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Highlights

We find a new sufficient condition for precautionary saving under two risks

We provide three economic interpretations for this condition

An interpretation focuses on the marginal effect of saving on total income variance

An interpretation focuses on the covariance between total income and saving returns

An interpretation focuses on the effect of saving on the utility premium

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New results on precautionary saving under two risks

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Abstract

This paper identifies a new sufficient condition for a prudent agent to have positive precautionary saving in the presence of labor income and interest rate risks of any size. We also provide three economic interpretations for this condition focusing respectively on the marginal effect of saving on total income variance, on the sign of the covariance between total income and the return of saving, and on the effect of saving on the utility premium.

Key words and phrases: Precautionary saving, interest rate risk, labor income risk, prudence, utility premium.

JEL Classification: D81, E21, D11

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1 Introduction

The effects of uncertainty on consumption and saving choices are the focus of precautionary saving theory. Leland (1968), Sandmo (1970) and Drèze & Modigliani (1972) made the first studies of the influence on saving of uncertainty, showing that the presence of labor income risk generates an extra–saving (called 'precautionary saving') when the third derivative of the utility function is positive. Kimball (1990) called this feature of agent preferences 'prudence'.[1]

Another strand of literature, originating in the seminal papers by Sandmo (1970) and Rothschild & Stiglitz (1971), investigates saving decisions when the interest rate (i.e. the return on saving) is uncertain. This literature shows that the existence of precautionary saving relies on the magnitude of the relative prudence index which must be higher than a threshold of 2.[2]

Recent papers have also considered the two types of risk jointly. In particular, Li (2012) finds that, under the hypothesis of positive quadrant dependent labor income risk and interest rate risk, the sufficient conditions for precautionary saving require the partial relative prudence index to be larger than 2. Baiardi et al. (2014) study the same problem removing the hypothesis of positive quadrant dependence but considering the case of small risks. They provide different sets of sufficient conditions for precautionary saving which include different threshold levels for partial relative prudence depending on the joint distribution of the two random variables representing the two risks.

These two papers derive sufficient conditions for a prudent agent to have positive precautionary saving, which require conditions on the joint distribution of the two risks, together with further conditions on agent preferences or on risk size.

In this work we re-examine precautionary saving in the presence of labor income and interest rate risks deriving a sufficient condition involving only the joint distribution of the two risks without requiring further conditions on agent's preferences or on risk size. We then provide three economic interpretations for this condition.

[1] See also Menegatti (2001).
[2] See also Eeckhoudt et al. (2005), p. 98.

The paper proceeds as follows. Section 2 introduces the model. Section 3 derives the results. Section 4 presents the interpretations. Section 5 concludes.

2 The model

We consider a two-period general framework. The consumer has a Von Neumann–Morgenstern utility function $u(x)$ in period 0 and $v(x)$ in period 1, where $x \in \mathbb{R}$. Functions u and v are assumed to be strictly increasing, strictly concave and three times continuously differentiable. The assumption of strict concavity denotes risk aversion. Functions u_1, u_{11}, u_{111} (respectively v_1, v_{11}, v_{111}) denote the first, second and third derivatives of u (respectively v). The subjective intertemporal discount rate is embedded in function $v(x)$.

In order to study precautionary saving we consider two different situations. In the first circumstance both labor income and the return on saving are random. The second circumstance is the simplified case where the consumer faces no uncertainty. We then compare the consumer's optimal choice in the presence of uncertainty with the optimal choice the consumer would make if labor income and the return on saving were not random.

We start examining the case with uncertainty. We denote by s the level of saving, which is our choice variable, and by $y_0 > 0$ first–period labor income. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the uncertain future level of labor income as $\tilde{y} \geq 0$, where $\mathbb{E}[\tilde{y}] = y_1 > 0$. We define further the uncertain level of the return on saving as $\tilde{R} \geq 0$ where $\mathbb{E}[\tilde{R}] = R > 0$. Therefore the consumer decision problem is:

$$\max_s u(y_0 - s) + \mathbb{E}[v(\tilde{y} + \tilde{R}s)]. \tag{1}$$

In the presence of two sources of uncertainty (i.e. income and interest rate risks), the optimal level of saving $s^{\star\star}$ is thus defined by the following first-order condition:

$$u_1(y_0 - s^{\star\star}) = \mathbb{E}[\tilde{R}v_1(\tilde{y} + \tilde{R}s^{\star\star})]. \tag{2}$$

If there were no uncertainty, both the certain levels of labor income and of the return on saving would coincide with their expected values y_1 and R.[3] Hence the consumer would face the following problem:

$$\max_s u(y_0 - s) + v(y_1 + Rs) \tag{3}$$

[3]This assumption is standard in the literature (see the standard books by Gollier, 2001 and Eeckhoudt et al., 2005 or the articles by Li, 2012 and Baiardi et al., 2014) and permits comparison between the pair of random variables $\tilde{y}$ and $\tilde{R}$, having positive variances, with the pair of degenerate random variables y_1 and R, having the same expected values but null variances. The difference between these two pairs involves only variances having either positive or null values, so the effect of variability in labor income and return on saving, i.e. the effect of uncertainty, on consumer choice is clearly identified.

In this case, the optimal level of saving $s^\star$ is defined by the first-order condition:

$$u_1\left(y_0 - s^\star\right) = Rv_1\left(y_1 + Rs^\star\right). \tag{4}$$

We can now define precautionary saving:

Definition 1. *Precautionary saving is the difference $s^{\star\star} - s^\star$, between the level of saving the consumer chooses when she faces random labor income and random return on saving and the level of saving she chooses when she faces no uncertanty.*

By comparing Conditions (2) and (4) we obtain, by second-order conditions, that there is a positive precautionary saving ($s^{\star\star} - s^\star \geq 0$) if and only if:

$$\mathbb{E}[\tilde{R}v_1(\tilde{y} + \tilde{R}s^{\star\star})] - Rv_1(y_1 + Rs^{\star\star}) \geq 0. \tag{5}$$

Condition (5) is standard in the analysis of precautionary saving in the presence of uncertain labor income and uncertain return on saving (see Li, 2012 and Baiardi et al., 2014). Starting from this, we derive a new sufficient condition for precautionary saving, removing some of the requirements introduced in the previous literature.

3 Results

Starting from Inequality (5), we obtain:

Proposition 1. *A prudent agent ($v_{111} > 0$) has positive precautionary saving ($s^{\star\star} \geq s^\star$) if*

$$cov[\tilde{y}, \tilde{R}] \leq -s^{\star\star} var[\tilde{R}]. \tag{6}$$

Proof. Since $\tilde{R} \geq 0$, the random variable $L = \frac{\tilde{R}}{R}$ is such that $\mathbb{P}(L \geq 0) = 1$, $\mathbb{P}(L > 0) > 0$ and $\mathbb{E}[L] = 1$. Hence, it is the Radon-Nikodym density of a probability measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ (that is $L = \frac{d\mathbb{Q}}{d\mathbb{P}}$), such that[4]

$$\mathbb{E}[\tilde{R}v_1(\tilde{y} + s^{\star\star}\tilde{R})] = R\mathbb{E}_{\mathbb{Q}}[v_1(\tilde{y} + s^{\star\star}\tilde{R})]. \tag{7}$$

Prudence ($v_{111} > 0$) and Jensen's inequality imply that

$$\mathbb{E}_{\mathbb{Q}}[v_1(\tilde{y} + s^{\star\star}\tilde{R})] \geq v_1\left(\mathbb{E}_{\mathbb{Q}}[\tilde{y} + s^{\star\star}\tilde{R}]\right). \tag{8}$$

[4]See for instance Jacod & Protter (2002), chap. 28.

Moreover

$$
\begin{aligned}
v_1\left(\mathbb{E}_{\mathbb{Q}}[\tilde{y}+s^{\star\star}\tilde{R}]\right) &= v_1\left(\frac{1}{R}\mathbb{E}[\tilde{R}(\tilde{y}+s^{\star\star}\tilde{R})]\right) \\
&= v_1\left(\frac{1}{R}[cov[\tilde{R},\tilde{y}]+Ry_1+s^{\star\star}(var[\tilde{R}]+R^2)]\right) \\
&= v_1\left(y_1+s^{\star\star}R+\frac{1}{R}[cov[\tilde{R},\tilde{y}]+s^{\star\star}var[\tilde{R}]]\right) \\
&\geq v_1\left(y_1+s^{\star\star}R\right)
\end{aligned}
$$

where the last inequality follows from condition (6) and risk-aversion ($v_{11} < 0$). Putting together this with (7) and (8), one obtains Inequality (5). □

This result, which is novel to the literature, complements the findings of other papers on the same issue (See Li, 2012 and Baiardi et al., 2014). A summary of these findings in comparison to ours is reported in the table below.

Table 1 about here

As shown in Table 1, we derive a condition on the joint distribution of labor income and interest rate risks which ensures positive precautionary saving for a prudent agent introducing neither further requirements on agent's preferences nor limitations on risk size.[5]

4 Economic Interpretations

Condition (6) has three different economic interpretations studied in the following subsections.

4.1 Saving and changes in the variance of total income

In our framework, consumer decisions on saving are related to both prudence and risk aversion. The literature shows that a prudent consumer wishes to increase her wealth in the period where she faces uncertainty (see Eeckhoudt and Schlesinger, 2006 and Menegatti, 2007). In our context, the level of uncertainty is measured by the variance of total income which is the variance of the sum of labor income and of the return of saving. The size of this variance determines the amount of wealth that the prudent consumer wishes to transfer from the first to the second period.

Consumer decision, however, is also influenced by risk aversion. Indeed, the decision to save/borrow exposes the agent to the interest rate risk, so that saving itself is a source

[5] Note that Proposition 1 is also a generalization of Corollary 6 in Baiardi et al. (2014).

of uncertainty. In particular, the marginal effect on uncertainty of an increase in $s^{\star\star}$ is given by its marginal effect on the variance of total income:

$$\frac{dvar[\tilde{y}+s^{\star\star}\tilde{R}]}{ds^{\star\star}} = \frac{d\left\{var[\tilde{y}]+(s^{\star\star})^2 var[\tilde{R}]+2s^{\star\star}cov[\tilde{y},\tilde{R}]\right\}}{ds^{\star\star}} = \\ = 2s^{\star\star}var[\tilde{R}]+2cov[\tilde{y},\tilde{R}]. \quad (9)$$

When the right–hand side of Equation (9) is negative, the effect of an additional unit of saving on total variance is negative as well. Therefore, a risk averse agent likes this increment in saving. Since Inequality (6) in Proposition 1 is the condition ensuring that the right-hand side of Equation (9) is negative, this conclusion provides a clear interpretation: Inequality (6) guarantees that an additional unit of saving reduces the variance of total income.

Result 1. *In the presence of labor income and interest rate risks, a prudent agent has a positive precautionary saving if the increase in saving reduces the variance of total income.*

4.2 Saving and the covariance between total income and interest rate risk

A second interpretation of Inequality (6) can be retrieved by considering the properties of covariance. As shown in the proof of Proposition 1, we have indeed that:

$$cov[\tilde{y}+\tilde{R}s^{\star\star},\tilde{R}] = cov[\tilde{y},\tilde{R}]+s^{\star\star}var[\tilde{R}]. \quad (10)$$

This implies that Inequality (6) is equivalent to

$$cov[\tilde{y}+\tilde{R}s^{\star\star},\tilde{R}] \leq 0. \quad (11)$$

We thus have that:

Result 2. *In the presence of labor income and interest rate risks, a prudent agent has a positive precautionary saving if the covariance between total income and the interest rate is negative.*

The meaning of this result is similar to that in Subsection 4.1. We are considering an agent who is risk averse and prudent. Prudence pushes the agent to move wealth toward the period where the agent bears the risk, raising her saving. But saving requires the purchase of a risky asset. If however this asset is negatively correlated with total income (the sum of labor income and saving return) also risk aversion pushes the agent to buy the asset.

This interpretation also provides an interesting parallelism with a well-known result in the consumption CAPM model. In that model a risky asset pays a spread with respect to the risk free asset whose sign depends on the covariance between the level of consumption and the asset return (see Mankiw and Shapiro, 1986 and Blanchard and Fisher, 1989 p.506). Since second-period consumption is in our framework equal to second-period total income, the covariance in Equation (11) is exactly the same used in consumption CAPM. In that framework a negative covariance implies that the risky asset pays a negative spread. In our framework, where we do not have an alternative risk free asset, a negative covariance implies a larger investment in the risky asset.

4.3 Saving and changes in the utility premium

A third interpretation of the result in Proposition (1) is provided by referring to the concept of utility premium. The utility premium, first introduced by Friedman and Savage (1948), is a measure of risk, alternative to the well-known Arrow-Pratt risk premium,[6] which, in our framework, can be defined as follow:

$$\varphi = v\,(y_1 + Rs) - \mathbb{E}[v(\tilde{y} + \tilde{R}s)]. \tag{12}$$

The variable is the difference between the utility in the certainty case and in the case with uncertainty, and is thus a measure of the disutility due to uncertainty. Finally note that, as shown by Stone (1970), the utility premium can be approximated by:[7]

$$\varphi \cong -v_{11}\,(y_1 + Rs)\, var[\tilde{y} + \tilde{R}s]. \tag{13}$$

The links between the utility premium and saving choice under uncertainty are shown in many papers. In particular Menegatti (2007) shows that, in the presence of labor income risk only, precautionary saving occurs when the utility premium is a decreasing function of wealth. Eeckhoudt & Schlesinger (2009) show that the same result holds in the presence of interest rate risk only. The interpretation of this conclusion is straightforward: the agent wishes to raise saving in the presence of uncertainty if the larger wealth in the second period, generated by saving, reduces her pain due to the presence of uncertainty.

A similar conclusion can be drawn in our framework. Indeed, by differentiating the right-hand side of Equation (13) with respect to s, we obtain

$$\frac{d\varphi}{ds} = -Rv_{111}(y + Rs^{\star\star})var[\tilde{y} + s^{\star\star}\tilde{R}] - 2v_{11}(y + Rs^{\star\star})[s^{\star\star}var[\tilde{R}] + cov[\tilde{y}, \tilde{R}]]. \tag{14}$$

Since $var[\tilde{y} + s^{\star\star}\tilde{R}] > 0$, if the agent is prudent ($v_{111}(x) > 0$) then $-Rv_{111}(y + Rs^{\star\star})var[\tilde{y} + s^{\star\star}\tilde{R}] < 0$. So the first term in the left-hand side of Equation (14) is

[6] The links between the utility premium and the risk premium are studied by Eeckhoudt & Schlesinger (2009) and by Menegatti (2011). On the utlity premium see also Li & Liu (2014).

[7] Note that this formulation defines the exact value of the utility premium only in a neighborhood of $y_1 + Rs$.

negative. Because of risk aversion, $v_{11}(y + Rs^{\star\star}) < 0$. Consequently, in order for the second term in the left-hand side of Equation (14) to be negative too, it is sufficient that Condition (6) holds. Inequality (6) is thus the condition that, given prudence, ensures the utility premium to be decreasing in the level of saving.

Result 3. *In the presence of labor income and interest rate risks, a prudent agent has a positive precautionary saving if an increase in saving reduces her utility premium.*

Similarly to the two cases studied by Menegatti (2007) and Eeckhoudt & Schlesinger (2009), the agent wishes to increase saving in the presence of two risks if this larger saving reduces the utility premium, i.e. the pain caused by the presence of uncertainty.

5 Conclusions

This note re-examines the determination of optimal saving for a prudent agent in the presence of labor income and interest rate risks. In this framework, we derive a sufficient condition for positive precautionary saving involving the comparison between the covariance between the two risks and the variance of the interest rate. We show that this condition allows to remove the requirements either in agent's preferences or in risk size introduced in the previous literature.

We then provide three different interpretations of our sufficient condition. The first interpretation is related to the fact that, in our framework, saving implies investment in a risky asset affecting income variability. We show that, for this reason, a prudent agent has a positive precautionary saving whenever a marginal increase in saving (meaning a larger investment in the risky asset) reduces the variance of total income. This lead us to the second interpretation, which involves the covariance between total income and the interest rate. When this covariance is negative, the incentive to increase future consumption (due to prudence) is strengthened by the desire (related to risk aversion) to invest in an asset which is negatively correlated with total income. Lastly, in the third interpretation, we focus on the utility premium, which is a measure of the pain caused by uncertainty in the presence of risk aversion, showing that a prudent agent has a positive precautionary saving whenever an increase in saving ensures a reduction in this pain.

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Table 1: Sufficient conditions for precautionary saving in case of prudence

	Li (2012)	**Baiardi et al. (2013)**	**This paper**
Conditions on agent's preferences	Absolute prudence index > 2	Absolute prudence index $> K$ (where $K < 2$)	–
Conditions on the joint distribution of $\tilde{y}$ and $\tilde{R}$	$\tilde{y}$ and $\tilde{R}$ positively quadrant dependent	Different sets of conditions involving $Cov(\tilde{y}, \tilde{R})$ and $Var(\tilde{R})$	$Cov(\tilde{y}, \tilde{R}) < -sVar(\tilde{R})$
Conditions on risk size	–	Small risks	–