = 0$, $\text{gcd}(A', B') = 1$ and with the property that every other pair $(A, B) \in \mathbb{Z}^2$ satisfying $Av_1 - Bu_1 = 0$ is of the form $(A, B) = h(A', B')$ for some $h \in \mathbb{Z}$. Since $(A, B) \neq (0, 0)$ we have $A'v_2 - B'u_2 \neq 0$. The first two equations of (7.2) are

$$\begin{cases} u'_I = \frac{1}{2} mku_I \mp 2\pi\sqrt{-1}e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)v_I \\ v'_I = \pm 2\pi\sqrt{-1}e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)u_I + \frac{1}{2} mkv_I, \end{cases}$$

which decouples after the substitution

$$\begin{pmatrix} u_I \\ v_I \end{pmatrix} = \begin{pmatrix} \mp\sqrt{-1} & \pm\sqrt{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_I \\ \zeta_I \end{pmatrix}.$$

Explicitly, we find the system

$$\begin{cases} \xi'_I = \left(\frac{1}{2}mk + 2\pi e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)\right) \xi_I \\ \zeta'_I = \left(\frac{1}{2}mk - 2\pi e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)\right) \zeta_I \end{cases}$$

whose solutions are

$$\begin{cases} \xi_I(z) = c_I e^{\frac{1}{2}mkz + 2\pi e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)} \\ \zeta_I(z) = d_I e^{\frac{1}{2}mkz - 2\pi e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)} \end{cases} \quad c_I, d_I \in \mathbb{C}.$$

So we have

$$\begin{cases} u_I = \mp\sqrt{-1}(\xi_I - \zeta_I) \\ v_I = \xi_I + \zeta_I \end{cases} \implies f_I = u_I + \sqrt{-1}v_I = \begin{cases} 2\sqrt{-1}\zeta_I & \text{for } b = 1, \\ 2\sqrt{-1}\xi_I & \text{for } b = -1. \end{cases}$$

We focus on the case $b = 1$, as the case $b = -1$ is similar. The Fourier expansion (3.2) simplifies to

$$f(x, y, z, t) = \sum_{h \in \mathbb{Z}} 2\sqrt{-1}d_h e^{\frac{1}{2}mkz - 2\pi e^{kz} \frac{1}{\delta} h(A'v_2 - B'u_2)} e^{2\pi\sqrt{-1} \frac{1}{\delta} h(A'v_2 - B'u_2)x}.$$

Now we exploit the fact that f is equivariant with respect to the action of the lattice, in particular we have that $f(x, z) = f(e^{kn\gamma}x, z + n\gamma)$ for every $\gamma \in \mathbb{Z}$ and we can write

$$f(e^{kn\gamma}x, z + n\gamma) = \sum_{h \in \mathbb{Z}} g_h^\gamma(z) e^{2\pi\sqrt{-1}\frac{1}{\delta}h(A'v_2 - B'u_2)x}.$$

The Fourier coefficients g_h^γ can be computed:

$$\begin{aligned} g_h^\gamma(z) &= \int_0^{\frac{\delta}{A'v_2 - B'u_2}} \sum_{l \in \mathbb{Z}} 2\sqrt{-1}d_l e^{\frac{1}{2}mk(z+n\gamma) - 2\pi e^{k(z+n\gamma)} \frac{1}{k\delta}l(A'v_2 - B'u_2)} e^{2\pi\sqrt{-1}\frac{1}{\delta}(A'v_2 - B'u_2)(e^{kn\gamma l - h})x} dx \\ &= \sum_{l \in \mathbb{Z}} 2\sqrt{-1}d_l e^{\frac{1}{2}mk(z+n\gamma) - 2\pi e^{k(z+n\gamma)} \frac{1}{k\delta}l(A'v_2 - B'u_2)} \int_0^{\frac{\delta}{A'v_2 - B'u_2}} e^{2\pi\sqrt{-1}\frac{1}{\delta}(A'v_2 - B'u_2)(e^{kn\gamma l - h})x} dx, \end{aligned}$$

and because of the equivariance we have $f_h(z) = g_h^\gamma(z)$ for every $\gamma \in \mathbb{Z}$. Let now $\gamma \rightarrow \pm\infty$ in such a way that $kn\gamma \rightarrow -\infty$: we have $f_h = \lim_{kn\gamma \rightarrow -\infty} g_h^\gamma$ and since

$$\begin{aligned} e^{\frac{1}{2}mk(z+n\gamma)} &\longrightarrow 0 \\ e^{-2\pi e^{k(z+n\gamma)} \frac{1}{k\delta}l(A'v_2 - B'u_2)} &\longrightarrow 1 \\ \int_0^{\frac{\delta}{A'v_2 - B'u_2}} e^{2\pi\sqrt{-1}\frac{1}{\delta}(A'v_2 - B'u_2)(e^{kn\gamma l - h})x} dx &\longrightarrow \begin{cases} 1 & \text{for } h = 0, \\ 0 & \text{for } h \neq 0 \end{cases} \end{aligned}$$

we deduce that this limit is zero. So $f = 0$ is the unique solution in this case.

As a consequence

$$\text{kod}(M(k), J_{(0, \pm 1, 0)}) = -\infty$$

also in this case.

- (4) Finally, let $a \neq 0$ and assume that $Av_1 - Bu_1 = 0$.

If $(A, B) \neq (0, 0)$ (observe that this is possible if and only if u_1 and v_1 are one a rational multiple of the other) then we have $Av_2 - Bu_2 \neq 0$ and so the determinant of the matrix of the system (7.2), which is

$$-4\pi^2 \left(\frac{1}{\delta} e^{kz} abh(A'v_2 - B'u_2) + b^2 C \right)^2 + \frac{1}{4} m^2 k^2 a^2 b^4,$$

is a function of z . It vanishes for at most a finite number of values of z . Where the determinant is non zero, the corresponding value of u_I and v_I must then be zero, and since both u_I and v_I are continuous we deduce that they must vanish identically.

We are then left with the case $(A, B) = (0, 0)$, where the determinant is

$$-4\pi^2 b^4 C^2 + \frac{1}{4} m^2 k^2 a^2 b^4.$$

For $C = 0$ this is non zero, hence $u_I = v_I = 0$. For $C \neq 0$ we deduce from the third equation of (7.2) that

$$v_I = \frac{a}{2\pi\sqrt{-1}C} u_I' \implies v_I' = \frac{a}{2\pi\sqrt{-1}C} u_I'',$$

so from the fourth equation we have

$$v_I' = -\frac{2\pi\sqrt{-1}C}{a} u_I \implies -\frac{2\pi\sqrt{-1}C}{a} u_I' = \frac{a}{2\pi\sqrt{-1}C} u_I'' \implies u_I'' = \frac{4\pi^2 C^2}{a^2} u_I.$$

We can solve this last equation, and find that

$$\begin{cases} u_I(z) = c_I e^{\frac{2\pi C}{a} z} + d_I e^{-\frac{2\pi C}{a} z} \\ v_I(z) = -\sqrt{-1} \left(c_I e^{\frac{2\pi C}{a} z} - d_I e^{-\frac{2\pi C}{a} z} \right). \end{cases}$$

for $c_I, d_I \in \mathbb{C}$. It follows that the Fourier coefficient $f_I(z)$ for f is

$$f_I(z) = u_I(z) + \sqrt{-1}v_I(z) = 2c_I e^{\frac{2\pi C}{a}z}.$$

It follows that

$$f = f(z, t) = \sum_{C \in \mathbb{Z} \setminus \{0\}} 2c_C e^{\frac{2\pi C}{a}z} e^{2\pi \sqrt{-1}Ct}$$

and since $f(z, t) = f(z + n\gamma, t)$ for every $\gamma \in \mathbb{Z}$ we deduce that

$$f_C(z) = f_C(z + n\gamma) \quad \text{for every } \gamma \in \mathbb{Z}.$$

Since this is possible only if $c_C = 0$, we deduce that we must have $f = 0$.

It then follows that

$$\text{kod}(M(k), J_{(a,b,0)}) = -\infty.$$

□

Remark 7.8 — We remark that the family of almost complex structures $\{J_r\}$ is obtained as particular values of a, b, c , namely: $a = \alpha(1-r)^2$, $b = -2\alpha r^2$ and $c = 2\alpha r(1-r)$. Precisely $\{J_r\}$ is a path in the Twistor space passing through J_0 ($r = 0$) and $-J_1$ ($r = 1$). However J_2 is not contained in this family.

7.3.2. *Kodaira dimension of $(\mathcal{N}, J_{(a,b,c)})$.* Consider $(\mathcal{N}, J_{(a,b,c)})$.

We have that

$$\bar{\partial}\varphi^1 = \frac{\sqrt{-1}(1-a^2)}{4}(\varphi^1 \wedge \bar{\varphi}^2 - \varphi^2 \wedge \bar{\varphi}^1)$$

and

$$\bar{\partial}\varphi^2 = \frac{b}{2}\varphi^1 \wedge \bar{\varphi}^1 + \frac{1}{4}(c - ab\sqrt{-1})\varphi^1 \wedge \bar{\varphi}^2 + \frac{1}{4}(c + ab\sqrt{-1})\varphi^2 \wedge \bar{\varphi}^1,$$

from which

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = -\frac{1}{4}((c + ab\sqrt{-1})\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1 + (1-a^2)\sqrt{-1}\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^2).$$

As a section of $\Omega_X^{0,1} \otimes \Omega_X^{2,0}$ this corresponds to $\bar{\partial}(\varphi^1 \wedge \varphi^2) = \omega \otimes (\varphi^1 \wedge \varphi^2)$, where

$$\omega = -\frac{1}{4}((c + ab\sqrt{-1})\bar{\varphi}^1 + (1-a^2)\sqrt{-1}\bar{\varphi}^2)$$

and so

$$\bar{\partial}((\varphi^1 \wedge \varphi^2)^{\otimes m}) = m\omega \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}.$$

Proposition 7.9 — *We have the following:*

$$\text{kod}(\mathcal{N}, J_{(a,b,0)}) = -\infty$$

for all a, b .

Proof. Following previous computation we have that a pluricanonical form $f(\varphi^1 \wedge \varphi^2)^{\otimes m}$ is pseudo-holomorphic with respect to $J_{(a,b,c)}$ if and only if:

$$\begin{cases} \mathcal{X}_1(f) - \frac{1}{4}mf(c + ab\sqrt{-1}) = 0 \\ \mathcal{X}_2(f) - \frac{1}{4}\sqrt{-1}m(1-a^2)f = 0. \end{cases}$$

Writing $f = u + \sqrt{-1}v$ previous system is equivalent to

$$\begin{cases} be_1(u) + ce_2(u) - ae_4(u) - e_3(v) - \frac{1}{2}m(cu - av) = 0 \\ be_1(v) + ce_2(v) + e_3(v) - ae_4(v) - \frac{1}{2}m(abu + cv) = 0 \\ ae_3(u) + ce_1(u) - be_2(u) - e_4(v) + \frac{1}{2}m(1-a^2)v = 0 \\ e_4(u) + ce_1(v) - be_2(v) + ae_3(v) - \frac{1}{2}m(1-a^2)u = 0. \end{cases}$$

By substituting Fourier coefficients $u_I(x)$ and $v_I(x)$, where $I = (A, B, C) \in \mathbb{Z}^3$, denoting ' the derivative with respect to x we get:

$$\begin{cases} bu'_I + [\pi c\sqrt{-1}(A + xB + \frac{1}{2}x^2C) - aC - \frac{1}{2}mc]u_I + [\frac{1}{2}mab - \pi\sqrt{-1}(xC + B)]v_I = 0 \\ bv'_I + [\pi c\sqrt{-1}(A + xB + \frac{1}{2}x^2C) - aC - \frac{1}{2}mc]v_I - [\frac{1}{2}mab - \pi\sqrt{-1}(xC + B)]u_I = 0 \\ cu'_I - \pi\sqrt{-1}[b(A + xB + \frac{1}{2}x^2C) - a(B + Cx)]u_I + [\frac{1}{2}m(1 - a^2) - \pi\sqrt{-1}C]v_I = 0 \\ cv'_I - \pi\sqrt{-1}[b(A + xB + \frac{1}{2}x^2C) - a(B + Cx)]v_I - [\frac{1}{2}m(1 - a^2) - \pi\sqrt{-1}C]u_I = 0. \end{cases}$$

If $c = 0$ the last two equations involve only u_I and v_I , the determinant of the matrix of this homogeneous system has imaginary part given by:

$$\pi Cm(1 - a^2)$$

which is zero if and only if $C = 0$. Moreover for $C = 0$ the real part of this determinant vanishes only for finite values of x and we can conclude that $u_I = v_I = 0$. So the statement. \square

7.3.3. *Kodaira dimension of $(\mathcal{M}(\lambda), J_{(a,b,c)})$.* Finally consider $(\mathcal{M}(\lambda), J_{(a,b,c)})$.

We have that

$$\bar{\partial}\varphi^1 = \frac{\lambda b}{2}\varphi^1 \wedge \bar{\varphi}^1 + \frac{1}{4}((1 + \lambda)c + ab(1 - \lambda)\sqrt{-1})\varphi^1 \wedge \bar{\varphi}^2 + (\lambda - 1)(c + ab\sqrt{-1})\varphi^2 \wedge \bar{\varphi}^1$$

and

$$\bar{\partial}\varphi^2 = \frac{(2 + \lambda)}{4}(-b + ac\sqrt{-1})\varphi^1 \wedge \bar{\varphi}^2 - \frac{1}{4}(b\lambda + ac(2 + \lambda)\sqrt{-1})\varphi^2 \wedge \bar{\varphi}^1 - \frac{c(1 + \lambda)}{2}\varphi^2 \wedge \bar{\varphi}^2,$$

from which

$$\bar{\partial}(\varphi^1 \wedge \varphi^2) = \frac{1}{4}((-b\lambda + ac(2 + \lambda)\sqrt{-1})\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1 + (c(1 + \lambda) - ab(1 - \lambda)\sqrt{-1})\varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^2).$$

As a section of $\Omega_X^{0,1} \otimes \Omega_X^{2,0}$ this corresponds to $\bar{\partial}(\varphi^1 \wedge \varphi^2) = \omega \otimes (\varphi^1 \wedge \varphi^2)$, where

$$\omega = \frac{1}{4}((-b\lambda + ac(2 + \lambda)\sqrt{-1})\bar{\varphi}^1 + (c(1 + \lambda) - ab(1 - \lambda)\sqrt{-1})\bar{\varphi}^2).$$

and so

$$\bar{\partial}((\varphi^1 \wedge \varphi^2)^{\otimes m}) = m\omega \otimes (\varphi^1 \wedge \varphi^2)^{\otimes m}.$$

Proposition 7.10 — *We have the following:*

$$\text{kod}(\mathcal{M}(\lambda), J_{(a,b,0)}) = -\infty$$

for all a, b .

Proof. Following previous computation we have that a pluricanonical form $f(\varphi^1 \wedge \varphi^2)^{\otimes m}$ is pseudo-holomorphic with respect to $J_{(a,b,c)}$ if and only if:

$$\begin{cases} \bar{\mathcal{X}}_1(f) + \frac{1}{4}mf(-b\lambda + ac\sqrt{-1}(2 + \lambda)) = 0 \\ \bar{\mathcal{X}}_2(f) + \frac{1}{4}mf(c(1 + \lambda) - ab(1 - \lambda)\sqrt{-1}) = 0. \end{cases}$$

Writing $f = u + \sqrt{-1}v$ and substituting Fourier coefficients $u_I(t)$ and $v_I(t)$, where $I = (A, B, C) \in \mathbb{Z}^3$ (cf. (3.10)), and denoting ' the derivative with respect to t we get:

$$\begin{cases} 2bu'_I + [4\pi\sqrt{-1}(ce^t\mu_I - av_Ie^{-(1+\lambda)t}) - mb\lambda]u_I - [4\pi\sqrt{-1}\lambda_Ie^{\lambda t} + mac(2 + \lambda)]v_I = 0 \\ 2bv'_I + [4\pi\sqrt{-1}(ce^t\mu_I - av_Ie^{-(1+\lambda)t}) - mb\lambda]v_I + [4\pi\sqrt{-1}\lambda_Ie^{\lambda t} + mac(2 + \lambda)]u_I = 0 \\ 2cu'_I - [4\pi\sqrt{-1}(b\mu_Ie^t - a\lambda_Ie^{\lambda t} - mc(1 + \lambda))]u_I + [-4\pi\sqrt{-1}\nu_Ie^{-(1+\lambda)t} + ab(1 - \lambda)m]v_I = 0 \\ 2cv'_I - [4\pi\sqrt{-1}(b\mu_Ie^t - a\lambda_Ie^{\lambda t} - mc(1 + \lambda))]v_I - [-4\pi\sqrt{-1}\nu_Ie^{-(1+\lambda)t} + ab(1 - \lambda)m]u_I = 0. \end{cases}$$

If $c = 0$ the last two equations involve only u_I and v_I , the determinant of the matrix of this homogeneous system has imaginary part given by:

$$-8\pi ab\nu_I m(1 - \lambda)e^{-(1+\lambda)t}$$

which is zero if and only if $a\nu_I = 0$. For $\nu_I = 0$ the real part of this determinant vanishes only for finite values of t and so we can conclude that $u_I = v_I = 0$. Finally, assume that $\nu_I \neq 0$ and $a = 0$. In this case we have $b = \pm 1$ and we can then deduce from the last two equations that

$$u_I = \pm \frac{\mu_I}{\nu_I} e^{(2+\lambda)t} v_I, \quad v_I = \mp \frac{\mu_I}{\nu_I} e^{(2+\lambda)t} u_I.$$

It follows that $u_I = \frac{\mu_I^2}{\nu_I^2} e^{2(2+\lambda)t} u_I$ and so $u_I = 0$, and similarly we deduce that $v_I = 0$.

Hence we get the statement. \square

8. NORDEN STRUCTURES

Norden structures were introduced by Norden in [Nor] and then studied also as almost complex structures with B -metric and anti Kählerian structures, they have applications in mathematics and in theoretical physics. We recall here the definition.

Definition 8.1 — Let (M, J) be an almost complex manifold and let g be a pseudo Riemannian metric on M such that J is a g -symmetric operator, (J, g) is called *Norden structure* on M and (M, J, g) is called a *Norden manifold*. If J is integrable then (M, J, g) is called a *complex Norden manifold*.

Let M one of the three four dimensional solvmanifolds $M(k)$, \mathcal{N} , $\mathcal{M}(\lambda)$.

By using previous notations consider the following natural neutral pseudo Riemannian metrics on M :

$$\begin{aligned} \tilde{g}_0 &:= e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4, \\ \tilde{g}_1 &:= e^1 \otimes e^1 - e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4, \\ \tilde{g}_2 &:= e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4. \end{aligned}$$

Direct computation gives the following.

Lemma 8.2 — $\tilde{g}_0(J_0 \cdot, J_0 \cdot) = \tilde{g}_0(\cdot, \cdot)$ and $\tilde{g}_0(J_i \cdot, \cdot) = \tilde{g}_0(\cdot, J_i \cdot)$ for $i = 1, 2$.

$$\tilde{g}_1(J_1 \cdot, J_1 \cdot) = \tilde{g}_1(\cdot, \cdot) \text{ and } \tilde{g}_1(J_i \cdot, \cdot) = \tilde{g}_1(\cdot, J_i \cdot) \text{ for } i = 0, 2.$$

$$\tilde{g}_2(J_2 \cdot, J_2 \cdot) = \tilde{g}_2(\cdot, \cdot) \text{ and } \tilde{g}_2(J_i \cdot, \cdot) = \tilde{g}_2(\cdot, J_i \cdot) \text{ for } i = 0, 1.$$

Hence we get:

Corollary 8.3 — (M, J_i, \tilde{g}_0) is a Norden manifold for $i = 1, 2$ and (M, J_0, \tilde{g}_0) is a pseudo Hermitian manifold.

(M, J_i, \tilde{g}_1) is a Norden manifold for $i = 0, 2$ and (M, J_1, \tilde{g}_1) is a pseudo Hermitian manifold.

(M, J_i, \tilde{g}_2) is a Norden manifold for $i = 0, 1$ and (M, J_2, \tilde{g}_2) is a pseudo Hermitian manifold.

Moreover we define:

$$\begin{aligned} \tilde{\omega}_0(\cdot, \cdot) &:= \tilde{g}_0(J_0 \cdot, \cdot) = e^1 \wedge e^2 - e^3 \wedge e^4, \\ \tilde{\omega}_1(\cdot, \cdot) &:= \tilde{g}_1(J_1 \cdot, \cdot) = e^1 \wedge e^3 + e^2 \wedge e^4, \\ \tilde{\omega}_2(\cdot, \cdot) &:= \tilde{g}_2(J_2 \cdot, \cdot) = e^1 \wedge e^4 - e^2 \wedge e^3, \end{aligned}$$

and we easily get:

Lemma 8.4 — *If $M=M(k)$ then $d\tilde{\omega}_0 = 0$, $d\tilde{\omega}_1 = -ke^2 \wedge e^3 \wedge e^4$, $d\tilde{\omega}_2 = ke^1 \wedge e^3 \wedge e^4$.
If $M=\mathcal{N}$ then $d\tilde{\omega}_0 = e^1 \wedge e^3 \wedge e^4$, $d\tilde{\omega}_1 = -e^1 \wedge e^2 \wedge e^3$, $d\tilde{\omega}_2 = 0$.
If $M=\mathcal{M}(\lambda)$ then $d\tilde{\omega}_0 = -e^1 \wedge e^3 \wedge e^4$, $d\tilde{\omega}_1 = \lambda e^1 \wedge e^2 \wedge e^4$, $d\tilde{\omega}_2 = (1+\lambda)e^1 \wedge e^2 \wedge e^3$.*

Finally, by direct computation, we obtain the following expressions for twin metrics:

$$\begin{aligned}\hat{g}_{01}(\cdot, \cdot) &:= \tilde{g}_0(\cdot, J_1 \cdot) = -e^1 \otimes e^3 - e^3 \otimes e^1 + e^2 \otimes e^4 + e^4 \otimes e^2, \\ \hat{g}_{02}(\cdot, \cdot) &:= \tilde{g}_0(\cdot, J_2 \cdot) = -e^1 \otimes e^4 - e^4 \otimes e^1 - e^2 \otimes e^3 - e^3 \otimes e^2, \\ \hat{g}_{10}(\cdot, \cdot) &:= \tilde{g}_1(\cdot, J_0 \cdot) = -e^1 \otimes e^2 - e^2 \otimes e^1 - e^3 \otimes e^4 - e^4 \otimes e^3, \\ \hat{g}_{12}(\cdot, \cdot) &:= \tilde{g}_1(\cdot, J_2 \cdot) = -e^1 \otimes e^4 - e^4 \otimes e^1 + e^2 \otimes e^3 + e^3 \otimes e^2, \\ \hat{g}_{20}(\cdot, \cdot) &:= \tilde{g}_2(\cdot, J_0 \cdot) = -e^1 \otimes e^2 - e^2 \otimes e^1 + e^3 \otimes e^4 + e^4 \otimes e^3, \\ \hat{g}_{21}(\cdot, \cdot) &:= \tilde{g}_2(\cdot, J_1 \cdot) = -e^1 \otimes e^3 - e^3 \otimes e^1 - e^2 \otimes e^4 - e^4 \otimes e^2.\end{aligned}$$

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