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# INVARIANT CONVEX SETS IN POLAR REPRESENTATIONS

LEONARDO BILIOTTI, ALESSANDRO GHIGI, AND PETER HEINZNER

ABSTRACT. We study a compact invariant convex set  $E$  in a polar representation of a compact Lie group. Polar representations are given by the adjoint action of  $K$  on  $\mathfrak{p}$ , where  $K$  is a maximal compact subgroup of a real semisimple Lie group  $G$  with Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ . If  $\mathfrak{a} \subset \mathfrak{p}$  is a maximal abelian subalgebra, then  $P = E \cap \mathfrak{a}$  is a convex set in  $\mathfrak{a}$ . We prove that up to conjugacy the face structure of  $E$  is completely determined by that of  $P$  and that a face of  $E$  is exposed if and only if the corresponding face of  $P$  is exposed. We apply these results to the convex hull of the image of a restricted momentum map.

The boundary of a compact convex set is the union of its faces. Among the faces, the simplest ones are the exposed ones. They are given by the intersection of the convex set with a supporting hyperplane. In [3, 4] we studied the convex hull  $\widehat{\mathcal{O}}$  of a  $K$ -orbit  $\mathcal{O}$  in  $\mathfrak{p}$ , where  $\mathfrak{p}$  is given by the Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  of a reductive Lie algebra  $\mathfrak{g}$  and  $K$  acts on  $\mathfrak{p}$  by the adjoint representation. In this paper we use the results of [4] and show that a substantial part of them holds for any  $K$ -invariant compact convex set  $E$  of  $\mathfrak{p}$ . More precisely we study the faces of  $E$ . We show in Proposition 1.2 that for a face  $F$  of  $E$  there exists a subalgebra  $\mathfrak{s} \subset \mathfrak{p}$  such that  $F$  is a subset of  $\mathfrak{p}^{\mathfrak{s}} = \{x \in \mathfrak{p} : [x, \mathfrak{s}] = 0\}$  and  $F$  is invariant with respect to the action of  $K^{\mathfrak{s}} = \{h \in K : \text{Ad}(h)(\mathfrak{s}) = \mathfrak{s}\}$ , where  $\text{Ad}$  denotes the adjoint representation.

If we fix a maximal abelian subalgebra  $\mathfrak{a} \subset \mathfrak{p}$ , then the set  $P = E \cap \mathfrak{a}$  is convex and invariant with respect to the action of the normalizer  $\mathcal{N}_K(\mathfrak{a}) = \{h \in K : \text{Ad}(h)(\mathfrak{a}) = \mathfrak{a}\}$  of  $\mathfrak{a}$  in  $K$ . The  $\mathcal{N}_K(\mathfrak{a})$ -action on  $P$  induces an action on the set of faces of  $P$ . Similarly  $K$  acts on the set of faces of  $E$ . Denote these sets by  $\mathcal{F}(P)$  respectively by  $\mathcal{F}(E)$ . If  $\sigma$  is a face of  $P$ , let  $\sigma^{\perp}$  denote the orthogonal complement in  $\mathfrak{a}$  of the affine hull of  $\sigma$  (see Section 1). Our main result is

**Theorem 0.1.** *The map  $\mathcal{F}(P) \rightarrow \mathcal{F}(E)$ ,  $\sigma \mapsto K^{\sigma^{\perp}} \cdot \sigma$  is well-defined and induces a bijection between  $\mathcal{F}(P)/\mathcal{N}_K(\mathfrak{a})$  and  $\mathcal{F}(E)/K$ .*

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An application of Theorem 0.1 is the following result.

**Theorem 0.2.** *The faces of  $E$  are exposed if and only if the faces of  $P$  are exposed.*

Interesting  $K$ -invariant compact subsets of  $\mathfrak{p}$  often arise as images of restricted momentum or gradient mappings. More precisely, let  $U$  be a compact connected Lie group which acts by biholomorphism and in a Hamiltonian fashion on a compact Kähler manifold  $Z$  with momentum map  $\mu : Z \rightarrow \mathfrak{u}$ . Let  $G \subset U^{\mathbb{C}}$  be a connected Lie subgroup of  $U^{\mathbb{C}}$  which is *compatible* with respect to the Cartan decomposition of  $U^{\mathbb{C}}$ . This means that  $G$  is a closed subgroup of  $U^{\mathbb{C}}$  such that  $G = K \exp(\mathfrak{p})$ , where  $K = U \cap G$  and  $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$  [13, 15]. Let  $X \subset Z$  be a  $G$ -invariant compact subset of  $Z$ . We have the restricted momentum map or the gradient map  $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$  in the sense of [13] (see also Section 3) and we denote by  $E = \widehat{\mu_{\mathfrak{p}}(X)}$  the convex hull of the  $K$ -invariant set  $\mu_{\mathfrak{p}}(X)$ . If  $\mathfrak{a}$  is a maximal abelian subalgebra of  $\mathfrak{p}$  and  $\pi$  is the orthogonal projection onto  $\mathfrak{a}$ , then  $\mu_{\mathfrak{a}} = \pi \circ \mu_{\mathfrak{p}} : X \rightarrow \mathfrak{a}$  is the gradient map with respect to  $A = \exp(\mathfrak{a})$ . Since  $P = E \cap \mathfrak{a} = \widehat{\mu_{\mathfrak{a}}(X)}$  is a convex polytope (Proposition 3.1), we deduce the following.

**Theorem 0.3.** *All faces of  $\widehat{\mu_{\mathfrak{p}}(X)}$  are exposed.*

A reformulation of Theorem 3.1 is that the faces of  $E$  correspond to maxima of components of the gradient map. This observation will be used to realize a close connection between the faces of  $E$  and parabolic subgroups of  $G$ . More precisely, for any face  $F \subset E$  let  $X_F := \mu_{\mathfrak{p}}^{-1}(F)$  and let  $Q^F = \{g \in G : g \cdot X_F = X_F\}$ . Then  $X_F$  is the set of maximum points of an appropriately chosen component of the gradient map and  $Q^F$  is a parabolic subgroup of  $G$ .

If  $X$  is a  $G$ -stable compact submanifold of  $Z$ , then for any face  $F$ , one can construct an open neighbourhood  $X_F^-$  of  $X_F$  in  $X$ , which is an analogue of an open Bruhat cell. Moreover there is a smooth deformation retraction of  $X_F^-$  onto  $X_F$ . See Theorem 3.1 for more details.

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## 1. GROUP THEORETICAL DESCRIPTION OF THE FACES

We start by recalling the basic definitions and results regarding convex bodies. For more details see e.g. [18]. Let  $V$  be a real vector space with scalar product  $\langle \cdot, \cdot \rangle$ . A *convex body*  $E \subset V$  is a convex compact subset of  $V$ . Let  $\text{Aff}(E)$  denote the affine span of  $E$ . The interior of  $E$  in  $\text{Aff}(E)$  is called the *relative interior* of  $E$  and is denoted by  $\text{relint } E$ . By definition a *face* of  $E$  is a convex subset  $F \subset E$  such that  $x, y \in E$  and  $\text{relint}[x, y] \cap F \neq \emptyset$

implies  $[x, y] \subset F$ . A face distinct from  $E$  and  $\emptyset$  is called a *proper face*. The *extreme points* of  $E$  are the points  $x \in E$  such that  $\{x\}$  is a face. We will denote by  $\text{ext } E$  the set of the extreme points of  $E$ . The set  $\text{ext } E$  completely determines the convex body  $E$  since the convex hull of  $\text{ext } E$  coincides with  $E$  and it is the smallest subset of  $E$  with this property. If  $F$  is a face of  $E$ , we denote by  $\text{Dir}(F)$  the vector subspace of  $V$  defined by  $\text{Aff}(F)$ , i.e.  $\text{Aff}(F) = p + \text{Dir}(F)$ . We call  $\text{Dir}(F)$  the *direction* of  $F$ . Every vector  $\beta \in V$  defines an *exposed face*  $F = F_\beta(E) = \{x \in E : \langle x, \beta \rangle = \max_{y \in E} \langle y, \beta \rangle\}$  with  $\text{Dir}(F_\beta(E)) \subset \{\beta\}^\perp$ . In general not all faces of a convex set are exposed, see Fig. 1 for an example. For any exposed face  $F$  the set

$$C_F = \{\beta \in V : F = F_\beta(E)\}, \quad (1)$$

is a convex cone. The faces of  $E$  are closed. If  $F_1$  and  $F_2$  are faces of  $E$  and they are distinct, then  $\text{relint } F_1 \cap \text{relint } F_2 = \emptyset$ . Moreover the convex body  $E$  is the disjoint union of the relative interiors of its faces (see [18, p. 62]).

We are interested in invariant convex bodies in polar representations. A theorem of Dadok [6] asserts that we can restrict ourselves to the following setting.

Let  $\mathfrak{g}$  be a semisimple Lie algebra with a Cartan involution  $\theta$  and let  $B$  be the Killing form of  $\mathfrak{g}$ . Then  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , is the eigenspace decomposition of  $\mathfrak{g}$  in 1 and  $-1$  eigenspaces of  $\theta$  and they are orthogonal under  $B$ . Moreover,  $B$  restricted to  $\mathfrak{k}$ , respectively  $\mathfrak{p}$ , is negative definite, respectively positive definite. In the sequel we denote  $\langle \cdot, \cdot \rangle = B|_{\mathfrak{p} \times \mathfrak{p}}$  which is a  $K$ -invariant scalar product. Our object of study will be a  $K$ -stable convex body  $E \subset \mathfrak{p}$ . For for any  $A, B \subset \mathfrak{p}$  we set

$$\begin{aligned} A^B &:= \{\eta \in A : [\eta, \xi] = 0, \text{ for all } \xi \in B\} \\ G^B &:= \{g \in G : \text{Ad } g(\xi) = \xi, \text{ for all } \xi \in B\}, \\ K^B &:= K \cap G^B. \end{aligned}$$

where  $\text{Ad}$  denotes the adjoint representation. In the sequel we denote by  $k \cdot x = \text{Ad}(k)(x)$  the action of  $K$  on  $\mathfrak{p}$  by linear isometries.

Faces of  $K$ -invariant convex bodies in  $\mathfrak{p}$  are closely connected to orbits of subgroups of  $K$  which are given as centralizers. More precisely for any nonzero  $\beta$  in  $\mathfrak{p}$  we have the Cartan decomposition  $\mathfrak{g}^\beta = \mathfrak{k}^\beta \oplus \mathfrak{p}^\beta$  of the Lie algebra of the centralizer  $G^\beta$  of  $\beta$  in  $G$ .

**Proposition 1.1.** *Let  $F = F_\beta(E)$  be an exposed face of  $E$ . Then*

- a)  $F \subset \mathfrak{p}^\beta$  and  $F$  is  $K^\beta$ -stable;
- b)  $\text{Dir}(F) \subset \beta^\perp$ , where  $\perp$  is in  $\mathfrak{p}$ .

*Proof.* If  $x \in F_\beta(E)$ , then  $\widehat{K \cdot x} \subset E$  since  $E$  is  $K$ -invariant. Moreover, we have

$$\max_{y \in E} \langle y, \beta \rangle = \max_{y \in \widehat{K \cdot x}} \langle y, \beta \rangle = \langle x, \beta \rangle.$$

Corollary 3.1 in [4] implies  $F_\beta(\widehat{K \cdot x}) \subset \mathfrak{p}^\beta$ . Therefore  $x \in \mathfrak{p}^\beta$ . This proves a). Part b) follows since  $F$  is contained in an affine hyperplane orthogonal to  $\beta$ .  $\square$

For an arbitrary face of  $E$  we have the following.

**Proposition 1.2.** *Let  $F \subset E$  be a face. Then there exists an abelian subalgebra  $\mathfrak{s} \subset \mathfrak{p}$  such that*

- a)  $F \subset \mathfrak{p}^\mathfrak{s}$  and  $F$  is  $K^\mathfrak{s}$ -stable;
- b)  $\text{Dir}(F) \subset \mathfrak{s}^\perp$ ;

*Proof.* We may fix a maximal chain of faces  $F = F_0 \subsetneq F_1 \subsetneq \dots \subsetneq F_k = E$  (see [3, Lemma 2]). If  $k = 0$ , then  $F = E$  and  $\mathfrak{s} = \{0\}$ . Assume the theorem is true for a face contained in a maximal chain of length  $k$ . Then the claim is true for  $F_1$  and consequently there exists  $\mathfrak{s}_1 \subset \mathfrak{p}$  such that  $F_1 \subset \mathfrak{p}^{\mathfrak{s}_1}$ ,  $F_1$  is  $K^{\mathfrak{s}_1}$ -stable and  $\text{Dir}(F_1) \subset \mathfrak{s}_1^\perp$ .  $F$  is an exposed face of  $F_1$ . Let  $\beta' \in \mathfrak{p}^{\mathfrak{s}_1}$  such that  $F = F_{\beta'}(F_1)$  and set  $\mathfrak{s} := \mathbb{R}\beta' \oplus \mathfrak{s}_1$ . Then  $F \subset \mathfrak{p}^\mathfrak{s}$ ,  $F$  is  $(K^{\mathfrak{s}_1})^{\beta'} = K^\mathfrak{s}$ -stable and  $\text{Dir}(F) \subset \mathfrak{s}^\perp$ .  $\square$

Let  $\mathfrak{a} \subset \mathfrak{p}$  be a maximal abelian subalgebra of  $\mathfrak{p}$  and let  $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$  be the orthogonal projection onto  $\mathfrak{a}$ . Then  $P = E \cap \mathfrak{a}$  is a convex subset of  $\mathfrak{a}$  which is  $\mathcal{N}_K(\mathfrak{a})$ -stable. The proof of the following Lemma is given in [7].

**Lemma 1.1.** *(i) If  $E \subset \mathfrak{p}$  is a  $K$ -invariant convex subset, then  $E \cap \mathfrak{a} = \pi(E)$  and  $K \cdot \pi(E) = E$ . (ii) If  $C \subset \mathfrak{a}$  is a  $\mathcal{N}_K(\mathfrak{a})$ -invariant convex subset, then  $K \cdot C$  is convex and  $\pi(K \cdot C) = C$ .*

**Lemma 1.2.** *Let  $U$  be a compact Lie group and let  $\mathfrak{g} \subset \mathfrak{u}^\mathbb{C}$  be a semisimple  $\theta$ -invariant subalgebra. Then any Lie subgroup with finitely many connected components and with Lie algebra  $\mathfrak{g}$  is closed and compatible.*

*Proof.* We fix an embedding  $U \hookrightarrow U(n)$  such that the Cartan involution  $X \mapsto (X^{-1})^*$  of  $\text{GL}(n, \mathbb{C})$  restricts to  $\theta$ . Then  $G$  is closed in  $\text{GL}(n, \mathbb{C})$  (see [16, p. 440] for a proof) and hence also in  $U^\mathbb{C}$ . Since  $\mathfrak{g}$  is  $\theta$ -invariant, also  $G$  is, and  $\theta$  restricts to the Cartan involution of  $G$ . This shows that  $G$  is compatible.  $\square$

If  $G \subset U^\mathbb{C}$  is compatible with Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ , then  $\mathfrak{g}$  is real reductive and there is a nondegenerate  $K$ -invariant bilinear form  $B : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  which is positive definite on  $\mathfrak{p}$ , negative definite on  $\mathfrak{k}$  and such that  $B(\mathfrak{k}, \mathfrak{p}) = 0$ . Indeed, fix a  $U$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{u}$ . Let  $\langle \cdot, \cdot \rangle$  denote also the inner product on  $i\mathfrak{u}$  such that multiplication by  $i$  be an isometry of  $\mathfrak{u}$  onto  $i\mathfrak{u}$ . Define  $B$  on  $\mathfrak{u}^\mathbb{C}$  imposing  $B(\mathfrak{u}, i\mathfrak{u}) = 0$ ,  $B = -\langle \cdot, \cdot \rangle$  on  $\mathfrak{u}$  and  $B = \langle \cdot, \cdot \rangle$  on  $i\mathfrak{u}$ . Therefore  $B$  is  $\text{Ad } U^\mathbb{C}$ -invariant and non-degenerate and its restriction to  $\mathfrak{g}$  satisfies the above conditions.

Let  $\mathfrak{q}$  be a  $K$ -invariant subspace of  $\mathfrak{p}$ . Then  $[\mathfrak{q}, \mathfrak{q}]$  is a  $K$ -invariant linear subspace of  $\mathfrak{k}$  and therefore an ideal of  $\mathfrak{k}$ . Since  $K$  is compact, we have the

following  $K$ -invariant splitting  $\mathfrak{k} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{k}'$ . In particular  $\mathfrak{k}'$  is an ideal of  $\mathfrak{k}$  commuting with  $[\mathfrak{q}, \mathfrak{q}]$ . Let  $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$  be a  $K$ -invariant splitting of  $\mathfrak{p}$ . Since

$$B([\mathfrak{q}, \mathfrak{q}'], \mathfrak{k}) = B(\mathfrak{q}, [\mathfrak{k}, \mathfrak{q}']) \subset B(\mathfrak{q}, \mathfrak{q}') = 0,$$

this shows that  $[\mathfrak{q}, \mathfrak{q}'] = 0$  and so  $[\mathfrak{q}', [\mathfrak{q}, \mathfrak{q}]] = [\mathfrak{q}, [\mathfrak{q}, \mathfrak{q}']] = 0$ . Moreover  $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{q}'$  implies that  $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$  and  $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{q}'$  are compatible  $K$ -invariant commuting ideal of  $\mathfrak{g}$ .

If a  $K$ -invariant linear subspace  $\mathfrak{q} \subset \mathfrak{p}$  is fixed, one gets decomposition of  $\mathfrak{g}$ , and so of  $G$ . This decomposition is the content of the next Proposition. We will need it in the case where  $F \subset \mathfrak{p}$  is a  $K$ -invariant convex body and  $\mathfrak{q}$  is such that  $\text{Aff}(F) = x_0 + \mathfrak{q}$ .

**Proposition 1.3.** *Let  $G \subset U^{\mathbb{C}}$  be a compatible subgroup with Lie algebra  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  and let  $\mathfrak{q} \subset \mathfrak{p}$  be a linear  $K$ -invariant subspace. Let  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$  where  $\mathfrak{h} = [\mathfrak{q}, \mathfrak{q}] \oplus \mathfrak{q}$  and  $\mathfrak{h}' = \mathfrak{h}^{\perp B}$ . Then the following hold.*

- a)  $\mathfrak{h}$  and  $\mathfrak{h}'$  are compatible  $K$ -invariant commuting ideal of  $\mathfrak{g}$ ;
- b) Let  $K_1$  be the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{k} \cap [\mathfrak{h}, \mathfrak{h}]$ . Then  $K_1 \exp(\mathfrak{q})$  is a connected compatible subgroup of  $G$  and any two maximal subalgebras of  $\mathfrak{q}$  are conjugate by an element of  $K_1$ .
- c) Let  $K_2$  be the connected Lie subgroup of  $G$  with Lie algebra  $\mathfrak{k} \cap [\mathfrak{h}', \mathfrak{h}']$ . Then any two maximal subalgebras of  $\mathfrak{q}'$  are conjugate by an element of  $K_2$ .

*Proof.* We have proved (a) in the above discussion. Let  $\mathfrak{b} := [\mathfrak{h}, \mathfrak{h}]$ . Then  $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{b}$  and  $\mathfrak{b}$  is semisimple. Denote by  $B$  the connected subgroup of  $U^{\mathbb{C}}$  with Lie algebra  $\mathfrak{b}$ . By Lemma 1.2  $B$  is a closed subgroup of  $U^{\mathbb{C}}$ . Set  $\mathfrak{z}_{\mathfrak{p}} := \mathfrak{z}(\mathfrak{h}) \cap \mathfrak{p}$  and  $\mathfrak{d} := \mathfrak{b} \oplus \mathfrak{a}$ . Then  $\mathfrak{d}$  is a reductive Lie algebra and  $\exp \mathfrak{a}$  is a compatible abelian subgroup commuting with  $B$ . Thus  $D := B \cdot \exp \mathfrak{a}$  is a connected closed subgroup with Lie algebra  $\mathfrak{d}$ . Moreover  $D \cap U = B \cap U$  and  $\exp(\mathfrak{b} \cap \mathfrak{p}) \cdot \exp \mathfrak{a} = \exp(\mathfrak{b} \cap \mathfrak{p} \oplus \mathfrak{a}) = \exp(\mathfrak{d} \cap \mathfrak{p})$ . This shows that  $D$  is compatible. Since  $D \cap U$  coincides with  $K_1$  and  $D$  is connected the last statement in (b) follows from standard properties of compatible subgroups (see e.g. Prop. 7.29 in [16]; note that a connected compatible subgroup is a reductive group in the sense of [16, p. 446]). This proves (b). For (c) the same argument applies more directly. It is enough to observe that the connected Lie subgroup  $H'' \subset G$  with Lie algebra  $[\mathfrak{h}', \mathfrak{h}']$  is semisimple, compatible and connected and that  $K_2 = H'' \cap U$ .  $\square$

**Remark 1.1.** *The compatible subgroup  $G$  in the previous Proposition is not assumed to be connected. Nevertheless the constructions in (b) and (c) depend only on  $G^0$ . Thus considering  $G^0$  in place of  $G$  makes no difference.*

**Lemma 1.3.** *Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$  be a reductive Lie algebra and  $\mathfrak{g}_i$  ideals. If  $\mathfrak{a} \subset \mathfrak{p}$  is a maximal subalgebra, then  $\mathfrak{a}_i := \mathfrak{a} \cap \mathfrak{p}_i$  is a maximal subalgebra of  $\mathfrak{p}_i$  and  $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ .*

If  $\sigma$  is a face of  $P$ , let  $\sigma^{\perp}$  denote the orthogonal (inside  $\mathfrak{a}$ ) to the direction of the affine hull of  $\sigma$ .

**Lemma 1.4.** *Let  $F$  be a face and let  $\mathfrak{s}$  be as in Proposition 1.2. Let  $\mathfrak{a} \subset \mathfrak{p}$  be a maximal abelian subalgebra containing  $\mathfrak{s}$ . Set  $\sigma := \pi(F)$ . Then  $\sigma$  is a face of  $P$ ,  $\sigma = F \cap \mathfrak{a}$  and  $F = K^{\sigma^\perp} \cdot \sigma$ . Moreover  $F$  is a proper face if and only if  $F \cap \mathfrak{a}$  is.*

*Proof.* By Proposition 1.2  $F \subset \mathfrak{p}^{\mathfrak{s}}$  is a  $K^{\mathfrak{s}}$ -stable convex set. By Lemma 1.1 we get  $\sigma = \pi(F) = F \cap \mathfrak{a}$  and this is a face  $P$  by [3, Lemma 11]. Since  $\text{Dir}(F)$  is contained in the orthogonal complement of  $\mathfrak{s}$ , and  $\text{Dir}(\sigma) \subset \text{Dir}(F)$ , we have  $\text{Dir}(\sigma) \subset \mathfrak{a} \cap \mathfrak{s}^\perp$ . Then  $\sigma^\perp \subset \mathfrak{s}$ . Hence  $K^{\sigma^\perp} \cdot \sigma \subset K^{\mathfrak{s}} \cdot \sigma \subset F$ . We prove the reverse inclusion. If  $y \in F$ , then  $F \cap \widehat{K \cdot y}$  is a face of  $\widehat{K \cdot y}$ . Set  $\tilde{\sigma} = \pi(F \cap \widehat{K \cdot y})$ . We have  $\tilde{\sigma} \subset \sigma$  and by Proposition 3.6 in [4] we also have that  $F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma}$ . On the other hand,  $\sigma^\perp \subset \tilde{\sigma}^\perp$ , so  $K^{\tilde{\sigma}^\perp} \subset K^{\sigma^\perp}$  and

$$F \cap \widehat{K \cdot y} = K^{\tilde{\sigma}^\perp} \cdot \tilde{\sigma} \subset K^{\sigma^\perp} \cdot \sigma.$$

This implies  $F = K^{\sigma^\perp} \cdot \sigma$ . Note that  $F$  is proper if  $\sigma$  is. It remains to prove that  $\sigma$  is proper, when  $F$  is proper.

Let  $\text{Aff}(E) = x_o + \mathfrak{q}_E$ . Note that  $\mathfrak{q}_E = \{x - y : x, y \in \text{Aff}(E)\}$  implies that  $\mathfrak{q}_E$  is  $K$ -invariant. Since  $K$  acts on  $\mathfrak{p}$  by isometries, we may assume that  $x_o$  is orthogonal to  $\mathfrak{q}$ . Note that  $x_o$  is uniquely defined by this condition. It follows that  $x_o$  is a  $K$  fixed point and  $E = x_o + E_1$ , where  $E_1$  is a  $K$ -invariant convex body of  $\mathfrak{q}_E$ . Proposition 1.3 applied to  $\mathfrak{q}_E$  yields  $K_1, K_2$  such that  $G_1 = K_1 \exp(\mathfrak{q}_E)$  is a connected compatible semisimple real Lie group,  $K = K_1 \cdot K_2$  and for any  $x \in E$  we have

$$K \cdot x = K \cdot (x_o + x_1) = x_o + K \cdot x_1 = x_o + K_1 \cdot x_1 = K_1 \cdot x.$$

since  $\mathfrak{q}_E$  is fixed pointwise by  $K_2$ . By Lemma 1.3,  $\mathfrak{a} = \mathfrak{a}_E \oplus \mathfrak{a}'_E$ , where  $\mathfrak{a}_E$  is a maximal abelian subalgebra of  $\mathfrak{q}_E$  and  $\mathfrak{a}'_E$  is a maximal abelian subalgebra of  $\mathfrak{q}'_E$ . Since  $\pi(E) = \pi(x_o) + \pi(E_1)$  and  $\text{Dir}(E_1) = \mathfrak{q}_E$ , it follows that the direction of  $\pi(E)$  is  $\mathfrak{a}_E$ . If  $\sigma = \pi(F) = \pi(E) = E \cap \mathfrak{a}$ , then  $\sigma^\perp = \mathfrak{a}'_E$  and so  $K_1 \subset K^{\mathfrak{a}'_E}$ . It follows that

$$F = K^{\mathfrak{a}'_E} \cdot (E \cap \mathfrak{a}) = K_1 \cdot (E \cap \mathfrak{a}) = K \cdot (E \cap \mathfrak{a}) = E.$$

where the last equality follows by Lemma 1.1. Hence, if  $F$  is proper, then  $\sigma = \pi(F) \subsetneq \pi(E) = E \cap \mathfrak{a}$ .  $\square$

**Proposition 1.4.** *Let  $F$  be a proper face and let  $\mathfrak{s}$  as in Proposition 1.2. Let  $\mathfrak{a} \subset \mathfrak{p}$  be a maximal abelian subalgebra containing  $\mathfrak{s}$ . Then  $F$  is exposed if and only if  $F \cap \mathfrak{a}$  is.*

*Proof.* Assume that there exists  $\beta \in \mathfrak{p}$  such that  $F = F_\beta(E)$ . Since  $F \cap \mathfrak{a} = \sigma$  is a proper face of  $P$ , the point  $\beta$  is not orthogonal to  $\mathfrak{a}$ . We have  $\beta = \beta_1 \oplus \beta_2$ , with  $\beta_1 \in \mathfrak{a}$  different from zero and  $\beta_2$  orthogonal to  $\mathfrak{a}$ . Therefore  $F_\beta(E) \cap \mathfrak{a} = F_{\beta_1}(E) \cap \mathfrak{a} = F_{\beta_1}(P) = \sigma$ . Now, assume that there exists  $\beta \in \mathfrak{a}$  such that  $\sigma = F_\beta(P)$ . Let  $F' := F_\beta(E)$ . By Proposition 1.1  $F' \subset \mathfrak{p}^\beta$ . Moreover  $\mathfrak{a} \subset \mathfrak{p}^\beta$  since  $\beta \in \mathfrak{a}$ . By Lemma 1.4 the intersection of a face with

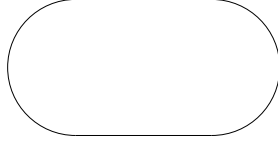


FIGURE 1.

$\mathfrak{a}$  determines the face. Since  $F' \cap \mathfrak{a} = F_\beta(P) = \sigma = F \cap \mathfrak{a}$  we conclude that  $F = F'$ . Thus  $F$  is exposed.  $\square$

**Remark 1.2.** *Given a Weyl-invariant convex body  $P \subset \mathfrak{a}$ , set  $E := K \cdot P$ . By Lemma 1.1  $E$  is a  $K$ -invariant convex body in  $\mathfrak{p}$  and  $P = E \cap \mathfrak{a}$ . Thus a general  $P$  can be realized as  $E \cap \mathfrak{a}$ . A general Weyl-invariant convex body  $P$  can have non-exposed faces. For example take  $G = U^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$  and  $K = \mathrm{SU}(2) \times \mathrm{SU}(2)$ . Then  $\mathfrak{a} = \mathbb{R}^2$  and the Weyl group is isomorphic to  $\mathbb{Z}/2 \times \mathbb{Z}/2$  where the generators are given by the reflections on the axes. The picture in Fig. 1 is a Weyl-invariant  $P$  with exactly 4 non-exposed faces. By the Proposition also the corresponding body  $E \subset \mathfrak{isu}(2) \oplus \mathfrak{isu}(2)$  has non-exposed faces.*

## 2. PROOF OF THE MAIN RESULTS

Let  $\mathfrak{a} \subset \mathfrak{p}$  and define the following map

$$\Upsilon : \mathcal{F}(P) \longrightarrow \mathcal{F}(E), \quad \sigma \mapsto K^{\sigma^\perp} \cdot \sigma$$

Since  $\sigma$  is  $\mathcal{N}_{K^{\sigma^\perp}}(\mathfrak{a})$ -invariant, it follows from Lemma 1.1 that  $\Upsilon(\sigma)$  is a face of  $E$ .

**Theorem 0.1.** *The map  $\Upsilon$  induces a bijection between  $\mathcal{F}(P)/\mathcal{N}_K(\mathfrak{a})$  and  $\mathcal{F}(E)/K$ .*

*Proof.* Set  $\mathcal{N} := \mathcal{N}_K(\mathfrak{a})$ . We first show that  $\Upsilon$  is  $\mathcal{N}$ -equivariant. Let  $w \in \mathcal{N}$ . Then  $\sigma' = w\sigma$  implies  $K^{\sigma'^\perp} = wK^{\sigma^\perp}w^{-1}$  and therefore  $\Upsilon(\sigma') = w\Upsilon(\sigma)$ . This means that the map

$$\tilde{\Upsilon} : \mathcal{F}(P)/\mathcal{N} \longrightarrow \mathcal{F}(E)/K, \quad [\sigma] \mapsto K^{\sigma^\perp} \cdot \sigma$$

is well-defined. Next, we prove that  $\tilde{\Upsilon}$  is injective. Assume for some  $g \in K$   $g \cdot F_1 = F_2$  where  $F_1 = \Upsilon(\sigma_1)$  and  $F_2 = \Upsilon(\sigma_2)$ . Since  $F_2 = K^{\sigma_2^\perp} \cdot \sigma_2$ , the face  $F_2$  is a  $K^{\sigma_2^\perp}$ -invariant convex body. Moreover  $\sigma_2 \subset \mathfrak{a} \subset \mathfrak{p}^{\sigma_2^\perp}$  and  $\mathfrak{p}^{\sigma_2^\perp}$  is  $K^{\sigma_2^\perp}$ -invariant. Therefore  $F_2$  is contained in  $\mathfrak{p}^{\sigma_2^\perp}$ . It follows that  $\mathrm{Aff}(F_2) = x_o + \mathfrak{q}_{F_2}$ , where  $\mathfrak{q}_{F_2}$  is a  $K^{\sigma_2^\perp}$  invariant subspace of  $\mathfrak{p}^{\sigma_2^\perp}$ ,  $x_o$  is a fixed  $K^{\sigma_2^\perp}$  point and it is orthogonal to  $\mathfrak{q}_{F_2}$ . We apply Proposition 1.3 to the group  $G^{\sigma_2^\perp}$  and  $\mathfrak{q}_{F_2}$ . Thus  $\mathfrak{h}_{F_2} = [\mathfrak{q}_{F_2}, \mathfrak{q}_{F_2}] \oplus \mathfrak{q}_{F_2}$  and its orthogonal complement in  $\mathfrak{g}^{\sigma_2^\perp}$ , that we denote by  $\mathfrak{h}'_{F_2}$ , are commuting ideal. The Proposition 1.3 also yields subgroups  $K_1, K_2 \subset K^{\sigma_2^\perp}$  such that any two maximal subalgebras in  $\mathfrak{q}_{F_2}$ , respectively  $\mathfrak{q}'_{F_2}$ , are interchanged by

$K_1$ , respectively  $K_2$ . Since  $\sigma_2 \subset \mathfrak{a}$ , also  $\text{Dir}(\sigma_2) \subset \mathfrak{a}$  and we may decompose  $\mathfrak{a} = \text{Dir}(\sigma_2) \oplus \sigma_2^\perp$ . But  $\text{Dir}(\sigma_2)$  is contained also in  $\mathfrak{q}_{F_2}$  since  $\sigma_2 \subset F_2$ . So  $\sigma_2^\perp \subset \mathfrak{q}_{F_2}^\perp \cap \mathfrak{p} = \mathfrak{q}'_{F_2}$ . By dimension  $\text{Dir}(\sigma_2)$  is a maximal subalgebra in  $\mathfrak{q}_{F_2}$  and  $\sigma_2^\perp$  is a maximal subalgebra in  $\mathfrak{q}'_{F_2}$ . On other hand from  $g \cdot F_1 = F_2$  it follows that  $g \cdot \text{Dir}(\sigma_1) \subset \mathfrak{q}_{F_2}$  and  $g \cdot \sigma_1^\perp \subset \mathfrak{q}'_{F_2}$ , and they are also maximal subalgebras in these spaces. By the Proposition 1.3 (b) and (c) there exist  $k_1 \in K_1, k_2 \in K_2$  such that

$$(k_1 g) \cdot \text{Dir}(\sigma_1) = \text{Dir}(\sigma_2)$$

$$(k_2 g) \cdot \sigma_1^\perp = \sigma_2^\perp.$$

Since  $x_0$  is fixed by the larger group  $K^{\sigma_2^\perp}$  it follows that  $k_1 g \sigma_1 = \sigma_2$ . Moreover  $k_1 k_2 = k_2 k_1$  since  $[\mathfrak{h}_{F_2}, \mathfrak{h}'_{F_2}] = 0$ . For the same reason  $\mathfrak{q}'_{F_2}$  is fixed pointwise by  $K_1$  and  $\mathfrak{q}_{F_2}$  is fixed pointwise by  $K_2$ . Set  $k = k_1 k_2$  and  $w = k g$ . Then  $k \in K^{\sigma_2^\perp}$  and  $w \in K$ . We get

$$w \cdot \text{Dir}(\sigma_1) = \text{Dir}(\sigma_2)$$

$$w \cdot \sigma_1^\perp = \sigma_2^\perp.$$

Thus  $w \cdot \mathfrak{a} = \mathfrak{a}$ , i.e.  $w \in \mathcal{N}$ . Since  $k \in K^{\sigma_2^\perp}$ ,  $w \cdot F_1 = (k g) \cdot F_1 = k \cdot F_2 = F_2$ . Since  $\sigma_1 = (x_0 + \text{Dir}(\sigma_1)) \cap F_1$  and similarly for  $F_2$ , we conclude that  $w \sigma_1 = \sigma_2$ . Finally we prove that  $\tilde{\Theta}$  is surjective. Let  $F \subset \hat{\mathcal{O}}$  be a face. Then  $F \subset \mathfrak{p}^\mathfrak{s}$  for some abelian subalgebra  $\mathfrak{s} \in \mathfrak{p}$ . Then there exists  $k \in K$  such that  $k \cdot \mathfrak{a} \subset \mathfrak{p}^\mathfrak{s}$ . Therefore  $k^{-1} \cdot F \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$  and  $\mathfrak{a} \subset \mathfrak{p}^{(k^{-1} \cdot \mathfrak{s})}$ . By Proposition 1.4,  $k \cdot F = K^{\sigma^\perp} \cdot \sigma$  where  $\sigma = (k \cdot F) \cap \mathfrak{a}$  and so  $\tilde{\Upsilon}$  is surjective.  $\square$

As an application of the above theorem and Proposition 1.4, we get the following result.

**Theorem 0.2.** *The faces of  $E$  are exposed if and only if the faces of  $P$  are exposed.*

*Proof.* By the above Theorem, the map  $\sigma \mapsto K^{\sigma^\perp} \cdot \sigma$  induces a bijection between  $\mathcal{F}(P)/\mathcal{N}$  and  $\mathcal{F}(E)/K$ . Hence, keeping in mind that if  $F_1 = k F_2$ , then  $F_1$  is exposed if and only if  $F_2$ , the result follows from Proposition 1.4.  $\square$

**Remark 2.1.** *We have proven Theorems 0.1 and 0.2 under the assumption that  $G$  is a connected real semisimple Lie group. From this it follows that both theorems hold true for any connected compatible subgroup of  $U^\mathbb{C}$ , since such a subgroup is real reductive in the sense of [16, p. 446] and thus it is the product of a semisimple connected subgroup and an abelian subgroup, see e.g. [16, p. 453].*

### 3. CONVEX HULL OF THE GRADIENT MAP IMAGE

Let  $U$  be a compact connected Lie group and  $U^\mathbb{C}$  its complexification. Let  $(Z, \omega)$  be a Kähler manifold on which  $U^\mathbb{C}$  acts holomorphically. Assume

that  $U$  acts in a Hamiltonian fashion with momentum map  $\mu : Z \rightarrow \mathfrak{u}^*$ . Let  $G \subset U^{\mathbb{C}}$  be a closed connected subgroup of  $U^{\mathbb{C}}$  which is compatible with respect to the Cartan decomposition of  $U^{\mathbb{C}}$ . This means that  $G$  is a closed subgroup of  $U^{\mathbb{C}}$  such that  $G = K \exp(\mathfrak{p})$ , where  $K = U \cap G$  and  $\mathfrak{p} = \mathfrak{g} \cap i\mathfrak{u}$  [13, 15]. The inclusion  $i\mathfrak{p} \hookrightarrow \mathfrak{u}$  induces by restriction a  $K$ -equivariant map  $\mu_{i\mathfrak{p}} : Z \rightarrow (i\mathfrak{p})^*$ . Using a fixed  $U$ -invariant scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{u}$ , we identify  $\mathfrak{u} \cong \mathfrak{u}^*$ . We also denote by  $\langle \cdot, \cdot \rangle$  the scalar product on  $i\mathfrak{u}$  such that multiplication by  $i$  be an isometry of  $\mathfrak{u}$  onto  $i\mathfrak{u}$ . For  $z \in Z$  let  $\mu_{\mathfrak{p}}(z) \in \mathfrak{p}$  denote  $-i$  times the component of  $\mu(z)$  in the direction of  $i\mathfrak{p}$ . In other words we require that  $\langle \mu_{\mathfrak{p}}(z), \beta \rangle = -\langle \mu(z), i\beta \rangle$ , for any  $\beta \in \mathfrak{p}$ . Then we view  $\mu_{i\mathfrak{p}}$  as a map

$$\mu_{\mathfrak{p}} : Z \rightarrow \mathfrak{p},$$

which is called the  $G$ -gradient map or restricted momentum map associated to  $\mu$ . For the rest of the paper we fix a  $G$ -stable compact subset  $X \subset Z$  and we consider the gradient map  $\mu_{\mathfrak{p}} : X \rightarrow \mathfrak{p}$  restricted on  $X$ . We also set

$$\mu_{\mathfrak{p}}^{\beta} := \langle \mu_{\mathfrak{p}}, \beta \rangle = \mu^{-i\beta}.$$

We will now study the convex hull of  $\mu_{\mathfrak{p}}(X)$ , that we denote by  $E$ . Let  $\mathfrak{a} \subset \mathfrak{p}$  be a maximal abelian subalgebra of  $\mathfrak{p}$  and let  $\pi : \mathfrak{p} \rightarrow \mathfrak{a}$  be the orthogonal projection onto  $\mathfrak{a}$ . Then  $\pi \circ \mu_{\mathfrak{p}} =: \mu_{\mathfrak{a}}$  is the gradient map associated to  $A = \exp(\mathfrak{a})$ . Let  $Z^A$  be the set of fixed points of  $A$ . We note that  $\mu_{\mathfrak{a}}$  is locally constant on  $Z^A$  since  $\text{Ker } d\mu_{\mathfrak{a}}(x) = (\mathfrak{a} \cdot x)^{\perp}$  (see [15]). Let  $\mathfrak{b}$  a subspace of  $\mathfrak{a}$  and let  $X^{\mathfrak{b}} = \{p \in X : \xi_X(p) = 0 \text{ for all } \xi \in \mathfrak{b}\}$ , where  $\xi_X$  is the vector field induced by the  $A$  action on  $X$ . Then the map  $\mu_{\mathfrak{b}} : X^{\mathfrak{b}} \rightarrow \mathfrak{b}$ , that is the composition of  $\mu_{\mathfrak{p}}$  with the orthogonal projection onto  $\mathfrak{b}$ , is locally constant ([11], Section 3). Since  $X^{\mathfrak{b}}$  is compact,  $\mu_{\mathfrak{b}}(X^{\mathfrak{b}})$  is a finite set. In [11] it also shown that for any  $y \in X^{(\mathfrak{b})} := \{p \in X : \mathfrak{a}_p = \mathfrak{b}\}$ , where  $\mathfrak{a}_p := \{\xi \in \mathfrak{a} : \xi_X(p) = 0\}$ , we have that  $\mu_{\mathfrak{a}}(A \cdot y) \subset \mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$  is an open subset of the affine space  $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$  (the orthogonal complements are taken in  $\mathfrak{a}$ ). Moreover  $\mu_{\mathfrak{a}}(A \cdot y)$  is a convex subset of  $\mu_{\mathfrak{a}}(y) + \mathfrak{b}^{\perp}$  (see [10]) and therefore  $\mu_{\mathfrak{a}}(\overline{A \cdot y}) = \overline{\mu_{\mathfrak{a}}(A \cdot y)}$  is a convex body.

Let  $P := \mu_{\mathfrak{a}}(X)$ . If  $\beta \in \mu_{\mathfrak{a}}(X)$  is an extremal point of  $P$ , and  $y \in \mu_{\mathfrak{a}}^{-1}(\beta)$ , then  $\mu_{\mathfrak{a}}(A \cdot y)$  is an open neighborhood of  $\mu_{\mathfrak{a}}(y)$  in  $\mu_{\mathfrak{a}}(y) + \mathfrak{a}_y^{\perp}$  and it is contained in  $\mu_{\mathfrak{a}}(X) \subset P$ . Since  $\mu_{\mathfrak{a}}(y)$  is an extremal point, it follows that  $\mathfrak{a}_y^{\perp} = \{0\}$  and so  $y$  is a fixed point of  $A$ . Since  $X$  is compact, the set  $X^A$  has finitely many connected components. Therefore  $P$  has finitely many extremal points, i.e. it is a polytope. We have shown the following.

**Proposition 3.1.** *Let  $X \subset Z$  be a  $G$ -invariant compact subset of  $Z$ . Then the image  $\mu_{\mathfrak{a}}(X^A)$  is a finite set  $\{c_1, \dots, c_p\}$  and  $P = \overline{\mu_{\mathfrak{a}}(X)}$  is the convex hull of  $c_1, \dots, c_p$ .*

As a corollary we get the following result.

**Theorem 0.3.** *Let  $X \subset Z$  be a  $G$ -invariant compact subset of  $Z$ . Then every face of  $E = \widehat{\mu_{\mathfrak{p}}(X)}$  is exposed.*

*Proof.* Since

$$\pi(E) = \pi(\widehat{\mu_{\mathfrak{p}}(X)}) = \widehat{\mu_{\mathfrak{a}}(X)},$$

by Lemma 1.1 (i) we conclude that  $E \cap \mathfrak{a} = \pi(E) = P$  and by Proposition 3.1, Remark 2.1 and Theorem 0.2 we get that every face of  $E$  is exposed.  $\square$

We call  $P$  the *momentum polytope*. If  $G = U^{\mathbb{C}}$  and  $X$  is a complex connected submanifold of  $Z$ , then  $P = \mu_{\mathfrak{a}}(X)$  by the Atiyah-Guillemin-Sternberg convexity theorem [1, 8]. The same holds for  $X$  an irreducible semi-algebraic subset of a Hodge manifold  $Z$  [17, 11, 5].

Since any proper face  $F$  of  $E$  is exposed, the set  $C_F$  defined in (1) is a non-empty convex cone in  $\mathfrak{p}$ . Set

$$K^F := \{g \in K : g \cdot F = F\}.$$

By Proposition 5 in [3] we have  $C_F^{K^F} := \{\beta \in C_F : K^F \cdot \beta = \beta\} \neq \emptyset$ . This means that for a proper face  $F$  one can find a  $K^F$ -invariant vector  $\beta$  such that  $F_{\beta}(E) = F$ . For  $\beta \in \mathfrak{p}$ , denote by  $X^{\beta}$  the set of points of  $X$  that are fixed by  $\exp(\mathbb{R}\beta)$ . If  $\beta \in C_F$ , let

$$X_{\max}^{\beta} := \{x \in X : \mu_{\mathfrak{p}}^{\beta}(x) = \max_X \mu_{\mathfrak{p}}^{\beta}\}.$$

Since the function  $\mu_{\mathfrak{p}}^{\beta}$  is  $K^{\beta}$ -invariant the set  $X_{\max}^{\beta}$  is  $K^{\beta}$ -invariant. Moreover  $X_{\max}^{\beta}$  is a union of finitely many connected components of  $X^{\beta}$  and  $X^{\beta}$  is  $G^{\beta}$ -stable. Every connected component of  $G^{\beta}$  meets  $K^{\beta}$ . This implies that  $G^{\beta}$  leaves  $X_{\max}^{\beta}$  invariant. Next we show that  $X_{\max}^{\beta}$  does not depend on the choice of  $\beta$  in  $C_F$ .

**Lemma 3.1.** *If  $\beta \in C_F$ , then  $X_{\max}^{\beta} = \mu_{\mathfrak{p}}^{-1}(F)$ . Moreover  $F$  is the convex hull of  $\mu_{\mathfrak{p}}(X_{\max}^{\beta})$ .*

*Proof.* Fix  $x \in X$ . Then  $\mu_{\mathfrak{p}}(x) \in F$  if and only if  $\langle \mu_{\mathfrak{p}}(x), \beta \rangle = \max_{v \in E} \langle v, \beta \rangle$ . Moreover  $\max_{v \in E} \langle v, \beta \rangle = \max_{v \in \mu_{\mathfrak{p}}(X)} \langle v, \beta \rangle = \max_X \mu_{\mathfrak{p}}^{\beta}$ . So  $x \in \mu_{\mathfrak{p}}^{-1}(F)$  if and only if  $x$  is a maximum of  $\mu_{\mathfrak{p}}^{\beta}(x)$  restricted to  $X$ . This shows that  $X_F^{\beta} = \mu_{\mathfrak{p}}^{-1}(F)$ . The inclusion  $\mu_{\mathfrak{p}}(X_F^{\beta}) \subset F$  follows from the definition and therefore  $\widehat{\mu_{\mathfrak{p}}(X_F^{\beta})} \subset F$ . By [3, Lemma 3]  $\widehat{\text{ext } F} = \text{ext } E \cap F$ , so  $\text{ext } F \subset \widehat{\mu_{\mathfrak{p}}(X) \cap F} = \widehat{\mu_{\mathfrak{p}}(X_F^{\beta})}$ . It follows that  $F = \widehat{\mu_{\mathfrak{p}}(X_F^{\beta})}$ .  $\square$

Motivated by the above Lemma we set  $X_F := X_{\max}^{\beta}$  where  $\beta$  is any vector in  $C_F$ . We also set

$$Q^F = \{g \in G : g \cdot X_F = X_F\}.$$

$Q^F$  is a closed Lie subgroup of  $G$ .

Given  $\beta \in \mathfrak{p}$  define the following subgroups:

$$\begin{aligned} G^{\beta+} &= \{g \in G : \lim_{t \rightarrow -\infty} \exp(t\beta)g \exp(-t\beta) \text{ exists}\}, \\ G^{\beta-} &= \{g \in G : \lim_{t \rightarrow +\infty} \exp(-t\beta)g \exp(t\beta) \text{ exists}\}, \\ R^{\beta+} &= \{g \in G : \lim_{t \rightarrow -\infty} \exp(t\beta)g \exp(-t\beta) = e\}, \\ R^{\beta-} &= \{g \in G : \lim_{t \rightarrow +\infty} \exp(-t\beta)g \exp(t\beta) = e\}. \end{aligned}$$

$G^{\beta+}$  (respectively  $G^{\beta-}$ ) is a parabolic subgroup,  $R^{\beta+}$  (respectively  $R^{\beta-}$ ) is its unipotent radical and  $G^\beta$  is a Levi factor. Therefore  $G^{\beta+} = G^\beta \rtimes R^{\beta+}$  (respectively  $G^{\beta-} = G^\beta \rtimes R^{\beta-}$ ).

**Lemma 3.2.**  $Q^F \cap K = K^F$ .

*Proof.* If  $g \in Q^F \cap K$ , then  $g \cdot X_F = X_F$ . Since  $\mu_{\mathfrak{p}}$  is a  $K$ -invariant map,  $g \cdot \mu_{\mathfrak{p}}(X_F) = \mu_{\mathfrak{p}}(X_F)$ . Taking the convex hull of both sides and using Lemma 3.1 we get that  $g \cdot F = F$ , thus  $g \in K^F$ . Conversely, if  $g \in K^F$ , the equivariance of  $\mu_{\mathfrak{p}}$  yields  $X_F = \mu_{\mathfrak{p}}^{-1}(F) = \mu_{\mathfrak{p}}^{-1}(g \cdot F) = gX_F$ , thus  $g \in Q^F$ .  $\square$

We are now ready to prove the connection between the set of the faces of  $E$  and parabolic subgroups of  $G$ .

**Proposition 3.2.**  $Q^F$  is a parabolic subgroup of  $G$ . Moreover  $Q^F = G^{\beta+}$  for every  $\beta \in C_F^{K^F}$ .

*Proof.* Observe that by definition  $Q^F$  is a closed subgroup of  $G$ . Let  $\beta \in C_F^{K^F}$ . Then  $F = F_\beta(E)$  and, by definition of  $K^F$ , we get  $K^F = K^\beta$ . The set  $X_F = \{x \in X : \mu_{\mathfrak{p}}^\beta(x) = \max_X \mu_{\mathfrak{p}}^\beta\}$  is  $G^\beta$ -stable. Fix  $p \in X_F$  and consider the orbit  $G \cdot p$ , which is a smooth submanifold contained in  $X$ . By Proposition 2.5 in [13] (see also Proposition 2.1 in [4]) we get that  $\xi_X(x) = 0$  for any  $\xi \in \mathfrak{r}^{\beta+}$  and for any  $x \in X_F$ . Therefore  $G^{\beta+} \cdot p \subset X_F$ . Hence  $G^{\beta+} \subset Q^F$  and the Lie algebra  $\mathfrak{q}^F$  of  $Q^F$  is parabolic. On the other hand by Lemma 3.2, we have  $\mathfrak{q}^F \cap \mathfrak{k} = \mathfrak{g}^{\beta+} \cap \mathfrak{k} = \mathfrak{k}^\beta$  and so by Lemma 3.7 [4] we conclude that  $\mathfrak{q}^F = \mathfrak{g}^{\beta+}$ . Since  $Q^F \subset N_G(\mathfrak{g}^{\beta+}) = G^{\beta+}$  we get  $Q^F = G^{\beta+}$ .  $\square$

**Remark 3.1.** If  $\beta' \in C_F^{K^F}$ , then  $Q_{F'} = G^{\beta'+} = G^{\beta+}$ . By Lemma 2.8 in [4], we have  $[\beta, \beta'] = 0$ ,  $G^\beta = G^{\beta'}$  and  $R^{\beta+} = R^{\beta'+}$ .

Let  $Q^{F-} = \Theta(Q^F)$ , where  $\Theta : G \rightarrow G$  denotes the Cartan involution. The subgroup  $Q^{F-}$  is parabolic and depends only on  $F$ . The subgroup  $L^F := Q^F \cap Q^{F-}$  is a Levi factor of both  $Q^F$  and  $Q^{F-}$ . Let  $\beta \in C_F^{K^F}$ . Then  $Q^F = G^{\beta+}$ ,  $L^F = G^\beta$  and we have the projection

$$\pi^{\beta+} : G^{\beta+} \rightarrow G^\beta, \quad \pi^{\beta+}(g) = \lim_{t \rightarrow +\infty} \exp(t\beta)g \exp(-t\beta),$$

respectively

$$\pi^{\beta+} : G^{\beta-} \longrightarrow G^{\beta}, \quad \pi^{\beta-}(g) = \lim_{t \rightarrow -\infty} \exp(t\beta)h \exp(-t\beta).$$

**Lemma 3.3.** *If  $\beta \in C_F^{K^F}$ , then the projections  $\pi^{\beta+}$  and  $\pi^{\beta-}$  depend only on  $F$ .*

*Proof.* Let  $g \in G^{\beta+}$ . We know that  $g = hr$ , where  $h \in G^{\beta}$  and  $r \in R^{\beta+}$ . Then

$$\pi^{\beta+}(g) = \lim_{t \rightarrow +\infty} \exp(t\beta)g \exp(-t\beta) = h \lim_{t \rightarrow +\infty} \exp(t\beta)r \exp(-t\beta) = h.$$

Since  $G^{\beta} = G^{\beta'}$  and  $R^{\beta+} = R^{\beta'+}$  the decomposition  $g = hr$  is the same for both groups and  $\pi^{\beta+}(g) = \pi^{\beta'+}(g)$ . The same argument works for  $\pi^{\beta-}$ .  $\square$

Now assume that  $X$  is a  $G$ -stable compact submanifold of  $Z$ .

For  $\beta \in C_F^{K^F}$  set  $X_F^{\beta-} := \{p \in X : \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \in X_F\}$ . Then the map

$$p^{\beta-} : X_F^{\beta-} \longrightarrow X_F, \quad p^{\beta-}(x) = \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot x \quad (2)$$

is well-defined,  $G^{\beta}$ -equivariant, surjective and its fibers are  $R^{\beta-}$ -stable.

More generally one can consider  $p^{\beta-}$  as a map from  $X^{\beta-} = \{y \in X : \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot y \text{ exists}\}$  to  $X^{\beta}$ . In general however this map is not even continuous [14, Example 4.2]. To ensure continuity and smoothness it is enough that the topological Hilbert quotient  $X^{\beta-}/G^{\beta}$  exists. Using the notation of [14] and choosing  $r = \max_X \mu_p^{\beta}$ , we have  $X_F = X_{\max}^{\beta} = X_r^{\beta}$  and  $X_r^{\beta-} = X_F^{\beta-}$ . Thus Prop. 4.4 of [14] applies and yields that  $X_F^{\beta-}$  is an open  $G^{\beta-}$ -stable subset of  $X$  and that (2) is smooth deformation retraction onto  $X_F$ . Using  $\pi^{\beta-}$  one defines an action of  $Q^{F-} = G^{\beta-}$  on  $X_F$  by setting  $g \cdot x = \pi^{\beta-}(g) \cdot x$ . This just depends on  $F$ . With respect to this action the map  $p^{\beta-}$  becomes  $Q^{F-}$ -equivariant.

**Lemma 3.4.** *The set  $X_F^{\beta-}$  and the map  $p^{\beta-}$  do not depend on the choice of  $\beta \in C_F^{K^F}$ .*

*Proof.* Set  $\Gamma = \exp(\mathbb{R}\beta)$ . If  $p \in X_F$  by the Slice Theorem [13, Thm. 3.1] there are open neighborhoods  $S_p \subset T_p X$  and  $\Omega_p \subset X$  and a  $\Gamma$ -equivariant diffeomorphism  $\Psi_p : S_p \longrightarrow \Omega_p$ , such that  $0 \in S_p$ ,  $p \in \Omega_p$ ,  $\Psi_p(0) = p$ . Since  $p$  is a maximum of  $\mu_p^{\beta}$  restricted to  $X$ , the following orthogonal splitting  $T_p X = V_0 \oplus V_-$  with respect to the Hessian of  $\mu_p^{\beta}$  holds. Here  $V_0$  denotes the kernel of the Hessian of  $\mu_p^{\beta}$  and  $V_-$  denotes the sum of eigenspaces of the Hessian of  $\mu_p^{\beta}$  corresponding to negative eigenvalues. We also point out that  $V_0 = T_p X_F$  and  $S_p = \{x_0 + x_- : x_0 \in S_p \cap V_0, x_- \in V_-\}$ , see [15]. It follows that  $\Omega_p \subset X_F^{\beta-}$ . Set  $\Omega := \bigcup_{p \in X_F} \Omega_p$ . By what we just proved,  $\Omega \subset X_F^{\beta-}$ . On the other hand  $\Omega$  is an open  $\Gamma$ -invariant neighbourhood of  $X_F$ , so  $X_F^{\beta-} \subset \Omega$ . So  $X_F^{\beta-} = \Omega$ . If  $\beta'$  is another vector of  $C_F^{K^F}$ , set

$B = \exp(\mathbb{R}\beta \oplus \mathbb{R}\beta')$ . This is a compatible abelian subgroup and  $X_F \subset X^B$ . So we may choose the open subsets  $\Omega_p$  above to be  $B$ -stable. Therefore we get  $X^{\beta'} = \Omega$  as well. This proves that  $X_F^{\beta^-} = X_F^{\beta'^-}$ .

Next we show that  $p^{\beta^-} = p^{\beta'^-}$ . First observe that  $p^{\beta^-}(y) = p^{\beta'^-}(y)$  if  $y \in \Omega$ . Indeed if  $y \in \Omega_p$  we can study the limit using the diffeomorphism  $\Psi_p : S_p \rightarrow \Omega_p$ . The decomposition  $T_p X = V_0 \oplus V_-$  is the same for  $\beta$  and  $\beta'$  since they commute and attain their maxima on  $X_F$ . Therefore if  $x = \Psi_p^{-1}(y) = x_0 + x_-$ , then

$$p^{\beta^-}(y) = \Psi_p(x_0) = p^{\beta'^-}(y). \quad (3)$$

If  $p \in X_F^{\beta^-}$  and  $q = \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \in X_F$ , there is  $t_1 \in \mathbb{R}$ , such that  $\exp(t\beta) \cdot p \in \Omega$ . Therefore

$$\begin{aligned} \lim_{t \rightarrow +\infty} \exp(t\beta') \cdot p &= \lim_{t \rightarrow +\infty} \exp(t\beta')(\exp(t_1\beta') \cdot p) \\ &= \lim_{t \rightarrow +\infty} \exp(t\beta)(\exp(t_1\beta') \cdot p) \quad (\text{by 3}) \\ &= \exp(t_1\beta') \left( \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p \right) \\ &= \lim_{t \rightarrow +\infty} \exp(t\beta) \cdot p. \end{aligned}$$

□

By the above Lemma if  $F$  is a face and  $\beta \in C_F^{K^F}$ , we can set  $X_F^- := X_F^{\beta^-}$  and  $p^{F^-} := p^{\beta^-} : X_F^- \rightarrow X_F$ .

**Theorem 3.1.** *For any face  $F \subset E$ , the set  $X_F$  is closed and  $L^F$ -stable,  $X_F^-$  is an open  $Q^{F^-}$ -stable neighborhood of  $X_F$  in  $X$  and the map  $p^{F^-}$  is a smooth  $Q^{F^-}$ -equivariant deformation retraction of  $X_F^-$  onto  $X_F$ .*

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