Convexity properties of gradient maps associated to real reductive representations

CONVEXITY PROPERTIES OF GRADIENT MAPS ASSOCIATED TO REAL REDUCTIVE REPRESENTATIONS

LEONARDO BILIOTTI

Abstract. Let $G = K\exp(\mathfrak{p})$ be a connected real reductive Lie group acting linearly on a finite dimensional vector space V over $\mathbb{R}$. This action admits a Kempf-Ness function and so we have an associated gradient map. If G is Abelian we explicitly compute the image of G orbits under the gradient map, generalizing a result proved by Kac and Peterson [37], see also [7]. If G is not Abelian, we explicitly compute the image of the gradient map with respect to $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra, of the gradient map restricted on the closure of a G orbit. We also describe the convex hull of the image of the gradient map, with respect to G, restricted on the closure of G orbits. Finally, we give a new proof of the Hilbert-Mumford criterion for real reductive Lie groups stressing the properties of the Kempf-Ness functions and applying the stratification theorem proved in [32].

1. Introduction

Let U be a compact connected Lie group and let $U^{\mathbb{C}}$ be its complexification. Let (Z, ω) be a Kähler manifold on which $U^{\mathbb{C}}$ acts holomorphically. Assume that U acts in a Hamiltonian fashion with momentum map $\mu : Z \longrightarrow \mathfrak{u}^*$. This means that ω is U-invariant, μ is U-equivariant and for any $\beta \in \mathfrak{u}$ we have $d\mu^\beta = i_{\beta_Z}\omega$, where $\mu^\beta(x) = \mu(x)(\beta)$ and β_Z denotes the fundamental vector field on Z induced by the action of U. It is well-known that the momentum map represents a fundamental tool in the study of the action of $U^{\mathbb{C}}$ on Z. Of particular importance are convexity theorems [1, 27, 34, 40, 43], which depend on the fact that the functions μ^β are Morse-Bott with even indices. Assume that U is a compact torus. If Z is compact, then Atiyah proved a convexity Theorem along $U^{\mathbb{C}}$ orbits [1]. Recently, Biliotti and Ghigi [16] proved a convexity Theorem along orbits in a very general setting using only so-called Kempf-Ness function. The original setting for Kempf-Ness function is the following: let V be a unitary representation of U. For $x = [v] \in \mathbb{P}(V)$ and $g \in U^{\mathbb{C}}$ we set $\Psi(x, g) = \log \frac{\|gv\|}{\|v\|}$ [39]. The behavior of the corresponding gradient map is encoded in the Ψ. In 1990 Richardson and Slodowoy [56] proved that the Kempf-Ness Theorem extends to the case of real reductive representations. This pioneering work has allowed to prove many results exploiting tools from geometric invariant theory. This is the perspective taken, amongst many others, in the papers [22, 24, 44, 45]. Recently Deré and Lauret [23] use nice convexity properties of the moment map for the variety of nilpotent Lie algebras to investigate

2010 *Mathematics Subject Classification.* 22E45,53D20; 14L24.

Key words and phrases. Gradient maps; real reductive representations, real reductive Lie groups, geometric invariant theory.

The author was partially supported by PRIN 2015 “Varietà reali e complesse: geometria, topologia e analisi armonica” and GNSAGA INdAM. .

which nilpotent Lie algebras admit a Ricci negative solvable extension. This motivated us to investigate convexity properties of gradient maps associated to real reductive representations. We point out that there exist several non equivalent definitions of real reductive Lie group in the literature ([4, 5, 6, 29, 41, 63]). Since we are interested in real reductive representations, we restricted ourselves to linear groups, i.e., subgroups of $\mathrm{GL}(V)$, where V is a finite dimensional real vector space. By a Theorem of Mostow [50], if $G \subset \mathrm{GL}(n, \mathbb{R})$ is closed under transpose then it is compatible with respect to the Cartan decomposition of $\mathrm{GL}(n, \mathbb{R})$. Hence we fix the following setup.

Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a faithful representation on a finite dimensional real vector space. We identify G with $\rho(G) \subset \mathrm{GL}(V)$ and we assume that G is closed and it is closed under transpose. This means there exists a scalar product $\langle \cdot, \cdot \rangle$ on V such that $G = K \exp(\mathfrak{p})$, where $K = G \cap \mathrm{O}(V)$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(V)$. Here we denote by $\mathrm{O}(V)$ the orthogonal group with respect to $\langle \cdot, \cdot \rangle$, by $\mathrm{Sym}(V)$ the set of symmetric endomorphisms of V and finally with $\mathfrak{g}$ the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition, that is $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Moreover, K is a maximal compact subgroup of G, the map $K \times \mathfrak{p} \mapsto G$, $(k, \xi) \mapsto k \exp(\xi)$, is a diffeomorphism, any two maximal Abelian subalgebras of $\mathfrak{p}$ are conjugate by an element of K and the decomposition $G = KTK$ holds, where $T = \exp(\mathfrak{t})$ is the connected Abelian subgroup corresponding to a maximal Abelian subalgebra $\mathfrak{t}$ contained in $\mathfrak{p}$ [35, 41]. In this setting, the function

$$\Psi : G \times V \longrightarrow \mathbb{R}, \qquad (g, x) \mapsto \frac{1}{2}(\langle gx, gx \rangle - \langle x, x \rangle).$$

is a Kempf-Ness function (see section 3) and the corresponding gradient map is given by

$$\mathfrak{F}_{\mathfrak{p}} : V \longrightarrow \mathfrak{p}^*, \qquad \mathfrak{F}_{\mathfrak{p}}(x)(\xi) = \langle \xi x, x \rangle.$$

If $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra, then $\Psi_{|_{A \times V}}$ is a Kempf-Ness function with respect to the $A = \exp(\mathfrak{a})$ action on V and the corresponding gradient map is given by $\mathfrak{F}_{\mathfrak{a}}(x) = \mathfrak{F}_{\mathfrak{p}}(x)_{|_{\mathfrak{a}}}$. The Kempf-Ness Theorem provides geometric criterion for the closedness of orbits of a representation of a real reductive Lie group and the existence of quotient [2, 18, 22, 31, 39, 49, 52, 56, 59]. Recently Böhm and Lafuente [19] proved the Kempf-Ness Theorem and the Hilbert Mumford criterion for linear actions of real reductive Lie groups avoiding any deep algebraic result. The basic tools are the notions of stable, semistable and polystable points. The numerical criteria for stability, semistability and polystability are defined from the Kempf-Ness functions and indeed if a reductive group G acting on a Hausdorff space admits a set of functions formally similar to the classical Kempf-Ness functions, we may define an analogue of the gradient map and the usual concepts of stability. The point of this construction is that one may develop the geometrical invariant theory and give numerical criteria for stability, semistability and polystabiliy. This is the perspective taken, amongst many others, by Biliotti, Ghigi, Raffero and Zedda in the papers [13, 14, 15, 16], motivated by an application to upperbounds for the first eigenvalue of the Laplacian acting on functions [8, 9, 21, 36], and Mundet and Teleman in the papers [52, 53, 54, 61]. Many of these ideas go back as far as Mumford [51, §2.2]. These techniques are basic in this paper.

In this paper we are able to explicitly compute the image of the gradient map corresponding to the A action on V restricted to A orbits (Theorems 16) generalizing a result due to Kac and Peterson in the complex setting [37], see also [7]. Roughly speaking, the gradient map restricted to $\overline{A \cdot x}$ is an homeomorphism onto a polyhedral C which is explicitly computed and determined from x. This homeomorphism sends the closure to a A orbit contained in $\overline{A \cdot x}$ into a face of C.

As an application we obtain the Hilbert-Mumford criterion and the algebraicity of the null cone for Abelian groups (Theorems 18 and 20). We also prove a convexity Theorem for the gradient map associated to A restricted on the closure of G orbits (Theorems 23). Applying results proved in [11, 12, 26], we completely describe the convex hull of the image of the gradient map, with respect to G, restricted on the closure of G orbits (Theorems 26). Finally, using in a different context original ideas from [25], which are themselves of some interest, and the slice Theorem [32], see also [33, 48, 55, 60], we give a probably new proof of the Hilbert-Mumford criterion for real reductive groups (Theorem 39).

This paper is organized as follows.

In the second section we recall basic notions of convex geometry. In particular we recall the definition of polyhedral, polytope, extremal points and exposed faces. In the third section we recall the abstract setting on which we are able to develop a geometrical invariant theory for actions of real reductive Lie groups. In the fourth section we consider a closed subgroup G of $\mathrm{GL}(V)$, where V is a finite dimensional real vector space endowed by a scalar product $\langle \cdot, \cdot \rangle$, which is also closed under transpose. Given an Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$, we explicitly compute the image of the gradient map, with respect to $A = \exp(\mathfrak{a})$, along A orbits and restricted on the closure of G orbits. As an application we get the Hilbert-Mumford criterion for Abelian groups and the algebraicity of the null cone. We also point out that we are able to explicitly the convex hull of the image of the gradient map, with respect to G, restricted to the closure of a G orbit. In the last section we give a proof of the Hilbert-Mumford criterion for real reductive groups.

2. Convex geometry

It is useful to recall a few definitions and results regarding convex sets. A good references, amongst many other, are [57, 58] (see also [10, 11, 12, 62]).

Let V be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$ and let $E \subset V$ be a convex subset. The *relative interior* of E, denoted $\operatorname{relint} E$, is the interior of E in its affine hull. A face F of E is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and $\operatorname{relint}[x, y] \cap F \neq \emptyset$, then $[x, y] \subset F$. The *extreme points* of E are the points $x \in E$ such that $\{x\}$ is a face. If E is closed and nonempty then the faces are closed [57, p. 62]. A face distinct from E and $\emptyset$ will be called a *proper face*. The *support function* of E is the function $h_E : V \to \mathbb{R}$, $h_E(u) = \max_{x \in E} \langle x, u \rangle$. If $u \neq 0$, the hyperplane $H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\}$ is called the *supporting hyperplane* of E for u. The set

$$F_u(E) := E \cap H(E, u) \tag{1}$$

is a face and it is called the *exposed face* of E defined by u. In general not all faces of a convex subset are exposed. A simple example is given by the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces.

A subset $E \subset V$ is called a *convex cone* if E is convex, not empty and closed under multiplication by non negative real numbers. It is easy to check that E is a convex cone if and only if E is closed under addition and under multiplication by non negative real numbers. The cone generated by the vectors $f_1, \ldots, f_n \in V$ is the set $C(f_1, \ldots, f_n) := \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_1 \geq 0, \ldots, \lambda_n \geq 0\}$. A cone arising in this way is called *finitely generated cone*. A *polytope* is the convex hull of a finite number of points of V. If $f_1, \ldots, f_n \in V$ then the set $P(f_1, \ldots, f_n) = \{\alpha_1 f_1 + \cdots + \alpha_n f_n : \alpha_1, \ldots, \alpha_n \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_n = 1\}$ is the polytope generated by $f_1, \ldots, f_n$. The following result goes back by Farkas, Minkowsky and Weyl (see [58] *p*.87 for a proof).

Theorem 2. *A convex cone is finitely generated if and only is the intersection of finitely many closed linear half spaces.*

The above theorem implies $C(f_1, \ldots, f_n)$ is closed. In the literature the intersection of finitely many closed linear half spaces is called polyhedral. Hence $C(f_1, \ldots, f_n)$ is a polyhedral. The concept of polytope and polyhedral are related and this statement is usually attribute to Minkowski (see [58] p. 89).

Theorem 3. *A convex set is a polytope if and only if it is a bounded polyhedral.*

If $f_1, \ldots, f_n \in V$, we denote by $C^o(f_1, \ldots, f_n) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_1 > 0, \ldots, \lambda_n > 0\}$. It is easy to check that $C^o(f_1, \ldots, f_n)$ is a convex cone satisfying $\overline{C^o(f_1, \ldots, f_n)} = C(f_1, \ldots, f_n)$. In particular $\operatorname{relint} C^o(f_1, \ldots, f_n) = \operatorname{relint} C(f_1, \ldots, f_n)$.

Lemma 4. *$C^o(f_1, \ldots, f_n)$ is closed if and only if $0 \in C^o(f_1, \ldots, f_n)$.*

Proof. If $0 \in C^o(f_1, \ldots, f_n)$, then there exist $\alpha_1, \ldots, \alpha_n > 0$ such that

$$0 = \alpha_1 f_1 + \cdots + \alpha_n f_n.$$

Let $v \in C(f_1, \ldots, f_n)$. Since $v = v + \alpha_1 f_1 + \cdots + \alpha_n f_n$ it follows $v \in C^o(f_1, \ldots, f_n)$ and so $C^o(f_1, \ldots, f_n) = C(f_1, \ldots, f_n)$ is closed. Vice-versa, assume $C^o(f_1, \ldots, f_n)$ is closed. Then $C^o(f_1, \ldots, f_n) = C(f_1, \ldots, f_n)$ and so $0 \in C^o(f_1, \ldots, f_n)$. □

Now, we investigate the boundary structure of a polyhedral by means of convex geometry.

Lemma 5. *Let F be an exposed proper face of $C(f_1, \ldots, f_n)$. Then there exists $u \in V - \{0\}$ such that*

$$F = \{x \in C(f_1, \ldots, f_n) : \max_{y \in C(f_1, \ldots, f_n)} \langle y, u \rangle = \langle x, u \rangle = 0\}.$$

Moreover, F is itself a polyhedral.

Proof. Let $u \in V$ such that $F = \{x \in C(f_1, \ldots, f_n) : \max_{y \in C(f_1,\ldots,f_n)} \langle y, u \rangle = \langle x, u \rangle = c\}$. Since $C(f_1, \ldots, f_n)$ is a cone, if $x \in C(f_1, \ldots, f_n)$, then $tx \in C(f_1, \ldots, f_n)$ for any positive t. Hence the maximum c must be zero and so F is a closed cone. We claim that there exists, $J = \{j_1, \ldots, j_s\} \subset \{1, \ldots, n\}$ such that $\langle f_{i_j}, u \rangle = 0$ per $j = 1, \ldots, s$ and $\langle f_r, u \rangle < 0$ for any $r \in \{1, \ldots, n\} - \{j_1, \ldots, j_s\}$. Otherwise there exist $\alpha, \beta \in \{1, \ldots, n\}$ such that $\langle f_\alpha, u \rangle \langle f_\beta, u \rangle < 0$ and so $\mathrm{relint}[f_\alpha, f_\beta] \cap F \neq \emptyset$ which is a contradiction. Now, it is easy to check that $F = C(f_{i_1}, \cdots, f_{i_s})$ and so the result is proved. □

Any face of a polytope is exposed [57]. The following statement proves that any face of a polyhedral is exposed as well. A proof is given in [58] *p*.101 (see also Proposition 1.22 *p*.6 in [62]).

Proposition 6. *Let F be an exposed face of $C(f_1, \ldots, f_n)$ and $F_1 \subset F$ be an exposed face of F. Then F_1 is an exposed face of $C(f_1, \ldots, f_n)$. Hence any face of a polyhedral is exposed.*

Remark 7. *Let E be a closed convex set of V. Let $F \subset E$ be face. If $F_1 \subset F$ is an exposed face of F, it is not true in general that F_1 is an exposed face of E. This means that the above result holds for a polyhedral but it in not true in general.*

3. Kempf-Ness functions

In this section we briefly recall the abstract setting introduced in [14] (see also [13, 15, 16]).

Let $\mathscr{M}$ be a Hausdorff topological space and let G be a connected real reductive group which acts continuously on $\mathscr{M}$. Observe that with these assumptions we can write $G = K \exp(\mathfrak{p})$, where K is a maximal compact subgroup of G. Starting with these data we consider a function $\Psi : \mathscr{M} \times G \to \mathbb{R}$, subject to four conditions.

$(P1)$ For any $x \in \mathscr{M}$ the function $\Psi(x, \cdot)$ is smooth on G.

$(P2)$ The function $\Psi(x, \cdot)$ is left–invariant with respect to K, i.e.: $\Psi(x, kg) = \Psi(x, g)$.

$(P3)$ For any $x \in \mathscr{M}$, and any $v \in \mathfrak{p}$ and $t \in \mathbb{R}$:

$$\frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(tv)) \geq 0.$$

Moreover:

$$\left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0} \Psi(x, \exp(tv)) = 0$$

if and only if $\exp(\mathbb{R}v) \subset G_x$.

$(P4)$ For any $x \in \mathscr{M}$, and any $g, h \in G$:

$$\Psi(x, g) + \Psi(gx, h) = \Psi(x, hg).$$

This equation is called the *cocycle condition.*

For $x \in \mathscr{M}$, we define $\mathfrak{F}(x) \in \mathfrak{p}^*$ by requiring that:

$$\mathfrak{F}(x) = \mathfrak{F}(x)(\xi) := \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \Psi(x, \exp(t\xi)).$$

We call $\mathfrak{F} : \mathscr{M} \longrightarrow \mathfrak{p}^*$ the *gradient map* of $(\mathscr{M}, G, K, \Psi)$. As immediate consequence of the definition of $\mathfrak{F}$ we have the following result.

Proposition 8. *The map $\mathfrak{F} : \mathscr{M} \to \mathfrak{p}^*$ is K-equivariant.*

Proof. It is an easy application of the cocycle condition and the left-invariance with respect to K of $\Psi(x, \cdot)$. Indeed,

$$\begin{aligned}\mathfrak{F}(kx)(\xi) &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(kx, \exp(t\xi)) \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, \exp(t\xi)k) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi(x, k^{-1}\exp(t\xi)k) \\ &= \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \Psi\left(x, \exp(t\mathrm{Ad}(k^{-1})(\xi))\right) = \mathrm{Ad}^*(k)(\mathfrak{F}(x))(\xi).\end{aligned}$$

□

The following definition summarizes the above discussion.

Definition 9. *Let G be a real reductive Lie group, K a maximal compact subgroup of G and $\mathscr{M}$ a topological space with a continuous G–action. A* Kempf-Ness function *for $(\mathscr{M}, G, K)$ is a function*

$$\Psi : \mathscr{M} \times G \to \mathbb{R},$$

that satisfies conditions (P1)–(P4).

Let $(\mathscr{M}, G, K)$ be as above and let Ψ be a Kempf-Ness function.

Definition 10. *Let $x \in \mathscr{M}$. Then:*

a) *x is* polystable *if $G \cdot x \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.*
b) *x is* stable *if it is polystable and $\mathfrak{g}_x$ is conjugate to a subalgebra of $\mathfrak{k}$.*
c) *x is* semistable *if $\overline{G \cdot x} \cap \mathfrak{F}_{\mathfrak{p}}^{-1}(0) \neq \emptyset$.*
d) *x is* unstable *if it is not semi–stable.*

Remark 11. *The four conditions above are G-invariant in the sense that if a point x satisfies one of them, then every point in the orbit of x satisfy the same condition. This follows directly from the definition for polystability, semi–stability and unstability, while for stability it is enough to recall that $\mathfrak{g}_{gx} = \mathrm{Ad}(g)(\mathfrak{g}_x)$.*

The following result establishes a relation between the Kempf-Ness function and polystable points. A proof is given in [14, p.2190] (see also [13, 61, 53]).

Proposition 12. *Let $x \in \mathscr{M}$. The following conditions are equivalent:*

a) *$g \in G$ is a critical point of $\Psi(x, \cdot)$;*
b) *$\mathfrak{F}(gx) = 0$;*
c) *$g^{-1}K$ is a critical point of ψ_x.*

Proposition 13. *If $\mathfrak{F}_\mathfrak{p}(x) = 0$, then the stabilizer of x, i.e., $G_x = \{g \in G : gx = x\}$ is compatible with respect to the Cartan decomposition of $G = K\exp(\mathfrak{p})$. Moreover, if $G = \exp(\mathfrak{p})$, with $\mathfrak{p}$ Abelian, then any stabilizer is compatible.*

Proof. The first part of the statement is well-known. A proof is given in [14], see also [31]. The second part is also easy to check. For the sake of completeness, we give a proof.

Let $x \in \mathscr{M}$ and let $g \in G_x$. Then $g = \exp(v)$ for some $v \in \mathfrak{p}$. Let $f(t) := \mathfrak{F}(\exp(tv)x)(v)$. Then $f(0) = f(1) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{dt}} f(t) = \frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}(\exp(tv)x)(v) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(tv)) \geq 0.$$

Therefore $\frac{\mathrm{d}^2}{\mathrm{dt}^2}\Psi(x,\exp(tv)) = 0$ for $0 \leq t \leq 1$. It follows from $(P3)$ that $\exp(tv)x = x$ for any $t \in \mathbb{R}$ and thus $\exp(t\xi) \in G_x$. $\square$

Given $\xi \in \mathfrak{p}$ for any $t \in \mathbb{R}$ we define $\lambda(x, \xi, t) = \mathfrak{F}(\exp(t\xi)x)(\xi)$. Applying the cocycle condition we get

$$\mathfrak{F}_\mathfrak{p}(\exp(t\xi)x)(\xi) = \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(t\xi))$$

and so

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathfrak{F}_\mathfrak{p}(\exp(t\xi)x)(\xi) = \frac{\mathrm{d}^2}{\mathrm{dt}^2} \Psi(x, \exp(t\xi)) \geq 0.$$

This means that

$$\lambda(x, \xi, t) = \mathfrak{F}_\mathfrak{p}(x)(\xi) + \int_0^t \frac{\mathrm{d}^2}{\mathrm{ds}^2} \Psi(x, \exp(s\xi))\mathrm{d}s$$

is a non decreasing function as a function of t. Moreover,

$$\Psi(x, \exp(t\xi)) = \int_0^t \lambda(x, \xi, \tau)\mathrm{d}\tau.$$

and so

$$\lambda(x, \xi) := \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \Psi(x, \exp(t\xi)) \in \mathbb{R} \cup \{\infty\}.$$

The function $\lambda(x, \cdot)$ is called maximal weight of x in the direction ξ. For a reference see, amongst many others, [13, 14, 38, 51, 53, 61]. We point out that the maximal weight is well defined for any convex function.

Let V be a finite dimensional real vector space and let $f : V \longrightarrow \mathbb{R}$ be a convex function. For any $\xi \in V$, the function $g(t) = f(t\xi)$ is convex and so

$$\lambda_f(\xi) = \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} f(t\xi) \in \mathbb{R} \cup \{\infty\}$$

is well-defined. The following lemma is well-known [38, 61] and it will be applied in the next section. For a sake of completeness we give a proof.

Lemma 14. *Let $f : V \longrightarrow \mathbb{R}$ be a convex function. Assume that for any $\xi \in V - \{0\}$, we have $\lambda_f(\xi) > 0$. Then f is an exhaustion and so it has a critical point which is a global minimum.*

Proof. We may assume that V is endowed by a scalar product $\langle \cdot,\cdot \rangle$. Denote by $\mathrm{S}(V)$ the unit sphere with respect to $\langle \cdot,\cdot \rangle$. Let $\xi \in \mathrm{S}(V)$. Since $\lambda_f(\xi) > 0$, keeping in mind that $\frac{\mathrm{d}^2}{\mathrm{dt}^2} f(\exp(tv)) \geq 0$, it follows there exist $t(\xi) > 0$ and $C_o > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{dt}} f(\exp(t(\xi)\xi) \geq C_o > 0,$$

for any $t \geq t(\xi)$. Hence there exists a neighborhood U_ξ of ξ in $\mathrm{S}(V)$ such that $\dfrac{\mathrm{d}}{\mathrm{dt}} f(t\nu) > \frac{C_o}{2}$ for any $t \geq t(\xi)$ and for any $\nu \in U_\xi$. By usual compactness argument, there exist two constants $C > 0$ and $t_o > 0$ such that $\dfrac{\mathrm{d}}{\mathrm{dt}} f(t\xi) \geq C$, for any $\xi \in \mathrm{S}(V)$ and for any $t \geq t_o$. Therefore, for any $v \in V$ such that $\parallel v \parallel \geq t_o$, we get

$$f(v) = f(t_o \frac{v}{\parallel v \parallel}) + \int_{t_o}^{\parallel v \parallel} \frac{\mathrm{d}}{\mathrm{dt}} f\left(t \frac{v}{\parallel v \parallel}\right) \mathrm{d}t$$

and so $f(v) \geq \min_{\parallel w \parallel = t_o} f(w)$. This means that f is an exhaustion and so it has a critical point which is a global minimum. □

4. Convexity Theorems for Abelian groups

Let G be a connected real reductive Lie group and let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a faithful representation on a finite dimensional real vector space V. We identify G with $\rho(G) \subset \mathrm{GL}(V)$ and we assume that G is closed and it is closed under transpose. This means that there exists a scalar product $\langle \cdot,\cdot \rangle$ on V such that $G = K \exp(\mathfrak{p})$, where $K \subset \mathrm{O}(V)$ and $\mathfrak{p} \subset \mathfrak{g} \cap \mathrm{Sym}(V)$. In the sequel, we denote by $\parallel \cdot \parallel = \sqrt{\langle \cdot,\cdot \rangle}$. We define

$$\Psi : V \times G \longrightarrow \mathbb{R} \qquad \Psi(x,g) = \frac{1}{2}\left(\parallel gx \parallel^2 - \parallel x \parallel^2\right).$$

Lemma 15. *$\Psi : V \times G \longrightarrow \mathbb{R}$ is a Kempf-Ness function and the corresponding gradient map $\mathfrak{F}_{\mathfrak{p}} : V \longrightarrow \mathfrak{p}^*$ is given by $\mathfrak{F}(x)(\xi) = \langle \xi x, x \rangle$.*

Proof. $(P1)$ and $(P2)$ are easy to check. Let $\xi \in \mathfrak{p}$ and let $f(t) = \Psi(x, \exp(t\xi))$. Then

$$f'(t) = \langle \exp(t\xi)\xi x, \exp(t\xi)x \rangle, \qquad f''(t) = \langle \exp(t\xi)\xi x, \exp(t\xi)\xi x \rangle.$$

Hence $f''(t) \geq 0$ and $f''(0) = 0$ if and only if $\xi x = 0$ and $\exp(\mathbb{R}\xi) \subset G_x$. Now,

$$\begin{aligned}
\Psi(x, hg) &= \frac{1}{2}\left(\parallel hgx \parallel^2 - \parallel x \parallel^2\right) \\
&= \frac{1}{2}\left(\parallel hgx \parallel^2 - \parallel gx \parallel^2\right) + \frac{1}{2}\left(\parallel gx \parallel^2 - \parallel x \parallel^2\right) \\
&= \Psi(gx, h) + \Psi(x, g),
\end{aligned}$$

proving the cocycle condition. Finally

$$\mathfrak{F}(x)(\xi) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \langle \exp(t\xi)x, \exp(t\xi)x \rangle = \langle \xi x, x \rangle,$$

concluding the proof. □

Let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra. It is easy to check that $\Psi_{|_{A\times V}}$ is a Kempf-Ness function with respect to A and $\mathfrak{F}_{\mathfrak{a}} := (\mathfrak{F}_{\mathfrak{p}})_{|_{\mathfrak{a}}}$ is the corresponding gradient map [14]. Since $\mathfrak{a}$ is an Abelian subalgebra of symmetric endomorphisms, they are simultaneously diagonalizable. Hence there exists an orthonormal basis $\{v_1, \dots, v_n\}$ of V and functionals $\alpha_1 \dots, \alpha_n \in \mathfrak{a}^*$ such that

$$\xi(v) = \sum_{i=1}^{n} \alpha_i(\xi)\langle v, v_i\rangle v_i.$$

This means that if $v = x_1 v_1 + \cdots + x_n v_n$, then

$$\exp(\xi)(v) = e^{\alpha_1(\xi)} x_1 v_1 + \cdots + e^{\alpha_n(\xi)} x_n v_n.$$

In particular,

$$\mathfrak{F}_{\mathfrak{a}}(v) = \sum_{i=1}^{n} \parallel x_i \parallel^2 \alpha_i,$$

and so the image of $\mathfrak{F}_{\mathfrak{a}}$ is contained in the polyhedral $C(\alpha_1, \dots, \alpha_n)$.

Let $x = x_1 v_1 + \cdots + x_n v_n$. We define $\mathrm{supp}_x = \{i \in \{1, \dots, n\} : x_i \neq 0\}$. If $I \subset \{1 \dots, n\}$, we denote by C_I the polyhedral generated by $\{\alpha_i : i \in I\}$, i.e.,

$$C_I = \left\{ \sum_{i\in I} s_i \alpha_i : i \in I \text{ and } s_i \geq 0 \right\}.$$

We also denote by $C_I^o = \left\{ \sum_{i\in I} s_i\alpha_i : i \in I \text{ and } s_i > 0 \right\}$. In the theory of real reductive representations, a fundamental problem is to compute the image of the gradient map. The following theorem generalizes a result proved by Kac-Peterson [37], see also [7], to the real case.

Theorem 16. *Let $x \in V$ and let $I = \mathrm{supp}_x$. Then the map $\mathfrak{F}_{\mathfrak{a}} : \overline{A \cdot x} \longrightarrow \mathfrak{a}^*$ satisfies:*

a) *$\mathfrak{F}_{\mathfrak{a}} : A \cdot x \longrightarrow C_I^o$ is a diffeomorphism onto;*

b) *$\mathfrak{F}_{\mathfrak{a}} : \overline{A \cdot x} \longrightarrow C_I$ is an homeomorphism and for any face $\sigma \subset C_I$ there exists a unique A-orbit Y such that $\mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma) = \overline{Y}$. Therefore $\mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot x})$ is a polyhedral.*

Proof. By Proposition 2.1 in [16], the map $\mathfrak{F}_{\mathfrak{a}} : A \cdot x \longrightarrow \mathfrak{a}^*$ is a diffeomorphism and its image is a convex open subset of $\mu(x) + \mathfrak{a}_x^o$, where $\mathfrak{a}_x^o = \{\varphi \in \mathfrak{a}^* : \varphi_{|_{\mathfrak{a}_x}} = 0\}$. It is easy to check that $\mathfrak{F}_{\mathfrak{a}}(A \cdot x)$ is invariant under multiplication of non negative real numbers and so it is an open convex cone contained in C_I^o. Now, we shall prove that $\mathfrak{F}_{\mathfrak{a}}(A \cdot x) = C_I^o$.

Let $\mathfrak{b} \subset \mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{a}_x \oplus \mathfrak{b}$. Let $c \in C_I^o$. Then $c = \sum_{i\in I} c_i \alpha_i$ with $c_i > 0$. We define

$$f : \mathfrak{b} \longrightarrow \mathbb{R} \qquad f(\xi) = \Psi(x, \exp(\xi)) - c(\xi).$$

The equation $\mathfrak{F}_{\mathfrak{a}}(\exp(\xi_o)x) = c$ means the existence of a critical point of f. We prove that f is strictly convex and an exhaustion.

Fix $v, w \in \mathfrak{b}$ with $w \neq 0$ and consider the curve $\gamma(t) = v + tw$. Set $u(t) = \mathfrak{F}_{\mathfrak{a}}(\gamma(t))$. It is easy to check that $u''(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \Psi(\exp(v)x, \exp(tw)) > 0$ since $w \notin \mathfrak{a}_x$. This proves f is a strictly convex function on $\mathfrak{b}$.

Let $\xi \in \mathfrak{b} - \{0\}$. Then

$$\frac{d}{dt} f(t\xi) = \sum_{i\in I} \langle (e^{t\alpha_i(\xi)} \parallel x_i \parallel^2 - c_i \rangle)\alpha_i(\xi).$$

Since $\xi \notin \mathfrak{a}_x$, it follows $\alpha_i(\xi) \neq 0$ for some $i \in I$. If $\alpha_i(\xi) > 0$, then

$$\lim_{t\mapsto+\infty} (e^{t\alpha_i(\xi)} \parallel x_i \parallel^2 - c_i)\alpha_i(\xi) = +\infty.$$

If $\alpha_i(\xi) < 0$, then $\lim_{t\mapsto+\infty} e^{t\alpha_i(\xi)} \parallel x_i \parallel^2 = 0$ and so, keeping in mind that $c_i > 0$, we get

$$\lim_{t\mapsto+\infty} (e^{t\alpha_i(\xi)} \parallel x_i \parallel^2 - c_i)\alpha_i(\xi) = -c_i\alpha_i(\xi) > 0.$$

Hence $\lambda_f(\xi) > 0$ for every $\xi \in \mathfrak{b} - \{0\}$. By Lemma 14 the function f is an exhaustion and so it has a critical point concluding the proof of item (a).

Let σ be a face of C_I. By Proposition 6, there exists $\xi \in \mathfrak{a}$ such that $\sigma = F_\xi(C_I)$. By Lemma 5 there exists $J \subset I$ such that $\alpha_i(\xi) = 0$ for $i \in J$ and $\alpha_i(\xi) < 0$ otherwise and $\sigma = C_J$. Then

$$\lim_{t\mapsto+\infty} \exp(t\xi)x = \sum_{i\in J} x_i v_i = \theta,$$

$\mathfrak{F}(\theta) \in \sigma$ and $\mathrm{supp}_\theta = J$. We shall prove

$$\mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma) = \{v \in \overline{A\cdot x} : \max_{z\in\overline{A\cdot x}} \mathfrak{F}_{\mathfrak{a}}(v)(\xi) = \mathfrak{F}_{\mathfrak{a}}(z)(\xi) = 0\} = \overline{A\cdot\theta}.$$

Let $u \in \mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma)$. Write $s(t) = \mathfrak{F}_{\mathfrak{a}}(\exp(t\xi)u)(\xi)$. The function s has a maximum in $t = 0$ and so $\dot{s}(0) = 2\langle \xi u, \xi u\rangle = 0$. This implies $\mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma) \subset \mathrm{Ker}\,\xi$. Now, the gradient flow of the contraction of the gradient map along ξ, i.e., the function $x \mapsto \mathfrak{F}_{\mathfrak{a}}(x)(\xi)$, is given by $\exp(t\xi)$. Therefore $\mathrm{Crit}\,\mathfrak{F}_{\mathfrak{a}}(\cdot)(\xi) = \mathrm{Ker}\,\xi$ and so $\mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma)$ is A-invariant as well. This implies $\overline{A\cdot\theta} \subset \mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma)$.

Let $z \in \mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma)$. Since $z \in \overline{A\cdot x}$, there exists $\{a_n\}_{n\in\mathbb{N}} \in A$ such that $a_n x \mapsto z$. The flow $\exp(t\xi)$ commutes with A and so for any $n \in \mathbb{N}$, we have

$$\lim_{t\mapsto+\infty} \exp(t\xi)(a_n x) = a_n\theta \in \mathrm{Ker}\,\xi.$$

This means $\lim_{t\mapsto+\infty} \exp(t\xi)a_n x = P(a_n x) = a_n P(x) = a_n\theta$, where $P : V \longrightarrow \mathrm{Ker}\,\xi$ is the orthogonal projection on $\mathrm{Ker}\,\xi$. Since $a_n x \mapsto z$ and $z \in \mathrm{Ker}\,\xi$, it follows $P(a_n x) = a_n\theta \mapsto z$. This implies $z \in \overline{A\cdot\theta}$ and so $\mathfrak{F}_{\mathfrak{a}}^{-1}(\sigma) = \overline{A\cdot\theta}$. Now, applying again item (a), $\mathfrak{F}_{\mathfrak{a}} : A\cdot\theta \longrightarrow C_J^o$ is a diffeomorphism. Since $\sigma = C_J$ and $\mathrm{relint}\sigma = \mathrm{relint}\, C_J^o$ it follows $\theta \in \mathrm{relint}\sigma$. By classical result of convex geometry [57, Theorem 2.1.2 p.62] C_I is the disjoint union of the relative interior of its faces. Summing up we have proved that $\mathfrak{F}_{\mathfrak{a}} : \overline{A\cdot x} \longrightarrow C_I$ is surjective. Now we prove it is injective.

Let $u, z \in \overline{A\cdot x}$ such that $\mathfrak{F}_{\mathfrak{a}}(u) = \mathfrak{F}_{\mathfrak{a}}(z)$. Let σ be the face of C_I such that $\mathfrak{F}_{\mathfrak{a}}(u) \in \mathrm{relint}(\sigma)$. By the above argument $\mathfrak{F}^{-1}(\sigma) = \overline{A\cdot v_\sigma}$ and so $u, z \in \overline{A\cdot v_\sigma}$. We claim $u, z \in A\cdot v_\sigma$. Indeed, let $X_n \in \mathfrak{a}$ such that $\exp(X_n)x \mapsto u$. Since $\lim_{t\mapsto+\infty} \mathfrak{F}_{\mathfrak{a}}(\exp(X_n)x) = \mathfrak{F}_{\mathfrak{a}}(u)$ and by the above item $\mathfrak{F}_{\mathfrak{a}} : A\cdot v_\sigma \longrightarrow C_J^o$, where $J = \mathrm{supp}\, v_\sigma$, is a diffeomorphism, it follows X_n has a limit and so $u \in A\cdot x$. The same holds for z. Therefore, again keeping in mind that $\mathfrak{F}_{\mathfrak{a}} : A\cdot v_\sigma \longrightarrow C_J^o$ is a diffemororphism, it follows $u = z$ concluding the proof. $\square$

Corollary 17. *Let $x \in V$ and let $I = \mathrm{supp}_x$. Let F be a face of C_I and let $J \subset I$ be the subset of I associated to F as in the proof of Theorem 16. Let $v_F = \sum_{i\in J} x_i v_i$. Then $v_F \in \mathrm{relint}(F)$ and*

$$\mathfrak{F}_{\mathfrak{a}} : A \cdot v_F \longrightarrow C_J^o,$$

is a diffeomorphism onto. Moreover, $\overline{A \cdot x}$ is the disjoint union of orbits $A \cdot v_F$, as F runs over the set of faces of the polyhedral C_I.

Proof. Let v_F the element associated to F. In Theorem 16 we have proved that $\mathfrak{F}_{\mathfrak{a}}(v_F) \in \mathrm{relint} F$, $\mathfrak{F}_{\mathfrak{a}} : A \cdot v_F \longrightarrow C_J^o$ is a diffeomorphism and $\mathfrak{F}_{\mathfrak{a}}^{-1}(F) = \overline{A \cdot v_F}$. Therefore $\overline{A \cdot x}$ is the disjoint union of orbits $A \cdot v_F$, as F runs over the set of faces of the polyhedral C_I. $\square$

As in [7] we prove that the Hilbert-Mumford criterion for Abelian group arises from the convexity properties of the gradient map.

Theorem 18 (Hilbert-Mumford criterion)**.** *Let $u \in \overline{A \cdot x}$. Then there exist $\xi \in \mathfrak{a}$ and $a \in A$ such that*

$$\lim_{t \mapsto +\infty} \exp(t\xi) a x = u.$$

Proof. By Corollary 17 there exists a unique face F such that $\mathfrak{F}(u) \in \mathrm{relint}\, F$. Therefore $u = a v_F$. Taking $\xi \in \mathfrak{a}$ such that $F = F_\xi(C_I)$ it follows

$$\lim_{t \mapsto +\infty} \exp(t\xi) a x = a v_F = u.$$

$\square$

Another consequence of Theorem 16 is the following well-known result.

Corollary 19. *$A \cdot x$ is closed if and only if $0 \in C_I^o$. Therefore $A \cdot x$ is closed if and only if $A \cdot x \cap \mathfrak{F}_{\mathfrak{a}}^{-1}(0) \neq \emptyset$.*

Proof. By Theorem 16, $A \cdot x$ is closed if and only if C_I^o is closed. By Lemma 4 we get $A \cdot x$ is closed if and only if $0 \in C_I^o$ and so if and only if $A \cdot x \cap \mathfrak{F}_{\mathfrak{a}}^{-1}(0) \neq \emptyset$. $\square$

In [34] the authors prove that the set $\{x \in V : 0 \in \overline{A \cdot x}\}$ is algebraic. Here we give another proof avoiding any algebraic result

Theorem 20. *The set $\{x \in V : 0 \in \overline{A \cdot x}\}$ is a real algebraic subset of V and so it is closed.*

Proof. By Theorem 18, $0 \in \overline{A \cdot x}$ if and only if 0 is a face of C_I, where $I = \mathrm{supp}_v$. Since there exist a finite numbers of C_I where $I \subset \{1, \ldots, n\}$, it follows that there exist $I_1, \cdots, I_k \subset \{1, \ldots, 1\}$ such that $0 \in \overline{A \cdot x}$ if and only if $\mathrm{supp}_x = I_j$ for some $j \in \{1, \ldots, k\}$. Since any face of C_I is exposed, there exist $\xi_1, \ldots, \xi_k \in \mathfrak{a}$ such that $0 \in \overline{A \cdot x}$ if and only if $\exp(t\xi_s)x \mapsto 0$ for some $s \in \{1, \ldots, k\}$. Now, $\exp(t\xi_s) \cdot x \mapsto 0$ if and only if $x = \sum_{i\in J} x_i v_i$ with $\alpha_i(\xi_s) < 0$ for any $i \in J$.

Let $Z_s = \{i \in \{1, \ldots, n\} : \alpha_i(\xi_s) < 0\}$, for $s = 1, \ldots, k$. Define $\mathcal{H}_s = \{x \in V : \langle x, v_k\rangle = 0$ for $k \in \{1, \ldots, n\} - Z_s\}$. It is easy to check $\exp(t\xi_s)x \mapsto 0$ if and only if $x \in \mathcal{H}_s$. Therefore

$$\{x \in V : 0 \in \overline{A \cdot x}\} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_k,$$

concluding the proof. $\square$

Now we investigate convexity Theorems for A-invariant subsets.

Proposition 21. *The image of $\mathfrak{F}_{\mathfrak{a}} : V \longrightarrow \mathfrak{a}^*$ is a polyhedral and the set $\{x \in V : \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(V)\}$ is an open and dense subset of V.*

Proof. Let $x = x_1 v_1 + \cdots + x_n v_n$ such that $x_i \neq 0$ for any $i = 1, \ldots, n$. Since $\mathrm{supp}_x = I = \{1, \ldots, n\}$, by Theorem 16 we get $\mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot x}) = C_I$. On the other hand, by definition of $\mathfrak{F}_{\mathfrak{a}}$, it follows $\mathfrak{F}_{\mathfrak{a}}(V) \subset C_I$ and so the image of $\mathfrak{F}_{\mathfrak{a}}$ is the polyhedral C_I.

Let $x \in V$ and let $I = \mathrm{supp}_x$. By Theorem 16, there exists a neighborhood U of x such that for any $y \in U$ we have $C_I^o \subset \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot y})$ for any $y \in U$. Therefore the set $\{x \in V : \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(V)\}$ is open and it contains an open dense subset of V. Therefore it is an open dense subset of V. □

The proof of next result uses original ideas from [7].

Proposition 22. *Let M be a closed real algebraic irreducible subset of V. Assume that M is A-invariant. Then $\mathfrak{F}_{\mathfrak{a}}(M) = C_I$ where $I = \mathrm{supp}_v$ for some $v \in M$ and so the image is a polyhedral.*

Proof. Let $v \in M$ and let $I = \mathrm{supp}_v$. Define $U_v := \{u \in M : C_I \subset C_{\mathrm{supp}_u}\}$. The set $\{u \in V : C_I \subset C_{\mathrm{supp}_u}\}$ is Zariski open and $v \in U_v$. Therefore U_v is Zariski open. Now, keeping in mind there exist finitely many subsets of $\{1, \ldots, n\}$, there exist a finite numbers of open subset U_v and so

$$M = U_{v_1} \cup \cdots \cup U_{v_k},$$

for some $v_1, \ldots, v_k \in M$. Since M irreducible, it follows that $U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset$, and so it is Zariski open. Therefore there exists $x \in U_{v_1} \cap \cdots \cap U_{v_k}$ and so $\mathfrak{F}_{\mathfrak{a}}(M) = \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot x}) = C_I$ where $I = \mathrm{supp}_x$. □

Theorem 23. *Let $x \in V$. There exists $v \in G \cdot x$ such that $\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot v})$ and so the image is a polyhedral.*

Proof. We give two proofs. In the first proof we assume that G is an algebraic real reductive group and $\rho : G \longrightarrow \mathrm{GL}(V)$ is a rational representation. Let $G^{\mathbb{C}}$ be the complexification of G acting on $V^{\mathbb{C}}$. Let $G_{\mathbb{R}} = G^{\mathbb{C}} \cap \mathrm{GL}(V)$. Then $\overline{G^{\mathbb{C}} \cdot x \cap V}$ is a closed real algebraic irreducible set and $G^{\mathbb{C}} \cdot x \cap V$ is a finite union of $G_{\mathbb{R}}$ orbits [3, 22, 64]. Moreover, any $G_{\mathbb{R}}^o$ orbit throughout an element $v \in G^{\mathbb{C}} \cdot x \cap V$ is open [3, Proposition 2.3]. Since $(G_{\mathbb{R}})^o = G$ [4, 6, 20] it follows that any G orbit throughout an element of $G^{\mathbb{C}} \cdot x \cap V$ is also open. Write $M = \overline{G^{\mathbb{C}} \cdot x \cap V}$. By Proposition 22 there exist $v_1, \ldots, v_k \in M$ such that $U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset$ and $M = U_{v_1} \cup \cdots \cup U_{v_k}$. Since $U_{v_1} \cap \cdots \cap U_{v_k}$ is Zarisky open, it follows $G \cdot x \cap U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset$, and so $\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot z})$ for some $z \in G \cdot x$.

The second proof works without any algebraic assumption. The main tool is a Theorem of Baire.

Let $y = y_1 v_1 + \cdots + y_n v_n$ with $y_1 \cdots y_n \neq 0$. If $y \in \overline{G \cdot x}$, then $\mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot y}) = \mathfrak{F}_{\mathfrak{a}}(V)$ and so the result is proved. Otherwise,

$$\overline{G \cdot x} = \overline{G \cdot x} \cap \{v \in V : \langle v, v_1 \rangle = 0\} \cup \cdots \cup \overline{G \cdot x} \cap \{v \in V : \langle v, v_n \rangle = 0\}.$$

By a Theorem of Baire, there exists $k \in \{1, \ldots, n\}$ such that $\overline{G \cdot x} \cap \{v \in V : \langle v, v_k \rangle = 0\}$ has an interior point. Therefore, there exists $y \in G \cdot x$ and a neighborhood V of the origin of the Lie algebra $\mathfrak{g}$ of G such that $\exp(\theta)y \in \{v \in V : \langle v, v_k \rangle = 0\}$ for any $\theta \in V$. Assume by contradiction that $G \cdot y$ is not contained in $V_k = \{v \in V : \langle v, v_k \rangle = 0\}$. Write $A = \{z \in G \cdot y : z \in V_k\}$. Since G is analytic and the G action on V is analytic, $G \cdot y$ is analytic. By the above discussion the interior of A, that we denote by A^o, is not empty. Let $z \in \partial A^o$. Let $\varphi : U \longrightarrow U'$ be a chart of z. Then $z \in V_k$ and so $\varphi^{-1}(U' \cap V_k)$ contains an open subset. Since φ is analytic it follows $\varphi(U) \subset V_k$. A contradiction. Therefore $\overline{G \cdot x} = \overline{G \cdot y} \subset V_k$. In particular, there exists a G-invariant subspace $Z \subset V_k$ containing $\overline{G \cdot x}$ such that $V = Z \oplus Z^{\perp}$ is a G-stable splitting. This follows by the Cartan decomposition $G = K \exp(\mathfrak{p})$ where $K \subset \mathrm{O}(V)$ and $\mathfrak{p} \subset \mathrm{Sym}(V)$. Moreover, if $x \in Z$, then

$$\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x}) = (\mathfrak{F}_{\mathfrak{a}})_{|Z}(\overline{G \cdot x}).$$

After a finite number of steps, there exists a G invariant subspace W such that $\overline{G \cdot x} \subset W$ and $\overline{G \cdot x}$ is not contained in any subspace of W, i.e., it is full. Hence, there exists $y \in \overline{G \cdot x}$ such that $\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot y}) = \mathfrak{F}_{\mathfrak{a}}(W)$. Moreover, since $\{z \in W : \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot z}) = \mathfrak{F}_{\mathfrak{a}}(W)\}$ is open and dense, we may choose $y \in G \cdot x$ such that $\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x}) = \mathfrak{F}_{\mathfrak{a}}(\overline{A \cdot y}) = \mathfrak{F}_{\mathfrak{a}}(W)$. □

Using an $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{p}$ we can identify $\mathfrak{p}$ with $\mathfrak{p}^*$ and so we may think the gradient map as a $\mathfrak{p}$-valued map. If $\mathfrak{a} \subset \mathfrak{p}$ is a an Abelian subalgebra, then $\mathfrak{F}_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mathfrak{F}_{\mathfrak{p}}$ is the gradient map associated to $A = \exp(\mathfrak{a})$, where $\pi_{\mathfrak{a}} : \mathfrak{p} \longrightarrow \mathfrak{a}$ is the orthogonal projection. Applying result proved in [11, 26], we completely describe the convex hull of the image of the closure of G orbits under the gradient map associated to G.

Theorem 24. *Let $x \in V$ and let $E = \mathrm{Conv}(\mathfrak{F}_{\mathfrak{p}}(\overline{G \cdot x}))$. Then E is a closed convex set and any face of E is exposed.*

Proof. By Theorem 23, $C = \mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x})$ is a polyhedral. Since $\mathfrak{F}_{\mathfrak{a}} = \pi_{\mathfrak{a}} \circ \mathfrak{F}_{\mathfrak{p}}$ and $\mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x})$ is convex, it follows $\pi_{\mathfrak{a}}(E) = \pi_{\mathfrak{a}}(\mathfrak{F}_{\mathfrak{p}}(\overline{G \cdot x})) = \mathfrak{F}_{\mathfrak{a}}(\overline{G \cdot x})$. By Lemma 7 in [26] we have $E = KC$. In particular E is closed, due to the fact that K is compact, and C is closed. By Theorem 5 any face of C is exposed and so, by Lemma 1.4 and Proposition 1.4 in [12], any face of E is exposed. □

Remark 25. *Lemma* 1.4 *and Proposition* 1.4 *in* [12] *have been statement under the assumption that E is compact. However the same proof works without the compactness assumption.*

Similarly, one may prove the following more general result.

Theorem 26. *Let M be an G-invariant closed algebraic irreducible subset of V. Let $\mathfrak{a} \subset \mathfrak{p}$ a maximal Abelian subalgebra. Then* $\mathrm{Conv}(\mathfrak{F}_{\mathfrak{p}}(M)) = KC$, *where C is a polyhedral. Therefore* $\mathrm{Conv}(\mathfrak{F}_{\mathfrak{p}}(M))$ *is closed and any face of* $\mathrm{Conv}(\mathfrak{F}_{\mathfrak{p}}(M))$ *is exposed.*

5. Hilbert-Mumford criterion for reductive groups

In this section we prove the Hilbert-Mumford criterion for real reductive groups. We use in a different context original ideas from [25].

The G action on V induces a G action on $\mathbb{P}(V)$. It is easy to check that the function

$$\tilde{\Psi} : G \times \mathbb{P}(V) \longrightarrow \mathbb{R} \qquad (g, [x]) \mapsto \log\left(\frac{\| gx \|}{\| x \|}\right),$$

is a Kempf-Ness function and the corresponding gradient map $\tilde{\mathfrak{F}}_{\mathfrak{p}} : \mathbb{P}(V) \longrightarrow \mathfrak{p}^*$ is given by

$$\tilde{\mathfrak{F}}([x])(\xi) = \frac{\langle \xi x, x\rangle}{\| x \|^2}.$$

We may fix $\mathrm{Ad}(K)$-invariant scalar product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is an orthogonal splitting. Hence we may think the gradient map as a $\mathfrak{p}$-valued map,

$$\tilde{\mathfrak{F}}_{\mathfrak{p}} : \mathbb{P}(V) \longrightarrow \mathfrak{p}.$$

We define

$$f : \mathbb{P}(V) \longrightarrow \mathbb{R}, \qquad [x] \mapsto \frac{1}{2}\langle \tilde{\mathfrak{F}}_{\mathfrak{p}}([x]), \tilde{\mathfrak{F}}_{\mathfrak{p}}([x])\rangle_{\mathfrak{g}}.$$

In the sequel if $\xi \in \mathfrak{g}$, we denote by $\xi^{\#}([x]) := \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_{t=0} \exp(t\xi)[x]$ the vector field induced by the G action.

Lemma 27. *The function f is analytic and its gradient is given by $\nabla f(x) = \tilde{\mathfrak{F}}_{\mathfrak{p}}^{\#}(x)$.*

Proof. We identify V with $\mathbb{R}^n$. Assume that $G = \mathrm{SL}(n)$. Since

$$\tilde{\mathfrak{F}}_{Sym(n)}([x]) = \frac{xx^T}{\| x \|^2} - \frac{1}{2n} Id,$$

$\tilde{\mathfrak{F}}_{Sym(n)}$ is a polynomial and so it is analytic. If $G \subset \mathrm{SL}(V)$, then $\tilde{\mathfrak{F}}_{\mathfrak{p}} = \pi_{\mathfrak{p}} \circ \tilde{\mathfrak{F}}_{Sym(n)}$, where $\pi_{\mathfrak{p}}$ is the orthogonal projection, and so it analytic as well. The second part of the statement is easy to check. □

The negative gradient flow of f throughout $y \in \mathbb{P}(V)$ is then solution of the differential equation

$$\begin{cases} \dot{x}(t) = -\tilde{\mathfrak{F}}_{\mathfrak{p}}^{\#}(x(t)) \\ x(0) = y \end{cases} \tag{28}$$

Since $\mathbb{P}(V)$ is compact, the solution is defined in all $\mathbb{R}$.

Lemma 29. *Let $g : \mathbb{R} \longrightarrow G$ be the unique solution of the differential equation*

$$\begin{cases} g^{-1}(t)\dot{g}(t) = \tilde{\mathfrak{F}}_{\mathfrak{p}}^{\#}(x(t)) \\ g(0) = e \end{cases}.$$

Then $x(t) = g^{-1}(t)y$.

Proof. The solution g is defined in all $\mathbb{R}$ (see [42] p. 69). Since $\dot{g^{-1}} = -g^{-1}\dot{g}g^{-1}$, it follows

$$\dot{x}(t) = -g^{-1}\dot{g}g^{-1}y = -\tilde{\mathfrak{F}}_{\mathfrak{p}}^{\#}(x(t)),$$

and so the result is proved. □

The following Theorem arises from Lojasiewicz gradient inequality, see [17, 47]. A proof is given in [25, Theorem 3.3 p.14].

Theorem 30. *In the above assumption, the limit* $\lim_{\mapsto+\infty} x(t) = x_\infty$ *exists. Moreover, there exists positive constants* α, c, β, $\frac{1}{2} < \gamma < 1$ *and* $T > 0$ *such that for any* $t > T$ *we have*

$$\begin{aligned} d(x(t), x_\infty) &\leq \int_t^\infty \parallel \dot{x}(s) \parallel \mathrm{d}s \\ &\leq \frac{\alpha}{1-\gamma}(f(x(t)) - f(x_\infty))^{1-\gamma} \\ &\leq \frac{c}{(t-T)^\beta}. \end{aligned}$$

The following Theorem is a consequence of the Stratification Theorem in [32] and results proved in [28, 46].

Theorem 31. *Let* $y \in \mathbb{P}(V)$ *and let* $x : \mathbb{R} \longrightarrow \mathbb{P}(V)$ *be the solution of 28. Let* $x_\infty = \lim_{t\mapsto+\infty} x(t)$. *Then*

$$\parallel \tilde{\mathfrak{F}}_\mathfrak{p}(x_\infty) \parallel = \inf_{z\in\overline{G\cdot y}} \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel .$$

Moreover, if $z \in \overline{G \cdot y}$ *satisfies* $\parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel = \inf_{z\in\overline{G\cdot y}} \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel$, *then* $z \in K \cdot x_\infty$.

Proof. Let $\mathcal{C}_\mathfrak{p}$ be the set of critical points of f. Set $\mathcal{B}_\mathfrak{p} = \tilde{\mathfrak{F}}_\mathfrak{p}(\mathcal{C}_\mathfrak{p})$. Since f is K-invariant, the sets $\mathcal{C}_\mathfrak{p}$ and $\mathcal{B}_\mathfrak{p}$ are K-invariant. By the Stratification Theorem [32], there exists $\beta \in \mathcal{B}_\mathfrak{p}$ such that $\tilde{\mathfrak{F}}_\mathfrak{p}(\overline{G \cdot y}) \cap \mathcal{B}_\mathfrak{p} = K\beta$ and $\inf_{z\in\overline{G\cdot y}} \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel = \parallel \beta \parallel$. Since x_∞ is a critical point of f it follows $\inf_{z\in\overline{G\cdot y}} \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel = \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(x_\infty) \parallel$. By Theorem 5.1 in [28], $G \cdot y$ collapses to a single K orbit under the negative gradient flow of f. Let C be the connected component of the critical set of f corresponding to β and let S_β the corresponding stratum. In [46], in the complex setting, the author proves that the map

$$\varphi_\infty : S_\beta \longrightarrow C \qquad \varphi_\infty(x) = x_\infty,$$

is a continuous retraction. This result follows by the Lojasiewicz gradient inequality (see Lemma 2.3. p.124). Therefore the same holds in our situation.

Let $g_n y \mapsto r$ be such that

$$\inf_{z\in\overline{G\cdot y}} \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(z) \parallel = \parallel \tilde{\mathfrak{F}}_\mathfrak{p}(r) \parallel .$$

Then

$$\lim_{n\mapsto+\infty} \varphi_\infty(g_n y) = \varphi_\infty(r) = r,$$

and so r belongs to $K \cdot x_\infty$. □

We define on G the left-Riemannian metric which agree with $\langle \cdot, \cdot \rangle_\mathfrak{g}$ on the tangent space $T_e G$. This metric is K-invariant with respect the right K-action. Let $\Phi_x : G \longrightarrow \mathbb{R}$, defined as

$$\Phi_x(g) = \log\left(\frac{\parallel g^{-1}x \parallel}{\parallel x \parallel}\right),$$

Lemma 32. *The differential of* Φ_x *is given by* $(\mathrm{d}\Phi_x)_g(v) = -\langle \tilde{\mathfrak{F}}_\mathfrak{p}(g^{-1}x), \mathrm{d}L_{g^{-1}}(v)\rangle$. *Therefore* $\nabla\Phi_x(g) = v_x(g)$ *where* $v_x(g) = -\mathrm{d}L_g(\tilde{\mathfrak{F}}_\mathfrak{p}(g^{-1}x))$, *Proof.* Let $g \in G$ and let $X \in \mathfrak{g}$. Then

$$(\mathrm{d}\Phi_x)_g(\mathrm{d}L_g(X)) = \frac{-\langle X(g^{-1}x), g^{-1}x\rangle}{\parallel g^{-1}x \parallel^2}.$$

If $X \in \mathfrak{p}$ then $(\mathrm{d}\Psi_x)_g(\mathrm{d}L_g(X)) = -\langle \tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x), X\rangle_{\mathfrak{g}}$. If $X \in \mathfrak{k}$, then

$$0 = (\mathrm{d}\Phi_x)_g(\mathrm{d}L_g(X)) = \langle \tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x), X\rangle_{\mathfrak{g}}.$$

This means

$$\begin{aligned}(\mathrm{d}\Phi_x)_g(\mathrm{d}L_g(X)) &= -\langle \tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x), X\rangle_{\mathfrak{g}} \\ &= -\langle \mathrm{d}L_g(\mathfrak{F}_{\mathfrak{p}}(g^{-1}x)), \mathrm{L}_g(X)\rangle,\end{aligned}$$

concluding the proof. □

Define $\Theta_x : G \longrightarrow G(x)$ as follows

$$\Theta_x(g) = g^{-1}x.$$

Lemma 33. *The map Θ_x intertwines the gradient of Φ_x and ∇f.*

Proof. Let $\xi \in \mathfrak{p}$. Since

$$\Theta_x(g\exp(t\xi)) = \exp(-t\xi)g^{-1}x,$$

we get

$$(\mathrm{d}\Theta_x)_g(\mathrm{d}L_g(\xi)) = -\xi^{\#}(g^{-1}x),$$

and so the result is proved. □

Since Φ_x is K-invariant, it descends to a smooth map $\Phi_x : G/K \longrightarrow \mathbb{R}$, that we also denote by Φ_x. A proof of the next Lemma is given in [25] (Lemma A.3 p.140).

Lemma 34. *Let $F : G/K \longrightarrow \mathbb{R}$ be a smooth function that is convex along geodesics. Let $c_0, c_1 : \mathbb{R} \longrightarrow M$ be the negative gradient flow of F and let $\rho(t) = d_M(c_0(t), c_1(t))$. Then $\rho(t)$ is nonincreasing function.*

Theorem 35. *Let $\Phi_x : G/K \longrightarrow \mathbb{R}$. Then*

a) *Φ_x is a Morse-Bott function and it is convex along geodesics;*
b) *The critical set, possibly empty, is the submanifold*
$$\{gK \in G/K : \tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x) = 0\}.$$
c) *If $c : I \longrightarrow M$ is a negative gradient flow of Φ_x, then*
$$\lim_{t\mapsto+\infty} \Phi_x(c(t)) = \inf_{x\in G/K}\Phi_x$$

Proof. By Lemma 6 in [14], see also Lemma 2.19 p. 1115, Φ_x is convex along geodesics. By Proposition 9 p.2191 it follows that a critical points of Φ_x are the elements $\pi(g) \in G/K$ such that $\tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x) = 0$. If $gK = \pi(g)$ is a critical point, then

$$\mathrm{Hess}(\Phi_x)(\mathrm{d}L_g(\xi), \mathrm{d}L_g(\xi)) = \left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0} \Phi_x(g\exp(tv)) = \left.\frac{\mathrm{d}^2}{\mathrm{dt}^2}\right|_{t=0} \log\left(\parallel \exp(-t\xi)g^{-1}x \parallel\right) \geq 0,$$

and it is 0 if and only if $\exp(t\xi) \subset G_{g^{-1}x}$. Let $g \in \mathrm{Crit}\Phi_x$. By Hadamard-Cartan Theorem, the map

$$\mathfrak{p} \longrightarrow G/K, \qquad \xi \mapsto g\exp(\xi),$$

is a diffemorphism. Theretofore $g\exp(\xi) \in \mathrm{Crit}\Phi_x$ if and only if $\tilde{\mathfrak{F}}_{\mathfrak{p}}(\exp(-\xi)g^{-1}x) = 0$. Since $\tilde{\mathfrak{F}}_{\mathfrak{p}}(g^{-1}x) = 0$, by Proposition 13 it follows that $\exp(t\xi) \in G_{g^{-1}x}$. This implies

$$\mathrm{Crit}\Phi_x = \{\pi(g\exp(t\xi)) : \exp(t\xi) \in G_{g^{-1}x}\}$$

and so it is submanifold and the Kernel of the Hessian. Therefore Φ_x is a Morse-Bott function.

The gradient of $\Psi_x : G \longrightarrow \mathbb{R}$ is given by $\nabla\Psi_x(\pi(g)) = \mathrm{d}\pi_g(v_x(g))$. Hence the negative gradient flow of $\Psi_x : G/K \longrightarrow \mathbb{R}$ lifts to the negative gradient flow of $\Psi_x : G \longrightarrow \mathbb{R}$. By Lemma 32 the map Θ_x intertwines the gradient of Ψ_x with ∇f. Therefore the negative gradient flow of $\Psi_x : G \longrightarrow \mathbb{R}$ satisfies the differential equation 28. Vice-versa if $g : \mathbb{R} \longrightarrow G$ satisfies the equation 28, then one may prove that $\pi \circ g$ is the negative gradient flow of $\Psi_x : G/K \longrightarrow \mathbb{R}$.

Let $c_1, c_2 : \mathbb{R} \longrightarrow M$ be negative gradient flow of Φ_x. Then there exist $g_0, g_1 : \mathbb{R} \longrightarrow G$ solution of 28 such that $c_0 = \pi \circ g_0$ and $c_1 = \pi \circ g_1$. Since $G = K\exp(\mathfrak{p})$, there exist $\xi : \mathbb{R} \longrightarrow \mathfrak{p}$ and $k : \mathbb{R} \longrightarrow K$ such that $g_1(t) = g_0(t)\exp(\xi(t))k(t)$. Write

$$H : \mathbb{R} \times \mathbb{R} \longrightarrow G/K, \qquad H(t,s) = \pi(g_0(t)\exp(s\xi(t)).$$

In the sequel we denote by $H_t(s) = H(t,s)$. The curve $s \mapsto H_t(s)$ is the unique geodetic joining $c_0(t)$ and $c_1(t)$. By Lemma 34 the function

$$\rho(t) = d_{G/K}(c_0(t), c_1(t)) = \parallel \xi(t) \parallel,$$

is nonincreasing. Assume $\mathrm{Crit}\Phi_x$ is not empty. Hence we may assume $g_0(0) \in \mathrm{Crit}\Phi_x$ and so the curve c_0 is constant. Since ρ is nonincreasing, the image c_1 is contained in a compact subset. This implies, keeping in mind that Φ_x is Morse-Bott, the limit $\lim_{t\mapsto+\infty} c_1(t) \in \mathrm{Crit}\Phi_x$ and so item (c) holds. In particular, every negative gradient flow converges to a critical point and so Φ_x has a global minimum. Now, assume that Φ_x does not have any critical point. Assume by contradiction

$$\lim_{t\mapsto+\infty} \Phi(c_0(t)) = a > \mathrm{Inf}_{G/K}\Phi_x.$$

Hence $\Phi(c_0(t))$ is bounded from below. We may choose c_0 such way $\Phi_x(c_1(0)) < a$. By Lemma 34, ρ is nonincreasing and so there exists $C > 0$ such that $\rho(t) = \parallel \xi(t) \parallel \leq C$. Hence

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{ds}}\right|_{s=0} \Phi_x(H_t(s)) &= (\mathrm{d}\Phi_x)_{c_0(t)}(\dot{H}_t(0)) \\
&= -\langle \tilde{\mathfrak{F}}_{\mathfrak{p}}(g_0(t)^{-1}x), \xi(t)\rangle \\
&- \geq \parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(g_0(t)^{-1}x) \parallel \parallel \xi(t) \parallel \\
&- \geq C \parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(g_0(t)^{-1}x) \parallel .
\end{aligned}$$

Since for t fixed, $H_t(s)$ is a geodesic, it follows that the function $s \mapsto \Phi_x(H_t(s))$ is convex and so its derivative $\frac{\mathrm{d}}{\mathrm{ds}}\Big|_s \Phi_x(H_t(s))$ increases. Hence

$$\begin{aligned}\Phi_x(c_1(t)) &= \Phi_x(H_t(1))) \\ &= \Phi_x(H_t(0)) + \int_0^1 \frac{\mathrm{d}}{\mathrm{ds}}\Big|_s \Phi_x(H_t(s))\mathrm{ds} \\ &\geq \Phi_x(c_0(t)) - C \parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(g_0(t)^{-1}x) \parallel .\end{aligned}$$

Now, the function $\Phi \circ c_1$ is bounded and $\frac{\mathrm{d}}{\mathrm{dt}}\Big|_t (\Phi_x \circ c_0) = - \parallel \mathfrak{F}_{\mathfrak{p}}(g_0(t)^{-1}x) \parallel$. Therefore, there exists a sequence $t_i \mapsto +\infty$ such that $\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=t_i} (\Phi_x \circ c_0) = - \parallel \mathfrak{F}_{\mathfrak{p}}(g_0(t_i)^{-1}x) \parallel$ goes to 0. This implies

$$\lim_{t_i \mapsto +\infty} \Phi_x(c_1(t_i)) \geq \Phi_x(c_0(t_i)) \geq a,$$

a contradiction since $(\Phi_x(c_o(t)) < a$ and so $\lim_{t_i \mapsto +\infty} \Phi_x(c_1(t_i)) < a$. □

Now, we are able to prove the Hilbert-Mumford criterion for real reductive groups. We start recalling a well-known numerical criterion, a proof is given in [14], and some technical lemmata.

Theorem 36. *Let $x \in \mathbb{P}(V)$. Then x is semistable if and only if for any $\xi \in \mathfrak{p}$, $\lambda(x, \xi) \geq 0$;*

Lemma 37. *Let $x \in \mathbb{P}(V)$ and let $\tilde{x} \in V$ such that $\pi(\tilde{x}) = x$. Let $\alpha_1 > \cdots > \alpha_k$ be the eigenvalues of ξ. We denote by V_i the corresponding eigenspaces. Write $\tilde{x} = v_1 + \cdots + v_k$. Then $\lambda(x, \xi) = \alpha_j$, where $j = \min\{1, \dots k\}$ such that $v_j \neq 0$. Moreover, $\lim_{t \mapsto +\infty} \exp(t\xi)\tilde{x} = 0$ if and only if $\alpha_j < 0$.*

Proof.

$$\begin{aligned}\lambda(x, \xi) &= \lim_{t \mapsto +\infty} \frac{\mathrm{d}}{\mathrm{dt}} \tilde{\Psi}(x, \exp(t\xi)) \\ &= \lim_{t \mapsto +\infty} \frac{\alpha_j e^{2\alpha_i t} \parallel v_j \parallel^2 + \cdots + \alpha_k e^{2\alpha_k t} \parallel v_k \parallel^2}{e^{2\alpha_j t} \parallel v_j \parallel^2 + \cdots + e^{2\alpha_k t} \parallel v_k \parallel^2} \\ &= \lim_{t \mapsto +\infty} \frac{\alpha_j \parallel v_j \parallel^2 + \alpha_{j+1} e^{2(\alpha_{j+1} - \alpha_j)t} \parallel v_{j+1} \parallel^2 + \cdots + \alpha_k e^{2(\alpha_k - \alpha_j)t} \parallel v_k \parallel^2}{\parallel v_j \parallel^2 + e^{2(\alpha_{j+1} - \alpha_j)t} \parallel v_{j+1} \parallel^2 \cdots + e^{2(\alpha_k - \alpha_j)t} \parallel v_k \parallel^2} \\ &= \alpha_j\end{aligned}$$

Since

$$\parallel \exp(t\xi)\tilde{x} \parallel^2 = \sum_{i=1}^k e^{2t\alpha_i} \parallel v_i \parallel^2,$$

it follows that $\exp(t\xi)\tilde{x} \mapsto 0$ if and only if $\alpha_j < 0$. Otherwise the limit does not exist or $\exp(t\xi) \in G_{\tilde{x}}$. □

Lemma 38. *Let $x \in \mathbb{P}(V)$ be a semistable point. Let $\tilde{x} \in V$ be such that $\pi(\tilde{x}) = x$. Then $0 \notin \overline{G \cdot \tilde{x}}$.*

Proof. Let $s : \mathbb{R} \longrightarrow \mathbb{P}(V)$ and let $g : \mathbb{R} \longrightarrow G$ so that $s(t) = g(t)^{-1}x$ is the solution of 28. By Theorem 30, the limit $\lim_{t\mapsto+\infty} s(t) = x_\infty$ exists and by Theorem 31 it satisfies

$$\parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(x_\infty) \parallel = \inf_{y\in\overline{G(x)}} \parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(y) \parallel .$$

Since x is semistable it follows that $\parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(x_\infty) \parallel = 0$. By the Lojasiewicz gradient inequality for f, there exist positive constant α, β and $\frac{1}{2} < \gamma < 1$ such that

$$\parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(x) \parallel^2 \leq 2f(x) \leq 2f(x)^\gamma \leq C \parallel \mathrm{grad}\, f \parallel = C \parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}^{\#}(x) \parallel .$$

Therefore

$$\parallel \tilde{\mathfrak{F}}_{\mathfrak{p}}(s(t)) \parallel^2 = -\frac{\mathrm{d}}{\mathrm{dt}} \Phi_{[x]} \circ c(t) \leq C \parallel \dot{s}(t) \parallel,$$

where $c(t) = \pi \circ g(t)$ is the negative gradient flow of $\Phi_x : G/K \longrightarrow \mathbb{R}$. By Theorem 30 the function $\parallel \dot{s}(t) \parallel$ is integrable over $\mathbb{R}^+$, and so the limit

$$\lim_{t\mapsto+\infty} \Phi_{[x]} \circ c(t) = a \in \mathbb{R}.$$

By Theorem 35 the function Φ_x is bounded below. Since $\Phi_{[x]}(g) = \log\left(\frac{\|g^{-1}\tilde{x}\|}{\|\tilde{x}\|}\right)$, we get $0 \notin \overline{G \cdot \tilde{x}}$ concluding the proof. □

Theorem 39. *Let $y \in V$ and assume that $0 \in \overline{G \cdot y}$. Then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t\mapsto+\infty} \exp(t\xi)y = 0$.*

Proof. Let $y \in V$ such that $0 \in \overline{G \cdot y}$. By Lemma 38 the point $\pi(y)$ is not semistable in $\mathbb{P}(V)$. By Theorem 36 there exists $\xi \in \mathfrak{p}$ such that $\lambda_x(\pi(y), \xi) < 0$. By Lemma 37 we get $\lim_{t\mapsto+\infty} \exp(t\xi)y = 0$ concluding the proof. □

Given $\beta \in \mathfrak{p}$, we define the following subgroups:

$$G^\beta = \{g \in G : \mathrm{Ad}(g)(\beta) = \beta\}$$
$$G^{\beta-} := \{g \in G : \lim_{t\mapsto+\infty} \exp(t\beta)g\exp(-t\beta) \text{exists}\}$$
$$R^{\beta-} := \{g \in G : \lim_{t\mapsto+\infty} \exp(t\beta)g\exp(-t\beta) = e\}$$

The next lemma is well-known. A proof is given in [11] (see also [40]).

Lemma 40. *If $g \in G^{\beta-}$ then $\lim_{t\mapsto+\infty} \exp(t\beta)g\exp(-t\beta) \in G^\beta$. Moreover, $G^{\beta-}$ is a parabolic subgroup of G with Lie algebra $\mathfrak{g}^{\beta-} = \mathfrak{g}^\beta \oplus \mathfrak{r}^{\beta-}$ and $G = G^{\beta-}K$. Every parabolic subgroup of G equals $G^{\beta-}$ for some $\beta \in \mathfrak{p}$. $R^{\beta-}$ is the unipotent radical of $G^{\beta-}$ and G^β is a Levi factor. Finally, $G = KG^{\beta-}$.*

Finally, applying the Slice Theorem proved in [32], we prove the Hilbert-Mumford criterion for reductive groups.

Theorem 41 (Hilbert-Mumford for reductive groups)**.** *Let $x \in V$ and let $u \in \overline{G \cdot x}$ be such that $G \cdot u$ is closed. Then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t\mapsto+\infty} \exp(t\xi)x$ lies in $G \cdot u$.*

Proof. Let $u \in \overline{G \cdot x}$ be such that $G \cdot u$ is closed. We may assume $u \in \mathfrak{F}_{\mathfrak{p}}^{-1}(0)$. By Proposition 13 $G_u = K_u \exp(\mathfrak{p}_u)$ is compatible and so the G_u action V is completely reducible [31, 14.9]. Therefore, there exists G_u stable decomposition $V = \mathfrak{g} \cdot u \oplus W$. By the Slice Theorem [32, Theorem 3.1], there exists a G-stable neighborhood Ω of u, a G_u-invariant open neighborhood S of $0 \in W$ and G-equivariant diffeomorphism $\Psi_u : G \times_{G_u} S \longrightarrow \Omega$, where $\Psi_u([e, 0]) = u$. In particular $G \cdot x \subset \Omega$ and $G \cdot x \cap S = G_u \cdot s$ for some $s \in G \cdot x \cap S$. This means that the condition $u \in \overline{G \cdot x}$ is equivalent to $0 \in \overline{G_u \cdot s}$. Moreover, since $K_u \subset \mathrm{O}(V)$, it follows that $0 \in \overline{G_u^o \cdot s}$. By Theorem 39 there exists $\nu \in \mathfrak{p}_u$ such that

$$\lim_{t \mapsto +\infty} \exp(t\nu)s = u.$$

Write, $s = gkx$, where $g \in G^{\nu-}$ and $k \in K$. Since

$$\exp(t\nu)gkx = (\exp(t\nu)g\exp(-t\nu))\, k^{-1} \exp(t\mathrm{Ad}(k^{-1})(\xi))x,$$

keeping in mind that $\lim_{t \mapsto +\infty} \exp(t\nu)g\exp(-t\nu)$ exists, it follows

$$\lim_{t \mapsto +\infty} \exp(t\mathrm{Ad}(k^{-1})(\xi))x \in G \cdot u.$$

□

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