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ON THE COEFFICIENTS OF TYZ EXPANSION OF LOCALLY HERMITIAN SYMMETRIC SPACES

ANDREA LOI AND MICHELA ZEDDA

ABSTRACT. In this paper we address the problem of studying those Kähler manifolds whose first two coefficients of the associated TYZ expansion vanish and we prove that for a locally Hermitian symmetric space this happens only in the flat case. We also prove that there exist nonflat locally Hermitian symmetric spaces where all the odd coefficients vanish.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let M be a n -dimensional complex manifold endowed with a Kähler metric g . Assume that there exists a holomorphic line bundle L over M such that $c_1(L) = [\omega]$, where ω is the Kähler form associated to g and $c_1(L)$ denotes the first Chern class of L (such an L exists if and only if ω is an integral form). Let $m \geq 1$ be a non-negative integer and let h_m be an Hermitian metric on $L^m = L^{\otimes m}$ such that its Ricci curvature $\text{Ric}(h_m) = m\omega$. Here $\text{Ric}(h_m)$ is the two-form on M whose local expression is given by

$$\text{Ric}(h_m) = -\frac{i}{2} \partial \bar{\partial} \log h_m(\sigma(x), \sigma(x)), \quad (1)$$

for a trivializing holomorphic section $\sigma : U \rightarrow L^m \setminus \{0\}$. In the quantum mechanics terminology L^m is called the *quantum line bundle*, the pair (L^m, h_m) is called a *geometric quantization* of the Kähler manifold $(M, m\omega)$ and $\hbar = m^{-1}$ plays the role of Planck's constant (see e.g. [2]). Consider the separable complex Hilbert space \mathcal{H}_m consisting of global holomorphic

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sections s of L^m such that

$$\langle s, s \rangle_m = \int_M h_m(s(x), s(x)) \frac{\omega^n}{n!} < \infty.$$

Define:

$$\epsilon_{mg}(x) = \sum_{j=0}^{d_m} h_m(s_j(x), s_j(x)), \quad (2)$$

where s_j , $j = 0, \dots, d_m$ ($\dim \mathcal{H}_m = d_m + 1 \leq \infty$) is an orthonormal basis of \mathcal{H}_m .

As suggested by the notation this function depends only on the metric mg and not on the orthonormal basis chosen or on the Hermitian metric h_m . Obviously if M is compact $\mathcal{H}_m = H^0(L^m)$, where $H^0(L^m)$ is the (finite dimensional) space of global holomorphic sections of L^m .

In the literature the function ϵ_{mg} was first introduced under the name of η -function by J. Rawnsley in [27], later renamed as θ -function in [5] followed by the *distortion function* of G. R. Kempf [14] and S. Ji [15], for the special case of Abelian varieties and of S. Zhang [32] for complex projective varieties.

In [31] Zelditch proved that if in the above setting M is compact, there exists a complete asymptotic expansion in the C^∞ category of Kempf's distortion function:

$$\epsilon_{mg}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \quad (3)$$

where $a_0(x) = 1$ and $a_j(x)$, $j = 1, \dots$ are smooth functions on M . This means that, for any nonnegative integers r, k the following estimate holds:

$$\|\epsilon_{mg}(x) - \sum_{j=0}^k a_j(x) m^{n-j}\|_{C^r} \leq C_{k,r} m^{n-k-1}, \quad (4)$$

where $C_{k,r}$ are constant depending on k, r and on the Kähler form ω and $\|\cdot\|_{C^r}$ denotes the C^r norm in local coordinates. The expansion (3) is called *Tian–Yau–Zelditch expansion* (TYZ in the sequel). Later on, Z. Lu [24], by means of Tian's peak section method, proved that each of the coefficients $a_j(x)$ is a polynomial of the curvature and its covariant derivatives at x of the metric g which can be found by finitely many algebraic operations. In

particular, he computed the first three coefficients which are given by:

$$\begin{cases} a_1(x) = \frac{1}{2}\rho \\ a_2(x) = \frac{1}{3}\Delta\rho + \frac{1}{24}(|R|^2 - 4|\text{Ric}|^2 + 3\rho^2) \\ a_3(x) = \frac{1}{8}\Delta\Delta\rho + \frac{1}{24}\text{divdiv}(R, \text{Ric}) - \frac{1}{6}\text{divdiv}(\rho\text{Ric}) + \\ \quad + \frac{1}{48}\Delta(|R|^2 - 4|\text{Ric}|^2 + 8\rho^2) + \frac{1}{48}\rho(\rho^2 - 4|\text{Ric}|^2 + |R|^2) + \\ \quad + \frac{1}{24}(\sigma_3(\text{Ric}) - \text{Ric}(R, R) - R(\text{Ric}, \text{Ric})), \end{cases} \quad (5)$$

where ρ , Ric , R , are, respectively, the scalar curvature, the Ricci tensor and the Riemann curvature tensor of (M, g) . The reader is also referred to [17] and [18] (see also formula (12) below) for a recursive formula of the coefficients a_j 's and an alternative computations of a_j for $j \leq 3$ using Calabi's diastasis function (see also [29] for a graph-theoretic interpretation of this recursive formula).

When M is noncompact, there is not a general theorem which assures the existence of an asymptotic expansion (3). Observe that in this case we say that an asymptotic expansion (3) exists if (4) holds for any compact subset $H \subset M$. M. Engliš in [10] showed that a TYZ asymptotic expansion exists in the case of strongly pseudoconvex bounded domain of \mathbb{C}^n with real analytic boundary, and proved that the first three coefficients are the same as those computed by Lu for compact manifolds (5). The reader is also referred to [26] (see also [20] and [23]) for the description of some curvature conditions which assures the existence of a TYZ expansion in the noncompact case. Due to Donaldson's work (cfr. [7], [8] and [3]) in the compact case and respectively to the theory of quantization in the noncompact case (see, e.g. [?, 5, ?, ?, ?, ?, ?]), it is natural to study metrics with the coefficients of the TYZ expansion being prescribed. In this regards Z. Lu and G. Tian [25] (see also [12] and [4] for the symmetric and homogenous case respectively) prove that the PDEs $a_j = f$ ($j \geq 2$ and f smooth function on M) are elliptic and that if the logterm of the Bergman and Szegő kernel of the unit disk bundle over M vanishes then $a_k = 0$, for $k > n$ (n being the complex dimension of M). The study of these PDEs makes sense despite to the existence of a TYZ expansion and so given any Kähler manifold (M, g) it makes sense to call the a_j 's the *coefficients associated to metric g* . In the noncompact case in [23] one can find a characterization of the flat metric as a Taub-Nut metric with $a_3 = 0$ while Feng and Tu [13] solve a conjecture formulated by the second authors in [30] by showing that the complex hyperbolic space is the only Cartan-Hartogs domain where the coefficient a_2 is constant.

In this paper we address the problem of studying those Kähler manifolds (M, g) such that the first two coefficients a_1 and a_2 associated to g vanish (obviously this happens when the metric g is flat, since in this case $a_j = 0$, for all $j \geq 2$). By the first of (5) a metric with $a_1 = 0$ is a scalar flat Kähler (SFK) metric. These metrics have been extensively analysed by several authors and we refer the reader to [16] for the main results about the existence of SFK metrics in the compact case (see also Example 2 below) and to [1] and reference therein for explicit construction of SFK in the noncompact settings. If a Kähler manifold is one dimensional then the condition $a_1 = 0$ is clearly equivalent to flatness while by the second of (5) the condition $a_2 = 0$ alone yields $\Delta\rho = 0$ and, so in the compact one-dimensional case, the scalar curvature is forced to be constant. Observe also that if the metric is not only SFK but also Einstein (and so Ricci flat) then, the condition $a_2 = 0$ implies $|R|^2 \equiv 0$, and so, also in this case, g is flat. On the other hand the following example shows that there exists Kähler metrics with $a_1 = a_2 = 0$ which are not flat.

Example 1. Let M be the blown-up of \mathbb{C}^2 at the origin and denote by E the exceptional divisor. Let (z_1, z_2) be the standard coordinates of \mathbb{C}^2 . In [28] Simanca constructs a SFK complete (not Ricci flat) metric g on M , whose Kähler potential on $M \setminus E = \mathbb{C}^2 \setminus \{0\}$ can be written as

$$\Phi(z) = |z|^2 + \log |z|^2, \quad |z|^2 = |z_1|^2 + |z_2|^2.$$

A straightforward computation gives:

$$|R|^2 = \frac{8}{(1 + |z|^2)^4}, \quad |\text{Ric}|^2 = \frac{2}{(1 + |z|^2)^4}.$$

Thus the coefficients a_1 and a_2 associated to g both vanishes.

The following example deals with the compact case of dimension 2.

Example 2. We start by recalling that the Euler characteristic of a real 4-dimensional compact manifold S is given by:

$$\chi(S) = \int_S [|R|^2 - 4|\text{Ric}|^2 + \rho^2] dS.$$

Therefore it follows by (5) that a necessary topological condition for the existence of Kähler metric on a compact complex surface with $a_1 = a_2 = 0$ is the vanishing of its Euler characteristic. The easiest example of compact surface which admits a SFK is the product $M = \Sigma_g \times \mathbb{C}P^1$ of a Riemann surface Σ_g of genus $g \geq 2$ with a metric of constant scalar curvature -1 and

the one dimensional complex projective space with the metric of constant scalar curvature 1. In this case the coefficient a_2 associated to this metric is different from zero. The Euler characteristic of M is equal to $4 - 4g$. LeBrun [16] proved that if one takes the blown up of M at an even number of points (in a suitable position) one gets a compact complex surface which still admits a SFK metric. Therefore by blowing up M at $4g - 4$ points one gets a compact complex surface with vanishing Euler characteristic and which admits a SFK metric. Unfortunately we were not able to compute a_2 .

By Example 1 it is natural to look for extra conditions, which together with the vanishing of a_1 e a_2 would imply the flatness of the Kähler manifold involved. The following theorem, which represents the first main result of this paper, shows that this is the case for locally Hermitian symmetric space.

Theorem 1. *Let (M, g) be a locally Hermitian symmetric space of complex dimension n . If the coefficients a_1 and a_2 vanish then g is flat.*

Throughout this paper by a *locally Hermitian symmetric space (LHSS* in the sequel) we shall mean a Kähler manifold (M, g) whose universal cover (with the Kähler metric induced by g) is an Hermitian symmetric space. It follows by the classification of Hermitian symmetric spaces that the universal cover of a LHSS is the Kähler product of irreducible Hermitian symmetric spaces of compact type, irreducible Hermitian symmetric spaces of noncompact type (namely bounded symmetric domains with a multiple of the Bergman metric) and the flat Euclidean space. Notice that if (M, ω) is a LHSS then also $(M, \lambda\omega)$ is a LHSS for each real number $\lambda > 0$.

We point out that one can construct examples (either compact or noncompact) of LHSS which are SFK and not flat (take for example the product $\Sigma_g \times \mathbb{C}P^1$ as in Example 2) and of LHSS with $a_2 = 0$ which are not SFK (see Example 3 below). Thus, by combining these observations with Example 1, we deduce that two of the three assumptions ($a_1 = 0$, $a_2 = 0$ and LHSS) in Theorem 1 are not sufficient to deduce the flatness of the metric involved.

The proof of the Theorem 1 is based on the the explicit expressions of the coefficients a_1 and a_2 associated to an irreducible bounded symmetric domain given by M. Engliš in [11] (see formula (7) below) which yields $a_2 - \frac{1}{2}a_1^2 < 0$ (see Lemma 4 below).

It could be interesting to extend our result to locally homogeneous Kähler spaces. Unfortunately we were not able to attack this case due to the lack of

knowledge of the coefficient a_1 and a_2 . Nevertheless we believe our theorem holds true also in this case.

From Theorem 1 it is natural to ask what happens when one imposes conditions on the other coefficients a_j , with $j \geq 3$. In the following theorem which represents our second and last result, we show that there exist nonflat LHSS where all the odd coefficients vanish.

Theorem 2. *Let (Ω, g) be a bounded symmetric domain, $g = \frac{1}{\gamma}g_B$, where g_B is its Bergman metric and γ its genus, and let (Ω^*, g^*) be its compact dual. If a_j (resp. a_j^*) are the coefficients of (Ω, g) (resp. (Ω^*, g^*)) one has:*

$$a_j = (-1)^j a_j^*. \quad (6)$$

Therefore the odd coefficients associated to the metric $g \oplus g^$ on $\Omega \times \Omega^*$ all vanish.*

The paper consists in two more sections. In the first one we describe the Kempf distortion function for irreducible bounded domains, we prove Lemma 4 and Theorem 1. The second one is dedicated to the proof of Theorem 2.

2. THE COEFFICIENTS OF BOUNDED SYMMETRIC DOMAINS AND THE PROOF OF THEOREM 1

Let (Ω, g_B) be an irreducible bounded symmetric domain of \mathbb{C}^d endowed with its Bergman metric g_B . Recall that g_B is the metric whose associated Kähler form is given by $\omega_B = \frac{i}{2} \partial \bar{\partial} \log K(z, z)$, where K is the reproducing kernel of the Hilbert space of holomorphic function on Ω which are L^2 bounded with respect to the Euclidean measure of \mathbb{C}^d .

A bounded symmetric domain (Ω, g_B) is uniquely determined by its rank r and its numerical invariants (a, b) , $a, b \geq 0$. In particular, $\{r, a, b\}$ determine the genus $\gamma = (r-1)a + b + 2$ and the dimension $d = r + \frac{r(r-1)}{2}a + rb$ of Ω . In [11, Sec. 5] M. Engliš computed the coefficients b_j of $(\Omega, \frac{1}{\gamma}g_B)$ up to the third. In particular the first two read:

$$\begin{cases} a_1 = -\frac{1}{2}\gamma d, \\ a_2 = \frac{1}{8}\gamma^2 d^2 - \frac{1}{6}\gamma^2 d + \frac{1}{24}q, \end{cases} \quad (7)$$

where

$$q = -\frac{r-1}{2}da^2 + \frac{r-1}{r}ad(d+r) + \frac{2d(d+r)}{r}.$$

Notice that the Kempf distortion function of an irreducible bounded symmetric domain $(\Omega, \frac{1}{\gamma}g_B)$ of rank r and numerical invariants a, b is a polynomial in m of degree d which reads as (see e.g. [11, Sec. 5]) :

$$\epsilon_m = \prod_{j=1}^r \frac{\Gamma(m - (j-1)\frac{a}{2})}{\Gamma(m - \frac{d}{r} - (j-1)\frac{a}{2})}.$$

Remark 3. Notice that in this case the TYZ asymptotic expansion (3) being a polynomial is finite and all the coefficients a_j associated to the metric $g = \frac{1}{\gamma}g_B$ are constants.

In the following lemma we prove an inequality needed in the proof of the main theorem.

Lemma 4. *Let $(\Omega, g = \frac{1}{\gamma}g_B)$ be an irreducible bounded symmetric domain. Then the coefficients a_1 and a_2 associated to g satisfies the following inequality:*

$$a_2 - \frac{1}{2}a_1^2 < 0. \quad (8)$$

Proof. By (7) we need to show that $\frac{1}{4}q < \gamma^2 d$, namely

$$-\frac{r-1}{2}a^2 + \frac{r-1}{r}a(d+r) + \frac{2(d+r)}{r} < 4\gamma^2.$$

Substituting $\gamma = (r-1)a + b + 2$, $d = r(r-1)\frac{a}{2} + rb + r$ we get:

$$\frac{1}{2}a^2(r-1)(r-2) < 4(r-1)^2a^2 + 4b^2 + 12 + 7(r-1)ab + 13(r-1)a + 14b,$$

which holds true since $a, b \geq 0$, $r \geq 1$ and $\frac{1}{2}(r-1)(r-2) < 4(r-1)^2$. \square

Lemma 5. *Let (M_1, g_1) and (M_2, g_2) be two Kähler manifolds and consider their product $(M_1 \times M_2, g_1 \oplus g_2)$. Denote by a_1, a_2, b_1, b_2 and c_1, c_2 the coefficients associated to the metrics g_1, g_2 and $g_1 \oplus g_2$ respectively. Then:*

$$c_1 = a_1 + b_1, \quad (9)$$

$$c_2 = a_2 + b_2 + a_1b_1. \quad (10)$$

Proof. The expression for c_1 follows by the first of (5) and by the fact that the scalar curvature of the $g_1 \oplus g_2$ is the sum of those of g_1 and g_2 . Equation (10) is more subtle and can be obtained as follows. For $j = 1, 2$ denote by Δ_j, R_j, Ric_j and ρ_j the Laplacian, the Riemannian tensor, the Ricci tensor and the scalar curvature of g_j , and by Δ, R, Ric and ρ those of $g_1 \oplus g_2$. Observe that by:

$$g_1 \oplus g_2 = \left(\begin{array}{c|c} g_1 & 0 \\ \hline 0 & g_2 \end{array} \right),$$

we get:

$$(g_1 \oplus g_2)^{-1} = \left(\begin{array}{c|c} g_1^{-1} & 0 \\ \hline 0 & g_2^{-1} \end{array} \right),$$

from which follows easily:

$$\begin{aligned} \Delta\rho &= \Delta_1\rho_1 + \Delta_2\rho_2, \\ |R|^2 &= \sum_{j,k,l,m,p,q,s,t} g^{\bar{j}k} g^{\bar{l}m} g^{\bar{p}q} g^{\bar{s}t} R_{k\bar{l}q\bar{s}} \overline{R_{j\bar{m}p\bar{t}}} = |R_1|^2 + |R_2|^2, \\ |\text{Ric}|^2 &= \sum_{j,k,l,m} g^{\bar{j}k} g^{\bar{l}m} \text{Ric}_{k\bar{l}} \overline{\text{Ric}_{j\bar{m}}} = |\text{Ric}_1|^2 + |\text{Ric}_2|^2. \end{aligned}$$

Thus, by the second of (5) we get:

$$c_2 = \Delta_1\rho_1 + \Delta_2\rho_2 + \frac{1}{24} (|R_1|^2 + |R_2|^2 - 4|\text{Ric}_1|^2 - 4|\text{Ric}_2|^2 + 3(\rho_1 + \rho_2)^2),$$

i.e. $c_2 = a_2 + b_2 + a_1 b_1$, as wished. \square

Example 3. Notice that there exist LHSS with vanishing a_2 which are not flat. Consider for example $(\mathbb{C} \times \mathbb{C}P^1, g_0 \oplus g_{FS})$ and denote by $c_1, c_2, a_1, a_2, b_1, b_2$ the coefficients associated to $g_0 \oplus g_{FS}, g_0$ and to g_{FS} respectively. Since $a_0 = 1, a_1 = a_2 = 0, b_0 = b_1 = 1, b_2 = 0$ by Lemma 5 one gets

$$c_0 = c_1 = 1, \quad c_2 = 0.$$

For a compact example just take $(\mathbb{T} \times \mathbb{C}P^1, g_0 \oplus g_{FS})$, where (\mathbb{T}, g_0) is the flat one-dimensional torus.

In order to prove Theorem 1 we notice that to any irreducible bounded symmetric domain $(\Omega, g = \frac{1}{\gamma} g_B)$ one can associate its compact dual (Ω^*, g^*) where g^* is the pull-back of Fubini–Study metric of $\mathbb{C}P^N$ via the Borel–Weil embedding $\Omega^* \rightarrow \mathbb{C}P^N$. In the affine chart $\Omega^* \setminus \text{Cut}_p(\Omega^*)$, where $\text{Cut}_p(\Omega^*)$ is the cut locus of the point $p \in \Omega^*$ w.r.t. g^* , one has:

$$\omega^* = -\frac{i}{2\gamma} \partial\bar{\partial} \log K(z, -\bar{z}). \quad (11)$$

(where $\omega_B = \frac{i}{2} \partial\bar{\partial} \log K(z, z)$). The reader is referred to [?, ?] for details and further results.

Proof of Theorem 1. Since the coefficients a_1 and a_2 are invariants by local isometry, without loss of generality we can assume that (M, g) is simply-connected and so, by the classification theorem for Hermitian symmetric spaces it can be written as:

$$(\Omega_1^* \times \cdots \times \Omega_s^* \times \Omega_{s+1} \times \cdots \times \Omega_k \times \mathbb{C}^m, \lambda_1 g_1^* \oplus \cdots \oplus \lambda_s g_s^* \oplus \lambda_{s+1} g_{s+1} \oplus \lambda_k g_k \oplus g_0),$$

where we denote by g_0 the flat metric on \mathbb{C}^m , $(\Omega_h^*, \lambda_h g_h^*)$, $h = 1, \dots, s$ are irreducible hermitian symmetric spaces of compact type ((Ω_h^*, g_h^*) is the compact dual of (Ω_h, g_h) , $h = 1, \dots, s$), (Ω_l, g_l) , $l = s+1, \dots, k$, are irreducible bounded symmetric domains and λ_h, λ_l are positive constants. Denote by c_1 and c_2 the coefficients associated to g , by $a_{1,h}^*, a_{2,h}^*$ the coefficients associated to g_h^* , $h = 1, \dots, s$, and by $a_{1,l}, a_{2,l}$ the coefficients associated to g_l , $l = s+1, \dots, k$. By a direct computation using (5) (or Corollary 8 below) one obtains that $a_{1,h}^* = -a_{1,h}$ and $a_{2,h}^* = a_{2,h}$, where $a_{1,h}$ and $a_{2,h}$ are the coefficients associated to the metric g_h , $h = 1, \dots, s$.

Then, by Lemma 5, one gets:

$$c_1 = \sum_{h=1}^s \frac{a_{1,h}^*}{\lambda_h} + \sum_{l=s+1}^k \frac{a_{1,l}}{\lambda_l} = -\sum_{h=1}^s \frac{a_{1,h}}{\lambda_h} + \sum_{l=s+1}^k \frac{a_{1,l}}{\lambda_l},$$

$$c_2 = \sum_{h=1}^s \frac{a_{2,h}^*}{\lambda_h^2} + \sum_{l=s+1}^k \frac{a_{2,l}}{\lambda_l^2} + \sum_{h,l} \frac{a_{1,h}^* a_{1,l}}{\lambda_h \lambda_l} = \sum_{u=1}^k \frac{a_{2,u}}{\lambda_u^2} + \frac{1}{2} \left[c_1^2 - \sum_{u=1}^k \frac{a_{1,u}^2}{\lambda_u^2} \right].$$

Since, by assumption $c_1 = 0$, one gets:

$$c_2 = \sum_{u=1}^k \frac{1}{\lambda_u^2} \left[a_{2,u} - \frac{1}{2} a_{1,u}^2 \right].$$

By (8) in Lemma 4 $a_{2,u} - \frac{1}{2} a_{1,u}^2 < 0$ for all u . Thus $c_2 = 0$ forces $k = 0$, i.e. $(M, \omega) = (\mathbb{C}^m, g_0)$. \square

3. THE PROOF OF THEOREM 2

A key ingredient in the proof of Theorem 2 is the following lemma which provide us with a recursive formula for the computation of the coefficients a_j associated to a Kähler manifold (M, g) (we refer the reader to [18] for details).

Lemma 6 (A. Loi [18]). *Let (M, g) be a Kähler manifold and let $a_j(x)$, $j = 0, 1, \dots$ be the coefficients associated to g and denote by $a_j(x, y)$ their almost analytic extension in a neighborhood U of the diagonal of $M \times M$. Then*

$$a_k(x) = c_k + \tilde{a}_k(x, x) + \sum_{r+j=k, r \geq 1, j \geq 1} C_r(\tilde{a}_j(x, y))|_{y=x}, \quad (12)$$

where $a_0 \equiv 1$ and for all $j = 1, 2, \dots$:

$$\tilde{a}_j(x, y) = \sum_{\alpha=0}^j a_\alpha(x, y) a_{j-\alpha}(y, x);$$

$$c_r(x) = C_r(1)(x); \quad (13)$$

$$C_r(f)(x) = \sum_{k=r}^{3r} \frac{1}{k!(k-r)!} L^k(f \det(g_{i\bar{j}}) S^{k-r})|_{y=x}; \quad (14)$$

where if we denote by $g^{\bar{j}k}$ the entries of the inverse matrix of the metric g , L^k is the operator defined by:

$$L^k \varphi = \sum_{j_1, \dots, j_k, i_1, \dots, i_k} g^{\bar{i}_1 j_1} \dots g^{\bar{i}_k j_k} \varphi_{j_1 \dots j_k \bar{i}_1 \dots \bar{i}_k}, \quad (15)$$

and

$$S_x(y) = -D_x(y) + \sum_{ij=1}^n g_{i\bar{j}}(y)(y_i - x_i)(\bar{y}_j - \bar{x}_j), \quad (16)$$

where $D_x(y)$ is the diastasis function centered at x .

Observe that the diastasis function centered at the origin for bounded symmetric domains (Ω, g) is given by:

$$D_0(z, \bar{z}) = \frac{1}{\gamma} \log(V(\Omega)K(z, z)), \quad (17)$$

where $V(\Omega)$ denotes the total volume of Ω with respect to the Euclidean measure of the ambient complex Euclidean space (see [21, Prop. 7]). Further, by the discussion above, the diastasis function centered at the origin for the compact dual (Ω^*, g_B^*) reads:

$$D_0^*(z, \bar{z}) = \frac{1}{\gamma} \log(V(\Omega)K(z, -z)). \quad (18)$$

We are now in the position of proving Theorem 2.

Proof of Theorem 2. Let (L, h) be a geometric quantization of (Ω^*, ω^*) , where ω^* is the integral Kähler form given by (11). Then it is not hard to see that for all positive integer m Kempf's distortion function ϵ_{mg} is defined and (by Riemann–Roch theorem) is a monic polynomial in m of degree n , namely:

$$\epsilon_{mg} = \sum_{j=0}^n a_j^* m^{n-j}, \quad a_0^* = 1, \quad (19)$$

Hence, as for the case of bounded symmetric domains, the coefficients a_j^* are constant. We start by proving (6) namely $a_j = (-1)^j a_j^*$, $j = 1, 2, \dots$

By (12), since a_k is constant for all k , we get:

$$a_k = a_k(0) = c_k(0) + \tilde{a}_k + \sum_{r+j=k, r \geq 1, j \geq 1} \tilde{a}_j c_r(0).$$

Observe that for $K = k_1 + \cdots + k_d$ and $J = j_1 + \cdots + j_d$, one has:

$$\frac{\partial^{K+J}}{\partial z_1^{k_1} \cdots \partial z^{k_d} \partial \bar{z}_1^{j_1} \cdots \partial \bar{z}_d^{j_d}} \det g|_0 = 0,$$

whenever $K \neq J$, as it follows by Prop. 7 in [21] once noticed that since g_B is a Kähler–Einstein metric, we have $\det(g) = \frac{1}{\gamma^d} e^{D_0(z, \bar{z})}$, where $D_0(z, \bar{z})$ is the diastasis function of (Ω, g_B) given in (17). Further it is easy to verify that $S_0(z, \bar{z})$, which by definition (16) reads:

$$S_0(z, \bar{z}) = -D_0(z, \bar{z}) + \sum_{ij=1}^d g_{i\bar{j}}(z, \bar{z}) z_i \bar{z}_j,$$

where d is the dimension of Ω , satisfies:

$$S_0(0) = 0, \quad \frac{\partial^{K+J}}{\partial z_1^{k_1} \cdots \partial z^{k_d} \partial \bar{z}_1^{j_1} \cdots \partial \bar{z}_d^{j_d}} S_0(z, \bar{z})|_0 = 0,$$

whenever $K = J$. Thus by (13) and (14), $c_r \neq 0$ iff $k - r = 0$ and we get:

$$c_r(0) = \frac{(-1)^r}{r!} L^r(\det(g)(y))|_{y=0}.$$

Let $D_0^*(z, \bar{z})$ be the diastasis function around the origin of g^* given by (18) and denote by S_0^* and c_r^* the operator in (16) and the coefficient (13) associated to g^* . By (11) one gets:

$$S_0^*(z, \bar{z}) = -D_0(z, -\bar{z}) - \sum_{ij=1}^d g_{i\bar{j}}(z, -\bar{z}) z_i \bar{z}_j,$$

and thus also S_0^* satisfies:

$$S_0^*(0) = 0, \quad \frac{\partial^{K+J}}{\partial z_1^{k_1} \cdots \partial z^{k_d} \partial \bar{z}_1^{j_1} \cdots \partial \bar{z}_d^{j_d}} S_0^*(z, \bar{z})|_0 = 0,$$

whenever $K = J$. Further, it follows by the Einstein equation that:

$$\det g^*(z, \bar{z}) = e^{-D_0^*(z, \bar{z})},$$

thus, by (11) we have:

$$\det(g)(z, \bar{z}) = \det(g^*)(z, -\bar{z}), \quad (20)$$

and we get:

$$\frac{\partial^{K+J} \det g^*(z, \bar{z})}{\partial z_1^{k_1} \cdots \partial z^{k_d} \partial \bar{z}_1^{j_1} \cdots \partial \bar{z}_d^{j_d}}|_0 = (-1)^J \frac{\partial^{K+J} \det g(z, -\bar{z})}{\partial z_1^{k_1} \cdots \partial z^{k_d} \partial (-\bar{z}_1)^{j_1} \cdots \partial (-\bar{z}_d)^{j_d}}|_0 = 0,$$

whenever $K \neq J$. Thus, also c_0^* satisfies:

$$c_r^*(0) = \frac{(-1)^r}{r!} L_*^r(\det(g)(y))|_{y=0},$$

where L_* denotes the operator in (15) for g^* , and by noticing that for $K = J = r$ from (20) follows:

$$\frac{\partial^{2r} \det g(z, \bar{z})}{\partial z_1^{k_1} \dots \partial z^{k_d} \partial \bar{z}_1^{j_1} \dots \partial \bar{z}_d^{j_d}} = (-1)^r \frac{\partial^{2r} \det g^*(z, -\bar{z})}{\partial z_1^{k_1} \dots \partial z^{k_d} \partial (-\bar{z}_1)^{j_1} \dots \partial (-\bar{z}_d)^{j_d}},$$

we get:

$$c_r(0) = (-1)^r c_r^*(0).$$

Thus by inductive hypothesis:

$$a_k = (-1)^k c_k^*(0) + \tilde{a}_k + \sum_{r+j=k, r \geq 1, j \geq 1} (-1)^{r+j} \tilde{a}_j^* c_r^*(0),$$

implies:

$$-a_k = (-1)^k c_k^*(0) + \sum_{\alpha, \beta} (-1)^k a_\alpha^* a_\beta^* + \sum_{r+j=k, r \geq 1, j \geq 1} (-1)^{r+j} \tilde{a}_j^* c_r^*(0).$$

Therefore formula (6), follows by

$$-a_k^* = c_k^*(0) + \sum_{\alpha, \beta} a_\alpha^* a_\beta^* + \sum_{r+j=k, r \geq 1, j \geq 1} \tilde{a}_j^* c_r^*(0).$$

The last part of the theorem, namely the vanishing of the odd coefficients associated to the metric $g \oplus g^*$, is a consequence of $a_j = (-1)^j a_j^*$ and of the following lemma and its corollary. \square

Lemma 7. *Let $(M_1, g_1), (M_2, g_2)$ be two Kähler manifolds with ω_1 and ω_2 integral. Then the Kempf distortion function $\epsilon_{1,2}$ of $(M_1 \times M_2, \omega_1 \oplus \omega_2)$ is given by:*

$$\epsilon_{1,2}(x, y) = \epsilon_1(x) \epsilon_2(y),$$

where ϵ_1 (resp. ϵ_2) is the Kempf distortion function associated to (M_1, g_1) (resp. (M_2, g_2)).

Proof. For $\alpha = 1, 2$ let (L_α, h_α) be the Hermitian line bundle over M_α such that $\text{Ric}(h_\alpha) = \omega_\alpha$ (cfr. (1) in the introduction) and let $(L_{1,2}, h_{1,2})$ be the holomorphic hermitian line bundle over $M_1 \times M_2$ such that $\text{Ric}(h_{1,2}) = \omega_1 \oplus \omega_2$. Let

$$\mathcal{H}_\alpha = \left\{ s \in H^0(L_\alpha) \mid \int_{M_\alpha} h_\alpha(s, s) \frac{\omega_\alpha^{n_\alpha}}{n_\alpha!} < \infty \right\},$$

where n_α is the complex dimension of M_α (if M_α is compact then $\mathcal{H}_\alpha \equiv H^0(L_\alpha)$). Let $\{s_j^1\}$ (resp. $\{s_k^2\}$) be a orthonormal basis for \mathcal{H}_1 (resp. \mathcal{H}_2) with respect to the L^2 -product induced by h_1 (resp. h_2). Fix a local trivialization $\sigma_\alpha : U_\alpha \rightarrow L_\alpha$ of L_α ($\alpha = 1, 2$) on an open and dense subset $U_\alpha \subset M_\alpha$ and let $\sigma : U_1 \times U_2 \rightarrow L_{1,2}$ be the local trivialization of

$L_{1,2}$ given by $\sigma_{1,2}(x, y) = \sigma_1(x)\sigma_2(y)$. Let $s_{j,k}$ be the global holomorphic sections of $L_{1,2}$ which in the trivialization $\sigma_{1,2}$ are given by: $s_{j,k}(x, y) = f_j^1(x)f_k^2(y)\sigma_{1,2}(x, y)$, where $s_j^1(x) = f_j^1(x)\sigma_1(x)$ and $s_k^2(y) = f_k^2(y)\sigma_2(y)$.

We claim that $s_{j,k}$ is an orthonormal basis for the Hilbert space:

$$\begin{aligned} \mathcal{H}_{1,2} &= \left\{ s \in H^0(L_{1,2}) \mid \int_{M_1 \times M_2} h_{1,2}(s, s) \frac{(\omega_1 \oplus \omega_2)^{n_1+n_2}}{(n_1+n_2)!} < \infty \right\} \\ &= \left\{ s \in H^0(L_{1,2}) \mid \int_{M_1 \times M_2} h_{1,2}(s, s) \frac{\omega_1^{n_1} \wedge \omega_2^{n_2}}{n_1!n_2!} < \infty \right\}. \end{aligned}$$

Indeed, in the the trivialization σ_α and $\sigma_{1,2}$ the Hermitian product h_α and $h_{1,2}$ are given respectively by $h_\alpha(\sigma_\alpha(x), \sigma_\alpha(x)) = e^{-\Phi_\alpha(x)}$ and $h_{1,2}(x, y) = e^{-(\Phi_1(x)+\Phi_2(y))}$, where $\Phi_\alpha : U_\alpha \rightarrow \mathbb{R}$, $\alpha = 1, 2$, is a Kähler potential for ω_α ($\omega_\alpha = \frac{i}{2}\partial\bar{\partial}\log\Phi_\alpha$). Thus

$$\begin{aligned} \int_{M_1 \times M_2} h_{1,2}(s_{j,k}, s_{l,m}) \frac{\omega_1^{n_1} \wedge \omega_2^{n_2}}{n_1!n_2!} &= \int_{M_1 \times M_2} e^{-(\Phi_1+\Phi_2)} f_j^1 f_k^2 \overline{f_l^1 f_m^2} \frac{\omega_1^{n_1} \wedge \omega_2^{n_2}}{n_1!n_2!} \\ &= \int_{M_1} h_1(s_j^1, s_l^1) \frac{\omega_1^{n_1}}{n_1!} \int_{M_2} h_2(s_k^2, s_m^2) \frac{\omega_2^{n_2}}{n_2!} = \delta_{jl}\delta_{km}, \end{aligned}$$

and the claim is proved. Hence

$$\epsilon_{1,2} = \sum_{j,k} h_{1,2}(s_{j,k}, s_{j,k}) = \sum_j h_1(s_j^1, s_j^1) \sum_k h_2(s_k^2, s_k^2) = \epsilon_1 \epsilon_2.$$

□

Corollary 8. *Let (M_1, g_1) , (M_2, g_2) be as in the lemma. Assume that for m sufficiently large there exist the TYZ expansions*

$$\epsilon_{1,m}(x) \sim \sum_{j=0}^{\infty} a_j(x) m^{n-j}, \quad \epsilon_{2,m}(y) \sim \sum_{j=0}^{\infty} b_j(y) m^{n-j},$$

$$\epsilon_{1,2,m}(x, y) \sim \sum_{j=0}^{\infty} c_j(x, y) m^{n-j},$$

of the Kempf distortion functions $\epsilon_{1,m}(x)$, $\epsilon_{2,m}(y)$ and $\epsilon_{1,2,m}(x, y)$ of (M_1, g_1) , (M_2, g_2) and $(M_1 \times M_2, g_1 \oplus g_2)$ respectively. Then

$$c_j(x, y) = \sum_{\alpha+\beta=j} a_\alpha(x) b_\beta(y).$$

Remark 9. We believe that the conclusion of the previous corollary holds true for the product of two Kähler manifolds without any assumption on the existence of a TYZ expansion.

Remark 10. It is worth pointing out that Kempf's function of a LHSS does not always exist. For instance, let $(\mathbb{C}P^1 \times \mathbb{C}P^1, \sqrt{2}g_{FS} \oplus g_{FS})$. Then the corresponding Kähler form $\lambda(\sqrt{2}\omega_{FS} \oplus \omega_{FS})$ is not integral for any $\lambda \in \mathbb{R}$. The integrality of the Kähler form is a necessary condition for Kempf's distortion function to exist, although it is not sufficient, as one can see for example when $(\Omega, \beta g_B)$ is a bounded symmetric domain endowed with its Bergman metric g_B , since βg_B is balanced iff $\beta > \frac{\gamma-1}{\gamma}$ (see [22, Th. 1]).

Notice also that when the Kempf's distortion function does exist, the rest in (4) is zero (i.e. the Kempf's distortion function is a polynomial in m) if and only if the LHSS M is simply connected. Indeed, when Kempf's distortion function is a polynomial in m , since its coefficients are constant it also is and in particular there exists an isometric and holomorphic immersion $f: M \rightarrow \mathbb{C}P^N$ for some $N \leq \infty$ (see [22] for details). Consider now the universal covering $\pi: \tilde{M} \rightarrow M$. Since \tilde{M} is a Hermitian symmetric space, there exists an *injective*, isometric and holomorphic immersion $F: \tilde{M} \rightarrow \mathbb{C}P^N$ into $\mathbb{C}P^N$ for some $N \leq \infty$ (see [19, Lemma 2.1] for the injectivity of F). On the other hand, the composition $f \circ \pi$, is also an isometric and holomorphic Kähler immersion of \tilde{M} into $\mathbb{C}P^N$, which is *not injective* unless M is simply connected. A contradiction then follows from Calabi rigidity's Theorem [6]. Viceversa, if M is simply connected and noncompact, then it is a polynomial (see e.g. [9, Ex. 2.14 p. 431] and references therein), while if it is simply connected and compact, then Kempf's distortion is constant and by Riemann–Roch theorem it is a polynomial.

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