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A Mellin transform approach to barrier option pricing

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Abstract

A barrier option is an exotic path-dependent option contract that, depending on terms, automatically expires or can be exercised only if the underlying asset ever reaches a predetermined barrier price. Using a partial differential equation approach, we provide an integral representation of the barrier option price via the Mellin transform. In the case of knock-out barrier options, we obtain a decomposition of the barrier option price into the corresponding European option value minus a barrier premium. The integral representation formula can be expressed in terms of the solution to a system of coupled Volterra integral equations of the first kind. Moreover, we suggest some possible numerical approaches to the problem of barrier option pricing.

1 Introduction

A European option is a financial derivative contract that gives the buyer the right to buy or sell a particular asset at a fixed maturity or expiry T and at a predetermined exercise or strike price E. In the case of a *barrier option*, this right is activated (*knock-in*) or extinguished (*knock-out*) when the underlying asset reaches a certain barrier price during the time interval $[0,T]$. If the option expires inactive or extinguishes, then it may be worthless or there may be a cash rebate R paid out. A barrier option has a lower premium than a similar European option without a barrier. Barrier options were created to provide the hedge of an option at a lower premium than a conventional option and are traded in large volumes.

Barriers are generally fixed, but time-dependent barriers can be considered as well. Moreover, time-dependent barriers arise naturally in financial markets even if the barriers in the option contract are constant. For instance, assuming a deterministic term structure of interest rates or a time-dependent volatility, by a change of variables we can move the time-dependence from the coefficients of the partial differential equation (PDE) modeling the price to the barriers. Furthermore, by referring to the forward price of the underlying, the option contract with constant barrier translates into a contract with time-dependent barriers. The assumption that the term structure of interest rates is deterministic is very common in the foreign exchange markets as the majority of barrier option contracts have short maturities of up to one year and have little dependence on the stochasticity of interest rates.

The academic literature on continuously monitored barrier options is vast and varied and dates back to at least the work of Merton [19], who presented a closed-form solution for the price of a continuously monitored down-and-out European call. Several approaches can be found. The first one, which mainly deals with constant barriers, identifies pathwise hedging strategies with European derivatives that either uniquely determine or provide an admissible range for the barrier option price (see, for instance, [5, 6, 7]). A static hedge using calls and puts for a time-dependent single barrier option is described in [1]. The result applies to linear diffusions with compound Poisson jumps, but the hedging strategy depends on knowing the values of the barrier contract to be hedged at certain times before expiry. A probabilistic approach using Laplace transforms for constant double barrier options in the Black-Scholes model is given in [10]. A method using the joint density of the stock, its maximum, and its minimum to find the price of time-dependent barrier options in the Black-Scholes model was pioneered in [16]. The article [21] used boundary crossing probabilities for Brownian motion to price single barrier options when the underlying asset price process has deterministic time-dependent drift and volatility. Lattice methods have been employed by several authors such as in [3, 23]. PDE-based methods for pricing continuously or discretely monitored barrier options are studied, among others, in [4, 27, 31] using finite difference and finite volume and infinite element methods.

More recent works on time-dependent double barrier options used analytic tools such as Fourier transforms [9], Green's functions [14], and complex integration [22]. Spectral methods were applied in [8] to find constant double barrier option prices in the class of CEV models. The use of the boundary element method to derive a suitable integral representation of the barrier option price has been explored in [2, 12, 13, 28]. Finally, [20] addressed the question of pricing time-dependent single and double barrier options when the underlying asset price process is a linear diffusion with mild regularity conditions on its volatility function. The approach in [20] is entirely probabilistic and yields a representation formula that is very close to ours, where the barrier premium can be expressed as a sum of integrals along the barriers of the option deltas that come from solving a system of Volterra integral equations of the first kind.

Another possible approach to pricing option contracts with time-dependent parameters (e.g., volatility, interest rate, and dividends) is by the Mellin transform. This approach has been applied to European options with general payoffs [25], American call and put options with approximate ordinary differential equations (ODEs) for the optimal exercise boundaries [24], European options where the underlying process is lognormal with jumps [17], and American options with general payoffs giving exact integral equations and approximate ODEs for the optimal exercise boundaries [26]. Yoon and Kim [30] use double Mellin transforms to study European vulnerable options under constant as well as stochastic interest rates assuming the Hull-White framework. Gzyl et al. [11] investigate the use of Mellin transforms to non-standard option pricing models, characterized by discontinuities in the terminal/boundary conditions and with time-varying parameters.

Transform methods and integral representation formulas allow the recovery of option prices in semi-analytic form as a solution of a nonsingular system of linear equations obtained by discretization of Volterra integral equations rather than from a numerical scheme for partial differential equations. An aim of this paper is to reformulate the single barrier problem (as well as the double barrier problem to be given later) into a more general framework which allows for an integral representation of the solution based on the Mellin transform. In the case of knock-out barrier options, this formula lends itself to the following interesting financial interpretation: the barrier option price is decomposed into the corresponding European option value minus a barrier premium. Under suitable assumptions, the integral formula is obtained after solving a system of coupled Volterra integral equations of the first kind for which some possible numerical approaches are suggested. Our approach applies to both single and double barrier problems with (i) time-dependent barriers, (ii) time-dependent parameters, (iii) possibly nonzero rebates and (iv) general payoffs.

For ease of exposition, let us consider the case of a single barrier option. Let $B(t) > 0$ denote the barrier at time $t \geqslant 0$ and $R \geqslant 0$ the rebate. Denote by v_{e} the European option pricing function such that at expiry $T > 0$ the payoff function is $x \mapsto v_{\mathrm{e}}(x,T)$, where $x \in \mathbb{R}_+ = [0,\infty)$. The *active domain* $I(t)$ is defined to be either $(0,B(t))$ or $(B(t),\infty)$. Outside the active domain, the option value is equal to R in the case of knock-out barriers and equal to the corresponding European option price in the case of knock-in barriers. Then the different types of barrier options are characterized by their specific active domain, payoff condition, and boundary condition (BC) as follows:

	$I(t)$	payoff	BC at $x = B(t)$
down-and-out	$(B(t),\infty)$	$v_{\mathrm{e}}(x,T)$	R
up-and-out	$(0,B(t))$	$v_{\mathrm{e}}(x,T)$	R
down-and-in	$(B(t),\infty)$	R	$v_{\mathrm{e}}(B(t),t)$
up-and-in	$(0,B(t))$	R	$v_{\mathrm{e}}(B(t),t)$

The PDE formulation for the single barrier problem takes the form of a final-boundary value problem. We have

$$\mathscr{L}v(x,t) = 0, \quad x \in I(t), \quad t \in [0,T), \tag{1.1}$$

where

$$\mathscr{L}v(x,t) = \frac{\partial v}{\partial t}(x,t) + \frac{1}{2}\sigma(t)^2 x^2 \frac{\partial^2 v}{\partial x^2}(x,t) + [r(t) - D(t)]x\frac{\partial v}{\partial x}(x,t) - r(t)v(x,t)$$

is the generalized Black-Scholes operator. Here, the risk-free rate r, the dividend yield D, and the volatility σ are continuous functions of t, where $\sigma(t) > 0$ and $D(t) \geqslant 0$ for all $t \in [0,T]$. The final condition is

$$v(x,T) = \begin{cases} v_{\mathrm{e}}(x,T) & \text{for knock-out options,} \\ R & \text{for knock-in options,} \end{cases} \quad x \in I(T). \tag{1.2}$$

The option value at the barrier is

$$v(B(t),t) = \begin{cases} R & \text{for knock-out options,} \\ v_{\mathrm{e}}(B(t),t) & \text{for knock-in options,} \end{cases} \quad t \in [0,T). \tag{1.3}$$

We shall assume that $t \mapsto B(t)$ is sufficiently regular and that (1.1)–(1.3) is a well-posed problem. Our goal here is to derive an integral representation of the solution, validated by numerical simulations, and leave the proof of well-posedness as future work.

The outline of the paper is as follows. In Section 2, we summarize preliminary results coming from the use of the Mellin transform in option valuation. In Section 3, we provide an integral representation formula for the single barrier problem and we pose suitable assumptions on the auxiliary functions f and g involved in the formulation. In Section 4, some possible numerical approaches to the solution of the Volterra integral equation are suggested and the validity of our representation formula is checked by numerical simulation. In Section 5, the extension to the double barrier problem is briefly sketched, and brief concluding remarks are given in Section 6. The proofs of theorems and other results are given in the Appendix.

2 Preliminary results

In this section, we summarize some results obtained as a consequence of using the Mellin transform in option valuation [17, 24, 25, 26]. We begin by defining useful auxiliary functions [24]. Let

$$\begin{aligned} z_1(x,t,u) &= \frac{\log x + \int_t^u [r(\tau) - D(\tau) + \sigma(\tau)^2/2]\,\mathrm{d}\tau}{[\int_t^u \sigma(\tau)^2\,\mathrm{d}\tau]^{1/2}}, \\ z_2(x,t,u) &= \frac{\log x + \int_t^u [r(\tau) - D(\tau) - \sigma(\tau)^2/2]\,\mathrm{d}\tau}{[\int_t^u \sigma(\tau)^2\,\mathrm{d}\tau]^{1/2}}. \end{aligned}$$

It follows that

$$x\mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau} N'(z_1(x,t,u)) - \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N'(z_2(x,t,u)) = 0,$$

where N is the cumulative distribution function of a standard normal random variable. The *Black-Scholes kernel* was defined in [25, 24] by

$$\mathscr{K}(x,t,u) = \frac{\mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau}}{[\int_t^u \sigma(\tau)^2\,\mathrm{d}\tau]^{1/2}} N'(z_2(x,t,u)) = \frac{x\mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau}}{[\int_t^u \sigma(\tau)^2\,\mathrm{d}\tau]^{1/2}} N'(z_1(x,t,u)).$$

It is straightforward to show by direct differentiation of the right-hand sides that

$$\begin{aligned} \mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-x\mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau} N\left(z_1\left(\frac{x}{y},t,u\right)\right)\right], \\ \frac{1}{y}\mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-\mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(z_2\left(\frac{x}{y},t,u\right)\right)\right]. \end{aligned} \tag{2.1}$$

The *Mellin transform* $\hat{f}$ at $\xi \in \mathbb{C}$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ is defined to be

$$\hat{f}(\xi) = \int_0^\infty x^{\xi-1} f(x)\,\mathrm{d}x,$$

provided the improper integral converges at ξ. Moreover, if we define the *Mellin convolution* $f * g$ of two functions f and g defined on $\mathbb{R}_+$ by

$$(f * g)(x) = \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y,$$

then it can be shown that a *convolution property* holds: $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

It was shown in [25] that the solution of the final value problem

$$\begin{aligned} &\mathscr{L} v_\mathrm{e}(x,t) = 0, \quad x \in \mathbb{R}_+, \quad t \in [0,T), \\ &v_\mathrm{e}(x,T) \text{ given}, \quad x \in \mathbb{R}_+, \end{aligned} \tag{2.2}$$

where $\mathscr{L}$ is the generalized Black-Scholes operator and $x \mapsto v_\mathrm{e}(x,T)$ is a given payoff function, is expressed as

$$v_\mathrm{e}(x,t) = \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, T\right) v_\mathrm{e}(y,T)\,\mathrm{d}y.$$

It follows that the European option pricing function is the Mellin convolution of the Black-Scholes kernel and the payoff function. Two particular important examples are the call and put vanilla options. For a call payoff $v_\mathrm{e}(x,T) = \max(x - E, 0)$, where $E > 0$ is the exercise price, with the aid of (2.1) we obtain that

$$\begin{aligned} v_\mathrm{call}(x,t;E,T) = {} & x\mathrm{e}^{-\int_t^T D(\tau)\,\mathrm{d}\tau} N\left(z_1\left(\frac{x}{E}, t, T\right)\right) \\ & - E\mathrm{e}^{-\int_t^T r(\tau)\,\mathrm{d}\tau} N\left(z_2\left(\frac{x}{E}, t, T\right)\right), \end{aligned} \tag{2.3}$$

while for a put payoff $v_\mathrm{e}(x,T) = \max(E - x, 0)$, we have

$$\begin{aligned} v_\mathrm{put}(x,t;E,T) = {} & E\mathrm{e}^{-\int_t^T r(\tau)\,\mathrm{d}\tau} N\left(-z_2\left(\frac{x}{E}, t, T\right)\right) \\ & - x\mathrm{e}^{-\int_t^T D(\tau)\,\mathrm{d}\tau} N\left(-z_1\left(\frac{x}{E}, t, T\right)\right). \end{aligned} \tag{2.4}$$

When r, D, and σ are constants, (2.3) and (2.4) are of course the well-known Black-Scholes formulas.

Remark 2.1. We observe that

$$\frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, T\right) = G(y,T;x,t),$$

where

$$\begin{aligned} G(y,u;x,t) = {} & \frac{\mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau}}{y[2\pi \int_t^u \sigma(\tau)^2\,\mathrm{d}\tau]^{1/2}} \\ & \times \exp\left(-\frac{\{\log(y/x) - \int_t^u [r(\tau) - D(\tau) - \sigma(\tau)^2/2]\,\mathrm{d}\tau\}^2}{2\int_t^u \sigma(\tau)^2\,\mathrm{d}\tau}\right) \end{aligned}$$

is the Green's function associated with the final value problem (2.2).

More generally, suppose that $(x,t) \mapsto f(x,t)$ and $x \mapsto g(x)$ are Mellin transformable functions with respect to x. Then it was shown in [26] that the formal solution of the nonhomogeneous final value problem

$$\begin{aligned} &\mathscr{L}v(x,t) = f(x,t), \quad x \in \mathbb{R}_+, \quad t \in [0,T), \\ &v(x,T) = g(x), \quad x \in \mathbb{R}_+ \end{aligned}$$

is given by

$$v(x,t) = v_0(x,t) - \int_t^T \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y},t,u\right) f(y,u)\, \mathrm{d}y\, \mathrm{d}u, \tag{2.5}$$

where

$$v_0(x,t) = \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y},t,T\right) g(y)\, \mathrm{d}y. \tag{2.6}$$

3 An integral representation formula for the single barrier problem

Recall from Section 1 that the single barrier problem can be expressed as

$$\begin{aligned} &\mathscr{L}v(x,t) = 0, \quad x \in I(t), \quad t \in [0,T), \\ &v(x,T) \text{ given}, \quad x \in I(T), \\ &v(B(t),t) \text{ given}, \quad t \in [0,T], \end{aligned} \tag{3.1}$$

where $I(t) = (0, B(t))$ or $I(t) = (B(t), \infty)$. We can embed the first two equations in (3.1) into the nonhomogeneous problem

$$\begin{aligned} &\mathscr{L}v(x,t) = f(x,t), \quad x \in \mathbb{R}_+, \quad t \in [0,T), \\ &v(x,T) = g(x), \quad x \in \mathbb{R}_+ \end{aligned} \tag{3.2}$$

provided that we assume the following:

(i) $f : \mathbb{R}_+ \times [0,T) \to \mathbb{R}$ is such that

$$f(x,t) = 0, \quad x \in I(t), \quad t \in [0,T). \tag{3.3}$$

(ii) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a suitable extension of the payoff condition such that

$$g(x) = v(x,T), \quad x \in I(T). \tag{3.4}$$

Although $x \mapsto v(x,T)$ is defined in (3.1) only on $I(T)$, in fact it can be extended to all of $\mathbb{R}_+$ since it is either the European payoff function $v_e(x,T)$ (knock-out options) or the rebate R (knock-in options). Alternatively, we can define $g(x) = \mathbf{1}_{I(T)}(x) v_e(x,T)$ for $x \in \mathbb{R}_+$, where $\mathbf{1}_A$ is the indicator function of the set A.

Our first result provides an analytical representation of the solution to (3.1). The proof of this theorem is given in Appendix A.

Theorem 3.1. *For $x \in I(t)$ and $t \in [0,T)$, define*

$$v(x,t) = v_0(x,t) - \int_t^T \int_{\mathbb{R}_+ \setminus I(u)} \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, u\right) h(y,u)\, \mathrm{d}y\, \mathrm{d}u,$$
$$v_0(x,t) = \int_0^\infty \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, T\right) g(y)\, \mathrm{d}y, \tag{3.5}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a Mellin transformable function that satisfies (3.4) *and $h : \mathbb{R}_+ \times [0,T) \to \mathbb{R}$ is Mellin transformable with respect to the first argument and satisfies the linear integral equation*

$$v_0(B(t),t) - v(B(t),t) = \int_t^T \int_{\mathbb{R}_+ \setminus I(u)} \frac{1}{y} \mathscr{K}\left(\frac{B(t)}{y}, t, u\right) h(y,u)\, \mathrm{d}y\, \mathrm{d}u, \quad t \in [0,T). \tag{3.6}$$

Then (3.5) *satisfies* (3.2) *with $f(x,t) = \mathbf{1}_{\mathbb{R}_+ \setminus I(t)}(x) h(x,t)$ and hence the single barrier problem* (3.1).

Remark 3.2. Consider the case of a down-and-out barrier option with constant barrier B and no rebate $R = 0$. By choosing

$$f(x,t) = x\, q(x,t) \delta(x - B), \quad g(x) = \mathbf{1}_{(B,\infty)}(x) v(x,T),$$

where $(x,t) \mapsto q(x,t)$ is a suitable auxiliary function and δ is the Dirac delta distribution, and using (2.5)–(2.6), we recover the integral formulation in [13]:

$$v(x,t) = \int_{\log B}^\infty p(z,T; \log x, t) v(\mathrm{e}^z, T)\, \mathrm{d}z - \int_t^T p(\log B, u; \log x, t) q(B,u)\, \mathrm{d}u.$$

Here, $p(z,u;\log x,t) = yG(y,u;x,t)$ is the transition probability density function of the log-price $z = \log y$ of the underlying asset. The proof is given in Appendix B.

We wish to find a more general formulation for the barrier option pricing problem that allows for different choices of the functions f and g, as well as a deeper financial understanding. Even though there are many ways to extend the function $x \mapsto v(x,T)$ outside $I(T)$ to give g, this should not matter since ultimately the integral equation (3.6) for h appropriately "adjusts" the expressions in (3.5).

In particular, in the case of knock-out barrier options, taking g equal to the payoff of the corresponding European option $v_\mathrm{e}(x,T)$, we see that formula (3.5) lends itself to the following interesting financial interpretation: the barrier option price is decomposed into the corresponding European option value v_0 minus a *barrier premium*. A similar result was obtained by a probabilistic approach in [20]. As an example, consider the call payoff $g(x,T) = \max(x - E, 0)$. Then the barrier option price is given by (3.5), where v_0 is the European call pricing function. For comparison, when r, σ, and B are all constant (with $D = 0$, $R = 0$, and $B < E$), the down-and-out call pricing function from [29] is

$$v(x,t) = v_0(x,t) - \left(\frac{x}{B}\right)^{-(2r/\sigma^2 - 1)} v_0\left(\frac{B^2}{x}, t\right).$$

This is of course valid only for $x \in (B, \infty)$ and $t \in [0, T]$.

It is clear therefore that the solution of the single barrier problem (3.1) reduces to the solution of the integral equation in (3.6), which is not tractable in general. To simplify further, suppose that h is linear in x and has time-dependent coefficients, i.e.,

$$h(x,t) = h_0(t) + x h_1(t), \quad x \in \mathbb{R}_+, \quad t \in [0, T).$$

This choice is partly motivated by analytical tractability, but also because in the American option pricing problem [26], whose formulation is related to the formulation of the barrier option pricing problem given here, this form for h is related to the value of the option pricing function in the exercise region.

We have the following consequence (proved in Appendix C) of Theorem 3.1 when h assumes a linear functional form in x:

Corollary 3.3. *Following the same notation as in Theorem 3.1, with v_0 defined as in* (3.5)*, suppose that $h(x,t) = h_0(t) + x h_1(t)$. Then h satisfies*

$$\begin{aligned} v_0(B(t),t) - v(B(t),t) = &\int_t^T \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(\pm z_2\left(\frac{B(t)}{B(u)}, t, u\right)\right) h_0(u)\,\mathrm{d}u \\ &+ B(t)\int_t^T \mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau} N\left(\pm z_1\left(\frac{B(t)}{B(u)}, t, u\right)\right) h_1(u)\,\mathrm{d}u, \end{aligned} \tag{3.7}$$

where the plus and minus signs correspond to the active regions $I(u) = (0, B(u))$ and $I(u) = (B(u), \infty)$, respectively. It follows from (3.5) *that the barrier option pricing function is*

$$\begin{aligned} v(x,t) = v_0(x,t) - &\int_t^T \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(\pm z_2\left(\frac{x}{B(u)}, t, u\right)\right) h_0(u)\,\mathrm{d}u \\ &- x\int_t^T \mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau} N\left(\pm z_1\left(\frac{x}{B(u)}, t, u\right)\right) h_1(u)\,\mathrm{d}u. \end{aligned} \tag{3.8}$$

Remark 3.4. In Corollary 3.3, we observe that (3.7) provides a single condition for the two functions h_0 and h_1; hence there remains an extra degree of freedom. We can consider two special cases:

(a) h does not depend on x;

(b) h depends on x but we impose the continuity condition $h(B(t),t) = 0$ across the boundary.

Case (a). Suppose that h does not depend on x, i.e., $h_1(t) = 0$ for $t \in [0,T)$. Then the barrier option pricing function using (3.8) is given by

$$v(x,t) = v_0(x,t) - \int_t^T h_0(u) \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(\pm z_2\left(\frac{x}{B(u)},t,u\right)\right) \mathrm{d}u, \tag{3.9}$$

where $h_0 : [0,T) \to \mathbb{R}$ satisfies the Volterra integral equation of the first kind

$$v_0(B(t),t) - v(B(t),t) = \int_t^T \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(\pm z_2\left(\frac{B(t)}{B(u)},t,u\right)\right) h_0(u)\,\mathrm{d}u. \tag{3.10}$$

Case (b). On the other hand, suppose that h depends on x but satisfies the continuity condition $h(B(t),t) = 0$ across the boundary. Thus $h_0(t) = -B(t)h_1(t)$ for $t \in [0,T)$ and the barrier option pricing function using (3.8) is expressed as

$$v(x,t) = \begin{cases} v_0(x,t) - \int_t^T h_1(u) v_{\mathrm{call}}(x,t;B(u),u)\,\mathrm{d}u & \text{if } I(t) = (0,B(t)), \\ v_0(x,t) + \int_t^T h_1(u) v_{\mathrm{put}}(x,t;B(u),u)\,\mathrm{d}u & \text{if } I(t) = (B(t),\infty), \end{cases} \tag{3.11}$$

where v_{call} and v_{put} are obtained by replacing E by $B(u)$ and T by u in (2.3) and (2.4), respectively. The function $h_1 : [0,T) \to \mathbb{R}$ satisfies the Volterra integral equation of the first kind

$$v_0(B(t),t) - v(B(t),t) = \begin{cases} \int_t^T h_1(u) v_{\mathrm{call}}(B(t),t;B(u),u)\,\mathrm{d}u & \text{if } I(t) = (0,B(t)), \\ -\int_t^T h_1(u) v_{\mathrm{put}}(B(t),t;B(u),u)\,\mathrm{d}u & \text{if } I(t) = (B(t),\infty). \end{cases} \tag{3.12}$$

4 Numerical approximations and results

Theorem 3.1 suggests some possible numerical approaches to the problem of barrier option pricing. From Remark 3.4, if h is assumed to be linear in x, then the problem reduces to solving Volterra integral equations of the first kind, i.e., either (3.10) or (3.12). Such integral equations have the general form

$$\int_t^T K(t,u)h(u)\,\mathrm{d}u = w(t), \quad t \in [0,T), \tag{4.1}$$

where w and K are given functions and the function h is to be approximated over $[0,T)$. We remark that although (4.1) does not necessarily hold at $t = T$, from the left-hand side we would expect that $\lim_{t\to T^-} w(t) = 0$. For instance, this condition is naturally met in (3.10) and (3.12) for knock-in barrier options when $g(x) = v_e(x,T)$. Standard regularity assumptions on $K(t,u)$ and $w(t)$ guarantee the existence and uniqueness of the solution to (4.1). In cases when the condition is not generally met, no classical solution can exist to both Volterra integral equations (3.10) and (3.12) and we have to look for a solution in distribution spaces.

It is well known that Volterra integral equations of the first kind are ill-posed mathematical problems whose solution is rather unstable, i.e., it is strongly dependent on the data in the sense that slight perturbations of the forcing function w may give rise to arbitrarily large variations in the solution h. Nevertheless, useful and meaningful solutions can be obtained with the aid of suitable stabilizing or regularizing procedures.

Because of their simplicity, direct methods, that is methods based on replacing the integral in (4.1) by a numerical quadrature, are usually preferred. Linz [18] showed that approximations to Volterra integral equations of the first kind can be obtained by certain simple numerical quadrature rules. However, many of the higher order quadrature methods lead to unstable algorithms so that lower order formulas are to be preferred.

Moreover, when considering numerical methods for integral equations, particular attention should be paid to the character of the kernel K, which is usually the main factor governing the choice of an appropriate quadrature formula or system of approximating functions. Assume for simplicity that the functions r, D, σ are constant and $B(t)$ is differentiable and analyze the nature of the integral equation into more details. In Case (a), the kernel

$$K(t,u) = \mathrm{e}^{-r(u-t)} N\left(\frac{(r-D-\sigma^2/2)\sqrt{u-t}}{\sigma}\right)$$

is a smooth function such that $K(t,t) = 1/\sqrt{2\pi} \neq 0$. This, in principle, allows us to convert the integral equation (3.10) by differentiation to one of the second kind and solve for $h(\cdot)$ by successive iterations or by more sophisticated iterative schemes based on Newton's method. For this type of equations, instabilities are appreciably less severe and the numerical solution presents less computational difficulty.

In Case (b), the kernel[1]

$$K(t,u) = v_{\mathrm{call}}(B(t),t;B(u),u)$$

is such that $K(t,t) = 0$. Moreover, suppose that we indicate by $\Delta_{\mathrm{call}}(x,t;E,T)$ and $\Theta_{\mathrm{call}}(x,t;E,T)$ the Delta and Theta at (x,t) of a European call option with strike

[1] Assuming $I(t) = (0, B(t))$; the case $I(t) = (B(t), \infty)$ is similar.

E and expiry T. The first derivative

$$K_t'(t,t) = \lim_{u\to t} \Delta_{\text{call}}(B(t),t;B(u),u)B'(t) + \Theta_{\text{call}}(B(t),t;B(u),u)$$

does not exist. This prevents us from reformulating the integral equation as one of the second kind and complicates the numerical solution of the integral equation (3.12).

Keeping all these considerations in mind, we propose a different numerical procedure and we solve the Volterra integral equation of the first kind (4.1) by a collocation method, paying special attention to the choice of the collocation points.

Consider a uniform decomposition of the time interval $[0,T]$ with time step length $\Delta t = T/M$ and M a positive integer:

$$t_k = k\Delta t, \quad k = 0,1,\ldots,M. \tag{4.2}$$

We choose piecewise continuous basis functions

$$\varphi_k(u) = H(u-t_{k-1}) - H(u-t_k), \quad u \in [0,T], \quad k = 1,\ldots,M, \tag{4.3}$$

where H denotes the usual Heaviside step function, to approximate h by

$$h(u) \simeq \sum_{k=1}^{M} c_k \varphi_k(u), \tag{4.4}$$

where the constants $c_1,\ldots,c_M$ are to be determined. Next we choose as collocation point $t_j^* = t_{j-1}$ on each subinterval $[t_{j-1},t_j]$ for $j = 1,\ldots,M$. Substituting the approximation (4.4) into (4.1) and evaluating it at the collocation points, we obtain the system of equations

$$\int_{t_j^*}^{T} K(t_j^*,u)\left[\sum_{k=1}^{M} c_k\varphi_k(u)\right] \mathrm{d}u = w(t_j^*), \quad j = 1,\ldots,M.$$

We can rewrite this system as

$$\sum_{k=1}^{M} a_{j,k}c_k = b_j, \quad j = 1,\ldots,M,$$

where

$$a_{j,k} = \int_{t_j^*}^{T} K(t_j^*,u)\varphi_k(u)\,\mathrm{d}u, \quad b_j = v(t_j^*), \quad j,k = 1,\ldots,M.$$

The coefficients $a_{j,k}$ can be simplified to

$$a_{j,k} = \int_{\max(t_{k-1},t_j^*)}^{t_k} K(t_j^*,u)\,\mathrm{d}u, \quad j,k = 1,\ldots,M \tag{4.5}$$

and numerically evaluated. Thus we generate a linear system $\mathbf{Ac} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,M} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,M} \\ 0 & 0 & a_{3,3} & \cdots & a_{3,M} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{M,M} \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_M \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} v(t_1^*) \\ \vdots \\ v(t_M^*) \end{pmatrix}.$$

Note that $\varphi_k(u) = 0$ for $u < t_{k-1}$ or $u > t_k$. For $j > k$, $a_{j,k} = 0$ because $t_j^* > t_k$ in (4.5) and therefore $\mathbf{A}$ is an upper triangular matrix.[2]

Once the linear system $\mathbf{Ac} = \mathbf{b}$ has been solved numerically for $c_1,\ldots,c_M$, the approximate solution of the integral equation (4.1) is expressed as in (4.4). This will be $h = h_0$ in (3.10) and $h = h_1$ in (3.12). Then the corresponding h is substituted into (3.9) and (3.11), respectively, to estimate the barrier option price through numerical quadrature.

To illustrate our results, let us consider the particular case of a put option with a single constant up-and-out barrier and no rebate. Thus

$$I(t) = (0,B), \quad g(x) = \max(E-x,0), \quad v(B,t) = 0.$$

We consider the two cases of Remark 3.4:

Case (a). Equations (3.9) and (3.10) simplify to

$$v(x,t) = v_{\mathrm{put}}(x,t;E,T) - \int_t^T h_0(u)\mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(z_2\left(\frac{x}{B},t,u\right)\right)\mathrm{d}u,$$

$$v_{\mathrm{put}}(B,t;E,T) = \int_t^T \mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau} N\left(z_2\left(1,t,u\right)\right) h_0(u)\,\mathrm{d}u.$$

Case (b). Equations (3.11) and (3.12) become

$$v(x,t) = v_{\mathrm{put}}(x,t;E,T) - \int_t^T h_1(u) v_{\mathrm{call}}(x,t;B,u)\,\mathrm{d}u,$$

$$v_{\mathrm{put}}(B,t;E,T) = \int_t^T h_1(u) v_{\mathrm{call}}(B,t;B,u)\,\mathrm{d}u.$$

[2] Moreover, if the kernel K depends only on the difference between its arguments, namely,

$$K(t_j^*,u) = K(|t_j^* - u|),$$

then the matrix entries are equal along the same diagonal (i.e., the matrix is of Toeplitz type) and therefore we can compute only the last column and solve the system simply by backward substitution.

As already remarked above, we would expect that $\lim_{t\to T^-} w(t) = 0$ in (4.1). However, when we choose g to be the payoff function of a European put option fulfilling the condition (3.4), it may be the case that $B < E$ and therefore $v_{\text{put}}(B,T;E,T) \neq 0$. Nevertheless, for the numerical approximation, we never solve equation (4.1) at $t = T$ but refine the mesh in (4.2) and solve (4.1) at a point t_M^* approaching T.

The previous issue suggests that we reconsider the fulfillment of the Volterra equation in a weak sense, and hence we consider a third case:

Case (c). Referring to (3.10), the starting boundary integral equation becomes

$$\int_0^T v_{\text{put}}(B,t;E,T)\,\psi(t)\,\mathrm{d}t = \int_0^T \psi(t)\int_t^T \mathrm{e}^{-\int_t^u r(\tau)\mathrm{d}\tau} N\left(z_2\left(1,t,u\right)\right) h_0(u)\,\mathrm{d}u\,\mathrm{d}t \tag{4.6}$$

for a suitable test function ψ. Then h_0 in (4.6) is approximated by a Galerkin method, e.g., by considering the same piecewise constant functions (4.3) as test functions: for $j = 1,\dots,M$, we have

$$\int_0^T v_{\text{put}}(B,t;E,T)\,\varphi_j(t)\,\mathrm{d}t = \int_0^T \varphi_j(t)\int_t^T \mathrm{e}^{-\int_t^u r(\tau)\mathrm{d}\tau} N\left(z_2\left(1,t,u\right)\right) h_0(u)\,\mathrm{d}u\,\mathrm{d}t.$$

With constant parameters and barrier, the analytical solution is known from [15] to be

$$v(x,t) = \begin{cases} E\mathrm{e}^{-r(T-t)}N(y_1 + (\sigma - 2\lambda\sigma)\sqrt{T-t}) \\ -x\mathrm{e}^{-D(T-t)}N(y_1 - 2\lambda\sigma\sqrt{T-t}) \\ +x\mathrm{e}^{-D(T-t)}(B/x)^{2\lambda}N(-y_1) \\ -E\mathrm{e}^{-r(T-t)}(B/x)^{2\lambda-2}N(-y_1+\sigma\sqrt{T-t}) & \text{if } B \leqslant E, \\ \\ v_{\text{put}}(x,t;E,T) + x\mathrm{e}^{-D(T-t)}(B/x)^{2\lambda}N(-y_2) \\ -E\mathrm{e}^{-r(T-t)}(B/x)^{2\lambda-2}N(-y_2+\sigma\sqrt{T-t}) & \text{if } B > E, \end{cases} \tag{4.7}$$

where $\lambda = (r - D + \sigma^2/2)/\sigma^2$ and

$$y_1 = \frac{\log(B/x)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \quad y_2 = \frac{\log(B^2/(xE))}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}.$$

Setting the option expiry at $T = 1$, the strike price $E = 3$, the interest rate $r = 0.1$, and the volatility $\sigma = 0.2$, we can check the validity of our numerical results at the evaluation time $t = 0$.

In Table 1, we want to observe the efficiency of the three different approaches, namely, Cases (a), (b), and (c), in relation to the position of the barrier with respect

B	Case (a)	Case (b)	Case (c)
4.0	8.4×10^{-10}	1.5×10^{-11}	1.7×10^{-09}
3.1	2.1×10^{-11}	4.6×10^{-09}	2.3×10^{-08}
3.0	5.3×10^{-07}	5.3×10^{-07}	3.5×10^{-06}
2.9	1.4×10^{-05}	1.4×10^{-05}	4.1×10^{-05}
2.0	1.4×10^{-04}	1.4×10^{-04}	6.1×10^{-05}

Table 1: Maximum absolute errors achieved over the interval $[0,B]$ by the three different approaches, for different barrier values above and below the strike price $E=3$.

to the strike price. The listed values represent the maximum errors at equally spaced points $x_i = i\Delta x$ with $\Delta x = 0.1$ and $i = 0,1,\ldots,B/\Delta x$, having considered a uniform time interval decomposition by $M = 2^{10}$ time steps. The approximation gets worse as the barrier diminishes and becomes lower than the strike price. The accuracy of the numerical results is comparable among the three cases but Case (c) requires a greater computational effort due to a further numerical integration that is not compensated by a meaningful gain in accuracy.

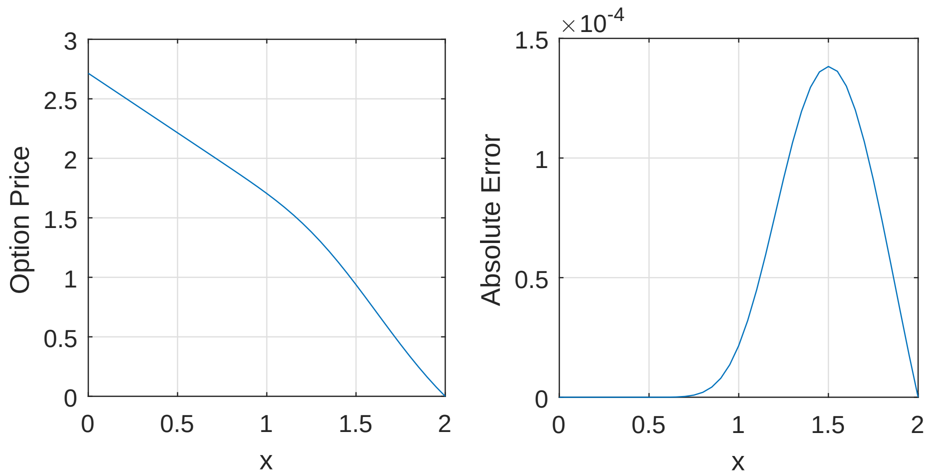

Figure 1: On the left: put up-and-out option value (numerical approximations and exact values overlap). On the right: absolute error produced in Case (a).

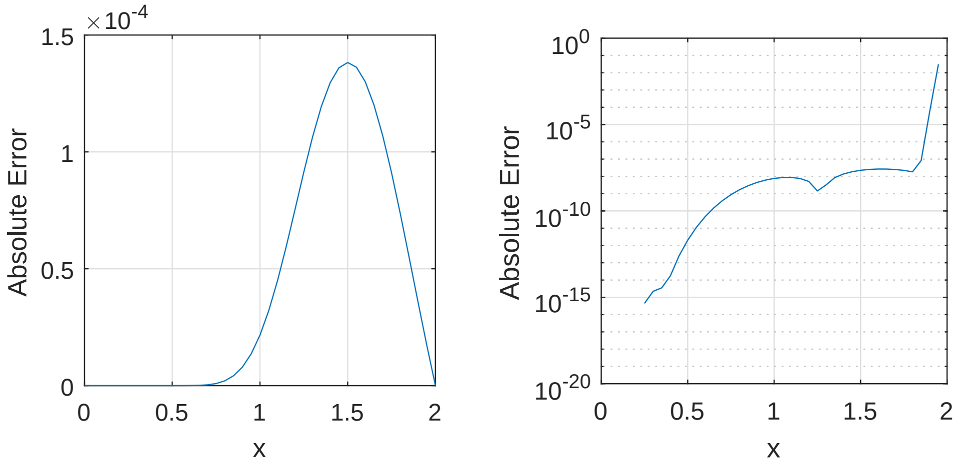

Figure 2: Absolute errors produced for Cases (b) and (c) on the left and on the right, respectively. The plot for Case (c) is displayed in logarithmic scale.

Figures 1 and 2 show the option value and the absolute errors obtained by the three approaches over the interval $[0, B)$ in the worst case when $B = 2$. While the errors achieved by the first two approaches are of the order of 10^{-4} for most of the asset values, in Case (c) the errors are overall smaller except close to the barrier where they become quite large.

In Table 2, giving the barrier the critical value $B = 2$, we want to observe the convergence towards the exact solution refining the time discretization. The listed values represent the maximum errors obtained over the x-grid $x_i = i\Delta x$, $i = 0, 1, \ldots, B/\Delta x$, with a smaller step $\Delta x = 0.05$, having considered a uniform time interval decomposition by an increasing number M of time steps. For all of the three methods the error is reduced as M grows but Case (c) is more affected by end effects: in Cases (a) and (b), halving Δt entails halved errors, while the convergence rate for Case (c) is strongly conditioned by the presence of the barrier. In fact, since the approach for Case (c) concentrates the greatest part of the error close to the barrier as shown in Figure 2, computing the errors at points over a finer x-grid gives worse results because the grid points x_i get closer to the barrier. This explains why the error worsens when refining the x-grid (compare the last row of Table 1 and the third row of Table 2).

As observed and numerically investigated also in [12], the extension to time-dependent parameters and/or barrier is straightforward as well as the application of these numerical approximations to call options and knock-in barriers. Moreover

M	Case (a)	Case (b)	Case (c)
2^8	5.5×10^{-04}	5.5×10^{-04}	1.6×10^{-01}
2^9	2.8×10^{-04}	2.8×10^{-04}	5.4×10^{-02}
2^{10}	1.4×10^{-04}	1.4×10^{-04}	3.1×10^{-02}
2^{11}	6.9×10^{-05}	6.9×10^{-05}	2.6×10^{-03}
2^{12}	3.5×10^{-05}	3.5×10^{-05}	7.6×10^{-04}

Table 2: Maximum absolute errors achieved by the three different approaches for $B = 2$ and $E = 3$, refining the time discretization grid.

the obtained numerical results validates the possibility of directly quantifying the gap between barrier options and vanilla option values (barrier premium).

5 An integral representation formula for the double barrier problem

In this section, we extend the results for the single barrier problem (3.1) to a double barrier problem. Let $a : [0,T] \to \mathbb{R}$ and $b : [0,T] \to \mathbb{R}$ be continuous functions such that $0 \leqslant a(t) < b(t) \leqslant \infty$ for all $t \in [0,T]$. Consider the final-boundary value problem

$$\begin{aligned} &\mathscr{L}v(x,t) = 0, \quad x \in (a(t),b(t)), \quad t \in [0,T), \\ &v(x,T) \text{ given}, \quad x \in (a(T),b(T)), \\ &v(a(t),t), v(b(t),t) \text{ given}, \quad t \in [0,T]. \end{aligned} \tag{5.1}$$

We can embed the first two equations in (5.1) into the nonhomogeneous problem

$$\begin{aligned} &\mathscr{L}v(x,t) = \mathbf{1}_{[0,a(t)]}(x)h^-(x,t) + \mathbf{1}_{[b(t),\infty)}(x)h^+(x,t), \quad x \in \mathbb{R}_+, \quad t \in [0,T), \\ &v(x,T) = g(x), \quad x \in \mathbb{R}_+, \end{aligned} \tag{5.2}$$

where $g(x) = v(x,T)$ for $x \in (a(T),b(T))$ and the functions $(x,t) \mapsto h^-(x,t)$ and $(x,t) \mapsto h^+(x,t)$ are to be determined. Using (2.5), (2.6), the solution of (5.2) for

$x \in (a(t), b(t))$ and $t \in [0, T)$ is

$$
\begin{aligned}
v(x,t) = v_0(x,t) &- \int_t^T \int_0^{a(u)} \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, u\right) h^-(y,u)\, \mathrm{d}y\, \mathrm{d}u \\
&- \int_t^T \int_{b(u)}^{\infty} \frac{1}{y} \mathscr{K}\left(\frac{x}{y}, t, u\right) h^+(y,u)\, \mathrm{d}y\, \mathrm{d}u, \qquad (5.3) \\
v_0(x,t) = \int_0^{\infty} \frac{1}{y} &\mathscr{K}\left(\frac{x}{y}, t, T\right) g(y)\, \mathrm{d}y.
\end{aligned}
$$

Evaluating (5.3) at $x = a(t)$ and $x = b(t)$ and using the third equation in (5.1), we see that h^- and h^+ satisfy the system of linear integral equations

$$
\begin{aligned}
v_0(a(t),t) - v(a(t),t) &= \int_t^T \int_0^{a(u)} \frac{1}{y} \mathscr{K}\left(\frac{a(t)}{y}, t, u\right) h^-(y,u)\, \mathrm{d}y\, \mathrm{d}u \\
&+ \int_t^T \int_{b(u)}^{\infty} \frac{1}{y} \mathscr{K}\left(\frac{a(t)}{y}, t, u\right) h^+(y,u)\, \mathrm{d}y\, \mathrm{d}u, \\
v_0(b(t),t) - v(b(t),t) &= \int_t^T \int_0^{a(u)} \frac{1}{y} \mathscr{K}\left(\frac{b(t)}{y}, t, u\right) h^-(y,u)\, \mathrm{d}y\, \mathrm{d}u \\
&+ \int_t^T \int_{b(u)}^{\infty} \frac{1}{y} \mathscr{K}\left(\frac{b(t)}{y}, t, u\right) h^+(y,u)\, \mathrm{d}y\, \mathrm{d}u.
\end{aligned} \qquad (5.4)
$$

Remark 5.1. The single barrier problem (3.1) can be obtained as a limiting case of the double barrier problem (5.1). More specifically, if $a(t) = B(t)$, $b(t) = \infty$, and $h^+(x,t) = 0$, then we have the single barrier problem when $I(t) = (B(t), \infty)$ if we assume that only $v(a(t),t)$ is given. Moreover, if $a(t) = 0$, $b(t) = B(t)$, and $h^-(x,t) = 0$, then this leads to the single barrier problem when $I(t) = (0, B(t))$ if we assume that only $v(b(t),t)$ is given.

As in the integral equation (3.6) for the single barrier problem, to simplify (5.4) we can make the assumption that h^- and h^+ are linear functions of x, namely, $h^{\pm}(x,t) = h_0^{\pm}(t) + x h_1^{\pm}(t)$, where the four functions $h_0^{\pm}$ and $h_1^{\pm}(t)$ are to be determined. Results analogous to those of Corollary 3.3 and Remark 3.4 can also be obtained in a straightforward manner, leading to a system of coupled Volterra integral equations of the first kind.

6 Concluding remarks

In this article, we used a Mellin transform approach to derive exact integral representations for single and double barrier option prices with time-dependent barriers and parameters. The integral representation formulas are expressed in terms of solutions of Volterra integral equations of the first kind. Numerical approaches were proposed to solve these types of integral equations, and results of simulations yielded excellent results when benchmarked with known exact pricing formulas for special cases.

Mellin transform techniques have been used to price European and American options under standard and jump-diffusion dynamics for the underlying asset [17, 24, 25, 26], as well as the inverse problem of implied volatility estimation [17]. In this article, they were applied to another class of exotic path-dependent options known as barrier options. Hence the Mellin transform can be seen as an emerging useful technique that expands the practitioners' toolbox.

Two future research directions that arise from our results here are (i) proof of well-posedness for single and double barrier problems as formulated here, (ii) application of Mellin transform techniques to other important classes of exotic and/or path dependent options.

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Appendix

A Proof of Theorem 3.1

Proof. Taking $f(x,t) = \mathbf{1}_{\mathbb{R}_+ \setminus I(t)}(x)h(x,t)$, where $\mathbf{1}_A$ is the indicator function of the set A, it is easy to see that (3.5) follows from (2.5), (2.6) as v, restricted to $x \in I(t)$ and $t \in [0,T]$, satisfies the first two equations in (3.1). Evaluating (3.5) at $x = B(t)$ and using the third equation in (3.1) yields (3.6). □

B Proof of Remark 3.2

Proof. Substituting f, g, and G in (3.5), and recalling Remark 2.1, we get

$$\begin{aligned} v(x,t) &= \int_B^\infty G(y,T;x,t)v(y,T)\,\mathrm{d}y - \int_t^T \int_0^\infty G(y,u;x,t)yq(y,u)\delta(y-B)\,\mathrm{d}y\,\mathrm{d}u \\ &= \int_B^\infty G(y,T;x,t)v(y,T)\,\mathrm{d}y - \int_t^T G(B,u;x,t)Bq(B,u)\,\mathrm{d}u. \end{aligned}$$

With the change of variable $z=\log y$, we obtain

$$v(x,t)=\int_{\log B}^{\infty} G(\mathrm{e}^z,T;x,t)v(\mathrm{e}^z,T)\mathrm{e}^z\,\mathrm{d}z-\int_t^T G(B,u;x,t)Bq(B,u)\,\mathrm{d}u.$$

But $\mathrm{e}^z G(\mathrm{e}^z,u;x,t)=p(z,u;\log x,t)$ and $BG(B,u;x,t)=p(\log B,u;\log x,t)$; hence

$$v(x,t)=\int_{\log B}^{\infty} p(z,T;\log x,t)v(\mathrm{e}^z,T)\,\mathrm{d}z-\int_t^T p(\log B,u;\log x,t)q(B,u)\,\mathrm{d}u.$$

□

C Proof of Corollary 3.3

Proof. Using the assumption for h, we see that

$$\begin{aligned}\int_{\mathbb{R}_+\setminus I(u)}\frac{1}{y}\mathscr{K}\left(\frac{B(t)}{y},t,u\right)h(y,u)\,\mathrm{d}y=h_0(u)&\int_{\mathbb{R}_+\setminus I(u)}\frac{1}{y}\mathscr{K}\left(\frac{B(t)}{y},t,u\right)\mathrm{d}y\\&+h_1(u)\int_{\mathbb{R}_+\setminus I(u)}\mathscr{K}\left(\frac{B(t)}{y},t,u\right)\mathrm{d}y.\end{aligned}\tag{C.1}$$

The identities in (2.1) give

$$\int_{\mathbb{R}_+\setminus I(u)}\frac{1}{y}\mathscr{K}\left(\frac{B(t)}{y},t,u\right)\mathrm{d}y=\mathrm{e}^{-\int_t^u r(\tau)\,\mathrm{d}\tau}N\left(\pm z_2\left(\frac{B(t)}{B(u)},t,u\right)\right)$$

and

$$\int_{\mathbb{R}_+\setminus I(u)}\mathscr{K}\left(\frac{B(t)}{y},t,u\right)\mathrm{d}y=B(t)\mathrm{e}^{-\int_t^u D(\tau)\,\mathrm{d}\tau}N\left(\pm z_1\left(\frac{B(t)}{B(u)},t,u\right)\right),$$

where the plus and minus signs correspond to the cases $I(u)=(0,B(u))$ and $I(u)=(B(u),\infty)$, respectively, and we use the property that $N(z)+N(-z)=1$ for all $z\in\mathbb{R}$. Substituting these integrals into (C.1) yields (3.7). □

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